Complexity, Holography and Geometry

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Ph.D. Thesis

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Złożoność, holografia i geometria

by

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May 2025

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A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Physics

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Abstract

Complexity, holography, and geometry are notions that in recent years often appear together, inextricably linked when the dynamics of high energy (and not only) systems are concerned. In this thesis I explore their interplay through the lens of Krylov complexity, a novel measure which has recently attracted the attention of the community. A key feature in this endeavour is the study of the underlying symmetries, which provide a rigorous framework through which these connections become manifest, allowing not only for a deeper understanding of the physical concepts themselves but also for their potential generalization. Starting from the definition of Krylov complexity, an understanding in terms of an associated symmetry algebra naturally emerges, endowing the dynamics of physical systems with a certain degree of universality and additionally with a geometric interpretation. Subsequently, this framework can be further utilized to study other quantities that characterize the dynamics of quantum systems, modified to describe systems with more sophisticated symmetry structures and even applied to the description of the behaviour of quantum systems under different kinds of evolution, such as modular flow.

Streszczenie

Złożoność, holografia i geometria to pojęcia, które w ostatnich latach często pojawiają się wspólnie, nierozerwalnie powiązane w kontekście dynamiki układów wysokoenergetycznych (choć nie tylko). W niniejszej pracy badam ich wzajemne relacje przez pryzmat złożoności Kryłowa (ang. Krylov complexity). Kluczowym elementem mojego podejścia jest analiza dynamicznych symetrii, które dostarczają ścisłych ram teoretycznych, w obrębie których powiązania te stają się wyraźne. Symetrie te umożliwiają nie tylko głębsze zrozumienie złożoności samych procesów fizycznych, lecz także jej potencjalne uogólnienie na układy o mniejszym stopniu symetrii. Wychodząc od definicji złożoności Kryłowa, w naturalny sposób pojawia się interpretacja oparta na odpowiadającej jej algebrze symetrii, nadająca dynamice układów fizycznych zarówno pewien stopień uniwersalności, jak i geometryczną interpretację. Tak zbudowane ramy teoretyczne wykorzystuję następnie do analizy innych, kwantowo-informacyjnych, wielkości opisujących dynamikę układów kwantowych, do badania układów o bardziej złożonych strukturach symetrii oraz do opisu złożoności różnych ewolucji kwantowych, takich jak np. ewolucja modularna (ang. modular flow).

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Acknowledgements

This part of my thesis has been on my mind for a very long time and I have thought about laying down these words over and over again, more than I should have anyway. Even before I got admitted to a PhD programme I was wondering what I was going to write in the acknowledgements section of the ultimate thesis, whom I would have to thank and how I would view my life looking back at it from the finish line of my studies. Sometimes I thought I should just keep it simple, just write a "thank you everyone" and be done with it, but I guess this is not really who I am and I only get one chance to write this so I might as well do it right. This means that my acknowledgements are going to be unusually long and full of personal details that are probably of little interest to most people, but are important to me since they are what made up my PhD years as much as the scientific work that went into them.

With that being said, I would like to start on the long list of people that I wish to thank with my supervisor Paweł Caputa. I am fairly sure he is not going to be a fan of my melodramatic tone, but this will be in all likelihood my last act of defiance and annoyance as his student and as such he will have to forgive me. Paweł never told me to do "this" or "that". He offered advice sparingly and with great care not to impose on my freedom to make choices for myself (or at least that is how I perceived it), even if that meant that many of my questions were answered by his famous eyebrow raise rather than with actual words. Instead, he always led by example and in the place of trivial, run of the mill advice, he gave me something I could look up to. His dedication, his work ethic, and quite importantly his kindness and openness has shown me what being a scientist is all about. At some point he had mentioned that the relationship between students and their supervisors is of the "monkey see, monkey do" type and by this token the way I approach science has been shaped to a great extent by him, which is something I could not be happier about.

Next I would like to thank the many wonderful collaborators with whom I have worked over the years and whose contributions have made it to a large extent possible for me to compile this thesis. I will start by thanking Javier Magan for his deep insights and contributions to our common works. He has always been helpful to me and I hope I get more chances to work with him in the future. Next I would like to express my heartfelt gratitude towards Watse Sybesma. Working with him has been a lot of fun since the very beginning and our approach to solving problems has always filled me with a profound sense of joy and satisfaction. Apart from our scientific endeavours together, Watse has also been a friend and mentor and has taught me a lot about how academia works and how one can find their place in it or outside it. Finally, I want to thank Bartek Czech who introduced me, among many other things, to the topic of holographic inequalities. Working with him has been an absolute pleasure predominantly due to his ability to make me feel that I am as smart as he is. Additionally, I would like to thank him for his amazing hospitality during my time in Beijing and for inviting me to Bali. Some of the fondest memories from my PhD are from these two places and for that I feel exceptionally grateful to him.

Even though our group in Warsaw has now been disbanded, it provided an excellent environment to work in and I am grateful for the chance to interact with the amazing people that were part of it. For that reason I would like to thank our postdocs Dongsheng Ge, Sinong Liu, and Bowen Chen for many interesting discussions and fun times together. A very special thanks goes to Jan Boruch,

the last remaining member of our group in Warsaw, but above all else a great friend and comrade in arms on the PhD battlefield. Jan is convinced that he is the main character in some sort of anime series and being the great friend that I am I will feed his delusion by informing him that I am in fact his archrival and academic nemesis. Jokes aside, Jan's friendship has had an important influence on me and my development as a scientist. Even more than that Jan has stood by my side when I needed it the most and on a number of occasions he was my interlocutor in long-winded, thought provoking conversations that have caused me to introspect, reevaluate, and on occasion change my opinion about important things in my life. I can only hope that I have done the same for him and wish him all the best in our citation race.

As I prefaced in the beginning the scientific portion of the past few years is only half of what made up my life (maybe more, maybe less, but who's counting?) and for the other half I have a number of people to thank, mainly for helping me maintain what little sanity I had to begin with. First and foremost, I want to thank Spyros Mouselinos for being my closest friend during these years. On many occasions it felt like if he weren't around, life in Warsaw would have been intolerable, so if I have made it to this point it is partially because he has always been someone that I could rely on for help and support. In the same light, I want to express my gratitude to Aditya Bawane, who has helped me tremendously with navigating my prospects within the academic landscape. I have valued the advice he has given me as much as I have enjoyed simply hanging out with him and convincing him to unleash his inner wizard by playing dungeons and dragons.

I would also like to say a big thank you to Elvira Khasanova and Julia Paczuska. We have had some great times together and I also feel indebted to them for pet-sitting during my many travels. I am sure Becca will miss you as much as I.

During my time in Warsaw, I tried out a bunch of new things, some of which I stuck with. One of my major undertakings parallel to my PhD has been learning Polish and I am incredibly thankful to my teacher Paulina Potasińska for the countless hours of entertainment and free therapy that she offered me, while teaching me Polish on the side. Another of my newfound interests has been bouldering, which I had the privilege to enjoy under the guidance of Kasia Hryniewiecka, the most multi-faceted PhD student I had the luck to meet. My climbing might have only improved marginally, but at least I got to hang out with one of the coolest people out there so I take that as a big win.

I also need to extend a warm thanks to Thanasis Milonas who is responsible for the striking cover of my thesis. After seeing how he transmuted a 15-minute description of my work into a visual cornucopia, I can only say that artistic interventions of this kind should be a much more desirable occurrence. It somehow made me realize how colourless my perception of my own work is and gave me inspiration to re-imagine it all through a new prism.

Of course, I would be remiss if I didn't take this opportunity to thank my family. My parents Manolis and Marina, and my siblings Myrto and Aris. Each one of them contributed in their own way to my development and have supported me fully during these years. I know it can't have been easy given the kind of person I am, so I feel incredibly grateful and proud that I have such a loving and supportive family and that amounts to a lot more than just getting a doctorate.

Last but not least, I would like to thank Becca. Now that feels rather strange because Becca is a cat and as such I cannot ascribe any agency to her actions. She simply demands to be fed at regular intervals, petted whenever she feels like it and access to the windows so that she can look outside. And yet somehow that is enough.

Thank you everyone

Chapter 1

Prolegomena

1.1 Weaving the web of complexity

I wanted to start this thesis with a rather light introduction to the notion of complexity, hopefully one that does not require a trained physicist or scientist to understand. The reason for it is that during the course of my studies friends and family would often ask me what is the object of my work and I would jokingly reply that I myself do not know. My intent of course was to be funny, but I knew that there was a pretty substantial fragment of truth in that statement. It is well known among the scientific community that there is no clear and unilateral definition of complexity as a quantity, so as a matter of fact nobody really knows what complexity is. This is an important point that I will elaborate on in later sections, however leaving that aside for the moment, I think it is also important to ask what complexity means as a word. Atypical as it may be to embark on such an etymological digression it is definitely worthwhile to clarify the associations between ordinary language and the mathematical formalism which will be predominant in the majority of this work.

Taking a look at the dictionary entry for complexity provided by wikipedia (perhaps not the most dependable of sources, but definitely the most easily accessible one) we are provided with the following definition

1. (uncountable) The state of being complex; intricacy; entanglement.

2. (countable) That which is and renders complex; intricacy; complication.

Obviously, that does not clarify much. In fact it leads to further confusion since complexity is presented as a synonym of entanglement! Within the framework of quantum mechanics, which is after all the arena in which we will be operating, these are thought to be two distinct concepts, although they might be closely related. The definition is therefore not much help, but what about the root of the word? Its latin ancestor appears to be *complexus* which is a compound of "*com-*" (with, together) and "*plecto*" (to weave, to braid) and had the meaning of embracing or entwining, from which I suppose it acquired the meaning of hand-to-hand combat (scuffle) and euphemistically used for sexual intercourse. Most Greeks have a desire to prove that all words originate from ancient Greek so I cannot pass the opportunity to point out that the latter constituent of the word is in fact related to the Greek "$\pi\lambda\acute{\epsilon}\kappa\omega$" [1] meaning precisely to weave. This is rather interesting since if I were to try and correlate the modern English version of the word complexity to modern Greek I would do it as follows: The prefix "*com-*" would become "$\sigma\upsilon\nu$-" (which also survives in a number of English words, e.g. synergy) and the latter part of the word, namely "*-plexity*" I would probably have to translate as "-$\pi\lambda\acute{\epsilon}\xi\eta$" producing the word "$\sigma\acute{\upsilon}\mu\pi\lambda\epsilon\xi\eta$" meaning intertwining or interweaving. The reason I mention this is because this is one of the words that have been proposed as a scientific

[1] One can possibly further track this down to the Proto-Indo-European pleḱ, meaning once again to weave or to fold

translation of the word entanglement, once again highlighting how closely these two notions are related.

To my dismay I found that the Polish word for complexity is "*złożoność*", even though the latin root does survive in words like "*skomplikowane*," meaning complicated. To the best of my understanding this originates from the protoslavic "*ložìti*", meaning "*to lay*", leaving me quite unable to make a connection with the notion of something being complex. Funnily enough though, if I were to take an educated guess (although I will leave the particulars to the linguists), I would say that this is where the word for bed, "łóżko," derives from, indicating perhaps that despite their different origins the words for complexity in these language groups are somehow connected to sex and thereby the universal truth that romantic relations are necessarily complicated?

This rather incoherent exploration of the word complexity is not without reason. In my opinion, it clearly shows that we physicists have not done a very good job at creating a terminology that can effectively relay the concepts we are trying to describe in this instance. In our defense, we do not really know what it is exactly that we are trying to describe in the first place. Even though these terms have been part of the scientific discourse for decades now (especially entanglement) and we have come to understand them a lot better over the years, there are still some aspects that remain quite mysterious. Therefore, as is often the case, the scientific jargon has twisted ordinary language in a manner that sometimes creates fictitious barriers between words that normally mean the same thing or merging the meaning of different words, or simply altering the usual sense in which words are used. There is not much to do about this other than accept it and move forward, since once a term "sticks" it is pretty much impossible to change it. So when it comes to communicating these concepts to the public, the only thing one can do is offer an explanation as to the original reasoning behind the nomenclature, point out the differences from ordinary language and move on.

So where does all that leave us? What is complexity after all? I truly do not know, but if there is one thing that this etymological journey taught me is that complexity is much more about the interplay between "things" rather than something about a "thing" itself. It is a quantitative notion which requires one to do some sort of counting, asking things like how many tangles, turns, connections or quanta? In that sense we can start to understand complexity as a number. A number which tells us how much or how little goes into creating or transforming something, even though as we will soon see that defining what is a lot and what is a little can be a very subjective endeavour.

Let us take as an example the different varieties of spider webs to stay on the topic of weaving. If I were to compare, say, a vertical orb web and a tangle web (pictures (e) and (f) in the figure [1] below) I would readily claim that the latter is more complex simply by inspection. I am inclined to believe that the majority of the readers would agree with me based on their instinct, but the question remains as to why that is the case. In other words, what is it that we are counting? Is it the number of nodes among the individual threads? It could be, but one can imagine a case in which what looks like the simpler web is actually much larger in volume than the other and so the number of nodes would lead one to believe that it is more complex. How about the number of nodes per unit of volume then? This improves our predicament but still one could have an example in which the weave of web (f) is much sparser than (e) and so on. One can play a similar game for different combinations of the webs in figure 1.1 (a-g), with different quantifiers such as the number of individual threads or a parameter related to the uniformity of the coordinates of the nodes etc. ever refining the definition such that it matches our intuition. Obviously this can be quite problematic for two reasons. First, if the definition itself becomes increasingly more precise we run the risk of rendering it useless outside the comparison we are performing and second, human intuition is famously misguided so we might be inventing a definition to justify something that is simply not true to begin with.

Of course, in daily life this is not an issue. When you type "how to..." in your favorite search engine, you expect the simplest answer to your inquiry to come up. If you are not satisfied, you adjust the criteria according to your specific preferences such as "how to... for free" or "how to...

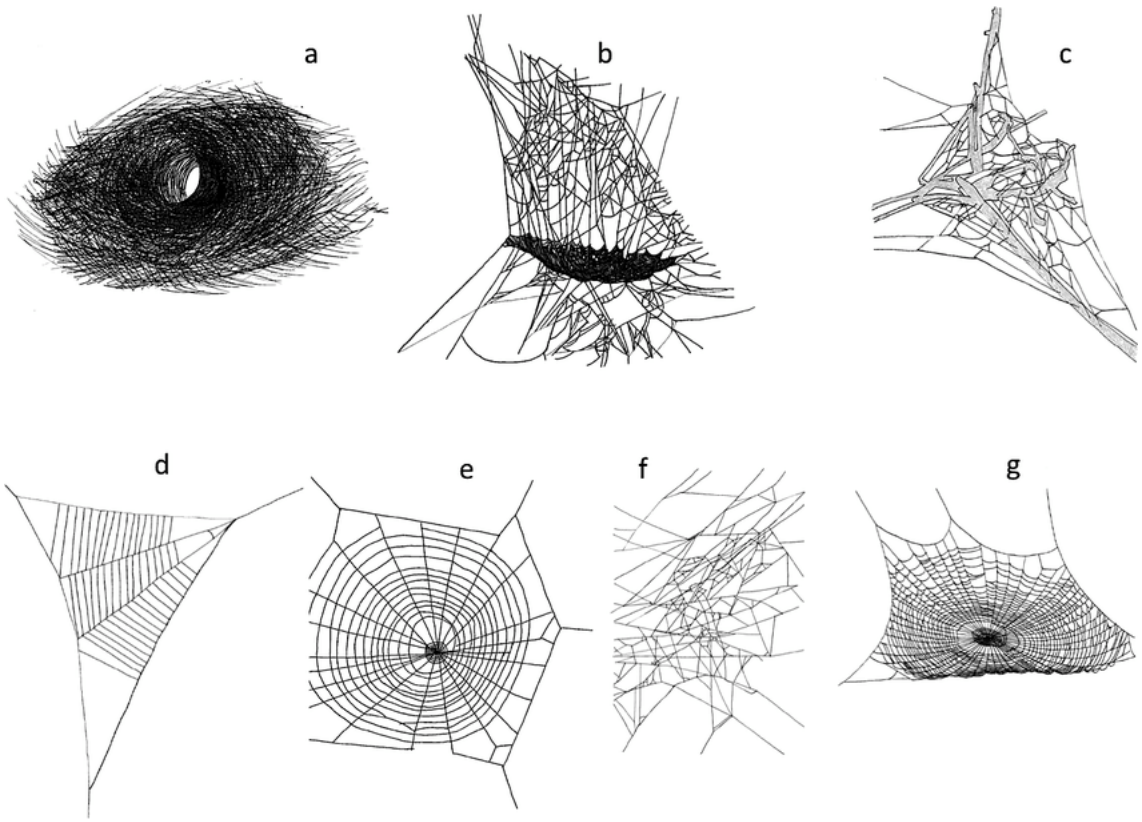

Figure 1.1: Collection of different types of spider webs. They all appear to be complex objects, but how does one compare their complexities?

without..." and problem solved. However, in science one is called to solve problems in a way that is convincing to others and can withstand their scrutiny and as such when one asks "how to..." it is difficult to discern whether an answer is simple or not. This is precisely the subject of the following section in which I will explain why this is the case and discuss the approach of a physicist attempting to characterize natural processes as simple or complex.

1.2 Exercises in futility?

Having done a rather general exposition on complexity it is time to move to a more formal approach that will provide the scientific perspective from which this work ultimately derives. The best starting point is the famous (or maybe infamous) millennium prize problem stated as "P vs NP". Here "P" stands for a class of problems that can be solved by a given algorithm in polynomial time and "NP" for a class of problems whose solutions can be checked in polynomial time but not necessarily found equally quickly. The question, simply put, is whether these two classes of problems are secretly the same, or in other words whether all problems whose solutions can be checked in polynomial time can also be solved in polynomial time. At the time of typing these words this question remains unanswered as it has been since its formal statement in 1971 by Stephen Cook.

However, this is only the tip of the iceberg as it implies that we cannot fundamentally characterize a certain task as being complex or simple, but rather any notion of complexity is reliant on its definition which is ultimately arbitrary. Of course, that has not deterred the scientific community which has produced no small number of definitions that can be useful depending on the problem at hand and I should stress that being able to quantify the difficulty of performing a certain task is in fact quite important. One would like for example to know how difficult it is to implement an

algorithm using a given assortment of hardware. Sure, this might not imply some deeper, universal truth about the difficulty of completing similar tasks, but it is an important issue to address even at this individual level. When it comes to physics, typically one wants to know how difficult it is to go from a given state of a physical system to a different one. Conventionally these are referred to as the reference and the target state respectively.

One might protest at this point that this appears to be quite the futile enterprise and certainly not something that should lead to the award of the Ph.D. title. Surely there must be a better approach than addressing every single problem individually and scientists should be smart enough to figure it out. Indeed, this is precisely the approach of the physicist and to a large extent the aim of this thesis. We might not be able to fundamentally characterize a task as being simple or complex and it might be incredibly tedious to address this issue for each problem individually, but we can hope to strike a balance of sorts by coming up with measures of complexity that are well-motivated and widely applicable. This somewhat utilitarian approach has led to important advances and in the chapters to follow I will be discussing different approaches to defining complexity in more detail.

But even if everyone agrees that this is a reasonable approach, there is a question that people still like to ask. "What is it good for?" In all honesty this is a question I am not particularly interested in. I myself find it an interesting mental exercise to engage in inquiries of this nature, futile though they may be. I am not so far removed from reality however as to pretend that this question is irrelevant. At the very least it is the kind of question that one has to answer in search of funding so in that sense I am very inclined to answer. Seeing that this is no plea to a funding agency though, I can afford myself some leeway in the manner of answering.

Scientists have a tendency to employ reductive reasoning in order to describe physical phenomena that are otherwise difficult to understand. This is in fact one of the most powerful weapons in our arsenal when faced with problems with many contributing variables and it has led to some of the most important insights in the history of physical sciences. Think of Galileo who (anecdotally) dropped cannon balls from the leaning tower of Pisa in order to show that the gravitational acceleration is independent of mass. His choice was motivated by the elimination of air resistance. Or think about Einstein and his famous elevator thought experiment which he used to argue for the equivalence principle. In both of these examples one has to perform the remarkable trick of stripping reality off all the inhibiting complications that arise in practice. They are in other words idealized versions of the physical phenomena that they seek to describe and for that reason they can only take us so far. The real world is after all a messy place.

Indeed, quite often we are faced with problems that are inherently complicated. One doesn't even need to look very hard to find a problem such as that. Just considering three bodies interacting with one another classically (that means with low enough energies that relativity doesn't come into play and at large enough scales that quantum mechanics is not important) leads to a problem that does not have an exact solution and is highly sensitive to the initial conditions. The famous three-body problem. In other words out of some very simple ingredients we have obtained a behavior that appears complex. Simply by adding additional ingredients then, one expects more and more sophisticated structures to emerge which can lead to new paradigms altogether. A very famous account of this notion of emergence is given in [2].

This entails a very simple yet crucial fact about the physicist's enterprise which is that if we hope to make sense of the real world we cannot rely solely on such simplified models of the natural world, distilled and filtered of their unpleasant properties. Instead, sometimes, we need to get our hands dirty and deal with all the gory details. That is why complexity, a notion precisely intended to describe these non-trivial aspects of physical systems, is worth exploring. If we hope to make sense out of concepts like quantum chaos, strongly interacting systems, and perhaps even the quantum nature of gravity it seems that complexity is either a necessary ingredient or an unavoidable obstacle, depending on your predisposition or in my case what day of the week it is.

As a final remark, I should comment on the state of the art from as general a viewpoint as I

can offer. In my opinion a good analogy about our current understanding of complexity and related matters is very similar to the understanding of thermodynamics during the 17th-19th centuries. People at the time had developed an understanding of notions like temperature, heat transfer, thermal equilibrium and even entropy without any knowledge of the inner workings of the systems they were considering. In the 20th century, with the advent of the theory of molecules, it became apparent that thermodynamics is an emergent statistical description of the interactions between a large number of microscopic constituents. If you think about it, it really is a triumph of the human intellect that we arrived at such conclusions as the laws of thermodynamics without knowing what the systems we were studying were made of, though in hindsight some aspects of the theory might look rather crude or ad-hoc. This is very much similar to the situation we are in right now with complexity. We have ways to quantify it, perhaps even measure it, as well as apparent laws that describe its behaviour, but at the same time we lack a proper fundamental understanding. I do not know whether something equivalent to molecules even exists or whether a solution to P vs NP is a prerequisite to get there. Perhaps such a solution is not possible as a matter of principle or it might require a few generations of scientists to find it. Regardless, I am optimistic that investigating and characterizing the inner workings of complex systems is less of an exercise in futility and more of a venture into the unknown.

Chapter 2

Introduction

2.1 Holographic complexity

One of the main motivations for this work has been holographic complexity, a notion that came to the forefront with the famous "entanglement is not enough" [3]. It is hard to provide the context for this statement without at least mentioning some milestones in the field of holography as a whole, which I will do without delving too much into the details. In its most general form the holographic principle posits that the degrees of freedom in a volume of spacetime can be encoded on its boundary which has one less dimension [4,5]. Therefore, one can describe the same physical system using two different theories that live in a different number of dimensions. This served as a possible way to understand some aspects of the mysterious nature of black holes and in particular one of their most peculiar features which is that their entropy scales with their surface area rather than their volume. This is captured by the famous Bekenstein-Hawking formula for the entropy of a black hole

$$S_{BH} = \frac{A}{4G} \ , \tag{2.1}$$

where A stands for the area and G is Newton's gravitational constant. This distinguishes in a rather radical manner black holes from something like an ordinary box filled with a gas whose entropy scales with the volume. This creates an apparent discrepancy as, in principle, a sufficiently large volume of gas can collapse, leading to the formation of a black hole. How does one reconcile these two states of a gas cloud then, one of which depends on its volume and the other on its surface area? This question hints to the very non-trivial idea that it might somehow be possible to describe the physics of a portion of spacetime using a theory with fewer degrees of freedom.

A concrete manifestation of such a duality between theories was provided by Maldacena in [6], where he offered evidence showing that the physics of an $AdS_5 \times S^5$ spacetime is dual to that of a $\mathcal{N} = 4$ super Yang-Mills conformal field theory. These results were quickly consolidated by a number of works (e.g. [7–10]) and thus the AdS/CFT conjecture essentially became its own field of research. The new paradigm that followed was based on the idea that an AdS spacetime in any number of dimensions is dual to a conformal field theory (CFT), so even if one does not know the specifics (namely the field content of the CFT) they can still make progress in understanding the duality due to the stringent symmetry constraints. In other words, it is not important to know the details on the field theory side, since important quantities like two-point and three-point functions are fixed by the symmetries and can be related to bulk (geometric) quantities on the side of AdS. This induced a gradual shift from a string theory focused approach to holography towards an attempt at a more comprehensive understanding of the duality.

One of the early seminal ideas in that direction, that ultimately shaped the last two decades of research is the Ryu-Takayanagi proposal [11] which relates the entanglement entropy of a boundary region to an area of spacetime. In its simplest form the proposal relates the length of a geodesic lying

on a timeslice to the entanglement entropy of the boundary interval its endpoints are anchored to. This was generalized in [12] to include covariant generalizations of entanglement entropy which correspond to minimal surfaces in the bulk, rendering geodesics a special case. Today this idea appears almost natural to the members of this community, but one has to realize the groundbreaking consequences of a quantitative relation between an information theoretical quantity, the entanglement entropy, and a purely geometric quantity. This is precisely what inspired slogans like "entanglement = geometry" and fueled a vast number of works related to "bulk reconstruction", that is the effort to reconstruct the bulk spacetime using data from the boundary theory. While this program has led to remarkable advances, such as the ability to reconstruct local operators in the bulk [13,14], curves and other geometric objects [15–17], entanglement wedge reconstruction [18, 19] and others, there is a geometric phenomenon that entanglement seems to be unable to capture. Namely, if one considers a two-sided wormhole embedded in AdS, it can be shown that the volume of its throat keeps growing while the entanglement entropy on the boundary saturates. Therefore it is evident that the behaviour of the entanglement entropy cannot possibly describe the growth of the wormhole, calling in turn for the introduction of a different quantity that can. Hence computational complexity was introduced as a potential candidate in [20].

The natural dual quantity in the gravitational theory appeared to be a volume, a proposal which was refined upon in [21]. This was dubbed the "complexity=volume" proposal and was followed by a host of others, the most prominent of which is the "complexity=action" [22, 23], positing that the dual quantity of complexity is the action of the Wheeler-de Witt patch. More recently these ideas have been extended to a large (in fact infinite) class of gravitational observables [24,25]. While there has been a large body of works which compute quantities in the bulk that are candidate duals of complexity, little progress has been made in precise matching between those with actual measures of complexity defined in the quantum theory of the boundary. I will reserve further discussion on this point for the final chapter of this thesis and instead, given that my work was predominantly concerned with the computation of quantum complexity and not gravity, I will discuss some of the different measures that are of interest to the community in the next chapter.

As a final remark on holographic complexity, it is worth mentioning that it is probably not the panacea that will resolve all the mysteries of the AdS/CFT correspondence and holography as a whole. Perhaps it seems like complexity is the only missing quantity that would allow one to fully describe a spacetime (at least semiclassically), given the data of a boundary theory. In my view, complexity is a natural quantity that emerges as a probe for the dynamics of a quantum system and as such it probably contains information about the dynamical evolution of a dual gravitational system. However, it seems improbable that it can describe all the intricacies of operator growth and spreading of quantum states and it is therefore reasonable to assume that it will also leave gaps in our knowledge of the dual process. For these reasons I believe we should treat it as yet another piece of the labyrinthine puzzle that is holography.

2.2 Examples of complexity measures

The main body of this thesis is devoted to the study of a measure called Krylov complexity. However, I thought that instead of making this a vacuum chamber where only this measure exists, I should start by mentioning different measures that people have studied previously. Hopefully, this will highlight the advantages and disadvantages of every approach and give the reader a sense of the challenges that one faces in this line of research. Additionally, in the articles that are presented in the following chapters we frequently make comparisons with other approaches without necessarily introducing them thoroughly, so this constitutes an attempt to provide the minimum background required for the ensuing discussions.

2.2.1 Circuit complexity

As explained previously the notion of computational complexity originated in the field of computer science where it is used to estimate the difficulty of performing a certain task by quantifying the amount of resources that are required to carry it out. This idea admits a natural generalization in the context of quantum mechanics as the difficulty of constructing a certain *target state* given an initial *reference state*. The process of transitioning from the reference state to the target state can be broken down to a number of elementary operations that we can perform on our states, otherwise called "gates" [26]. Typically, one wants to pick such gates to be certain unitary operators. This allows us to construct a *circuit*, endowed with a notion of time and a Hamiltonian evolution. One can then further endow this circuit with a notion of complexity by assigning a cost to the gates that are being implemented. Of course the choices of gates and their costs are arbitrary, but they can be well motivated. For example if one is considering an experimental protocol that involves certain manipulations of a quantum state, it is natural to assign a low cost to operations that are easier to perform in the lab. Similar considerations apply to other cases, but it should be evident that using this approach entails an inherent degree of arbitrariness that cannot be avoided.

One may illustrate the concepts above by drawing a simple graph, like the one found in [27], which I have included in figure 2.1. Quantum circuits of this type are of paramount importance

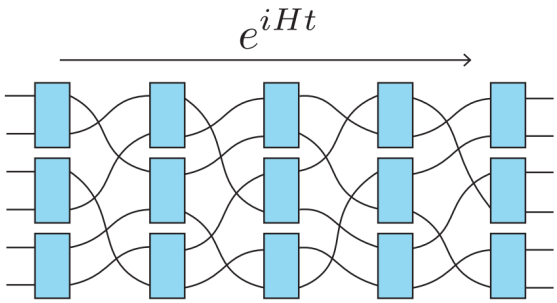

Figure 2.1: Illustration of a quantum circuit with time evolution generated by a 2-local Hamiltonian. In the simplest case the blocks represent qubits, but more generally they can be qudits or arbitrary subsystems of the quantum system under study.

for much more than just quantum computation though. Their dynamics can be tweaked to closely mimic the behaviour of generic quantum systems and more importantly black holes. According to Hayden and Preskill [28] this should be realized by picking gates through an ensemble of random unitaries, which ensures the reproduction of the maximal mixing property that's characteristic of black holes.

In order to generalize this idea to quantum field theories one needs to replace the discretum of a circuit of the form discussed above with a continuum. An indispensable tool for this change of perspective is the geometric approach to complexity due to Nielsen and his collaborators [29–31]. The seminal concept is that instead of thinking in terms of the application of discrete gates we can view the transition from the reference state to the target state as a trajectory in the space of states (or equivalently in the space of unitary operators). This does not fix the choice for a cost function, although some standard choices in the literature are the Fubini-Study distance and the $\mathcal{F}_1$ cost function, defined respectively as

$$\mathcal{F}_{FS}(\sigma)d\sigma = \sqrt{\langle\psi_R| dU(\sigma)^\dagger dU(\sigma) |\psi_R\rangle - |\langle\psi_R| U(\sigma)^\dagger dU(\sigma) |\psi_R\rangle|^2} \tag{2.2}$$

$$\mathcal{F}_1(\sigma)d\sigma = \left|\langle\psi_R| U(\sigma)^\dagger dU(\sigma) |\psi_R\rangle\right| . \tag{2.3}$$

Here $|\psi_R\rangle$ denotes the reference state and $U(\sigma)$ is a function defining a path in the space of unitaries.

This allows a description of complexity in terms of the geometric features of the latter, thus providing the basis for a definition of complexity in QFT. In [32, 33] the authors introduce these notions and set a paradigm wherein complexity acquires a meaning as the geodesic distance between the reference and target states.

This paradigm has also been made further use of within the context of CFTs [34, 35], where it is evident that the coadjoint orbits of the conformal and Virasoro symmetry groups respectively play a prominent role in the description of complexity. While the notion of a coadjoint orbit perhaps seems like an exotic mathematical object, it often arises when one considers the representations of a symmetry group and the state spaces they define. In later chapters we will see that this is a common point between Nielsen and Krylov complexities. While the exact relationship between the two is still being discussed, this serves to show there are perhaps certain universal features of any measure of quantum complexity.

2.2.2 The OTOC

The out-of-time-order correlator (OTOC) is a tool predominantly thought of as a probe for quantum chaos, even though it was initially introduced as part of the calculation of a vertex correction for the current of a superconductor [36]. Its definition is in terms of two Hermitian operators V, W and reads

$$C(t) = -\langle [W(t), V(0)]^2 \rangle \ , \tag{2.4}$$

where $\langle \cdot \rangle$ denotes thermal averaging. A very instructive argument for its relation to chaos is presented in [37] where the authors consider the OTOC for a generalized pair of position and momentum operators q, p. Namely, they notice that the commutator $[q(t), p]$ in the semiclassical limit is expressed in terms of the Poisson bracket $i\hbar\{q(t), p\} = i\hbar \frac{\partial q(t)}{\partial q(0)}$. It is known that in classically chaotic theories this quantity grows exponentially with a characteristic *Lyapunov* exponent as $\frac{\partial q(t)}{\partial q(0)} \sim e^{\lambda_L t}$, so the OTOC should grow as $C(t) \sim \hbar^2 e^{2\lambda_L t}$.

There is one immediately obvious limitation of the OTOC which stems from the fact that its interpretation as a probe of chaos relies on the existence of a semiclassical limit from which a relation with a Lyapunov exponent is inferred. Not all quantum systems conform to this condition, as a semiclassical limit might not be well defined, but the OTOC is unarguably an important diagnostic in the cases it can be used, with the additional advantage of relating quantum and classical chaos directly.

So far though it does not seem like the OTOC quantifies complexity, at least in the same straightforward manner that circuit complexity does. However, it is related to the notion of *operator growth* which can be interpreted as a quantifier of the complexity of operators. More concretely, operator growth refers the increasing number of degrees of freedom of a system that an operator affects. An initially simple operator, which is in some sense localized, can grow in terms of affecting more degrees of freedom as the system evolves. It is conjectured that for chaotic systems the rate of this growth is maximal. This is indeed a reasonable assumption given that one of the quintessential features of chaos is the "short memory" of a system, meaning how quickly its evolution loses its dependence on the initial state. This is closely related to the notion of *scrambling*, which quantifies the redistribution of information among the degrees of freedom of a system, even though their precise relation is for the time being rather obscure.

One of the seminal conclusions of [37] is the introduction of the chaos bound, which restricts the maximal growth of operators by the Lyapunov exponent

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar} \ . \tag{2.5}$$

It is believed that this bound is observed by black holes as prime examples of quantum chaotic systems that also have a well defined semiclassical limit. The saturation of this bound has also

served as a signature of chaos. This has been a development of profound importance as it is quite often not feasible to check the spectral statistics of complex systems directly.

2.2.3 Path integral optimization

The notion of path integral optimization was introduced in [38] as a means to estimate holographic complexity by utilizing the putative connection between the formulation of tensor networks and Anti-de-Sitter spacetime (introduced in [39]). This is a method heavily relying on the structure of conformal field theory as it renders the computation of the relevant path integrals tractable. The premise is that by finding the optimal discretization of the path integral leading to the correct vacuum CFT one can introduce a notion of complexity by counting the number of tensors in the emerging tensor network description. Let us unpack this statement following the original work [38] and a few that followed it [40, 41].

The ground state wave function, which is a functional of the boundary condition $\tilde{\varphi}(x)$, of a CFT in d dimensions is computed via the path integral

$$\Psi_{\text{CFT}}(\tilde{\varphi}(x)) = \int \left(\prod_x \prod_{\epsilon<z<\infty} \mathcal{D}\varphi(z,x) \right) e^{-S_{\text{CFT}}} \prod_x \delta(\varphi(\epsilon,x) - \tilde{\varphi}(x)) \ , \tag{2.6}$$

where the Euclidean time τ is given by $z = -(\tau - \epsilon)$ and x represents the rest of the spatial dimensions. This path integral is discretized in a lattice by the introduction of the UV cutoff ϵ leading to the flat metric

$$ds^2 = \epsilon^{-2}(dz^2 + dx_i dx^i) \ . \tag{2.7}$$

The optimization procedure involves changing the geometry of the lattice regularization, which pictorially corresponds to figure 2.2 taken from [38]. In 2 dimensions, where this procedure is most

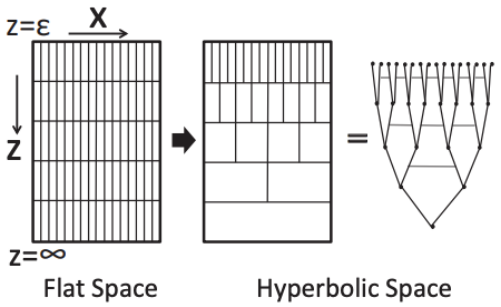

Figure 2.2: Optimization of the ground state wave functional computed from the Euclidean path integral. Note the analogy with the tensor network form on the right.

easily carried out analytically, this amounts to rewriting the metric with a conformal factor to be associated with ϵ as

$$ds^2 = e^{2\phi(z,x)}(dz^2 + dx^2), \quad e^{2\phi(z=\epsilon,x)} = \frac{1}{\epsilon^2} \ . \tag{2.8}$$

It can then be shown that

$$\Psi_{g_{ab}=e^{2\phi}\delta_{ab}}(\tilde{\varphi}(x)) = e^{S_L[\phi]-S_L[0]}\Psi_{g_{ab}=\delta_{ab}}(\tilde{\varphi}(x)) \ , \tag{2.9}$$

where S_L is the Liouville action

$$S_L[\phi] = \frac{c}{24\pi}\int_{-\infty}^{\infty} dx \int_{\epsilon}^{\infty} dz[(\partial_x\phi)^2 + (\partial_z\phi)^2 + \mu e^{2\phi} + R_0\phi] \ , \tag{2.10}$$

with R_0 being the Ricci scalar of the original space, which for this setup is zero. The constant c denotes the central charge of the CFT and μ is a constant that can be absorbed by a shift of ϕ. The argument is then that the optimization is achieved by minimizing the normalization $e^{S_L[\phi]}$, which is equivalent to requiring the equations of motion of the Liouville action

$$4\partial_w \partial_{\overline{w}} \phi = e^{2\phi} \ , \tag{2.11}$$

with $w = z + ix$ and $\overline{w} = z - ix$. Different solutions of the equations of motion will lead to different geometries corresponding to the appropriate time slices of the gravity dual. This approach has also been extended to states other than the vacuum and namely, states excited by primary operators and the thermofield double. The complexity arising from path integral optimization is inherently holographic, something that was expanded on in [41] using Hartle-Hawking wavefunctions, both in AdS_3 and higher dimensional spacetimes. Moreover, this method admits a generalization to Lorentzian path integrals that are related to de-Sitter spacetimes instead.

It should be evident that this definition of complexity adopts a radically different approach compared to the ones described previously. Namely, this measure is tailored to the language of CFTs and the preparation of a state using a path integral, rather than studying the time evolution of a generic quantum system. This change in perspective makes it interesting in its own right, but the reason I chose to highlight this is because it is even more intriguing to consider whether it is related to the other definitions in some non-obvious way. To my knowledge, there is no sharp answer yet, but I will offer some speculative comments in the final section of this thesis.

2.3 Operator growth and Krylov complexity

The notion of Krylov complexity was introduced in [42] as a way to quantify operator growth in a generic quantum system. In this section I will provide a detailed introduction to its definition and basic properties, presented through the lens of my own involvement in the topic. As before, the goal is to quantify how complex an operator (or state) becomes under time evolution. However, instead of assigning certain costs to certain operations one takes a different approach that assigns a notion of complexity from the perspective of the system itself. This produces an inherent quantity that is uniquely determined for a quantum system, given a Hamiltonian and an initial state. In a somewhat limited analogy one could compare this to the notion of time (or length) in relativity. While different observers will assign different values to these quantities, there exists a unique proper time (length). In that sense Krylov complexity is a kind of "proper" complexity.

2.3.1 The Lanczos algorithm

Let us see how one can obtain it simply starting with a generic quantum system for an initial operator that I will denote $\mathcal{O}$ and a Hamiltonian H. The time evolution of this operator is then given by the Heisenberg equation

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt} \ , \tag{2.12}$$

which can be expanded into the form

$$\mathcal{O}(t) = 1 + it[H, \mathcal{O}(0)] + \frac{(it)^2}{2}[H, [H, \mathcal{O}(0)]] + ... \tag{2.13}$$

First, it is convenient to define the Liouvillian super-operator

$$\mathcal{L} = [H, \cdot] \ . \tag{2.14}$$

It is a super-operator in the sense that it does not act on the usual Hilbert space of states, but rather on the Hilbert space of operators. Using this we can rewrite the above equation as

$$\mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O}(0) = 1 + it\mathcal{L}\mathcal{O}(0) + \frac{(it)^2}{2}\mathcal{L}^2\mathcal{O}(0) + ... \tag{2.15}$$

Henceforth, I will follow the standard convention found in the literature and use $|\mathcal{O})$ to emphasize that operators are thought of as states in the Hilbert space of operators in contrast to the usual Hilbert space of states for which I will reserve the usual Dirac notation. Going back to equation (2.15) one then finds that the time evolved operator can be thought of as a linear combination of states and namely the ones created by acting with consecutive powers of the Liouvillian on the initial state. Generally, a set of vectors generated by the action of consecutive powers of an operator is called a *Krylov space*, which is where the nomenclature derives from. If we can then write

$$|\mathcal{O}(t)) = \sum_n \varphi_n |K_n) \ , \tag{2.16}$$

where the index n counts the power of the Liouvillian and $|K_n)$ denotes a properly orthonormalized set of vectors in the Krylov space, we are immediately provided with a notion of complexity. The more times one needs to apply the Liouvillian in order to reach a state the more complex it is, or in other words the higher the statistical weight of states with large n in the above linear combination, the higher the complexity. Before making this more precise though, it is imperative to understand how one goes from (2.15) to (2.16). Namely, how does one turn the set of states $\{|\mathcal{O}), \mathcal{L}|\mathcal{O}), \mathcal{L}^2|\mathcal{O}), ...\}$ to the orthonormalized basis I labeled as $|K_n)$?

The canonical way of doing so is called the Lanczos algorithm [43], which is nothing but the Gram-Schmidt orthogonalization scheme applied to this particular set of vectors. The way it proceeds is as follows:

1. Start with the initial operator $|\mathcal{O}) = |K_0)$ and apply the Liouvillian to obtain $\mathcal{L}|K_0) = |\mathcal{O}_1)$. Then normalize such that $|K_1) = b_1^{-1}|\mathcal{O}_1)$, with $b_1^2 = (\mathcal{O}_1|\mathcal{O}_1)$.

2. Apply the Liouvillian on the state obtained previously to obtain $\mathcal{L}|K_1) = |\mathcal{O}_2) + b_1|K_0)$. The final term is included to ensure that the state we are generating is going to be orthogonal to $|K_1)$ but it is not $|K_0)$. We then define $|K_2) = b_2^{-1}|\mathcal{O}_2)$, with $b_2^2 = (\mathcal{O}_2|\mathcal{O}_2)$.

3. Proceed in a similar fashion, generating $|\mathcal{O}_n) = \mathcal{L}|K_{n-1}) - b_{n-1}|K_{n-2})$ and defining $|K_n) = b_n^{-1}|\mathcal{O}_n)$, with $b_n = (\mathcal{O}_n|\mathcal{O}_n)$. The algorithm stops if $b_n = 0$ for some n, but there are also cases where it goes on indefinitely.

The output of the algorithm is the orthonormal basis $|K_n)$ which is called the Krylov basis and the set of normalization constants b_n called the Lanczos coefficients. Before continuing there are some points worth addressing. In the description of the Lanczos algorithm, I used the notion of an inner product between operator states without defining it. There are different inner products that one can use, but the most common is the Wightmann inner product defined as

$$(A|B) = \langle e^{H\beta/2}A^\dagger e^{-H\beta/2}B\rangle_\beta \ . \tag{2.17}$$

However, depending on the problem at hand it might be more convenient to use a different definition. For example when dealing with systems with finite Hilbert space, one can use the Frobenius norm defined as

$$(A|B) = \frac{1}{D}\,\mathrm{Tr}\Big\{A^\dagger B\Big\}, \quad ||A||^2 = \frac{1}{D}\,\mathrm{Tr}\Big\{A^\dagger A\Big\} \ , \tag{2.18}$$

as was done for instance in [44]. A judicious choice of inner product can be instrumental in the simplification of the calculations involved in the Lanczos algorithm. This statement relates to the next point which is that the action of the Liouvillian on the Krylov states is of the form

$$\mathcal{L}|K_n) = b_{n+1}|K_{n+1}) + b_n|K_{n-1}) \ . \tag{2.19}$$

More generally, one can consider the inclusion of a diagonal term $a_n|K_n)$ in the expression above, which then generates an extra set of coefficients through the Lanczos algorithm. However, when dealing with Hermitian operators and using the Wightmann inner product the coefficients a_n vanish which makes matters simpler.

Another way of looking at the Lanczos algorithm is as a method to tridiagonalize the Liouvillian, thus bringing it into the form

$$\mathcal{L}_{ij} = \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 \\ b_1 & 0 & b_2 & \ddots & \vdots \\ 0 & b_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & 0 \end{pmatrix} , \tag{2.20}$$

where b_n are the Lanczos coefficients. This raises yet another important question as to why the Liouvillian assumes such a simple form since all we are doing is implementing the Gram-Schmidt orthogonalization. That is to say that typically the Gram-Schmidt matrix is not constrained to be tridiagonal so what makes this case special? The answer lies in the equation for the generation of a new element of the Krylov basis. This process is described by

$$|\mathcal{O}_n) = \mathcal{L}|K_{n-1}) - b_{n-1}|K_{n-2}) \ , \tag{2.21}$$

where we see that the generated element has to be orthogonal only to the two previous ones and not the full set of vectors as one would expect for typical Gram-Schmidt. The short explanation for this phenomenon is that this we are not dealing with a generic set of vectors, but rather a special set generated by powers of an operator [45]. That is, if one assumes

$$|\mathcal{O}_3) = \mathcal{L}|K_2) - b_1|K_1) - c|K_0) \ , \tag{2.22}$$

by demanding that $(K_0|\mathcal{O}_3) = 0$ it can be easily verified that it has to be $c = 0$. Similarly, one can show that all off-diagonal elements vanish, except for b_n.

A different approach that we might take makes use of the moments of the Liouvillian

$$\mu_n = (K_0|(i\mathcal{L})^n|K_0) \ . \tag{2.23}$$

Notice that the relationship of the moments to the problem at hand is through the autocorrelator (sometimes called the survival amplitude) $S(t) = \langle\mathcal{O}(t)|\mathcal{O}(0)\rangle = \varphi_0^*(t)$, which serves as the generating function of the moments

$$\mu_n = \frac{d^n}{dt^n} S(t)\Big|_{t=0} \ . \tag{2.24}$$

These can then be related to the Lanczos coefficients through (2.33), highlighting that all that all the information contained in the Krylov basis is packaged inside the autocorrelator. This method is particularly useful in cases where the spectrum of the Hamiltonian (Liouvillian) is known, or the system of interest is of finite size which allows for the efficient use of numerical methods. The precise relationship between moments and Lanczos coefficients is through the determinants of the Hankel matrix [45]

$$b_1^2 ... b_n^2 = \det(\mu_{i+j}) \ . \tag{2.25}$$

2.3.2 The Krylov chain

At this point we have obtained the Krylov basis $|K_n)$, but if we want to express the time evolved state as in (2.16) we still need to gain access to the expansion coefficients φ_n. As it turns out there is an easy way to relate the latter to the Lanczos coefficients by taking a time derivative of (2.16) in two different ways

$$\frac{d}{dt}e^{i\mathcal{L}t}|\mathcal{O}_0) = i\mathcal{L}|\mathcal{O}(t)) = i\sum_n \varphi_n(t)[b_{n+1}|K_{n+1}) + b_n|K_{n-1})] \tag{2.26}$$

$$\frac{d}{dt}|\mathcal{O}(t)) = \sum_n \frac{d\varphi_n(t)}{dt}|K_n) \ . \tag{2.27}$$

These must be equal, which further implies

$$\boxed{-i\frac{d\varphi_n(t)}{dt} = b_{n+1}\varphi_{n+1}(t) + b_n\varphi_{n-1}(t)} \ . \tag{2.28}$$

This equation is the cornerstone of the study of Krylov complexity, which is why it deserves to be placed in a special box. By an internal redefinition of the wavefunctions of the form $\varphi_n = i^n\phi_n$, it can be recast in the form

$$\frac{d\phi_n(t)}{dt} = b_n\phi_{n-1} - b_{n+1}\phi_{n+1} \ , \tag{2.29}$$

which also appears in later chapters.

There are many comments that I would like to make about this expression, but first let us complete the framework by finally defining a measure of complexity. Once we have solved the Lanczos algorithm we have at our disposal the b_n, which we can use to solve this equation iteratively, using as an initial condition that $\varphi_n(0) = \delta_{n,0}$. The initial condition is nothing more than the requirement that at time $t = 0$ the operator as expressed in the Krylov basis, is the initial operator. Solving for all φ_n leads to the probability distribution satisfying $\sum_n |\varphi_n(t)|^2 = 1$. Krylov complexity is then defined as the mean of that distribution

$$\mathcal{C}_K = \sum_n n|\varphi_n(t)|^2 \ . \tag{2.30}$$

Heuristically this quantity measures how many times on average one has to apply the Liouvillian on the initial state, in order to be able to effectively describe it at a later time. Of course in any non-trivial case one expects this number to grow, at least initially. Therefore a primary concern is not the growth itself but instead the rate of growth, which can be used to distinguish between different classes of systems.

A concept that is closely related to Krylov complexity is the Krylov or K-entropy, first introduced in [46]. It is defined in terms of the probability distribution over the Krylov basis as

$$S_K = \sum_n |\varphi_n(t)|^2 \log|\varphi_n(t)|^2 \ . \tag{2.31}$$

K-entropy is not to be confused with the entanglement entropy which is obtained by considering the reduced density matrix associated to a subsystem of the total state. Instead, it is more insightful to interpret it directly in terms of its information content. Namely, K-entropy is the Shannon entropy related to the probability distribution over the Krylov basis and in this sense it constitutes a measure of uncertainty of the possible states of the system. In other words, it characterizes the uniformity of the probability distribution, which is an important diagnostic of the dynamics, complementing the information about the mean of the distribution that is provided by Krylov complexity.

Returning to (2.28) one may notice that it can be interpreted as a discrete Schrödinger equation for a particle hopping between nearest neighbor sites on a semi-infinite chain. The Lanczos coefficients play the role of the probability of hopping from one site to the next and the φ_n can be interpreted as the amplitudes of finding the particle at any given site. This interpretation has been dubbed the Krylov chain picture and has led to significant insights. The Krylov complexity can then be interpreted as the average position of the particle on the chain as a function of time. If one takes a moment to appreciate this statement they will realize that what we have achieved is quite remarkable. We started with the seemingly complex problem of tracking down how an operator evolves in a particular subspace whose dimension grows as the system evolves and we mapped it to tracking the average position of a particle living on a one-dimensional chain. Not bad.

In fact, this picture has been an important heuristic tool in a number of recent works. As an example, one can take a continuous limit [47–49] of (2.28) by setting $x = \epsilon n$, leading to

$$-(\partial_t + ia)\Phi = \epsilon b'\Phi + 2\epsilon b\Phi' + O(\epsilon^2) \ , \tag{2.32}$$

where now b has been promoted to a continuous function playing the role of velocity. While this approach leads to certain simplifications, there are also certain subtleties involved in taking this limit that are beyond the scope of this thesis and are discussed in detail in [49].

2.3.3 Krylov complexity for states and the optimality of the Krylov basis

The framework discussed in the previous subsections can be extended to the study of quantum states rather than operators. While there are certain subtleties related to switching between the two, for all practical purposes it can be thought of as simply switching from the Heisenberg to the Schrödinger picture of quantum mechanics. The framework for the Krylov complexity of states, or spread complexity as it is often called, was established in [50]. Here I will simply review the relevant differences between operators and states, and discuss why the Krylov basis is special to begin with.

For the evolution of states one makes use of the Hamiltonian instead of the Liouvillian, for which a Lanczos algorithm can be applied in order to bring it in a tridiagonal form as before. Thus, the related expression for the action of the Hamiltonian on the Krylov basis is

$$H\left|K_n\right\rangle = a_n\left|K_n\right\rangle + b_n\left|K_{n-1}\right\rangle + b_{n+1}\left|K_{n+1}\right\rangle \ . \tag{2.33}$$

Unlike case of operators, there is no standard way of setting $a_n = 0$, which is why these coefficients will be explicit from now on. Ultimately this is related to the Schrödinger equation

$$i\partial_t\psi_n(t) = a_n\psi_n(t) + b_{n+1}\psi_{n+1}(t) + b_n\psi_{n-1}(t) \ . \tag{2.34}$$

The goal is the same as before, namely to decompose the time evolved state as

$$\left|\psi(t)\right\rangle = \sum_n \psi_n(t)\left|K_n\right\rangle \ , \tag{2.35}$$

and subsequently compute $\mathcal{C}_\mathcal{K} = \sum_n n|\psi_n(t)|^2$. The computation of the Lanczos coefficients is once again achieved either by performing the Lanczos algorithm, or equivalently by using the moments of the Hamiltonian $\mu_n = \frac{d^n}{dt^n}\left\langle\psi(t)\right| H^n\left|\psi(0)\right\rangle|_{t=0}$.

Generally, one could ask what is special about the Krylov basis to begin with. After all, the process of obtaining it through the Lanczos algorithm is a rather ad hoc procedure which seemingly does not arise from any deeper principles. Following the line of reasoning in [50] one can define a more general function of the form

$$C_\mathcal{B}(t) = \sum_n c_n|\langle\psi(t)|B_n\rangle|^2 \ , \tag{2.36}$$

with $|B_n\rangle$ being an element of a complete, ordered, orthonormal basis $\mathcal{B} = \{|B_n\rangle, n = 0, 1, 2, ...\}$ and c_n a positive, increasing sequence of positive numbers. In [50] the authors show that minimization of complexity over the basis

$$C(t) = \min_{\mathcal{B}} C_{\mathcal{B}}(t) \ , \tag{2.37}$$

is achieved when $\mathcal{B}$ is the Krylov basis. In that sense this basis is a special choice which by default produces the minimal number for complexity. This method of proof does not appear to exclude cases where a choice of different basis leads only to a difference of an additive or multiplicative constant, not affecting the scaling, which is after all the most relevant feature. While the Krylov basis might then strictly minimize complexity, it could still be useful to use a convenient basis from which the scaling can be inferred. A discussion in terms of a concrete example is offered in chapter 5.

2.3.4 Krylov complexity and quantum chaos

Krylov complexity was first introduced as part of the "Universal operator growth hypothesis" of [42] which purports that an exponential growth of complexity signifies quantum chaos. In order to understand this argument, it is relevant to discuss the relationship between complexity, the Lanczos coefficients, the moments and the spectral function. As explained previously, all the information about the Lanczos coefficients and the expansion of a state in the Krylov basis is packaged in the autocorrelation function

$$S(t) = (\mathcal{O}(t)|\mathcal{O}(0)) \ . \tag{2.38}$$

The Fourier transform of this quantity is the spectral function

$$\Phi(\omega) = \int_{-\infty}^{\infty} S(t) e^{-i\omega t} dt \ . \tag{2.39}$$

The moments can also be obtained from it as

$$\mu_n = \int \omega^n \Phi(\omega) d\omega \ . \tag{2.40}$$

Given the relations between these quantities one can establish an upper bound for the growth of the Lanczos coefficients, based on the behavior of the spectral function which follows [42, 51]

$$\Phi(\omega) \sim e^{-\left|\frac{\omega}{\omega_0}\right|^{\frac{1}{\delta}}}, \quad 1 \geq \delta > 0 \ . \tag{2.41}$$

It is believed that an exponential decay of the spectral function can serve as a predictor of chaos (see for example [52]). The above expression is equivalent to

$$b_n \sim n^{\delta} \ . \tag{2.42}$$

This behaviour of the spectral function is based on the assumption of a k-local Hamiltonian. In turn it implies the maximal growth of the Lanczos coefficients is at most linear in n[1]. This further implies an exponential growth of Krylov complexity.

It has already been shown that the universal operator growth is not consistent with the dynamics of quantum field theories. This was first noticed in [53] and further elaborated on in [54, 55]. Essentially, the authors verify that Krylov complexity can grow exponentially even for free theories which are clearly not chaotic and hence the hypothesis does not apply. While Krylov complexity is not a good probe for chaos in this regime, the evolution in Krylov space still appears to efficiently

[1] There are additional subtleties for 1-D systems that lead to a sub-linear maximal growth, which I will not examine here

track the dynamics of QFTs and exhibits interesting behavior such as persistent staggering effects of the Lanczos coefficients [54, 55].

A rigid consensus has not yet been reached on the usefulness of Krylov complexity as a probe of chaos. Nevertheless, the utility of tracking the dynamics of a quantum system through the Krylov space is evident. In order for a system to be classified as chaotic it seems that exponential growth of Krylov complexity is a necessary condition, but does not appear to be a sufficient one. In other words there seems to be no correlation between the exponential growth of Krylov complexity and chaotic dynamics. However, different aspects of the Krylov formalism might be sensitive to these phenomena. For example in [49] the authors argue that it is possible to diagnose chaos through the close link between the transition probability (survival amplitude) and the spectral form factor, first reported in [50].

In the latter it was already shown that for random matrices the Krylov complexity exhibits a characteristic behaviour containing a ramp-peak-slope-plateau pattern that in a way mirrors the spectral form factor. Moreover, it was shown that removing the correlations between the matrix eigenvalues removes the peak-slope portion of the evolution. This clearly indicates a differentiation between chaotic dynamics, which are strongly linked to the correlations between random matrix eigenvalues, and non-chaotic dynamics in their absence. Subsequently in [49] it was further shown that fundamentally (and perhaps not surprisingly) it is the transition probability that determines how the system behaves and in fact the existence of a long ramp in the transition probability gives rise to the peak in Krylov complexity.

One final noteworthy feature of Krylov complexity is that it serves as an upper bound for a number of other measures (dubbed q-complexities in [42]), among which the most notable is the OTOC. In fact, Krylov complexity respects the chaos bound of [37] for which it admits the same value as the OTOC. This introduces an interesting point of comparison of the two quantities which has not yet been fully understood.

Chapter 3

Geometry of Krylov complexity

Extended summary

In the following work we developed a framework to treat systems that are characterized by symmetries described by certain Lie groups. This led to several insights, including the emergence of a geometric picture for Krylov complexity, an inherent algebraic structure that we dubbed the complexity algebra, and the computation of other quantum informational quantities that can be used to probe the evolution of an operator of a quantum system.

I think it is interesting to make some remarks about the state of the art at the time when we started working on this project and how our results brought a new perspective to the field. As explained in detail in the introduction, in order to compute Krylov complexity one has to first go through the Lanczos algorithm and subsequently solve the resulting discrete Schrödinger equation iteratively. This is usually a very tedious process and finding a system for which these tasks can be carried out analytically seemed nothing short of a miracle. One prominent example in which that is possible is the large q limit of the SYK model, which was studied in [42]. After some painstaking calculations the authors arrive at the formula for the wavefunctions

$$\varphi_n(t) = \sqrt{\frac{\Gamma(\eta+n)}{n!\Gamma(\eta)}}\frac{\tanh^n(\alpha t)}{\cosh^\eta(\alpha t)}\ , \tag{3.1}$$

where the constants α, η originate in the autocorrelation function $C(t) = \cosh^{-\eta}(\alpha t)$.

We noticed that this formula is suspiciously similar to the so called squeezed states that are well known objects in quantum optics. This of course was not an accident. These seemingly unrelated systems share the same symmetry structure and namely they can both be interpreted as generalized coherent states of the group SL(2,R). The implication then became apparent. If the Liouvillian is a linear combination of the generators of a Lie algebra, the time evolved state $e^{-i\mathcal{L}t}|\mathcal{O}(0))$ can be interpreted as a generalized coherent state[1]. This not only allowed us to analytically compute Krylov complexity for a number of examples involving other Lie groups, but it also highlighted that these results are universal for each one of them. In other words the dynamics of the system, as far as Krylov complexity is concerned, are completely determined by the symmetry structure. Additionally, coherent states are well studied objects and it is known that they give rise to geometrical spaces which we used to interpret Krylov complexity in this new light.

Apart from the consolidation of the results regarding the large q limit of SYK, our main results included the computation of the Lanczos coefficients and Krylov complexity for the groups SL(2,R), SU(2) and Heisenberg-Weyl. In particular, we computed the Lanczos coefficients and Krylov wavefunctions analytically and showed that their respective complexities follow the universal behavior

$$K_{SL(2,R)} = 2h\sinh^2(\alpha t), \quad K_{SU(2)} = 2j\sin^2(\alpha t), \quad K_{HW} = \alpha^2 t^2\ , \tag{3.2}$$

[1]Simply put a generalized coherent state can be thought of as a group element acting on an initial state.

where h, j denote the weight of the SL(2,R) and SU(2) representations and in each case α is a non-universal constant that encodes the physical details of a system. All these groups are associated with a rank-1 Lie algebra, simply meaning that they are comprised of a raising and a lowering operator and a diagonal operator. We identified that this structure is an inherent property of the generation of the Krylov space through the Lanczos algorithm which is what allowed us to perform the matching of symmetries. Additionally, it allowed us to form a simplicity hypothesis which concerns the closure of the algebra and showed that, if it is respected, it leads to the linear bound of the growth of the Lanczos coefficients. We dubbed this structure as the "complexity algebra" and provided the groundwork to interpret it from the point of view of the phase space.

Indeed, for each one of these groups, the evolution of the system can be thought of as a trajectory in the space of coherent states and depending on its geometric features we obtain a different universality class of results. These also distinguish between systems that exhibit exponential growth and as such can potentially be chaotic and systems that are inadvertently non-chaotic.

Finally, we discuss the extension of these notions in 2D CFT, with applications in holography in mind. We show that since the global part of the conformal symmetry is SL(2,R), the dynamics generated that it generates conform to the universal behaviour previously derived. In this way, we lay the foundations for more detailed explorations of the full conformal symmetry as well as potential holographic interpretations.

Overall, this work provided a more complete picture of the notion of Krylov complexity and a framework that facilitates its computation for systems with symmetry. Our paradigm has proved useful in establishing a number of subsequent results, for example proving the speed limits of operator growth [56,57], the relationship with orthogonal polynomials [47], the spread complexity of states [50] and more recently a duality between the rate of change of complexity and the momentum of an infalling particle [58].

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Abstract

We develop a geometric approach to operator growth and Krylov complexity in many-body quantum systems governed by symmetries. We start by showing a direct link between a unitary evolution with the Liouvillian and the displacement operator of appropriate generalized coherent states. This connection maps operator growth to a purely classical motion in phase space. The phase spaces are endowed with a natural information metric. We show that, in this geometry, operator growth is represented by geodesics and Krylov complexity is proportional to a volume. This geometric perspective also provides two novel avenues towards computation of Lanczos coefficients and sheds new light on the origin of their maximal growth. We describe the general idea and analyze it in explicit examples among which we reproduce known results from the Sachdev-Ye-Kitaev model, derive operator growth based on SU(2) and Heisenberg-Weyl symmetries, and generalize the discussion to conformal field theories. Finally, we use techniques from quantum optics to study operator evolution with quantum information tools such as entanglement and Renyi entropies, negativity, fidelity, relative entropy and capacity of entanglement.

3.1 Introduction

The study of classical and quantum chaos is both an exciting and inherently complicated subject. In the classical regime, there is an accepted definition of chaos, based on the behavior of nearby trajectories in phase space and the sensitivity to initial conditions i.e., the butterfly effect [59]. For chaotic Hamiltonians, trajectories that started infinitesimally close to each other separate exponentially fast with a characteristic Lyapunov exponent. Difficulties in translating such definitions into quantum mechanics have ended up producing many complementary probes of non-equilibrium systems with signatures of quantum chaos [36, 60, 61].

Many recent approaches to quantum chaos, that are in one way or another motivated by holography [6], have focused on analyzing characteristic features of the Heisenberg evolution of quantum operators. The main goal there is to quantify the spread/growth of the initial "simple" operator into "complex" operators of the model as time evolves. Sensible measures of this "operator growth" or "operator complexity" are expected to grow exponentially fast in time allowing the definition of different Lyapunov exponents. A related task is to classify families of many-body models from the perspective of operator growth and specify the necessary as well as sufficient conditions for a holographic gravity dual.

When quantifying the operator growth, we face two types of problems: conceptual and technical. Conceptually, the notion of operator size is, to a large extent, arbitrary and it seems unlikely that a single universal quantity will play this role. Indeed, up to date, various "witnesses" of the growth have been employed. Among them stand out the out-of-time-ordered correlators (OTOC) appearing when computing variances of commutators [36, 37, 62–64]. Their analysis culminated in [37], where a universal upper bound on the Lyapunov exponent, saturated by black holes, was found. This approach is closely related to the eigenstate thermalisation hypothesis [65] and to scrambling [62, 66–69]. Later, a natural definition of operator size in the Sachdev-Ye-Kitaev (SYK) model [64,70] was analyzed in [71,72]. More recently, quantum information definitions were employed in [73–75] and a broader approach based on the GNS construction was described in [76]. Finally, the idea of Krylov complexity, that will be our main topic, was put forward in [42]. In all these approaches, various notions of size were shown to evolve exponentially fast and Lyapunov-type exponents were extracted. On the technical side, we need to solve the Heisenberg dynamics in the models of interest. Unfortunately, solving the evolution equation is out of reach in most chaotic quantum systems and thus we are only able to perform numerics in specific models.

In what follows, we will focus on the approach to operator growth and Krylov complexity proposed in [42]. This work is rooted in the Lanczos algorithm approach to many-body dynamics [45] (that we review below) used as a systematic way of constructing a basis in the space of complex operators (Krylov basis). In the same framework, authors introduced a notion of operator complexity, dubbed "Krylov complexity", that can be computed in many-body systems including quantum field theories. In addition, using explicit numerical and analytical examples, the authors put forward a "universal operator growth hypothesis", arguing that in maximally chaotic systems, Krylov complexity grows at most exponentially fast with a characteristic Lyapunov exponent. Moreover, they pointed that the operator growth hypothesis might ultimately lead to a new physical proof and understanding of the chaos bound, see also [65].

Various aspects of this hypothesis were already investigated in [46, 53, 76–78, 78–84] and Krylov complexity became a good candidate for a universal notion of complexity in interacting quantum field theories. Nevertheless, its physical as well as the operational meaning remain mysterious. On the same footing, the relation to more established notions of complexity is an open problem. On the other hand, despite the relatively unambiguous definition, computing Krylov complexity requires numerics and understanding its universal features becomes very complicated. These conceptual and technical drawbacks are our main motivations to explore and develop it further in this work.

To make progress, it will be fruitful to focus on certain classes of chaotic models such as those

appearing in the context of the AdS/CFT correspondence [6], where, due to conformal symmetry, many-body quantum states are efficiently described geometrically. Indeed, black holes in holography are often seen as collection of qubits (the so-called "central dogma") described by Hamiltonians that show signatures of maximal quantum chaos. The SYK model [64, 70] described by two-dimensional Anti-de Sitter (AdS_2) gravity is the canonical modern example. Moreover, the quantum information "revolution" that started with holographic entanglement entropy [11] and continues with holographic complexity [21–23, 85, 86] brought new intuitions that allow us to connect seemingly unrelated concepts from quantum information and computation to geometry (see e.g. reviews [87, 88]). For instance, microscopic measures of operator growth and complexity are believed to encode subtle information about near horizon geometries of black holes [76, 82, 89–96].

In this light, we develop a geometric approach to Krylov complexity. Our work will explore the underlying symmetries controlling the system dynamics, although certain observations will be more general. We will be led to the field of generalized coherent states and their associated information geometry. This geometrization will clarify the definition of the operator complexity from a physical standpoint. More concretely, we will find a precise interpretation of the Krylov complexity as a volume in the information geometry. We will also find the relation between the symmetry algebra governing the operator growth and isometries of this geometry. At the same time, we will see how this approach simplifies the technical analysis opening new avenues towards the computation of defining aspects of operator growth, such as Lanczos coefficients or Lyapunov exponents in various chaotic and integrable setups. We also notice that the present approach provides a new geometric take on an old field, namely the Lanczos approach to non-equilibrium dynamics, connecting it with the field of generalized coherent states.

This article is organised as follows. In sec II we review the Lanczos algorithm and its recent applications to maximmally chaotic systems. In sec III we describe our main idea that, for symmetry scenarios, the Liouvillian operator can be written in terms of algebra generators as a sum of "ladder" operators. This naturally connects with generalized coherents states and their associated geometry. In sec IV we illustrate these ideas in four canonical examples, SL(2,R) (or SU(1,1)), SU(2), Heisenberg-Weyl and 2d CFTs. As highlights, the Lanczos coeffients for SYK, first derived in [42] using involved techniques, will acquire a simple and more transparent meaning, and we will determine the geometric roles played by Krylov complexity and the operator wavefunction. In sec V we arrive at the Lanczos coefficients in yet another way, by enforcing the closure of the ladder operator algebra. In sec VI we formulate operator dynamics in terms of a purely classical motion, allowing connections with classical chaos and geometric approaches to complexity. In sec VII, using the two-mode representation of coherent states from quantum optics, we introduce several quantum information tools to probe operator growth: operator entanglement/Renyi entropies, negativity, capacity, fidelity and relative entropy. Finally, in sec VIII we discuss generalizations of Krylov complexity in CFTs and relations to known tools used in discussions of complexity and chaos. Four appendices provide more technical details complementing the discussion in the main part.

3.2 Operator Growth and Krylov Complexity

We begin with a brief review of the Lanczos approach [45] to operator dynamics in many-body systems, leading to the definition of Krylov complexity. We also review previous results for the SYK model that will be reproduced in the following part of the article using our novel approach. Since this topic may not be familiar to a broader audience, we explain it in a slightly pedagogical manner. Readers familiar with the subject may proceed directly to the next section.

3.2.1 Operator Growth

The Lanczos approach starts with a quantum Hamiltonian H and a time-dependent Heisenberg operator $\mathcal{O}(t)$ in a given model. The operator can have more labels, such as position, spin, etc., but for the present purposes, only the time-dependence will be explicitly denoted. The evolution of the operator is governed by the Heisenberg equation

$$\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)], \tag{3.3}$$

where $[A, B] = AB - BA$, is the commutator. This equation is formally solved by

$$\mathcal{O}(t) = e^{iHt}\, \mathcal{O}(0)\, e^{-iHt}, \tag{3.4}$$

and in what follows, we will denote $\mathcal{O}(0) = \mathcal{O}$. The previous expression can be expanded in a formal power series in t as

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \tilde{\mathcal{O}}_n, \tag{3.5}$$

where $\tilde{\mathcal{O}}_n$ are nested commutators of $\mathcal{O}$ with the Hamiltonian

$$\tilde{\mathcal{O}}_0 = \mathcal{O}, \quad \tilde{\mathcal{O}}_1 = [H, \mathcal{O}], \quad \tilde{\mathcal{O}}_2 = [H, [H, \mathcal{O}]], ... \tag{3.6}$$

Knowing the result of these commutators is equivalent to solving the operator dynamics. Unfortunately, this is rarely the case in generic physical systems.

Despite this technical obstruction, we would like to have a notion of growth or complexity of the Heisenberg operator as a function of time. Intuitively, if the Hamiltonian governing the dynamics is sufficiently "chaotic", even if we start from a "simple" operator $\mathcal{O}$, the result of these commutators will be given by increasingly complex operators. In other words, the more "chaotic" the Hamiltonian H, the faster the operator $\mathcal{O}$ will mix with other operators of the theory. The main objective is then to quantify such a mixing in a precise manner.

3.2.2 Lanczos Algorithm and Krylov Basis

In order to sharpen the previous intuitions it will be useful to switch to a better suited formalism and define the Liouvillian super-operator $\mathcal{L}$ (see e.g. [45]) as

$$\mathcal{L} = [H, \cdot], \qquad \mathcal{O}(t) \equiv e^{i\mathcal{L}t}\mathcal{O}, \tag{3.7}$$

and by super-operator we just mean a linear map in the space of operators of the theory. In this language, the operators $\tilde{\mathcal{O}}_n$ in (3.5) are results of the repeated action of the Liouvillian $\mathcal{L}$ on $\mathcal{O}$ such that $\tilde{\mathcal{O}}_n \equiv \mathcal{L}^n \mathcal{O}$.

This view suggests interpreting (3.5) as an "operator's wavefunction", and the Liouvillian $\mathcal{L}$ as a Hamiltonian in the Schrodinger formulation. However, we cannot qualify the coefficients of t^n associated with operators $\tilde{\mathcal{O}}_n$ as "amplitudes". One transparent reason is that the sum of their modulus squared is not conserved in time. The precise reason though is that to use the operator algebra as a Hilbert space (in which we expand vectors unambiguously in an orthonormal basis), we need to introduce an inner product. The choice of such an inner product is one of the ambiguities (features) of this approach. In this work, we will follow the most canonical one used in the physics literature.

More concretely, associating $|\mathcal{O})$ with the Hilbert space vector corresponding to operator $\mathcal{O}$, the following family of inner products was described in [45]

$$(A|B)^g_\beta = \int_0^\beta g(\lambda)\, \langle e^{\lambda H} A^\dagger e^{-\lambda H} B \rangle_\beta \, d\lambda. \tag{3.8}$$

In this formula, the bracket $\langle\rangle_\beta$ denotes the thermal expectation value

$$\langle A\rangle_\beta = \frac{1}{Z}\,\mathrm{Tr}\left(e^{-\beta H}A\right), \qquad Z = \mathrm{Tr}\left(e^{-\beta H}\right). \tag{3.9}$$

Also, for this definition to be a proper inner-product, $g(\lambda)$ has to satisfy the following conditions

$$g(\lambda)\geq 0, \quad g(\beta-\lambda)=g(\lambda), \quad \frac{1}{\beta}\int_0^\beta d\lambda g(\lambda)=1. \tag{3.10}$$

In this work, following [42], we will mainly focus on the Wightman inner product

$$(A|B) = \langle e^{H\beta/2}A^\dagger e^{-H\beta/2}B\rangle_\beta, \tag{3.11}$$

which corresponds to $g(\lambda)=\delta(\lambda-\beta/2)$. This is a physical choice that amounts to taking the expectation value of the operators in the thermofield double state, with operators A and B inserted in the two different copies. In any case, once the dynamics is solved for one specific choice of inner product, the behaviour associated with other choices can be found (see e.g. App A in [76]).

Once we have chosen an inner product, the arbitrary choice of basis in which to expand our evolving operator does not affect the physics of the problem. However, some choices are more convenient than others. Here we will follow the Lanczos approach to non-equilibrium dynamics, which uses the canonical basis generated by the $|\tilde{\mathcal{O}}_n)$. More precisely, starting from $|\tilde{\mathcal{O}}_n)$ and using the Gram–Schmidt orthogonalization procedure we arrive at an orthonormal basis, known as the Krylov basis $|\mathcal{O}_n)$. In a certain precise sense, this is the "optimal" choice since the operators $|\tilde{\mathcal{O}}_n)$ are the only ones appearing in (3.5).

The Krylov basis is defined recursively using the following algorithm (also known as Lanczos algorithm). We start by noticing that the first two operators in $|\tilde{\mathcal{O}}_n)$ are always orthogonal with respect to the previous inner products (3.8). Therefore we can directly include them in our basis

$$|\mathcal{O}_0) := |\tilde{\mathcal{O}}_0) = |\mathcal{O}), \qquad |\mathcal{O}_1) := b_1^{-1}\mathcal{L}|\tilde{\mathcal{O}}_0), \tag{3.12}$$

where $b_1 = (\tilde{\mathcal{O}}_0\mathcal{L}|\mathcal{L}\tilde{\mathcal{O}}_0)^{1/2}$ normalizes the vector. The next states are constructed iteratively by first computing

$$|A_n) = \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2}), \tag{3.13}$$

and then normalizing

$$|\mathcal{O}_n) = b_n^{-1}|A_n), \qquad b_n = (A_n|A_n)^{1/2}. \tag{3.14}$$

This way, we arrive at an orthonormal basis $(\mathcal{O}_n|\mathcal{O}_m)=\delta_{n,m}$ that has been generated by the set $\{\mathcal{L}^n\mathcal{O}\}$. We can now use it to expand any element of this set and the evolving operator $|\mathcal{O}(t))$. Notice that in addition to the Krylov basis states $|\mathcal{O}_n)$, this algorithm yields the so-called Lanczos coefficients b_n. Finding these coefficients for the system under consideration amounts to solving for the dynamics and it is one of the technical challenges in this approach, see [45]. Let us also point that the above algorithm can be generalized to include diagonal terms in the Liouvillian (see e.g. Appendix A).

We now expand the time-dependent operator in the Krylov basis as

$$|\mathcal{O}(t)) = \sum_n i^n\varphi_n(t)|\mathcal{O}_n)\,. \tag{3.15}$$

In this expansion, the amplitudes $\varphi_n(t)$ turn out to be real. Generally, their modulus squared defines probabilities whose sum is conserved in time

$$\sum_n |\varphi_n(t)|^2 \equiv \sum_n p_n(t) = 1. \tag{3.16}$$

These amplitudes are determined by solving a "Schrodinger equation", that descends from the original Heisenberg equation satisfied by $\mathcal{O}(t)$. To derive this equation, notice that the previously defined Liouvillian $\mathcal{L}$ plays the role of the Hamiltonian in the new Hilbert space spanned by the Krylov basis $|\mathcal{O}_n)$. In particular, the state representing $\mathcal{O}(t)$ is given by

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}). \tag{3.17}$$

Computing the time derivative

$$\partial_t|\mathcal{O}(t)) \quad = \quad i\mathcal{L}|\mathcal{O}(t)), \tag{3.18}$$

or equivalently, using (3.15) we arrive at

$$\partial_t|\mathcal{O}(t)) = \sum_n i^n \partial_t \varphi_n(t)|\mathcal{O}_n). \tag{3.19}$$

Next, from the Lanczos algorithm (3.13), we find the action of the Liouvillian on the Krylov basis vectors

$$\mathcal{L}|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1}). \tag{3.20}$$

From this expression it is clear that the Liovillian is tridiagonal in the Krylov basis (generally we may have a diagonal term in (3.20)). This fact will play an important role in the following sections. Applying this to (3.18) and shifting the summation appropriately, we derive

$$\partial_t|\mathcal{O}(t)) = \sum_n i^n \left(b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)\right)|\mathcal{O}_n). \tag{3.21}$$

Comparing the coefficients of (3.19) and (3.21), we arrive at the discrete Schrodinger equation determining the time evolution of the amplitudes $\varphi_n(t)$

$$\partial_t\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)\ . \tag{3.22}$$

With this equation, once we derive the Lanczos coefficients b_n, we can solve for the amplitudes $\varphi_n(t)$ with initial condition $\varphi_n(0) = \delta_{n0}$ and determine the operator wavefunction (3.15). The operator's wavefunction then completely determines the growth of the operator that, as we will describe below, can be measured using tools of quantum mechanics, quantum information, or quantum complexity.

Before we discuss operator's complexity, we note that a very special role in the Krylov approach is played by the so-called auto-correlation function

$$C(t) \equiv (\mathcal{O}(t)|\mathcal{O}) = \varphi_0(t)\ . \tag{3.23}$$

Indeed, as reviewed in [42], starting from $C(t)$ and/or its appropriate transforms we can obtain the Lanczos coefficients b_n and operator wavefunction. In this work, it will be more instructive to develop our physical understanding of the Liouvillian instead. This will allow us to easily extract both $C(t)$ and b_n.

3.2.3 Krylov Complexity

We now describe how to quantify operator complexity in this framework. Using physical intuition, we can first interpret the dynamics in equation (3.22) as that of a particle moving on a one-dimensional chain, where the sites with label n are in one-to-one correspondence with the Krylov basis vectors (see also [80] for a Toda chain perspective). This suggests a natural measure of operator complexity, dubbed Krylov complexity [42], defined to be the average position in the chain

$$K_{\mathcal{O}} \equiv \sum_n n\, p_n(t) = \sum_n n\,|\varphi_n(t)|^2\ . \tag{3.24}$$

Formally, this quantity can be written as the expectation value in the evolving state $|\mathcal{O}(t))$ of the following "Krylov complexity operator"

$$\hat{K}_{\mathcal{O}} = \sum_n n|\mathcal{O}_n)(\mathcal{O}_n| \,, \tag{3.25}$$

such that Krylov complexity reads

$$K_{\mathcal{O}} = (\mathcal{O}(t)|\hat{K}_{\mathcal{O}}|\mathcal{O}(t)) \,. \tag{3.26}$$

Intuitively, this position operator (3.25) in the chain can also be interpreted as a "number operator". Unlike the Liouvillian, it is diagonal in the Krylov basis.

Clearly, as with the choice of the inner product, there is a certain ambiguity in this definition of operator complexity. Indeed, several definitions of operator complexities that have appeared in the literature can always be written in such a way, see [42,76]. However, as we will see in this work, this "minimal" choice acquires a simple geometric interpretation.

The recent interest in the Krylov approach to operator complexity has various origins. First, modulo simple physical assumptions, it is a well defined and concrete approach, potentially applicable to QFTs. These features make it appealing from the point of view of holography. Second, based on various explicit numerical as well as analytical examples, [42] conjectured a maximal possible growth of Lanczos coefficients in quantum systems, namely a linear growth:

$$b_n \leq \alpha n + \gamma + O(1), \tag{3.27}$$

where α is the operator growth rate and γ is a non-universal constant that depends on the details of the operator. In particular, for this type of Lanczos coefficients, i.e., systems saturating the bound, the Krylov complexity grows exponentially fast with an exponent given by $\lambda = 2\alpha$. In several examples, some of which will be described below, at finite temperature $T = 1/\beta$ one arrives at $\alpha = \pi/\beta$, and this was conjectured to bound the Lyapunov exponent, as defined by out-of-time ordered correlation functions [37].

3.2.4 SYK example

As the key example of the behaviour (3.27), the SYK model [64,70], which is a modern playground for quantum chaos [37,64], was analyzed in [42]. The SYK model [64,70] is a model of N Majorana fermions interacting with all-to-all random couplings. For random q-body interactions, the Hamiltonian is of the form

$$H = i^{q/2} \sum_{1\leq i_1<i_2<\cdots<i_q\leq N} J_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q} \,, \tag{3.28}$$

This model has been at the center of attention for the past years for several important reasons, namely exact solvability at large N, conformal phase at low energies, and maximal chaos in the sense of [37].

Operator growth for this system was considered in [71], using a natural notion of growth arising from the exact Majorana fermion formulation of the model. An advantage of such an approach is that it was naturally related to out-of-time ordered correlation functions, see also [72]. A disadvantage is that such a definition does not seem to find a natural extension to higher dimensions and QFTs.

Operator growth for this system was also reconsidered in [42] using the Lanczos approach. As explained above, the starting point of this approach can be taken to be the autocorrelation function. For SYK at low temperatures this is

$$C(t) = \cosh^{-\eta}\left(\frac{\pi t}{\beta}\right). \tag{3.29}$$

In this case, the Lanczos coefficients can be obtained analytically [42] (see also [80]) and are given by

$$b_n = \frac{\pi}{\beta}\sqrt{n(\eta+n-1)}\,. \tag{3.30}$$

The operator wavefunction can then be found by solving (3.22) and reads

$$\varphi_n(t) = \sqrt{\frac{\Gamma(\eta+n)}{n!\Gamma(\eta)}}\frac{\tanh^n(\alpha t)}{\cosh^\eta(\alpha t)}. \tag{3.31}$$

The probabilities $p_n(t) = |\varphi_n(t)|^2$ from this solution correspond to the negative binomial distribution. The evolution of these probabilities depicts a one-dimensional diffusion process over the Krylov basis. The time evolution of the mean position in this chain, or equivalently the evolution of Krylov complexity, is of exponential type. It is controlled by the maximal Lyapunov exponent $\lambda = 2\pi/\beta$. More explicitly

$$K_{\mathcal{O}} = \eta\sinh^2(\alpha t) \sim \frac{\eta}{4}e^{2\alpha t} = e^{2\alpha\left(t-\frac{1}{2\alpha}\log\left(\frac{4}{\eta}\right)\right)}\,, \tag{3.32}$$

where we have written the coefficient of the exponent in an analogous way to the scrambling time in the OTOC. Observe that, while the exponential growth is "more universal" than the usual Lyapunov growth (it does not receive stringy corrections for example in the context of holography), the "scrambling time" for a given operator is by construction less universal. Nevertheless, it depends on the scaling dimension of the initial perturbation and may also be a good probe for the operator growth.

Before moving forward we want to make a couple of remarks. First, from a technical standpoint, the derivation of the operator wavefunctions in both [71] and [42] is quite involved. This feature makes it difficult to extrapolate to other systems, in particular to higher dimensions. On the other hand, readers familiar with the SYK model and the arguments that lead to the derivation of the correlator (3.29) (using large-N techniques, see [64, 70]) may recall it was the conformal symmetry appearing in the low energy Schwinger-Dyson equations that was responsible for the form of this two-point function. In other words, the fermions behave as primaries transforming in specific representations of the SL(2,R) algebra. In particular, for the q-body interaction, the associated scaling dimension is $h = 1/q$. We might expect a deeper and simpler understanding of operator dynamics and wavefunction when such a feature is included in the analysis.

Second, from a more holographic standpoint, the relation between Krylov complexity and the actual physics of the problem is far from clear. In the light of recent discussions on near horizon symmetries in black hole physics and their potential connections with operator complexity [76, 82, 89, 91–93], we would like to have a better understanding of the Krylov complexity operator.
In the following sections, we will explore a geometric avenue towards both problems, which more broadly can be seen as a new perspective on the Lanczos approach.

3.3 Liouvillian and symmetry: General Idea

In this section, we describe a general paradigm that we will follow through the rest of the article. The main idea is simple yet powerful, and we describe it below. From the zoo of complicated quantum systems, we focus our attention on models governed by symmetry. By this, we mean systems for which the Liouvillian operator belongs to the Lie algebra of a given symmetry group. In the context of the usual Shrodinger evolution, this is quite a common lore. For example, in QFT or CFT the Hamiltonian belongs to the Lie algebra of the Poincaré group or the conformal group, respectively. This idea is old and well explored in Hamiltonian dynamics (see e.g. review [97]). Here, we import it to the physics of operator evolution, instead of state evolution, where the Liouvillian plays the role of the Hamiltonian in the Krylov basis.

With symmetries in mind, our key observation is that the action of the Liouvillian on the Krylov basis (3.20) can be interpreted as the action of the sum of abstract "raising" and "lowering" ladder operators L_+ and L_-, namely

$$\mathcal{L} = \alpha \left(L_+ + L_- \right) . \tag{3.33}$$

In this expression, α is a proportionality factor, not fixed by symmetry. It will depend on the details of the physical setup, such as the choice of the inner-product, etc. Its meaning will become clearer in the examples below.

With such Liouvillians, the Krylov basis states will naturally furnish representations of the appropriate symmetry group. This is again analogous to relativistic QFT or CFT, where states are organized through representations of the Poincaré or conformal group. The only difference here is that we apply such a structure to operator dynamics on the Krylov basis.

In the light of symmetry, the previously described quantities associated with the Lanczos approach take a more transparent meaning. First, since the action of the ladder operators in a certain representation is fixed by the symmetry group, this approach allows us to read off the Lanczos coefficients immediately. More precisely, they are simply determined from the action of ladder operators in the Krylov basis

$$\alpha L_+ |\mathcal{O}_n) = b_{n+1} |\mathcal{O}_{n+1}), \quad \alpha L_- |\mathcal{O}_n) = b_n |\mathcal{O}_{n-1}). \tag{3.34}$$

We will also see that, under certain conditions, the Lie group approach leads to quadratic algebraic equations for Lanczos coefficients. This will ensure that, at least in our examples, they will not grow faster than n, in agreement with the maximal operator growth hypothesis [42].

Moreover, the above paradigm allows us to make a powerful connection with generalized coherent states [98–100]. This comes from the fact that the Liouvillian time evolution in the Krylov basis with (3.33) can be seen as a particular instance of a generalized displacement operator $D(\xi)$ for a Lie group. These displacements operators typically take the form

$$D(\xi) \equiv e^{\xi L_+ - \bar{\xi} L_-} , \tag{3.35}$$

for some complex ξ, its conjugate $\bar{\xi}$ and the same abstract ladder operators $L_\pm$. We will make all these formulas precise when analyzing specific examples in the next section. The coherent state can now be written as the action of the displacement operator on some reference state $|\Psi_0\rangle$, usually chosen to be the highest weight state of the representation. It is clear that unitary time evolution, as generated by the Liouvillian (3.33), is just a displacement operator with $\xi = i\alpha\, t$. In other words, we can interpret the operator dynamics and its growth in the Krylov basis as a trajectory through the Hilbert space of coherent states. This way, after associating $|\Psi_0\rangle$ with our initial operator $|\mathcal{O})$, and expanding the coherent states in an orthonormal basis, we will be able to read off the amplitudes $\varphi_n(t)$ and the Krylov basis vectors $|\mathcal{O}_n)$.

The link with coherent states further allows us to geometrize Krylov complexity. This formulation is rooted in the well-known connection between coherent states and information metric (Fubini-Study metric) on the Hilbert space, abstractly defined for the coherent state $|z\rangle$ as

$$ds^2_{FS} = \langle dz|dz\rangle - \langle dz|z\rangle\langle z|dz\rangle. \tag{3.36}$$

This metric is also associated with the coadjoint orbit of the relevant group (see e.g. [97]). As we will see, the Krylov complexity will be universally proportional to the "Volume" in this geometry. In addition, both the Liouvillian $\mathcal{L}$ as well as the Krylov complexity operator $\hat{K}_\mathcal{O}$ can be related to isometry generators in these information geometries. Indeed they form a "complexity algebra" isomorphic to the algebra of isometries and we will show how it determines Lanczos coefficients.

Finally, the association of the coherent state complex label ξ with real-time suggests that we are secretly discussing a classical motion in phase space. This interpretation is indeed correct and it paves a way towards understanding the relations between Krylov complexity and circuit complexity.

3.4 Liouvillian and symmetry: Examples

In this section, we analyze explicit examples of the general idea above. From a physical perspective, the most interesting one is that of SL(2,R) and its generalizations to Conformal Field Theories (CFT). These have applications to classical and quantum chaos and the physics of black holes. We will also discuss the examples of SU(2) and the Heisenberg-Weyl group, which will help us gain more intuition about the relation between Krylov complexity, group theory, and geometry.

3.4.1 Example I: SL(2,R)

The first example is operator evolution governed by SL(2,R). In this case, we will re-derive the SYK results of [42] using the above general paradigm.
We start from the commutation relations for the SL(2,R) algebra

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \qquad [L_1, L_{-1}] = 2L_0, \tag{3.37}$$

and consider a discrete series representation labeled by a positive integer h. This representation is typically expanded by orthonormal vectors $|h, n\rangle$, for n a non-negative integer, satisfying $\langle h, m|\, h, n\rangle = \delta_{n,m}$. The basis vectors are eigenstates of the L_0 operator as well as the Casimir operator $C_2 = L_0^2 - \frac{1}{2}(L_{-1}L_1 + L_1 L_{-1})$ with eigenvalue $h(h-1)$. The full action of the SL(2,R) generators in this basis is given by

$$\begin{aligned} L_0\,|h,n\rangle &= (h+n)\,|h,n\rangle\,, \\ L_{-1}\,|h,n\rangle &= \sqrt{(n+1)(2h+n)}\,|h,n+1\rangle\,, \\ L_1\,|h,n\rangle &= \sqrt{n(2h+n-1)}\,|h,n-1\rangle\,, \end{aligned} \tag{3.38}$$

which in particular implies that

$$|h,n\rangle = \sqrt{\frac{\Gamma(2h)}{n!\Gamma(2h+n)}} L_{-1}^n\,|h\rangle\ . \tag{3.39}$$

The same Hilbert space can be also expanded by means of generalized coherent states, see [99], that are defined by using the displacement operator

$$|z,h\rangle \equiv D(\xi)\,|h\rangle\,, \qquad D(\xi) = e^{\xi L_{-1} - \bar{\xi} L_1}, \tag{3.40}$$

where the relation between the complex variables is

$$z = \frac{\xi}{|\xi|}\tanh(|\xi|), \qquad |\xi| = \sqrt{\xi\bar{\xi}}. \tag{3.41}$$

It is useful to introduce polar coordinates $\xi = \frac{1}{2}\rho e^{i\phi}$, such that z parametrizes the unit disc

$$z = \tanh\left(\frac{\rho}{2}\right) e^{i\phi}, \qquad |z| < 1. \tag{3.42}$$

Using the action of the SL(2,R) generators on the primary state, in particular relation (3.39), we can write these so-called SU(1,1) Perelomov coherent states more explicitly as

$$|z,h\rangle = (1-|z|^2)^h \sum_{n=0}^{\infty} z^n \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}}\,|h,n\rangle\,. \tag{3.43}$$

Now we will follow the general paradigm described in the previous section. First, from (3.38), we note that L_{-1} is playing the role of the abstract raising operator L_+ and L_1 of the lowering operator

L_-. This way, the Liouvillian governing the SL(2,R) operator dynamics in the Krylov basis is given by

$$\mathcal{L} = \alpha \left(L_{-1} + L_1 \right) . \tag{3.44}$$

As reviewed above, the operator wavefunction (3.15) is obtained by applying the unitary evolution with $\mathcal{L}$, so that

$$|\mathcal{O}(t)) = e^{i\alpha(L_{-1}+L_1)t} \, |h\rangle \, . \tag{3.45}$$

Returning to the definition of the coherent state (3.40), we make the key observation that our operator's wavefunction is nothing but the Perelomov coherent state with $\xi = i\alpha t$. More explicitly we have the relation

$$|\mathcal{O}(t)) = |z = i \tanh(\alpha t), h = \eta/2\rangle \, , \tag{3.46}$$

as well as the identification between the Krylov basis and the basis vectors associated with representation h of the SL(2,R) group

$$|\mathcal{O}) = |h\rangle \, , \qquad |\mathcal{O}_n) = |h, n\rangle \, . \tag{3.47}$$

Arguably the most elegant consequence of this map is the fact that from the action of the ladder operators (3.38), we immediately get the Lanczos coefficients

$$b_n = \alpha \sqrt{n(2h + n - 1)}. \tag{3.48}$$

We can indeed check that the wavefunctions (3.31) are just coefficients of the coherent state (3.43) with $z = i \tanh(\alpha t)$ and solve the Schrodinger equation (3.22) with the Lanczos coefficients above. The Krylov complexity is then proportional to the highest weight h and grows exponentially with time, with the Lyapunov exponent $\lambda = 2\alpha$

$$K_{\mathcal{O}} = (\mathcal{O}(t)|n|\mathcal{O}(t)) = 2h \sinh^2(\alpha t). \tag{3.49}$$

Moreover, the $n = 0$ amplitude is the SYK auto-correlation function (3.29) with $\eta = 2h$.

A more appropriate interpretation of the identification $\xi = i\alpha t$ is that operator dynamics in this setup is mapped to a particular classical trajectory in the phase space of coherent states. In polar coordinates, this trajectory corresponds to setting $\rho = 2\alpha t$ and $\phi = \pi/2$. We will return to this interpretation in section VII.

We now introduce the "information geometry" associated with generalized coherent states and use it to interpret operator growth and Krylov complexity geometrically. To this end, we recall that in a quantum theory, the space of coherent states has an associated geometry described by the Fubini-Study metric (also dubbed information metric). For our states (3.43) this becomes the standard metric on the hyperbolic disc. In complex coordinates $(z, \bar{z})$ as well as in (ρ, ϕ) it reads

$$ds^2_{FS} = \frac{2h dz d\bar{z}}{(1 - z\bar{z})^2} = \frac{h}{2} \left(d\rho^2 + \sinh^2(\rho) d\phi^2 \right) . \tag{3.50}$$

With this geometry at our disposal, we want to make several comments. Firstly, the identification used to describe the growth, namely $\rho = 2\alpha t$ and $\phi = \pi/2$, defines a geodesic in this geometry. In other words, the operator growth process gets mapped to a geodesic motion in a hyperbolic geometry (3.50). This will be made more precise in section VII. Secondly, we can interpret the Krylov complexity operator as a generator of translations in the ϕ direction (an isometry generator). This is seen from the explicit form of the coherent state, and the fact that $-i\partial_\phi$ produces a factor n. We will discuss more precisely the relation between the isometries of this information geometry and the complexity algebra generated by the Liouvillian and the Krylov complexity operator in a later section.

Thirdly, motivated by the recent developments concerning the geometric approach to complexity, we note that the actual Krylov complexity is proportional to the volume enclosed by the geodesic radius $\rho = \alpha t$, i.e., it is proportional to the volume of the region from the origin $\rho = 0$ up to $\rho = 2\alpha t$ (see Fig. 3.1). The explicit computation gives

$$V_t = \int_0^{2\alpha t} d\rho \int_0^{2\pi} d\phi \sqrt{g} = 2\pi h \sinh^2(\alpha t) = \pi K_{\mathcal{O}}. \tag{3.51}$$

This is one of the main new results of our work. We will show that this relation holds more generally in other examples.

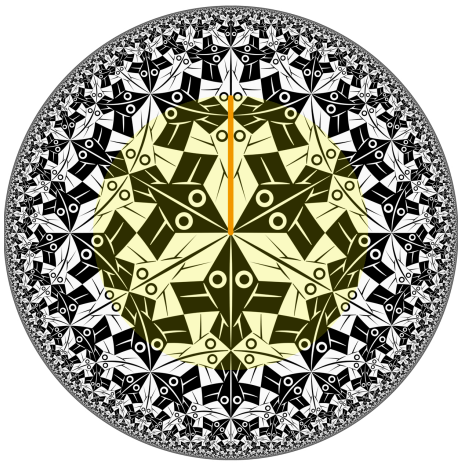

Figure 3.1: Cartoon of the operator growth and Krylov complexity for SL(2,R). Coherent states allow us to map the operator evolution to a geodesic (in orange) on the hyperbolic disc. Volume (in yellow) of the region enclosed by the particle's position $\rho = 2\alpha t$ at $\phi = \pi/2$ is proportional to the Krylov complexity.

Based on the intuition from Nielsen's approach to circuit complexity, to be described later, one may have naively expected a relation between the geodesic length and complexity. However, the geodesic distance between two arbitrary points (ρ_i, ϕ_i) and (ρ_f, ϕ_f) in geometry (3.50) is given by

$$\cosh(L/l) = \cosh(\rho_f)\cosh(\rho_i) - \cos(\Delta\phi)\sinh(\rho_f)\sinh(\rho_i), \tag{3.52}$$

where the radius of the hyperbolic space is denoted as $l^2 = h/2$. This way, if we measure it from the center of the disc $\rho_i = 0$, the geodesic length is $L = \rho_f$. For our geodesic motion, we have $\rho_f = \alpha t$, which only grows linearly in t. We will also return to this point in a later section, where we will see the more direct relation to Nielsen's complexity.

Last but not least, geometry (3.50) is negatively curved. The Ricci scalar is related to the highest weight state h as

$$R = -\frac{4}{h}, \tag{3.53}$$

and it decreases for large h.

There have already been several discussions, both in classical and quantum chaos as well as in the complexity literature, about the role of negatively curved information geometry [101–105]. In the context of black holes, for example, perturbations in the near horizon region can be described in this way, see [66, 68, 106, 107]. The present example is a precise contribution to this intuition. Indeed, we will see in the following examples that the sign of the curvature is correlated with the nature of Krylov complexity growth.

3.4.2 Example II: SU(2)

We now analyze the example in which the Liouvillian belongs to the SU(2) algebra. This will give us a more general intuition about the Krylov approach in non-chaotic systems. In particular, we will see the consequences of working with a finite-dimensional Hilbert space and having non-maximal Lanczos coefficients on the Krylov complexity and its geometry.

We start with the familiar SU(2) Lie algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \tag{3.54}$$

and introduce the ladder operators $J_\pm = J_1 \pm i J_2$. Renaming $J_3 \to J_0$ the previous algebra transforms into

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0. \tag{3.55}$$

Using the ladder operators we can build the usual basis for representation $j = 0, \frac{1}{2}, 1, \cdots$, namely $|j, n\rangle$, with $-j \leq n \leq j$. In order to make the connection with operator growth, it will be convenient to re-label the basis vectors as $n \to j + n$, so that $n = 0, ..., 2j$. This way, the $2j + 1$ orthonormal basis vectors can be written as

$$|j, -j + n\rangle = \sqrt{\frac{\Gamma(2j - n + 1)}{n!\Gamma(2j + 1)}} J_+^n |j, -j\rangle \ . \tag{3.56}$$

In this basis, the action of the Lie algebra generators is

$$\begin{aligned} J_0 |j, -j + n\rangle &= (-j + n) |j, -j + n\rangle , \\ J_+ |j, -j + n\rangle &= \sqrt{(n + 1)(2j - n)} |j, -j + n + 1\rangle , \\ J_- |j, -j + n\rangle &= \sqrt{n(2j - n + 1)} |j, -j + n - 1\rangle . \end{aligned} \tag{3.57}$$

As before, we will choose the highest weight state $|j, -j\rangle$, annihilated by J_-, as our initial state. Equivalently we could have started from $J_+ |j, j\rangle = 0$ but we chose to follow the usual convention [99].

Following previous steps, we build the so-called spin coherent states by applying the displacement operator

$$|z, j\rangle = D(\xi) |j, -j\rangle , \qquad D(\xi) = e^{\xi J_+ - \bar{\xi} J_-}, \tag{3.58}$$

where now we have the complex coordinate

$$z = \tan\left(\frac{\theta}{2}\right) e^{i\phi}, \tag{3.59}$$

that parametrizes a spherical geometry.

More explicitly, the spin coherent states are written in the orthonormal basis as

$$|z, j\rangle = (1 + z\bar{z})^{-j} \sum_{n=0}^{2j} z^n \sqrt{\frac{\Gamma(2j + 1)}{n!\Gamma(2j - n + 1)}} |j, -j + n\rangle . \tag{3.60}$$

To analyze the operator growth we repeat the same steps as in the previous example. The SU(2) Liouvillian takes the form

$$\mathcal{L} = \alpha(J_+ + J_-) \ , \tag{3.61}$$

and we find finite-dimensional Hilbert spaces, with dimensions $2j + 1$, where Krylov basis states are associated with orthonormal vectors in an obvious way

$$|\mathcal{O}_n) = |j, -j + n\rangle , \quad n = 0, ..., 2j. \tag{3.62}$$

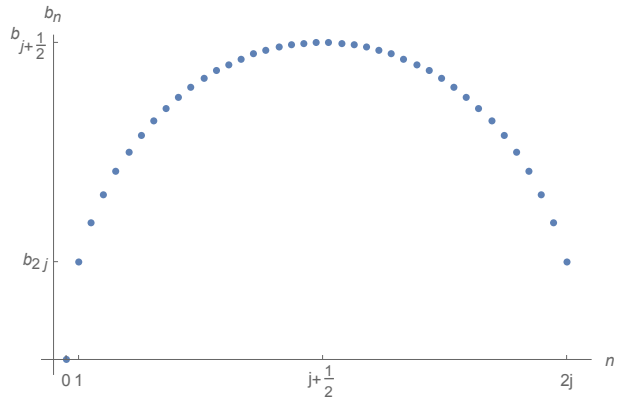

Figure 3.2: Distribution of the SU(2) Lanczos coefficients. Sample plot for $j = 20$.

With this identification, the action of the lowering operator J_- in (3.57) automatically allows us to read off the b_n's

$$b_n = \alpha\sqrt{n(2j-n+1)}. \tag{3.63}$$

These Lanczos coefficients grow slower than their SL(2,R) cousins, namely as $\alpha\sqrt{2j\,n}$ up to a maximum value

$$n_{max} = j + \frac{1}{2}, \quad b_{j+\frac{1}{2}} = \alpha\left(j+\frac{1}{2}\right) = \alpha n_{max}, \tag{3.64}$$

and then come back down to the final value (see example on Fig. 3.2)

$$b_{2j} = \alpha\sqrt{2j} = b_1. \tag{3.65}$$

Using the above form of the spin coherent states, we find the Heisenberg operator wavefunction $|\mathcal{O}(t))$ in the Krylov space by replacing $\theta = 2\alpha t$ and $\phi = \pi/2$. More precisely

$$|\mathcal{O}(t)) = |z = i\tan(\alpha t), j\rangle = e^{i\alpha(J_+ + J_-)t}\,|j,-j\rangle\,. \tag{3.66}$$

The SU(2) wavefunction arising from this identification

$$\varphi_n(t) = \frac{\tan^n(\alpha t)}{\cos^{-2j}(\alpha t)}\sqrt{\frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)}}, \tag{3.67}$$

satisfies the Schrodinger equation (3.22) with Lanczos coefficients (3.63). To get intuition about the shape of these functions, we plot the example of $j = 5$ in Fig (3.3). In this SU(2) case, the probabilities $p_n(t)$ form the binomial distribution

$$p_n(t) = |\varphi_n(t)|^2 = \binom{2j}{n}\lambda^n(1-\lambda)^{2j-n}, \tag{3.68}$$

with $\lambda = \sin(\alpha t)$. The auto-correlation function for SU(2), from which one obtains the return probability, is given by

$$C(t) = \varphi_0(t) = \frac{1}{\cos^{2(-j)}(\alpha t)}. \tag{3.69}$$

This correlation function appears e.g. when analyzing a free harmonic oscillator at finite temperature. The two-point function in Euclidean time for such an oscillator is (see e.g. [108])

$$G(\tau) = \frac{1}{2\omega}\frac{\cosh[(\beta/2-\tau)\omega]}{\sinh(\beta\omega/2)}\,. \tag{3.70}$$

After doing the usual analytic continuation $\tau \to it$, and then the analytic continuation towards the inner product $t \to t - i\beta/2$ we find

$$C(t) = \frac{1}{2\omega} \frac{\cos(t\omega)}{\sinh(\beta\omega/2)} , \tag{3.71}$$

which is the previous $SU(2)$ result for $j = 1/2$ and $\alpha = \omega$, up to operator normalization (which should be fixed at initial times). Other j are e.g. achieved by considering a different number of uncoupled harmonic oscillators with the same frequencies.

We also remark that in this class of models, the right assignation of α is temperature independent. This implies that e.g. introducing temperature in a free system does not change the complexity/operator growth. It just changes the correct operator normalization at the initial time, but not the operator wavefunction. This is similar to the computation of Lyapunov exponents at high temperature in SYK [109] (free regime), where it only depends on the coupling constant, the only scale-dependent parameter of the theory.

Using the operator wavefunction (3.67), we can now compute the Krylov complexity

$$K_{\mathcal{O}} = \sum_{n=0}^{2j} n|\varphi_n(t)|^2 = 2j \sin^2(\alpha t). \tag{3.72}$$

Clearly, there is no exponential growth of complexity in this case. This fits well with the fact that b_n's are not linear in n. More precisely, the complexity grows quadratically $K_{\mathcal{O}} \sim 2j\alpha^2 t^2$ at early times and reaches its maximum at $t = \pi/(2\alpha)$ given by $K_{\mathcal{O}}^{max} = 2j$. After that it reduces back to zero at $t = \pi/\alpha$. This is the expected behaviour for a complexity measure. The reason is that $t = \pi/(2\alpha)$ is the furthest point in the complexity geometry, as we are going to show shortly. Passing that point in phase space we begin our trip back to the initial state.

As in the case of SL(2,R), we can better observe these complexity features geometrically, by deriving the information metric associated with (3.60). This is the spherical metric

$$ds^2 = \frac{2j dz d\bar{z}}{(1+|z|^2)^2} = \frac{j}{2}\left(d\theta^2 + \sin^2\theta d\phi^2\right) . \tag{3.73}$$

We conclude that this particular non-chaotic operator dynamics is associated with geometry of constant positive curvature

$$R = \frac{4}{j}. \tag{3.74}$$

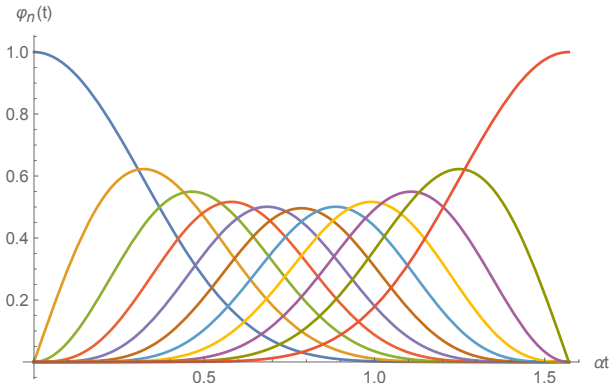

Figure 3.3: All the 11 wavefunctions $\varphi_n(t)$ for spin $j = 5$ plotted between $\alpha t \in (0, \pi/2)$. Different wavefunctions are peaked at later values of αt symmetrically, reflecting the symmetry of b_n's.

As before, for large spin j the curvature decreases. The operator growth again gets mapped to a geodesic in this geometry.

Last but not least, we can evaluate the volume in this information geometry up to $\theta = 2\alpha t$

$$V_t = \int_0^{2\alpha t} d\theta \int_0^{2\pi} d\phi \sqrt{g} = 2\pi j \sin^2(\alpha t) = \pi K_{\mathcal{O}} \,, \tag{3.75}$$

confirming our proposed relation between the volume in the information geometry and the Krylov complexity.

3.4.3 Example III: Heisenberg-Weyl

The next example is somewhat in between the previous two. It concerns the Heisenberg-Weyl algebra and its associated standard coherent states. We start with the usual creation ($a^\dagger$) and annihilation (a) operators, the identity 1 and the number operator ($\hat{n} = a^\dagger a$). These operators define the following algebra

$$[a, a^\dagger] = 1, \qquad [\hat{n}, a^\dagger] = a^\dagger, \quad [\hat{n}, a] = -a, \tag{3.76}$$

with all other commutators vanishing. The infinite dimensional Hilbert space is expanded in the usual orthonormal basis

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle \,, \tag{3.77}$$

on which the ladder operators $a^\dagger$ and a act as

$$a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle \,, \qquad a |n\rangle = \sqrt{n}\, |n-1\rangle \,, \tag{3.78}$$

and n is the eigenvalue of the number operator.

Following the general paradigm, these relations allow us to identify the Heisenberg-Weyl Liouvillian, the infinite dimensional Krylov basis and the Lanczos coefficients. These are given by

$$\mathcal{L} = \alpha(a^\dagger + a), \quad |\mathcal{O}_n) = |n\rangle \,, \quad b_n = \alpha\sqrt{n}. \tag{3.79}$$

The standard coherent states are defined by the action of the displacement operator on the vacuum state

$$|z\rangle = D(z) |0\rangle \,, \quad D(z) = e^{z a^\dagger - \bar{z} a}, \tag{3.80}$$

with complex coordinate $z = re^{i\phi}$. Using the previous algebra one finds

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \,. \tag{3.81}$$

We can now find the operator wavefunction by exploring the relation between the unitary evolution with the Liouvillian and the displacement operator. In particular, by setting $z = i\alpha t$, or $r = \alpha t$ and $\phi = \pi/2$, we write the Heisenberg's operator state in the Krylov space

$$|\mathcal{O}(t)) = |z = i\alpha t\rangle = e^{i\alpha(a^\dagger + a)t} |0\rangle \,, \tag{3.82}$$

from where the operator wavefunction is

$$\varphi_n(t) = e^{-\alpha^2 t^2/2} \frac{\alpha^n t^n}{\sqrt{n!}}, \qquad \sum_{n=0}^{\infty} |\varphi_n|^2 = 1. \tag{3.83}$$

It solves the Schrodinger equation (3.22) with the above b_n's and corresponding probabilities form the Poisson distribution. Examples are plotted on Fig. 3.4.

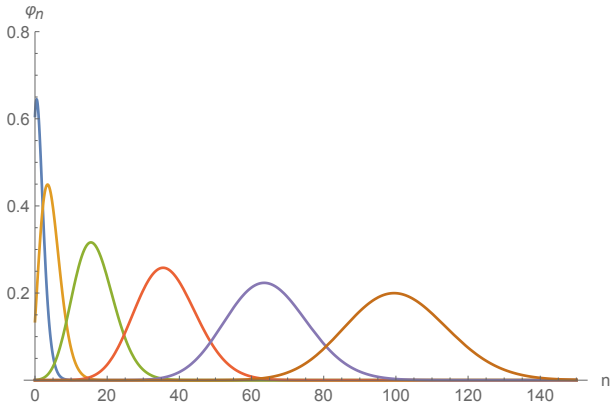

Figure 3.4: Wavefunctions from the Weyl-Heisenberg coherent states as functions of n for $\alpha t = 1, 2, 4, 6, 8, 10$, from left to right.

Here, the basis is infinite-dimensional but the growth of b_n's is not maximal. Also, the auto-correlation function, in this case, is exponentially decaying

$$C(t) = \varphi_0(t) = \exp\left(-\alpha^2 t^2/2\right). \tag{3.84}$$

With the explicit solution we can compute the Krylov complexity

$$K_{\mathcal{O}} = \sum_{n=0}^{\infty} n|\varphi_n(t)|^2 = \alpha^2 t^2 \,. \tag{3.85}$$

Similarly to SL(2,R) and SU(2), the early time growth is universally proportional to t^2 (quadratic), but here it continues as such for all times.

Finally, the information metric is the flat ($R = 0$) complex plane

$$ds^2_{FS} = dz d\bar{z} = dr^2 + r^2 d\phi^2. \tag{3.86}$$

The classical trajectory $r = \alpha t$ describing the operator dynamics is again a geodesic in this manifold. Moreover, we find the same universal relation between the Volume in the information geometry and Krylov complexity

$$V_t = \int_0^{\alpha t} dr \int_0^{2\pi} d\phi \sqrt{g} = \pi \alpha^2 t^2 = \pi K_{\mathcal{O}}. \tag{3.87}$$

In the next example, we proceed towards settings more akin to holography.

3.4.4 Example IV: Conformal Field Theories in 2d

As our last example we consider the case of global symmetry of 2d CFTs [110, 111]. This is an extension of the single $SL(2, R)$ described above and it follows by considering the growth defined by two copies of $SL(2, R)$, corresponding to the global part of the conformal group. From this perspective, the SYK example is equivalent to a chiral CFT with a single SL(2,R). The Lie algebra is then given by

$$\begin{aligned} [L_0, L_{\pm 1}] &= \mp L_{\pm 1}, \qquad [L_1, L_{-1}] = 2L_0, \\ \left[\bar{L}_0, \bar{L}_{\pm 1}\right] &= \mp \bar{L}_{\pm 1}, \qquad \left[\bar{L}_1, \bar{L}_{-1}\right] = 2\bar{L}_0 \,. \end{aligned} \tag{3.88}$$

We will begin with the highest weight state $|h, \bar{h}\rangle$ that is an eigenstate of the CFT Hamiltonian

$$H\,|h, \bar{h}\rangle = (L_0 + \bar{L}_0)\,|h, \bar{h}\rangle = (h + \bar{h})\,|h, \bar{h}\rangle\,, \tag{3.89}$$

This state arises by acting with the mode $\mathcal{O}_{-h,-\bar{h}}$ of the primary operator $\mathcal{O}$ with conformal dimension $\Delta = h + \bar{h}$ and spin $s = h - \bar{h}$ on the CFT vacuum

$$|h, \bar{h}\rangle = \mathcal{O}_{-h,-\bar{h}} \, |0, 0\rangle \, . \tag{3.90}$$

Generalization of the coherent states to 2d CFT is now straightforward. We just use two copies of the SL(2,R) displacement operator

$$|z, h; w, \bar{h}\rangle = D(\xi)\bar{D}(\zeta) \, |h, \bar{h}\rangle \, , \tag{3.91}$$

where z and w are related to ξ and ζ respectively as in (3.41). In this case, we need to work with the sum of two Liouvillians (see appendix A) with generally two different coefficients

$$\mathcal{L} = \alpha_+ \, (L_{-1} + L_1), \quad \bar{\mathcal{L}} = \alpha_- \, (\bar{L}_{-1} + \bar{L}_1) \, . \tag{3.92}$$

For example, in a general (charged) thermal state inner-product with different left and right temperatures $\beta_\pm = T_\pm^{-1} = \beta(1 \pm \Omega)$ (e.g. with angular momentum and chemical potential Ω) we may associate $\alpha_\pm = \pi/\beta_\pm$.

Unitary evolution with the Liouvillian is again a displacement of the initial state. By setting

$$z = i \tanh(\alpha_+ t), \qquad w = i \tanh(\alpha_- t) \, , \tag{3.93}$$

this leads to the coherent state or operator wavefunction

$$\begin{aligned} |\mathcal{O}(t)) &= |z = i \tanh(\alpha_+ t), h; w = i \tanh(\alpha_- t), \bar{h}\rangle \\ &= \sum_{n,m=0}^{\infty} \varphi_{n,m}(t) \, |h, n; \bar{h}, m\rangle \, . \end{aligned}$$

The total wavefunction $\varphi_{n,m}(t) = \varphi_n^{\alpha_+}(t)\varphi_m^{\alpha_-}(t)$ is a product of the "left" and "right" wavefunctions:

$$\begin{aligned} \varphi_n^{\alpha_+}(t) &= \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}} \frac{\tanh^n(\alpha_+ t)}{\cosh^{2h}(\alpha_+ t)} \\ \varphi_m^{\alpha_-}(t) &= \sqrt{\frac{\Gamma(2\bar{h}+m)}{m!\Gamma(2\bar{h})}} \frac{\tanh^m(\alpha_- t)}{\cosh^{2\bar{h}}(\alpha_- t)} \, . \end{aligned} \tag{3.94}$$

The evolution of $\varphi_{n,m}(t)$ is again described by the Schrodinger equation with two pairs of "left" and "right" SL(2,R) Lanczos coefficients. Details are given in appendix A.

This product solution and its $n = m = 0$ components lead to a consistent two-point function in CFT

$$\begin{aligned} C(t) &= (\mathcal{O}(t)|\mathcal{O}(0)) \\ &\simeq \left(\cosh \frac{\pi t}{\beta_+}\right)^{-2h} \left(\cosh \frac{\pi t}{\beta_-}\right)^{-2\bar{h}} . \end{aligned} \tag{3.95}$$

Recall that, for holographic CFTs, this two point correlator can be computed in the standard way from gravity by the exponent of the length of a geodesic that stretches between the two sides of the eternal black hole (see e.g. [112]).

The Krylov complexity is now the sum of the expectation values of the position in the left and right chains (see also next section)

$$K_{\mathcal{O}} = \sum_{n,m} (n + m) |\varphi_{n,m}(t)|^2 . \tag{3.96}$$

In terms of the total conformal dimension $\Delta = h + \bar{h}$ and the spin $s = h - \bar{h}$ this becomes

$$\begin{aligned} K_{\mathcal{O}} &= \Delta \left[\sinh^2\left(\frac{\pi t}{\beta_+}\right) + \sinh^2\left(\frac{\pi t}{\beta_-}\right) \right] \\ &+ s \left[\sinh^2\left(\frac{\pi t}{\beta_+}\right) - \sinh^2\left(\frac{\pi t}{\beta_-}\right) \right]. \end{aligned} \tag{3.97}$$

Clearly, for $\beta_+ \neq \beta_-$, Krylov complexity is sensitive to the operator's spin s.

Finally, the information geometry consists of two copies of the Euclidean Poincaré disc (3.50). The classical trajectory corresponds to a geodesic in this product manifold and the Krylov complexity is proportional to the volume as in previous examples.

Note that this generalization to 2d CFTs seems completely determined by symmetries (see more discussion in [53]). This universality is a simple consequence of the operators that we chose to describe the growth of. For a free CFT, we could have chosen a momentum mode instead, and the Krylov approach would look like the case of SU(2), instead of SL(2,R). We can also imagine considering composite CFT operators and/or consider the thermal CFT on the circle. Such setups will require more detailed information about the CFT spectrum etc, and they will distinguish between chaotic and non-chaotic CFTs.

Moreover, in 2d CFTs, we could also study the less universal Liouvillian dynamics based on the Virasoro algebra. Even though we leave this as an interesting future direction, in appendix B we consider a simpler but non-trivial example of SL(2,R) subalgebras of the Virasoro algebra given by $\{L_{-k}, L_0, L_k\}$ for some fixed k [113]. Already in this case we end up with Krylov complexity that depends on the central charge c of the CFT

$$K_{\mathcal{O}} = 2h_k \sinh^2(\alpha_k t), \tag{3.98}$$

where

$$h_k = \frac{c}{24}\left(k - \frac{1}{k} + \frac{24h}{ck}\right), \qquad \alpha_k = k\alpha, \tag{3.99}$$

and Lanczos coefficients are also asymptotically linear $b_n \simeq \alpha_k n$.

We will return to discussion of CFT generalizations at the very end.

3.5 Complexity algebra and geometry

Previously we have analyzed specific examples related to different groups. In this section, we come back to a more general discussion of the Lanczos coefficients in the light of symmetry. We argue that there exists a natural algebra associated with operator dynamics and Krylov complexity, and that the closure of this algebra on different levels provides another way towards finding potential sets of Lanczos coefficients. In particular, we will again reproduce our previous results from this angle.

The logic proceeds as follows. As described above, the action of the Liouvillian in the Krylov basis yields two terms (3.20) and suggests a definition of "generalized ladder operators"

$$\mathcal{L} = \tilde{L}_+ + \tilde{L}_-, \tag{3.100}$$

where for simplicity we absorbed α into the ladder operators of the previous section $\tilde{L}_\pm = \alpha L_\pm$ such that

$$\tilde{L}_+|\mathcal{O}_n) = b_{n+1}|\mathcal{O}_{n+1}), \quad \tilde{L}_-|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1}). \tag{3.101}$$

The algebra generated by the generalized ladder operators $\tilde{L}_+$ and $\tilde{L}_-$ is simply equivalent to the algebra generated by the Liouvillian and the operator $\mathcal{B}$, defined as

$$\mathcal{B} = \tilde{L}_+ - \tilde{L}_- \,. \tag{3.102}$$

By definition, action of this anti-Hermitian operator on the Krylov basis is

$$\mathcal{B}|\mathcal{O}_n) = -b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1}) \ . \tag{3.103}$$

We now want to explore the following question. What happens when we start commuting these two operators? From their definitions, we can easily derive the action of the commutator, that we name $\tilde{K}$, in the Krylov basis. Using (3.20) and (3.103) we obtain

$$\tilde{K} \equiv [\mathcal{L}, B]|\mathcal{O}_n) = 2(b_{n+1}^2 - b_n^2)|\mathcal{O}_n). \tag{3.104}$$

This operator turns out to be diagonal in the Krylov basis with eigenvalues $\tilde{k}(n) = 2(b_{n+1}^2 - b_n^2)$.

Given this generic algebraic structure, we now entertain a "simplicity" hypothesis. This hypothesis demands that these three operators close an algebra that we may call a "complexity algebra". This enforces the following constraint on the commutator eigenvalues

$$\tilde{k}(n) = An + B \ , \tag{3.105}$$

for some constants A and B, implying that $\tilde{k}(n)$ grows at most linearly in n. We then conclude that this hypothesis (closure of the algebra) provides a recurrence equation for the Lanczos coefficients

$$2(b_{n+1}^2 - b_n^2) = An + B. \tag{3.106}$$

A general solution to this equation is given by (the positive root)

$$b_n = \sqrt{\frac{1}{4}An(n-1) + \frac{1}{2}Bn + C} \ , \tag{3.107}$$

with C also being an arbitrary constant. Furthermore, requiring $b_0 = 1$, which holds for any operator growth, fixes this constant to $C = 0$. This family of Lanczos coefficients was also derived in [80] from the Toda chain approach.

We see that the hypothesis does not allow the Lanczos coefficients to grow faster than n. It would be interesting to see if imposing the closure of the algebra at a later level, by allowing the complexity algebra to include more operators generated by $\mathcal{L}$ and $\mathcal{B}$, still enforces the universal linear bound.

Note that the examples considered earlier all fall within the simplicity hypothesis. For instance, for $SL(2, R)$ we have that

$$\mathcal{L} = \alpha(L_{-1} + L_1), \quad \mathcal{B} = \alpha(L_{-1} - L_1), \quad \tilde{K} = 4\alpha^2 L_0, \tag{3.108}$$

and hence the eigenvalue

$$\tilde{k}_{sl(2,R)}(n) = 4\alpha^2(n + h). \tag{3.109}$$

Moreover, we can observe a simple relation between the commutator $\tilde{K}$ and the Krylov complexity operator, namely

$$\tilde{K} = 4\alpha^2(\hat{K}_{\mathcal{O}} + h) \ . \tag{3.110}$$

They are the same up to a constant and a proportionality factor. In particular, they both grow exponentially with the same growth rate/Lyapunov exponent. This suggests that the operator L_0 (or more generally the energy in CFTs) may also be a good candidate for operator complexity or a witness of the operator growth. This proposal was put forward in [91] and the present results provide a firmer ground for this idea. Nevertheless, the definition of the operator (3.25) seems more robust, especially from the point of view of generic systems that we can analyze only numerically.

The geometric interpretation of these complexity algebra generators is also very elegant. They are just related to the Killing vectors of the information metric (3.50), that in our coordinates become

$$\begin{aligned} L_0 &= i\partial_\phi, \\ L_{-1} &= -ie^{-i\phi}\left[\coth(\rho)\partial_\phi + i\partial_\rho\right], \\ L_1 &= -ie^{i\phi}\left[\coth(\rho)\partial_\phi - i\partial_\rho\right]. \end{aligned} \tag{3.111}$$

The operators $(\mathcal{L}, \mathcal{B}, \tilde{K})$ are built from these generators and satisfy the same algebra. They are therefore associated with the isometries of the information metric. In particular, $\tilde{K}$, almost equal to the Krylov complexity operator, generates translations in ϕ. Since the difference between $\tilde{K}$ and K is a constant, which just produces non-physical overall phases, we conclude that the Krylov complexity operator is also the generator of translations in ϕ. In addition, the geometric picture shows that (absolute value of) the expectation value of the operator $\mathcal{B}$ also grows exponentially in the course of operator dynamics (i.e., with $\rho = 2\alpha t$ and $\phi = \pi/2$).

In complete analogy, for SU(2) we have

$$\mathcal{L} = \alpha(J_+ + J_-), \quad \mathcal{B} = \alpha(J_+ - J_-), \quad \tilde{K} = -4\alpha^2 J_0, \tag{3.112}$$

and the eigenvalues of $\tilde{K}$ becomes

$$\tilde{k}_{su(2)}(n) = -4\alpha^2(n - j). \tag{3.113}$$

Again this implies a simple relation with the Krylov complexity operator

$$\tilde{K} = -4\alpha^2(\hat{K}_{\mathcal{O}} - j) \ , \tag{3.114}$$

and both $\tilde{K}$ and the Krylov complexity operator generate rotations in the information metric.

Finally, for the Heisenberg-Weyl algebra the appropriate assignation is

$$\mathcal{L} = \alpha(a^\dagger + a), \quad \mathcal{B} = \alpha(a^\dagger - a), \quad \tilde{K} = 2\alpha^2 1, \tag{3.115}$$

providing the eigenvalue

$$\tilde{k}_{HW}(n) = 2\alpha^2. \tag{3.116}$$

In this case, the commutator is proportional to the identity. Therefore, the relation to Krylov complexity operator is not just a simple constant shift and appears less natural.

We believe that this new perspective will serve as a solid starting point for a systematic approach to the classification of various operator dynamics. This classification could start by specifying the number of operators we need to add to the Liouviilian and the operator $\mathcal{B}$ to close an algebra. This program may follow by analyzing the possible representations of such a complexity algebra. Certainly, new examples associated with other Lie groups as well as their deformations will serve as important data points in this direction.

3.6 Relation to Geometric Complexity: Particle on a group

In this section, we provide a bridge between operator dynamics and Krylov complexity, and the geometric approach to computational/circuit complexity. This approach stands out in its similarities to the way physicists think. It was pioneered by Nielsen and collaborators [29–31] and, more recently, attracted significant attention with prospective applications to holographic complexity (see e.g. [32–34, 114–119] as well as [41, 120–122] for some of the alternative definitions). The main elegant idea in Nielsen's works is to think about quantum circuits as paths in the manifold of unitary transformations. These paths are determined by different choices of instantaneous quantum gates, characterised by time-dependent Hamiltonians.

On one hand, linking these two ideas is important from a physics point of view, given the recent activity concerning the relation between complexity and black hole physics [21–23, 85, 89, 91]. On the other hand, this connection can sharpen the operational meaning of the operator wavefunction and the Krylov complexity.

The bridge is built upon the previously described connection to generalized coherent states. To make our points clear, we start from the transition amplitude between a coherent state $|z_i\rangle$ at some initial time t_i and a coherent state $|z_f\rangle$ for time t_f, defined as

$$T(z_f, t_f; z_i, t_i) = \langle z_f | \exp\left(-iH(t_f - t_i)\right) |z_i\rangle \, . \tag{3.117}$$

We can write a path integral representation of these transition amplitudes, see [123]

$$T(z_f, t_f; z_i, t_i) = \int d\mu[z(t)] e^{iS}, \tag{3.118}$$

where $d\mu[z(t)]$ is an appropriate functional measure (an invariant measure on the coset space associated with the problem) and S is an action functional for the paths

$$S = \int_{t_i}^{t_f} L(z(t), z'(t), \bar{z}(t), \bar{z}'(t)) dt. \tag{3.119}$$

For our discussion on complexity, we only need the classical approximation to this propagator and the classical equations of motion from S. In particular, see [123], the Lagrangian above takes the following standard form

$$L = \langle z | i\partial_t - H | z \rangle \equiv \langle z | i\partial_t | z \rangle - \mathcal{H}(z, \bar{z}). \tag{3.120}$$

In this formula, the first term is simply $\partial_t = z'\partial_z + \bar{z}'\partial_{\bar{z}}$. It acts on the coherent state before the overlap is computed (see appendix C). In the second term we have defined the "classical Hamiltonian" $\mathcal{H}$ as

$$\mathcal{H}(z, \bar{z}) = \langle z | \, H \, | z \rangle \,, \tag{3.121}$$

usually referred to as the "symbol" of the quantum Hamiltonian H. For now, we keep H general, but we will consider particular choices momentarily.

The classical Euler-Lagrange equations of motion for the two variables $z(t)$ and $\bar{z}(t)$ associate with this action are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial z'}\right) - \frac{\partial L}{\partial z} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \bar{z}'}\right) - \frac{\partial L}{\partial \bar{z}} = 0. \tag{3.122}$$

Equivalently, following the Hamiltonian approach we can write the same equations in terms of Poisson brackets

$$z'(t) = \{z, \mathcal{H}(z, \bar{z})\}, \qquad \bar{z}'(t) = \{\bar{z}, \mathcal{H}(z, \bar{z})\}. \tag{3.123}$$

It is now straightforward to check that for our three main examples, with general classical Hamiltonian $\mathcal{H}(z, \bar{z})$, the previous equations of motion can be written in terms of the following Poisson brackets. For the group $SL(2, R)$, corresponding to the hyperbolic phase space geometry, we define

$$\{A, B\} = i\frac{(1 - |z|^2)^2}{2h}\left(\frac{\partial A}{\partial \bar{z}}\frac{\partial B}{\partial z} - \frac{\partial A}{\partial z}\frac{\partial B}{\partial \bar{z}}\right). \tag{3.124}$$

For $SU(2)$, the phase space is the sphere and we have

$$\{A, B\} = i\frac{(1 + |z|^2)^2}{2j}\left(\frac{\partial A}{\partial \bar{z}}\frac{\partial B}{\partial z} - \frac{\partial A}{\partial z}\frac{\partial B}{\partial \bar{z}}\right). \tag{3.125}$$

Finally, the standard flat one for the Weyl-Heisenberg scenario reads

$$\{A, B\} = i\left(\frac{\partial A}{\partial \bar{z}}\frac{\partial B}{\partial z} - \frac{\partial A}{\partial z}\frac{\partial B}{\partial \bar{z}}\right). \tag{3.126}$$

Let us now return to the quantum Hamiltonians and their classical counterparts. Since the appropriate group acts transitively in phase space, the Hamiltonian must be an element of the Lie algebra (see [97] for a review of this type of symmetry dynamics). This condition preserves the coherence of the states through evolution. A general Hamiltonian for the SL(2,R) example is then written as

$$H_{SL(2,R)} = aL_0 + bL_1 + cL_{-1}, \tag{3.127}$$

where a, b, c are arbitrary constants. They can be time-dependent. Indeed, in connections with circuit/Nielsen complexity, these parameters might depend non-trivially on circuit time, defining the instantaneous gate at each time. However, for the discussion in this section, it is sufficient to consider them constant.

We can now compute the expectation value of such a generic operator, "the symbol" (see App.D) and obtain the following classical Hamiltonian in phase space

$$\mathcal{H}(z,\bar{z}) = \frac{h}{1-|z|^2}\left(a(1+|z|^2)+2bz+2c\bar{z}\right). \tag{3.128}$$

The classical Hamilton equations of motion associated with this Hamiltonian turn out to be the complex Riccati equations and, after we express everything in terms of ρ and ϕ coordinates, they become

$$\begin{aligned}\rho'(t) &= i\left(be^{i\phi(t)}-ce^{-i\phi(t)}\right),\\ \phi'(t) &= -a-\coth(\rho(t))\left(be^{i\phi(t)}+ce^{-i\phi(t)}\right).\end{aligned} \tag{3.129}$$

Our previous solution $\rho = 2\alpha t$ and $\phi = -\pi/2$ describing the operator wavefunction is indeed a solution of these equations with $a = 0$ and $c = b = \alpha$. This is consistent with the expected Hamiltonian $H = \alpha(L_{-1} + L_1)$. The minus sign difference in $\phi = \pm\pi/2$ can be traced back to the forward vs backward time evolution with Hamiltonian vs Liouvillian. Dynamics with such Hamiltonians was also discussed in [124] and this context is may related to our choice of the inner product. We leave more detailed explorations in this direction for future works.

Similarly, for the SU(2) case, the most general coherence-preserving Hamiltonian is written as

$$H_{SU(2)} = aJ_0 + bJ_+ + cJ_-. \tag{3.130}$$

Taking the expectation value in the coherent state basis we obtain the classical Hamiltonian

$$\mathcal{H}_{SU(2)} = -ja\cos(\theta) + j\sin(\theta)(be^{-i\phi}+ce^{i\phi}), \tag{3.131}$$

leading to the following equations of motion

$$\begin{aligned}\theta'(t) &= i\left(ce^{i\phi(t)}-be^{-i\phi(t)}\right),\\ \phi'(t) &= -a-(be^{-i\phi(t)}+ce^{i\phi(t)})\cot(\theta(t)).\end{aligned} \tag{3.132}$$

Again, the solution described earlier is a solution to these equations corresponding to $a = 0$ and Hamiltonian $H = \alpha(J_+ + J_-)$, as it should be.

Finally, for standard Heisenberg-Weyl symmetry we generically have

$$H_{HW} = a\hat{n} + b\hat{a}^\dagger + c\hat{a} + d1, \tag{3.133}$$

with arbitrary constants a, b, c, d. The classical Hamiltonian is then

$$\mathcal{H}_{HW} = ar^2 + r(be^{-i\phi}+ce^{i\phi}) + d, \tag{3.134}$$

and the equations follow

$$\begin{aligned}r'(t) &= -\frac{i}{2}(be^{-i\phi(t)}-ce^{i\phi(t)}),\\ \phi'(t) &= -a-\frac{1}{2r}(be^{-i\phi(t)}+ce^{i\phi(t)}).\end{aligned} \tag{3.135}$$

Setting $\phi = -\pi/2$ as well as $a = 0$ and $b = c = \alpha$ (or $r = \alpha t$) corresponds to motion of a particle on this phase space with Hamiltonian $H_{HW} = \alpha(a^\dagger + a)$.

We conclude that the operator wavefunctions considered earlier, including the example of SYK, can be simply mapped to classical motions i.e., solutions of the classical Hamilton equations of motion in the appropriate generalized coherent state phase space. The classical Hamiltonians above follow directly by taking the expectation value in the generalized coherent states of our proposed form of the Liouvillian $\mathcal{L} = \alpha(L_+ + L_-)$. This way, we can not only think about operator growth geometrically but also naturally regard unitary Liouvillian evolution as a quantum circuit

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}). \tag{3.136}$$

With these results in mind, let us return to Nielsen's approach. In this framework, after assigning a particular (highly non-unique) cost function to the instantaneous gates, one can estimate the computational complexity of the task by finding the length of the minimal geodesic in the geometry of unitaries. The ambiguity in the cost functions somewhat parallels the freedom in choosing the inner product to turn the operator algebra into a Hilbert space. However, the geodesic length is not obviously related to the operator complexity. Naively, one is tempted to identify the information geometry (Fubini-Study metric) with Nielsen's metric, but as we saw before the geodesic length between the origin and $\rho = 2\alpha t$ (at fixed $\phi = \pi/2$) grows only linearly in time. Indeed, as we saw before, it is the phase space Volume in the Fubini-Study metric that Krylov complexity measures.

Still, one can interpret Krylov complexity in terms of a geodesic length. This fact comes from the universal relation between the $\mathcal{F}_1$ norm and Krylov complexity (see (3.260) in Appendix D). Indeed, for phase-space displacements in the angular direction we have

$$\mathcal{F}_1 = |\langle z|\delta z\rangle| = K_{\mathcal{O}} d\phi, \tag{3.137}$$

This can be interpreted as the Nielsen complexity, defined with $\mathcal{F}_1$ cost functions [91, 114, 125, 126], of the circuit that takes us from trajectory $(\rho = 2\alpha t, \phi = \pi/2)$ to a nearby geodesic with $(\rho = 2\alpha t, \phi = \pi/2 + \delta\phi)$. This in turn is very closely related to the definition of classical chaos that we discussed in the introduction.

3.7 Quantum Information tools for operator growth

This last section is devoted to contrasting the evolution of Krylov complexity with more conventional quantum information tools. For this, we step again on the connection between operator dynamics and coherent states. More concretely, a certain two-mode representation of the displacement operator will allow us to derive a density matrix associated with the evolving operator. Then, instead of quantifying complexity with expectation values of operators, such as the Krylov complexity, we will explore it with different quantum information tools. As new outcomes, we will discuss traces of the operator growth in entanglement measures, define a notion of operator proper temperature that connects to the physics of black holes and to quantum optics. This last outcome will suggest a way to contrast these theoretical problems with experiments.

Below we will concentrate on the "chaotic" example of SL(2,R), and focus on the time dependence of the different quantities, comparing their growth with Krylov complexity. Most of these results hold for other coherent states as well and here we only survey the most important findings. Further details with non-chaotic examples will be described in [127].

We start by representing the SL(2,R) generators in terms of two oscillator modes as

$$L_{-1} = a_1^\dagger a_2^\dagger, \quad L_1 = a_1 a_2, \quad L_0 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1), \tag{3.138}$$

where the creation $a_i^\dagger$ and annihilation a_i operators satisfy the Weyl-Heisenberg algebra. In this representation, the displacement operator $D(\xi)$ becomes the standard two-mode squeezing operator

frequently used in theoretical as well as experimental quantum optics, see e.g. [128]. Then we consider the so-called k-photon added/subtracted states

$$|z,k\rangle \equiv \mathcal{N}(a_2)^k S(\xi)\,|0,0\rangle \ , \tag{3.139}$$

in which there is a difference of k-excitations between the two modes. Using the standard Bogoliubov transformation we can expand this state in the two oscillator Fock space basis as

$$|z,k\rangle = (1-|z|^2)^{\frac{k+1}{2}} \sum_{n=0}^{\infty} z^n \sqrt{\frac{\Gamma(k+1+n)}{n!\Gamma(k+1)}}\,|n+k,n\rangle\,. \tag{3.140}$$

In the amplitudes of this state we recognise those of the SL(2,R) coherent states. One just needs to perform the identification $k+1=\eta=2h$, while the phase space coordinates z's remain unchanged. In this squeezeed representation, the coherent states are entangled states. Also, in this form, the Krylov basis is the standard two-oscillator Fock space

$$|\mathcal{O}_n) = |n+k,n\rangle = \frac{(a_1^\dagger)^{n+k}}{\sqrt{(n+k)!}}\frac{(a_2^\dagger)^n}{\sqrt{n!}}\,|0,0\rangle\,. \tag{3.141}$$

In this two-mode representation, we can think about the operator wavefunction $|\mathcal{O}(t))$ as a "perturbed" thermofield double state. Tracing out the second oscillator we arrive at the following density matrix

$$\rho_1^{(k)} = Tr_2\left(|z,k\rangle\,\langle z,k|\right) = \sum_{n=0}^{\infty}\lambda_n\,|n+k\rangle\langle n+k|\,. \tag{3.142}$$

Its eigenvalues are precisely the probabilities in the Krylov basis

$$\lambda_n = |\varphi_n(t)|^2 = \frac{\Gamma(2h+n)}{n!\Gamma(2h)}(1-|z|^2)^{2h}|z|^{2n}, \tag{3.143}$$

where we remind that in order to describe operator growth we need to assign $z = i\tanh(\alpha t)$.

This description of the operator growth process allows us to assign a "proper temperature" to the operator. To this end, and for simplicity, we analyze the special case of $k=0$ (or $h=1/2$). Then defining

$$e^{-\beta(t)\omega} = \tanh^2(\alpha t), \tag{3.144}$$

the mixed state $\rho_1^{(k)}$ is just the thermal state of the harmonic oscillator with inverse temperature $\beta(t)$. This is a temperature $T(t)=1/\beta(t)$ naturally associated with operator growth. At large times we find that this operator temperature behaves as

$$T(t) \xrightarrow[\alpha t\gg 1]{} \frac{e^{2\alpha t}}{4}\ , \tag{3.145}$$

growing exponentially fast with the right Lyapunov exponent. Quite interestingly, this is the expected behaviour of proper temperatures/energies of infalling perturbations into a black hole, which is just universally controlled by the near-horizon redshift, determined by the time-time component of the black hole metric.

This density matrix representation of the operator growth also allows us to explore it using tools from quantum information. First of all, the K-entropy, defined in [46], is just the standard von-Neumann entropy of $\rho_1^{(k)}$, now appearing as an entanglement entropy

$$S_{\mathcal{O}} = -\sum_n |\varphi_n|^2 \log\left(|\varphi_n|^2\right), \tag{3.146}$$

between the two modes of the squeezed state. The analytic answer is obtained, for example, by setting $k = 0$ and and it grows linearly at late times

$$S_{\mathcal{O}} = 2\log\left(\cosh(\rho)\right) - 2\log\left(\tanh(\rho)\right)\sinh^2(\rho) \simeq 2\alpha t, \tag{3.147}$$

with a proportionality factor equal to the Lyapunov exponent. This behaviour is reminiscent of that of a Kolmogorov-Sinai entropy, whose rate of growth is upper bounded by the sum of Lyapunov exponents.

More generically we can compute Renyi entropies

$$S_{\mathcal{O}}^{(q)} = \frac{1}{1-q}\log\left(\sum_n |\varphi_n|^{2q}\right), \tag{3.148}$$

and for $k = 0$ these are given by

$$S_{\mathcal{O}}^{(q)} = \frac{1}{q-1}\log\left(\cosh^{2q}(\alpha t)(1-\tanh^{2q}(\alpha t))\right). \tag{3.149}$$

One more interesting quantity that has been studied recently in various contexts [129–131] is the capacity of entanglement

$$\mathcal{C}_{\mathcal{O}} = \lim_{q\to 1} q^2\partial_q^2\left[(1-q)S_{\mathcal{O}}^{(q)}\right]. \tag{3.150}$$

In particular, in [131] it was shown to be a useful new probe of local operators. For $k = 0$, we can easily compute it analytically from (3.149) and it becomes

$$\mathcal{C}_{\mathcal{O}} = \sinh^2(2\alpha t)(\log(\tanh(\alpha t)))^2. \tag{3.151}$$

This way, capacity of entanglement also grows for early times $\alpha t \simeq 0$ as

$$\mathcal{C}_{\mathcal{O}} \simeq 4\alpha^2 t^2\log(\alpha t)^2, \tag{3.152}$$

but then saturates to 1 exponentially fast as $t \to \infty$ (with twice the Lyapunov exponent)

$$\mathcal{C}_{\mathcal{O}} \xrightarrow[\alpha t\gg 1]{} 1 - \frac{4}{3}e^{-4\alpha t}. \tag{3.153}$$

The physical interpretation of this saturation ("thermalisation") is not yet clear to us but it indicates that not all the probes must necessarily grow/decay (linearly or exponentially) in order to extract certain universal features from their evolution.

Related to entanglement entropy, a useful measure of "quantumness" of a state considered in quantum optics is entanglement negativity [132]. For operator growth, it can be written in terms of the operator wavefunctions as

$$E_{\mathcal{N}}(\rho) = 2\log\left(\sum_n |\varphi_n|\right). \tag{3.154}$$

In the simplest case with $k = 0$, it becomes precisely

$$E_{\mathcal{N}}(\rho) = 2\alpha t, \tag{3.155}$$

resembling the growth of a geodesic length from the origin of the hyperbolic disc.

Finally, when quantifying complexity of quantum states it is natural to expect that different distance measures also play a significant role (see e.g. [133–135]). For example, relative entropy between $\sigma = |k\rangle\langle k|$ and $\rho_1^{(k)}$ is given by

$$\begin{aligned} S(\sigma|\rho_1^{(k)}) &= Tr(\sigma\log(\sigma)) - \mathrm{Tr}\left(\sigma\log\left(\rho_1^{(k)}\right)\right) \\ &= -2\log(|\varphi_0(t)|) = 4h\log(\cosh(\alpha t)), \end{aligned} \tag{3.156}$$

growing linearly with time for late times. Another example is the fidelity

$$\begin{aligned} F(\sigma,\rho) &= Tr\left(\rho^{1/2}\sigma\rho^{1/2}\right) = |\varphi_0(t)| \\ &= \cosh^{-2h}(\alpha t) \simeq 2^{2h}e^{-2h\alpha t}, \end{aligned} \tag{3.157}$$

that again decays with time. Interestingly, both of these measures are simply expressed by $\varphi_0(t)$.

The main message from this section is that these QI tools can be directly written in terms of $\varphi_n(t)$'s and are sensitive to the Lyapunov exponent 2α. The example of proper temperatures and their relation to black hole physics is quite intriguing. In addition, from the geometric perspective, entanglement negativity or the late time relative entropy are bounded by the geodesic length from the origin up to the radius $\rho(t)$. More discussion and detailed analysis of the quantum information tools for the operator growth will be presented in [127].

3.8 Discussion: CFT Generalizations

After analyzing our examples, it is clear that most of the discussion has been constrained by the universal kinematics of the underlying symmetries. This was already discussed in [53]. Of course, for tests and applications to genuine holography in two and higher dimensions, we will need to focus on less universal setups/operators that are sensitive to more detailed aspects of the spectrum and dynamics.

Some generalizations were already put forward in the original work [42]. These come under the name of q-complexities. In fact, the authors argued that Krylov complexity bounds the growth of the OTOC, that arise by choosing inner products of the type $([V,\mathcal{O}(t)]|[V,\mathcal{O}(t)])$, for arbitrary operators $\mathcal{O}(t)$ and V. Another interesting generalization was studied in [82] and focused on a certain fixed energy bandwidth. Below, we discuss more of such new directions that are natural from the perspective of CFTs and holography.

Firstly, extending the symmetry discussion in 2d CFTs to the full Virasoro algebra as well as the conformal algebra in higher dimensions would be very interesting. This would parallel the circuit complexity setup proposed in [114], see also [34, 126]. For instance, investigating the Lanczos coefficients and their scaling with n for the full Virasoro group, as well as W_n or BMS, would teach us important lessons about holographic CFTs. We hope to report on these developments in future works.

A more immediate generalization arises as follows. We can consider analyzing the growth, not of a single operator $\mathcal{O}(t)$, but a product of operators $\mathcal{O}_1(t)\mathcal{O}_2(t)$ with generally different conformal dimensions. This problem requires more detailed non-universal information about the CFT. For instance, the auto-correlation function is now a four-point function

$$\begin{aligned} C(t) &= (\mathcal{O}_1(t)\mathcal{O}_2(t)|\mathcal{O}_1(0)\mathcal{O}_2(0)) \\ &\equiv \langle e^{H\beta/2}\mathcal{O}_1(t)\mathcal{O}_2(t)^\dagger e^{-H\beta/2}\mathcal{O}_1(0)\mathcal{O}_2(0)\rangle_\beta, \end{aligned} \tag{3.158}$$

which opens a path to relate Krylov complexity and the more conventional approach to quantum chaos based on the OTOC correlators in QFTs. To study the growth of $\mathcal{O}_1(t)\mathcal{O}_2(t)$ we may resort to the Operator Product Expansion (OPE) in CFT. Two CFT operators can be fused and written as a sum of local operators. For our purposes we need to take the limit of coincident points in the OPE. Given that the coefficient of a certain operator $\mathcal{O}$ of dimension Δ in the OPE of $\mathcal{O}_1(x)\mathcal{O}_2(y)$ is down by powers of $|x-y|^{2\Delta}$ as $x\to y$, the fusion is controlled by the operator $\mathcal{O}(t)$ with lowest dimension appearing in the OPE. In this limit we can write

$$\mathcal{O}_1(t')\mathcal{O}_2(t) \underset{t'\to t}{\longrightarrow} F(t-t')\,\mathcal{O}(t). \tag{3.159}$$

The proportionality factor $F(t-t')$ will go away once we normalize the initial state. We conclude that the growth of the composite operator $\mathcal{O}_1(t)\mathcal{O}_2(t)$ is described by the growth of the single operator $\mathcal{O}'$ with the lowest Δ appearing in the OPE. To analyze it, we can use the results of the previous section.

Another natural generalization emerges by analyzing the Krylov complexity of $\mathcal{O}_1(0)\mathcal{O}_2(t)$, as time evolves, i.e., by considering composite operators at non-coincident locations. In this setup, the OPE does not collapse to a single operator but is expanded in terms of OPE blocks, see [16,111,136]. This suggests considering first the Krylov complexity of OPE blocks. In the light of the present article, the block Liouvillian is expected to take the generic form (3.33) and the block complexity may be related to volumes in kinematic space [16, 136]. We leave these interesting problems for future work as well.

Last but not least, we may be interested in building the Krylov basis and Krylov complexity for the time evolution of more general initial states (not necessarily simple operators). These include time evolution of boundary and TFD states of CFTs, evolution of states dual to local operators in the bulk [137], or microscopic SYK high energy states such as the ones considered in [138]. In the first two examples, we naturally encounter objects related to the return amplitudes [139]

$$\mathcal{F}(t) = (\Psi|e^{i\mathcal{L}t}|\Psi), \tag{3.160}$$

that in turn are closely related to spectral form factors (see e.g. [140]) used in the studies of chaos and operator growth. Using them as inputs (generalizing the auto-correlation function) of the Lanczos algorithm and studying their Krylov complexity is definitely one the most important future directions.

3.9 Conclusions

In the context of operator and circuit complexity, and their applications to theoretical physics, the are two problems that stand out. The first, and in our opinion probably the most important, is to frame the context in a way that transparently connects to the "physicist perspective". The second is to have a formulation that allows for progress on a technical level.

The objective of this article has been to develop a unifying approach to these and other related problems. Concerning the complexity of operators we advocated for the Krylov complexity and the Lanczos approach as a natural starting point. Concerning the physics, it has been crucial to analyze systems from the perspective of their symmetry, generalized coherent states and their associated geometries. These two, a priori distinct, fields share a common ground when one notices that the Liouvillian/Hamiltonian of the system can be written in the Lanczos basis as the sum of ladder operators (and more generally diagonal symmetry generators). Dynamics of the system with that type of "symmetry" is typically equivalent to the classical evolution of a "particle" in the appropriate phase space.

This equivalence allows us to make progress in various directions. It provides a more transparent, physical meaning to Krylov complexity operator. It turns out to be related to the vector fields generating isometries of the classical phase space geometry. Equivalently, it is an element of the Lie algebra of the symmetry group of the system. Moreover, its expectation value is the volume defined by the classical motion on the geometry. From a more technical perspective, the present identification simplifies enormously the computation of Lanczos coefficients.

Finally, we want to end with several open questions. First, there is an interesting consequence of the relation between operator complexity and classical evolution in appropriate phase spaces. In several cases of interest the problem can be stated in a language that makes contact with the field of quantum optics. One might enjoy the thought that the classical process that faithfully represents the operator growth wavefunction in the SYK model could be analyzed experimentally using photons. This promising avenue deserves more development and we hope to come back to it soon.

From a more theoretical perspective, the relation between Krylov complexity and phase space volumes calls for a deeper understanding. In parallel with generalized coherent states [99], it will be interesting to consider operator dynamics governed by Liouvillians that are arbitrary combinations of the algebra elements and explore the relation with volume (operator) further. The role of the volume certainly resonates with the holographic complexity proposal made in [21], and we expect it will teach us new insights about black hole interiors.

Finally, the relation between operator growth and classical motion described above, together with notions of quantum chaos and Lyapunov exponents based on the out-of-time correlators may indeed allow for a more physical derivation of the bound on quantum chaos [37, 65].

Acknowledgements. We wish to thank Jan Boruch, Diptarka Das, Felix Haehl, Sinong Liu, Wolfgang Muck, Pratik Nandy, Milosz Panfil, Krzysztof Wohlfeld, Onkar Parrikar, Tadashi Takayanagi, Claire Zukowski for discussions and comments and especially, Anatoly Dymarsky, Dongsheng Ge and Joan Simon for discussions and comments that advanced this work. The work of P.C and D.P is supported by NAWA "Polish Returns 2019" and NCN Sonata Bis 9 grants. The work of J.M is supported by a DOE QuantISED grantDE-SC0020360 and the Simons Foundation It From Qubit collaboration (385592).

3.10 Appendix A: Lanczos Algorithm

In this first appendix, we carry on explicitly the Lanczos algorithm with Liouvillian built from the elements of the Lie algebra. Instead of recurring from the intuition of coherent states, we will proceed carefully through each step of the iterative process.

The algorithm starts with the initial operator and the second element of the Lanczos basis

$$|\mathcal{O}_0) := |\mathcal{O}), \qquad |\mathcal{O}_1) := b_1^{-1}\mathcal{L}|\mathcal{O}_0), \tag{3.161}$$

where $b_1 = (\mathcal{O}_0\mathcal{L}|\mathcal{L}\mathcal{O}_0)^{1/2}$. The algorithm then constructs an orthogonal basis of states iteratively as

$$|A_n) = \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2}), \tag{3.162}$$

which are then normalized

$$|\mathcal{O}_n) = b_n^{-1}|A_n), \qquad b_n = (A_n|A_n)^{1/2}. \tag{3.163}$$

In addition to the basis states $|\mathcal{O}_n)$, this algorithm outputs the so-called Lanczos coefficients b_n.

For simplicity, we will analyze in detail the example of the SL(2,R) algebra (3.37) with associated Liouvillian (3.44). In the first step of the Lanczos algorithm we choose an initial operator/state

$$|\mathcal{O}_0) = |\mathcal{O}) = |h\rangle\,. \tag{3.164}$$

This is the highest weight state

$$L_0\,|h\rangle = h\,|h\rangle\,, \qquad L_1\,|h\rangle = 0. \tag{3.165}$$

We find it conveneint to think about this example as a "chiral CFT" where

$$|h\rangle = \mathcal{O}_{-h}\,|0\rangle\,, \tag{3.166}$$

where $|0\rangle$ is the vacuum and $\mathcal{O}_{-h}$ is a mode of a primary operator with (chiral) dimension h. These primary field modes satisfiy the following commutation relations with the generators of the conformal algebra

$$[L_m, \mathcal{O}_n] = ((h-1)m - n)\mathcal{O}_{n+m}. \tag{3.167}$$

In particular

$$[L_m, \mathcal{O}_{-h}] = (h(m+1) - m)\mathcal{O}_{-h+m}, \tag{3.168}$$

where for us $m \in \{-1, 0, 1\}$. These field modes also satisfy their own algebra, generally written for two quasi-primary operators $\mathcal{O}^{(i)}$ and $\mathcal{O}^{(j)}$ as

$$\begin{aligned} [\mathcal{O}_m^{(i)}, \mathcal{O}_n^{(j)}] &= \delta_{m+n,0} d^{ij} \binom{m+h_i-1}{2h_i-1} \\ &+ \sum_k C_k^{ij} p_k^{ij}(m,n) \mathcal{O}_{m+n}^{(k)}, \end{aligned} \tag{3.169}$$

where d^{ij} are the two-point function coefficients (that are usually set to δ_{ij} by normalizing the operators), C_k^{ij} are the three-point function coefficients and $p_k^{ij}(m,n)$ are some universal polynomials specified by the CFT data (see e.g. [110]).

The next step is to normalize the second basis state of the Lanczos basis, namely

$$|\mathcal{O}_1) = b_1^{-1} \mathcal{L} |\mathcal{O}_0) = b_1^{-1} \alpha L_{-1} |h\rangle . \tag{3.170}$$

The normalization, i.e. the first Lanczos coefficient, is computed from the commutation relation of the SL(2,R) algebra. It is given by

$$b_1 = (\mathcal{O}_0 \mathcal{L} | \mathcal{L} \mathcal{O}_0)^{1/2} = \alpha \sqrt{\langle h | L_1 L_{-1} | h \rangle} = \alpha \sqrt{2h}. \tag{3.171}$$

This way we have

$$|\mathcal{O}_1) = \frac{1}{\sqrt{2h}} L_{-1} |h\rangle \equiv |h, 1\rangle . \tag{3.172}$$

Notice that in this chiral CFT we could equivalently interpret this second basis state in terms of the commutator of the mode $\mathcal{O}_{-h}$ with the following "Hamiltonian"

$$\mathcal{L}\mathcal{O}_{-h} = [H, \mathcal{O}_{-h}], \qquad H = \alpha(L_{-1} + L_1). \tag{3.173}$$

Indeed, from the commutation relations (3.168) it is clear that

$$[H, \mathcal{O}_{-h}] = \alpha \left(\mathcal{O}_{-(h+1)} + (2h-1)\mathcal{O}_{-h+1} \right), \tag{3.174}$$

hence

$$|\mathcal{L}\mathcal{O}_0) = [H, \mathcal{O}_{-h}] |0\rangle = \alpha \mathcal{O}_{-(h+1)} |0\rangle . \tag{3.175}$$

The normalization of this state is obtained from (3.169) (with $d^{ij} = \delta_{ij}$)

$$b_1 = \alpha \langle 0 | [\mathcal{O}_{h+1}, \mathcal{O}_{-(h+1)}] | 0 \rangle^{1/2} = \alpha \sqrt{2h}. \tag{3.176}$$

We now move to the third step and find

$$\begin{aligned} |A_2) &= \mathcal{L}|\mathcal{O}_1) - b_1 |\mathcal{O}_0), \\ &= \frac{\alpha}{\sqrt{2h}} \left(L_{-1}^2 |h\rangle + L_1 L_{-1} |h\rangle \right) - \alpha \sqrt{2h} |h\rangle , \\ &= \frac{\alpha}{\sqrt{2h}} L_{-1}^2 |h\rangle , \end{aligned} \tag{3.177}$$

where in the second equality we computed the product of $L_1 L_{-1}$ using the SL(2,R) commutator. This canceled the last term. Computing the Lanczos coefficient

$$b_2 = (A_2 | A_2)^{1/2} = \frac{\alpha}{\sqrt{2h}} \langle h | L_1^2 L_{-1}^2 | h \rangle^{1/2}, \tag{3.178}$$

we obtain

$$b_2 = \alpha\sqrt{2(2h+1)}, \tag{3.179}$$

where we have used the general result

$$\langle h|L_1^n L_{-1}^n|h\rangle = n!\frac{\Gamma(2h+n)}{\Gamma(2h)}. \tag{3.180}$$

We then arrive at the third basis vector

$$|\mathcal{O}_2) = b_2^{-1}|A_2) = \frac{1}{\sqrt{4h(2h+1)}}L_{-1}^2|h\rangle \equiv |h,2\rangle\,. \tag{3.181}$$

Continuing this procedure we can derive the full $SL(2,R)$ result for the Lanczos coefficients

$$b_n = \alpha\sqrt{n(2h+n-1)}, \tag{3.182}$$

as well as the Krylov basis vectors

$$|\mathcal{O}_n) = |h,n\rangle = \sqrt{\frac{\Gamma(2h)}{n!\Gamma(2h+n)}}L_{-1}^n|h\rangle\,, \tag{3.183}$$

These vectors are orthonormal $(\mathcal{O}_m|\mathcal{O}_n) = \delta_{n,m}$.

It is also useful to repeat the Lanczos Algorithm for the two-mode representation of the SL(2,R) Liouvillian that we used in the main text

$$\mathcal{L} = \alpha\left(a_2^\dagger a_1^\dagger + a_2 a_1\right). \tag{3.184}$$

This operator can be written in terms of two oscillator modes with standard commutation relations

$$[a_i, a_i^\dagger] = 1, \qquad [a_i, a_j^\dagger] = 0 \quad \text{for} \quad i \neq j, \tag{3.185}$$

with $i,j = 1,2$. In what follows, the two-mode vacuum state is anihilated by both operators a_i

$$a_i|0,0\rangle = 0, \tag{3.186}$$

and we introduce the n-particle orthonormal states

$$|n\rangle_i = \frac{(a_i^\dagger)^n}{\sqrt{n!}}|0\rangle_i\,. \tag{3.187}$$

Without causing confusion, we will drop the subscript from the states, with the understanding that operators act on their appropriate Hilbert spaces. We then have

$$a_i|n\rangle = \sqrt{n}\,|n-1\rangle\,, \qquad a_i^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{3.188}$$

In this version, the Lanczos algorithm starts from a super-operator

$$\mathcal{O}^{(k)} = \frac{(a_1^\dagger)^k}{\sqrt{k!}}, \tag{3.189}$$

and the corresponding two-mode state

$$|\mathcal{O}_0^{(k)}) = \mathcal{O}^{(k)}|0,0\rangle = |k,0\rangle\,. \tag{3.190}$$

In quantum optics, this state is usually called the k-photon added state since the difference between the excitation numbers in each mode is k. We then proceed with the algorithm and apply the Liouvillian to define the second basis vector

$$|O_1^{(k)}) = b_1^{-1}\mathcal{L}|O_0^{(k)}) = \frac{\alpha\sqrt{k+1}}{b_1}\,|k+1,1\rangle\,, \tag{3.191}$$

where

$$|k+1,1\rangle = a_1^\dagger a_2^\dagger\,|k,0\rangle\,. \tag{3.192}$$

The normalization (the first Lanczos coefficient) is now given by

$$b_1 = \sqrt{(O_0^{(k)}\mathcal{L}|\mathcal{L}O_0^{(k)})} = \alpha\sqrt{k+1}, \tag{3.193}$$

so that

$$|O_1^{(k)}) = |k+1,1\rangle\,. \tag{3.194}$$

In the next step we write

$$\begin{aligned} |A_2) &= \mathcal{L}\,|k+1,1\rangle - b_1\,|k,0\rangle\,, \\ &= \alpha\sqrt{2(k+2)}\,|k+2,2\rangle + \alpha\sqrt{k+1}\,|k,0\rangle - b_1\,|k,0\rangle \\ &= \alpha\sqrt{2(k+2)}\,|k+2,2\rangle\,, \end{aligned} \tag{3.195}$$

where

$$|k+2,2\rangle = \frac{(a_1^\dagger)^2}{\sqrt{2!}}\frac{(a_2^\dagger)^2}{\sqrt{2!}}\,|k,0\rangle\,. \tag{3.196}$$

The normalization fixes the second Lanczos coefficient to

$$b_2 = (A_2|A_2)^{1/2} = \alpha\sqrt{2(k+2)}, \tag{3.197}$$

and finally we get the third basis vector

$$|\mathcal{O}_2^{(k)}) = |k+2,2\rangle\,. \tag{3.198}$$

This way, repeating the above procedure, we construct an orthonormal Krylov basis

$$|\mathcal{O}_n^{(k)}) = |k+n,n\rangle = \frac{(a_1^\dagger)^n}{\sqrt{n!}}\frac{(a_2^\dagger)^n}{\sqrt{n!}}\,|k,0\rangle\,, \tag{3.199}$$

with Lanczos coefficients

$$b_n = \alpha\sqrt{n(k+n)}. \tag{3.200}$$

The relation to the previous representation arises by the simple assignation $2h = k+1$.

Analogous computations can be done for general 2d CFTs. In this scenario the symmetry group is $SU(2,R)_L \otimes SU(2,R)_R$ and we have two Liouvillians; left (chiral) $\mathcal{L}$ and right (anti-chiral) $\bar{\mathcal{L}}$. The starting point now is the highest weight state

$$|\mathcal{O}^{h,\bar{h}}) = \mathcal{O}_{-h,-\bar{h}}\,|0\rangle_L \otimes |0\rangle_R \tag{3.201}$$

and we are interested in computing the wavefunction in the Krylov basis that describes the evolution of the operator

$$\begin{aligned} |\mathcal{O}^{(h,\bar{h})}(t)) &= e^{i(\mathcal{L}+\bar{\mathcal{L}})t}|\mathcal{O}^{h,\bar{h}}), \\ &= \sum_{n,m=0}^{\infty} i^{n+m}\varphi_{n,m}(t)|\mathcal{O}_{n,m}^{(h,\bar{h})}). \end{aligned} \tag{3.202}$$

The Lanczos algorithm can be followed exactly as before, but with two sets of Lanczos coefficients defined from the actions of the Liouvillians in the Krylov basis

$$\begin{aligned}\mathcal{L}|\mathcal{O}^{(h,\bar{h})}_{n,m}) &= b_n|\mathcal{O}^{(h,\bar{h})}_{n-1,m}) + b_{n+1}|\mathcal{O}^{(h,\bar{h})}_{n+1,m}),\\ \bar{\mathcal{L}}|\mathcal{O}^{(h,\bar{h})}_{n,m}) &= \bar{b}_m|\mathcal{O}^{(h,\bar{h})}_{n,m-1}) + \bar{b}_{m+1}|\mathcal{O}^{(h,\bar{h})}_{n,m+1}).\end{aligned} \tag{3.203}$$

The Schrodinger equation becomes

$$\begin{aligned}\partial_t\varphi_{n,m}(t) &= b_n\varphi_{n-1,m}(t) - b_{n+1}\varphi_{n+1,m}(t)\\ &+ \bar{b}_m\varphi_{n,m-1}(t) - \bar{b}_{m+1}\varphi_{n,m+1}(t).\end{aligned} \tag{3.204}$$

We can check that for

$$b_n = \alpha\sqrt{n(n+2h-1)}, \qquad \bar{b}_m = \bar{\alpha}\sqrt{m(m+2\bar{h}-1)}, \tag{3.205}$$

this equation is solved by the product of the wavefunctions (3.94).
Let us finally point that we can start the Lanczos algorithm from a more general initial state where the Liouvillian has a non-zero expectation value i.e., diagonal part. Equivalently, one may be interested in constructing the Krylov basis for a general Hamiltonian evolution [45]. In such cases, the algorithm can be repeated similarly with

$$|A_{n+1}) = (\mathcal{L}-a_n)|\mathcal{O}_n) - b_n|\mathcal{O}_{n-1}), \quad |\mathcal{O}_n) = b_n^{-1}|A_n), \tag{3.206}$$

and the two sets of coefficients are defined as

$$a_n = (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_n), \qquad b_n = (A_n|A_n)^{1/2}. \tag{3.207}$$

From these expressions we can easily determine the action of the Liouvillian in the Krylov basis

$$\mathcal{L}|\mathcal{O}_n) = a_n|\mathcal{O}_n) + b_{n+1}|\mathcal{O}_{n+1}) + b_n|\mathcal{O}_{n-1}), \tag{3.208}$$

and derive a discrete Schrodinger equation for the wavefunctions

$$\partial_t\phi_n(t) = i\left(a_n\phi_n + b_n\phi_{n-1} + b_{n+1}\phi_{n+1}\right), \tag{3.209}$$

that define the general state

$$|\mathcal{O}(t)) = \sum_n \phi_n|\mathcal{O}_n). \tag{3.210}$$

In symmetry setups such as SL(2,R), a_n's can e.g. be naturally associated with eigenvalues of L_0.

3.11 Appendix B: $SL(2,R)$ subalgebras of Virasoro

In this appendix we want to consider another example of our symmetry proposal. This is a natural variant of the SL(2,R) algebra scenario. Namely, instead of the global part of the conformal group, let us now consider the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \tag{3.211}$$

and focus on a different subset of three generators, namely L_0 and two Virasoro modes L_k and L_{-k} for a fixed integer $k>0$. Using (3.211), we see these three generators form a closed algebra since

$$[L_k, L_{-k}] = 2kL_0 + \frac{c}{12}k(k^2-1), \tag{3.212}$$

as well as

$$[L_0, L_{\pm k}] = \mp k L_{\pm k}. \tag{3.213}$$

By redefining

$$\tilde{L}_0 = \frac{1}{k}\left(L_0 + \frac{c}{24}(k^2-1)\right), \qquad \tilde{L}_{\pm 1} = \frac{1}{k} L_{\pm k}, \tag{3.214}$$

this is the standard $SL(2,R)$ algebra (3.37).

Given this observation, we can consider the following displacement operator

$$D_k(\xi) = e^{\xi L_{-k} - \bar{\xi} L_k}. \tag{3.215}$$

Using the BCH formula and $\xi = re^{i\phi}$ this can be written in factorized form

$$\begin{aligned} D_k(\xi) &= e^{e^{i\phi}\frac{\tanh(kr)}{k} L_{-k}} e^{-\frac{2}{k}\log(\cosh(kr))\left(L_0 + \frac{c}{24}(k^2-1)\right)} \\ &\times e^{-e^{-i\phi}\frac{\tanh(kr)}{k} L_k}. \end{aligned} \tag{3.216}$$

Applying it to the eigenstate $|h\rangle$ satisfying

$$L_0 |h\rangle = h |h\rangle, \qquad L_k |h\rangle = 0, \tag{3.217}$$

we derive

$$D_k(\xi) |h\rangle = \frac{1}{\cosh^{2h_k}(kr)} \sum_{n=0}^{\infty} e^{in\phi} \frac{\tanh^n(kr)}{n! k^n} L_{-k}^n |h\rangle. \tag{3.218}$$

where

$$h_k = \frac{1}{k}\left(h + \frac{c}{24}(k^2-1)\right). \tag{3.219}$$

Using the following result

$$\langle h| L_k^n L_{-k}^n |h\rangle = n! k^{2n} \frac{\Gamma(2h_k + n)}{\Gamma(2h_k)}, \tag{3.220}$$

we can introduce the orthonormal basis

$$|h, nk\rangle \equiv \sqrt{\frac{\Gamma(2h_k)}{n! k^{2n} \Gamma(2h_k + n)}} L_{-k}^n |h\rangle, \tag{3.221}$$

such that

$$\langle h, nk | h, mk\rangle = \delta_{n,m}. \tag{3.222}$$

We can now write the coherent state (3.218) in this basis and obtain

$$|z, h, k\rangle = \sum_{n=0}^{\infty} e^{in\phi} \frac{\tanh^n(kr)}{\cosh^{2h_k}(kr)} \sqrt{\frac{\Gamma(2h_k + n)}{n!\Gamma(2h_k)}} |h, nk\rangle. \tag{3.223}$$

To connect with the Lanczos approach we first notice that

$$\begin{aligned} L_0 |h, nk\rangle &= (h + nk) |h, nk\rangle, \\ L_{-k} |h, nk\rangle &= k\sqrt{(n+1)(2h_k + n)} |h, (n+1)k\rangle, \\ L_k |h, nk\rangle &= k\sqrt{n(2h_k + n - 1)} |h, (n-1)k\rangle. \end{aligned} \tag{3.224}$$

This structure now allows us to define the Liouvillian and Krylov basis

$$\mathcal{L}_k = \alpha(L_{-k} + L_k), \qquad |\mathcal{O}_n) = |h, nk\rangle, \tag{3.225}$$

together with the Lanczos coefficients

$$b_n = k\alpha\sqrt{n(2h_k + n - 1)}. \tag{3.226}$$

Let us point out that Hamiltonians of the type (3.225) also appeared recently in [141] but the more precise link remains to be understood.
These coefficients satisfy the algebraic relation

$$b_{n+1}^2 - b_n^2 = 2k^2\alpha^2(h_k + n). \tag{3.227}$$

For large n they grow as

$$b_n \simeq k\alpha n + k\alpha\frac{2h_k - 1}{2} + O(1/n). \tag{3.228}$$

Consequently, we can compute the Krylov complexity in this example

$$K_{\mathcal{O}} = \sum_n n|\varphi_n(t)|^2 = 2h_k \sinh^2(k\alpha t), \tag{3.229}$$

and conclude that the growth rate is now generalized to $\alpha_k = k\alpha$. Interestingly, this result is sensitive to the central charge c of the CFT. More physical aspects of complexity and entanglement in these states will be considered in [142].

3.12 Appendix C: CFT Correlators

In this appendix, we review/collect the basic material about thermal two-point correlators in 2d CFTs that appeared in the main text.

In Euclidean CFTs in 2d, the basic objects of interest are correlation functions of quasi-primary operators. These operator transform under chiral $z \to f(z)$ and anti-chiral conformal transformations $\bar{z} \to \bar{f}(\bar{z})$ as

$$\mathcal{O}(z, \bar{z}) \to (f'(z))^h(\bar{f}'(\bar{z}))^{\bar{h}}\mathcal{O}(f(z), \bar{f}(\bar{z})), \tag{3.230}$$

where the coefficients h and $\bar{h}$ are related to the conformal dimension Δ and the spin s of the operator by

$$\Delta = h + \bar{h}, \qquad s = h - \bar{h}. \tag{3.231}$$

From these finite transformations we can infer their infinitesimal counterparts. Under $z \to f(z) \simeq z + \epsilon(z)$ (and similarly for $\bar{z}$) we have

$$\delta_{\epsilon,\bar{\epsilon}}\mathcal{O}(z, \bar{z}) = (h\partial_z\epsilon + \epsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\mathcal{O}(z, \bar{z}). \tag{3.232}$$

The two-point functions of the quasi-primary operators $\mathcal{O}_i$ and $\mathcal{O}_j$ in the CFT vacuum state (this state is the state annihilated by the Virasoro modes $L_k|0\rangle = 0$ for $k \geq -1$)

$$G(z_1, z_2) = \langle 0|\mathcal{O}_i(z_1, \bar{z}_1)\mathcal{O}_j(z_2, \bar{z}_2)|0\rangle, \tag{3.233}$$

are fully determined by global conformal symmetry. Indeed, the invariance under inifinitesimal transformations with $\epsilon(z) = z^{k+1}$ for $k = -1, 0, 1$ forces them to satisfy the following differential equations

$$\sum_i \partial_{z_i} G = \sum_i (z_i\partial_{z_i} + h_i)G = \sum_i (z_i^2\partial_{z_i} + 2h_i z_i)G = 0, \tag{3.234}$$

and a similar set for the anti-chiral variables $\bar{z}_i$. These equations can be solved by

$$G(z_1, z_2) = \frac{d_{ij}\delta_{h_i,h_j}\delta_{\bar{h}_i,\bar{h}_j}}{z_{12}^{2h_i}\bar{z}_{12}^{2\bar{h}_i}}, \tag{3.235}$$

with $z_{ij} = z_i - z_j$. In most of examples one normalizes the operators appropriately such that $d_{ij} = \delta_{ij}$.

CFT's two-point functions on the Euclidean cylinder can now be obtained by the exponential conformal map, together with the transformation rule of the quasi-primary operators (3.230). This procedure allows us to obtain a universal two-point function for a CFT on a circle (finite size). Similarly, if we take the time of the cylinder to be compact with period β, we can use the same trick and arrive at the universal correlator for a CFT on a line at finite temperature.

More concretely, considering the two exponential conformal maps

$$f(z) = \exp\left(\frac{2\pi}{\beta_+}z\right), \qquad \bar{f}(\bar{z}) = \exp\left(\frac{2\pi}{\beta_-}\bar{z}\right), \tag{3.236}$$

the two-point correlator in a CFT on a line at chiral temperature $T_+ = 1/\beta_+$ and anti-chiral $T_- = 1/\beta_-$ becomes

$$\begin{aligned}\langle \mathcal{O}(z_1,\bar{z}_1)\mathcal{O}(z_2,\bar{z}_2)\rangle &= \left(\frac{\beta_+}{\pi}\sinh\left(\frac{\pi z_{12}}{\beta_+}\right)\right)^{-2h} \\ &\times \left(\frac{\beta_-}{\pi}\sinh\left(\frac{\pi \bar{z}_{12}}{\beta_-}\right)\right)^{-2\bar{h}}.\end{aligned} \tag{3.237}$$

All these results concern Euclidean CFTs. However, when studying operator growth we employed the result for the Lorentzian two-point function. This Lorentzian continuation appeared in the survival amplitude or autocorrelation function of the Lanczos approach

$$\begin{aligned}(\mathcal{O}(t)|\mathcal{O}) &\equiv \langle e^{\frac{\beta}{2}H}\mathcal{O}(t)e^{-\frac{\beta}{2}H}\mathcal{O}\rangle_\beta, \\ &= \langle e^{\frac{\beta}{2}H}e^{iHt}\mathcal{O}e^{-iHt}e^{-\frac{\beta}{2}H}\mathcal{O}\rangle_\beta,\end{aligned} \tag{3.238}$$

where in the second step we used the definition of the Heisenberg operator. The reason why this correlation function comes about was described in detail in section II.

This Lorentzian correlator can be obtained by analytic continuation of the Euclidean one. More concretely we just write

$$(\mathcal{O}(t)|\mathcal{O}) = \langle e^{\left(\frac{\beta}{2}+it\right)H}\mathcal{O}e^{-\left(\frac{\beta}{2}+it\right)H}\mathcal{O}\rangle_\beta. \tag{3.239}$$

This is just the previous euclidean correlator (3.237), after the identifications $z = x + i\tau$, $\bar{z} = x - i\tau$ and substitution $\tau \to \beta/2 + it$, so that

$$z_{12} = x_{12} + i\left(\frac{\beta_+}{2} + it\right), \qquad \bar{z}_{12} = x_{12} - i\left(\frac{\beta_-}{2} + it\right). \tag{3.240}$$

3.13 Appendix D: Coherent States: Formulas

In this last appendix, we collect several formulas from generalized coherent states used in the main text. There are many excellent books on this subject. We refer the reader to e.g. [99, 100].

We used three main families of coherent states, based on algebra SU(1,1) (or SL(2,R)) i.e., the Perelomov coherent states

$$|z,h\rangle = (1-|z|^2)^h \sum_{n=0}^{\infty} z^n \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}}\,|h,n\rangle\,, \tag{3.241}$$

the spin coherent states based on SU(2) algebra

$$|z,j\rangle = (1+z\bar{z})^{-j} \sum_{n=0}^{2j} z^n \sqrt{\frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)}}\,|j,-j+n\rangle\,, \tag{3.242}$$

and the standard coherent states based on the Heisenberg-Weyl algebra

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle\,. \tag{3.243}$$

Coherent states form an over-complete basis and are not orthogonal. Namely, in three examples above, for different values of the complex variables we have

$$\langle z_2,h|z_1,h\rangle = \frac{(1-|z_1|^2)^h(1-|z_2|^2)^h}{(1-z_1\bar{z}_2)^{2h}}, \tag{3.244}$$

$$\langle z_2,j|z_1,j\rangle = \frac{(1+|z_1|^2)^{-j}(1+|z_2|)^{-j}}{(1+z_1\bar{z}_2)^{-2j}}, \tag{3.245}$$

$$\langle z_2|z_1\rangle = \exp\left(-\frac{|z_1|^2+|z_2|^2}{2}+z_1\bar{z}_2\right). \tag{3.246}$$

In the part where the classical motion in phase space was studied and the Lagrangian, was derived, we used

$$\langle z,h|i\partial_t|z,h\rangle = i\frac{h(\bar{z}z'-z\bar{z}')}{1-|z|^2}, \tag{3.247}$$

for SU(1,1), then

$$\langle z,j|i\partial_t|z,j\rangle = i\frac{j(\bar{z}z'-z\bar{z}')}{1+|z|^2}, \tag{3.248}$$

for SU(2), and finally

$$\langle z|i\partial_t|z\rangle = \frac{i}{2}\left(\bar{z}z'-z\bar{z}'\right), \tag{3.249}$$

for Weyl-Heisenberg coherent states. Moreover, for the computation of symbols of Hamiltonians, we used the expectation values of individual generators of the Lie algebras in the coherent states, which for SL(2,R) take the form

$$\begin{aligned}
\langle z,h|\,L_0\,|z,h\rangle &= h\frac{1+|z|^2}{1-|z|^2},\\
\langle z,h|\,L_{-1}\,|z,h\rangle &= \frac{2h\bar{z}}{1-|z|^2},\\
\langle z,h|\,L_1\,|z,h\rangle &= \frac{2hz}{1-|z|^2}.
\end{aligned} \tag{3.250}$$

From the above relations we determine the expectation values of the complexity algebra generators in these coherent states

$$\begin{aligned}
\langle z,h|\,\tilde{K}_{\mathcal{O}}\,|z,h\rangle &= 4h\alpha^2\cosh(\rho),\\
\langle z,h|\,\mathcal{L}\,|z,h\rangle &= 2h\alpha\cos(\phi)\sinh(\rho),\\
\langle z,h|\,\mathcal{B}\,|z,h\rangle &= -2ih\alpha\sin(\phi)\sinh(\rho).
\end{aligned} \tag{3.251}$$

On our trajectory, $\rho = 2\alpha t$ and $\phi = \pi/2$ we see that both, $\tilde{K}_{\mathcal{O}}$ and the anti-Hermitian $\mathcal{B}$ grow exponentially while the Liouvillian vanishes.

Next, the Euler-Lagrange equations where written in terms of Poisson bracket that for (ρ, ϕ) coordinates of SU(1,1) became

$$\{A, B\} = \frac{1}{h \sinh(\rho)} \left(\frac{\partial A}{\partial \rho} \frac{\partial B}{\partial \phi} - \frac{\partial A}{\partial \phi} \frac{\partial B}{\partial \rho} \right). \tag{3.252}$$

For SU(2) we similarly have

$$\begin{aligned} \langle z, j | J_0 | z, j \rangle &= -j \frac{1 - |z|^2}{1 + |z|^2}, \\ \langle z, j | J_+ | z, j \rangle &= j \frac{2\bar{z}}{1 + |z|^2}, \\ \langle z, j | J_- | z, j \rangle &= j \frac{2z}{1 + |z|^2}. \end{aligned} \tag{3.253}$$

Then, in the (θ, ϕ) coordinates the SU(2) Poisson bracket reads

$$\{A, B\} = \frac{1}{j \sin(\theta)} \left(\frac{\partial A}{\partial \theta} \frac{\partial B}{\partial \phi} - \frac{\partial A}{\partial \phi} \frac{\partial B}{\partial \theta} \right). \tag{3.254}$$

For Heisenberg-Weyl Lagrangian and Hamiltonian we used

$$\langle z | a | z \rangle = a, \quad \langle z | a^\dagger | z \rangle = \bar{z}, \quad \langle z | \hat{n} | z \rangle = |z|^2. \tag{3.255}$$

Then, in the r, ϕ coordinates of the plane, the Weyl-Heisenberg Poisson bracket is

$$\{A, B\} = \frac{1}{2r} \left(\frac{\partial A}{\partial r} \frac{\partial B}{\partial \phi} - \frac{\partial A}{\partial \phi} \frac{\partial B}{\partial r} \right). \tag{3.256}$$

Next, the symbols for SL(2,R) are related to embedding coordinates of the hyperbolic space as follows

$$\begin{aligned} \langle z, h | L_0 | z, h \rangle &= \sqrt{2h}\, X_0, \\ \langle z, h | L_{-1} | z, h \rangle &= \sqrt{2h}\, X_+, \\ \langle z, h | L_1 | z, h \rangle &= \sqrt{2h}\, X_-, \end{aligned} \tag{3.257}$$

and satisfy

$$-X_0^2 + X_+ X_- = -h/2. \tag{3.258}$$

We can further parametrize them in terms of ρ and ϕ coordinates

$$\{X_0, X_+, X_-\} = \sqrt{\frac{h}{2}} \{\cosh(\rho), e^{-i\phi} \sinh(\rho), e^{i\phi} \sinh(\rho)\},$$

so that the induced metric becomes

$$ds^2 = \frac{h}{2} \left(d\rho^2 + \sinh^2 \rho d\phi^2 \right). \tag{3.259}$$

Similar relations hold for the other two symmetry groups.
Last but not least, we discussed the role played by $\mathcal{F}_1$ norms that are given by

$$\langle z, h | \delta z, h \rangle = i2h \sinh^2 \left(\frac{\rho}{2} \right) d\phi, \tag{3.260}$$

$$\langle z, j | \delta z, j \rangle = i2j \sin^2 \left(\frac{\theta}{2} \right) d\phi, \tag{3.261}$$

$$\langle z | \delta z \rangle = i r^2 d\phi, \tag{3.262}$$

in SL(2,R), SU(2) and Heisenberg-Weyl examples respectively. For our trajectories $\rho = 2\alpha t$, $\theta = 2\alpha t$ and $r = \alpha t$, they all have the universal form

$$\langle z|\delta z\rangle = iK_{\mathcal{O}}d\phi. \tag{3.263}$$

Chapter 4

Probing the entanglement of operator growth

Extended summary

This article builds on the results reported in the previous chapter and namely it expands on the calculation of quantum information tools that probe the dynamics of operator growth using the Krylov formalism. In order to achieve that, I had to make use of the features of two-mode coherent states which are expanded in terms of a tensor product of Fock states that takes the form $|n\rangle \otimes |m\rangle = |n, m\rangle$. This implies the factorization of the Hilbert space in subsystems, which in turn allows one to explicitly trace out part of the degrees of freedom to gain access to entanglement negativity as well as Renyi entropies and associated quantities such as the entanglement entropy and capacity of entanglement.

I computed these quantities for the two-mode coherent states associated with SU(1,1) and SU(2) and showed that they provide additional nuances to the evolution of operator growth that can be used in tandem with Krylov complexity in order to get a more complete picture of this phenomenon. Since obtaining a general analytic expression for SU(1,1) was not tractable, I elected to make a late time approximation which allowed for an exact solution to be extracted. The assumption hinges on the idea that for late enough times, the support of the wavefunction on the Krylov chain will be at large site number n and as such the binomial coefficient entering the probability distribution over the Krylov basis can be approximated by

$$\binom{n+n_0}{n_0} \simeq \frac{n^{n_0}}{n_0!} \ , \tag{4.1}$$

where n_0 is the weight of the representation. The above assumption is based on the fact that one expects such a spread of the wavefunction to be present for systems that are sufficiently complex and their evolution generic. I also verified that the approximation is valid by comparing the late time limit of cases where an analytical solution could be found for all times, with the approximate expressions.

The behavior of the capacity of entanglement is of particular interest since it contains additional structure compared to the rest. Namely, it exhibits local extrema at times when the other quantities are not sensitive to. This could potentially indicate that the capacity of entanglement contains information about some novel physical process related to operator growth, which however I have no additional insight for.

The main message of this work is that the probability distribution over the Krylov basis contains a wealth of information and even though Krylov complexity, which is the mean of this distribution, is an important quantity, there are many facets of the dynamics that it overlooks. Moreover, it takes

a step in understanding the differences between measures of entanglement and complexity which can be directly compared within this framework.

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Abstract

In this work we probe the operator growth for systems with Lie symmetry using tools from quantum information. Namely, we investigate the Krylov complexity, entanglement negativity, entanglement entropy and capacity of entanglement for systems with SU(1,1) and SU(2) symmetry. Our main tools are two-mode coherent states, whose properties allow us to study the operator growth and its entanglement structure for any system in a discrete series representation of the groups under consideration. Our results verify that the quantities of interest exhibit certain universal features in agreement with the universal operator growth hypothesis. Moreover, we illustrate the utility of this approach relying on symmetry as it significantly facilitates the calculation of quantities probing operator growth. In particular, we argue that the use of the Lanczos algorithm, which has been the most important tool in the study of operator growth so far, can be circumvented and all the essential information can be extracted directly from symmetry arguments.

4.1 Introduction

Quantum information tools are becoming an increasingly important resource for the study of complex systems. In fact, quantities from quantum information theory have found applications in a variety of topics ranging from condensed matter physics to the study of quantum gravity. Among those, one of the most well-studied is entanglement entropy which has been at the center of many pioneering works. For example, in the context of the AdS/CFT correspospondence [6], the seminal works of Ryu and Takayanagi [143,144] have established the duality between the entanglement entropy of intervals in the CFT and the length of bulk geodesics. This development has sparked many subsequent works further exploring the relationship between entanglement entropy and the structure of spacetime [16, 87, 88, 145–149]. Considerations of entanglement entropy for many-body and condensed matter systems have been equally important in furthering our ability to understand them and predict their behavior, as reviewed in [150].

In recent years this discourse has grown especially in the direction of systems with chaotic dynamics. More specifically, what one finds in such cases is that entanglement entropy might not be sufficient for a complete description of the system. This observation holds particular relevance in the study of black holes, where it appears that more refined probes are required to describe their late time dynamics. In [3], it was suggested that a quantity which could serve this purpose is computational complexity, which appears to be correctly capturing the features of the black hole evolution at late times. Since then significant effort has been devoted to finding appropriate measures of complexity and studying their properties [22, 23, 27, 32–34, 114, 118, 120, 122, 126, 151].

The main topic of this work is the study of operator growth, which has been investigated from different angles in a number of recent works [42,46,53,71,74,76,78,81,82,152–156]. In particular, the seminal work [42] introduced the concept of Krylov complexity (also referred to as K-complexity) as a probe for operator growth and furthermore asserted that the latter exhibits universal behavior. The authors dubbed this “the universal operator growth hypothesis" which we will discuss in more detail in the following section. Subsequent works made use of the concept of Krylov complexity in a number of different settings in order to characterize the operator growth and whether it is universal. For example, in [46] the authors study the evolution of Krylov complexity in systems with finite degrees of freedom for different timescales with emphasis on those much larger than scrambling time. Furthermore, they define the notion of K-entropy, which is an additional tool that we also make use of in this work. In [53] the authors compute Krylov complexity for different CFT models and

provide both analytical and numerical results that are consistent with an exponential growth for all the theories under study. Thus, they argue that Krylov complexity cannot be used to differentiate between chaotic and integrable or even free CFTs. In [154] the authors make use of symmetry arguments to facilitate the computation of Krylov complexity and characterize the operator growth for quantum systems with Lie symmetry. Subsequently in [155] the authors study the particulars of 2-dimensional CFTs that are characterized by Virasoro symmetry. While important advances have been made, a consensus on whether operator growth obeys some type of universality has not been reached and a complete characterization of the processes involved is still missing.

As the name operator growth suggests, operators in quantum systems tend to "grow" in the sense that they become more complicated as the system evolves in time, even if we start with an operator that is initially simple. This is particularly prominent in chaotic systems for which this growth is believed to be maximal. We will make these statements more precise in following sections but the question we ought to answer from the outset is in what way can one probe this type of process. We advocate that certain tools from quantum information are exceptionally well-suited candidates as they highlight universal features of operator growth. Additionally, we show that computing these probes can be greatly facilitated for systems with symmetry. More concretely, our goal will be to characterize operator growth for systems with SU(1,1) and SU(2) symmetry. We will achieve this by employing the generalized coherent states associated to each group, which will allow us to compute their Krylov complexity, entanglement entropy, negativity, and capacity of entanglement. The main point of comparison between these two cases will be the chaotic and integrable dynamics that they exhibit respectively. We show that it is possible to obtain analytic results for the aforementioned quantities for any discrete series representation of the Lie groups under study thus expanding what had already been shown about the trivial representations in the author's previous work [154]. Furthermore, we provide an in-depth analysis of the less well-studied capacity of entanglement, which reveals some intriguing behavior that could signal some novel physical phenomenon. In particular, we observe certain oscillating structures that indicate that the capacity of entanglement is sensitive to information that the rest of the quantities we studied are not able to detect.

The article is structured as follows. In section 2 we provide a brief review on the topic of operator growth and how it can be characterized using Krylov complexity. Section 3 reviews the construction of two-mode coherent states which serve as the starting point for the calculations that we need to carry out. Section 4 contains a brief introduction to the other quantum information tools mentioned above and our main results for SU(1,1) and SU(2). Finally, in section 5 we summarize our conclusions and provide additional comments.

4.2 Operator growth and Krylov complexity

Studying the time evolution of operators in quantum systems is hardly a new problem. However, recently there has been a renewed interest in the subject mainly in the context of chaotic systems. One of the developments that has been highly influential is the universal operator growth hypothesis [42], which asserts that there are certain features of operator growth that depend only on the class of model under study and not on its specific details. In this section we will briefly review the notion of operator growth in general and follow the arguments leading up to the universal operator growth hypothesis. A quantity that arises naturally in this endeavor is Krylov complexity, which will also prove relevant to our discussions in later sections by virtue of it being a quantum information tool of particular interest. There are a number of works with pedagogical introductions to this topic [42, 152–154], so here we restrict ourselves to the basic notions required for the ensuing discussions.

Let us start by stating the problem that we wish to study. Suppose we are given a quantum system described by the Hamiltonian H and a time-dependent operator $\mathcal{O}(t)$. For simplicity we will

assume that the Hamiltonian is local and the operator Hermitian. More general setups have been considered for example in [152], however for the purposes of the present work it suffices to restrict to the above. We shall now try to determine the time evolution of the operator $\mathcal{O}(t)$ which in the Heisenberg picture is given by

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} \ , \tag{4.2}$$

or equivalently as an expansion

$$\mathcal{O}(t) = 1 + it[H, \mathcal{O}(0)] + \frac{(it)^2}{2}[H, [H, \mathcal{O}(0)]] + ... \tag{4.3}$$

We would like to have an expression for $\mathcal{O}(t)$ at arbitrary times and obviously having to perform all the nested commutators is not the optimal way of doing so. A particular way to approach this problem is by using the Lanczos algorithm [43, 45]. This is simply an iterative procedure that using some initial data can generate for us approximate answers. The main idea behind the Lanczos algorithm is providing an orthonormal basis in terms of the nested commutators of the Hamiltonian, which we can then use to express $\mathcal{O}(t)$ at an arbitrary time. To make this more precise let us first define the Liouvillian super-operator

$$\mathcal{L} = [H, \cdot] \ . \tag{4.4}$$

This then implies [1]

$$\mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O}(0) = 1 + it\mathcal{L}\mathcal{O}(0) + \frac{(it)^2}{2}\mathcal{L}^2\mathcal{O}(0) + ... \tag{4.5}$$

We seek to construct a basis out of the elements

$$\mathcal{O}(0) = |\tilde{\mathcal{O}}), \quad \mathcal{L}\mathcal{O}(0) = |\tilde{\mathcal{O}}_1), \quad \mathcal{L}^2\mathcal{O}(0) = |\tilde{\mathcal{O}}_2)... \tag{4.6}$$

To do so it is convenient to view the above as the basis vectors (states) in an abstract operator Hilbert space provided that we can orthogonalize them with respect to each other. We denote the orthonormalized vectors as $|\mathcal{O}_n)$. We must also specify the inner product between these vectors. The standard choice used throughout the literature is the Wightman inner product

$$(A|B) = \langle e^{H\beta/2}A^\dagger e^{-H\beta/2}B\rangle_\beta \ , \tag{4.7}$$

where $\langle\rangle_\beta$ is to be understood as the thermal expectation value at inverse temperature β. Having laid down the groundwork, one only needs to provide a scheme for the orthogonalization of the vectors $|\mathcal{O}_n)$ which is precisely the purpose of the Lanczos algorithm. Starting with the initial state $|\mathcal{O})$ we subsequently obtain

$$|\mathcal{O}_1) = b_1^{-1}\mathcal{L}|\mathcal{O}) \ , \tag{4.8}$$

where b_1 is the normalization constant. The algorithm proceeds iteratively as follows

$$|A_n) = \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2}) \ , \tag{4.9}$$

and then normalizing

$$|\mathcal{O}_n) = b_n^{-1}|A_n), \qquad b_n = (A_n|A_n)^{1/2} \ . \tag{4.10}$$

In this manner one orthogonalizes each vector with respect to the previous one in the sequence and ultimately obtains an orthonormal basis, referred to as the Krylov basis. The normalization constants b_n are called the Lanczos coefficients and as we will see below they encode very useful information about the behavior of the system. Depending on the system of interest the Lanczos algorithm may

[1] The first term of the RHS should be $\mathcal{O}(0)$, but this typo made it to the published version

or may not terminate, providing a finite or infinite dimensional Krylov space respectively. Once this procedure has been completed the operator $\mathcal{O}(t)$ can be expanded using the Krylov basis as

$$|\mathcal{O}(t)) = \sum_n i^n \varphi_n(t) |\mathcal{O}_n) \ , \tag{4.11}$$

where $\varphi_n(t)$ are some appropriate coefficients that satisfy the condition

$$\sum_n |\varphi_n(t)|^2 = 1 \ . \tag{4.12}$$

Of course determining these coefficients is not a menial task. However, with a few lines of algebra one can find a formula that relates them to the Lanczos coefficients (for a detailed derivation see [154]). This expression takes the form of a discrete Schrodinger equation that is solved iteratively

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1} \varphi_{n+1}(t) \ . \tag{4.13}$$

Already from (4.5) it is easy to see that as the system evolves in time we require more terms of the expansion to accurately describe the operator. From the perspective of the Krylov basis this can be interpreted as the operator "growing" in Krylov space and as such requiring more basis vectors for its decomposition. A natural candidate that can serve as a quantitative measure of this growth is Krylov complexity, simply defined as

$$K_{\mathcal{O}} \equiv \sum_n n \, |\varphi_n(t)|^2 \ . \tag{4.14}$$

Equation (4.13) can be thought of as describing a particle moving on a one-dimensional chain, where φ_n are the different sites and b_n are the hopping coefficients between them. This entails an elegant interpretation of Krylov complexity as the average position of the particle on the chain.

Before concluding this section a few remarks are in order. We have described a procedure that enables us to extract information about the evolution of Heisenberg operators in principle for any quantum system. More importantly it is possible to take an extra step and classify quantum systems according to certain universal features that they exhibit. Namely, for chaotic systems one expects that the growth of operators will be maximal, whereas for other generic cases such as integrable or free theories the growth will proceed at a slower rate. Hopefully it will become clearer why that should be the case in later sections, but for now let us rely on our physical intuition to justify these claims. As we mentioned previously the problem of operator growth can be mapped to a particle hopping on a one-dimensional chain. From this point of view one expects that it becomes increasingly easier for the particle to hop on every next site, since that would be the picture consistent with an operator that exhibits maximal growth. Hence, the Lanczos coefficients b_n that serve as the hopping amplitudes should grow with increasing n. The question is then "how fast do they grow"? The simple answer provided by the universal operator growth hypothesis for maximally chaotic theories is "as fast as locality permits". More concretely, in [42] the authors show that, following this hypothesis, the growth of b_n is of the linear form $b_n \sim an + \gamma$, which in turn implies an exponential growth of Krylov complexity with a characteristic exponent that depends on the specifics of the system. This result is consistent with the notion of classical chaos where one also expects an exponential growth with a characteristic Lyapunov exponent. Similar conclusions can be drawn for integrable and free theories where the b_n have been shown to grow as $b_n \sim \sqrt{n}$ and remain constant respectively. More recently, the authors of [53] have shown that certain free field QFTs attain the maximal growth for the Lanczos coefficients and hence they have argued that this should not be necessarily regarded as a sign of chaos. While this shows that the universal operator growth hypothesis cannot immediately discriminate between chaotic and non-chaotic theories, it still provides a framework within which

one can have a universal description of different classes of systems according to the time evolution of their operators.

However, such powerful techniques always come with certain limitations. In this case we are restricted by the iterative nature of this approach. Even though it is very well-tailored for obtaining numerical results, it is generally hard to go beyond that. In a handful of cases it is possible to obtain a closed form for the Lanczos coefficients, but even for those the process is far from simple.

Recently, this obstacle was partially circumvented for systems whose symmetry is described by a Lie group. More specifically, in [154] it was shown that it is possible to obtain the Lanczos coefficients as well as the φ_n and quantities related to those directly from symmetry arguments without appeal to the Lanczos algorithm itself. The way to do so is through coherent states, which are objects intimately connected to the symmetry of the problem. In the sections to come we will further explore this direction and show how we can use it to not only compute the Krylov complexity, but other quantum information tools as well.

A final point that may be worth stressing is that the Lanczos coefficients b_n computed starting from such symmetry arguments have been shown to be universal for each symmetry group (at least as far their discrete series representations are concerned). In particular, it has been shown that they assume the general forms

$$b_n = \alpha\sqrt{n(2h+n-1)}\ , \tag{4.15}$$

for SU(1,1) and

$$b_n = \alpha\sqrt{n(2j-n+1)}\ , \tag{4.16}$$

for SU(2). In both cases α is a system specific parameter and h, j label the representation of the group. For large n the Lanczos coefficients associated to SU(1,1) grow linearly, whereas their SU(2) counterparts grow as $\sqrt{n}$ up to their maximum value in agreement with the universal operator growth hypothesis. Any calculations that we carry out in following sections do not affect their behavior and any deviation from universality that we find is a sign that the Lanczos coefficients do not entirely capture the behavior of the system.

4.3 Two-mode coherent states

In this section we review the construction of two-mode coherent states in the context of the Lie groups that are examined in this work. This will allow us to compute several quantities of interest in the following sections. The key fact in this discussion is that for each group under consideration there exist discrete series representations that can be obtained by expressing the generators of the group in terms of a pair of bosonic ladder operators. Let us make this assertion more concrete by first examining the properties of SU(1,1) (a more rigorous discussion can be found in [99]).

The Lie algebra of SU(1,1) has three generators K_0, K_1, K_2 satisfying the commutation relations

$$[K_1, K_2] = -iK_0, \quad [K_2, K_0] = iK_1, \quad [K_0, K_1] = iK_2\ . \tag{4.17}$$

We can define a set of ladder operators by simply changing the basis as follows

$$K_\pm = \pm i(K_1 \pm iK_2), \quad K_0\ . \tag{4.18}$$

The appropriate basis vector on which these operators act is of the form $|k, \mu\rangle$. For the discrete series representations k takes integer and half-integer values ($k = 1/2, 1, 3/2, ...$) and μ is the eigenvalue of K_0. We can further modify this basis by the identifications

$$K_+ = a^\dagger b^\dagger, \quad K_- = ab, \quad K_0 = \frac{1}{2}(a^\dagger a + b^\dagger b + 1)\ , \tag{4.19}$$

where a, b are bosonic ladder operators that satisfy the usual commutation relations $[a, a^\dagger] = [b, b^\dagger] = 1$. It is straightforward to check that this new basis still satisfies the SU(1,1) commutation relations and naturally the associated basis vector has the form of a two-mode state

$$|m, n\rangle = (m!n!)^{-\frac{1}{2}} (a^\dagger)^m (b^\dagger)^n \, |0, 0\rangle \ . \tag{4.20}$$

The characteristic k of the representation is related to the above expression through $k = \frac{1}{2}(1 + |n_0|)$, where $n_0 = m - n$. Therefore, it is possible to obtain any representation of the discrete series by considering the appropriate two-mode state.

Using the Fock space that each representation defines one can construct an associated family of coherent states. This is achieved by the action of a displacement operator on the appropriate vacuum state. The former is defined in terms of the algebra generators as:

$$D(\xi) = e^{\xi K_+ - \bar{\xi} K_-} = e^{\xi a^\dagger b^\dagger - \bar{\xi} ab} = e^{zK_+} e^{\eta K_0} e^{-\bar{z} K_-} \ , \tag{4.21}$$

where in the last equality we used the BCH formula to bring the operator in a normal form with

$$z = \frac{\xi}{|\xi|} \tanh |\xi|, \quad \eta = 2 \ln \cosh |\xi| \ . \tag{4.22}$$

It is useful to introduce polar coordinates $\xi = re^{i\phi}$, such that z parametrizes the unit disc

$$z = e^{i\phi} \tanh r, \qquad |z| < 1 \ . \tag{4.23}$$

The coherent states that we are about to construct are actually well studied objects in the field of quantum optics (for a pedagogical introduction see [128]). They are referred to as two-mode photon added (or subtracted) squeezed states and despite their long name are actually rather simple. The most common species among them are the single photon added states which arise from the action of the above displacement (squeezing) operator on the shifted vacuum $|1, 0\rangle$. Here we want to consider their more exotic cousins that are generated from an arbitrarily shifted vacuum $|n_0, 0\rangle$. Thus, the coherent states we are seeking are given by

$$|z\rangle_{n_0} = \mathcal{N}_{n_0} (b)^{n_0} D(z) \, |0, 0\rangle \ , \tag{4.24}$$

where $\mathcal{N}$ is a normalization constant. We would like to express $|z\rangle$ as a linear combination of Fock states, for which we need the Bogoliubov transformation [2]

$$b(z) = D^\dagger(z) b D(z) = b \cosh(r) + a^\dagger e^{i\phi} \sinh(r) \ . \tag{4.25}$$

Using the above it is straightforward to obtain

$$|z\rangle_{n_0} = \mathcal{N}_{n_0} D(z) e^{in_0\phi} \sinh^{n_0} r \sqrt{n_0!} \, |n_0, 0\rangle \ . \tag{4.26}$$

The normalization is chosen as

$$\begin{aligned} \mathcal{N}_{n_0} &= \langle z| \, (b^\dagger)^{n_0} (b)^{n_0} \, |z\rangle^{-1/2} = \langle 0, 0| \, D^\dagger(z) (b^\dagger)^{n_0} (b)^{n_0} D(z) \, |0, 0\rangle^{-1/2} \, , \\ &= \langle 0, 0| \, (b^\dagger(z))^{n_0} (b(z))^{n_0} \, |0, 0\rangle^{-1/2} \end{aligned} \tag{4.27}$$

and an explicit computation yields

$$\mathcal{N}_{n_0} = \frac{1}{\sqrt{n_0!} \sinh^{n_0} r} \ . \tag{4.28}$$

[2] The first term of the RHS should be $a \cosh r$

We can now act with the displacement operator on the shifted vacuum to get

$$D(z)e^{in_0\phi}\left|n_0,0\right\rangle = e^{in_0\phi}e^{e^{i\phi}\tanh(r)a^\dagger b^\dagger}e^{-\log(\cosh(r))(a^\dagger a+b^\dagger b+1)}e^{-e^{i\phi}\tanh(r)ab}\left|n_0,0\right\rangle \tag{4.29}$$

$$= e^{in_0\phi}\frac{1}{(\cosh r)^{n_0+1}}\sum_{n=0}^{\infty}e^{in\phi}(\tanh r)^n\sqrt{\binom{n+n_0}{n_0}}\left|n_0+n,n\right\rangle \ . \tag{4.30}$$

Therefore, neglecting the constant phase factor $e^{in_0\phi}$, which as we shall see is irrelevant for our subsequent calculations, we can write

$$\left|z\right\rangle_{n_0} = \sum_{n=0}^{\infty}e^{in\phi}\varphi_n\left|n_0+n,n\right\rangle \ , \tag{4.31}$$

with

$$\varphi_n = \frac{\tanh^n r}{(\cosh r)^{n_0+1}}\sqrt{\binom{n+n_0}{n_0}} \ . \tag{4.32}$$

The choice of the symbol φ to represent the coefficients of the decomposition is of course not a coincidence. In [154] it is rigorously shown that these coefficients match precisely the wavefunctions φ that arise from the Lanczos algorithm as was explained in the previous section. Furthermore, the variable r is taken to be proportional to time $r = \alpha t$, which allows an interpretation of the operator growth in terms of a motion in the classical phase space of the problem. In this work we will also treat r as time even though we will not denote it explicitly. For an alternative derivation of these states using the techniques of 2d CFT see appendix 4.5.

Let us now turn our attention to the construction of the two-mode coherent states for SU(2) [157]. The process is practically the same as before, so for the sake of brevity we will skip the tedious steps and focus on the essentials. The Lie algebra of SU(2) is characterized by 3 generators that we label J_0, J_+, J_- satisfying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0 \ . \tag{4.33}$$

Once again we seek to express those in terms of a pair of bosonic ladder operators which we will now label α_1, α_2 to keep them distinct from the SU(1,1) case. To build a representation of SU(2) the appropriate relation between the bosonic operators and the J_0, J_+, J_- is given by the following identifications

$$J_+ = \alpha_1^\dagger\alpha_2, \quad J_- = \alpha_2^\dagger\alpha_1, \quad J_0 = \frac{1}{2}(\alpha_1^\dagger\alpha_1 + \alpha_2^\dagger\alpha_2) \ . \tag{4.34}$$

The coherent states in this basis are given by

$$\left|z_1,z_2\right\rangle = \sum_{N_1,N_2}F_{N_1,N_2}\left|N_1,N_2\right\rangle \ , \tag{4.35}$$

where the function F is

$$F_{N_1,N_2} = \sqrt{\frac{(N_1+N_2)!}{N_1!N_2!}}(\cos\chi e^{i\beta_1})^{N_1}(\sin\chi e^{i\beta_2})^{N_2} \ . \tag{4.36}$$

The coordinates $0 \le \chi \le \pi/2, \quad 0 \le \beta_1 \le 2\pi, \quad 0 \le \beta_2 \le 2\pi$ parametrize the 3-sphere and the occupation numbers N_1, N_2 have to satisfy the property $N_1 + N_2 = 2j$ where j is the quantum number associated to the typical representation of SU(2) mentioned above. Thus we can rewrite the expression for F as:

$$F_{N_2} = \sqrt{\frac{(2j)!}{(2j-N_2)!N_2!}}(\cos\chi e^{i\beta_1})^{2j-N_2}(\sin\chi e^{i\beta_2})^{N_2} \ . \tag{4.37}$$

This will allow us to express our results in terms of the more familiar quantum number j. Notice that this implies that our states can be rewritten as

$$|z_1, z_2\rangle = \sum_{N_2=0}^{2j} F_{N_2} |N_2, 2j - N_2\rangle \ . \tag{4.38}$$

4.4 Quantum information tools

Before diving into the computations that utilize the technology developed above, we provide some generalities about each of the quantities that we will be computing in this section. Krylov complexity was reviewed in its own right in section 2, so here we will be concerned with negativity, entanglement entropy and capacity of entanglement.

Negativity is a measure of entanglement that, given a mixed state ρ, quantifies by how much the partial transpose ρ^{PT} fails to be positive definite [158]. It is defined as

$$\mathcal{N} \equiv \frac{||\rho^{PT}|| - 1}{2} \ , \tag{4.39}$$

where

$$||\rho^{PT}|| = \mathrm{Tr}\left\{\sqrt{\rho^{PT\dagger}\rho^{PT}}\right\} \ . \tag{4.40}$$

A common variation (and the one we will be primarily concerned with) that originates from the above definition is the logarithmic negativity

$$E_{\mathcal{N}}(\rho) = \log_2(1 + 2\mathcal{N}(\rho)) \ . \tag{4.41}$$

This particular probe of entanglement is widely used in quantum optics where it is common to know explicitly the density matrix. In other fields it is standard practice to use the natural logarithm instead of the logarithm with base 2, which simply results in a difference by some constant.

For a general two mode state the computation is as follows. The density matrix is of the form

$$\rho = \sum_{n,m} c_n c_m^* |n, n\rangle \langle m, m| \ . \tag{4.42}$$

The partial transpose is then given by transposing the elements of only one of the states obtaining

$$\rho^{PT} = \sum_{n,m} c_n c_m^* |n, m\rangle \langle m, n| \ . \tag{4.43}$$

In [128] the negativity is computed by bringing the partial transpose to its diagonal form and reading off its eigenvalues. However, a straightforward calculation of the trace norm is more illuminating, albeit slightly longer. First, let us compute the Hermitian conjugate of ρ^{PT}

$$\rho^{PT\dagger} = \sum_{n,m} [c_n c_m^* |n, m\rangle \langle m, n|]^\dagger = \sum_{n,m} c_n^* c_m \left[(|n\rangle \otimes |m\rangle)(\langle n| \otimes \langle m|)\right]^\dagger = \sum_{n,m} c_n^* c_m |m, n\rangle \langle n, m| \ . \tag{4.44}$$

Computing the trace norm is then simply done as shown below

$$\left|\left|\rho^{PT}\right|\right| = \sum_{k,l} \sqrt{\sum_{n,m} |c_n c_m|^2 \langle k, l|m, n\rangle \langle m, n|k, l\rangle} \tag{4.45}$$

$$= \sum_{k,l} |c_k c_l| = \left(\sum_k |c_k|\right)^2 \tag{4.46}$$

and therefore the logarithmic negativity assumes the simple form

$$E_{\mathcal{N}}(\rho) = \log_2 \left(1 + \sum_{n \neq m} |c_n c_m| \right) = \log_2 \left(\sum_n |c_n| \right)^2 . \tag{4.47}$$

Given a density matrix ρ the entanglement entropy is defined as

$$S \equiv - \operatorname{Tr} \rho_A \ln \rho_A \ , \tag{4.48}$$

however, in practice one usually obtains it as the $q \to 1$ limit of the Rényi entropy

$$S^{(q)} = \frac{1}{1-q} \ln \operatorname{Tr} \rho_A^q \ . \tag{4.49}$$

Finally, the capacity of entanglement is a concept recently gaining popularity in the context of the black hole information paradox [130, 159]. This is because it is a probe that is more sensitive to the intricate phenomena that take place during black hole evaporation compared to the entanglement entropy which is the quantity that had been extensively studied in the past. An elaborate review on the capacity of entanglement was given in [129]. There are several different definitions, but the one that is best suited for our purposes is given again in terms of the Rényi entropy as

$$C = \left. \frac{q^2 d^2 [(1-q) S^{(q)}]}{dq^2} \right|_{q=1} . \tag{4.50}$$

In computing the Rényi entropies we will be taking advantage of the two-mode representation of our states. Namely, the density matrices of the states we are interested in are of the form $\rho = \sum_{n,m} \varphi_n \varphi_m^* \, |n + n_0, n\rangle \langle m, m + n_0|$ and so by tracing over one of the two modes we have $\rho_A = \sum_n |\varphi_n|^2 \, |n + n_0\rangle \langle n + n_0|$. The $|\varphi_n|^2$ can then be interpreted as the eigenvalues of the reduced density matrix, which in turn allows us to directly compute the Rényi entropies by the following substitution

$$S^{(q)} = \frac{1}{1-q} \ln \operatorname{Tr} \rho^q = \frac{1}{1-q} \ln \sum_n |\varphi_n|^{2q} \ . \tag{4.51}$$

Notice that all the quantities we are concerned with require as input the norm $|\varphi_n|$, which implies that any phase factors in our expressions are rendered irrelevant. This is the reason we are being cavalier about keeping track of phase factors throughout this work, although it would be interesting to consider in the future whether they contain any non-trivial information.

4.4.1 SU(1,1)

We begin with the easiest quantity to compute which is the Krylov complexity. By its definition, we simply have to perform the sum

$$K = \sum_{n=0}^{\infty} n |\varphi_n|^2 = \sum_{n=0}^{\infty} n \frac{(\tanh r)^{2n}}{(\cosh r)^{2(n_0+1)}} \binom{n+n_0}{n_0} = (1+n_0) \sinh^2 r \ . \tag{4.52}$$

This clearly shows that for large r (late times) the Krylov complexity grows exponentially as expected. This is depicted for a few different choices of n_0 in figure 4.1.

We proceed by computing the logarithmic negativity associated to the two-mode coherent states (4.31). We showed that in general the former is given by

$$E_{\mathcal{N}}(\rho) = \log_2 \left(\sum_n |\varphi_n| \right)^2 , \tag{4.53}$$

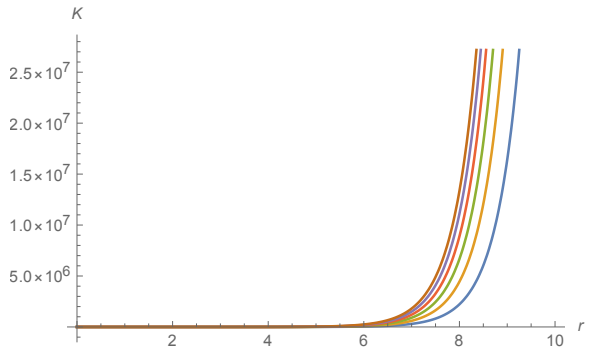

Figure 4.1: Krylov complexity for systems with SU(1,1) symmetry as a function of r. The curves depicted are for $n_0 = 0, 1, 2, 3, 4, 5$. We observe that for bigger n_0 the exponential rise of the complexity starts sooner.

and substituting the coefficients φ_n it can be written out explicitly as

$$E_{\mathcal{N}}(\rho) = \log_2 \left(\sum_{n=0}^{\infty} \frac{\tanh^n r}{\cosh^{n_0+1} r} \sqrt{\binom{n+n_0}{n_0}} \right)^2 . \tag{4.54}$$

For general n_0 an analytic solution does not appear to be possible and for that reason we will have to perform an approximation based on the behaviour of the φ_n as functions of r. More concretely, we will show that for sufficiently large r the functions have support in large enough n to justify keeping only the leading order contribution of the binomial coefficient. In figure 4.2 it is shown that as r increases the relevant n also increase. Therefore, one can expand the binomial coefficient for

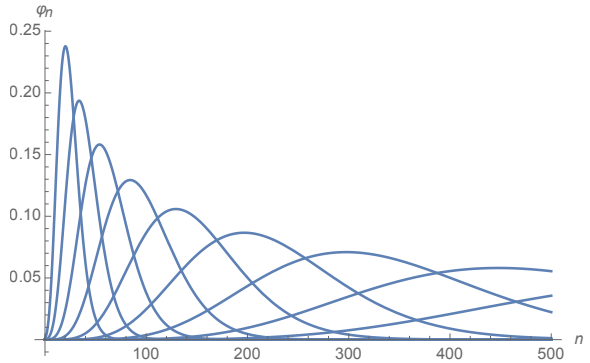

Figure 4.2: φ_n as function of n. The chosen parameters for this graph are $n_0 = 2$ and r takes values from 1 to 2.6 in steps of 0.2. This is essentially the spreading of the wavefunction that was observed in earlier works.

large n, which yields

$$\binom{n+n_0}{n_0} \simeq \frac{n^{n_0}}{n_0 \Gamma(n_0)} . \tag{4.55}$$

For $n_0 = 0, 1$ it is possible to obtain the analytic answers so let us compare them with the results of this approximation. For $n_0 = 0$ the approximation is exact so the comparison is trivial, but for $n_0 = 1$ we find out that the agreement for large r is remarkable (as shown in figure 4.3), thus justifying our initial arguments for using the large n approximation. The solution for the negativity

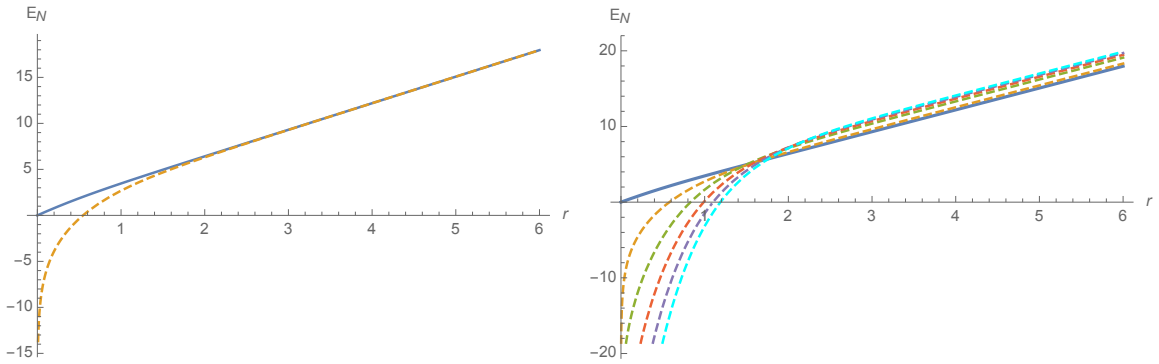

Figure 4.3: Left: Comparison of the logarithmic negativity as a function of r for $n_0 = 1$ between the exact solution (continuous curve) and the approximated solution (dashed curve). We maintain this way of representing exact results using continuous curves and approximate using dashed throughout the rest of this work. Right: Logarithmic negativity as a function of r for $n_0 = 1, 2, 7, 12, 17, 22$. Clearly, the features of the approximated functions cannot be trusted for early times as they become negative. However, for late times when our approximation becomes relevant, we observe a linear trend both for the exact and approximated solutions.

that we obtain for arbitrary n_0 using this approximation is

$$\log_2\left(\sum_{n=0}^{\infty}\frac{\tanh^n r}{\cosh^{n_0+1} r}\sqrt{\binom{n+n_0}{n_0}}\right)^2 \simeq \log_2\left(\frac{\mathrm{Li}_{-\frac{n_0}{2}}(\tanh r)}{\cosh(r)^{(1+n_0)}\sqrt{n_0\Gamma(n_0)}}\right)^2 . \tag{4.56}$$

Since we are interested in the regime of large n, by virtue of our approximation, we can employ an asymptotic expansion of the negativity to obtain a more palatable expression in the case that n_0 is even. The answer we obtain is

$$n_0 = 2p \Rightarrow E_{\mathcal{N}} = \log_2\left(\frac{4^{1-p}}{((\frac{3}{2})_{p-1})^2}e^{2r}\right) , \tag{4.57}$$

which makes more apparent that the negativity at late times follows a linear trend. In figure 4.3 one can see the logarithmic negativity resulting from (4.56) for different values of n_0. It is evident that these different cases exhibit a universal linear behaviour for large r, which is consistent with our expectations from the universal operator growth hypothesis.

Within the regime of the aforementioned approximation, one can compute the associated entanglement entropy and capacity of entanglement. We begin by computing the Rényi entropies as shown below

$$S^{(q)} = \frac{1}{1-q}\ln\sum_n |\varphi_n|^{2q} \simeq \frac{1}{1-q}\ln\left(\frac{\mathrm{Li}_{-qn_0}(\tanh^{2q} r)}{(\cosh r)^{2q(1+n_0)}(n_0\Gamma(n_0))^q}\right) . \tag{4.58}$$

It is then straightforward to obtain the entanglement entropy and capacity of entanglement, although the resulting expressions are quite lengthy and for that reason their presentation here is omitted. For the sake of completeness we have included the resulting formulae in appendix B. However, we can still generate graphical data from them which are of particular value and interest. These are shown in figure 4.4. An interesting observation is that the capacity of entanglement exhibits some non-universal features at early times. In particular, we find that as n_0 grows the plots develop local minima and maxima, for which a physical explanation does not appear to be straightforward. Even though the quantitative features of these plots are approximate, we have confidence in the qualitative ones even for early times. One can easily confirm this claim by employing an approach that is in

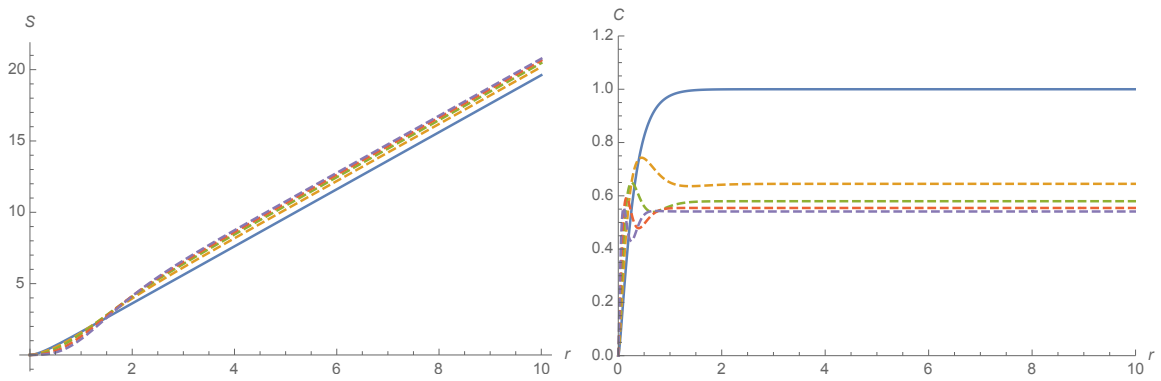

Figure 4.4: Left: entanglement entropy as function of the parameter r for $n_0 = 0, 1, 2, 3, 4$. Once again we observe a universal linear trend for late times, consistent with our expectations. Right: Capacity of entanglement as a function of the parameter r for $n_0 = 0, 1, 2, 3, 4$. The late time behavior is again universal but its quantitative precision is unclear. Also notice that for early times we observe the development of local minima and maxima, which we comment on further in the main text.

some sense the reverse of our approximation and namely by numerically performing the sum in 4.58 up to some finite value of n. By doing so we can capture the correct qualitative behavior for early times as justified by figure 4.2 and indeed verify the existence of these mysterious local extrema. The entanglement entropy also exhibits some non-universal features at early times albeit much more subtle and much less puzzling.

Another important comment regarding the capacity of entanglement has to do with the differences in the saturation value for different n_0. Even though we do not necessarily expect these values to be accurate, it is interesting to consider whether their differences are a feature of the various SU(1,1) representations rather than a byproduct of our approximation. It would also be interesting to consider whether this behavior can be captured in more physical terms, for example by the quasi-particle picture provided in [131].

4.4.2 SU(2)

For the case of SU(2) we will simply repeat the process we illustrated above using the appropriate coherent states which we presented in section 3. Note that, unlike the SU(1,1) case, the sums we will have to perform are always finite, because the discrete series representations of SU(2) have a finite number of Fock states. This makes things significantly more simple, as there will be no approximations required, but rather an exact result can be obtained for any SU(2) representation.

Once again we start by focusing on Krylov complexity, for which by definition we have

$$K = \sum_{N_2=0}^{2j} N_2 |F_{N_2}|^2 = 2j \sin^2 \chi \ . \tag{4.59}$$

The Krylov complexity is shown for different values of j in figure 4.5.

Proceeding to the computation of negativity, following our results for SU(1,1), we simply have to perform the sum

$$E_{\mathcal{N}} = 2 \log_2 \sum_{N_2=0}^{2j} |F_{N_2}| \ . \tag{4.60}$$

For general j the sum does not assume a closed form, but since it is always finite one can easily obtain an answer for any given value of j. Below we present the results for a few of them in figure

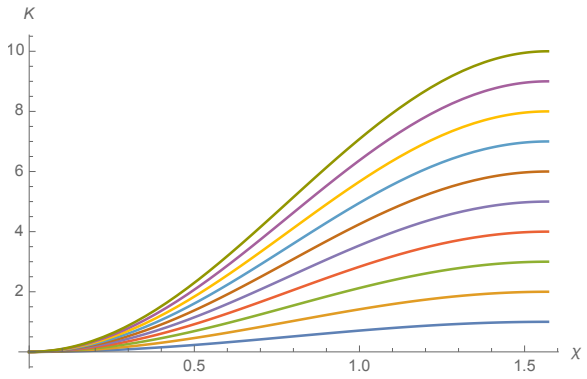

Figure 4.5: Krylov complexity as a function of χ for $\frac{1}{2} \leq j \leq 5$. We observe that in all case the Krylov Complexity grows to its maximum value at $\chi = \frac{\pi}{2}$ as expected.

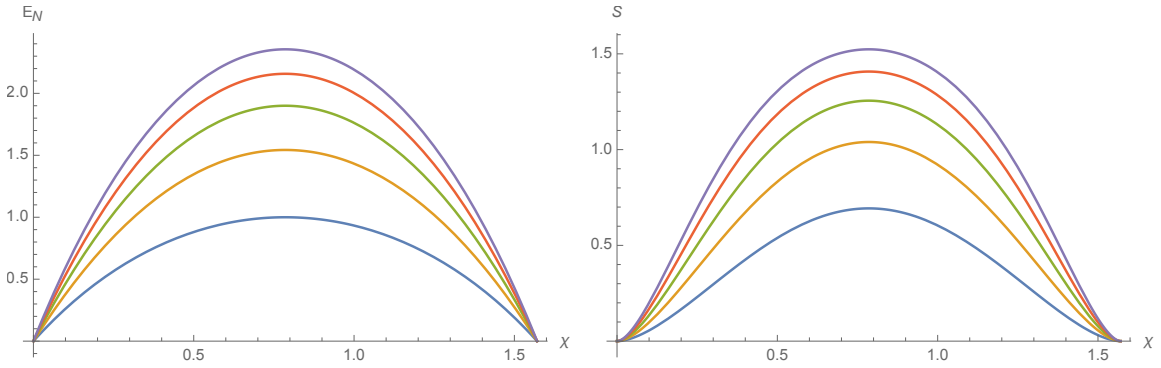

Figure 4.6: Left: Logarithmic negativity as a function of χ for $\frac{1}{2} \leq j \leq \frac{5}{2}$. Right: Entanglement entropy as a function of the parameter χ for $\frac{1}{2} \leq j \leq \frac{5}{2}$.

4.6. The entanglement entropy and capacity of entanglement can be straightforwardly obtained from the corresponding definitions and in figure 4.7 we present the resulting plots for different values of j.

Similarly to the case of SU(1,1) it is apparent that for all the quantum information tools under study there are certain universal features that characterize them. The important distinction with the SU(1,1) example is that we observe a finite instead of an infinite growth. This difference originates from the fact that for SU(2) there is a finite number of Fock states that we are using to describe this

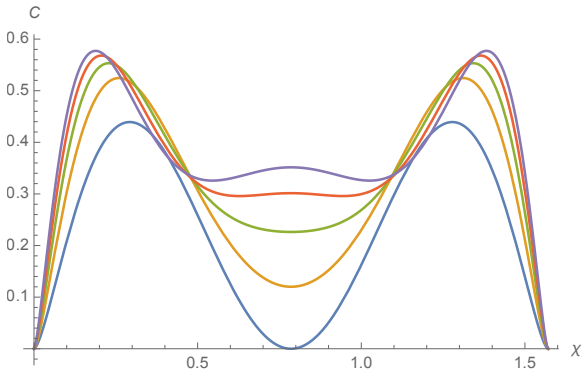

Figure 4.7: Capacity of entanglement as a function of the parameter χ for $\frac{1}{2} \leq j \leq \frac{5}{2}$

process. Having explained the identification between the coherent states and Krylov basis it follows that the operator only has a "limited space" in which it can grow. This is clearly reflected in the above figures, in support of this picture. An important comment on our results concerns the capacity of entanglement which, similarly to the SU(1,1) case, exhibits some sensitivity to representation specific information. Namely, it is evident that for the representations with $j = \frac{1}{2}, 1, \frac{3}{2}$ there is a minimum at $\chi = \frac{\pi}{4}$, whereas for representations with $j \geq 2$ this turns to a local maximum. The question of whether there is some physical picture that can explain this behavior is also relevant here and once again an answer could potentially be provided in terms of quasi-particle entanglement between EPR pairs [131].

4.5 Conclusions and discussion

We have considered the computation of several quantum information tools for systems with Lie symmetry. More specifically, we have shown that for systems with SU(1,1) and SU(2) symmetry we can easily obtain the associated Krylov complexity, negativity, entanglement entropy and capacity of entanglement for any discrete series representation by studying the properties of the corresponding two-mode coherent states. Furthermore, we illustrated that all of these quantities exhibit certain universal features in agreement with the universal operator growth hypothesis. For the SU(1,1) case we used an approximation which we argued is accurate for late times. Despite that, we were also able to obtain some qualitative features of the early time behavior of such systems which would be interesting to explain from a physical perspective. Our approach was similar in the case of SU(2) for which there was no need for an approximation and hence all results are exact. Once again we concluded that apart from the universal aspects of the growth that we can observe, there are certain features that differ between the various representations. Namely, the capacity of entanglement seems to be sensitive to this kind of information and in fact exhibits a rather interesting transition for $j = 2$. To clarify, these observations are still consistent with the operator growth hypothesis, as the operator of any system in a given representation of the groups we considered will behave in the same way. However, it appears that there are certain traits in the quantities we studied that would allow one to discern which particular representation the system is in, which in turn implies that this aspect of operator growth is not universal.

There are several directions that are worthwhile to investigate in the future. In particular, here we have considered only two cases of Lie groups, so it would be interesting to determine whether this approach works more generally. However, not all Lie groups admit a two-mode coherent state representation, as is the case for the Heisenberg-Weyl group for example. One could still assert that the quantities we defined using the two-mode states would be valid in such cases, however, a more detailed analysis is required to prove this claim rigorously. Making progress in that direction would be an important step forward, as there are already works that have considered Krylov complexity in other setups using the Lanczos algorithm. For example in [53] the authors consider different models of CFTs, such as 2d CFTs, free field and holographic models. We know the symmetry groups of these theories, so in principle it is possible to use our approach in order to compare the results for Krylov complexity and possibly complement the picture by computing the other quantum information tools that we have discussed. Recently there have also been works on Nielsen complexity that are similar in spirit in their use of coherent states or the symmetry of the system in general [34,105,114,115,160]. It would be interesting to further explore the connection between these two approaches to complexity given their similarities and determine the point at which they diverge and what this can teach us about the field as a whole.

As we have stressed in previous sections, here we have only considered the discrete series representations of the groups under study. So, another question that naturally arises from our tools from symmetry prescription is whether it can be extended to include other representations as well. This would require constructing the coherent states for these representations and using them to define

the quantum information quantities we are interested in. It is already well known how to construct the states (as reviewed in [99] for example), although the process is more involved compared to the discrete series case, consequently leading to several subtleties.

In light of some recent advances in the field of dS/CFT [161, 162] we would like to point to the potential relevance of our results for SU(2) in this direction. In particular, the authors of [163] advocate that the CFT dual of dS_3 is given by an SU(2) Wess-Zumino-Witten model in the large central charge limit. As such, it is very intriguing to consider whether our approach provides a natural candidate for exploring holographic complexity in dS/CFT.

Additionally, as shown in [154], Krylov complexity can be interpreted as a volume on the space of coherent states. Therefore, it would be interesting to consider whether the other quantum information tools that we have discussed also admit an interpretation in terms of the coherent state geometry. Given that we are using the coherent state properties to compute these quantities, it appears intuitively plausible that they should indeed possess some geometric interpretation. However, at the time of writing it is not clear how these identifications can be performed in a rigorous manner.

Finally, some questions of exceptional interest concern the moments, cumulants and generating functions of n for the coherent states as one-parameter extensions of Krylov complexity. These tools could potentially reveal some interesting behavior in the Krylov chain picture. Progress in this direction has already been made for example in [155] with the use of what the authors define as K-variance (which is simply the variance of the probability distribution related to the coherent states of the system) and in [44] where the authors argue that they can diagnose differences in the spectral statistics of the system through the spreading patterns of the wavefunction on the Krylov chain. These results are still in early stages and as such it is important to delve further into the subject by exploring different quantities that can probe these processes and more importantly that can provide information about the spectral statistics.

Acknowledgements

The author wishes to thank Aditya Bawane, Jan Boruch, Dongsheng Ge and Javier Magan for helpful discussions and comments. Special thanks to Paweł Caputa for his continuous guidance through all the stages of this work. DP is supported by NAWA “Polish Returns 2019".

CFT construction of SU(1,1) cohererent states

This appendix is devoted to an alternative construction of the coherent states presented in section 2 using CFT techniques. This is feasible as the global symmetry group of 2d CFTs is SU(1,1) and hence the symmetry arguments that were presented previously are implicitly encoded in the CFT formalism that we use below. For a similar construction of the coherent states associated with the full Virasoro symmetry of 2d CFTs see [155]. Our starting point is a state of highest weight h, $|h\rangle = \mathcal{O}_{-h}|0\rangle$, where $\mathcal{O}_{-h}$ is a mode of a chiral primary operator with dimension h. This means that there is an expansion of the form

$$\mathcal{O}(z)|0\rangle = \sum_{n=0}^{\infty} z^n \mathcal{O}_{-h-n}|0\rangle \ , \tag{4.61}$$

which we will make use of. Furthermore, we know that the modes have the following properties

$$[L_m, \mathcal{O}_{-h}] = (h(m-1)-m)\mathcal{O}_{-h+m}, \quad m = \{-1, 0, 1\} \ , \tag{4.62}$$

$$[\mathcal{O}_m^{(i)}, \mathcal{O}_n^{(j)}] = \delta_{m+n,0} d^{ij} \binom{m+h-1}{2h-1} + \sum_k C_k^{ij} p_k^{ij}(m,n) \mathcal{O}_{m+n}^k \ , \tag{4.63}$$

where we have adopted the usual notation for the SU(1,1) generators in the CFT language in terms of L_m and d^{ij} are structure constants that can be set to δ^{ij}.

We are interested in the action of the displacement operator on these states, which as we argued previously is given by

$$D(\xi) = e^{\xi L_{-1} - \bar{\xi} L_1} \ . \tag{4.64}$$

Using the BCH formula, the action of the displacement operator on the states can be written as

$$e^{zL_{-1}} e^{aL_0} e^{\bar{z}L_1} |h\rangle = e^{zL_{-1}} e^{aL_0} e^{\bar{z}L_1} \mathcal{O}_{-h} |0\rangle \ , \tag{4.65}$$

where a, z are numbers to be determined. It is easy to verify that this can be reduced to

$$e^{ah} e^{zL_{-1}} \mathcal{O}_{-h} |0\rangle \ , \tag{4.66}$$

as commuting L_1 acting on $|h\rangle$ yields zero and $L_0 |h\rangle = h |h\rangle$. Thus, by inserting the identity we obtain

$$e^{ah} e^{zL_{-1}} \mathcal{O}_{-h} e^{-zL_{-1}} e^{zL_{-1}} |0\rangle = \tag{4.67}$$

$$e^{ah} \sum_{n=0}^{\infty} z^n \mathcal{O}_{-h-n} |0\rangle = e^{ah} \mathcal{O}(z) |0\rangle = |z, h\rangle \ . \tag{4.68}$$

The factor e^{ah} can be simply regarded as the normalization of the state and can be obtained by imposing the condition

$$\langle 0| \mathcal{O}^\dagger(z) e^{2ah} \mathcal{O}(z) |0\rangle = 1 \Rightarrow e^{2ah} = \frac{1}{\langle \mathcal{O}^\dagger(z) \mathcal{O}(z) \rangle} \ . \tag{4.69}$$

In order to compute the correlator we will make use of the relation

$$\mathcal{O}^\dagger(z) = \bar{z}^{-2h} \mathcal{O}(1/\bar{z}) \tag{4.70}$$

and the standard result for two-point functions in 2d CFT

$$\langle \mathcal{O}(z) \mathcal{O}(w) \rangle = \frac{1}{(z-w)^{2h}} \ , \tag{4.71}$$

which then imply

$$\bar{z}^{-2h} \langle \mathcal{O}(1/\bar{z}) \mathcal{O}(z) \rangle = \bar{z}^{-2h} (\frac{1}{\bar{z}} - z)^{-2h} = (1 - |z|^2)^{-2h} \ . \tag{4.72}$$

Substituting this result in (4.69) we obtain

$$e^{ah} = \frac{1}{\sqrt{\langle \mathcal{O}^\dagger(z) \mathcal{O}(z) \rangle}} = (1 - |z|^2)^h \ . \tag{4.73}$$

Let us now consider the coefficients given by the projection of the states $|z, h\rangle$ on the modes of $\mathcal{O}(z)$

$$\psi_n = \langle 0| \mathcal{O}^\dagger_{-h-n} |z, h\rangle = \frac{\langle \mathcal{O}_{h+n} \mathcal{O}(z) \rangle}{\sqrt{\langle \mathcal{O}^\dagger(z) \mathcal{O}(z) \rangle}} \ . \tag{4.74}$$

The numerator on the RHS is

$$\langle \mathcal{O}_{h+n} \mathcal{O}(z) \rangle = \langle [\mathcal{O}_{h+n}, \mathcal{O}(z)] \rangle + \langle \mathcal{O}(z) \mathcal{O}_{n+h} \rangle \tag{4.75}$$

$$= \langle [\mathcal{O}_{h+n}, \sum_m z^m \mathcal{O}_{-h-m}] \rangle = z^n \binom{2h+n-1}{2h-1} \ , \tag{4.76}$$

and by the appropriate substitutions we can rewrite (4.74) as

$$\psi_n = (1-|z|^2)^h z^n \binom{2h+n-1}{2h-1} \,. \tag{4.77}$$

Using the identifications $z = e^{i\phi}\tanh r$ and $2h-1 = n_0$ these precisely match the coefficients we computed in section 2, with $\psi_n = e^{in\phi}\varphi_n$.

In summary, one can construct the same set of coherent states that we derived in section 2 by thinking exclusively in terms of 2d CFT quantities. In particular, the coherent states themselves can be obtained by the action of a chiral primary operator on the vacuum state and the coefficients of their decomposition in terms of modes are given by correlators whose form is fixed by the conformal symmetry.

4.6 Appendix: Analytic evaluation of entanglement entropy and capacity of entanglement

As mentioned in the main text, the evaluation of the entanglement entropy and capacity of entanglement using the late time approximation for the SU(1,1) case is straightforward. Using the definitions referenced in the main text and (4.58) we obtain the following.

$$\begin{aligned} S = \frac{\cosh(r)^{-2(1+n_0)}}{\Gamma(1+n_0)} \big[&\mathrm{Li}_{-n_0}(\tanh^2 r)(\log(n_0) + 2(1+n_0)\log(\cosh r) + \log(\Gamma(n_0))) \\ &- 2\mathrm{Li}_{-1-n_0}(\tanh^2 r)\log(\tanh r) - n_0 \frac{\partial}{\partial n_0}\left(\mathrm{Li}_{-n_0}(\tanh^2 r)\right) \big] \end{aligned} \tag{4.78}$$

and for the capacity of entanglement

$$\begin{aligned} C = \frac{1}{(\mathrm{Li}_{-n_0}(\tanh^2 r))^2} \big[&-4(\mathrm{Li}_{-1-n_0}(\tanh^2 r))^2(\log(\tanh r))^2 + 4\mathrm{Li}_{-2-n_0}(\tanh^2 r)\mathrm{Li}_{-n_0}(\tanh^2 r)(\log(\tanh r))^2 \\ &- 4n_0\mathrm{Li}_{-1-n_0}(\tanh^2 r)\log(\tanh r)\frac{\partial}{\partial n_0}\left(\mathrm{Li}_{-n_0}(\tanh^2 r)\right) + n_0[-n_0\left(\frac{\partial}{\partial n_0}\left(\mathrm{Li}_{-n_0}(\tanh^2 r)\right)\right)^2 \\ &+ \mathrm{Li}_{-n_0}(\tanh^2 r)\left(4\log(\tanh r)\frac{\partial}{\partial n_0}\left(\mathrm{Li}_{-1-n_0}(\tanh^2 r)\right) + n_0\frac{\partial^2}{\partial n_0^2}\left(\mathrm{Li}_{-n_0}(\tanh^2 r)\right)\right)]\big] \,. \end{aligned} \tag{4.79}$$

We stress again that even though these expressions are convoluted we can easily generate graphical data from them that contain very useful information about the evolution of the system.

Chapter 5

Krylov complexity in a natural basis for the Schrödinger algebra

Extended summary

This work is an attempt to generalize the methodology of extracting a measure of complexity from the generalized coherent states for Lie groups that are not semi-simple and their respective algebras have rank higher than 1. To achieve this, we study the Schrödinger group which in 1+1 dimensions reduces to a semi-direct product of the Heisenberg-Weyl and SL(2,R) groups. In simple terms, the Schrödinger algebra [1] is made up from the usual raising and lowering operators of the harmonic oscillator and their bilinears.

This structure is relevant for a number of reasons. Firstly, it is the only algebra (up to potential isomorphisms) that contains higher powers of the raising and lowering operators and is still closed. If one includes powers of 3 or higher they will find that the algebra does not close unless it is extended to include an infinite number of generators. This makes it a useful tool to study problems where an expansion in terms of the ladder operators is required. One may simply truncate the expansion at second order and still be able to use the powerful structure provided by symmetry. This has been made use of in molecular dynamics and quantum optics. Furthermore, this symmetry describes non-relativistic systems which have recently drawn attention due to their relevance in flat space holography, but can be more generally thought of as the low energy limit of more general theories. In that regard, it is important to make sure that results in this limit remain consistent with our theoretical expectations.

Since the Schrödinger algebra does not have the same rank as the complexity algebra it is not clear whether there exists a mapping between the two. Moreover, even though the Liouvillian is in principle tridiagonalizable, there is no apparent way of bringing it in this form due to the fact that it is an infinite matrix. These two conditions led us to develop an approach that relies on the notion of a "natural" measure of complexity, that is a measure which can be immediately obtained from the symmetry structure. More concretely, we utilized the generalized coherent states of the Schrödinger group and their expansion in the Fock basis, arguing that we obtain a similar picture with the Krylov chain that contains next-to-nearest neighbor hopping. We showed that in certain limits this picture reduces to the usual results for Krylov complexity that were previously derived for the Heisenberg-Weyl and SL(2,R) groups up to a multiplicative constant.

More concretely, our approach makes use of the "natural" orthonormal basis provided by the unitary representation of the symmetry group. By defining the time evolution operator as a symmetry

[1] In some references it is also known as the two-photon algebra

element of the group of the form

$$e^{i\mathcal{L}t} = e^{i[\alpha(a^\dagger + a) + \frac{\beta}{2}((a^\dagger)^2 + a^2)]t} \ , \tag{5.1}$$

we can make use of the coherent state structure of the system. This allows the decomposition of the time evolved state as

$$e^{i\mathcal{L}t} |0\rangle = \sum_{k=0}^{\infty} \phi_k |k\rangle \ , \tag{5.2}$$

with

$$\phi_k = \frac{1}{\sqrt{k!}} \left(\frac{1}{2} \frac{\overline{w}}{|w|} \tanh |w| \right)^{\frac{k}{2}} H_k(s) \phi_0 \ , \tag{5.3}$$

where the variables in that expression are appropriate functions of α, β, t. This ultimately allows the computation of complexity

$$K_{\mathcal{O}} = \alpha^2 t^2 + \sinh^2 \beta t + \alpha^2 \left[\frac{4 \cosh \beta t \sinh^2 \frac{\beta t}{2}}{\beta^2} - t^2 \right] \ . \tag{5.4}$$

The three terms correspond to the portions of the dynamics due to HW, SL(2,R) and a mix of the two respectively. From this expression one can easily verify that for $\alpha = 0$ or $\beta = 0$, the complexity reduces to the correct form for either of these two groups (up to a multiplicative constant). Finally, we discuss the behavior of the autocorrelation function and the scrambling time which grant further insight on the interplay between the different degrees of freedom of a system characterized by Schrödinger symmetry.

Krylov complexity in a natural basis for the Schrödinger algebra

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Abstract

We investigate operator growth in quantum systems with two-dimensional Schrödinger group symmetry by studying the Krylov complexity. While feasible for semisimple Lie algebras, cases such as the Schrödinger algebra which is characterized by a semi-direct sum structure are complicated. We propose to compute Krylov complexity for this algebra in a *natural* orthonormal basis, which produces a *penta*diagonal structure of the time evolution operator, contrasting the usual tridiagonal Lanczos algorithm outcome. The resulting complexity behaves as expected. We advocate that this approach can provide insights to other non-semisimple algebras.

5.1 Introduction

Krylov complexity [42] is a measure introduced in order to quantify how quantum operators change under time evolution. In complex quantum systems one generically expects that, due to the non-trivial dynamics, a typical operator undergoing time evolution will tend to "spread" such that it affects a larger number of the degrees of freedom of the system thereby increasing in complexity. In [154] it was shown that symmetry plays an important role in determining the features of this growth and following similar methods in this work we will be proposing a more general framework that includes symmetries described by non-semisimple Lie groups.

Krylov complexity is conjectured to grow at a maximal rate for chaotic systems, although there are subtleties that need to be taken into account, especially for quantum field theories due to their infinite degrees of freedom in contrast to ordinary quantum mechanical systems, see discussions in [49, 53–57, 164, 165]. When one considers quantum many body systems such as spin chains it is more straightforward to produce evidence that Krylov complexity can for example distinguish between integrable and chaotic dynamics as was argued in [44, 166, 167].

However, despite its apparent relevance in the discourse regarding quantum chaos, one might ask why is it worth studying Krylov complexity over all the different available measures of which there has been a profound proliferation in recent years? We believe that there are two main reasons. First, Krylov complexity can be applied to any quantum system making it computationally available, at least in principle, for a plethora of different cases including but not limited to condensed matter and many-body systems [80, 167–170], quantum and conformal field theories [53–55, 155, 171, 172], open systems [173–177], topological phases of matter [178, 179] and many other topics related to aspects of the above and not only [180–183]. Second, it is related by its construction to inherent properties and characteristic parameters of the system, namely the Hamiltonian and the Hilbert space that it defines. The reason why this is important is because it removes the arbitrariness which is to a large extent present in other definitions of complexity. Namely, one usually needs to define a set of elementary operations that can be performed on the system and assign a cost to them according to

some justifiable albeit arbitrary criteria. For a review on this topic see for example [27, 88]. In the context of Krylov complexity these choices are reduced to the choice of an inner product using which the Lanczos algorithm can map the dynamics of a quantum mechanical system onto a semi-infinite one-dimensional chain with nearest neighbor hopping. Krylov complexity is then simply defined as the average position on the chain as a function of time. In the current note we review some important examples of semisimple algebras for which one can analytically compute Krylov complexity: the case of the Heisenberg-Weyl algebra, corresponding to states expressible in the one-dimensional harmonic oscillator basis, and $SL_2(\mathbb{R})$, which for example represents the two-dimensional conformal algebra.

Subsequently, we will go beyond studying the Krylov complexity of semisimple algebras, which as we will discuss poses certain analytic challenges. We focus on the specific example of the two-dimensional Schrödinger algebra. This algebra is the maximal symmetry algebra for a two-dimensional Schrödinger equation with a quadratic potential or no potential at all and appears as a subalgebra of the three-dimensional conformal algebra $SO(3,2)$, see e.g. [184]. As the two-dimensional Schrödinger equation is relevant for, e.g., cold atom traps and optical systems [100], having a prediction for Krylov complexity can yield insights in both directions. Moreover, having analytic results for systems with such a symmetry structure can potentially set the groundwork to study more complex systems that are relevant in high energy physics.

However, the usual Krylov approach for the two-dimensional Schrödinger group is problematic from an analytic point of view and only numerical approximations are available [185], making it difficult to work with. In this note we propose an alternative method to probe the Krylov subspace that relies on making use of the semi-direct sum structure of the symmetry algebra and methods used in the study of coherent states. We find that the semi-direct sum structure presents us with an orthogonal basis that translates to a picture of hopping on a semi-infinite chain where *next-to-nearest* neighbor interactions are allowed, contrasting conventional Krylov complexity approaches. We advocate that generically in the case where symmetry algebras are found to allow for a semi-direct sum structure, it can be more fruitful to use a "natural" orthogonal basis that as a result induces more interactions from the chain point of view, but can allow for direct evaluation.

While this approach in itself is novel, there have been several works already where an orthonormal basis for the Krylov subspace is not obtained through the Lanczos algorithm and as such the Liouvillian (Hamiltonian) is not tridiagonal. These include [155, 186], where (similarly to this work) the symmetry algebra is considered for the full Virasoro group and $SL_n(\mathbb{R})$ respectively. For the former this leads to a block diagonal structure of the Liouvillian and for the latter to a "trivially" non-tridigonal structure along the same lines with $SL_2(\mathbb{R})$ which we examine in more detail in the following sections. Importantly, in both cases, this leads to analytically tractable results, although there does not appear to be a complete understanding of the precise difference between the implementation of this approach as compared to the Lanczos algorithm. Another instance where an explicitly non-tridiagonal form of the Liouvillian is required in order to make progress arises in open quantum systems (see e.g. [175]). The reason is that the Liouvillian for these systems is not a Hermitian operator anymore and as such the Lanczos algorithm is insufficient, so a more general orthogonalization scheme has to be implemented instead. In particular one can use the Arnoldi iteration which puts the Liouvillian in Hessenberg form (triangular matrix with non-zero entries in the first subdiagonal). The common element in all of these considerations is that while it is possible to obtain an orthonormal basis for the Krylov subspace in different ways, the differences between these bases and the associated complexities are unclear. We comment further on this point from the perspective of our results in the discussion section.

The remainder of this note is organized in the following way. In Section 5.2 we review Krylov complexity and establish notation. In Section 5.3 we introduce the Schrödinger algebra and present our results. Finally, in section 5.4 we present a discussion on our obtained results.

5.2 Krylov complexity

In this section we introduce Krylov complexity and establish notation through a series of examples.

5.2.1 Krylov basis

There are many works that include a pedagogical introduction to the notion of the Krylov basis, to which we refer the reader for a detailed derivation [42, 46, 153, 154, 156]. The central concept is the expansion of a time evolved operator $\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt} = e^{i\mathcal{L}t}\mathcal{O}$ in the Krylov subspace $\mathcal{H}_\mathcal{O}$ defined as the linear span of the action of the Liouvillian on the initial operator

$$\mathcal{H}_\mathcal{O} = \text{span}\{\mathcal{L}^n\mathcal{O}\}_{n=0}^{+\infty} \, . \tag{5.5}$$

In other words one seeks to decompose an operator at some arbitrary time as

$$|\mathcal{O}(t)) = \sum_n \phi_n |K_n) \, . \tag{5.6}$$

Here we adopt the curly bracket notation to denote a state in the operator Hilbert space. Therefore, $|K_n)$ denotes an orthonormal basis for the Krylov subspace and ϕ_n the appropriate coefficients satisfying $\sum_n |\phi_n|^2 = 1$. The most common way of obtaining such a basis is through the Lanczos algorithm which applies the Gram-Schmidt orthogonalization scheme to the elements $|\mathcal{O}_n) = \mathcal{L}^n|\mathcal{O})$ [45]. This has the effect of tridiagonalizing the Liouvillian such that

$$\mathcal{L}|\mathcal{O}_n) = b_{n+1}|\mathcal{O}_{n+1}) + b_n|\mathcal{O}_{n-1}). \tag{5.7}$$

This tridiagonal structure manifests explicitly when the elements of the Liouvillian are represented in matrix form.

$$\mathcal{L} = \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 \\ b_1 & 0 & b_2 & \ddots & \vdots \\ 0 & b_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_n \\ 0 & \cdots & 0 & b_n & 0 \end{pmatrix} \, . \tag{5.8}$$

Note that the diagonal elements are zero due to the properties of the operator inner product, which we assume (as is standard) to involve a trace and given that we are interested in the growth of Hermitian operators. Having obtained the basis one can then study different aspects of the probability distribution provided by ϕ_n to probe the growth of the operator in the Krylov subspace and assess its resulting complexity. In particular the average of that distribution $\sum_n n|\phi_n|^2$ is dubbed Krylov complexity and quantifies, albeit in a somewhat crude manner, the growth of the operator. A natural question that arises is why should one consider the average as a measure of complexity rather than any other moment of the distribution. There are some formal arguments to be made in favour of that choice as is done for example in [50], but here we will take a more heuristic approach that will prove useful in our subsequent considerations. The so-called "Krylov chain" picture was pointed out in the seminal work of [42] and it has since been explored by several others [80, 167, 187]. This picture arises from the realization that the dynamics of the Krylov subspace can be mapped to that of a particle hopping on a semi-infinite, one-dimensional chain. This identification is made by the discrete Schrödinger equation that relates the Lanczos coefficients b_n with the coefficients ϕ_n as

$$-i\frac{d\phi_n}{dt} = b_{n+1}\phi_{n+1} + b_n\phi_{n-1} \, . \tag{5.9}$$

In this picture the b_n play the role of the hopping coefficients between two adjacent sites and ϕ_n the amplitude of finding the particle at each site. Krylov complexity can be interpreted as the

average position on the chain at a particular time. This is indeed a very useful piece of information, especially in cases where access to other aspects of the probability distribution defined by the ϕ_n is limited. However, there is much more refined information that one can extract from said probability distribution if it can be obtained analytically. For instance one can study the Krylov variance [155], entropy [46, 188, 189], logarithmic negativity and capacity of entanglement [187].

Our main motivation is to study the Krylov subspace as it arises from the symmetry of the quantum system of interest in the spirit of [154]. Assuming that this symmetry is described by a Lie group and that the system is closed, then the Liouvillian can be written as a linear combination of the algebra generators. For three-dimensional Lie algebras that admit a representation in terms of some generalized ladder operators one can write the Liouvillian in the following form

$$\mathcal{L} = \xi L_+ - \bar{\xi} L_- \ . \tag{5.10}$$

The action of such a Liouvillian on a general Fock state $|n\rangle$ is then by construction identical to how the Liouvillian acts on the Krylov basis and by an appropriate choice of ξ one can identify the two in a precise mathematical manner. This enables us to identify the operator $e^{i\mathcal{L}t}$ as a group element, thus leading to an interpretation of the time evolved operator

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}(0)) = \sum_n \phi_n \, |n\rangle \ , \tag{5.11}$$

as a coherent state. The latter have been studied extensively [99] and are very well known for a number of different symmetry groups. This is particularly useful given that one can readily obtain the coefficients ϕ_n from which the Krylov complexity can be extracted, but it additionally endows these results with a certain degree of universality. More specifically, this construction implies that any system with the same type of symmetry will exhibit the same behaviour in terms of its Krylov complexity. As mentioned in the introduction our aim here is to study a group with more general structure for which the Liouvillian is not automatically tridiagonal. Our vehicle in that endeavour is the Schrödinger group which in 1+1 dimensions is the semi-direct product of the Heisenberg-Weyl and $SL_2(\mathbb{R})$ groups. For that reason, we review the methods and results for these two groups in the following subsections.

5.2.2 Heisenberg-Weyl algebra

The Heisenberg-Weyl algebra (HW) is defined by (we borrow the notation employed in [154])

$$[a, a^\dagger] = 1 \, , \quad [\hat{n}, a^\dagger] = a^\dagger \, , \quad [\hat{n}, a] = -a \, , \tag{5.12}$$

with all other commutators vanishing, $\hat{n} = a^\dagger a$ and the usual creation and annihilation operators $a^\dagger$ and a. Defining a vacuum state $|0\rangle$ via $a|0\rangle = 0$ we can consider the following orthonormal basis for the corresponding Hilbert space:

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \, , \tag{5.13}$$

such that

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \, , \quad a|n\rangle = \sqrt{n}|n-1\rangle \, . \tag{5.14}$$

To compute the associated Krylov complexity it was shown [154] that we can simply make the following identifications

$$\mathcal{L} = \alpha(a^\dagger + a) \, , \quad |\mathcal{O}_n) = |n\rangle \, . \tag{5.15}$$

This allows us the write the Heisenberg operator state in Krylov space as

$$|\mathcal{O}(t)) = e^{i\alpha t(a^\dagger + a)}|0\rangle = \sum_{n=0}^{\infty}(it)^n \phi_n(t)|n\rangle \, , \quad \phi_n = e^{-\alpha^2 t^2}\frac{\alpha^n t^n}{\sqrt{n!}} \, , \quad \sum_{n=0}^{\infty}|\phi_n|^2 = 1 \, , \tag{5.16}$$

where at the second equality sign we simply wrote the exponent in its series representation and worked out that $(a^\dagger)^n|0\rangle = \frac{1}{\sqrt{n!}}|n\rangle$. This allows us to compute Krylov complexity $K_\mathcal{O}$:

$$K_\mathcal{O} = \sum_{n=0}^{\infty} n|\phi_n(t)|^2 = \alpha^2 t^2 \,. \tag{5.17}$$

While the HW group appears elementary it actually arises in cases much more sophisticated than that of the harmonic oscillator with which it is usually associated. For example in [190] the authors discuss how the symmetry of SYK in the triple scaling limit is described by HW and how that leads to the universal Krylov complexity result presented above.

5.2.3 $SL_2(\mathbb{R})$ algebra

We will now consider the $SL_2(\mathbb{R})$ (or the isomorphic $SU(1,1)$) algebra and compute complexity in two different bases: the Heisenberg-Weyl basis and the 'natural' $SL_2(\mathbb{R})$ basis. The algebra is defined as

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1} \,, \quad [L_1, L_{-1}] = 2L_0 \,, \tag{5.18}$$

where we define a vacuum $|h\rangle$ via $L_1|h\rangle = 0$ and $L_0|h\rangle = h|h\rangle$ where positive integer h labels the different states that are allowed. An orthonormal basis is given by

$$|h, n\rangle = \sqrt{\frac{\Gamma(2h)}{n!\Gamma(2h+n)}} L_{-1}^n |h\rangle \,, \tag{5.19}$$

such that

$$L_0|h,n\rangle = (h+n)|h,n\rangle \,, \quad L_\pm|h,n\rangle = \sqrt{\left(n + \frac{1 \mp 1}{2}\right)\left(2h + n - \frac{1 \pm 1}{2}\right)}|h, n \pm 1\rangle \,. \tag{5.20}$$

To compute the associated Krylov complexity $K_\mathcal{O}$ we can make the following identifications [154]

$$\mathcal{L} = \beta(L_{-1} + L_1) \,, \quad |\mathcal{O}_n) = |h, n\rangle \,. \tag{5.21}$$

It then follows that

$$\phi_n(t) = \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}} \frac{\tanh^n(\beta t)}{\cosh^{2h}(\beta t)} \,, \quad K_\mathcal{O} = 2h \sinh^2(\beta t) \,. \tag{5.22}$$

What happens when we express the above in the HW basis instead? After some algebra we establish the mapping

$$L_0 = \frac{1}{4}(a^\dagger a + a a^\dagger) \,, \quad L_{+1} = \frac{1}{2}a^2 \,, \quad L_{-1} = \frac{1}{2}(a^\dagger)^2 \,, \tag{5.23}$$

with value of $h = 1/4$, which we treat as a continuation from the usual integer values. As a result the exponentiated Liouvillian takes the form of a so-called squeeze operator which one encounters quite frequently in quantum optics [128]. In that case it is known that [99]

$$|\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \frac{\sqrt{(2n)!}}{2^m m!} \frac{\tanh^n(\beta t)}{\sqrt{\cosh(\beta t)}} |2n\rangle \,, \tag{5.24}$$

which leads to the same complexity as previously computed, but restricted to $h = 1/4$.

5.3 Two-dimensional Schrödinger algebra

5.3.1 Schrödinger symmetries

The Schrödinger group is the maximal symmetry group corresponding to the free Schrödinger equation [191] and is isomorphic to the maximal symmetry group of the Schrödinger equation with a harmonic potential [192]. In general, from a group perspective the Schrödinger group can be viewed as the extension of the Galilean group. The generators of the Galilean group consist of time (T) and spatial translations (P_i), Galilean boost (G_i) and spatial rotations. The Galilei algebra admits a central extension M of the commutator of the boost generator G_i and the spatial translation generator P_i. Adding this extension yields the Bargmann algebra. In order to reach the Schrödinger algebra we take into account a dilatation operator D under which time scales twice as fast as space and a special conformal symmetry generator K. We tailored the names of these generators to the case of the free Schrödinger equation. In two dimensions the only non-vanishing commutators of this algebra are given by (where we dropped the index i as it runs over the spatial indices)

$$[P, G] = -iM\,, \tag{5.25}$$

$$[D, T] = -2T\,, \quad [D, K] = 2K\,, \quad [T, K] = D\,, \tag{5.26}$$

$$[D, P] = -P\,, \quad [D, G] = G\,. \tag{5.27}$$

In (5.25) we recognize a Heisenberg-Weyl sub-algebra and in (5.26) we recognize an $SL_2(\mathbb{R})$ sub-algebra. The remaining brackets in (5.27) are between elements of either sub-algebra. We conclude that the two-dimensional Schrödinger algebra is a semi-direct sum $sl_2(\mathbb{R}) \ltimes hw$.

Using the insight from Section 5.2 in which we showed that $sl_2(\mathbb{R})$ can be written in terms of hw generators we cast all the algebra elements of the two-dimensional Schrödinger in terms of ladder operators:

$$P = -\frac{i}{\sqrt{2}}(a - a^\dagger)\,, \quad G = \frac{1}{\sqrt{2}}(a^\dagger + a)\,, \quad M = aa^\dagger - a^\dagger a\,, \tag{5.28}$$

$$T = -\frac{1}{4}(a - a^\dagger)^2\,, \quad K = -\frac{1}{4}(a + a^\dagger)^2\,, \quad D = \frac{1}{2}(a^2 - (a^\dagger)^2)\,. \tag{5.29}$$

Therefore, the Liouvillian expressed as a general element of the Lie algebra can be cast in the form

$$\mathcal{L} = \alpha(a^\dagger + a) + \frac{\beta}{2}((a^\dagger)^2 + a^2)\,, \quad \alpha, \beta \in \mathbb{R}\,, \tag{5.30}$$

and accordingly the element of the Schrödinger group that results from its exponentiation can be parametrized as

$$e^{i\mathcal{L}t} = S(v, w) = e^{(va - \overline{v}a^\dagger)} e^{(\frac{w}{2}a^2 - \frac{\overline{w}}{2}(a^\dagger)^2)}\,, \tag{5.31}$$

where we have not taken into account bilinears of a and $a^\dagger$ that act diagonally on the Fock states in either of the two expressions involving the Liouvillian, since they just produce a phase that can be included in the normalization. We provide an explicit mapping between the parameters α, β and v, w in subsection 5.3.4.

The Schrödinger group, apart from describing the symmetries of a non-relativistic CFT [193] and thus being relevant for a non-relativistic limit of AdS/CFT [6, 184], finds applications in systems with experimental realizations. In particular, it is used to describe specific processes in molecular dynamics and two-photon processes [97, 100]. Namely, within the study of coherent states, squeezing refers to saturating the uncertainty relation and in quantum optics, to reach such a state, one needs to take into account pairs of photons on top of single photon states. Two-photon processes and the collision of molecules can be modelled with a Hamiltonian expanded in a power series and truncated at quadratic order thus producing an effective Hamiltonian of the form

$$H = \hbar\omega(aa^\dagger + \frac{1}{2}) + f_2(t)(a^\dagger)^2 + f_2^*(t)a^2 + f_1(t)a^\dagger + f_1^*(t)a\,. \tag{5.32}$$

In the Liouvillian presented above the coefficients α, β are time independent, but generally they can be promoted to time-dependent as in this Hamiltonian to capture squeezing that is not necessarily linear in time.

5.3.2 Computation of coherent states

Acting with this general element on a quantum state of a system with Schrödinger symmetry provides a family of generalized coherent states in line with Perelomov [99]. Let us discuss the construction of those coherent states following [194]. We first consider the Baker-Campbell-Hausdorff (BCH) decomposition of the displacement operator combined with a complex phase factor θ

$$\begin{aligned} S(v,w) &= \theta e^{(va-\overline{v}a^\dagger)} e^{(\frac{w}{2}a^2-\frac{\overline{w}}{2}(a^\dagger)^2)} \\ &= \theta e^{-\frac{|v|^2}{2}} e^{-\overline{v}a^\dagger} e^{va} e^{-\frac{1}{2}\frac{\overline{w}}{|w|}\tanh(|w|)(a^\dagger)^2} e^{-\ln(\cosh(|w|))(a^\dagger a+\frac{1}{2})} e^{\frac{1}{2}\frac{w}{|w|}\tanh(|w|)a^2} , \end{aligned} \tag{5.33}$$

where we have used the standard results for the BCH decomposition of the two exponential terms, see e.g. [100]. One might easily guess that expanding the action of that operator on a general quantum state is impractical. Instead one can obtain a recurrence formula for the coefficients $\phi_k = \langle k| S |0\rangle$. However, before that we need to calculate ϕ_0 which will serve as normalization.

$$\begin{aligned} \phi_0 = \langle 0| S |0\rangle &= \theta \frac{e^{-\frac{|v|^2}{2}}}{\cosh(|w|)^{\frac{1}{2}}} \langle 0| e^{va} e^{-\frac{1}{2}\frac{\overline{w}}{|w|}\tanh|w|(a^\dagger)^2} |0\rangle \\ &= \theta \frac{e^{-\frac{|v|^2}{2}}}{\cosh(|w|)^{\frac{1}{2}}} e^{-\frac{1}{2}v^2\frac{\overline{w}}{|w|}\tanh|w|} . \end{aligned} \tag{5.34}$$

Subsequently one can compute ϕ_k by observing that

$$\langle k| Sa |0\rangle = \langle k| SaS^{-1}S |0\rangle = 0 . \tag{5.35}$$

Inserting above the following Bogoliubov transformation (i.e., the result of commuting a with S^{-1})

$$SaS^{-1} = \cosh|w| a + \frac{\overline{w}}{|w|}\sinh|w| a^\dagger + \overline{v}\cosh|w| + v\frac{\overline{w}}{|w|}\sinh|w| , \tag{5.36}$$

we obtain the following relation

$$\sqrt{k+1}\cosh|w|\phi_{k+1} + \sqrt{k}\frac{\overline{w}}{|w|}\sinh|w|\phi_{k-1} + (\overline{v}\cosh|w| + v\frac{\overline{w}}{|w|}\sinh|w|)\phi_k = 0 . \tag{5.37}$$

This three-term recurrence relation can be solved in terms of Hermite polynomials, thus producing an analytical expression for all ϕ_k [194]

$$\phi_k = \frac{1}{\sqrt{k!}}\left(\frac{1}{2}\frac{\overline{w}}{|w|}\tanh|w|\right)^{\frac{k}{2}} H_k(s)\phi_0 , \tag{5.38}$$

where

$$s = -\frac{1}{\sqrt{2}}\left(v\left(\frac{\overline{w}}{|w|}\right)^{\frac{1}{2}}\sqrt{\tanh|w|} + \overline{v}\left(\frac{w}{|w|}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\tanh|w|}}\right) . \tag{5.39}$$

Here, the appearance of orthogonal polynomials is reminiscent of [47], where the authors investigate the relationship of Krylov complexity and orthogonal polynomials. While this hints at a possible connection, the orthogonal polynomials here enter the expression for ϕ_k whereas in [47] they play the role of the operator states themselves. It is unclear at this time whether there is an overarching

description of this problem in terms of orthogonal polynomials or this is simply a coincidence. Moving past this digression, we have achieved our initial goal of obtaining a distribution over the Fock basis of the Perelomov coherent states for the two-dimensional Schrödinger group in the form

$$S\left|0\right\rangle = \sum_{k=0}^{\infty} \phi_k \left|k\right\rangle \ , \tag{5.40}$$

with the coefficients ϕ_k specified above.

5.3.3 Krylov complexity in a natural basis

As we showed previously, the probability distribution over the Fock basis is given by (5.38). First, let us show that indeed the condition $\sum_k |\phi_k|^2 = 1$ is satisfied. For that we will need to use Mehler's formula

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{n!}\left(\frac{z}{2}\right)^n = (1-z^2)^{-\frac{1}{2}} e^{\frac{2xyz-(x^2+y^2)z^2}{1-z^2}} \ , \tag{5.41}$$

which for $x = s$ and $y = \overline{s}$ simplifies to

$$\sum_{n=0}^{\infty} \frac{H_n(s)H_n(\overline{s})}{n!}\left(\frac{z}{2}\right)^n = (1-z^2)^{-\frac{1}{2}} e^{\frac{2|s|^2 z-(s^2+\overline{s}^2)z^2}{1-z^2}} \ . \tag{5.42}$$

The sum expression $|\phi_n|^2$ in terms of (5.38), using that $\overline{H}_n(z) = H_n(\overline{z})$, simply becomes

$$\sum_{k=0}^{\infty} \frac{H_k(s)H_k(\overline{s})}{k!}\left(\frac{\tanh|w|}{2}\right)^k |\phi_0|^2 = \cosh|w| e^{\frac{2|s|^2\tanh|w|-(s^2+\overline{s}^2)\tanh^2|w|}{1-\tanh^2|w|}} |\phi_0|^2 = 1 \, , \tag{5.43}$$

where we used the definition of s from (5.39) to show the last step. In computing Krylov complexity we consider rewriting the definition as follows

$$\begin{aligned} K_{\mathcal{O}} &= \sum_{k=0}^{\infty} k|\phi_k|^2 \\ &= |\phi_0|^2 z\partial_z \sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k H_k(s)\overline{H}_k(s) - |\phi_0|^2 z\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k \partial_z\left[H_k(s)\overline{H}_k(s)\right] \\ &= |\phi_0|^2 z\partial_z |\phi_0|^{-2} - |\phi_0|^2 y\partial_y |\phi_0|^{-2} \\ &= |v|^2 + \sinh^2|w| \, , \end{aligned} \tag{5.44}$$

where $z = \tanh|w|$ and $y = (v/|v|)^2$ and in the third equality we used the identity

$$z\partial_z s = y\partial_y s \, . \tag{5.45}$$

In fact, using induction one can easily arive at the conclusion

$$K_{(n)} = \sum_{k=0}^{\infty} k^n |\phi_k|^2 = |\phi_0|^2 \left(\frac{\sinh(2|w|)}{2}\partial_{|w|}\right)^n |\phi_0|^{-2}\Big|_s \ , \tag{5.46}$$

where n is any integer and $|_s$ indicates that when taking the derivative with respect to $|w|$, s is kept fixed.

The variance $\sigma^2 = \langle k^2\rangle - \langle k\rangle^2$ can be computed using the identity in (5.46) and reads

$$\sigma^2 = |v|\cosh 2|w| + \sinh|w|\cosh|w|\left(\sinh 2|w| - \frac{\overline{v}^2 w + v^2\overline{w}}{|w|}\right) . \tag{5.47}$$

To get some more feeling for expression (5.44), it is instructive to reproduce the limit in which either the Heisenberg-Weyl part or the $SL_2(\mathbb{R})$ part are turned off by hand. If we take $v = -\overline{v} = i\alpha t$ and $w = \overline{w} = 0$ we find

$$K_{\mathcal{O}} = \alpha^2 t^2 \tag{5.48}$$

where we assumed α to be real and we indeed reproduce the correct Krylov complexity from (5.17). In the opposite case we consider $v = \overline{v} = 0$ and $w = -\overline{w} = i\beta t$, where β is real, and we find

$$K_{\mathcal{O}} = \sinh^2 \beta t \tag{5.49}$$

which indeed reproduces (5.22) (with $h = 1/4$) up to a factor of 2. This can be attributed to the fact that the chain picture that we obtain for the Schrödinger group has double the number of sites compared to $SL_2(\mathbb{R})$. Setting $v = 0$ has the effect of the probability vanishing on odd sites, which however are still counted by the complexity. On the contrary the way that the Krylov basis is built for $SL_2(\mathbb{R})$ (using the harmonic oscillator realization of the algebra presented previously) is such that only the even sites are taken into account to begin with and this subtle difference leads to this factor of two discrepancy. This effect is captured graphically in figure 5.1.

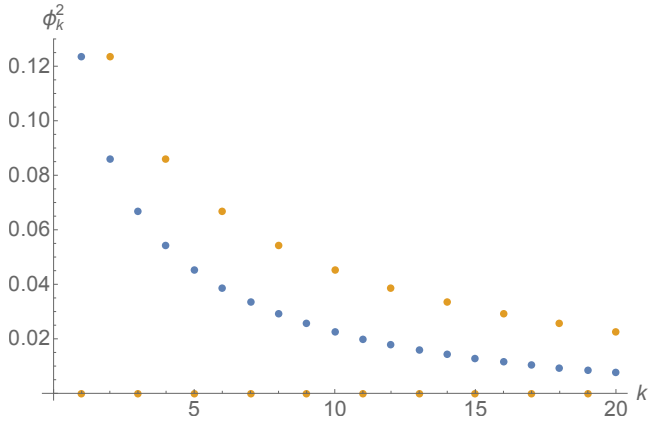

Figure 5.1: In this figure we compare the probabilities for the Schrödinger group (yellow dots) and $SL_2(\mathbb{R})$ (blue dots) as a function of their index k. Notice that for odd k the probabilities for the Schrödinger group vanish, while for even k they are pairwise equal to $SL_2(\mathbb{R})$, but shifted.

5.3.4 Interpreting complexity

The displacement operator $S(v, w)$ introduced in (5.33) leads to Perelomov coherent states for the two-dimensional Schödinger group. In order to interpret the result for complexity we need to relate the displacement operator $S(v, w)$ to the Liouvillian

$$\mathcal{L} = \alpha(a^\dagger + a) + \frac{\beta}{2}((a^\dagger)^2 + a^2)\,, \quad \alpha, \beta \in \mathbb{R}\,, \tag{5.50}$$

such that complex parameters v and w inherit the appropriate time dependence. To relate these quantities we will use the identity

$$e^{i[\alpha(a^\dagger+a)+\frac{\beta}{2}((a^\dagger)^2+a^2)]t} = \theta e^{(va-\overline{v}a^\dagger)} e^{(\frac{w}{2}a^2-\frac{\overline{w}}{2}(a^\dagger)^2)}\,, \tag{5.51}$$

$$v = \frac{\alpha}{\beta}(1 - \cosh\beta t) + i\frac{\alpha}{\beta}\sinh\beta t\,, \quad w = i\beta t\,, \tag{5.52}$$

and we pick up a phase factor $\theta = \exp\left[i\frac{\alpha^2}{\beta^2}(\sinh\beta t - \beta t)\right]$. To obtain the above identity we circumvented using the BCH formula by adopting an explicit matrix representation [100] that allows us to perform ordinary matrix exponentiation:

$$\eta\left(a^\dagger a + \frac{1}{2}\right) + \delta + R(a^\dagger)^2 + La^2 + ra^\dagger + la \quad \mapsto \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ r & \eta & 2R & 0 \\ -l & -2L & -\eta & 0 \\ -2\delta & -l & -r & 0 \end{pmatrix}, \tag{5.53}$$

where η, δ, R, L, r, l are complex numbers.

Now that we know $v(t)$ and $w(t)$ we can interpret complexity (5.44) as a function t:

$$K_{\mathcal{O}} = \alpha^2 t^2 + \sinh^2\beta t + \alpha^2\left[\frac{4\cosh\beta t\sinh^2\frac{\beta t}{2}}{\beta^2} - t^2\right], \tag{5.54}$$

where the first term reproduces the HW complexity ($\beta = 0$) and the second term reproduces the $SL_2(\mathbb{R})$ complexity ($\alpha = 0$), for a comparison see Section 5.2. The third term in square brackets, which we dub the interaction term, vanishes in either cases and we interpret it as arising from the fact that elements of either sub-algebras do not commute. When expanding the hyperbolic functions in the interaction term we get an infinite series in even power that cancels the single t^2, showing that the interaction term is always positive. This means that the Schrödinger complexity is more than the sum of the separate complexities of HW and $SL_2(\mathbb{R})$. While for non-zero α, β the interaction term is always present, the relative size of these parameters determines the character of the system. In other words if $\frac{\alpha}{\beta}$ is small then the system behaves closer to pure $SL_2(\mathbb{R})$ and conversely when $\frac{\alpha}{\beta}$ is large then the behavior resembles pure HW, as seen in figure 5.2.

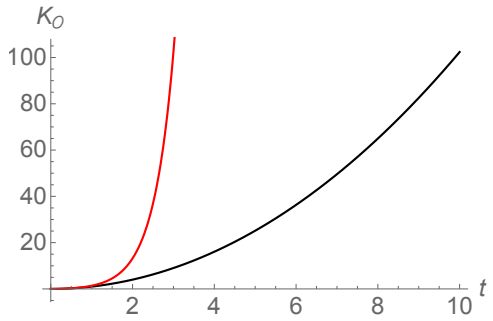

Figure 5.2: This figure contains the plots of complexity as a function of time for the cases where $\alpha = 0.01, \beta = 1$ (red curve) and $\alpha = 1, \beta = 0.01$ (black curve). We observe that the red curve clearly exhibits the exponential behavior characteristic of $SL_2(\mathbb{R})$ whereas the black curve is closer to the quadratic behavior of HW.

Let us consider early and late time limits to explore the effects of the interaction term. For early time we find

$$K_{\mathcal{O}} = (\alpha^2 + \beta^2)t^2 + \mathcal{O}(t^3), \tag{5.55}$$

where the β^2 comes from the $\sinh^2$ and the interaction term does not contribute. The effects of the interaction term becomes apparent at late times

$$K_{\mathcal{O}} = \left(\frac{1}{4} + \frac{1}{2}\frac{\alpha^2}{\beta^2}\right)e^{\beta t} = e^{\beta(t-t_s)}, \tag{5.56}$$

which implies that the Lyapunov exponent remains unaltered, but the scrambling time t_s, which can be a probe for operator growth, does receive a correction:

$$t_s = \frac{1}{\beta}\log\frac{4\beta^2}{\beta^2 + 2\alpha^2}. \tag{5.57}$$

For $\alpha > 0$, the scrambling time decreases as compared to the $SL_2(\mathbb{R})$ case and for $\alpha > \beta\sqrt{3/2}$ the scrambling time becomes negative, as can also happen for the pure $SL_2(\mathbb{R})$ case for different representations, i.e., values of h in (5.22). This is to be expected as it signals that one needs more information than just the leading order term for such cases. We furthermore point out that the first and second derivative of the complexity with respect to time are positive for $t \geq 0$.

The variance in (5.47) combined with $v(t)$ and $w(t)$ and normalized with $K_{\mathcal{O}}^2$ yields

$$\sigma^2 = \left\{\alpha^2 t^2 + \frac{1}{2}\sinh^2 2\beta t + \frac{\alpha^2}{2\beta^2}\left[4\left(\cosh\beta t + \cosh 3\beta t\right)\sinh^2\frac{\beta t}{2} - 2\beta^2 t^2 \right.\right. \\ \left.\left. +32\cosh\frac{\beta t}{2}\cosh^{3/2}\beta t\sinh^4\frac{\beta t}{2}\right]\right\} \Big/ \left[\sinh^2\beta t + 4\frac{\alpha^2}{\beta^2}\cosh\beta t\sinh^2\frac{\beta t}{2}\right]^2 , \tag{5.58}$$

where the first two terms correspond to the variance of purely HW and $SL_2(\mathbb{R})$ respectively. The term in the numerator within brackets is positive for $t > 0$. In order to appreciate the effect of the interaction term it is useful to investigate the plots of the probabilities ϕ_k^2 which appear in figure 5.3.

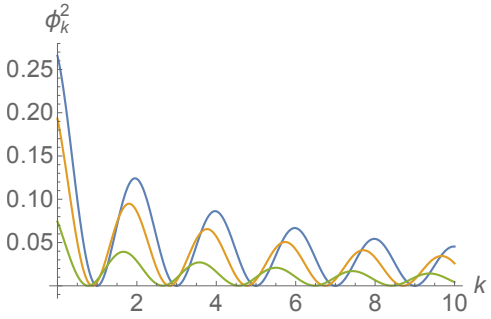

Figure 5.3: Probabilities for the Schrödinger group as functions of their index k (made into a continuous variable for illustration purposes) for $\alpha = 0$ (blue curve), $\alpha = 0.25$ (yellow curve) and $\alpha = 0.5$ (green curve). We notice that the magnitude of the probability drops for larger values of α. Since the probability functions are normalized. This illustrates the faster spreading that occurs due to the interaction term in (5.58).

It is also worth examining in more detail the form of $|\phi_0|^2$, which is the autocorrelator (survival amplitude) of the system

$$|(\mathcal{O}(0)|\mathcal{O}(t))|^2 = |\phi_0(t)|^2 = \frac{e^{-\frac{\alpha^2(e^{2\beta t}-1)^2}{8\beta^2}+2\beta t}}{\cosh(2\beta t)} , \tag{5.59}$$

where we have restored the time dependence using (5.51). We remind the reader that the information contained in the autocorrelation function is equivalent to the Lanczos coefficients, or in our case the hopping coefficients. We observe that the leading contribution to the decay of the autocorrelation function is of the form $e^{-\alpha^2 e^{2\beta t}}$, that is doubly exponential. This is unlike any of the semisimple groups that were studied so far, although it corroborates the enhanced spreading of the wavefunction as argued previously. We illustrate this result by comparing to the results for pure HW and $SL_2(\mathbb{R})$ in figure 5.4.

5.4 Discussion

In this paper we proposed an approach to computing Krylov complexity for the two-dimensional Schrödinger group in what we dub to be a *natural* orthonormal basis, which does not involve

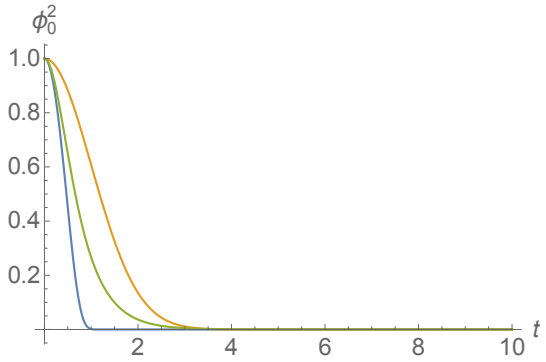

Figure 5.4: Comparison for the autocorrelator functions for $\alpha = 1, \beta = 0$ (yellow curve), $\alpha = 0, \beta = 1$ (green curve), $\alpha = 1, \beta = 1$ (blue curve). We observe that the autocorrelator for the Schrödinger group decays faster than both the HW and $SL_2(\mathbb{R})$ groups.

the usual tridiagonal Liouvillian, but a pentadiagonal one instead. We argue that regardless the same Krylov subspace is probed. The naturalness of the basis arises from the fact that the two-dimensional Schrödinger group is generated by a non-semisimple algebra that can be written as the semi-direct sum of the Heisenberg-Weyl algebra and $SL_2(\mathbb{R})$, which can both be naturally expressed in the Heisenberg-Weyl basis. We advocate that this approach might provide insights to other non-semisimple algebras for which the usual Krylov basis is technically non-attainable.

We find that the Krylov complexity in its natural basis is greater than the sum of the Krylov complexity of the separate sub-algebras and the same holds for its variance. At late time we recover the same Lyapunov exponent as for $SL_2(\mathbb{R})$, but we do find a naive scrambling time of smaller value. This is an intriguing aspect of our results as it creates the following picture. At late times the dynamics of the system are dominated by the $SL_2(\mathbb{R})$ degrees of freedom, which is why we observe the exponential growth of Krylov complexity with the same Lyapunov exponent. However, the spreading of the wave function, or in other words the distribution of the state over the Krylov subspace happens at a faster rate as shown by our variance calculations and the smaller scrambling time. While a rigorous interpretation of this phenomenon is outside of our reach for the moment, this highlights the importance of probing multiple aspects of the probability distribution rather than just the average. We believe this behavior makes sense from an intuitive point of view: due to its semi-direct sum structure, elements in different sub-algebras have non-trivial commutation relations and as such there should be an increase of the total complexity compared to the naive sum of the sub-algebras.

Our complexity grows convexly, which contrasts the numerical findings of [185], who considered Krylov complexity in an approximate tridiagonal basis. They furthermore explicitly find that the complexity of the two-dimensional Schrödinger algebra is *less* than its separate sub-algebras. It would be interesting to understand this discrepancy, especially since in the end both methods probe the same Krylov subspace.

Doing computations in the natural basis, from the perspective of the harmonic oscillators basis $a^\dagger$ and a, amounts to taking into account generators $(a^\dagger)^2$ and a^2 in the Liouvillian. One might wonder: why stop at quadratic order? It turns out that higher orders, say, $a^{\dagger 3}$ and a^3 do not close the algebra. Commuting these generators yields fourth order generators in terms of the oscillator basis and one is led to conclude that the algebra does not close for a finite number of generators.

The autocorrelator computed using the Krylov basis for $SL_2(\mathbb{R})$ can be reproduced using a thermofield double setup in the context of holography. The Schrödinger algebra is non-Lorentzian and its holographic manifestation is less well understood [184, 195]. Perhaps the here presented results, the autocorrelator that exhibits doubly-exponential scrambling specifically, can provide a

bench mark for further developing Schrödinger holography.

Finally, it is important to ask what our results imply for the bigger picture and how do they fit into our understanding of Krylov complexity and its relation to other complexity measures. Here we have established that one can obtain analytic results for systems with symmetry that go beyond the case of semisimple Lie algebras. The key ingredient in this endeavour is the use of generalized coherent states, which appear to play a prevalent role not only when it comes to Krylov complexity, but also in approaches to complexity that rely on geometry as is the case in [34, 114, 125, 196–198] for example. This is due to the natural organization of coherent states in metric spaces as explained in detail in [99]. It would be interesting to think how our results fit into that framework as the Schrödinger group provides an incremental step in tackling more complicated problems like the Virasoro group which, being centrally extended, also falls into the category of non-semisimple groups. Due to its relevance in AdS/CFT this group has been studied extensively and there are some results with regard to complexity that take advantage of the associated geometry and are possibly implicitly related to certain classes of coherent states [114]. Recently there have also been advances in understanding Krylov complexity from a holographic perspective [186, 190, 199], although there are still many open questions that one would hope to address. Thus, we believe that our work is not only important for a non-relativistic limit of AdS/CFT, but also provides the groundwork for more general considerations.

Acknowledgements

It is a pleasure to acknowledge the scientific program "Reconstructing the Gravitational Hologram with Quantum Information", which took place at the INFN Galileo Galilei Institute (GGI) for Theoretical Physics, National Centre for Advanced Studies in the summer of 2022, in Florence, where this work was initiated. We furthermore wish to thank Paweł Caputa for insightful discussions and support and Jose Barbon, Jan Boruch, Sinong Liu and Javier Magan for helpful comments. DP is supported by "Polish Returns 2019" grant of the National Agency for Academic Exchange (NAWA) PPN/PPO/2019/1/00010/U/0001 and Sonata Bis 9 2019/34/E/ST2/00123 grant from the National Science Center, NCN. WS is supported by the Icelandic Research Fund via the Grant of Excellence titled "Quantum Fields and Quantum Geometry" and by the University of Iceland Research Fund. WS furthermore thanks Paweł and Dimitris for their unrivalled hospitality in Warsaw.

Chapter 6

Krylov complexity of modular Hamiltonian evolution

Extended summary

The main idea of this work is the implementation of the techniques developed within the framework of Krylov complexity on the modular evolution of states. That is, we generalize the notion of Krylov complexity to a case where the evolution does not take place with regard to real time, but rather an arbitrary parameter which can be thought of as the time of an auxiliary circuit. Our main approach utilizes the modular partition function which can be related to the notion of Renyi entropies with an analytically continued index. Namely, by defining the modular partition function

$$\tilde{Z}(n) = \mathrm{Tr}\{(\rho_A^n)\} = \sum_j e^{-n\mathcal{E}_j} \ , \tag{6.1}$$

we can express the survival amplitude in the form

$$S(s) = \tilde{Z}(1 - is) \ , \tag{6.2}$$

where s is the circuit time. Then, following the standard procedure of expansion of the survival amplitude in terms of moments, we show that we can derive Lanczos coefficients that are explicitly related to the Renyi entropies $S_A^{(1-is)}$, thus acquiring a meaning as quantum information quantities themselves. Subsequently we can obtain the Krylov wavefunctions and define a notion of modular spread complexity in line with the original definition. Moreover, we are able to explicitly calculate the modular spread complexity for a number of examples.

These examples contain the evolution of primary and local operators of a 2D CFT, random Hamiltonians, and entangled qubit states such as GHZ and W-states analogues. Once again, the symmetries present in these setups play an important role, as the universal results we obtained in [154] are emerging in this novel setting. This is easily highlighted for the qubit state $|\Psi_0\rangle = \sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle$ for which we analytically derive the modular spread complexity to be

$$\mathcal{C}(s) = 4p(1-p)\sin^2\left(\frac{s}{2}\log\frac{1-p}{p}\right) \ . \tag{6.3}$$

Note that its overall behaviour is consistent with the universal result predicted by the SU(2) symmetry present in the system. Another interesting aspect is the relationship to entanglement. It is easy to confirm that for a maximally entangled state (i.e a Bell pair) the complexity vanishes, which can be interpreted in light of the relationship of the Lanczos coefficients with the entanglement spectrum. In particular b_1^2 is identified with the capacity of entanglement and as such a maximally

entangled state leads to $b_1 = 0$ terminating the Lanczos algorithm at the first step. This interplay between entanglement and modular complexity is the fulcrum of this work which is further explored in the more sophisticated examples of the article.

In particular, following the same methodology we compute the modular spread complexity for a system of two coupled harmonic oscillators whose reduced density matrix after tracing out one of the oscillators has eigenvalues

$$\lambda_k = (1-\xi)\xi^k, \quad 0 < \xi < 1 \ . \tag{6.4}$$

This ultimately leads to the derivation of the spread complexity

$$\mathcal{C}(s) = \frac{4\xi}{(1-\xi)^2} \sin^2(\frac{s}{2}\log\xi) \ . \tag{6.5}$$

Similarly, we consider the case of an interval in a 2D CFT for which the analytically continued partition function is

$$\tilde{Z}(1-is) = \exp\left\{\left(-\frac{s^2 W}{s^2+1} + i\frac{s(s^2+2)W}{s^2+1}\right)\right\}, \quad W = \frac{c}{6}\log\left(\frac{u-v}{\epsilon}\right) \ , \tag{6.6}$$

with c being the central charge of the CFT, $(u-v)$ the length of the interval and ϵ a UV cutoff. Based on this we show that the modular spread complexity grows (for large W) proportionally to the entanglement entropy as $\mathcal{C}(s) \sim S_A s^2$.

In order to study the case of random modular Hamiltonians we consider the thermodynamic limit, rendering the Krylov chain a continuuous line and the Lanczos coefficients functions $a(x), b(x)$ depending on the position on the line. In this limit we can perform an approximation by dividing the line into segments of length L for which the Lanczos coefficients are constants, leading to a block approximation of the modular Hamiltonian matrix. In this regime the spectrum is given by

$$E_k = 2b\cos\left(\frac{k\pi}{L+1}\right) + a \ , \tag{6.7}$$

eventually leading to the probability distribution over the Krylov basis

$$\overline{p}(x) = N I_0(2b(x)) e^{-a(x)} \ , \tag{6.8}$$

where N is the size of the matrix.

For the local operators in 2D CFT the crucial ingredient is the return amplitude, whose form is fixed by conformal symmetry to be

$$S(s) = \frac{\langle \mathcal{O}^\dagger(0,u_1)\mathcal{O}(s,u_2)\rangle}{\langle \mathcal{O}^\dagger(0,u_1)\mathcal{O}(0,u_2)\rangle} \ , \quad u_1 = l + i\epsilon, \, u_2 = l - i\epsilon \tag{6.9}$$

where l is the size of the interval. From this expression we can derive the modular spread complexity

$$\mathcal{C}(s) = \frac{\beta_0^{loc}(l)^2}{\epsilon^2} \sinh^2 \pi s \ , \tag{6.10}$$

where the local temperature β_0^{loc}, critically depends on the size of the subsystem under consideration. This for example determines the scrambling time, which is an important aspect of the dynamics.

These results lead to the most important lesson from this work, which goes beyond the notion of symmetry. Namely, due to the relationship between modular spread complexity and the entanglement spectrum it becomes possible to describe the system dynamics using the latter, in the regime where the entanglement entropy itself fails. Said differently, even though entanglement is not enough, there are cases for which the entanglement spectrum is. Here our approach relies on prior knowledge of the entanglement spectrum which is more than on usually has at their disposal, but it serves as a strong proof of concept regarding the possibility of studying modular dynamics using Krylov methods.

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Abstract

We investigate the complexity of states and operators evolved with the modular Hamiltonian by using the Krylov basis. In the first part, we formulate the problem for states and analyse different examples, including quantum mechanics, two-dimensional conformal field theories and random modular Hamiltonians, focusing on relations with the entanglement spectrum. We find that the modular Lanczos spectrum provides a different approach to quantum entanglement, opening new avenues in many-body systems and holography. In the second part, we focus on the modular evolution of operators and states excited by local operators in two-dimensional conformal field theories. We find that, at late modular time, the spread complexity is universally governed by the modular Lyapunov exponent $\lambda_L^{mod} = 2\pi$ and is proportional to the local temperature of the modular Hamiltonian. Our analysis provides explicit examples where entanglement entropy is indeed not enough, however the entanglement spectrum is, and encodes the same information as complexity.

6.1 Introduction and summary

In recent years quantum complexity has become a new exciting area within quantum many-body systems, quantum gravity and quantum field theory, see e.g. [27,86,88]. It provides a new perspective on the structure of quantum states as well as quantum dynamics, complementing that of quantum information. It is also instrumental in understanding black holes in holography [6] and quantum gravity [21, 22, 200, 201]. In this last context, it was argued by Susskind that "entanglement is not enough" [3], in particular if one seeks to understand aspects of the long time regime of chaotic systems and black holes. Complexity measures were then proposed as fine-grained probes of dynamics at late times. For this reason, most research in this direction has focused on properties of a real time, Hamiltonian evolution of complexity measures, trying to test their supremacy to entanglement measures. On the other hand, it is known that entanglement entropy contains only a small fraction of information about bipartite entangled states, and that the entanglement spectrum is a much more fine-grained measure of their structure [202]. In this vein, it is natural to ask for more direct relations between complexity and entanglement, and better characterize what types of entanglement measures are enough for the study of holography and quantum black holes. This is one of our main motivations in this work.

To this end, we focus on arguably the most promising (in relation to the problems mentioned above) definition of complexity called, Krylov [42] or spread complexity [50], that can be applied both to quantum operators and states. This measure was inspired by pioneering works on operator size, quantum chaos and thermalisation in many-body systems [71, 72], and it has produced a burst of activity and interest in recent years [46, 47, 49, 53–57, 76, 77, 82, 154–156, 164, 165, 167–169, 171, 172, 178, 179, 187, 190, 198, 199, 203–207, 207–209, 209, 210, 210–213]. As shown in [50], Krylov or spread complexity defines complexity as the minimal amount of spread of the wavefunction in the Hilbert space. Such minimization is universally accomplished for a finite amount of time by the so-called Krylov basis, that arises via the Lanczos recursion method [214] (to be reviewed below). Some highlights of Krylov complexity in many body systems are the demonstration of the exponential growth with the universal Lyapunov exponent, together with the idea that Krylov complexity bounds the growth of out-of-time-ordered correlators [42, 215], the derivation of the linear growth regime

of complexity [46], the geometric approach and connection to generalized coherent states [154], the ability to codify quantum chaos and fine-grained properties of the spectrum [49, 50, 164], and the recent holographic demonstrations that it can reproduce the volume of black hole interiors, a.k.a. the volumes of Einstein-Rosen bridges, see [49, 190, 206] for the JT gravity cases, and [199] for the case of general relativity in general dimensions.

Given these recent developments, and with the aim of exploring the relation between entanglement and complexity, in this work we expand the Krylov or Lanczos approach in two new directions. First, generalising previous work on the time evolution of the thermofield double state (TFD) [50], we define and study the complexity of modular evolution in generic bipartite entangled states. We show that this measure is controlled by the entanglement spectrum of the reduced density matrix. Equivalently it is controlled by the modular Lanczos spectrum, which interestingly contains the very same information, and might be taken as a new characterization of the entanglement structure. Indeed the first modular Lanczos coefficient is the entanglement entropy itself, while the second is the (square) of capacity of entanglement [129, 216]. We will then analyze this quantity in several examples. For random states, the modular Hamiltonian is random, and we discuss how entanglement entropy is codified in the plateau of the modular complexity evolution, and how modular complexity is also sensitive to the Page curve [217]. In the second part we will discuss modular growth and evolution of quantum operators. In holography this paves a way towards a precise measure of complexity of bulk reconstruction. Exploiting again the power of generalised coherent states as well as modular two-point correlators in two-dimensional (2d) CFTs, we will derive a universal growth of spread complexity of modular evolution characterised by Lyapunov exponent $\lambda^{mod} = 2\pi$ and the scrambling time governed by an effective local temperature of the modular Hamiltonian for a single interval as well as two intervals in free fermion CFT.

Overall, this approach makes it clear why "entanglement is not enough" [3], while at the same time it also suggests that a slight but insightful modification may solve the issue at stake, and that indeed entanglement spectrum is enough.

6.2 Spread Complexity

For completeness, we begin with a brief review of the spread complexity [50]. The starting point of the discussion is the unitary evolution of an initial quantum state $|\psi_0\rangle$ with time-independent Hamiltonian H

$$|\Psi(t)\rangle = e^{-iHt}\,|\psi_0\rangle\,. \tag{6.11}$$

Generically, this evolution spreads the state $|\psi_0\rangle$ in the Hilbert space of the model, making it more complex. While the amount of the spread depends on the choice of basis, we can quantify the complexity of this process by minimizing the spread of the wavefunction over all choices of basis. The result of this minimization, at least for a finite period of time, brings us to the so-called Krylov basis. This basis, denoted below by $|K_n\rangle$, is obtained via the Gram-Schmidt orthogonalisation procedure on the subspace of all the powers of H applied to $|\psi_0\rangle$. The iterative procedure to achieve this is called the Lanczos algorithm [214] and it can be written as

$$|A_{n+1}\rangle = (H - a_n)\,|K_n\rangle - b_n\,|K_{n-1}\rangle\,, \tag{6.12}$$

where $|K_n\rangle = b_n^{-1}\,|A_n\rangle$, $b_0 = 0$ and the first vector coincides with our initial state $|K_0\rangle = |\psi_0\rangle$. The key role in this story is played by Lanczos coefficients a_n and b_n that control the dynamics and are defined as

$$a_n = \langle K_n|\,H\,|K_n\rangle\,, \qquad b_n = \langle A_n|A_n\rangle^{1/2}. \tag{6.13}$$

The algorithm stops as soon as any of the $b_n = 0$, which signifies that no more independent basis vectors can be constructed. After running this iterative algorithm, we can expand the state in the

Krylov basis

$$|\Psi(t)\rangle = \sum_n \psi_n(t)\,|K_n\rangle\,. \tag{6.14}$$

By construction, the coefficients of this expansion satisfy a discrete Schrodinger equation

$$i\partial_t\psi_n(t) = a_n\psi_n(t) + b_n\psi_{n-1}(t) + b_{n+1}\psi_{n+1}(t), \tag{6.15}$$

that also highlights the fact that the Hamiltonian is tridiagonal in the Krylov basis, with tridiagonal elements given by the Lanczos coefficients. Finally, if we are able to solve this equation, the spread complexity is computed as the average value of n in the probability distribution $p_n(t) \equiv |\psi_n(t)|^2$, namely

$$\mathcal{C}(t) = \sum_n n\,p_n(t). \tag{6.16}$$

Clearly, solving (6.15) is the main step and it requires the knowledge of the Lanczos coefficients. They are in fact encoded in the return amplitude (the Loschmidt amplitude)

$$S(t) = \langle\Psi(t)|\Psi(0)\rangle = \langle\psi_0|\,e^{iHt}\,|\psi_0\rangle = \sum_n \mu_n\frac{t^n}{n!}. \tag{6.17}$$

Its moments $\mu_n = \langle\psi_0|(iH)^n|\psi_0\rangle$ allow us to extract Lanczos coefficients that are related via polynomial equations e.g., the first two are (see more in Appendix A)

$$a_0 = -i\mu_1, \qquad b_1^2 = \mu_1^2 - \mu_2. \tag{6.18}$$

Inversely, the knowledge of the Lanczos coefficients allows the computation of the moments of the Hamiltonian. Therefore, since the Lanczos coefficients play such a pivotal role, it is important to understand their physical meaning and how different phenomena are encoded in their scaling with n.

We conclude this introduction with two remarks. Firstly, an important class of initial states $|\psi_0\rangle$ is given by the TFD state [218]. Denoting by $|n\rangle$ the eigenstate of the Hamiltonian with energy E_n this state reads

$$|\psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\frac{\beta}{2}E_n}\,|n\rangle_L \otimes |n\rangle_R\,, \tag{6.19}$$

where $Z(\beta)$ is the partition function at temperature $T = 1/\beta$. The TFD state is the canonical purification of the thermal density matrix $\rho = e^{-\beta H}$. It is then interesting to consider the time evolution of (6.19) with the Hamiltonian of a single copy, say H_L, especially in the context of black holes [112,219]. For this evolution the return amplitude becomes the analytically continued partition function

$$S(t) = \frac{Z(\beta - it)}{Z(\beta)}, \tag{6.20}$$

whose modulus squared is the spectral form factor, a key object in the field of quantum chaos [220]. This way, the Lanczos coefficients as well as spread complexity are directly probing the spectrum of the evolving Hamiltonian, and they codify the fine-grained aspects such as spectral rigidity and the universality class of the chaotic model [49,50,164]. Indeed, for chaotic systems with no degeneracies the Lanczos spectrum of this process contains exactly the same information as the spectrum itself. The main idea of this work is to generalise this TFD example to reduced density matrices and modular Hamiltonian evolution.

Secondly, the Krylov complexity of the operator growth [42] can be studied in a complete analogy with the discussion above. The only non-trivial step is the choice of the inner-product in the space of operators that allows us to map Heisenberg evolution of an operator $\mathcal{O}(t)$ to a state $|\mathcal{O}(t))$. The crucial information about the operator growth is then captured by the return amplitude that corresponds to a two-point correlator $(\mathcal{O}|\mathcal{O}(t))$. Along these lines, below we will consider operator growth as well as the dynamics of CFT states excited by local operators under modular Hamiltonian evolution. They will involve return amplitudes based on modular two-point functions in 2d CFTs.

6.3 Modular Spread Complexity

We now consider spread complexity of modular Hamiltonian evolution. As reviewed above, we start with a pure state $|\Psi_0\rangle$ in some Hilbert space $\mathcal{H}$. We then pick a sub-system A and its complement A^c, and assume a Hilbert space decomposition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$, so that we can write $|\Psi_0\rangle$ in the Schmidt form

$$|\Psi_0\rangle = \sum_j \sqrt{\lambda_j}\, |j\rangle_A\, |j\rangle_{A^c}\,, \tag{6.21}$$

where $|j\rangle$ are basis vectors in A (and the complement). As usual, we define the reduced matrix ρ_A of the sub-region A as well as the modular Hamiltonian H_A by

$$\rho_A = \mathrm{Tr}_{A^c}\left(|\Psi_0\rangle\langle\Psi_0|\right) \equiv e^{-H_A}. \tag{6.22}$$

The Schmidt coefficients in (6.21) describe the spectrum λ_j of ρ_A or the spectrum $\mathcal{E}_j$ of the modular Hamiltonian H_A

$$\lambda_j \equiv e^{-\mathcal{E}_j}, \qquad \sum_j \lambda_j = 1, \tag{6.23}$$

and, by analogy with thermal states, we can define the modular partition function at inverse temperature $\beta = n$ as

$$\tilde{Z}(n) = \mathrm{Tr}(\rho_A^n) = \sum_j e^{-n\mathcal{E}_j}. \tag{6.24}$$

Conventionally, we normalise $\mathrm{Tr}(\rho_A) = \tilde{Z}(1) = 1$.

Finally, we define the modular evolution of the initial state (6.21) as

$$|\Psi(s)\rangle = e^{-isH_A\otimes 1_{A^c}}\, |\Psi_0\rangle\,, \tag{6.25}$$

where s is the modular time. Note that we perform this evolution with $H_A \otimes 1_{A^c}$ and not with the total modular Hamiltonian $H_{mod} = H_A \otimes 1_{A^c} - 1_A \otimes H_{A^c}$; indeed, $|\Psi_0\rangle$ is invariant under the evolution with H_{mod} [221]. By analogy with the TFD state (evolution with H_L vs $H_L - H_R$), this leads to a non-trivial evolution of the state $|\Psi_0\rangle$. In the following, our goal will be to quantify the spread complexity of this state in various models and shed light on the Lanczos coefficients in this evolution.

For that we use the Lanczos algorithm to construct an orthonormal basis $|K_n\rangle$ and expand our state as in (6.14), where the expansion coefficients $\psi_n(s)$ satisfy (6.15) with Lanczos coefficients a_n and b_n encoded in the modular return amplitude

$$S(s) \equiv \langle\Psi(s)|\Psi_0\rangle = \sum_j \lambda_j^{1-is} = \tilde{Z}(1-is). \tag{6.26}$$

This object is closely related to the Renyi entropies of the reduced density matrix ρ_A defined for integer n as

$$S_A^{(n)} = \frac{1}{1-n}\log(\mathrm{Tr}\,\rho_A^n), \tag{6.27}$$

and we have the relation to the analytically continued Renyi with replica index $n = 1 - is$

$$S(s) = \exp\left(is\, S_A^{(1-is)}\right). \tag{6.28}$$

We conclude that the Lanczos procedure, based on the moments of $S(s)$, will involve interesting combinations of quantum information measures. Indeed, already from (6.18), we can see that for the modular Hamiltonian H_A, the moments a_0 and b_1^2 will be simply the von Neumann entropy S_A and the capacity of entanglement C_E [129, 131, 159, 183, 216, 222] respectively. At the conceptual

level, since spread complexity is a functional of the survival amplitude, and this is a functional of the entanglement spectrum, we conclude that, while entanglement is not enough (it is just a_0), entanglement spectrum is enough. Going in the reverse direction, since the Renyi entropies can be found from the modular survival amplitude, and this is a functional of the modular Lanczos spectrum, we also conclude that Lanczos spectrum is enough. This construction then provides a solid bridge between entanglement and complexity, as we further develop below.

6.4 Examples

It is useful to consider a few simple, analytical examples. Let us start from a qubit state $|\Psi_0\rangle = \sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle$ where A and A^c are the first and second spins respectively and $p \in [0,1]$. Tracing out the second Hilbert space we obtain the return amplitude

$$S(s) = \mathrm{Tr}\left(\rho_1^{(1-is)}\right) = p^{(1-is)} + (1-p)^{(1-is)}, \tag{6.29}$$

with moments (see definition (6.17))

$$\mu_k = (-i)^k \left(p\log^k(p) + (1-p)\log^k(1-p)\right). \tag{6.30}$$

From them we extract the non-vanishing Lanczos coefficients

$$\begin{aligned} a_0 &= -p\log(p) - (1-p)\log(1-p) = S_1, \\ b_1^2 &= p(1-p)\left(\log(1-p) - \log(p)\right)^2 = C_E(\rho_1), \\ a_1 &= -p\log(1-p) - (1-p)\log(p), \end{aligned} \tag{6.31}$$

and confirm the relation with entanglement entropy and capacity of entanglement. At present, we do not have a sharp quantum information interpretation for a_1 and we hope to return to this issue in the future.

Next, we derive the two solutions of the Schrodinger equation (6.15) with these Lanczos coefficients and they are

$$\begin{aligned} \psi_0(s) &= p^{1+is} + (1-p)^{1+is} = S(s)^*, \\ \psi_1(s) &= \mp\sqrt{p(1-p)}((1-p)^{is} - p^{is}), \end{aligned} \tag{6.32}$$

with $\mp$ corresponding to $\pm$ in b_1 (recall that b_n's (6.13) are always positive so this sign depends on the difference between $\log(1-p)$ and $\log(p)$). By construction, the coefficient $\psi_0(s)$ and $S(s)$ are related by the simple complex conjugation. Finally, the modular spread complexity (6.16) is given by

$$\mathcal{C}(s) = 4p(1-p)\sin^2\left(\frac{s}{2}\log\frac{1-p}{p}\right). \tag{6.33}$$

In this simple example with Krylov space dimension equal to 2, we have a relation between the modular spread complexity and modular spectral form factor: $\mathcal{C}(s) = |\psi_1(s)|^2 = 1 - |\psi_0(s)|^2$, which is not true in general for long times [49]. Clearly, the complexity growth is determined by the value of p. In particular, for maximally entangled state $p = 1/2$, the b_1 as well as $C(s)$ vanish (see Appendix B for another example). More generally for flat entanglement spectrum we have

$$\mathrm{Tr}(\rho_A^n) = \dim(\mathcal{H}_A)^{1-n}, \tag{6.34}$$

and we only get non-trivial $a_0 = S_A$ and all $b_n = 0$. This is a physically sensible result. In this context we only need one number to understand the structure of the state. In the thermodynamic limit of physical systems, such as those appearing in quantum gravity, it might seem that we have flat

entanglement spectrum at micro-canonical sectors. This is only an artefact of the thermodynamic limit. In reality the spectrum is chaotic, and the eigenvalues, although close to the average flat value, show no degeneracy and resemble the spectrum of a random matrix. In this scenario the Lanczos spectrum is completely different, as we discuss below.

Another simple example consists of two coupled harmonic oscillators [223]. After tracing one of them, we get the entanglement spectrum and modular partition function

$$\lambda_k = (1-\xi)\xi^k, \qquad \tilde{Z}(n) = \frac{(1-\xi)^n}{1-\xi^n}, \tag{6.35}$$

where $0 < \xi < 1$ is related to the details of the coupling between the oscillators. Following the above procedure, we can derive a general form for the Lanczos coefficients

$$\begin{aligned} a_n &= -n\frac{1+\xi}{1-\xi}\log(\xi) - \log(1-\xi) - \frac{\xi}{1-\xi}\log(\xi), \\ b_n &= n\frac{\sqrt{\xi}}{1-\xi}\log(1/\xi), \end{aligned} \tag{6.36}$$

where again $a_0 = S_1$ and $b_1^2 = C_E(\rho_1)$. Observe that these Lanczos coefficients are governed by the SL(2,R) symmetry algebra and our modular evolution of the state can be mapped to a coherent state of this Lie group. This allows us to recycle the derivations in [154] and derive the modular spread complexity

$$\mathcal{C}(s) = \frac{4\xi}{(1-\xi)^2}\sin^2\left(\frac{s}{2}\log(\xi)\right). \tag{6.37}$$

The entanglement spectrum is equivalent to the thermal spectrum of a single oscillator, i.e. writing $\xi = \exp(-\beta\omega)$ we have $\lambda_k = e^{-\beta E_k}/Z(\beta)$ with E_k being the energy of a single harmonic oscillator with frequency ω. Even though we have an infinite dimensional Krylov basis, modular spread complexity oscillates. However, we can formally send $\omega \to i\tilde{\omega}$ (complex $\log(\xi)$) and observe exponential growth of the modular spread complexity.

Finally, we consider 2d CFT where the trace of the reduced density matrix of a single interval $A = [u, v]$ can be computed using the replica trick as a correlator of twist operators inserted at the end-points of A [224]

$$\tilde{Z}(n) = \langle \sigma_n(u)\tilde{\sigma}_n(v)\rangle = \exp(-(n-1/n)\,W), \tag{6.38}$$

where W contains the CFT central charge c and details of the interval as well as geometry of the underlying CFT and is directly related to entanglement entropy $S_A = 2W$ (e.g. $W = \frac{c}{6}\log((u-v)/\epsilon)$ for the vacuum in a line). For our discussion, we neglected an overall non-universal constant in (6.38). However, it is crucial that we keep the cut-off ϵ small, but finite. The analytic continuation gives the modular partition function

$$\tilde{Z}(1-is) = \exp\left(-\frac{s^2 W}{s^2+1} + i\frac{s(s^2+2)W}{s^2+1}\right), \tag{6.39}$$

therefore, the corresponding modular spectral form factor $|\tilde{Z}(1-is)|^2$ decays to a plateau with value $\exp(-S_A)$. By expanding Lanczos coefficients for large W (or large central charge c, see Appendix C), we can show that spread complexity grows quadratically for initial modular time, proportionally to the entanglement entropy S_A

$$\mathcal{C}(s) \sim S_A\, s^2. \tag{6.40}$$

For later times, at finite cut-off ϵ, we also expect a period of linear growth and saturation to a plateau (analogous to the spectral form factor (6.39)). Verifying this expectation numerically would be interesting and we leave it for future work. Next we move to more general qualitative arguments in the context of random matrix theory.

6.5 Random Modular Hamiltonians

Further relations between entanglement entropy and entanglement spectrum on one hand, and spread complexity and the Lanczos spectrum on the other, arise by considering the example of random pure states. Given a pure state and a bipartition of the system into A and A^c, a putative ensemble of pure states (defining the particular notion of random state) naturally defines an ensemble of modular Hamiltonians H_A. This ensemble defines a particular notion of random modular Hamiltonian.

The analysis of the Lanczos approach for random matrices was recently developed in [164, 225]. The application of these constructions to modular evolution goes as follows. We first notice that, in the context of random states, the Lanczos coefficients of a reduced subsystem are random parameters, and the first goal is to compute their statistics. This can be accomplished with two assumptions. First we need take the thermodynamic limit, where the dimension N of the subsystem A goes to infinity. Without loss of generality we assume that this dimension is smaller than the dimension of A^c. In this limit the average values reliably inform us of the typical values associated with individual instances of the random modular Hamiltonian. Second we need to choose as initial state the vector $(1, 0, \cdots, 0)$. The reason is that for this state we know how to compute the Jacobian of the transformation between the original form of the random modular Hamiltonian and the tridiagonal form. It is given by [164]

$$J = \prod_{n=1}^{N-1} b_n^{(N-n)\beta-1} \,, \tag{6.41}$$

where β is the Dyson index of the ensemble of random matrices. This Jacobian should be thought as the analogue of the Vandermonde determinant for the change of variables that takes us to the diagonal form of the matrix. Equivalently, if the ensemble is invariant under a certain group of unitaries, we are free to take any initial state that follows from the previous one by applying a unitary belonging to such a group.

In the thermodynamic limit, see [164,225–227], it becomes natural to label the Lanczos coefficients in terms of $x \equiv n/N$, namely as $a(x) \equiv a_{n=xN}$ and $b(x) \equiv b_{n=xN}$. The reason is that in this limit, on average over the ensemble, the Lanczos coefficients become continuous functions in the interval $x \in [0, 1]$. We can now obtain the relation between these functions and the modular spectrum. We cut the Krylov chain into shorter segments of a given length L, such that $L \to \infty$ and $L/N \to 0$ in the thermodynamic limit. This is a block approximation of the Hamiltonian whose density of states is the sum of the densities of each block. Given the continuity assumption, a_n and b_n can be taken as constants in each block, equal to $a(x)$ and $b(x)$.

The different Hamiltonian blocks are then Toeplitz matrices of size L, with diagonal elements given by certain a and off-diagonal elements given by certain b. These matrices have eigenvalues $E_k = 2\,b\,\cos(k\pi/(L+1)) + a$, with $k = 1, \cdots, L$, and their density of states read

$$\rho_{a,b}(E) = \frac{1/L}{|dE_k/dk|} = \frac{H(4\,b^2 - (E-a)^2)}{\pi\,\sqrt{4\,b^2 - (E-a)^2}} \,. \tag{6.42}$$

Here $H(x)$ is the Heaviside step function and we normalized the density of states by dividing by L. The total (normalized) density of states is the sum over all blocks. In the thermodynamic or continuum limit this becomes [164]

$$\rho(E) = \int_0^1 dx\, \frac{H(4\,b(x)^2 - (E-a(x))^2)}{\pi\,\sqrt{4\,b(x)^2 - (E-a(x))^2}} \,. \tag{6.43}$$

This formula relates the average Lanczos coefficients to the modular spectrum, in particular to the modular density of states, where we remind that $\lambda = e^{-E}$ (see (6.23)). Deviations from this formula were also found in [164], further providing a relation between the average Lanczos coefficients and the potential defining the ensemble of random matrices.

Generically, in chaotic systems the wavefunction in the Krylov basis (6.14) reaches a stationary regime. In this regime the probabilities fluctuate around a mean value $\bar{p}(x)$. For special initial states we might have $\bar{p}(x) = 1$, namely constant in x, but this is not the generic situation as can be established numerically in simple scenarios [50, 164]. It is thus natural to inquire for the form of the stationary distribution $\bar{p}(x)$. Indeed, in terms of the distribution of energies of the initial state

$$|\langle\psi|E\rangle|^2 \equiv P(E)\,, \tag{6.44}$$

this is derived in [225] as follows. Assume $P(E)$ is a continuous function of the energy. For the modular state evolution that we are considering, this implies a continuous entanglement spectrum with small fluctuations around the average. Using (6.42), the number of states in the interval between x and $x + dx$ and in the interval between E and $E + dE$ is

$$\tilde{\rho}(x, E)\, dx\, dE = \frac{N\, dx\, dE}{\pi\sqrt{4b(x)^2 - (E - a(x))^2}}. \tag{6.45}$$

The long-time average probability distribution in the Krylov basis is just the the convolution of this density with the distribution of energies of the initial state (which is conserved in time). This reads

$$\bar{p}(x) = \int dE\, P(E)\, \tilde{\rho}(x, E)\,. \tag{6.46}$$

For the modular evolution of states the initial state was (6.21). The distribution of energies in the initial state is then $P(E) = \lambda = e^{-E}$, and we arrive at

$$\bar{p}(x) = N\, I_0(2b(x))e^{-a(x)}\,. \tag{6.47}$$

The plateau of the modular spread complexity and the Shannon entropy H_{Shannon} in the Krylov basis (dubbed K-entropy in [46]) follow from this probability distribution $\bar{p}(x)$. This is an explicit function once we have derived the Lanczos spectrum from the modular density of states using (6.43). It turns out that the result for the Shannon entropy is quite insensitive to the specific ensemble of random modular Hamiltonian, i.e on the specific Lanczos coefficients $a(x)$ and $b(x)$. Indeed

$$H_{\text{Shannon}} = -\sum_n \bar{p}_n \ln \bar{p}_n \approx \ln N, \tag{6.48}$$

up to subleading corrections in the thermodynamic limit. This means that the dimension of the Hilbert space explored by the random modular evolution is the same as the number of non-zero eigenvalues in the reduced density matrix, counted by its leading density of states. Notice that this same result applies for the complementary subsystem A^c. Although we assumed the dimension of A was smaller than that of A^c, the modular Hamiltonian and modular spectrum are the same up to zeros. In particular the number of non-zero eigenvalues is the same, and the saturation will happen at $\log N$ as well for A^c, where we remind that N is the dimension of the smaller subsystem A.

Finally, we can turn things around. Starting from the Lanczos coefficientsf $a(x)$ and $b(x)$, we can find the stationary distribution of the modular spread complexity $\bar{p}(x)$ and from there we can obtain the initial probability distribution in the energy basis as

$$P(E) = \int dx\, \bar{p}(x)\, \tilde{\rho}(x, E)\,. \tag{6.49}$$

We are led to the following conclusions. The first is that we could use these results in the context of the Page curve [217] (recall also that the relevance of the capacity of entanglement, that is our Lanczos coefficient b_1, to the Page curve was already discussed in [159, 222]). In this scenario, for random states drawn from the Haar measure the modular density of states is known and of

compact support [217]. Although (6.43) cannot be solved in closed form in this case, the Lanczos $b(x)$ coefficients decay to zero as they should and the modular spread complexity follows the regimes described in [50]. In particular, the spread complexity will saturate at a value controlled by the dimension of the smallest subsystem. For example, the entropy will be precisely $\log N$ in the leading approximation, where N is such dimension. The plateau of modular spread complexity then draws a complexity Page curve in the same way as the entanglement entropy.

The second conclusion concerns the slogan "entanglement is not enough" [3]. This was put forward to motivate the introduction of the notion complexity in quantum gravity. The present construction transparently shows why this is true when for the word "entanglement" we more precisely understand entanglement entropy itself. The reason is that entanglement entropy is the first entry of the Lanczos spectrum. But one needs the full spectrum of Lanczos coefficients to predict the long time dynamics of the wavefunction of the system. Spread complexity, which serves to characterize these dynamics, is also a functional of the whole spectrum. Clearly then, entanglement entropy is not enough. It is however not true if we slightly, but insightfully, modify the slogan so that it refers to the entanglement spectrum. As we have derived, there is a precise relation (one follows from the other and vice-versa), between the entanglement or modular spectrum, the modular Lanczos coefficients, the modular survival amplitude and the modular spread complexity. In this precise sense, we reach again the conclusion that the entanglement spectrum seems to be enough in the context of quantum gravity. The Lanczos modular spectrum and associated survival amplitude and modular complexity are enough as well.

6.6 Modular Growth and Evolution of Primary Operators

In this final section we discuss the operator growth and spread complexity of operators under the modular flow with the total modular Hamiltonian $H_{mod} = H_A \otimes 1_{A^c} - 1_A \otimes H_{A^c}$ (A^c being the complement of A). Namely, we consider the following modular evolution [221, 228, 229]

$$\mathcal{O}(s) = e^{iH_{mod}s}\mathcal{O}(0)e^{-iH_{mod}s} \equiv e^{i\mathcal{L}_{mod}s}\mathcal{O}(0), \tag{6.50}$$

where the modular Liouvillian (super-operator) acts on operators by taking the commutator $\mathcal{L}_{mod} \equiv [H_{mod}, \cdot]$. This modular flow of operators has been a central topic in a variety of recent works in QFT and holography [96, 230–242] but, to our knowledge, its complexity remains relatively unexplored.

To make progress, for simplicity, we first consider H_{mod} for static, universal examples where A is a single interval in the vacuum of a 2d CFT defined either on the line or on the circle, leaving more complicated cases to future works (see more in Appendix D). In the final part of this section we also mention the result for two disjoint intervals on the line for the free massless Dirac field in its ground state.

We start with the modular evolution of the highest weight state $|h\rangle$ (eigenstate of the CFT Hamiltonian i.e., $|h\rangle \equiv \lim_{z\to 0}\mathcal{O}(z)|0\rangle$ in radial quantisation of the Euclidean formalism) with the total modular Hamiltonian of an interval A in 2d CFT

$$|\psi(s)\rangle = e^{-isH_{mod}}|h\rangle\,. \tag{6.51}$$

The total modular Hamiltonian is a well-defined operator in the continuum and, in 2d CFTs, it can be written as a linear combination of the SL(2,R) generators (see e.g. [17, 243])

$$H_{mod} = \sigma_{-1}L_{-1} + \sigma_0 L_0 + \sigma_1 L_1 + a.c., \tag{6.52}$$

where the anti-chiral (a.c.) part is similarly expressed in terms of global $\bar{L}$'s (for simplicity, we will focus on the chiral part) and the coefficients depend on the CFT and interval geometry (see Appendix D). For this reason (6.51) is simply a coherent state and falls into the Lie-algebra symmetry

examples considered in [50, 154] where, using the Baker–Campbell–Hausdorff formula, the spread complexity can be evaluated as a simple function of general σ_i's (see (6.104) in Appendix D). Before we write it down, note that we may think about this state simply in the context of spread complexity of states [50] or as a state representing operator growth [42] with a particular choice of the inner product that corresponds to the return amplitude

$$S(s) = \langle h|e^{isH_{mod}}|h\rangle\,. \tag{6.53}$$

By using the procedures discussed in [50, 154], we find that Lanczos coefficients from (6.53) have the SL(2,R) form: $a_n = \gamma(n+\Delta)$ and $b_n = \alpha\sqrt{n(n+2\Delta-1)}$ with $\alpha = \sqrt{\sigma_1\sigma_{-1}}$ and $\gamma = \sigma_0$. Interestingly, in all the examples where (6.52) holds, the coefficients satisfy $\sigma_1\sigma_{-1} - \sigma_0^2/4 = \pi^2$ and this combination is directly linked to the Lyapunov exponent defined from the Krylov complexity [42]. For example, for a single interval $A = [a, b]$ in 2d CFT on a circle of size L we obtain

$$\mathcal{C}(s) \simeq \frac{2h}{\sin^2\left(\frac{\pi(b-a)}{L}\right)}\sinh^2(\pi s). \tag{6.54}$$

Clearly, at late modular time $s \gg 1$, the spread complexity grows exponentially with Lyapunov exponent $\lambda_L^{mod} = 2\pi$. We will see below that this is in fact a universal behaviour also for local operator growth. The size of the entangling interval $b-a$ governs the scrambling time at late time (see also below). We should also point that, this result that uses σ_i's from [17, 243] in the general formula (6.104), does not seem to have a well-defined (naive) limit of $L \to \infty$. As already pointed out in [154, 155], the spread complexity of coherent states can be written as an expectation value of L_0. When passing from the cylinder to the plane, the derivative of the exponential map will bring the appropriate factor of L that cures this (that is why we used $\simeq$).

Next, for 2d CFTs, we consider modular Hamiltonian evolution of states locally excited by a primary operator $\mathcal{O}(l)$ of conformal dimension h placed inside the interval $A = [a, b]$, i.e., $l \in A$. This state is defined as

$$|\psi(s)\rangle = \mathcal{N}e^{-iH_{mod}s}e^{-\epsilon H}\mathcal{O}(l)\,|0\rangle\,, \tag{6.55}$$

where $|0\rangle$ is the ground state of the entire system bipartite as $A \cup A^c$ and H_{mod} is the total modular Hamiltonian associated with A in this state. We remark that, again for the sake of simplicity, we only consider a chiral part of the 2d CFT. Note that the local operator is first smeared with the CFT Hamiltonian by an amount ϵ in Euclidean time such that the energy of the excitation is finite $E_{\mathcal{O}} \sim h/\epsilon$ and factor $\mathcal{N}$ is the normalisation of this initial state with the operator. The standard Hamiltonian evolution of these states has been extensively studied in the past [244–247] (see for corresponding spread complexity in Appendix D) but here we will be interested in the modular evolution instead.

Before we proceed, it is important to point that, since the operators $\mathcal{O}(l)$ are inserted in A, the actions of H_{mod} and $H_A \otimes 1_{A^c}$ on them are identical. Hence, our discussion in the following also holds for (6.55) with H_{mod} replaced by H_A and we will use them interchangeably in our formulas. In particular, the modular correlators that will be used in our return amplitudes (see below) are identical for these two modular evolutions.

Let us then recall a few basic facts about H_A. In the chiral 2d CFTs and for some particular states and bipartitions (e.g. when the CFT is defined either on the line or on the circle and is in its ground state), the modular Hamiltonian can be written as

$$H_A = 2\pi\int_a^b \beta_0(u)T(u)du, \qquad \beta_0(u) = \frac{1}{w'(u)}, \tag{6.56}$$

where $T(u)$ is the chiral component of the 2d CFT energy-momentum tensor and the weight function $\beta_0(u)$ (often called local inverse temperature) encodes the dependence on the state and of the

bipartition for the specific cases we are considering. For instance, for the ground state of a CFT on the line or on a circle of length L, we have respectively

$$w(u) = \log\left(\frac{u-a}{b-u}\right), \quad w(u) = \log\left(\frac{\sin[\pi(u-a)/L]}{\sin[\pi(b-u)/L]}\right). \tag{6.57}$$

The modular evolution generated by (6.56) for a primary operator $\mathcal{O}$ of conformal dimension h is

$$\mathcal{O}(s,u) \equiv e^{isH_A}\mathcal{O}(u)e^{-isH_A}, \tag{6.58}$$

and it can be written as [230, 231, 238]

$$\mathcal{O}(s,u) = \left(\frac{\beta_0(\xi(s,u))}{\beta_0(u)}\right)^h \mathcal{O}\big(\xi(s,u)\big), \tag{6.59}$$

where $\mathcal{O}(u)$ is the initial configuration of the field at $s = 0$ and $\xi(s,u)$ satisfies the following differential equation

$$\partial_s\xi(s,u) = 2\pi\beta_0(x)\,\partial_u\xi(s,u), \qquad \xi(0,u) = u. \tag{6.60}$$

The solution of this equation reads

$$\xi(s,u) \equiv w^{-1}\big(w(u) + 2\pi s\big), \tag{6.61}$$

in terms of $w(u)$ defined in (6.56) and its inverse function.
Then, the modular evolution (6.59) can be expanded in powers of s as follows

$$\mathcal{O}(s,u) = \sum_{n=0}^{\infty} \frac{(2\pi\, s)^n}{n!}\, \widetilde{\mathcal{O}}_n(u). \tag{6.62}$$

By employing (6.60), the first three (non-trivial) operators in this expansion are

$$\begin{aligned}
\widetilde{\mathcal{O}}_1(u) &= \beta_0(u)\,\mathcal{O}'(u) + h\,\beta_0'(u)\,\mathcal{O}(u), \\
\widetilde{\mathcal{O}}_2(u) &= \beta_0(u)^2\,\mathcal{O}''(u) + (2h+1)\beta_0(u)\beta_0'(u)\,\mathcal{O}'(u) \\
&\quad + h\big[h\,\beta_0'(u)^2 + \beta_0(u)\beta_0''(u)\big]\mathcal{O}(u), \\
\widetilde{\mathcal{O}}_3(u) &= \beta_0(u)^3\,\mathcal{O}'''(u) + 3(h+1)\beta_0(u)^2\beta_0'(u)\,\mathcal{O}''(u) \\
&\quad + \beta_0(u)\big[(3h^2+3h+1)\beta_0'(u)^2 \\
&\qquad\qquad + (3h+1)\beta_0(u)\beta_0''(u)\big]\,\mathcal{O}'(u) \\
&\quad + h\big[h^2\beta_0'(u)^3 + (3h+1)\beta_0(u)\beta_0'(u)\beta_0''(u) \\
&\qquad + \beta_0(u)^2\beta_0'''(u)\big]\mathcal{O}(u).
\end{aligned} \tag{6.63}$$

It is straightforward to write also $\widetilde{\mathcal{O}}_n(u)$ with $n > 3$, but their expressions are rather complicated to be reported here.
Clearly, the growth of the operator (in operator space) due to the modular evolution (6.58) is determined also by the weight function $\beta_0(u)$ occurring in the modular Hamiltonian (6.56) and its non-trivial derivatives provide additional contributions of the initial field configuration $\mathcal{O}(u)$ into $\widetilde{\mathcal{O}}_n(u)$. Indeed, setting $\beta_0(u) = \text{const}$ in (6.63) simplifies the expressions in a considerable way. On the other hand, the actual operator size and Krylov complexity [42] is usually computed based on the return amplitude that, after an appropriate choice of the inner-product, may become a two-point correlator (computable with (6.59)). Below, we will add to these intuitions by computing the spread complexity of (6.55) and find that it indeed depends on the local temperature $\beta_0(u)$.

Now, let us get back to the computation of the modular spread complexity of (6.55). The crucial ingredient is again the return amplitude that can be written as a special modular two-point correlator

$$S(s) = \frac{\langle \mathcal{O}^\dagger(0,u_1)\mathcal{O}(s,u_2)\rangle}{\langle \mathcal{O}^\dagger(0,u_1)\mathcal{O}(0,u_2)\rangle}, \tag{6.64}$$

which satisfies $S(0) = 1$ by construction. The insertion points of the operators in the initial state are (see Appendix D)

$$u_1 = l + i\epsilon, \qquad u_2 = l - i\epsilon. \tag{6.65}$$

The two-point correlators of the operators after modular flow can be found e.g. in [230,231,241,242]. Their general form is

$$\frac{\langle \mathcal{O}(s_1,u_1)\mathcal{O}(s_2,u_2)\rangle}{\langle \mathcal{O}(0,u_1)\mathcal{O}(0,u_2)\rangle} = \left(\frac{e^{w(u_1)} - e^{w(u_2)}}{e^{w(u_1)+\pi s_{12}} - e^{w(u_2)-\pi s_{12}}}\right)^{2h}, \tag{6.66}$$

where $s_{12} \equiv s_1 - s_2$ and $w(u)$ is defined in (6.56). The modular correlator (6.66) satisfies the KMS condition with inverse temperature $\beta_{KMS} = 1$. Using (6.66), we can write our return amplitude with general $w(u)$ as

$$S(s) = \left(\frac{e^{-\pi s}(1-B)}{e^{-2\pi s} - B}\right)^{2h}, \qquad B = e^{w(u_2)-w(u_1)}, \tag{6.67}$$

where B, via $w(u)$, depends on the details of the bipartition. This return amplitude again falls into the SL(2,R) symmetry class and we can derive universal Lanczos coefficients a_n and b_n for arbitrary B (or $w(u)$) and compute the modular spread complexity. To derive correct Lanczos coefficients, it is important to perform this computation for general B and take small ϵ in (6.65) only at the end (instead of first expansing $S(s)$ in ϵ and then trying to derive moments). The Lanczos coefficients are

$$\begin{aligned} a_n &= \frac{2\pi i(B+1)}{B-1}(n+h), \\ b_n &= \frac{2\pi\sqrt{B}}{\sqrt{-(B-1)^2}}\sqrt{n(n+2h-1)}, \end{aligned} \tag{6.68}$$

where the factor of i and the signs are chosen such that they are real for our physical insertion points (6.65). Finally, the spread complexity for finite ϵ can be written compactly for arbitrary $w(u)$ as

$$\mathcal{C}(s) = \frac{2h}{\sin^2\left(\frac{i(w(u_1)-w(u_2))}{2}\right)}\sinh^2(\pi s). \tag{6.69}$$

Comparing with (6.54), we see that the modular evolution with $\sinh^2(\pi s)$ is universally the same but the pre-factor in the present case involves the details of the insertion of the local operator $\mathcal{O}(l)$. Interestingly, the small ϵ expansion leads to

$$\mathcal{C}(s) = 2h\frac{\beta_0(l)^2}{\epsilon^2}\sinh^2(\pi s) + O(\epsilon^0), \tag{6.70}$$

which depends on $\beta_0(l)$, while the sub-leading orders contain also the derivatives of $\beta_0(l)$ (see (6.120)).

We remark that the universal dependence on s is consistent with the analyticity properties of the two-point function (6.66) and, since the KMS inverse temperature is $\beta_{KMS} = 1$ for the modular evolution, it can be understood as $\sinh^2(\pi s/\beta_{KMS})$. Moreover, at late modular time s, we find

$$C(s) \sim e^{\lambda_L^{mod}(s-s_*)}, \tag{6.71}$$

where the modular Lyapunov exponent λ_L^{mod} and scrambling time s_* for the local operator are determined by the local temperature of the modular Hamiltonian respectively as

$$\lambda_L^{mod} = 2\pi, \qquad s_* = \frac{1}{\pi} \log\left(\sqrt{\frac{2}{h}} \frac{\epsilon}{\beta_0(l)}\right). \tag{6.72}$$

It is interesting to point that, since the spatial bipartition is symmetric w.r.t. the center of the interval, the coefficient of the modular spread complexity $\beta_0(l)^2$ (or the scrambling time) is maximal for l in the middle of the entangling region $l = (a+b)/2$ whereas it is suppressed (vanishes) close to the boundary points of the entangling interval (see e.g. (6.110)). These are our main results in this section. Similarly to the bound of the Lyapunov exponent from Krylov complexity [42, 215] we conjecture that our modular exponent provides a bound on the modular chaos (see e.g. [96]).

For more intervals, general modular Hamiltonians become more complicated and non-universal so the analysis is beyond the scope of this work. Nevertheless, for the free massless Dirac fermion in the vacuum, the modular Hamiltonian of disjoint intervals and the dicorresponding two-point modular correlators are known explicitly [231]. More precisely, we can consider local fermion operator $\Psi(l)$ with $h = 1/2$, in either of the two intervals $[a_1, b_1] \cup [a_2, b_2]$, evolved with the modular Hamiltonian of this union region. The two-point function of modular flow of Ψ is known in this case [231, 241] (see also e.g. [242]) and, somewhat surprisingly, it turns out that the corresponding return amplitude can again be written as (6.67) with

$$w(u) = \log\left(-\frac{(u-a_1)(u-a_2)}{(u-b_1)(u-b_2)}\right). \tag{6.73}$$

This is sufficient to determine the modular spread complexity that, in the leading ϵ, becomes

$$\mathcal{C}(s) = \frac{\beta_0^{loc}(l)^2}{\epsilon^2} \sinh^2(\pi s), \qquad \beta_0^{loc}(u) = \frac{1}{w'(u)}, \tag{6.74}$$

in terms of (6.73). In fact the local part of this modular Hamiltonian (that also contains a non-local piece) of these two disjoint intervals can again be written in the form (6.56) with $\beta_0^{loc}(u)$; hence it governs the scrambling time.

6.7 Discussion and Outlook

In this work we have expanded Krylov complexity technology to the context of modular evolution. In particular, we have studied the relations between the entanglement spectrum, the Lanczos spectrum, and the notions of Krylov and spread complexity in various concrete examples. On one hand, this construction transparently shows why "entanglement is not enough" [3]. In fact, from the complexity perspective of this story, entanglement entropy is just the first Lanczos coefficient, namely $a_0 = S_E$. However, to understand the evolution of the wavefunction, and consequently of spread complexity, we need to know the full modular Lanczos spectrum. One can make an analogous statement about the TFD state, where the thermal entropy is related to the first Lanczos coefficient, but all the higher coefficients are also crucial to determine the evolution of complexity. On the other hand, the full Lanczos spectrum is obtained from the entanglement spectrum, providing concrete evidence that entanglement spectrum may be enough in certain scenarios.

From a different standpoint, from the Lanczos spectrum one can determine the entanglement spectrum up to degeneracies. In fact one can obtain all the moments of the modular Hamiltonian, and therefore the modular flow and all Renyi entropies. This way, the analysis of the modular Lanczos coefficients opens up a new window on the study of entanglement measures. As we show, the entanglement entropy is the first Lanczos coefficient while the capacity of entanglement is the

second. It would be interesting if also the higher n Lanczos coefficients contain similar information theoretic interpretations (perhaps along the lines of entanglement monotones [248]) and we leave this problem for the future investigation.

Then, we found that the modular growth of operators exhibits universal modular Lyapunov exponent $\lambda_L^{mod} = 2\pi$, related to the β_{KMS} and analyticity of the return amplitude, as well as the scrambling time sensitive to the local temperature of the CFT modular Hamiltonians. Going beyond our universal examples is certainly very important. For example, numerics for modular Hamiltonians in lattice models [249–251], would clarify the aforementioned relation between entanglement, complexity and modular chaos.

In addition, we remark that in our analysis of spread complexity, by definition, we work with the standard, natural inner product in Hilbert space. For Krylov complexity of operators [42], the freedom of choosing a different inner product (e.g. Wightman) provides a different modular Krylov complexity. A systematic study and better understanding of sensitivity of modular complexity and modular chaos to these choices is an interesting open problem.

Another important direction concerns holography. In holographic theories we expect the relation [233]

$$H_{mod}^{bdy} = \frac{\hat{A}_{ext}}{4G_N} + \hat{S}_{\text{Wald-like}} + H_{mod}^{bulk} \,, \tag{6.75}$$

for the boundary modular Hamiltonian in terms of bulk quantities ($\hat{A}_{ext}$ computes the area of the Ryu-Takayanagi [11] extremal surface $\mathcal{S}$ in the bulk and $\hat{S}_{\text{Wald-like}}$ can be expressed by expectation values of local operators on $\mathcal{S}$ [233]). In the semiclassical limit, we also have that in the entanglement wedge of region R_b the commutators $[H_{mod}^{bdy}, \phi_R]$ and $[H_{bulk}^{bdy}, \phi_R]$ are the same, for any local operator ϕ_R in R_b. Equivalently, in the low energy limit (in the code-subspace) we have

$$\phi_R(s) = e^{is\mathcal{L}_{bdy}}\phi_R = e^{is\mathcal{L}_{bulk}}\phi_R, \tag{6.76}$$

and the Krylov subspace with modular Liouvillians $\mathcal{L}_{bdy}$ or $\mathcal{L}_{bulk}$ will be the same. It is then interesting to study the complexity of bulk reconstruction joining the results developed in [234], the analysis of the Lanczos approach for generalized free fields described in [76], and the present techniques. This might naturally be extended to the complexity of extracting information from the black hole interior, following the islands construction [252].

Finally, it would also be interesting to extend our discussion to the analysis of the black hole microstates put forward in [199, 253], which are insightful examples of the so-called PETP states [254]. We hope to report on it in the near future.

Acknowledgements

We are grateful to Vijay Balasubramanian, Jan Boruch, Anatoly Dymarsky, Nima Lashkari, Sinong Liu, Joan Simon, Qingyue Wu and Claire Zukowski for many conversations on Krylov and spread complexity, and useful comments on the present draft. We also wish to thank Roberto Auzzi, Shira Chapman, Aldo Cotrone, Dongsheng Ge, Francesco Gentile, Mihail Mintchev, Giuseppe Mussardo, Giuseppe Policastro, Domenico Seminara. PC and DP are supported by NAWA "Polish Returns 2019" PPN/PPO/2019/1/00010/U/0001 and NCN Sonata Bis 9 2019/34/E/ST2/00123 grants. The work of JM is supported by CONICET, Argentina. ET is grateful to the Henri Poincaré Institute (Paris) and to the CTP at MIT (Boston) for hospitality and financial support during part of this work.

6.8 Appendix A: Lanczos coefficients and moments

Here we briefly explain how the Lanczos coefficients can be computed in a simple way (for relatively low n). By definition the moments of the return amplitude $S(t)$ (for real or modular times t or s

respectively) are related to the expectation value of the evolving Hamiltonian

$$\mu_n = \langle\psi_0|(iH)^n|\psi_0\rangle. \tag{6.77}$$

Also, by construction, the initial state is the first state in the Krylov basis $|\psi_0\rangle = |K_0\rangle$ in which H is tri-diagonal with Lanczos coefficients a_n on the diagonal and off-diagonal b_n's. The recursive algorithm is a version of a Markov process where μ_n for some fixed n is expressed only in terms of lanczos coefficients a_n and b_n with labels only up to that fixed n. This way, we can find polynomial relations between μ_n's and Lanczos coefficients directly from (6.77) by simply generating a tri-diagonal matrix H of at least size n, taking its n-th power and extracting the 00-element. This gives

$$\mu_1 = ia_0, \qquad \mu_2 = -a_0^2 - b_0^2, \tag{6.78}$$

and

$$\mu_3 = -i(a_0^3 + 2a_0b_1^2 + a_1b_1^2), \tag{6.79}$$

and so on. Then we just solve these relations remembering that b_n are the positive normalisations of the Krylov basis states. This gives

$$a_0 = -i\mu_1, \qquad b_1^2 = \mu_1^2 - \mu_2, \tag{6.80}$$

and

$$a_1 = i\frac{\mu_1^3 - 2\mu_1\mu_2 + \mu_3}{\mu_1^2 - \mu_2}, \tag{6.81}$$

$$b_2^2 = \frac{\mu_2^3 + \mu_3^2 + \mu_1^2\mu_4 - 2\mu_1\mu_2\mu_4 - \mu_2\mu_4}{(\mu_1^2 - \mu_2)^2}. \tag{6.82}$$

For simple return amplitudes (e.g. fixed by dynamical Lie algebra symmetry) we can often guess a general form after first several steps of this procedure and verify that it holds for higher n's.

The polynomial relations above are just part of the algorithm, and hold for arbitrary μ_n, a_n and b_n in the Lanczos algorithm. For a more detailed recursive derivation of the Lanczos coefficients in terms of the survival amplitude and moments, see [50].

6.9 Appendix B: GHZ vs W-states

To gain more intuition for modular spread complexity in multi-partite setups, here we give one more example in quantum mechanics with tripartite entangled states of class GHZ and W. Let us start from slightly more general states parametrized as

$$|GHZ\rangle_p = \sqrt{p}\,|000\rangle + \sqrt{1-p}\,|111\rangle\,, \tag{6.83}$$

and

$$|W\rangle_{p_1,p_2} = \sqrt{p_1}\,|100\rangle + \sqrt{p_2}\,|010\rangle + \sqrt{1-p_1-p_2}\,|001\rangle\,. \tag{6.84}$$

If we trace one of the spins in the first case, we end up with the reduced density matrix of the two spins with modular eigenvalues $\{p, 1-p\}$ (studied in the main text) and spread complexity

$$\mathcal{C}_p(s) = 4p(1-p)\sin^2\left(\frac{s}{2}\log\frac{1-p}{p}\right). \tag{6.85}$$

For the actual GHZ state with $p = 1/2$, the complexity vanishes.

On the other hand, tracing out the first or second spin in the W-class states brings again the reduced density matrix for two spins with modular eigenvalues $\{p_i, 1-p_i\}$, $i = 1, 2$ and modular

spread complexity (6.85) with p_i. Moreover, integrating over the third spin gives modular spectrum $\{p_1+p_2, 1-(p_1+p_2)\}$ and modular spread complexity

$$\mathcal{C}_{p_1,p_2}(s) = 4(p_1+p_2)(1-p_1-p_2)\sin^2\left(\frac{s}{2}\log\frac{1-p_1-p_2}{p_1+p_2}\right). \tag{6.86}$$

For the W-state with $p_1 = p_2 = 1/3$ we get

$$C(s) = \frac{8}{9}\sin^2\left(\frac{s}{2}\log(2)\right). \tag{6.87}$$

Clearly, the structure of entanglement is more susceptible to increase in complexity for the W-state that is not maximally entangled. It may be interesting to repeat this analysis more generally in the Hilbert space of 3 qubits for a state where we sum over all the basis vectors (probably numerically) with appropriate coefficients.

6.10 Appendix C: Lanczos coefficients for large W

In the main text, we have used the expansion of Lanczos coefficients coming from (6.39) for large values of W. This was done as follows. First we can compute several coefficients exactly and they read

$$a_0 = 2W, \quad a_1 = 2W+3, \quad b_1^2 = 2W, \quad b_2^2 = 4W+3, \tag{6.88}$$

however, from a_2 they get more complicated:

$$\begin{aligned} a_2 &= \frac{8W^2+30W+15}{4W+3}, \\ b_3^2 &= 6\frac{16W^3+48W^2+45W+12}{(4W+3)^2}, \end{aligned} \tag{6.89}$$

and so on. What we can do in practice is to take these exact solutions (up to some large say $n \sim 15$) and expand them for large W. From this we analytically find general answer for the first couple of orders

$$\begin{aligned} b_n &= \sqrt{2W}\sqrt{n}\left[1+\frac{3(n-1)}{8W}-\frac{9(n-1)^2}{128W^2}+O(W^{-3})\right], \\ a_n &= 2W\left[1+\frac{3n}{2W}-\frac{3n(n-1)}{16W^2}+O(W^{-3})\right]. \end{aligned} \tag{6.90}$$

We can see that for initial $n \ll W$ we have constant $a_n \sim 2W$'s and $b_n \sim \sqrt{2W}\sqrt{n}$. We can then solve the Schrodinger equation (6.15) for $\psi_n(s)$ in this regime by first noting that, for constant $a_n = a$, we can simply substitute

$$\psi_n(s) = e^{-ias}\phi_n(s), \tag{6.91}$$

with $\phi_s(s)$ satisfying

$$i\partial_s\phi_n(s) = b_n\phi_{n-1}(s) + b_{n+1}\phi_{n+1}(s). \tag{6.92}$$

More generally, for $b_n = \alpha\sqrt{n}$, the solution of this equation is simply the same as for the Heisenberg-Weyl algebra [154]

$$\phi_n(s) = \frac{(-i\alpha s)^n}{\sqrt{n!}}e^{-\frac{1}{2}\alpha^2 s^2}, \tag{6.93}$$

and the spread complexity grows quadratically in this initial regime with coefficient specified by the entanglement entropy $S_A = 2W$

$$\mathcal{C}(s) = \alpha^2 s^2 = 2W\, s^2 = S_A\, s^2. \tag{6.94}$$

When $n \sim W$, this expansion breaks down. Beyond this regime a transition to linear growth of complexity followed by a saturation is expected because the modular spectral form factor saturates. It would be very interesting to verify these two behaviours explicitly with numerics and we leave it as an important future problem.

6.11 Appendix D: Local Operator Evolution

In this appendix we provide more details for the spread complexity of local operators under modular evolution. In order to gain some intuition and perspective on this computation, we first evaluate spread complexity of a state locally excited by a primary operator and evolved with the Hamiltonian of a 2d CFT. This setup has been extensively studied before [244–247] as a milder version of a local quench.

6.11.1 Hamiltonian evolution

The starting point is a quantum state locally excited by a primary operator of conformal dimensions $\Delta = h + \bar{h}$ inserted in position l. The density matrix can be written as

$$\rho_0 = \mathcal{N} e^{-\epsilon H} \mathcal{O}(l) \left|0\right\rangle \left\langle 0\right| \mathcal{O}^\dagger(l) e^{-\epsilon H} \equiv \left|\psi_0\right\rangle\!\left\langle\psi_0\right|, \tag{6.95}$$

where we regulate (smear) the operator in Euclidean time with cut-off ϵ that makes the energy of the excitation finite ($E_{\mathcal{O}} \sim \Delta/\epsilon$). The normalisation is chosen such that $\mathrm{Tr}(\rho)$=1. In the previous studies, one was interested in the real time evolution of $\rho(t) = e^{-iHt}\rho_0 e^{iHt}$ and dynamics of entanglement or correlation functions in this protocol. Here we first focus on the spread complexity of the associated state

$$\left|\psi(t)\right\rangle = e^{-iHt}\left|\psi_0\right\rangle, \tag{6.96}$$

which can be computed from the return amplitude

$$S(t)^* = \mathrm{Tr}(\rho_0 e^{-iHt}) = \frac{\langle \mathcal{O}^\dagger(z_1,\bar{z}_1)\mathcal{O}(z_2(t),\bar{z}_2(t))\rangle}{\langle \mathcal{O}^\dagger(z_1,\bar{z}_1)\mathcal{O}(z_2(0),\bar{z}_2(0))\rangle}, \tag{6.97}$$

where we used complex coordinates $(z,\bar{z}) = (x+i\tau, x-i\tau)$ in which the insertion points are

$$\begin{aligned} &z_1 = l + i\epsilon, \quad \bar{z}_1 = l - i\epsilon, \\ &z_2(t) = l - i(\epsilon + it), \quad \bar{z}_2(t) = l + i(\epsilon + it). \end{aligned} \tag{6.98}$$

If we start from a two-point correlator in a CFT on a line

$$\langle \mathcal{O}^\dagger(z_1,\bar{z}_1)\mathcal{O}(z_2,\bar{z}_2)\rangle = z_{12}^{-2h}\bar{z}_{12}^{-2\bar{h}}, \tag{6.99}$$

with $z_{ij} = z_i - z_j$, the return amplitude becomes

$$S(t) = \left(1 - \frac{it}{2\epsilon}\right)^{-2\Delta}. \tag{6.100}$$

Analogously, using the two-point function in a CFT on a circle of size L

$$\langle \mathcal{O}^\dagger(z_1,\bar{z}_1)\mathcal{O}(z_2,\bar{z}_2)\rangle = \left(\frac{L}{\pi}\sin\left(\frac{\pi z_{12}}{L}\right)\right)^{-2h}(\ldots)^{-2\bar{h}}, \tag{6.101}$$

yields the return amplitude

$$S(t) = \left(\frac{\sinh\left(\frac{2\pi\epsilon}{L}\left(1 - \frac{it}{2\epsilon}\right)\right)}{\sinh\left(\frac{2\pi\epsilon}{L}\right)}\right)^{-2\Delta}. \tag{6.102}$$

It oscillates in time and, it reduces to (6.100) in the infinite L limit. Since both correlators are translationally invariant, the dependence on the insertion point l cancels. We could also introduce a UV cut-off ϵ_{UV} to the correlators but it would also cancel in the normalisation of the return amplitude. However, the return amplitude clearly depends on the operator regulator ϵ. Finally, we can also use the two-point correlator at finite temperature that is formally obtained by taking $L = i\beta$. We will then evaluate spread complexity from (6.102) and simply extract answers for the CFT on the line and at finite temperature by the above limit and the substitution. The last important remark is that, since we are interested in the evolution for all times, we should derive Lanczos coefficients and compute the spread complexity with finite ϵ and consider small ϵ only at the end of the computation. The other order of limits, taking first small ϵ expansion of $S(t)$ and then computing moments and Lanczos coefficients, is simply incorrect.

Following the algorithm [50, 214] we can compute the moments and extract Lanczos coefficients analytically

$$\begin{aligned} a_n &= \frac{2\pi\,(n+\Delta)}{L\tanh\left(\frac{2\pi\epsilon}{L}\right)}, \\ b_n &= \frac{\pi}{L\sinh\left(\frac{2\pi\epsilon}{L}\right)}\sqrt{n(n+2\Delta-1)}, \end{aligned} \tag{6.103}$$

and they correspond to the Lanczos coefficients governed by the SL(2,R) algebra labeled by the highes weight representation Δ that have a general form $a_n = \gamma(n+\Delta)$ and $b_n = \alpha\sqrt{n(n+2\Delta-1)}$ [154]. In that case the spread complexity is a general function of α and γ and becomes [50]

$$\mathcal{C}(t) = \frac{2\Delta}{1-\frac{\gamma^2}{4\alpha^2}}\sinh^2\left(t\sqrt{\alpha^2-\frac{\gamma^2}{4}}\right). \tag{6.104}$$

For our coefficients (6.103), we then have

$$\mathcal{C}(t) = 2\Delta\frac{\sin^2\left(\frac{\pi t}{L}\right)}{\sinh^2\left(\frac{2\pi\epsilon}{L}\right)}. \tag{6.105}$$

Clearly it oscillates in time, with period L, and in the small ϵ limit it is proportional to the energy of the excitation. For large L it reproduces the answer for the CFT on a line and grows quadratically with time

$$\mathcal{C}(t) = \frac{\Delta}{2\epsilon^2}t^2. \tag{6.106}$$

This is also consistent with the limiting behaviour of α and γ that satisfy $\gamma = 2\alpha$ when $L \to \infty$. Lastly, continuing to $L \to i\beta$ gives exponentially growing spread complexity

$$\mathcal{C}(t) = 2\Delta\frac{\sinh^2\left(\frac{\pi t}{\beta}\right)}{\sin^2\left(\frac{2\pi\epsilon}{\beta}\right)}. \tag{6.107}$$

At late times, this complexity is characterized by the same the Lyapunov exponent $\lambda = 2\pi/\beta$ as obtained from the Krylov complexity of the operator growth [42].

6.11.2 Modular Hamiltonian evolution

We can now move to the modular Hamiltonian evolution. The starting point will be the same as above i.e., locally excited state $|\psi_0\rangle$, but we consider the insertion l of the local operator $\mathcal{O}$ to be inside an interval $A = [a, b]$ and perform the evolution with the modular Hamiltonian of A of the initial state. For the vacuum excitations, the state of our interest will be

$$|\psi(s)\rangle = \mathcal{N}e^{-iH_A s}e^{-\epsilon H}\mathcal{O}(l)\,|0\rangle\,, \tag{6.108}$$

where we denote the modular time as s and evolution is done with the modular Hamiltonian of A

$$H_A^{\pm} = \pm 2\pi \int_a^b \beta_0(u_\pm) T_\pm(u_\pm) du_\pm, \qquad \beta_0(u) = \frac{1}{w'(u)}, \tag{6.109}$$

where $T_\pm$ are chiral and anti-chiral components of the energy-momentum tensor. For simplicity we will consider chiral operators and drop the $\pm$ notation. The kernel $\beta_0(u)$ stands for effective (inverse) temperature and e.g. for the CFT on a circle of size L and in its ground state is given by

$$\beta_0(u) = \frac{L}{\pi} \frac{\sin\frac{\pi(b-u)}{L} \sin\frac{\pi(u-a)}{L}}{\sin\frac{\pi(b-a)}{L}}. \tag{6.110}$$

Note that for the operator $\mathcal{O}(l)$ inside A we could evolve with the total modular Hamiltonian of A and its complement A^c defined as $H_{mod} = H_A - H_{A^c}$ that has the same action on operators in A but is a well-defined operator in continuum QFT. In 2d CFT, this total modular Hamiltonian for a single interval can be written in terms of global SL(2,R) generators (see e.g. [17, 243])

$$H_{mod} = \sigma_{-1} L_{-1} + \sigma_0 L_0 + \sigma_1 L_1 + a.c., \tag{6.111}$$

where the coefficients σ_i depend on the end-points of the interval and the size (or temperature) and $a.c.$ stands for the anti-chiral part in terms of $\bar{L}_n$'s. For example, the coefficients for a single interval $A = [a, b]$ in CFT on a circle of size L are

$$\sigma_0 = -2\pi \cot\frac{\pi(b-a)}{L}, \qquad \sigma_{\pm 1} = \frac{2\pi \cot\frac{\pi(b-a)}{L}}{e^{\pm\frac{2\pi i}{L}b} + e^{\pm\frac{2\pi i}{L}a}}. \tag{6.112}$$

Generally, these types of SL(2,R) Hamiltonians describe inhomogeneous 2d CFTs and can be understood geometrically as different quantisation (than e.g. the usual radial used in CFTs).

The important ingredient is again the return amplitude

$$S(s) = \langle\psi_0| e^{iH_A s} |\psi_0\rangle = \frac{\langle \mathcal{O}^\dagger(0,u_1)\mathcal{O}(s,u_2)\rangle}{\langle \mathcal{O}^\dagger(0,u_1)\mathcal{O}(0,u_2)\rangle}, \tag{6.113}$$

where $\mathcal{O}(s,u) \equiv e^{isH_A}\mathcal{O}(u)e^{-isH_A}$ is the chiral flow of the operator $\mathcal{O}(u)$ and $u_1 = l + i\epsilon$ and $u_2 = l - i\epsilon$. The relevant two-point correlators of the operators after modular flow can be found e.g. in [235]. Their general form is given in (6.66) and they satisfy the KMS condition with periodicity $s + i$ so are analytic on a strip of size $\beta_{KMS} = 1$. This way, we can write our return amplitude as

$$S(s) = \left(\frac{e^{-\pi s}(1-B)}{e^{-2\pi s} - B}\right)^{2h}, \qquad B = e^{w(u_2)-w(u_1)}, \tag{6.114}$$

where B depends on the details of the bipartition via $w(u)$. Interestingly, we can show that moments of this amplitude again correspond to the SL(2,R) Lanczos coefficients that are real, and can be written in terms of B as

$$\begin{aligned} a_n &= \frac{2\pi i(B+1)}{B-1}(n+h), \\ b_n &= \frac{2\pi\sqrt{B}}{\sqrt{-(B-1)^2}}\sqrt{n(n+2h-1)}. \end{aligned} \tag{6.115}$$

Equivalently, one can obtain these coefficients also by applying the procedure described in [53] to the modular correlators (with the convention that our b_n corresponds to b_{n+1} in that work; for us

$b_0 = 0$).
This allows us to read off α and γ as before and check that they satisfy

$$\alpha^2 - \gamma^2/4 = \pi^2, \quad 1 - \frac{\gamma^2}{4\alpha^2} = \frac{-(B-1)^2}{4B} = \sin^2\left(\frac{iw_{12}}{2}\right), \tag{6.116}$$

with $w_{12} = w(u_1) - w(u_2)$. For our insertion points $u_1 = l + i\epsilon$ and $u_2 = l - i\epsilon$, we then find

$$1 - \frac{\gamma^2}{4\alpha^2} = w'(l)^2\epsilon^2 + O(\epsilon^4). \tag{6.117}$$

The coefficient of the first term is given by the effective temperature evaluated at the operator's insertion point $w'(l) = 1/\beta_0(l)$. Combining the two identities (6.116), we get the universal result for the modular spread complexity of local operators for small ϵ quoted in the main text

$$\mathcal{C}(s) = 2h\frac{\beta_0(l)^2}{\epsilon^2}\sinh^2(\pi s). \tag{6.118}$$

For finite ϵ we simply get

$$\mathcal{C}(s) = \frac{8h\,B}{-(B-1)^2}\sinh^2(\pi s) = \frac{2h}{\sin^2\left(\frac{iw_{12}}{2}\right)}\sinh^2(\pi s). \tag{6.119}$$

We can expand it in ϵ as (suppressing l dependance)

$$\mathcal{C}(s) = \left[\frac{\beta_0^2}{\epsilon^2} + \frac{1 + 2\beta_0'^2 - \beta_0\beta_0''}{3} + O(\epsilon^2)\right] 2h\sinh^2(\pi s), \tag{6.120}$$

so we see that the sub-leading corrections also depend on derivatives of $\beta_0(l)$.
The finite ϵ expression can be also written explicitly in our examples, e.g. for finite size we have

$$\frac{1}{\sin^2\left(\frac{iw_{12}}{2}\right)} = \frac{4\left|\sin\left(\frac{\pi(b-l+i\epsilon)}{L}\right)\right|^2\left|\sin\left(\frac{\pi(l-a+i\epsilon)}{L}\right)\right|^2}{\sin^2\left(\frac{\pi(b-a)}{L}\right)\sinh^2\left(\frac{2\pi\epsilon}{L}\right)}. \tag{6.121}$$

The finite temperature case is recovered by $L = i\beta$ and for the CFT on a line and in the vacuum we simply get

$$\frac{1}{\sin^2\left(\frac{iw_{12}}{2}\right)} = \frac{((b-l)^2+\epsilon^2)((l-a)^2+\epsilon^2)}{(b-a)^2\epsilon^2}. \tag{6.122}$$

Chapter 7

Conclusion

In this final chapter I will summarize the main points of the results presented in the preceding articles and explain how they fit into the bigger picture of the field. Of course our understanding of complexity is far from complete. Its role in holography, its relation to quantum chaos, its universality, and many other of its aspects are topics we have just begun to understand. This leaves a lot of room for questions, speculations and potential new projects some of which I will detail in the following sections.

7.1 Summary of results

This thesis has been mainly devoted to the study of Krylov complexity and its relationship with symmetry. It is clear that this novel measure of complexity is becoming a staple of the field and the works contained herein have contributed both in bettering our conceptual understanding and providing a practical framework of computation with a wide range of applicability.

In chapter 3 I presented our main work regarding the connections between Krylov complexity, symmetry and geometry. These results set the groundwork for much of the research that ensued as they provided a novel framework for the computation of Krylov complexity and associated quantities. Apart from elucidating some of the pre-existing results of the literature we were able to capture some essential features of the Krylov methods, such as the existence of a rank-1 Lie algebra we dubbed the complexity algebra. This structure has produced many results, but on equal measure it has created questions regarding the limitations of the Krylov space in describing the dynamics of a given quantum system.

In chapter 4 I included my single-authored work on using quantum information quantities to probe the entanglement of operator growth. I utilized the Krylov space structure of systems with symmetry and in particular the two-mode coherent states arising from the latter in order to improve our understanding of the processes taking place during a system's evolution. I concluded that quantities such as the Krylov entropy, entanglement negativity and capacity of entanglement do in fact contain similar and perhaps additional information compared to Krylov complexity. These results and are limited due to their reliance on the formulation of two-mode states, but they highlight the multifaceted information that the Krylov space can provide. Moreover, the interpretation, while clear from a mathematical point of view, remains physically unclear.

Chapter 5 is devoted to the implementation of the techniques developed in previous chapters on systems with the Schrödinger symmetry. We find that while mapping the complexity algebra to the Schrödinger algebra does not appear feasible, we can still define a natural notion of complexity based on the Fock basis that the system comes equipped with. This measure reduces to Krylov complexity for SL(2,R) and HW in the appropriate limits and according to our analysis it effectively captures their interplay in the intermediate regime. These results illustrate that even for systems

we lack the control required to compute Krylov complexity analytically, we can still make progress by defining related measures that for all practical purposes describe the dynamics of the system in the same way.

Finally, in chapter 6 I presented our work on defining and computing the spread complexity of states under modular evolution. We find that under this definition a connection between complexity and the entanglement spectrum emerges, suggesting that the latter suffices for the description of aspects of the dynamics that entanglement entropy by itself cannot. While we illustrated this in a variety of examples, there are still many promising setups in which these ideas can be further tested.

7.2 Open questions and future directions

Since its introduction, Krylov complexity has become a very popular tool in the study of dynamics of complex systems. More generally though, the study of complexity is a rapidly evolving field with a deluge of updates and breakthroughs every year. This means that in the time between the works that comprise this thesis were published and now, there have been important advances, which are summarized in the following two review articles [255, 256], the former focusing on Krylov complexity in particular and the latter on the field of complexity as a whole. That said, there are still many gaps in our understanding. Part of the reason is definitely the fact that there is no unique approach to complexity, though there are important questions that ought to be answered even in the context of a specific measure. That is, the relation of complexity and chaos, the notion of scrambling, thermalization, the holographic interpretation of complexity etc.

In more concrete terms, one would for example like to understand the relationship between Krylov complexity and OTOCs, the former being related to a two-point function (the autocorrelator) and the latter to a 4-point function. While there is a general understanding of their behaviour in terms of [42] where the authors posit that Krylov complexity serves us an upper bound of OTOC and despite more recent progress in [257] there is still a lack of understanding regarding the different aspects of the dynamics that they capture. Clearly the OTOC is a finer probe, so it saturates faster when the system is strongly interacting. It remains to be seen what one can then infer for the late time behaviour of a system using Krylov methods that the OTOC is not sensitive to, or whether there is no such advantage over it.

In the same light, it is not completely understood how Krylov complexity is related to other measures. Perhaps the most intriguing relation is to Nielsen complexity whose geometric formulation is suggestively similar to the results presented in this thesis. This is apparent in [34] and even though it has been investigated to a degree in [198, 258] a clear geometric framework containing both of them is still lacking. While there is no evident relation between Krylov complexity and path integral optimization it is intriguing to think how the two describe different aspects of a system and what conclusions they lead to. A comment of speculative nature that I can offer is that perhaps there exists a way to redefine Krylov complexity in terms of preparing a state using path integrals rather than tracking time evolution. In this way it would produce a notion of complexity which is in the spirit of path integral optimization and can be thought perhaps as a complexity of formation. The main challenge of course is the reformulation of the Lanczos algorithm and the generation of a Krylov subspace in terms of a Euclidean evolution, which is by its definition non-unitary. While there are several results that have been mentioned previously regarding the non-unitary evolution of quantum systems, there appears to be a qualitative difference between Lindbladian and Euclidean evolution.

Related to that is the connection between Krylov complexity and circuit models where one can quantify the notion of operator growth through its size, in the spirit of [71, 94]. Once again there is a hint that these two notions are related, but yet again a concrete framework is lacking. This is potentially a very fruitful avenue of further exploration given the claims regarding the holographic duality between the rate of change of complexity and the momentum of a particle falling into a

black hole [259]. While steps have been taken to verify this conjecture both using Krylov and circuit methods, namely in [94] and [58] where the setups are strikingly similar, there is no unifying picture between the two. This further enhances the intuition (and I should stress that this is merely speculation on my part) that possibly many of these measures of complexity contain similar information about the system, just packaged in a different way. Perhaps indeed, Krylov complexity is a "proper" measure, simply describing the dynamics from the perspective of the system itself while other measures are related to it by a change of perspective.

One of the main directions that has been pursued relates to finding the holographic dual of Krylov complexity, which could potentially verify some of the claims surrounding the late time evolution of holographic systems. Initially, some results of this nature were presented in [190] positing that the Krylov complexity computed in the triple-scaling limit of SYK is equal to the bulk wormhole length in the dual JT gravity theory. This is based on the formulation of the bulk Hilbert space of double-scaled SYK (DSSYK) in terms of chords [206]. The key ingredient of this approach is the identification of the chord and Krylov bases, which is suggested by the tridiagonal form of the transition matrix of the former. It was recently followed up in [260] where the authors make the claim that Krylov complexity is generally equal to the expectation value of a length operator, as defined in the dual of DSSYK. A different holographic approach was given in [58], where the authors find that the rate fo change of spread complexity of CFT states excited by local operators on the boundary is equal to the momentum of a particle in the bulk. Both of these approaches are consistent with different expectations for holographic complexity, but at the time of writing it is unclear whether they are related through some underlying principle or not.

Expanding on the holographic implications of Krylov complexity are the directions of flat space and dS holography. These versions of holography are much less mature compared to the AdS/CFT correspondence, but given that their symmetry structure is known to a large extent, one can be hopeful that our methods are applicable. Our work in chapter 5 already provides some hints in that direction by exploring the Schrödinger symmetry which in 1+1 dimensions is closely related to the Galilelean and Carroll symmetries. This provides a path, although a rather winding and crooked one, to the conformal Carroll symmetries relevant for flat space holography. It is our hope that we can shed light on this topic in the near future. When it comes to dS holography, at the time of writing, there are no clear routes or conclusive results, which makes it all the more intriguing to think about, despite its inherent difficulties.

A somewhat less explored avenue is the realization and measurement of complexity in experiments. I believe this to be a direction of great importance moving forward, as the advent of quantum computing platforms allows for a number of increasingly sophisticated experiments. There are already protocols regarding the measurement of OTOCs (see for instance [261]), so it would be interesting and fruitful to employ similar means to get a more physical picture of Krylov complexity through experiment.

In conclusion, the study of the dynamics of complex systems using the methods detailed in this thesis is still at its infancy. There are many aspects of the phenomena involved that are for the time being a mystery which we are tasked with unraveling.

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