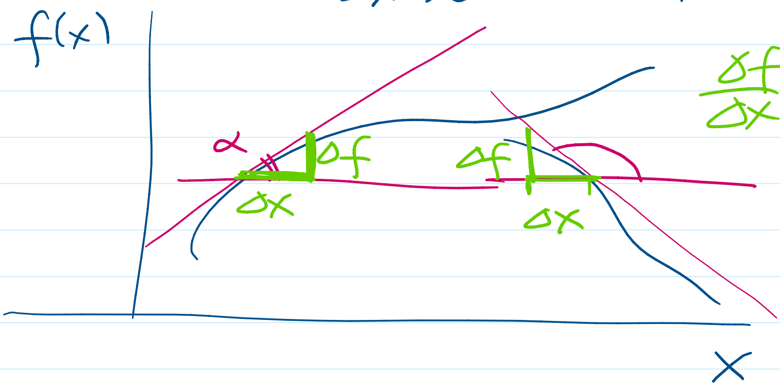
Wykład 16

czwartek, 29 kwietnia 2021 13:10

OPERATORY RÓŻNICZKOWE

$f(x)$ przypomnienie:

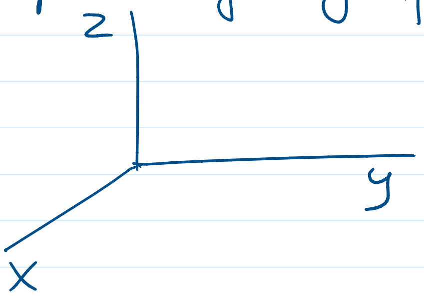
$$\frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\overbrace{f(x+\Delta x) - f(x)}^{\Delta f}}{\Delta x}$$



$\frac{\Delta f}{\Delta x} \to \operatorname{tg}\alpha$

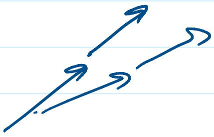
$\Delta f < 0$

W fizyce potrzebujemy funkcje wielu zmiennych



$\underbrace{\text{przestrzeń } R^3 \text{ (3-wym. przestrzeń)} + R^1 \text{ (czas)}}_{\text{czasoprzestrzeń}}$

np. pole $\bar{E}$



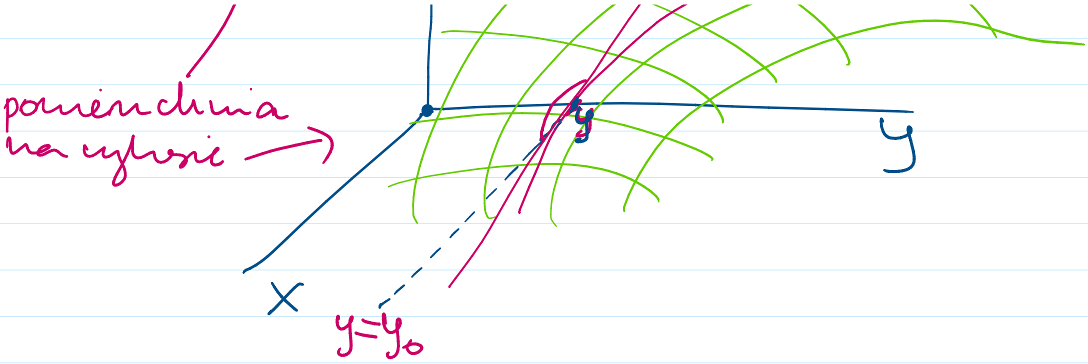
$$\bar{E}(x,y,z,t)$$

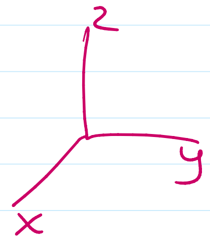
$$\bar{E} = \left(E_x(x,y,z,t),\ E_y(x,y,z,t),\ E_z(x,y,z,t)\right)$$

Rozszerzenie pojęcia pochodnej $f(x)$ na większą liczbę zmiennych

$$f(x) \to f(x,y,z) \quad ?$$

dla $\boxed{f(x,y)}$





Pochodna cząstkowa:

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

tzn. że $y = const$ w tym rachunku

$$\frac{\partial f(x,y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

tu: $x = const$

Proste przejście do $f(x,y,z)$: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x}$$

etc.

Przykłady:

$$f(x,y,z) = 2x^2y + 3xz$$

$\frac{\partial f}{\partial x} = 4xy + 3z$ $\qquad \frac{\partial^2 f}{\partial x^2} = 4y$ $\qquad \boxed{\frac{\partial^2 f}{\partial y\, \partial x} = 4x}$

$\frac{\partial f}{\partial y} = 2x^2$ $\qquad \frac{\partial^2 f}{\partial y^2} = 0$ $\qquad \frac{\partial^2 f}{\partial z\, \partial x} = 3$

$\frac{\partial f}{\partial z} = 3x$ $\qquad \frac{\partial^2 f}{\partial z^2} = 0$ $\qquad \frac{\partial^2 f}{\partial z\, \partial y} = 0$

Druga pochodna cząstkowa

$f(x,y,z)$: $\qquad \frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z}$

$$\frac{\partial^2 f}{\partial x^2} \qquad \frac{\partial^2 f}{\partial y^2} \qquad \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y} \qquad \frac{\partial^2 f}{\partial x\, \partial z}$$

$$\left\{ \begin{matrix} \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z} \end{matrix} \right\}$$

drugie pochodne mieszane

~~6~~ ~~2~~ 3 różne

Tw: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ etc.

kolejność różniczkowania w pochodnych mieszanych nie odgrywa roli

Extra: $\frac{\partial^3 f}{(\ )(\ )(\ )}$ $\quad \frac{\partial^3 f}{\partial x^3}$ $\quad \frac{\partial^3 f}{\partial y \partial x \partial x}$ $\quad \frac{\partial^3 f}{\partial y\, \partial x\, \partial z}$

$\frac{}{\partial y^2 \partial x}$ $\quad \frac{}{\partial z\, \partial y\, \partial x}$

$(\hat{\ })$

Operator różniczkowy: argumentem jest funkcja

np. $\hat{D} = \frac{d}{dx}$ dla f. jednej zmiennej

argumentem jest funkcja, tu: $f(x)$

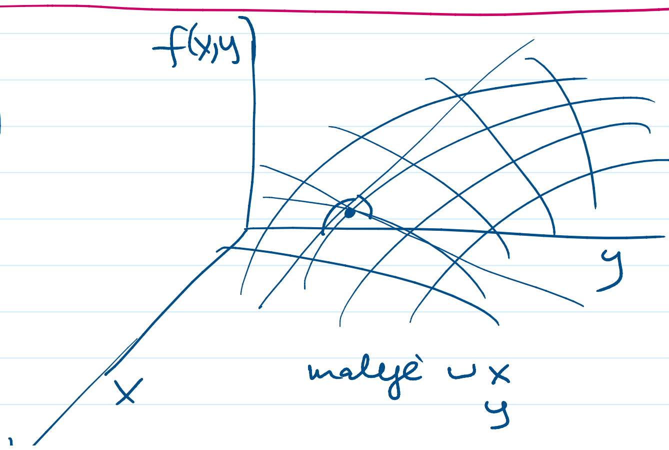
$\boxed{\hat{D} f = \frac{df}{dx}}$ (z "przykładamy" f prawej strony)

Gradient: grad lub ∇ (nabla)

$\underbrace{f(x,y,z)}_{\text{skalar}} \xrightarrow{grad}$ przyporządkowuje wektor

$$\boxed{\text{grad } f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)} \qquad \nabla f$$

$f(x,y)$



$tg\alpha_x \quad tg\alpha_y$

$<0 \quad <0$

wektor grad:

maleje w x, y

Własności gradientu: wynikają z własności pochodnych i wektorów

$f(x,y,z)$, $g(x,y,z)$

(1) $\operatorname{grad}(af+bg) = a \operatorname{grad} f + b \operatorname{grad} g$ ←

gradient jest operatorem liniowym

$$\operatorname{grad}(\quad) = \left(a\frac{\partial f}{\partial x} + b\frac{\partial g}{\partial x}, \dots\dots\right) =$$

$$= a\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) + b\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$$

(2) $\operatorname{grad}(f_1 \cdot f_2) = \left(\frac{\partial}{\partial x}(f_1 \cdot f_2), \dots\dots\right) = \left(f_2\frac{\partial f_1}{\partial x} + f_1\frac{\partial f_2}{\partial x}, \dots\dots\right)$

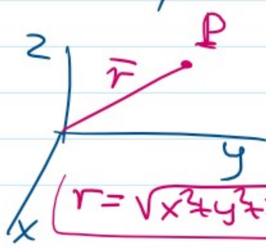
$$= f_2\left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_1}{\partial z}\right) + f_1\left(\frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f}{\partial z}\right) =$$

$$= f_2 \operatorname{grad} f_1 + f_1 \operatorname{grad} f_2$$

(3)

$$\operatorname{grad}\left(\frac{1}{f}\right) = \left(\frac{\partial}{\partial x}\left(\frac{1}{f}\right), \dots\dots\right) = \left(-\frac{1}{f^2}\cdot\frac{\partial f}{\partial x}, \dots\dots\right)$$

$$= -\frac{1}{f^2} \operatorname{grad} f$$



$\boxed{r = \sqrt{x^2+y^2+z^2}}$

$(x,y,z) = \bar{r}$

(4)

$$f(x,y,z) = \sqrt{x^2+y^2+z^2} = r$$

$$\frac{\partial f}{\partial x} = \frac{1}{\cancel{2}\sqrt{\quad}} \cdot \cancel{2}x = \frac{x}{\sqrt{\quad}} = \frac{x}{r}$$

$$\frac{\partial f}{\partial y} = \qquad = \frac{y}{r}$$

$$\frac{\partial f}{\partial z} = \qquad = \frac{z}{r}$$

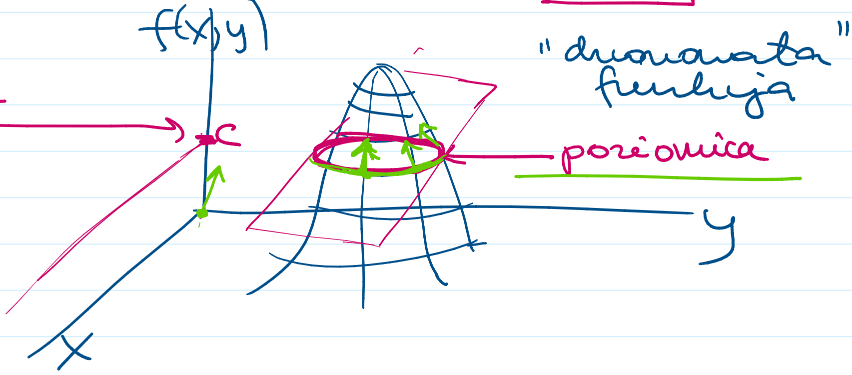
$r(x,y,z)$

$$\operatorname{grad} r = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{1}{r}(x,y,z) = \frac{\bar{r}}{r} = \bar{e}_r.$$

$$\boxed{\mathrm{grad}\, \underset{r(x,y,z)}{r} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \frac{1}{r}(x,y,z) = \frac{\vec{r}}{r} = \vec{e}_r}$$

(5) gdy $f = \mathrm{const} = C$

$$\boxed{\mathrm{grad}\, f = 0}$$



$\mathrm{grad}\, f \perp$ poziomicy

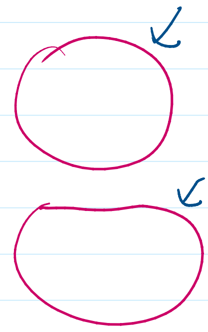
(6) Niech f zależy od r $\quad f(x,y,z) = f(r) = f(r(x,y,z))$

$$\mathrm{grad}\, f(r) = \left(\frac{\partial f}{\partial x}, \ldots\ldots \right) = \left(\frac{df}{dr}\cdot\underset{\frac{x}{r}}{\underset{\parallel}{\frac{\partial r}{\partial x}}}, \frac{df}{dr}\underset{\frac{y}{r}}{\frac{\partial r}{\partial y}}, \frac{df}{dr}\underset{\frac{z}{r}}{\frac{\partial r}{\partial z}}\right)$$

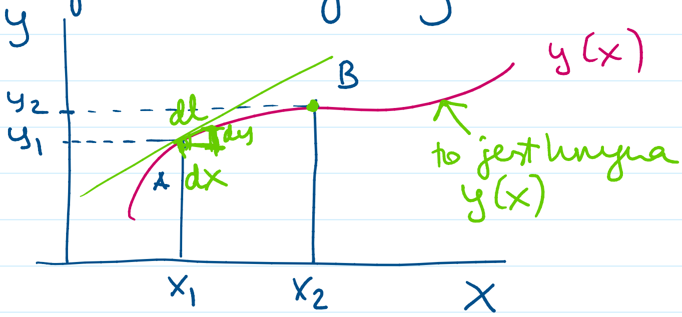
$$\frac{df}{dr}\ \frac{x}{r}$$

$$\boxed{\mathrm{grad}\, f(r) = \left(\frac{df}{dr}\cdot\frac{x}{r}, \frac{df}{dr}\cdot\frac{y}{r}, \frac{df}{dr}\cdot\frac{z}{r}\right) = \frac{df}{dr}\underbrace{\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)}_{\vec{e}_r} = \frac{df}{dr}\vec{e}_r}$$

(7) $f(r) = \frac{1}{r}$



Dygresja: długość krzywej



Jaka jest długość krzywej między A i B $= L_{AB}$

$$dl = \sqrt{dx^2 + dy^2}$$

$$\boxed{L_{AB} = \int^{B} dl = \int^{B}\sqrt{dx^2+dy^2} = \int^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}$$

$$L_{AB} = \int_{A} dl = \int_{A} \sqrt{dx^2 + dy^2} = \int_{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$