Transilvania University of Brasov

Beyond Riemann: Finsler Spacetimes and the Geometry of Cosmic Acceleration

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Contents

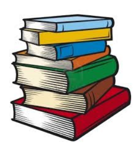
Talk based mainly on:

[1] M. Hohmann, C. Pfeifer, N. Voicu, *Mathematical foundations for field theories on Finsler spacetimes*, J. Math. Phys 63, 032503 (2022).

[2] M. Hohmann, C. Pfeifer, N. Voicu, *Relativistic kinetic gases as direct sources of gravity*, Phys. Rev. D 101, 024062 (2020).

[3] M. Hohmann, C. Pfeifer, N. Voicu, *Finsler gravity action from variational completion*, Phys. Rev. D 100, 064035 (2019).

[4] C. Pfeifer, N. Voicu, A. Friedl-Szász, E. Popovici-Popescu, *From kinetic gasses to an exponentially expanding universe*, JCAP 10, 050 (2025).



1 Finsler: Why, for Physics' sake?

Geometric alternative to dark energy/dark matter.

Example (this talk): Natural Finslerian model for gravitating **kinetic gas:**

exponentially expanding vacuum universe, no need for dark energy

Other reasons:

Metric-affine gravity theories with variational autoparallels.

✓ Minimal extension of Ehlers-Pirani-Schild axioms

✓ Quantum gravity phenomenology (*modified dispersion relations*)

✓ Description of wave propagation in media

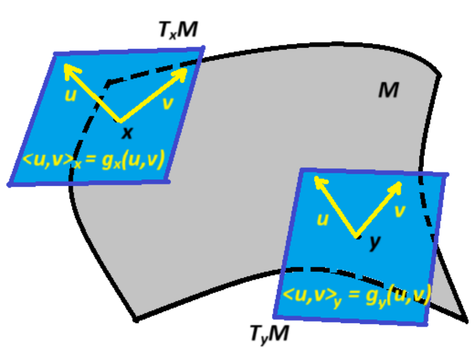
2 Finsler spacetime geometry in a nutshell

2.1 The intuitive idea of Finsler space

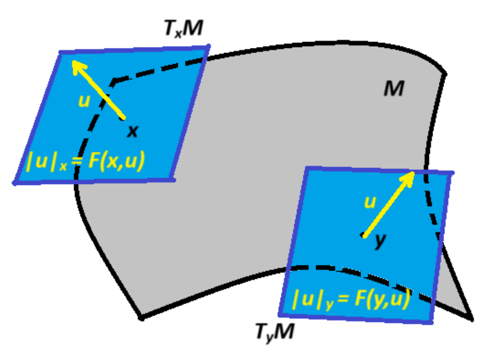
$\longrightarrow$ a smooth manifold M, with a **generalized notion of arc length:**

$$ds = F\left(x, dx\right).$$

Equivalently: each T_xM - equipped with a '*norm*' of vectors:



Riemann



Finsler

2.2 A minimalist definition of Finsler spacetimes

Denote M - smooth manifold, $\overset{\circ}{TM} := TM \backslash \{0\}$.

Finsler spacetime:$=$ a pair (M, L), where M - 4-dim. smooth manifold and

$$L = L(x, \dot{x}) : \mathcal{T} \to \mathbb{R}_{+}^{*},$$

where $\mathcal{T} \subset \overset{\circ}{TM}$ (*conic subbundle** with connected fibers), is a smooth map s.th.:

(i) $L(x, \alpha\dot{x}) = \alpha^2 L(x, \dot{x})$, $\forall \alpha > 0$;

(ii) $g_{\mu\nu}(x, \dot{x}) = \dfrac{1}{2}\dfrac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu}(x, \dot{x})$ has $(+, -, -, -)$ signature;

(iii) $L_{|\partial \mathcal{T}} = 0$ - continuous prolongation.

**Conic subbundle*:=an open subset $\mathcal{T} \subset \overset{\circ}{TM}$ with non-empty fibers $\mathcal{T}_x = \mathcal{T} \cap T_x M$ and such that $(x, \dot{x}) \in \mathcal{T} \Rightarrow (x, \alpha\dot{x}) \in \mathcal{T}$, $\forall \alpha > 0$.

Interpretation: $\mathcal{T}$ - set of **future-directed timelike vectors**.

Note: L is generally defined and smooth on a larger conic subset $\mathcal{A} \subset TM$.
But: L - not necessarily > 0, g - not necessarily Lorentzian outside $\mathcal{T}$.

Arc length element:

$$ds^2 = L(x, dx) = g_{\mu\nu}(x, \dot{x})\, dx^\mu dx^\nu. \tag{1}$$

Length of a curve $\gamma : [a, b] \to M$:

$$\ell(\gamma) = \int_\gamma \sqrt{|L(\gamma(t), \dot{\gamma}(t))|}dt, \tag{2}$$

independent on the choice of the parameter t.

$$\boxed{\text{Finsler geometry - most general notion of geometric clock:}}$$

$$\mathbf{ds = cd\boldsymbol{\tau}}. \tag{3}$$

Most common examples:

$\surd$ **(Pseudo-)Riemannian = quadratic in $\dot{x}$:**

$$L(x,\dot{x}) = a_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}$$

$\surd$ *Randers* ($\rightarrow$ linearized perturbations around Riemannian background, GW):

$$L(x,\dot{x}) = \sqrt{|a_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}|} + b_{\rho}(x)\dot{x}^{\rho}$$

$\surd$ *Bogoslovsky/Kropina (Very Special Relativity - Cohen&Glashow):*

$$L(x,\dot{x}) = |a_{\mu\nu}(x)\dot{x}^{\mu}\dot{x}^{\nu}|^{1-q}(b_{\rho}(x)\dot{x}^{\rho})^{2q}$$

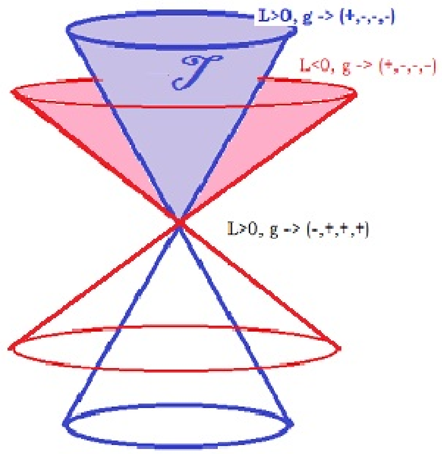
$\surd$ 4-*th root* ($\rightarrow$ *birefringence - Pfeifer&Wohlfarth, Perlick etc.):*

$$L(x,\dot{x}) = \sqrt{|(a_{\mu\nu}(x)\,\dot{x}^{\mu}\dot{x}^{\nu})\,(h_{\rho\sigma}(x)\,\dot{x}^{\rho}\dot{x}^{\sigma})|}$$

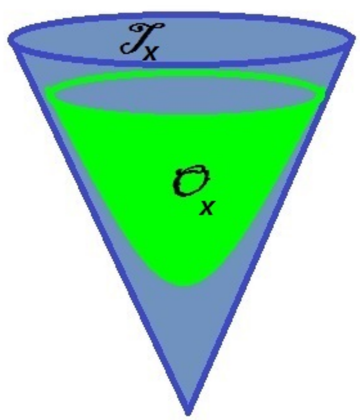
"Finsler geometry is Riemannian geometry without the quadratic restriction"

✓ **Set of normalized unit timelike vectors ("mass shell"/"observer space"):**

$$\mathcal{O} = \{(x, \dot{x}) \in \mathcal{T} \mid L(x, \dot{x}) = 1\} . \tag{4}$$



future-pointing timelike cone (blue)



timelike cone and observer space

2.3 Geodesics, canonical nonlinear connection, curvature

Geodesic equations of (M, L) :

$$\begin{aligned} \ddot{\gamma}^{\mu}(s) + 2G^{\mu}(\gamma(s), \dot{\gamma}(s)) &= 0 \quad (5) \\ &\Leftrightarrow \\ \ddot{\gamma}^{\mu}(s) + G^{\mu}{}_{\nu}(\gamma(s), \dot{\gamma}(s))\dot{\gamma}^{\nu}(s) &= 0. \quad (6) \end{aligned}$$

where $G^{\mu} = G^{\mu}(x, \dot{x})$, $G^{\mu}{}_{\nu} = G^{\mu}{}_{\nu}(x, \dot{x})$ - defined by:

$$2G^{\mu} := \frac{1}{2} g^{\mu\nu} \left(\dot{x}^{\rho} \frac{\partial^2 L}{\partial \dot{x}^{\nu} \partial x^{\rho}} - \frac{\partial L}{\partial x^{\nu}} \right), \qquad G^{\mu}{}_{\nu} = \frac{\partial G^{\mu}}{\partial \dot{x}^{\nu}}. \quad (7)$$

Proposition (Characterization of Finslerian geodesics):

$\gamma : [a,b] \to M, \quad s \mapsto \gamma(s)$ is a geodesic $\Leftrightarrow$ any of the following holds:

1. Its extension $(\gamma, \dot{\gamma})$ to TM is an integral curve of the **canonical spray**

$$S = \dot{x}^\mu \partial_\mu - 2G^\mu \dot{\partial}_\mu \in \mathcal{X}(\mathcal{A}) \,. \tag{8}$$

2. The extension $(\gamma, \dot{\gamma})$ is an autoparallel (horizontal) curve w.r.t. the **canonical nonlinear connection** N**:**

$$T\mathcal{A} = H\mathcal{A} \oplus V\mathcal{A} \tag{9}$$

with local coeffs $G^\mu{}_\nu$.

Local adapted basis $\{\delta_\mu, \dot{\partial}_\mu\}$ for $\mathcal{X}(\mathcal{A})$ - given by the *horizontal derivatives:*

$$\delta_\mu := \partial_\mu - G^\nu{}_\mu \dot{\partial}_\nu. \tag{10}$$

✓ **Particular case:** If L - **Riemannian** $\Rightarrow$

$$2G^\mu = \overset{\circ}{\Gamma}{}^\mu{}_{\rho\nu}(x)\,\dot{x}^\rho \dot{x}^\nu \tag{11}$$

$$G^\mu{}_\rho = \overset{\circ}{\Gamma}{}^\mu{}_{\rho\nu}(x)\,\dot{x}^\nu \tag{12}$$

✓ Generally: $G^\mu{}_\nu = G^\mu{}_\nu(x, \dot{x})$ - **nonlinear** in $\dot{x}$.

Geodesic deviation equation

$$(\nabla\nabla V^h)_{|(\gamma,\dot{\gamma})} = \mathbf{R}(\dot{\gamma}^h, V^h), \tag{13}$$

where ∇ - **dynamical covariant derivative:**

$$\nabla : \mathcal{X}^h(\mathcal{A}) \to \mathcal{X}^h(\mathcal{A}), \quad \nabla(X^\mu \delta_\mu) = (S(X^\mu) + G^\mu{}_\nu X^\nu)\,\delta_\mu \tag{14}$$

Finsler-Ricci scalar: $R := trace(\mathbf{R}\left(\dot{\gamma}^h, \cdot\right))$

$$R := R^\mu{}_{\mu\rho}(x, \dot{x})\,\dot{x}^\rho \tag{15}$$

Particular case: If L - Riemannian $\Rightarrow$

$$\begin{aligned} R^\mu{}_{\nu\rho} &= \overset{\circ}{R}_\tau{}^\mu{}_{\nu\rho}(x)\dot{x}^\tau \\ R &= \overset{\circ}{R}_{\tau\rho}(x)\dot{x}^\tau\dot{x}^\rho \quad \rightsquigarrow \quad Raychaudhuri\ eq. \end{aligned}$$

2.4 Non-Riemannian quantities

Q1. How non-Riemannian (non-quadratic in $\dot{x}$) is L?

Cartan tensor $C : \mathcal{A} \to T^0_3 M,\ C = C_{\mu\nu\rho}(x, \dot{x})\, dx^\mu \otimes dx^\nu \otimes dx^\rho$:

$$C_{\mu\nu\rho}(x, \dot{x}) := \frac{1}{2}\frac{\partial g_{\mu\nu}(x, \dot{x})}{\partial \dot{x}^\rho} = \frac{1}{4}\frac{\partial^3 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu \partial \dot{x}^\rho}.$$

Main property:

$$L - quadratic\ (Riemannian) \Leftrightarrow C = 0$$

Q2: Canonical connection of L : How nonlinear is it?

That is: How $\dot{x}$-dependent are

$$G^{\mu}{}_{\nu\rho}(x,\dot{x}) = \frac{\partial G^{\mu}{}_{\nu}}{\partial \dot{x}^{\rho}}(x,\dot{x})?$$

Berwald tensor $B : \mathcal{A} \to T^1_3 M$, with local comps:

$$B^{\mu}{}_{\nu\rho\tau} = \frac{\partial G^{\mu}{}_{\nu\rho}}{\partial \dot{x}^{\tau}}$$

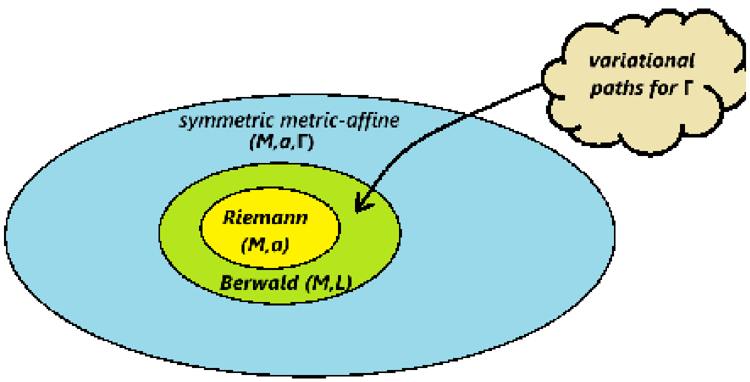
Berwald (affinely connected) spaces: $B = 0$, i.e.

$$G^{\mu}{}_{\nu\rho}(x,\dot{x}) = G^{\mu}{}_{\nu\rho}(x) =: \Gamma^{\mu}{}_{\nu\rho}(x)\,,$$

i.e., G descends into a *symmetric affine connection* Γ *on spacetime*.

Q2': If (M, a, Γ) - metric affine, when is it Finsler-Berwald metrizable?

Finsler metrizability <–> parametrization-invar. var. principle for paths of Γ



Necessary and sufficient conds.+precise form of L known for:

✓ $SO(3)$-invariant connections

✓ affine connections with vectorial nonmetricity

Q3: How different are local (T_xM) geometries from one another?

Landsberg tensor

$$P = \nabla C : \mathcal{A} \to T^0_3 M$$

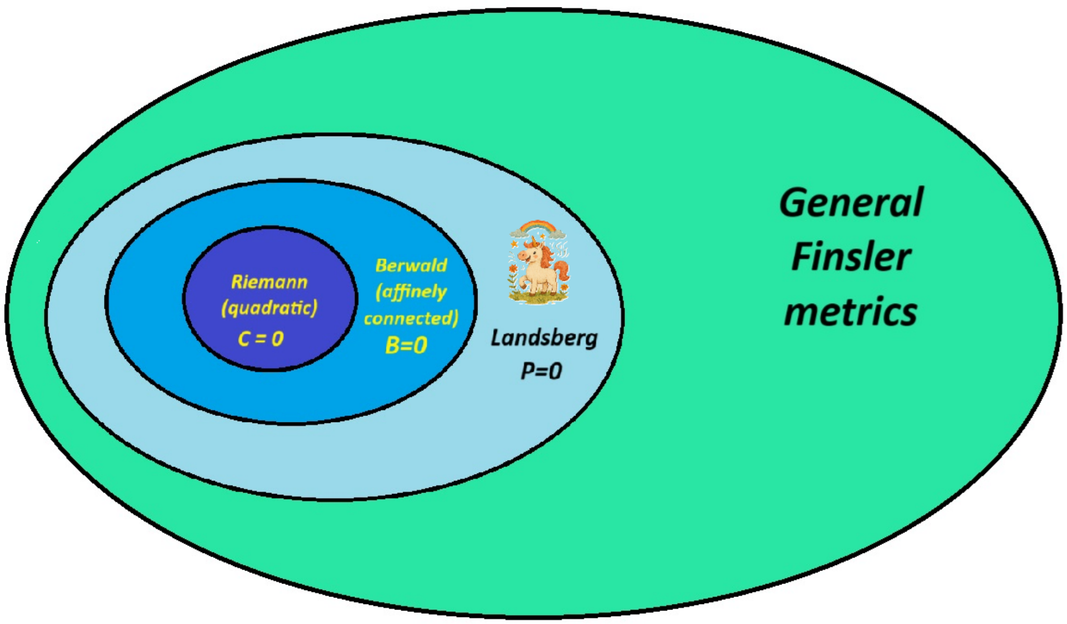
Intuitively: $P \simeq$ rate of change of C as one moves along geodesics of (M, L).

Landsberg spaces - defined by:

$$P = 0. \tag{16}$$

✓ L - Landsberg $\Rightarrow$ tangent spaces $T_xM \simeq T_yM$ are **isometric**

Geometric ladder - from Riemann to general Finsler:



3 Finsler-based field theories - and a concrete model

3.1 What are Finslerian geometric objects?

Denote: $\overset{\circ}{TM} := TM \backslash \{0\}$.

✓ $L, g, C, B, P, ...$ - are all sections of $\pi^*_{TM}(T^p_q(M))$ - i.e., mappings

$$\mathbb{T} : \mathcal{A} \to T^p_q(M); \quad (x, \dot{x}) \mapsto \mathbb{T}_{(x,\dot{x})} \in T^p_q(M). \tag{17}$$

✓ They are all homogeneous of some degree in $\dot{x}$, i.e., **equivariant** under

$$\chi_\alpha : \overset{\circ}{TM} \to \overset{\circ}{TM}, \quad \chi_\alpha(x, \dot{x}) = (x, \alpha\dot{x}) \qquad (\alpha > 0)$$

$=$ flow of the *Liouville vector field*

$$\mathbb{C} = \dot{x}^\mu \dot{\partial}_\mu \in \mathcal{X}(\mathcal{A}) \tag{18}$$

✓ The canonical connection N - also homogeneous.

Main task: construct general configuration bundles (Y, Π, X), allowing

- k-homogeneous Finslerian geometric objects as sections**;**

- well defined fibered automorphisms;

- compactly supported variations;

$\Rightarrow$ **Base space:**

$$X := PTM^{+}. \tag{19}$$

3.2 The projective sphere bundle PTM^+

1. On arbitrary manifolds M, dim $M = n$. Define:

$$PTM^+ := \overset{\circ}{TM}_{/\sim} \tag{20}$$

where:

$$(x, \dot{x}) \ \sim \ (x, u) \Leftrightarrow \exists \alpha > 0 : \ u = \alpha \dot{x}. \tag{21}$$

✓ PTM^+ - smooth, *orientable* $(2n-1)$-dim. manifold
✓ PTM^+ - natural bundle over M, with *fibers* $\simeq \mathbb{S}^{n-1}$.
✓ $(\overset{\circ}{TM}, \pi^+, PTM^+, \mathbb{R}^*_+)$ - principal bundle, with projection:

$$\pi^+ : \overset{\circ}{TM} \to PTM^+, \ (x, \dot{x}) \mapsto [(x, \dot{x})]. \tag{22}$$

✓ *Homogeneous local coords* of $[(x, \dot{x})]$: $(x^\mu, \dot{x}^\mu)$

2. **On Finsler spacetimes** (M, L)**:** The set of future-pointing *timelike directions:*

$$\mathcal{T}^{+} = \pi^{+}(\mathcal{T}) \subset PTM^{+}$$

has a **contact structure** - **the Hilbert form**

$$\omega^{+} = \dot{\partial}_{\mu} F dx^{\mu}, \qquad F = \sqrt{L}. \tag{23}$$

Canonical volume form:

$$d\mathbf{\Sigma}^{+} := -\frac{1}{3!}\omega^{+} \wedge (d\omega^{+})^{3} = \frac{|\det g|}{L^{2}} \mathbf{i}_{\mathbb{C}} (dx \wedge d\dot{x}) \tag{24}$$

Theorem, [1]: (Mass shell $\mathcal{O}$ vs. set of timelike directions $\mathcal{T}^{+}$):
1. $\pi^{+} : \mathcal{O} \to \mathcal{T}^{+}$ is a diffeomorphism.
2. If $\rho^{+} \in \Omega_{7}(\mathcal{T}^{+})$ - compactly supported and $\rho := \left(\pi^{+}\right)^{*} \rho^{+}$, then:

$$\int_{\mathcal{T}^{+}} \rho^{+} = \int_{\mathcal{O}} \rho. \tag{25}$$

Structure of fibered manifolds over PTM^+

Consider: (M, L) - Finsler spacetime, (Y, Π, PTM^+) - fibered manifold $\Rightarrow$

$$Y \xrightarrow{\Pi} PTM^+ \xrightarrow{\pi_M} M. \tag{26}$$

Fibered automorphisms of (Y, Π, PTM^+) :

$$\begin{tikzcd}
Y \arrow[r, "\Phi"] \arrow[d, "\Pi"'] & Y \arrow[d, "\Pi"] \\
PTM^+ \arrow[r, "\phi"] \arrow[d, "\pi_M"'] & PTM^+ \arrow[d, "\pi_M"] \\
M \arrow[r, "\phi_0"] & M
\end{tikzcd}$$

3.3 The configuration bundle Y

A *k-homogeneous (Finslerian) geometric object*= a local section:

$$\overset{\circ}{\gamma}\ : \mathcal{A} \to \overset{\circ}{Y}, \quad (x, \dot{x}) \mapsto (x, \dot{x}, y(x, \dot{x})),$$

of some fiber bundle $(\overset{\circ}{Y}, \overset{\circ}{\Pi}, T\overset{\circ}{M}, Z)$ obeying:

$$\overset{\circ}{\gamma}(x, \alpha\dot{x}) = \left(x, \alpha\dot{x}, \alpha^k y\right), \quad \forall\alpha > 0.$$

Necessary cond.: $\exists$ an *action* $H : \mathbb{R}_+^* \times \overset{\circ}{Y} \to \overset{\circ}{Y}$ by fibered automorphisms:

$$H(\alpha, \cdot) = H_\alpha \in Aut(\overset{\circ}{Y}), \qquad H_\alpha(x, \dot{x}, y) = \left(x, \alpha\dot{x}, \alpha^k y\right), \tag{27}$$

Then: k-homogeneity = equivariance:

$$H_\alpha \circ \overset{\circ}{\gamma} = \overset{\circ}{\gamma} \circ \chi_\alpha. \tag{28}$$

★ **Idea: "factor away" the action of $\mathbb{R}^*_+$ from both** $\overset{\circ}{Y}$ and $T\overset{\circ}{M}$ $\Rightarrow$

$$Y = \overset{\circ}{Y}_{/\sim} \tag{29}$$

✓ Y fiber bundle over PTM^+, with typical fiber Z and projection:

$$\Pi : Y \to PTM^+, \qquad \Pi\,[x, \dot{x}, y] = [x, \dot{x}]\,.$$

✓ *k-homogeneous* sections $\overset{\circ}{\gamma} : \mathcal{Q} \to \overset{\circ}{Y} \leftrightarrow$ local sections $\gamma : \pi^+(\mathcal{Q}) \to Y$.

$$\begin{tikzcd} V \times \mathbb{R}^*_+ \arrow[r] \arrow[d, "\overset{\circ}{\Pi}"'] & (U^+ \times \mathbb{R}^*_+) \times Z\,. \arrow[dl, "\mathrm{proj}_1"] \\ U^+ \times \mathbb{R}^*_+ & \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r] \arrow[d, "\Pi"'] & U^+ \times Z\,. \arrow[dl, "\mathrm{proj}_1"] \\ U^+ & \end{tikzcd}$$

Figure: local trivializations on $\overset{\circ}{Y}$ *and* Y.

Fibered homogeneous coordinates on Y (unique up to positive rescaling):

$$[x, \dot{x}, y] \mapsto (x^\mu, \dot{x}^\mu, y^\sigma) \tag{30}$$

$:=$ local coords of an arbitrarily chosen representative of the class $[x, \dot{x}, y]$.

Examples:

1. **Finsler (2-homogeneous) functions** $L = L(x, \dot{x}) : \mathcal{A} \to \mathbb{R}$ - sections of

$$\overset{\circ}{Y} = T\overset{\circ}{M} \times \mathbb{R}, \qquad H_\alpha(x, \dot{x}, L) = (x, \alpha\dot{x}, \alpha^2 L), \quad \forall \alpha > 0. \tag{31}$$

2. 0-**homogeneous metric d-tensors** $g : \mathcal{A} \to T_2^0(T\overset{\circ}{M})$ - sections of

$$\overset{\circ}{Y} = T_2^0(T\overset{\circ}{M}), \qquad H_\alpha(x, \dot{x}, y) = (x, \alpha\dot{x}, y), \quad \forall \alpha > 0.$$

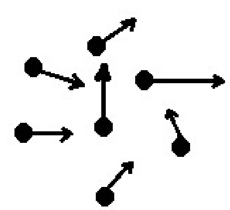
Other examples: *d-tensors, connections*.

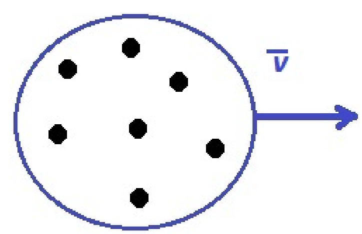
A Finslerian model of gravity

✓ Main idea: Treat L **as the fundamental variable.**

✓ Finslerity = non-quadratic "warping" between space and time.

Main motivation: gravitational field of kinetic gases:





3.4 Kinetic gases

$\boxed{\textbf{Kinetic gas}}$ = a large number N of interacting point particles, described by a smooth **1-particle distribution function:**

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}(x, \dot{x}) \tag{32}$$

✓ Particles move on normalized **geodesics** γ, that is: $(\gamma, \dot{\gamma}) \in \mathcal{O}$.

✓ **Role of** φ : counts particle trajectories $(\gamma, \dot{\gamma})$ crossing a hypersurface $\sigma \subset \mathcal{O}$

$$N_\sigma = \int\limits_\sigma \varphi d\Omega, \tag{33}$$

where $d\Omega$ - canonical volume form on σ.

Kinetic gases in General Relativity: Einstein-Vlasov equations (ref: Sarbach&Zannias, CQG 2014)

★ **Energy-momentum tensor:=second moment** of φ

$$T^{\mu\nu}(x) = \int\limits_{\mathcal{O}_x} m\varphi \dot{x}^\mu \dot{x}^\nu d\mathbf{\Sigma}_x. \tag{34}$$

★ Couple $T^{\mu\nu}$ to geometry via Einstein equations:

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}. \tag{35}$$

Comments:

1. $T^{\mu\nu}$ obtained by *averaging* over all 4-velocities $\rightarrow$ loss of information.
2. **What about all other moments** of φ :

$$T^{\mu_1\cdots\mu_k}(x) = \int\limits_{\mathcal{O}_x} m\varphi \dot{x}^{\mu_1}...\dot{x}^{\mu_k} d\mathbf{\Sigma}_x \quad ? \tag{36}$$

Want full velocity distribution? Think Finsler!

Basic idea: couple gravity directly to φ (no averaging over velocities):

$\varphi = \varphi(x, \dot{x})$	$\longleftrightarrow$	$L = L(x, \dot{x})$	$\longrightarrow$	$\mathfrak{G}(x, \dot{x}) = \kappa\varphi(x, \dot{x})$
kinetic gas 1PDF		Finsler function (*dynamical variable*)		gravitational field equation

–> all living on the same same space: the tangent bundle

$$TM \equiv \{(x, \dot{x}) \,|\, x \in M,\ \dot{x} \in T_xM\} \tag{37}$$

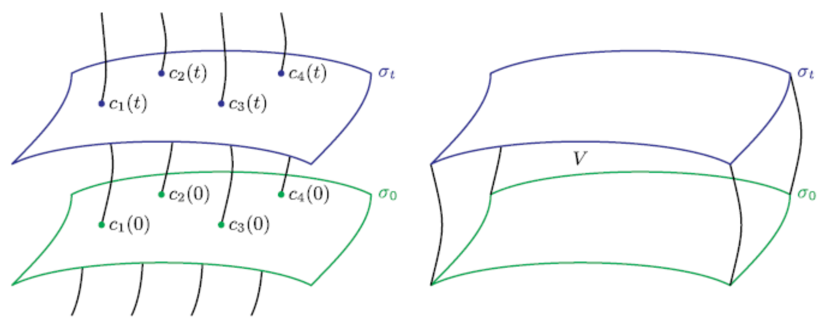
3.5 Kinetic gas action, [2]

Action for the kinetic gas (particles of same mass m):

$$S_{gas} = Nmt = mN\int_0^t ds = m\int_V \varphi d\boldsymbol{\Sigma}, \qquad V = [0,t]\times\sigma, \tag{38}$$

where: $d\boldsymbol{\Sigma}$ - canonical volume form on set of timelike directions $\mathcal{T}^+ \simeq \mathcal{O} \Rightarrow$

$$S_{gas} = m\int_V \varphi \; d\boldsymbol{\Sigma}. \tag{39}$$



3.6 Vacuum action, [3]

Starting principle (Pirani):
In vacuum: trace of geodesic deviation operator = 0
$\Leftrightarrow$ "seed" equation (Rutz 1990):

$$R = 0 \tag{40}$$

$\longrightarrow$ **not** variational $\Rightarrow$???

Solution: **variational completion** technique (NV, Krupka 2015):
✓ canonical (Vainberg-Tonti) Lagrangian function for (40): $\mathbb{L} = \dfrac{R}{L}$
✓ Adding to the lhs of (40) its 'obstructions from variationality'
$\Rightarrow$ **Vacuum action**

$$S_L = \frac{1}{2\kappa^2} \int_V \frac{R}{L}\, d\Sigma \tag{41}$$

3.7 Total action and field equation

Total action: $S := S_L + S_{gas} \Rightarrow$

Field equation:

$$3L^{-2}R - \frac{1}{2}g^{\mu\nu}\dot{\partial}_\mu\dot{\partial}_\mu R + \mathrm{div}(\mathcal{P}) = \kappa\varphi \tag{42}$$

where $\mathcal{P}$ - obtained from the Landsberg tensor:

$$\mathcal{P} = P^\mu\delta_\mu + (\nabla P^\mu)\,\dot{\partial}_\mu, \qquad P^\mu = g^{\mu\nu}P^\tau{}_{\nu\tau}. \tag{43}$$

3.8 Averaged equations: recovering lost moments

✓ Second moments:

$\underbrace{\int\limits_{\mathcal{O}_x} \mathfrak{G} \frac{\dot{x}^{\mu}\dot{x}_{\nu}}{L^{-1}} d\mathbf{\Sigma}_x}_{\mathfrak{G}^{\mu}{}_{\nu}} = -\kappa^2 \mathcal{T}^{\mu}_{\nu}$	$\overset{a}{R}{}^{\mu}{}_{\nu} - \overset{a}{R}\delta^{\mu}{}_{\nu} = \frac{8\pi G}{c^4} T^{\mu}_{\nu}$
averaged Finslerian	Einstein-Vlasov

✓ General n-th moment equations:

$$\mathfrak{G}^{\mu_1 \cdots \mu_n}(x) := \int\limits_{\mathcal{O}_x} \frac{\dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}}{L^{n/2}} \mathfrak{G}(x, \dot{x})\, d\mathbf{\Sigma} = \kappa \int\limits_{\mathcal{O}_x} \frac{\dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n}}{L^{n/2}} \varphi(x, \dot{x})\, d\mathbf{\Sigma}$$

Finslerian approach - considers full velocity distribution of gas particles.

4 Why Is the Universe accelerating? A Finslerian answer

4.1 Cosmologically symmetric Finsler structures

Generators of cosmological symmetry - same as in GR, [5].

Cosmologically symmetric Finsler spacetime functions

$$L = L(t, \dot{t}, \ell) = \dot{t}^2 h\,(t, s)\,, \tag{44}$$

where

$$\begin{aligned} \ell^2 &= \frac{\dot{r}^2}{1 - kr^2} + r^2 \left(\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2\right), \quad \text{- spatial line element} \qquad (45) \\ s &= \frac{\ell}{\dot{t}} \text{ - } \textbf{spatial speed} \end{aligned}$$

General features:

✓ Spatial geometry: Riemannian (same as FLRW);

✓ Difference: h - non-quadratic in $s \Rightarrow$ more general intertwining between space and time

Investigating the special classes of Finsler spaces:

✓ Cosmological+Berwald ⇒ Riemannian (FLRW) or stationary, [5].

✓ There exists a unique cosmological Landsberg (non-Berwaldian) metric, [6]:

$$L(t,\dot{t},\ell) = e^{\sigma(t)}\frac{1}{-s_1 s_2}(\dot{t}s_1-\ell)^{\frac{2s_1}{s_1-s_2}}(\ell-\dot{t}s_2)^{-\frac{2s_2}{s_1-s_2}}e^{c_3}\,, \tag{46}$$

where $s_1 < 0,\ s_2 > 0$ - constant.

Drawback: the corresponding φ has a singularity at $s = 0$.

Solution: general Finsler (beyond Landsberg/Berwald) metrics.

4.2 The Finsler-Friedmann equation, [4]

Separated variables (conformal time) Ansatz:

$$L = \dot{\eta}^2 a^2(\eta) f^2(s), \tag{47}$$

where $t =: \eta$ - conformal time

FLRW	general Finsler
$f(s) = \sqrt{1 - s^2}$	$f = f(s)$ - arbitrary

Field equation in conformal time =: the *Finsler-Friedmann equation, [5]:*

$$\frac{k}{a^2(\eta)}\mathcal{G}_k(s) + \frac{\mathcal{H}^2(\eta)}{a^2(\eta)}\mathcal{G}_{\mathcal{H}}(s) + \frac{\dot{\mathcal{H}}(\eta)}{a^2(\eta)}\mathcal{G}_{\dot{\mathcal{H}}}(s) = \kappa\varphi, \tag{48}$$

where:

$$\mathcal{H}(\eta) = \frac{\dot{a}(\eta)}{a(\eta)}, \tag{49}$$

$$\mathcal{G}_k = \frac{2(f - sf')\left(-sf'^2 + f(f' + 2sf'')\right)}{f^3 f' f''}$$

$$\begin{aligned}
\mathcal{G}_{\dot{\mathcal{H}}} = {} & \frac{1}{4}\frac{f'^2(3sf'^2f'' + sff'f''' - 4sff''^2 - 8ff'f'')}{sf^4f''^3} \\
& + \frac{1}{4}\frac{20sf'f''^3 - 6s^2f''^4 - f'^2(2f''^2 + s^2f'''^2 + sf''(4f''' - sf''''))}{s^2f^2f''^4} \\
& + \frac{1}{2}\frac{2s^2f''^4 + s^2f'f''^2f''' - 2s^2f'^2f''f'''' + 2s^2f'^2f'''^2 - 2sf'f''^2 - 2f'^2f''^2}{s^2ff'^2f''^3} \\
& + \frac{1}{4}\frac{f'(4f''^3 - 3s^3f'''^3 + 4s^3f''f'''f'''' + f''^2(2sf''' - s^3f'''''))}{s^3ff''^5}.
\end{aligned}$$

$$\mathcal{G}_{\mathcal{H}} = \frac{3}{4}\frac{f'^4(4ff'f'''s - 15ff''^2s + 2sf'^2f'' + 4ff'f'')}{sf^5f''^4} \tag{C.2}$$

$$+\frac{1}{4}\frac{f'^2(-98s^2f''^4 + 4sf'f''^2(11f'' + 13sf''') - 6f''^2 - 18s^2f'''^2 - sf''(16f''' - 7sf''''))}{s^2f^3f''^5} \tag{C.3}$$

$$+\frac{1}{2}\frac{33sf''^2 - 26sf'f'' - 22f'f''}{sf^2f''^2} + \frac{1}{4}\frac{f'^2(14f''^2 + 55s^2f'''^2 + sf''(32f''' - 27sf''''))}{s^2f^2f''^4} \tag{C.4}$$

$$-\frac{1}{4}\frac{f'^3(24s^2f'''^2 - 3sf''f'''(-4f''' + 7sf'''') + f''^2(4f''' - 4sf'''' + 3s^2f'''''))}{s^2f^2f''^6} \tag{C.5}$$

$$-\frac{1}{2}\frac{2s^2f''^4 - 2s^2f'f''^2f''' - 11s^2f'^2f''f'''' + 17s^2f'^2f'''^2 - 2sf'f''^3 - 5f'^2f''^2}{s^2ff'^2f''^3} \tag{C.6}$$

$$+\frac{1}{2}\frac{f'(-8f''^3 + 27s^3f'''^3 - 28s^3f''f'''f'''' + f''^2(-6sf''' + 5s^3f'''''))}{s^3ff''^5} \tag{C.7}$$

$$+\frac{1}{4}\frac{f'^2(6f''^4 - 30s^4f'''^4 + 42sf''f'''^2f'''' + f''^2(6s^2f''''^2 - 6s^4f'''''^2 - 9s^4f'''f''''''))}{s^4ff''^7} \tag{C.8}$$

$$+\frac{1}{4}\frac{f'^2(8f''' - 2sf'''' + s^3f'''''')}{s^3ff''^4}. \tag{C.9}$$

4.3 Vacuum solutions $\varphi = 0$

A. Solving for the scale factor a. Vacuum equation:

$$k\mathcal{G}_k(s) + \mathcal{H}^2(\eta)\,\mathcal{G}_{\mathcal{H}}(s) + \dot{\mathcal{H}}(\eta)\,\mathcal{G}_{\dot{\mathcal{H}}}(s) = 0 \tag{50}$$

$\Rightarrow$ we can separate the time dependence:

$$\frac{d}{d\eta}\left(\frac{\ddot{\mathcal{H}}}{\mathcal{H}\dot{\mathcal{H}}}\right) = 0 \Rightarrow \quad \dot{\mathcal{H}} = c_1\mathcal{H}^2 + c_2. \tag{51}$$

The case $c_2 = 0$: solution in cosmological time

Cosmological time t - defined by:

$$a d\eta = dt. \tag{52}$$

We find:

★ For $c_1 = 1 \Rightarrow$ **exponential evolution:**

$$a(t) = d_2 e^{d_1 t}, \quad d_1, d_2 \in \mathbb{R}. \tag{53}$$

★ For $c_1 \neq 1 \Rightarrow$ **power law:**

$$a(t) = (d_1 t + d_2)^{\frac{1}{1-c_1}}, \quad d_1, d_2 \in \mathbb{R}. \tag{54}$$

$\boxed{\text{Accelerated expansion - no need for dark energy/extra geometric objects!}}$

B. Causal structure of the exponentially expanding vacuum:

Set:

$$k = 0, \quad c_1 = 1, \;\; c_2 = 0.$$

Vacuum field equation becomes:

$$\mathcal{G}_{\mathcal{H}}(s) + \mathcal{G}_{\dot{\mathcal{H}}}(s) = 0: \tag{55}$$

✓ 6th order, nonlinear ODE $\Rightarrow$ no analytic solution found.

✓ Take a power series ansatz $f(s) = \sum_{i=0}^{\infty} f_i s^i \overset{(!)}{=} \sum_{i=0}^{\infty} f_{2i} s^{2i} =: \sum_{i=0}^{\infty} Q_i X^i.$

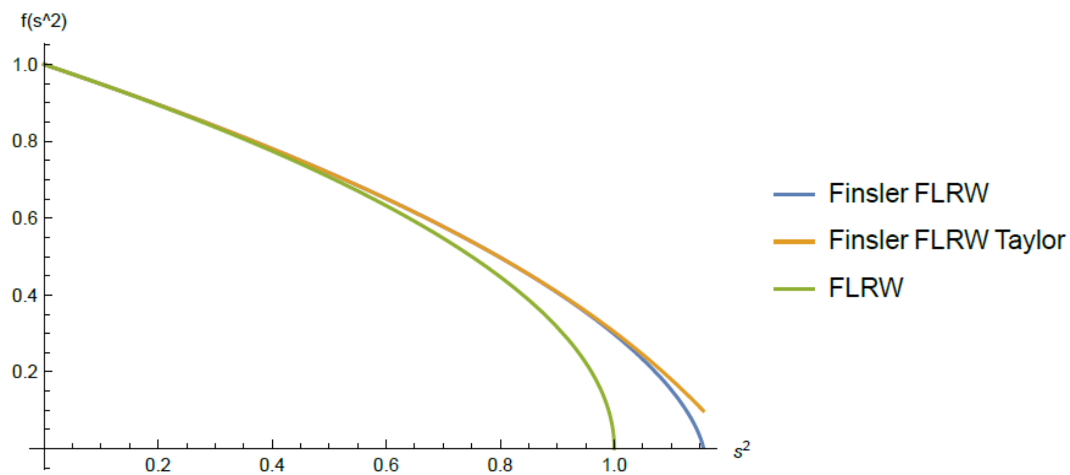
✓ Use $Q_0, .., Q_5$ as initial conds for a **numerical solution.**

✓ For Q_0, Q_1 : compare to FLRW $f(s) = \sqrt{1 - s^2} \simeq 1 - \frac{s^2}{2} + \mathcal{O}\left(s^4\right) \Rightarrow$

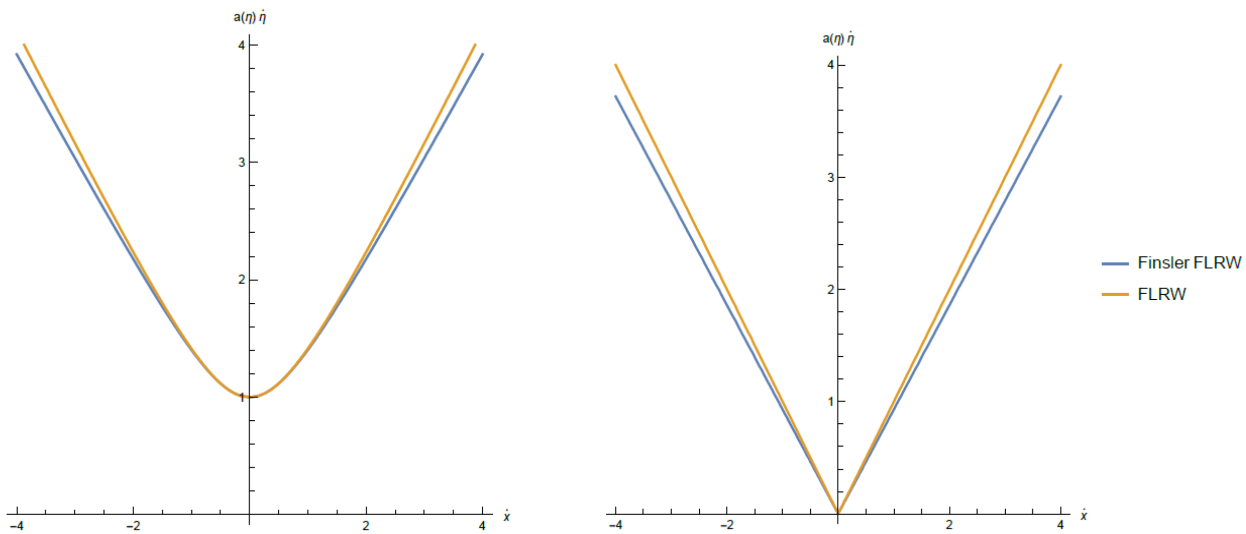
$$Q_0 = 1, \;\; Q_1 = -\frac{1}{2}.$$

✓ Solve field eq. order by order $\Rightarrow Q_2, \ldots, Q_5$.

Numerical solution for f (using Mathematica):



Mass shell $L = 1$ and light cones:



Remark: For small s, the causal structures are nearly indistinguishable. Deviations - at large s.

5 Conclusion

Finslerian spacetime framework:

✓ Generalization of Riemannian one, based on a most general notion of arc length (proper time).

✓ Light cone structure, geodesic structure - well defined.

✓ Local Lorentz symmetries are *deformed.*

✓ A Finslerian geometry of spacetime supports a (polynomially or exponentially) expanding vacuum.

Ongoing work:

✓ Use a cosmologically symmetric, *collisionless kinetic gas* 1PDF $\varphi = \varphi(t, s)$ as gravity source and solve the field equation for it.

✓ The Newtonian limit of Finsler gravity $\rightsquigarrow$ Poisson equation.

✓ Find *exact solutions* $f = f(s)$ in cosmology.

✓ Finsler/Riemann metrizability conditions for *general torsion-free affine connections*.

✓ Berwald-Finsler, non-Ricci flat $SO(3)$-symmetric models for compact objects.

Talk based on:

[1] M. Hohmann, C. Pfeifer, N. Voicu, *Mathematical foundations for field theories on Finsler spacetimes*, J. Math. Phys 63, 032503 (2022).

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[9] N. Voicu, C. Pfeifer, S. Cheraghchi, *Birkhoff Theorem for Berwald Finsler spacetimes*, Phys. Rev. D 108, 104060 (2023).
[10] S. Cheraghchi, C. Pfeifer, N. Voicu, *Four-dimensional SO(3)-spherically symmetric Berwald-Finsler spaces*, IJGMMP 20 (11), 2350190 (2023).

Thank you for your attention!

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