UNIWERSYTET WARSZAWSKI

ROZPRAWA DOKTORSKA

Topologicznie trywialne i nietrywialne extremalne, izolowane horyzonty

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“*W połowie drogi naszego żywota*
W pośród ciemnego znalazłem się lasu,
Albowiem z prostej zbłąkałem się ścieżki.
O jakże ciężko teraz wypowiedzieć,
Jak ten las dziki, gęsty i ponury; —
Wspomnienie samo wznawia strach okropny,
Że śmierć zaledwie okropniejszą będzie . . .”

Piekło: Pieśń I autorstwa Dantego Alighieri,
przełożona przez Antoniego R. Stanisławskiego [1]

Streszczenie

Czarne dziury znajdują się w centrum uwagi relatywistyki i całej fizyki teoretycznej, jednakże istniejące rozwiązania równań Einsteina nie tylko nie opisują wielu ciekawych, fizycznych aspektów tychże obiektów, ale stwarzają też szereg problemów praktycznych i konceptualnych. Te ostatnie dotyczą zasady lokalności, jako że żaden fizyk nie jest w stanie obserwować całego wszechświata jednocześnie. Problemy praktyczne natomiast biorą się z ogólności wielu podejść, które nie zapewniają wystarczającej struktury matematycznej do przeprowadzania obliczeń. Aby poradzić sobie z tymi trudnościami powstało wiele różnych sposobów opisu czarnych dziur i ich horyzontów. Ten rozpatrywany przez nas nazywa się quasi-lokalnym.
Nakładamy więzy na geometrię hiperpowierzchni zerowej – znikanie jej ekspansji i niezależność jej kowariantnej koneksji od czasu. Takie ogólne pojęcie stabilności dostatecznie dobrze opisuje rzeczywiste czarne dziury. Więzy te pozwalają opisać horyzont za pomocą metryki jego cięcia, pewnej jednoformy i tensora energii-pędu. Jeśli grawitacja powierzchniowa takiej hiperpowierzchni znika (co zahacza o wiele złożonych problemów matematycznych), to nazywamy taki horyzont ekstremalnym i izolowanym. Przypadek ten jest znów ciekawy z punktu widzenia nie tylko matematyka, ale i fizyka, gdyż wiele supermasywnych, obracających się czarnych dziur jest blisko ekstremalności. Powyższa konstrukcja może być przeprowadzona w sposób abstrakcyjny, poprzez zdefiniowanie horyzontu jako abstrakcyjnej, trójwymiarowej rozmaitości.
Równania Einsteina-Maxwella indukują na horyzoncie układ równań, nazywanych równaniami geometrii przyhoryzontowej. Pomimo, że topologia horyzontu (czyli struktura jego wiązki) może być skomplikowana, to równania te są od niej niezależne (pomijając warunki brzegowe). W tej rozprawie prezentujemy rozwiązania równań geometrii przyhoryzontowej. Udało nam się znaleźć te rozwiązania dla przypadku trywialnych topologii $\mathbb{R} \times \mathbb{S}_2$ i $\mathbb{R} \times \mathbb{T}_2$, które wyczerpują wszystkie interesujące przypadki (zgodnie z twierdzeniem o sztywności horyzontu). Interesująca jest również analiza bardziej egzotycznych topologii ($\mathbb{S}_3$), które odpowiadają nietrywialnym wiązkom. Rzeczywiście, tego typu struktury znaleźć można w czasoprzestrzeniach z parametrem NUT-a. W tym przypadku też rozwiązaliśmy równania geometrii przyhoryzontowej, z założeniem o symetrii osiowej.
Następnie znaleźliśmy zanurzenie wspomnianych ekstremalnych, izolowanych horyzontów w znanych czasoprzestrzeniach czarnodziurowych Reissnera-Nordströma-(anty-)de Sittera, Kerra-Newmana-(anty-)de Sittera i Plebańskiego-Demiańskiego.

UNIVERSITY OF WARSAW

DOCTORAL THESIS

Topologically trivial and non-trivial extremal, isolated horizons

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"*Midway upon the journey of our life*
I found myself within a forest dark,
For the straightforward pathway had been lost.
Ah me! how hard a thing it is to say
What was this forest savage, rough, and stern,
Which in the very thought renews the fear.
So bitter is it, death is little more ..."

Inferno: Canto I by Dante Alighieri,
translated by Henry W. Longfellow [2]

Abstract

Black holes remain in the limelight of relativity and theoretical physics as a whole, however, not only do the existing solutions of Einstein's equaitions fail to describe many interesting physical situations, but they can also pose practical and conceptual problems. The latter pertain in particular to the principle of locality, as no physicist is able to observe the whole universe, while the former have to do with the fact, that many general approaches fail to supply enough structure to conduct much of mathematical analysis. In order to help alleviate the latter, one can use many different description of a black hole or its horizon. The one we use in this thesis is so-called quasi-local description.

We impose geometrical constraints on the null hypersurface – vanishing expansion and independence of its intrinsic covariant connection from time. Such a general notion of stability is suitable to approximate the description of black holes. These constraints are enough to describe this hypersurface by the metric of its cross-section, a certain one-form and energy-momentum tensor. If such a hypersurface has vanishing surface gravity (thus touching on interesting and complex mathematical issues), we call it an extremal isolated horizon. It is again interesting case, not only mathematically, but also physically, as many supermassive, spinning black holes appear to be close to such extremality. The construction above can be carried out in an abstract way, assuming that horizon is an abstract three-dimensional manifold.

Einstein-Maxwell equations induce on a horizon a system of equations – near-horizon geometry equations. While the topological structure (or bundle structure) of such a horizon can be complicated, these equations are independent of that (bar boundary conditions). In this dissertation, we present solutions to these equations. We were able to find solutions for the horizons with trivial topology $\mathbb{R} \times \mathbb{S}_2$ and $\mathbb{R} \times \mathbb{T}_2$, which (as per rigidity theory for extremal isolated horizons) exhausts all interesting horizons for this topology. Theanalysis of more exotic topologies ($\mathbb{S}_3$), which corresponds to non-trivial bundle, is also very interesting. Indeed such structures are found in spacetimes with NUT parameter. Here too we have solved near-horizon geometry equations, assuming axial symmetry.

Consequently we were able to find embeddings of such extremal, isolated horizon into known black hole solutions: Reissner-Nordström-(anti-)de Sitter, Kerr-Newman-(anti-)de Sitter and Plabański-Demiański spacetimes.

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List of Abbreviations

EE	**E**instein's **E**quations
IH	**E**xtremal **I**solated **H**orizon
EIH	**I**solated **H**orizon
NEH	**N**on-**E**xpanding **H**orizon
NHG	**N**ear-**H**orizon **G**eometry
NHGE	**N**ear-**H**orizon **G**eometry **E**quation
NP	**N**ewman-**P**enrose
RN	**R**eissner-**N**ordström spacetime
RNadS	**R**eissner-**N**ordström-(**a**nti-)**d**e **S**itter spacetime
KNadS	**K**err-**N**ewman-(**a**nti-)**d**e **S**itter spacetime
PB	**P**lebański-**D**emiański spacetime

List of Symbols

$\mathcal{M}$	spacetime – n-dimensional Lorentzian manifold
α, β, $\gamma\ldots$	indices of the spacetime $\mathcal{M}$ (not necessarly in holonomic basis)
g, $g_{\mu\nu}$	metric of $\mathcal{M}$
$\mathcal{H}$	null, codimension 1 hypersurface in $\mathcal{M}$
a, b, $c\ldots$	indices of the null hypersurface $\mathcal{H}$
q, q_{ab}	metric of $\mathcal{H}$
ℓ	null normal generator of $\mathcal{H}$
$T(\mathcal{H})$	tangent bundle of $\mathcal{H}$
$\theta^{(\ell)}$, $\kappa^{(\ell)}$, $\omega^{(\ell)}$	surface gravity, shear and vorticity of a congruence generated by ℓ
$\mathcal{S}$	cross-section of $\mathcal{H}$
A, B, $C\ldots$	indices of the cross-section of null hypersurface $\mathcal{S}$
q, q_{AB}	metric of $\mathcal{S}$
$\omega^{(\ell)}$, ω	rotation one-form on $\mathcal{S}$
R	Ricci scalar or the radius of a cross-section $\mathcal{S}$
K	Gaussian curvature of $\mathcal{S}$
$R_{\alpha\beta\gamma\delta}$	Riemann tensor
$R_{\alpha\beta}$	Ricci tensor
R	Ricci scalar
Λ	cosmological constant
F	Maxwell tensor
A	electromagnetic potential
Φ_1	complex function describing Maxwell tensor in Newman-Penrose formalism
c_1, c_2, $\ldots$	constants of integration

Many of the symbols above are used for different quantities because of widespread custom in the existing literature. We hope that the context we have given will suffice to differentiate between them.

Chapter 1

Introduction

Isolated horizons – models of black hole horizon – are null hypersurfaces with vanishing expansion and intrinsic connection independent of the flow of its generators. They can be thought of as both abstract and embedded manifolds, and the connection between one and the other is of interest in physical considerations. For a given spacetime they appear as Killing horizons. Extremal, isolated horizons are isolated horizons with additional, special property. Einstein-Maxwell equations induce constraints on their metrics called near-horizon geometry equations, that can be solved to all geometrical quantities describing such horizons. While for trivial cases such horizons have product topology (that is when principal bundle is trivial), there are more exotic ones, when the generators of hypersurface are not the fibres of the bundle of a horizon. In this thesis we consider both cases. We find solutions to near-horizon geometry equations in both aforementioned cases, and embeddings in known spacetimes, thus connecting abstract and embeddable horizons. Below we present the structure of the whole thesis.

In the Chapter 2 we have introduced the most important definitions for this thesis, such as non-expanding and isolated horizons and the extremality of isolated horizons. We describe those horizons both as abstract manifolds and embeddable hypersurfaces. We elucidate on extremality/degeneracy of the horizon, and also present near-horizon geometry equations. We describe an extremal isolated horizon with its metric and rotation one-form ω. In the Chapters 3 and 4 we have described the main results of this thesis.

In the Chapter 3 we were considering topologically trivial extremal, isolated horizons, in the presence of electromagnetic field and cosmological constant; and an appropriate boundary conditions. We were able to solve near-horizon geometry equation for horizons of topology $\mathbb{R}\times\mathbb{S}_2$, and embedd non-rotating ones into Reissner-Nordström-(anti-)de Sitter spacetime, and rotating ones into Kerr-Newman-(anti-)de Sitter spacetime. For horizons of topology $\mathbb{R}\times\mathbb{T}_2$ we have shown, that the only solutions to near-horizon geometry equation is the trivial one – flat metric, and vanishing ω. That, as per the theorem on the rigidity of extremal, isolated horizons, exhaust all possible topologies for non-trivial horizon.

In the Chapter 4 we consider extremal, isolated horizons with non-trivial topology $\mathbb{S}_3$. This non-triviality nevertheless leads us to the same near-horizon equation, but with different boundary conditions. We were able to find the axisymmetric solution to the problem, and embedd it into Plebański-Demiański class of spacetime. We have also considered the limiting solutions, for vanishing acceleration.

In the last Chapter we summarize our findings.

In the Appendix A we have rederived near-horizon geometry equation for a spacetime, defined in the neighbourhood of a degenerate Killing horizon.

In the Appendix B we have introduced Newman-Penrose formalism.

In the Appendix C we have presented the detailed method of solving one of near-horizon geometry equations, describing the metric on a horizon.

Chapter 2

Isolated Horizons

In this chapter we recall some basic definitions, such as non-expanding horizon, isolated horizon and extremal isolated horizon. We write down so-called near horizon geometry equations, which are the focal point of the whole work. We also take a look the basic properties of some known black hole spacetimes, and their horizons.

2.1. Introduction

Since their conception at the beginning of the twentieth century, as solutions to Einstein's Equations, black holes become one of the most interesting objects of research in general relativity, and in theoretical physics in general. While the first examples of these objects were found to exist in Schwarzschild spacetime in 1958 [3], and later in other solutions to Einstein's equations, there have been created many *more general* definitions of black holes, and their horizons. In asymptotically flat spacetimes (although this notion can be generalized beyond flatness alone), they can be thought of as regions, where no null curve ever reaches future null infinity [4, 5]. They can be defined in terms of the marginally outer trapped horizon (see [6] and references therein), and such an approach resulted in the famous theorem by Penrose [7]. One can also use notions of the Killing horizon (see e.g. [8]).

Research inspired by black holes has dominated several areas of gravitational physics since the early seventies, and there have been many results pertaining to both black-hole spacetimes, and the more general definitions we have just mentioned. We will now present some of them (see [9] and references therein).

There are simple laws characterizing and connecting their energy, area, surface gravity, angular velocity and momentum, and electrostatic potential and charge (see [10], and review in [11]). They seem parallel to classical laws of equilibrium thermodynamics, but in reality, they are emerging as thermodynamical laws governing a quantum system [12]. They can be furthermore connected to string theory, for some supersymmetric, four- and five-dimensional extremal black holes [13–16]. These results were established for extremal black holes and lay on the assumption of the uniqueness of these spacetimes, which is well-established in four dimensions [17]. The entropy of the extremal black hole was succesfully calculated from microstates of field on its horizon [18–20]. As it turns out, this state function is, to a considerable degree, independent of the content of particular string theory defined on its horizon [21–26]. AdS/CFT theory [27–30] asserts a fully non-perturbative equivalence of classical gravity in anti-de Sitter spacetimes and the strongly coupled regime in the conformally invariant quantum field theory in one lower spatial dimension. This is an explicit realization of the *holographic principle* [31] underlying quantum gravity. It provides a precise framework to analyze the microscopic description of black holes in terms of well-defined quantum field theories. Such gauge/gravity dualities are believed to hold more generally [32, 33]. Ideas from the gauge/gravity duality have been used to model certain phase transitions in condensed-matter systems, such as superfluids or superconductors [34, 35]. The key motivation for this line of research, in contrast to the above, is to use knowledge of the gravitational system to learn about strongly coupled field theories. As our understanding of gravity, and fundamental interactions is not yet full, we can expect black holes (extremal ones in particular [36]) to be useful in testing and elucidating new theories. Modern astronomy tells us that many accreting stellar-mass black holes are spinning close to extremality (see e.g. [37]).

The classification of higher-dimensional stationary black-hole solutions to Einstein's equations is a major open problem in higher dimensional general relativity (see [17, 38, 39], and references therein). Its study is of intrinsic value for both a physicist and mathematician alike.

The fruitfulness of the multitude of approaches cannot be downplayed, nonetheless, they still display some conceptual and computational problems. Some assume the complete knowledge of the whole of spacetime, which is not only practically unfeasible but also against the notion of *locality* of physics, that has been deeply ingrained into modern science. Others, on the other hand, do not supply us with enough structure to put forward and interesting theorems, or to conduct meaningful calculations. We will therefore elucidate and employ one more description of a black hole horizon, called *quasi-local approach.*

2.2. Basic definitions

2.2.1 Non-expanding horizons

Let us consider spacetime $(\mathcal{M}, g)$, where g has signature $(-,+,+,+)$, satisfying **Einstein's equations (EE)**

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa_0 T_{\mu\nu}\,. \tag{2.1}$$

Let us also make some assumptions about momentum-energy tensor. It is said to obey **dominant energy conditions** if for every future-pointing, causal v, the vector $-T_{\mu\nu}v^{\mu}$ is also future-pointing and causal. It is a rather mild constraint and is satisfied by Maxwell field, Klein-Gordon field, Yang-Mills field, and many others.

A null hypersurface $\mathcal{H} \subset \mathcal{M}$ is a codimension one hypersurface, with null normal vector field ℓ. The restriction of a metric g to $\mathcal{H}$ will be denoted q_{ab}, with signature $(0,+,+)$. Vector field ℓ is tangent to $\mathcal{H}$ and it spans the null direction of a spacetime metric:

$$\ell^a q_{ab} = 0\,. \tag{2.2}$$

Vector field ℓ – generator of null hypersurface – is defined up to multiplication by a smooth function, defining an equivalence class $[\ell]$:

$$\ell \sim \alpha\ell \qquad \forall\alpha : \alpha \in C^{\infty}(\mathcal{H})\,,\; \alpha \neq 0\,. \tag{2.3}$$

Vector field ℓ also satisfies the geodesic equation

$$\ell^{\nu}\nabla_{\nu}\ell^{\mu} = \kappa^{(\ell)}\ell^{\mu} \tag{2.4}$$

on $\mathcal{H}$, where $\kappa^{(\ell)}$ is called a **surface gravity**. That makes $\mathcal{H}$ into a null congruence, or rather the whole family of congruences, as we ought to always use equivalence class $[\ell]$. On such a null congruence one can define the usual optical scalars: an expansion $\theta^{(\ell)}$, shear $\sigma^{(\ell)}$ and vorticity $\omega^{(\ell)}$. These scalars do depend on the choice of ℓ. In particular, the rescaling (2.3) changes expansion in the following way

$$\theta^{(\alpha\ell)} = \alpha\theta^{(\ell)} \qquad \forall\alpha : \alpha \in C^{\infty}(\mathcal{H})\,,\; \alpha \neq 0\,. \tag{2.5}$$

A degenerate metric q on $\mathcal{H}$, also called its first fundamental form, does not have a unique inverse. We define the latter as any tensor q^{ab} such that

$$q^{ab}q_{ac}q_{bd} = q_{cd}\,. \tag{2.6}$$

These tensors also constitute an equivalence class $[q^{-1}]$:

$$q^{ab} \sim \left(q^{ab} + \ell^{(a}X^{b)}\right) \qquad \forall X \in T(\mathcal{H})\,. \tag{2.7}$$

A covariant derivative on $\mathcal{H}$ is not uniquely defined by q, as opposed to intrinsic covariant derivatives on time- or space-like hypersurfaces. Nevertheless, if the pull-back of $\nabla_{\mu}\ell_{\nu}$ to $\mathcal{H}$ vanishes then it preserves $T(\mathcal{H})$, and defines a connection ∇_a on $\mathcal{H}$. Such ∇_a is torsion-less and it preserves the metric

$$\nabla_a q_{bc} = 0\,. \tag{2.8}$$

The Raychaudhuri equation, together with the null energy conditions, ensure that it is satisfied if

$$\theta^{(\ell)} = 0\,. \tag{2.9}$$

The null hypersurface with the properties described above can be used to model the horizon of the black hole, and so we will now state two definitions characterizing such a hypersurface. The first one describes a horizon embedded in the spacetime:

Definition 2.2.1 *[40, 41] Let there be a null, codimension one hypersurface $(\mathcal{H}, q)$, embedded in four-dimensional spacetime $(\mathcal{M}, g)$ generated by a nowhere-vanishing, null, and future-directed vector field ℓ; such that:*

1. *The expansion $\theta^{(\ell)}$ vanishes.*
2. *Einstein's equations hold on $\mathcal{H}$, with energy-momentum tensor such that $-T^{\mu}{}_{\nu}\ell^{\nu}$ is causal and future-directed on $\mathcal{H}$.*

Such a hypersurface is called an ***embedded non-expanding horizon****.*

The second definition lets us make away with the direct connection to $\mathcal{M}$, and lets us describe a horizon in an abstract way, taking into account only the intrinsic geometry of a hypersurface:

Definition 2.2.2 *[41, 42] Let there be a three-dimensional manifold $(\mathcal{H}, q, \nabla_a)$, with*

1. *A degenerate metric tensor q of the signature $(0, +, +)$.*
2. *A torsion-free covariant derivative ∇_a, that preserves the degenerate metric tensor*

$$\nabla_a q_{bc} = 0\,. \tag{2.10}$$

Such a manifold is called an ***abstract non-expanding horizon****.*

Naturally every embedded non-expanding horizon also defines an abstract non-expanding horizon. We will now elucidate on the geometric structure of the latter [40]. The metric q is preserved by ℓ

$$\mathcal{L}_\ell q = 0\,, \tag{2.11}$$

which can also be called *time independence* of q. Because ∇_a preserves q_{bc}, the derivative of ℓ is proportional to ℓ, that is

$$\nabla_a \ell^b = \omega_a^{(\ell)} \ell^b\,. \tag{2.12}$$

We will be calling $\omega^{(\ell)}$ a **rotation one-form**. it transforms under rescaling of ℓ

$$\omega_a^{(\alpha\ell)} = \omega_a^{(\ell)} + \nabla_a \log\alpha\,, \tag{2.13}$$

but induced curvature

$$d\omega_{ab}^{(\ell)} = 2\nabla_{[a}\omega^{(\ell)}{}_{b]} \tag{2.14}$$

is gauge-independent. One-form $\omega^{(\ell)}$ is also connected to surface gravity

$$\kappa^{(\ell)} = \omega_a^{(\ell)} \ell^a\,. \tag{2.15}$$

2.2.2 Isolated horizons

The definition of a non-expanding horizon does assume time-independence of the metric (2.11), but not of the connection. An even stronger notion of stability of a horizon is to have a connection conserved by the flow of ℓ:

Definition 2.2.3 *[43] The abstract non-expanding horizon is called* ***abstract isolated horizon*** *if*

$$[\mathcal{L}_\ell, \nabla_a]\, V^b = 0 \qquad \forall V \in T(\mathcal{H})\,, \tag{2.16}$$

for a null, nowhere-vanishing vector field ℓ on $\mathcal{H}$.

In contrast to the definition of a NEH, for an isolated horizon, the vector field ℓ is defined up to a scaling only by a non-zero constant:

$$\ell \sim \alpha \ell \qquad \alpha \in \mathbb{R} \setminus \{0\} \,. \tag{2.17}$$

The **embedded isolated horizon** is an embedded non-expanding horizon satisfying these conditions, as an abstract NEH for the proper choice of ℓ. Every Killing horizon is an embedded isolated horizon. One can show [44] that on an isolated horizon

$$\mathcal{L}_\ell \omega_a^{(\ell)} = 0 \,, \qquad \kappa^{(\ell)} = \text{const.} \tag{2.18}$$

Moreover, in the case of Einstein-Maxwell theory, on an embedded isolated horizon

$$\ell^a F_{ab} = \ell^a \star F_{ab} = 0 \implies \mathcal{L}_\ell F = \mathcal{L}_\ell \star F = 0 \,, \tag{2.19}$$

where F_{ab} is a pull-back of the four-dimensional Maxwell tensor, and $\star F_{ab}$ is a pull-back of the four-dimensional Hodge dual of the Maxwell tensor.

One can also define covariant phase space on an isolated horizon [43, 44], and introduce the notions of energy $E_\mathcal{H}$ or angular momentum of a horizon $J_\mathcal{H}$. In particular one can show that if

$$d\omega = 0 \,, \tag{2.20}$$

then $J_\mathcal{H}$ must vanish. It lets us introduce the notion of **rotating** and **non-rotating** horizon.

The constancy of surface gravity provides the proof of the zeroth law of the thermodynamics of black holes. If $\kappa^{(\ell)}$ vanishes, the $\mathcal{H}$ is called an **extremal/degenerate isolated horizon (EIH)**, otherwise it is non-extremal/non-degenerate. These two cases differ considerably in their mathematical properties, both as abstract and as embedded hypersurfaces. There have been many important results pertaining to both kinds of Killing horizons (every Killing horizon is also IH, as you will see in (3.1.1)), but we want to concentrate on degenerate ones. Extremal IHs are of particular interest because many important theorems and results have only been proven for non-degenerate Killing horizons. For example, any spacetime representing the asymptotic final state of a black hole formed by gravitational collapse may be assumed to possess a bifurcate Killing horizon or a degenerate Killing horizon [17, 45]. We know now that, under reasonable global conditions, the domains of dependence of analytic, stationary, asymptotically-flat electrovacuum black-hole spacetimes with a connected non-degenerate horizon belong to the Kerr-Newman family [17]. No such uniqueness theorem can be proven in the degenerate case. The horizons in Israel-Wilson-Perjés metrics [46, 47] (possible candidates for generalizations of the Majumdar-Papapetrou black holes) are necessarily non-rotating and degenerate [17]. By the laws of black hole mechanics, when surface gravity vanishes, the horizon does not radiate. It should signify a simpler quantum-mechanical description of such a system [9]. Furthermore, effective gravitational field theory may break down near horizon [48], which would make it a laboratory for testing new theories of gravity. Many important results, like the higher-dimensional version of the rigidity theorem [49], are derived assuming that the horizon is not extremal. Meanwhile an extremal version of such a (four-dimensional) *no-hair heorem* was proven much later [50]. These, and many more results, facilitate focusing on the degenerate case to generate similar results in the quasi-local formulation and to push research on black hole physics forward.

Consider now any smooth map

$$\sigma : \mathcal{S} \longrightarrow \mathcal{H} \,, \qquad \dim(\mathcal{S}) = 2 \,; \tag{2.21}$$

transversal to ℓ, that is

$$\sigma\left(T(\mathcal{S})\right) \oplus \text{span}(\ell) = T(\mathcal{H}) \,. \tag{2.22}$$

We call such a map **local section**. One can consider a pull-back of the metric, rotation one-form and energy-momentum tensor

$$q^{(\mathcal{S})} = \sigma^\star q^{(\mathcal{H})} \,, \qquad \omega^{(\mathcal{S})} = \sigma^\star \omega^{(\mathcal{H})} \,, \qquad T^{(\mathcal{S})} = \sigma^\star T^{(\mathcal{H})} \,. \tag{2.23}$$

The metric $q_{AB}^{(\mathcal{S})}$ is non-degenerate due to condition (2.22). The knowledge of ω, q, T, and other quantities on section, allows us to reconstruct them along all integral curves of ℓ crossing $\sigma(\mathcal{S})$.

In the case of extremal horizon, the Einstein-Maxwell equations induce the following constraints on $\mathcal{S}$ [40, 51]:

$$\nabla_{(A}\omega^{(\mathcal{S})}{}_{B)} + \omega^{(\mathcal{S})}_A\omega^{(\mathcal{S})}_B - \frac{1}{2}R^{(\mathcal{S})}_{AB} + \frac{1}{2}\Lambda q^{(\mathcal{S})}_{AB} + \frac{1}{2}\kappa_0 T^{(\mathcal{S})}_{AB} = 0\,, \tag{2.24}$$

where stress-energy tensor on $\mathcal{S}$ is given by

$$T^{(S)}_{AB} = 2|\Phi_1|^2 q^{(S)}_{AB}\,, \tag{2.25}$$

complex function Φ_1 encodes Maxwell field, and Maxwell equations reduce on $\mathcal{S}$ to a single equation

$$\bar{\eth}\Phi_1 + 2\pi\Phi_1 = 0\,; \tag{2.26}$$

see Appendix B. We call (2.24), together with Maxwell equation (2.26) for Φ_1, the **near-horizon geometry equations (NHGE)**.

In general the flow of ℓ can we very wild, so it is very difficult to write this equation globally. In the following chapters we will be considering NHGE in different situations.

2.2.3 Generalization of NHGE

It is worth mentioning that once can consider the generalized version of (2.24). In [52] authors considered the following n-dimensional equation:

$$\nabla_{(A}X_{B)} + aX_AX_B - bR_{AB} = \lambda q_{AB}\,, \qquad a, b, \lambda = \text{const.}\,, \tag{2.27}$$

which they called the generalized Ricci soliton equations. Currently the equations of the form

$$R_{AB} = \frac{1}{m}X_AX_B - \nabla_{(A}X_{B)} + \lambda q_{AB}\,, \tag{2.28}$$

are called quasi-Einstein equations, and has roused a big interest [53, 54]. In the case of $m = 2$ Equation (2.28) is a vacuum version of NHGE. For $m = 2 - n$ ($n \geq 3$) it defines the Einstein-Weyl structure, while for $m = 1 - n$ and $\lambda = 0$, there exists a relation to projective geometry.

2.3. Near-horizon Geometry

Let us consider an extremal, topologically trivial Killing horizon. The general form of the metric g in its neighborhood can be expressed in the following way [9]:

$$g = 2dv\left(dr + rh_A(r,x)dx^A + \frac{1}{2}r^2F(r,x)dv\right) + \gamma_{AB}(r,x)dx^Adx^B\,. \tag{2.29}$$

We introduce coordinate transformation

$$v \to \frac{v}{\epsilon}\,, \qquad r \to \epsilon r\,; \tag{2.30}$$

which is a one-parameter diffeomorphism of g. With our choice of metric, the limit of $\epsilon \to 0$ is always smooth, and results in a near-horizon metric

$$g^2 = 2dv\left(dr + rh_A(x)dx^A + \frac{1}{2}r^2F(x)dv\right) + \gamma_{AB}(x)dx^Adx^B\,. \tag{2.31}$$

Vectors

$$\ell = \frac{\partial}{\partial v}\,, \qquad n = -\frac{\partial}{\partial r}\,; \tag{2.32}$$

are null, perpendicular vectors on the surface $r = 0$, which defines a horizon. This procedure was conducted for the first time by James Bardeen and Gary T. Horowitz [55]. There are results pertaining to uniqueness [56, 57] and stability [58] of such geometries.

If the original metric satisfied Einstein-Maxwell equations, then this one also satisfies them with the limiting momentum-energy tensor. The study of these spacetimes is equivalent to the study of (topologically trivial) degenerate Killing horizons. Many results present in this work can be derived using this approach. Reader can familiarize himself with this in Appendix A, where we have derived Equation (2.24) using this approach.

2.4. Spacetimes with isolated horizons

2.4.1 Static, spherically symmetric black holes

The most general form of a static, spherically symmetric metric is of the form

$$g = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \qquad \lim_{r\to\infty} f(r) = \lim_{r\to\infty} h(r) = 1 . \tag{2.33}$$

The Killing vector $X = \partial_t$ generates one-parameter group of isometries, and is responsible for staticity of the metric. We will specialize the form of (2.33) into

$$g = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \tag{2.34}$$

as this is the form of solutions that are of interest to us (the equality of f and h in these cases is imposed by EE). One-form $\hat{X}$ (corresponding to vector X) is hypersurface-orthogonal if

$$\hat{X} \wedge d\hat{X} = 0 \implies \frac{\partial_r f}{f^3} = 0 , \qquad \hat{X} = \hat{X}_\alpha dx^\alpha = \left(g_{\alpha\beta}X^\beta\right) dx^\alpha . \tag{2.35}$$

Such a hypersurface, where X does not vanish, yet is null

$$X^2 = -f = 0 \tag{2.36}$$

defines black hole horizon at

$$f(r_0) = 0 . \tag{2.37}$$

Metric (2.34) becomes singular on the horizon, but we can use either ingoing or outgoing Eddington-Finkelstein coordinates

$$du = dt \pm \frac{dr}{f} , \tag{2.38}$$

that yield

$$g = -f(r)dt^2 \mp 2dudr + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) . \tag{2.39}$$

One can see, that at a horizon, surface gravity is

$$\kappa^{(\ell)} = -\frac{1}{2}\frac{X^\mu\partial_\mu X^2}{X^2} = -\frac{1}{2}\partial_r f , \tag{2.40}$$

so an extremal horizon is at $r = r_0$ such that

$$f(r_0) = 0 , \qquad \partial_r f(r_0) = 0 . \tag{2.41}$$

The Kretschmann scalar

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \partial_r^2 f + 4\left(\frac{\partial_r f}{r}\right)^2 + 4\left(\frac{f-1}{r^2}\right)^2 \tag{2.42}$$

is regular at EIH, it is therefore not a true coordinate singularity at $r = r_0$ (as opposed to the true singularity at $r = 0$). The metric of a cross-section of an extremal, isolated horizon of the metric (2.34) will then be

$$\widetilde{q} = r_0^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) . \tag{2.43}$$

The vacuum solution discovered by Karl Schwarzschild [59, 60] describes spherically symmetric, static, and asymptotically flat spacetime. It was the first example of a black hole spacetime, that is a spacetime possessing a horizon. In the following year, a maximally symmetric, vacuum solution to Einstein's equations was discovered by Willem de Sitter [61, 62] (and independently by Tullio Levi-Civita [63]). Its metric can be fully described by a constant curvature, which is proportional to the cosmological constant. This class of spacetimes also possesses a horizon, usually called *cosmological*. While both Schwarzschild and (anti–)de Sitter spacetimes have horizons, these cannot be made isolated and extremal. To do that one needs to combine the two spacetimes into one. Such a generalization was discovered by Friedrich Kottler [64], Hermann Weyl [65], and Erich Trefftz [66], and is called Schwarzschild-de Sitter metric, which reads

$$g = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \tag{2.44}$$

in its usual coordinates. Kretschmann scalar

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6} + \frac{8\Lambda^2}{3} \tag{2.45}$$

is singular only for $r = 0$, which tells us, that any other singularity is superfluous one, caused by the unsatisfactory choice of coordinate system. One can use this scalar to check for singularities in a similar way in other spacetimes, letting us to *place* a horizon. The horizon becomes extremal for

$$f = f' = 0 , \tag{2.46}$$

that is

$$r_0 = \sqrt{\frac{1}{\Lambda}} . \tag{2.47}$$

It is also possible to define Schwarzschild–anti-de Sitter metric, with an extremal, isolated horizon for $r_0 = |\Lambda|^{-\frac{1}{2}}$, but its cross-section would have the topology of hyperbolic space $\mathbb{H}_2$ [56], which is not compact. We will therefore exclude it from further considerations.

The Reissner-Nordström spacetime spacetime was found independently by Reissner [67], Weyl [68] and Nordström [69], and describes a charged black hole. It can be thought of as a Schwarzshild solution, modified by introducing a spherically-symmetric, sourceless electromagnetic field, described by the potential

$$A = -\frac{e}{r}dt, \tag{2.48}$$

which results in a static, spherically symmetric, and asymptotically flat metric

$$g = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{e^2}{r^2}} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{2.49}$$

with Kretschmann scalar

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 8\frac{\left(6m^2r^2 - 12e^2mr + 7e^2\right)}{r^6} \tag{2.50}$$

Horizon is extremal for

$$f = f' = 0 \implies r_0 = m = |e| . \tag{2.51}$$

Parameter e can be interpreted as the charge. To do this, we calculate the following asymptotic, Gaussian flux [70]:

$$\frac{1}{4\pi}\oint \star F = e, \tag{2.52}$$

because

$$F = dA = \frac{e}{r^2} dr \wedge dt \implies \star F = e \sin\theta d\theta \wedge d\varphi \tag{2.53}$$

Finally we can generalize RN solution to include cosmological constant. The Reissner-Nordström-(anti-)de Sitter spacetime is described by the metric

$$g = -Q(r)dt + \frac{dr^2}{Q(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \tag{2.54}$$

where

$$Q(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} - \frac{\Lambda}{3} r^2 . \tag{2.55}$$

The electro-magnetic potential is the same as in (2.48). We introduce Eddington-Finkelstein coordinate V such that

$$dv = dt + \frac{dr}{Q} \implies g = -q dv^2 + 2 dv dr + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) . \tag{2.56}$$

Now the vector field X

$$X = \partial_v + Q\partial_r \tag{2.57}$$

is null on the horizon at $r = r_0$, where $Q(r_0) = 0$. We can calculate its surface gravity

$$X^\mu \nabla_\mu X^\alpha = \kappa^{(X)} X^\alpha \implies \kappa^{(X)} = \frac{1}{2} \partial_r Q , \tag{2.58}$$

and so an extremal horizon is placed at $r = r_0$ such that

$$Q(r_0) = 0 , \qquad \partial_r Q(r_0) = 0 ; \tag{2.59}$$

that is

$$\begin{cases} -\frac{\Lambda}{3} r_0^4 + r_0^2 - 2 m r_0 + e^2 = 0 , \\ \qquad -\frac{\Lambda}{3} r_0^4 + m r_0 - e^2 = 0 . \end{cases} \tag{2.60}$$

First consequence of (2.60) is that

$$m r_0 = \frac{1}{3} \left(r_0^2 + e^2\right) \geq 0 , \tag{2.61}$$

so it follows that $r_0 > 0$, as we only consider $m > 0$. The other consequence is

$$\Lambda r_0^4 - r_0^2 + e^2 = 0 \implies r_0^2 = \frac{1 \pm \sqrt{1 - 4\Lambda e^2}}{2\Lambda} \tag{2.62}$$

Now if $\Lambda < 0$, then

$$r_0^2 = \frac{1 - \sqrt{1 - 4\Lambda e^2}}{2\Lambda} \tag{2.63}$$

We can calculate the following limits

$$\lim_{\Lambda \to 0^-} \frac{1 - \sqrt{1 - 4\Lambda e^2}}{2\Lambda} = \lim_{\Lambda \to 0^-} \frac{-\frac{1}{2}\left(1 - 4\Lambda e^2\right)^{-\frac{1}{2}} \left(-4e^2\right)}{2} = e^2 , \tag{2.64}$$

$$\lim_{\Lambda \to -\infty} \frac{1 - \sqrt{1 - 4\Lambda e^2}}{2\Lambda} = 0 . \tag{2.65}$$

This leads us to a conclusion, that

$$r_0^2 \in (0, e^2) \iff \frac{r_0^2}{e^2} \in (0, 1) . \tag{2.66}$$

For $\Lambda > 0$, there is

$$r_0^2 = \frac{1 \pm \sqrt{1 - 4\Lambda e^2}}{2\Lambda}\,, \qquad \Lambda e^2 \leq \frac{1}{4}\,, \qquad \Lambda r_0^2 \leq 1\,. \tag{2.67}$$

2.4.2 Kerr-Newman-(anti-)de Sitter spacetime

The generalization of Schwarzschild spacetime, which describes a rotating black hole was found by Kerr in 1963 [71]. It is stationary, axially symmetric, and asymptotically flat. This solution was further generalized to include aligned electromagnetic field [72], and cosmological constant (see [73] and references therein). In the standard Boyer-Lindquist-type coordinates the metric reads

$$g = -\frac{\Delta_r}{\Xi^2 \rho^2}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\Xi^2\rho^2}\left(a dt - \left(r^2 + a^2\right)d\varphi\right)^2\,, \tag{2.68}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \tag{2.69}$$

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{1}{3}\Lambda r^2\right) - 2mr + e^2\,, \tag{2.70}$$

$$\Delta_\theta = 1 + \frac{1}{3}\Lambda a^2\cos^2\theta\,, \tag{2.71}$$

$$\Xi = 1 + \frac{1}{3}\Lambda a^2\,. \tag{2.72}$$

For this metric to have a proper signature, it follows from (2.71), that

$$\Lambda a^2 > -3\,. \tag{2.73}$$

Electro-magnetic potential is

$$A = \frac{er}{\rho^2 \Xi}\left[dt - a\sin^2\theta d\varphi\right]\,. \tag{2.74}$$

In particular limits

$$e \to 0 \quad \text{and} \quad \Lambda \to 0 \tag{2.75}$$

exist, and correspond to Kerr–(anti–)de Sitter, and Kerr–Newman spacetimes respectively.

In Kerr–Newman–(anti-)de Sitter spacetime, null tetrad takes the form

$$\ell = \frac{\Xi}{\Delta_r}\left[(r^2 + a^2)\partial_t + \frac{\Delta_r}{\Xi}\partial_r + a\partial_\varphi\right]\,, \tag{2.76}$$

$$n = \frac{\Xi}{2\rho^2}\left[(r^2 + a^2)\partial_t - \frac{\Delta_r}{\Xi}\partial_r + a\partial_\varphi\right]\,, \tag{2.77}$$

$$m = \frac{\Xi}{\sqrt{2\Delta_\theta}(r + ia\cos\theta)}\left[a\sin\theta\partial_t + \frac{\Delta_\theta}{\Xi}\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi\right]\,; \tag{2.78}$$

and spin coefficients of it are

$$\kappa = \nu = \sigma = \lambda = \varepsilon = 0\,, \tag{2.79}$$

$$\varrho = -\frac{1}{r - ia\cos\theta}\,, \tag{2.80}$$

$$\tau = -i\sqrt{\frac{\Delta_\theta}{2}}\frac{a\sin\theta}{\rho^2}\,, \tag{2.81}$$

$$\mu = -\frac{1}{2}\frac{1}{\rho^2}\frac{1}{r - ia\cos\theta}\Delta_r\,, \tag{2.82}$$

$$\gamma = \frac{\Delta_r'}{4\rho^2} + \mu\,, \tag{2.83}$$

$$\pi = i\sqrt{\frac{\Delta_\theta}{2}}\frac{a\sin\theta}{(r-ia\cos\theta)^2}\,, \tag{2.84}$$

$$\beta = \sqrt{\frac{\Delta_\theta}{2}}\frac{1}{2(r-ia\cos\theta)}\left(\frac{\Delta'_\theta}{2\Delta_\theta}+\cot\theta\right)\,, \tag{2.85}$$

$$\alpha = \pi - \overline{\beta}\,. \tag{2.86}$$

One can see, that vanishing of Δ_r makes expansion, and vorticity vanish (see Appendix B). The spin coefficient contributing to NHGE is

$$\widetilde{\Phi}_1 = \frac{e}{\sqrt{2\kappa_0}(r-ia\cos\theta)^2}\,. \tag{2.87}$$

Killing horizons are situated at the roots of the quartic function $\Delta_r(r)$. Finding the number of such horizons and their characteristic is an elementary, but time-consuming endeavour. It can be shown that [8]:

1. For $\Lambda > 0$ function $\Delta_r(r)$ has two root $r_0^{(i)}$ such, that

$$r_0^{(1)} < 0 < r_0^{(2)}\,, \tag{2.88}$$

or four root such that

$$r_0^{(1)} < 0 < r_0^{(2)} \leq r_0^{(3)} \leq r_0^{(3)}\,. \tag{2.89}$$

2. For $\Lambda < 0$ function $\Delta_r(r)$ has either no roots, one double root, or two positive roots.

It turns out that $r = r_0$ corresponds to an extremal horizon if r_0 is a double root of $\Delta_r(r)$, that is it is also a root of a cubic function $\partial_r\Delta_r(r)$, which we will also proceed to show. To see it one can rewrite metric (2.68) in the coordinates (v, r, θ, ϕ), where

$$v = t + \int \frac{\Xi(r^2+a^2)}{\Delta_r}dr\,, \tag{2.90}$$

$$\phi = \varphi + \int \frac{\Xi a}{\Delta_r}dr\,. \tag{2.91}$$

It makes it possible to smoothly extend the metric through the horizons. We retrieve the following metric:

$$\begin{aligned} g = &-\frac{\Delta_r - a^2\sin^2\theta\Delta_\theta}{\Xi^2\rho^2}dv^2 + \frac{2}{\Xi}dvdr - 2a\sin^2\theta\frac{(r^2+a^2)\Delta_\theta-\Delta_r}{\Xi^2\rho^2}dvd\phi \\ &- 2\frac{a\sin^2\theta}{\Xi}drd\phi + \sin^2\theta\frac{(r^2+a^2)^2\Delta_\theta - a^2\Delta_r\sin^2\theta}{\Xi^2\rho^2}d\phi^2\,, \end{aligned} \tag{2.92}$$

that is well defined for

$$v \in \mathbb{R}\,, \qquad r \in (0,\infty)\,. \tag{2.93}$$

Let us next consider a vector field of the form [8]:

$$X = \nabla r = g^{\mu\nu}\nabla_\mu r\partial_\nu = \frac{1}{\rho^2}\left[\Xi(r^2+a^2)\partial_v + \Delta_r\partial_r + \Xi a\partial_\phi\right]\,, \tag{2.94}$$

whose norm is

$$\|X\| = \sqrt{g\,(X,X)} = \sqrt{\frac{\Delta_r}{\rho^2}}\,. \tag{2.95}$$

The norm of X vanishes on the horizon ($\Delta_r = 0$) and its integral curves (generators) are null geodesic on the horizon. If a surface $\mathcal{H}$ is a horizon at $r = r_0$, then

$$X|_{\mathcal{H}} = \frac{1}{\rho^2}\left[\Xi(r^2+a^2)\partial_v + \Xi a\partial_\phi\right]\,. \tag{2.96}$$

Notice also, that the vector field

$$\ell = \partial_v + \Omega\partial_\phi\,, \qquad \Omega = \frac{a}{r^2+a^2} \tag{2.97}$$

is null and tangent to the generators of $\mathcal{H}$. We call Ω the *angular velocity* of the horizon. Using (2.4) we can express surface gravity by

$$\kappa^{(\ell)} = \frac{\partial_r \Delta_r(r_0)}{2(r_0^2+a^2)\Xi^2}\,. \tag{2.98}$$

We can see that $r = r_0$ is an extremal horizon if and only if $\partial_r\Delta_r = 0$, that is only if r_0 is a double root of the polynomial $\Delta_r(r)$. Moreover, the fact that $\kappa^{(\ell)}$ is constant on the whole horizon confirms the fact, that the surface $r = r_0$ indeed describes an isolated horizon. From the solutions of $\Delta_r(r_0) = 0$ and $\partial_r\Delta_r(r_0) = 0$ we can get

$$m = r_0\left(1 - \frac{1}{3}\Lambda\left(a^2 + 2r_0^2\right)\right)\,, \tag{2.99}$$

$$e^2 = r_0^2 - \Lambda r_0^4 - a^2\left(1 + \frac{1}{3}\Lambda r_0^2\right)\,. \tag{2.100}$$

The explicit expression for r_0 is

$$r_0^2 = \frac{1}{2\Lambda}\left[\left(1 - \frac{1}{3}a^2\Lambda\right) \pm \sqrt{\left(1 - \frac{1}{3}a^2\Lambda\right)^2 - 4\Lambda(a^2+e^2)}\right]\,, \qquad \Lambda \neq 0\,. \tag{2.101}$$

There are constraints on physical parameters imposed by extremity. By eliminating mass parameter m (with the use of the fact, that the discriminant of Δ_r must vanish), we have [74]:

1. For $\Lambda > 0$:
$$a^2 \in \left[0, \frac{21 - 12\sqrt{3}}{\Lambda}\right]\,, \qquad e^2 \in \left[0, \frac{\Lambda^2 a^4 - 42\Lambda a^2 + 9}{36\Lambda}\right]\,. \tag{2.102}$$
2. For $\Lambda < 0$:
$$a^2 \in \left[0, \frac{3}{|\Lambda|}\right)\,, \qquad e^2 \in [0, \infty)\,. \tag{2.103}$$

Such horizons have an induced metric on their cross-sections

$$\widetilde{q} = \frac{r_0^2 + a^2\cos^2\theta}{1 + \frac{1}{3}\Lambda a^2\cos^2\theta}d\theta^2 + \frac{\left(1 + \frac{1}{3}\Lambda a^2\cos^2\theta\right)\left(a^2 + r_0^2\right)^2\sin^2\theta}{\left(r_0^2 + a^2\cos^2\theta\right)\left(1 + \frac{1}{3}\Lambda a^2\right)^2}d\varphi^2\,. \tag{2.104}$$

Charge e appears only in r_0, so the geometry of Kerr–Newman EIH can be retrieved by setting $\Lambda = 0$. Then the horizon is situated on

$$r_0 = m\,, \quad \text{where} \quad m^2 = a^2 + e^2\,. \tag{2.105}$$

Chapter 3

Topologically trivial horizons

In this chapter we shall construct all abstract extremal, isolated horizons with topology of $\mathbb{R}\times\mathbb{S}_2$, and $\mathbb{R}\times\mathbb{T}_2$, with electromagnetic field and cosmological constant. In [50, 75–77] it was shown that Killing vector has to exist, so we can assume axial symmetry without the loss of generality. We are also analysing the embeddability of such horizons into Reissner-Nordström-(anti-)de Sitter spacetimes and Kerr-Newman-(anti-)de Sitter spacetimes.

3.1. Trivial principal fibre bundle

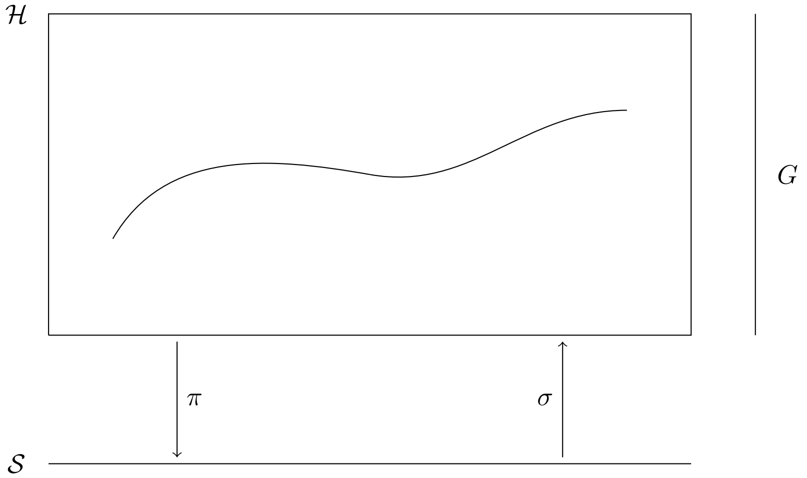

FIGURE 3.1: Principal fibre bundle $\mathcal{H}$.

Let us assume that the isolated horizon $\mathcal{H}$ is diffeomorphic to the product

$$\mathcal{H}\simeq\mathcal{S}\times\mathbb{R}\,, \tag{3.1}$$

where $\mathcal{S}$ is a closed two-dimensional manifold. The fibers of a projection

$$\pi:\mathcal{H}\simeq\mathcal{S}\times\mathbb{R}\longrightarrow\mathcal{S} \tag{3.2}$$

are null curves in $\mathcal{H}$. Such an isolated horizon can be thought of as a principal fiber bundle [41, 78]. The flow of null generators ℓ defines a free, transitive, and fibre-preserving action of the group $G=\mathbb{R}$:

$$\mathcal{S}\times G\ni(p,t)\longmapsto\Phi_t(p)\in\mathcal{S}\,. \tag{3.3}$$

This group is Abelian, so the flow is both left- and right-acting. The fibers are integral curves of ℓ and also orbits of the action. All of that makes $\mathcal{H}$ into a principal fiber bundle over the space of null generators

$$G\hookrightarrow\mathcal{H}\xrightarrow{\pi}\mathcal{S}\,. \tag{3.4}$$

Rotation one-form on a non-degenerate IH can be associated with a connection one-form A, namely

$$A = \frac{\omega}{\kappa^{(\ell)}} \otimes \ell^{\star} , \qquad A(\ell) = 1 ; \tag{3.5}$$

where $\ell^{\star}$ is an element of the Lie algebra of the group G corresponding to ℓ. You can see the whole structure of the principal bundle in Figure 3.1. When IH is extremal, the situation is different. All quantities q, ω and T, can be expressed as pull-back by π of the objects on S

$$q^{(\mathcal{H})} = \pi^{*} q^{(\mathcal{S})} , \qquad \omega^{(\mathcal{H})} = \pi^{*} \omega^{(\mathcal{S})} , \qquad T^{(\mathcal{H})} = \pi^{*} T^{(\mathcal{S})} . \tag{3.6}$$

By locally choosing a cross-section σ of the projection, one can check, that $q^{(\mathcal{S})}$, $\omega^{(\mathcal{S})}$ and $T^{(\mathcal{S})}$ satisfy NHGE on the whole $\mathcal{S}$.

3.1.1 Topological and geometric constraints

Some properties of the solutions of (2.24) can be deduced from the general considerations. Let us take the trace of (2.24), and assume that electromagnetic field vanishes. For two-dimensional surfaces, we have

$$R_{AB} = K g_{AB} , \tag{3.7}$$

where K is Gaussian curvature of the surface. We are left with one, scalar equation

$$\mathrm{div}\omega + \omega^2 - K + \Lambda = 0 , \tag{3.8}$$

which can be now integrated over $\mathcal{S}$. Gauss-Bonnet theorem [79] for the compact manifolds states that

$$\int_{\mathcal{S}} K = 2\pi\chi(\mathcal{S}) , \tag{3.9}$$

where $\chi(\mathcal{S})$ is Euler characteristic of $\mathcal{S}$, that is connected to its genus (see e.g. [80]), namely

$$\chi(\mathcal{S}) = 2 \left(1 - \mathrm{Genus}(\mathcal{S})\right) . \tag{3.10}$$

Using this theory gets us

$$\frac{4\pi}{\mathrm{Area}(\mathcal{S})} \left(1 - \mathrm{Genus}(\mathcal{S})\right) = \frac{1}{\mathrm{Area}(\mathcal{S})} \int \omega^2 + \Lambda \geq \Lambda . \tag{3.11}$$

This inequality has a bearing on the possible topology of $\mathcal{S}$ [81]:

1. For a sphere (vanishing genus), all values of Λ are possible.
2. For a torus (genus equal to one), the cosmological constant must be less than or equal to zero [82]:

   $$\Lambda \leq 0 . \tag{3.12}$$

   Moreover for vanishing Λ we must have

   $$\int \omega^2 = 0 \Longrightarrow \omega^2 = 0 \Longrightarrow \omega = 0 . \tag{3.13}$$

3. For a higher genus surface, the cosmological constant must be strictly negative

   $$\Lambda < 0 . \tag{3.14}$$

For a compact manifold $\mathcal{S}$ even stronger results may be proven [76], namely that the solution to NHGE with electromagnetic field, there is either

$$\omega^{(\mathcal{S})} = 0 , \qquad R^{(\mathcal{S})} = \mathrm{const.} , \qquad F^{(\mathcal{S})}_{AB} = \mathrm{const.} \cdot \epsilon^{(\mathcal{S})}_{AB} ; \tag{3.15}$$

or $\mathcal{S}$ must be a sphere $\mathbb{S}_2$.

Finally let us state the rigidity theorem for extremal, isolated horizons:

Theorem 3.1.1 *[50, 75–77] Let $\mathcal{S}$ be a two dimensional, compact, orientable Riemannian manifold equipped with Riemannian metric q_{AB}, one-form ω_A and a complex function Φ_1 satisfying (2.24) with Maxwell field. Then one of the following holds:*

1. *Metric q_{AB} has constant curvature and*
$$\omega_A = 0\,, \qquad \Phi_1 = const. \tag{3.16}$$
2. *$\mathcal{S}$ is a sphere and there exists a symmetry $U(1)$ of the data generated by a Killing vector field K^A such that*
$$\mathcal{L}_K q = 0\,, \qquad \mathcal{L}_K\omega = 0\,, \qquad \mathcal{L}_K\Phi_1 = 0\,. \tag{3.17}$$

There are partial extensions of these results to higher dimensions. This rigidity theorem limits further investigation of possible two-dimensional IHs to axisymmetric manifolds.

3.2. $\mathbb{R}\times\mathbb{S}_2$ topology

3.2.1 Symmetries, adapted coordinates and potentials

The metric any two-dimensional, oriented Riemannian manifold is conformally flat, and can be expressed as
$$q = R^2P^2(y,\varphi)\left(dy^2+d\varphi^2\right)\,, \qquad R = \text{const.} \tag{3.18}$$

As we have already mentioned in Theorem 3.1.1, IH must have at least one Killing vector field. Therefore, a metric can be characterized by either one Killing vector ∂_φ, say, or by three Killing vectors, that is, it can be either axially or spherically symmetric. We choose to adapt our coordinates to suit ∂_φ symmetry, and set
$$P = P(y)\,. \tag{3.19}$$

In the usual manner, the coordinate φ will be periodic, with period 2π. We may now transform the coordinates
$$dx = P^2(y)dy \Longleftrightarrow x = \int P^2\left(y\right)dy\,. \tag{3.20}$$
and retrieve
$$q = R^2\left(\frac{1}{P^2}dx^2 + P^2d\varphi^2\right)\,, \quad P = P(x)\,. \tag{3.21}$$

In the NP formalism, we can write the null tangent and cotangent frame as
$$m^A\partial_A = \frac{1}{\sqrt{2}R}\left(P\partial_x + i\frac{1}{P}\partial_\varphi\right)\,, \qquad \overline{m}_A dx^A = \frac{R}{\sqrt{2}}\left(\frac{1}{P}dx - iPd\varphi\right)\,; \tag{3.22}$$
and express metric as
$$q = 2m_{(A}\overline{m}_{B)}\,. \tag{3.23}$$

Parameter R will serve as a *radius* of our horizon, that is
$$\text{Area}(\mathcal{S}) = \int_{\mathcal{S}}\sqrt{\det q}\,dxd\varphi = R^2\int_{x_1}^{x_2}dx\int_0^{2\pi}d\varphi = 4\pi R^2\,. \tag{3.24}$$

The coordinate x can take values between arbitrary constants x_1 and x_2, differing by 2, as per (3.24), which we will fix, so that
$$x\in[-1,1]\,. \tag{3.25}$$

The endpoints of this range will be poles of the coordinate system on $\mathcal{S}$. Gaussian curvature of metric (3.21) is now given by
$$K = -\frac{1}{2}\frac{1}{R^2}\frac{\partial^2P^2}{\partial x^2}\,. \tag{3.26}$$

Similarly to metric, the Maxwell tensor F also has to be axially symmetric:

$$\mathcal{L}_{\partial_\varphi} F = 0\,. \tag{3.27}$$

The only equation for the component of momentum-energy tensor in the NP formalism (see Appendix B for more details) is

$$\overline{\delta}\Phi_1 + 2\pi\Phi_1 = 0\,, \quad \overline{\delta} = \overline{m}^A \partial_A\,; \tag{3.28}$$

and spin coefficient π is given by

$$\pi = \frac{P}{\sqrt{2}R}\left(-i\partial_x U + \frac{\partial_x B}{B}\right)\,. \tag{3.29}$$

Equation (3.28) can be integrated in the general form to yield

$$\Phi_1 = \frac{E_0}{B^2} e^{i2U}\,, \qquad E_0 \in \mathbb{C} \implies |\Phi_1|^2 = \frac{|E_0|^2}{B^4}\,. \tag{3.30}$$

Using Hodge decomposition[1], we can describe a rotation one-form with two scalar potentials

$$\omega = \star dU + d\log B = \frac{\partial_x B}{B} dx + P^2 \partial_x U d\varphi\,, \tag{3.31}$$

where U is defined up to an additive constant U_0, and B up to multiplicative constant B_0. We can change these constants without the loss of any generality, and so we shall call this *gauge freedom* in the choice of both potentials. Both potentials are functions of x only, as they are to be axially symmetric. Notice also that freedom of choice of constant B allows us to assume, that B is a positive function. Now the equation (2.24) reduces to a system of equations for functions $B(x)$ and $U(x)$ of the form

$$\partial_x^2 B - (\partial_x U)^2 B = 0\,, \tag{3.32}$$

$$\partial_x\left(B^2 \partial_x U\right) = 0\,; \tag{3.33}$$

and an equation

$$\frac{1}{2}\partial_x^2 P^2 + \frac{\partial_x B}{B}\partial_x P^2 + \left[\frac{1}{B}\partial_x^2 B + (\partial_x U)^2\right] P^2 + \Lambda R^2 + \frac{2\kappa_0 R^2 |E_0|^2}{B^4} = 0\,. \tag{3.34}$$

The equation (3.33) can be integrated to yield

$$B^2 \partial_x U = \widetilde{\Omega} = \text{const.} \tag{3.35}$$

The constant Ω can be either zero or non-zero, which will lead us to two different solutions (these are the only solutions to (3.35), as B cannot vanish). In the former case there is

$$\widetilde{\Omega} = 0 \implies U = U_0 = \text{const.}\,, \tag{3.36}$$

where, using the gauge freedom, we can set $U_0 = 0$ without the loss of generality. It is then easy to integrate (3.32), and the potentials have the form

$$U = U_0 = \text{const.}\,, \qquad B = B_1 x + B_2\,; \tag{3.37}$$

where B_1 and B_2 are constants. One-form ω has only ω_x component for such potentials. The second solution to (3.35) is

$$\widetilde{\Omega} \neq 0 \implies U \neq \text{const.} \tag{3.38}$$

We can eliminate function U from equation (3.32), with the (3.33), to get

$$B^3 \partial_x^2 B = \widetilde{\Omega}^2\,. \tag{3.39}$$

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[1] It states, that every k-form can be uniquely decomposed into exact, co-exact and harmonic form (see e.g. [83]). In this work we are using the fact that harmonic forms vanish on (topological) spheres.

To solve it, we can multiply by $\partial_x B$ and divide by B^3 from both sides, and get

$$\partial_x\left[(\partial_x B)^2+\widetilde{\Omega}^2\frac{1}{B^2}\right]=0 \implies (\partial_x B)^2+\widetilde{\Omega}^2\frac{1}{B^2}=c_1=\text{const.} \tag{3.40}$$

We can transform it to

$$\left(\partial_x B^2\right)^2=4\left(c_1B^2-\widetilde{\Omega}^2\right) \implies \partial_x B^2=\pm 2\sqrt{c_1B^2-\widetilde{\Omega}^2}\,. \tag{3.41}$$

Now integration of

$$\int\frac{dB^2}{\sqrt{c_1B^2-\widetilde{\Omega}^2}}=\pm 2\int dx \tag{3.42}$$

leaves us with

$$\sqrt{B^2-\widetilde{\Omega}^2}=\pm c_1x+c_2 \implies B^2=c_1^2\left(x+\frac{c_2}{c_1}\right)^2+\widetilde{\Omega}^2\,. \tag{3.43}$$

Knowing B it is easy to integrate (3.33). To summarize, potentials take the following form [84]:

$$U=\arctan\left(\frac{x-x_0}{\Omega}\right)+U_0\,, \tag{3.44}$$

$$B^2=B_0^2\left[\Omega^2+(x-x_0)^2\right]\;; \tag{3.45}$$

where we have defined the following:

$$c_1=B_0\,,\qquad \Omega^2=\frac{\widetilde{\Omega}^2}{B_0}\,,\qquad x_0=-\frac{c_2}{c_1}\,. \tag{3.46}$$

The constants B_0 and U_0 correspond to the multiplicative and additive freedom in the choice of the potentials. One-form ω has the following form for such potentials:

$$\omega=\frac{x-x_0}{\Omega^2+(x-x_0)^2}dx+P^2\frac{\Omega^2}{\Omega^2+(x-x_0)^2}d\varphi\,. \tag{3.47}$$

Notice that it is regular in the poles – its axial component vanishes (because P^2 is zero at the poles), and its second component is well-defined . In the limit of $\Omega\to 0$ we get

$$U\longrightarrow U_0\,, \tag{3.48}$$

$$B\longrightarrow B_1x+B_2\,; \tag{3.49}$$

which is identical to (3.37).

Notice that the imaginary part of the scalar Φ_1 is

$$\Im(\Phi_1)=\frac{E_0}{B^2}\sin(2U)\,, \tag{3.50}$$

while the differential of the rotation one-form is

$$d\omega=2\Im(\Psi_2)\eta=\partial_x\left(P^2\partial_x^2U\right)dx\wedge d\varphi\,. \tag{3.51}$$

So if U is constant (which is equivalent to vanishing by gauge freedom), then by (2.20), the angular momentum of a horizon is zero.

3.2.2 Constraints on poles

We require our metric to be free from conical singularities. For it to be well-defined on poles $x=\pm1$, there must be

$$P^2(x=\pm1)=0\,, \tag{3.52}$$

while its derivative must be such, that

$$\left.\frac{\partial P^2}{\partial x}\right|_{x=\pm 1} = \mp 2\,, \tag{3.53}$$

see [84]. These conditions are made obvious by looking at the general form of axially-symmetric metric in the familiar system of spherical coordinates (θ,φ)

$$q = \Sigma^2(\theta)\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\,. \tag{3.54}$$

By a simple coordinate transformation

$$d\theta = \frac{R^2}{\Sigma^2 \sin\theta} dx \tag{3.55}$$

we get back

$$q = R^2\left(\frac{R^2}{\Sigma^2\sin^2\theta}dx^2 + \frac{\Sigma^2\sin^2\theta}{R^2}d\varphi^2\right) \implies P^2 = \frac{\Sigma^2\sin^2\theta}{R^2}\,. \tag{3.56}$$

For a metric (3.54) to be well-defined on poles, it must be continuous, so it must vanish in two poles if we account for axial symmetry, hence equation (3.52). Furthermore, we demand, that our metric should not have any angle deficiency around poles. Let us consider small loops, circles of radius Δx around poles. The ratio of their lengths to their radii must be equal to 2π, that is

$$\frac{2\pi P\left(\pm 1 \mp \Delta x\right)}{\mp\int_{\mp 1}^{\pm 1 \mp \Delta x} P^{-1} dx} \xrightarrow{\Delta x \to 0} \mp\frac{2\pi \partial_x P(\pm 1)}{P^{-1}(\pm 1)} = 2\pi \implies \partial_x P^2(x=\pm 1) = \mp 2\,. \tag{3.57}$$

This justifies Equation (3.53). If the ratio of length to radius is not 2π, then we call it an *angle deficit (or excess)*, and say, that metric has *conical singularity.* We shall analyse such a possibility in the Chapter 4. Equations (3.52)–(3.53) also assure, that the axial component of one-form ω, and two-form $d\omega$ vanish on the poles.

3.3. Topologically spherical EIH for $\Omega = 0$

3.3.1 General solution

If we were to choose $\Omega = 0$ (which is equivalent to $\widetilde{\Omega} = 0$) in (3.35), then it follows that potentials of ω have the form (3.37), that is

$$U = U_0 = \text{const.}\,, \qquad B = B_1 x + B_2\,. \tag{3.58}$$

Notice that the constant B_2 cannot vanish because the function $\log B$ in (3.31) would be ill-defined for some values of x. Equation (3.34) is then reduced to

$$\frac{1}{2}\partial_x^2 P^2 + \frac{B_1}{B_1 x + B_2}\partial_x P^2 + \Lambda R^2 + \frac{2\kappa_0 R^2 |E_0|^2}{(B_1 x + B_2)^4} = 0\,. \tag{3.59}$$

Its solution is the function

$$P^2 = \frac{c_1}{B_1(B_1 x + B_2)} + c_2 - \frac{\Lambda R^2}{3}\frac{(B_1 x + B_2)^2}{B_1^2} - \frac{2\kappa_0 R^2 |E_0|^2}{B_1^2 (B_1 x + B_2)^2}\,, \tag{3.60}$$

where we have set $B_1 \neq 0$. We postpone the analysis of the $B_1 = 0$ case to the later part of this section. In the generic case – non-vanishing Λ and E_0 – the boundary conditions (3.52) requires that

$$c_1 = -\frac{2B_2}{3B_1}\frac{\Lambda R^2\left(B_1^2 - B_2^2\right)^2 - 6\kappa_0|E_0|^2R^2}{B_1^2 - B_2^2}\,, \tag{3.61}$$

$$c_2 = \frac{1}{3B_1^2}\frac{\Lambda R^2\left[(B_1^2 - B_2^2)^2 - 4B_2^4\right] + 6\kappa_0|E_0|^2R^2}{B_1^2 - B_2^2}\,; \tag{3.62}$$

and also

$$B_2 \pm B_1 > 0 \Longleftrightarrow B_2 > |B_1|\,, \tag{3.63}$$

because B must be positive. From the last inequality it also follows that

$$B_2 > 0\,. \tag{3.64}$$

With that, the solution (3.60) takes the form

$$P^2 = \frac{1 - x^2}{B_1x + B_2}\left[\frac{\Lambda R^2}{3}(B_1x + 3B_2) - \frac{2\kappa_0R^2|E_0|^2}{(B_1^2 - B_2^2)(B_1x + B_2)}\right]\,. \tag{3.65}$$

Furthermore conditions (3.53) result in the additional constraints

$$\frac{\Lambda R^2(B_1 - 3B_2)}{3(B_1 - B_2)} - \frac{2\kappa_0R^2|E_0|^2}{(B_1^2 - B_2^2)(B_1 - B_2)^2} = 1\,, \tag{3.66}$$

$$\frac{\Lambda R^2(B_1 + 3B_2)}{3(B_1 + B_2)} - \frac{2\kappa_0R^2|E_0|^2}{(B_1^2 - B_2^2)(B_1 + B_2)^2} = 1\,; \tag{3.67}$$

which can be simplified to a system

$$\frac{B_1B_2}{(B_1^2 - B_2^2)^2}\left(\Lambda R^2 - \frac{3}{2}\right) = 0\,, \tag{3.68}$$

$$\frac{B_1B_2}{(B_1^2 - B_2^2)^3}\left(4\kappa_0R^2E_0^2 + \frac{1}{3}\Lambda R^2(B_1^2 - B_2^2)^2\right) = 0\,. \tag{3.69}$$

We see that, as per (3.63), either $B_1 = 0$ or $B_2 = 0$, but for the latter, the potential

$$B = B_1x \tag{3.70}$$

is ill-defined at $x = 0$. Let us consider now the case of $B_1 = 0$, where equation (3.34) takes the form

$$\partial_x^2P^2 + 2\Lambda R^2 + \frac{4\kappa_0R^2E_0^2}{B_2^2} = 0\,. \tag{3.71}$$

It is trivial to integrate, and applying conditions (3.52)–(3.53) results in the following solution

$$P^2 = 1 - x^2\,, \tag{3.72}$$

where

$$\Lambda R^2 + \frac{2\kappa_0R^2|E_0|^2}{B_2^4} = 1\,. \tag{3.73}$$

This equation constrains both the cosmological constant and the electromagnetic field. Moreover, as both potentials are constant, the rotation one-from must also vanish:

$$\begin{cases} B = \text{const.} \\ U = \text{const.} \end{cases} \Longrightarrow \ \omega = 0\,. \tag{3.74}$$

3.3.2 Embedding in Reissner-Nordström-(anti-)de Sitter spacetime

In the Section (2.4.1) we have described Reissner-Nordström-(anti-)de Sitter spacetimes, together with conditions (2.66) and (2.67) that have to be satisfied for an extremal horizon. In this Section we will show, that every abstract, non-rotating, isolated horizon, described by

(3.72)–(3.74), can be embedded in RNdS spacetime. From the form of the metric (3.72) on a cross-section of EIH, it follows that

$$R^2 = r_0^2 \,. \tag{3.75}$$

Let us introduce the notation

$$\mathcal{E}^2 = 2\kappa_0 \frac{|E_0|^2}{B_1^4} \tag{3.76}$$

now the condition (3.73) can be expressed as

$$\Lambda R^2 + \mathcal{E}^2 R^2 = 1 \,. \tag{3.77}$$

By comparing spin coefficients for both abstract EIH and RNadS spacetime (for potential (2.48))

$$|\Phi_1|^2 = \frac{\mathcal{E}^2}{2\kappa_0} \,, \qquad |\widetilde{\Phi}_1|^2 = \frac{e^2}{2\kappa_0 r_0^2} \,; \tag{3.78}$$

we see, that there has to be

$$\mathcal{E}^2 = \frac{e^2}{r_0^4} \tag{3.79}$$

We will now continue with the analysis of spacetimes with $\Lambda < 0$. We can write

$$\frac{r_0^2}{e^2} = \frac{1}{r_0^2 \mathcal{E}^2} = \frac{1}{R^2 \mathcal{E}^2} = \frac{1}{1 - \Lambda R^2} \,. \tag{3.80}$$

By studying the limits

$$\lim_{\Lambda \to -\infty} \frac{r_0^2}{e^2} = 0 \,, \tag{3.81}$$

$$\lim_{\Lambda \to 0^-} \frac{r_0^2}{e^2} = 1 \,; \tag{3.82}$$

we retrieve the conditions (2.66) from Section 2.4.1. In the case of $\Lambda > 0$ there is

$$\Lambda R^2 \in (0, 1) \,. \tag{3.83}$$

Notice the following:

$$\Lambda e^2 = \Lambda \mathcal{E}^2 r_0^4 = \Lambda \mathcal{E}^2 R^4 = \Lambda R^2 \cdot \mathcal{E}^2 R^2 = \Lambda R^2 \left(1 - \Lambda R^2\right) = \frac{1}{4} - \left(\Lambda R^2 - \frac{1}{2}\right)^2 \,, \tag{3.84}$$

which means, that

$$\Lambda e^2 \in \left(0, \frac{1}{4}\right) \,, \tag{3.85}$$

which, together with (3.83), is consistent with (2.67), from Section 2.4.1.

3.3.3 Vanishing electromagnetic field ($E_0 = 0$)

When the electromagnetic field vanishes, the NHG equation (3.59) is

$$\frac{1}{2}\partial_x^2 P^2 + \frac{B_1}{B_1 x + B_2} \partial_x P^2 + \Lambda R^2 = 0 \,. \tag{3.86}$$

Its solution has the form

$$P^2 = \frac{c_1}{B_1 (B_1 x + B_2)} + c_2 - \frac{\Lambda R^2}{3} \frac{(B_1 x + B_2)^2}{B_1^2} \,, \tag{3.87}$$

which is equal to (3.65) with $E_0 = 0$. Boundary conditions (3.52) result in

$$c_1 = -\frac{\Lambda R^2}{3}\frac{2B_2(B_1^2 - B_2^2)}{B_1}\,, \tag{3.88}$$

$$c_2 = \frac{\Lambda R^2}{3}\frac{(B_1^2 + 2B_2^2)}{B_1^2}\,. \tag{3.89}$$

Similarly to the general case, conditions (3.53) force us to take $B_1 = 0$, to keep the potential B well-defined. Then the NHG equation simplifies to

$$\partial_x^2 P^2 + 2\Lambda R^2 = 0\,, \tag{3.90}$$

which can be integrated to yield

$$P^2 = c_1 + c_2 x - \Lambda R^2 x^2\,, \tag{3.91}$$

which isin turn well-defined for all x only if

$$c_1 = \Lambda R^2\,, \qquad c_2 = 0\,, \qquad \Lambda R^2 = 1\,. \tag{3.92}$$

To summarize [81],

$$P^2 = 1 - x^2\,, \qquad \omega = 0\,. \tag{3.93}$$

This EIH is embeddable in an extremal Schwarzschild-de Sitter spacetime by setting

$$x = -\cos\theta\,, \quad R = \sqrt{\frac{1}{\Lambda}}\,. \tag{3.94}$$

3.3.4 Vanishing cosmological constant ($\boldsymbol{\Lambda = 0}$)

When the cosmological constant vanishes, the NHG equation (3.59) is

$$\frac{1}{2}\partial_x^2 P^2 + \frac{B_1}{B_1 x + B_2}\partial_x P^2 + \frac{2\kappa_0 R^2 |E_0|^2}{(B_1 x + B_2)^4} = 0\,, \tag{3.95}$$

with the solution

$$P^2 = \frac{c_1}{B_1(B_1 x + B_2)} + c_2 - \frac{\Lambda R^2}{3}\frac{(B_1 x + B_2)^2}{B_1^2} - \frac{2\kappa_0 R^2 |E_0|^2}{B_1^2 (B_1 x + B_2)^2}\,, \tag{3.96}$$

what is consistent with (3.65). The boundary conditions (3.52) set the integration constants to

$$c_1 = \frac{4B_2 \kappa_0 R^2 |E_0|^2}{B_1(B_1^2 - B_2^2)}\,, \tag{3.97}$$

$$c_2 = \frac{2\kappa_0 R^2 |E_0|^2}{B_1^2(B_1^2 - B_2^2)}\,; \tag{3.98}$$

which yields

$$P^2 = (1 - x^2)\frac{4\kappa E_0^2 R^2}{(B_1^2 - B_2^2)(B_1 x + B_2)^2}\,. \tag{3.99}$$

This time conditions (3.53) would be satisfied for

$$B_2 = 0\,, \tag{3.100}$$

$$B_1^4 = 4\kappa E_0^2 R^2\,; \tag{3.101}$$

but it would result in the solution

$$P^2 = \frac{1 - x^2}{x^2}\,, \tag{3.102}$$

that is ill-defined at $x = 0$. Because of that, and by the same arguments as in the previous subsections, we will set $B = B_2 = \text{const.}$, to yield the following NHG equation:

$$\frac{1}{2}\partial_x^2 P^2 + \frac{2\kappa_0 R^2 |E_0|^2}{B_2^4} = 0\,, \tag{3.103}$$

with the solution

$$P^2 = -\frac{2\kappa_0 |E_0|^2 R^2}{B_2^4} x^2 + c_1 x + c_2\,. \tag{3.104}$$

It can be easily seen, that conditions (3.52)–(3.53) result in

$$c_1 = 0\,, \qquad c_2 = \frac{2\kappa_0 |E_0|^2 R^2}{B_2^4}\,, \qquad B_2^2 = 2\kappa_2 |E_0|^2 R^2\,. \tag{3.105}$$

The final solution is then

$$P^2 = 1 - x^2\,, \tag{3.106}$$

$$\omega = 0\,. \tag{3.107}$$

This EIH is embeddable in an extremal Reissner-Nordström spacetime by setting

$$x = -\cos\theta\,, \tag{3.108}$$

$$R = m = |e|\,. \tag{3.109}$$

3.3.5 Vacuum ($E_0 = \Lambda = 0$)

Interestingly, there is no solution in the case of a complete vacuum. Equation (3.59) is reduced to

$$\frac{1}{2}\partial_x^2 P^2 + \frac{B_1}{B_1 x + B_2}\partial_x P^2 = 0\,, \tag{3.110}$$

and yields

$$P^2 = c_1 - \frac{c_2}{B_1(B_1 x + B_2)}\,, \tag{3.111}$$

where c_1 and c_2 are integration constants. There are no values of these constants, besides unphysical

$$c_1 = c_2 = 0\,, \tag{3.112}$$

that would satisfy conditions (3.52)–(3.53). It is consistent with the fact that there are no extremal isolated horizons in Schwarzschild spacetime.

3.4. Topologically spherical EIH for $\Omega \neq 0$

3.4.1 General solution

If we were to choose $\Omega \neq 0$ (which is equivalent to $\widetilde{\Omega} \neq 0$) in (3.35), then it follows that potentials of ω have the form (3.44)–(3.45). With that (3.34) can be rewritten as

$$\partial_x^2 P^2 + \frac{2(x-x_0)}{(x-x_0)^2 + \Omega^2}\partial_x P^2 + \frac{4\Omega^2}{\left[(x-x_0)^2+\Omega^2\right]^2} P^2 + 2\Lambda R^2 + \frac{2E^2}{\left[(x-x_0)^2+\Omega^2\right]^2} = 0\,, \tag{3.113}$$

where

$$E^2 = \frac{2\kappa_0 R^2 |E_0|^2}{B_0^4}\,. \tag{3.114}$$

The general solution to (3.113) is

$$P^2 = -\frac{E^2 + \frac{1}{3}\Lambda R^2 x^2\left[(x-2x_0)^2 + (3\Omega^2 - x_0^2)\right] + c_1\left[(x-x_0)^2 - \Omega^2\right] - 2c_2\Omega(x-x_0)}{(x-x_0)^2+\Omega^2}\,, \tag{3.115}$$

where c_1 and c_2 are integration constants. The detailed derivation can be found in Appendix C. Applying boundary conditions (3.52)–(3.53) allows us to differentiate between two branches of our solution. For the first one, there is

$$x_0^2 + \Omega^2 = 1 \tag{3.116}$$

and

$$c_1 = -\frac{1}{2}\,, \qquad c_2 = -\frac{x_0}{2\Omega}\,, \qquad \Lambda R^2 = \frac{3}{2}\,, \qquad E^2 = 2(x_0^2 - 1)\,. \tag{3.117}$$

Notice however, that by (3.116) there must be

$$x_0^2 = 1 - \Omega^2 \implies |x_0| < 1 \implies E^2 = 2(x_0^2 - 1) < 0\,, \tag{3.118}$$

which is a contradiction. For the second branch – the well-defined solution – there is

$$x_0^2 + \Omega^2 \neq 1\,. \tag{3.119}$$

Now boundary condition (3.52) results in

$$c_1 = \frac{E^2 + \frac{1}{3}\Lambda R^2\left(3\Omega^2 - x_0^2 + 1\right)}{x_0^2 + \Omega^2 - 1}\,, \tag{3.120}$$

$$c_2 = -\frac{x_0}{\Omega}\,\frac{E^2 + \frac{1}{3}\Lambda R^2\left(5\Omega^2 + x_0^2 - 1\right)}{x_0^2 + \Omega^2 - 1}\,; \tag{3.121}$$

while (3.53) makes us set

$$x_0 = 0\,, \tag{3.122}$$

and

$$E^2 = \left(\Omega^2 + 1\right)\left(\Omega^2 - 1 - \Lambda R^2\left(\Omega^2 - \frac{1}{3}\right)\right)\,. \tag{3.123}$$

Function P^2 now has the form

$$P^2 = (1 - x^2)\left(\frac{\left(\Omega^2 + 1\right)\left(1 - \frac{1}{3}\Lambda R^2\right)}{x^2 + \Omega^2} + \frac{1}{3}\Lambda R^2\right)\,, \tag{3.124}$$

The positivity of functions (3.123) and (3.124) creates constraints on our parameters, namely

$$\Omega^2 - 1 - \Lambda R^2\left(\Omega^2 - \frac{1}{3}\right) \geq 0\,, \tag{3.125}$$

$$\Omega^2 + 1 + \frac{1}{3}\Lambda R^2(x^2 - 1) > 0 \quad \text{for} \quad |x| < 1\,. \tag{3.126}$$

The first inequality (3.125) should be considered in two subcases. For $\Omega^2 > \frac{1}{3}$ we get

$$\Lambda R^2 \leq \frac{\Omega^2 - 1}{\Omega^2 - \frac{1}{3}} = 1 - \frac{2}{3\Omega^2 - 1}\,, \tag{3.127}$$

which means, that $\Lambda R^2 < 1$. On the other hand, for $\Omega^2 < \frac{1}{3}$ we get

$$\Lambda R^2 \geq \frac{\Omega^2 - 1}{\Omega^2 - \frac{1}{3}} = 1 - \frac{2}{3\Omega^2 - 1}\,, \tag{3.128}$$

which allows for $\Lambda R^2 > 3$. Notice that every of

$$\Omega^2 = \frac{1}{3} \quad \text{or} \quad \Lambda R^2 = 1 \quad \text{or} \quad \Lambda R^2 = 3 \tag{3.129}$$

lead to a contradiction. To sum it up, an admissible cosmological constant is

$$\Lambda R^2 \in (-\infty, 1) \cup (3, \infty)\,, \tag{3.130}$$

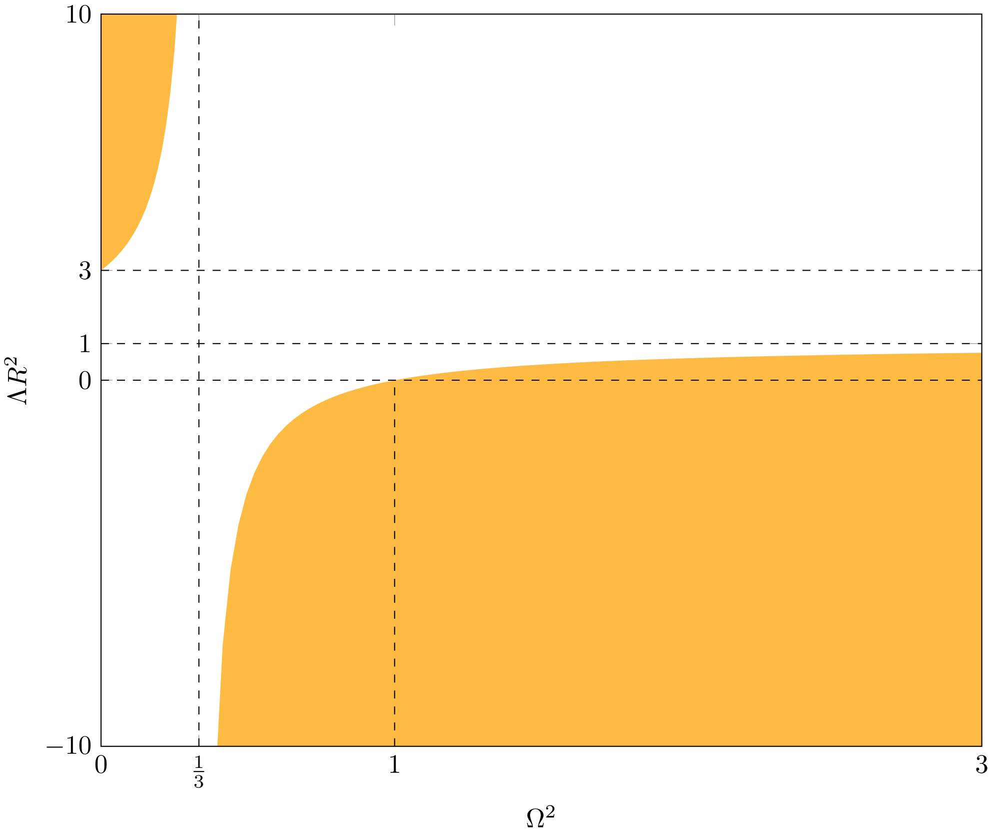

FIGURE 3.2: Possible values of ΛR^2 with regards to Ω^2 in the general solution of topologically trivial NHG equation, after accounting for constraint (3.125).

which is depicted in Figure 3.2. There is a horizontal asymptote for $\Lambda R^2 = 1$ and vertical one for $\Omega^2 = \frac{1}{3}$.

The condition (3.126) comes down to examining the positivity of a second-degree polynomial in x of the form

$$p(x) = \frac{1}{3}\Lambda R^2 x^2 + \left(\Omega^2 + 1 - \frac{1}{3}\Lambda R^2\right) , \tag{3.131}$$

in the range $x \in [-1, 1]$. The sign of the multiplicator of x^2 is, of course, crucial in our analysis, and we can divide its whole possible range into three subranges:

$$(i) \quad \Lambda R^2 \in (-\infty, 0) \,, \tag{3.132}$$

$$(ii) \quad \Lambda R^2 \in (0, 1) \,, \tag{3.133}$$

$$(iii) \quad \Lambda R^2 \in (3, \infty) \,. \tag{3.134}$$

The values between $[1, 3]$ are not permissible as per (3.125). In the case (i), we know that polynomial p reaches its maximum at $x = 0$, and its minimal values at $x = \pm 1$, but

$$f(\pm 1) = \Omega^2 + 1 > 0 \implies p > 0 \,. \tag{3.135}$$

In the case (ii), we must have

$$\Omega^2 > \frac{1}{3}\Lambda R^2 - 1 \,, \tag{3.136}$$

but notice that

$$1 - \frac{1}{3}\Lambda R^2 > 0 \,, \tag{3.137}$$

so inequality (3.136) always holds, and p is positive. Lastly, in the case (iii) the polynomial p is positive only if (3.136) is satisfied, but then, by inequality (3.125) there must be

$$\Omega^2 \leq \frac{\frac{1}{3}\Lambda R^2 - 1}{\Lambda R^2 - 1} < \frac{1}{3}\Lambda R^2 - 1\,, \tag{3.138}$$

which leads to a contradiction. To summarize, the allowed values of the cosmological constant are

$$\Lambda R^2 \in (-\infty, 1)\,, \tag{3.139}$$

and constant Ω must satisfy inequality (3.125)

$$\Omega^2 \geq \frac{1 - \frac{1}{3}\Lambda R^2}{1 - \Lambda R^2}\,. \tag{3.140}$$

Viable ΛR^2, relative to Ω^2 is shown in the Figure 3.3. As before there are asymptotes for $\Lambda R^2 = 1$ and $\Omega^2 = \frac{1}{3}$. Notice that there must be

$$\Omega^2 > \frac{1}{3}\,. \tag{3.141}$$

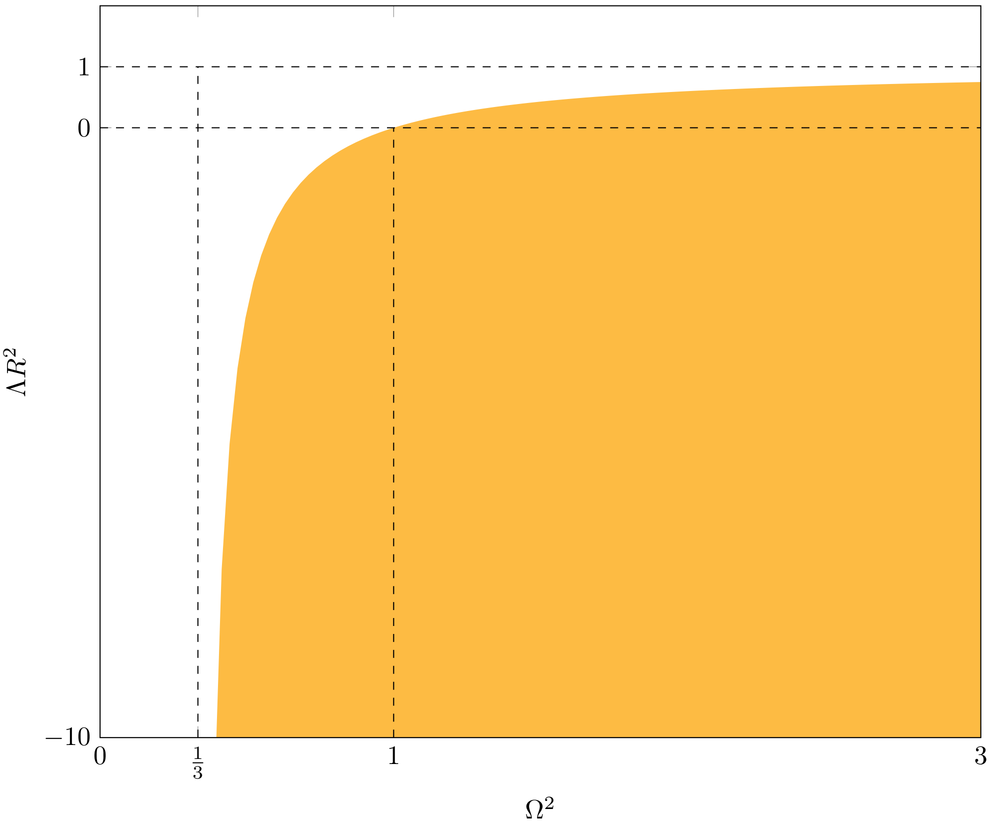

FIGURE 3.3: Possible values of ΛR^2 with regards to Ω^2 in the general solution of the topologically trivial NHG equation, after accounting for all constraints.

Using (3.123) we can rewrite (3.124) as

$$P^2 = \frac{1 - x^2}{3\Omega^2 - 1}\left[\frac{2\Omega^2(\Omega^2 + 1) + E^2}{x^2 + \Omega^2} - \frac{E^2 + (1 - \Omega^4)}{1 + \Omega^2}\right], \tag{3.142}$$

together with the constraint (3.140) in the form

$$\frac{E^2(3\Omega^2-1)}{3E^2+2(\Omega^2+1)}\geq 0\,. \tag{3.143}$$

Function P^2 in this parametrization is well defined for

$$|x|\leq 1\,,\qquad E\in\mathbb{R}\,,\qquad \Omega^2>\frac{1}{3}\,. \tag{3.144}$$

The cosmological constant can be expressed as

$$\Lambda R^2=\frac{(\Omega^4-1)-E^2}{(\Omega^2+1)\left(\Omega^2-\frac{1}{3}\right)}<1\,. \tag{3.145}$$

We can also calculate scalar Φ_1 of our horizon. Using trigonometric identities (see Appendix C for details) we express scalar Φ_1 as

$$\Phi_1=\frac{E_0}{B_0^2\left(\Omega^2+x^2\right)^3}\left[(\Omega^2-x^2)+i2\Omega x\right]\implies|\Phi_1|^2=\frac{|E_0|^2}{B^4}=\frac{E^2}{2\kappa_0R^2\left(\Omega^2+x^2\right)^4}\,. \tag{3.146}$$

3.4.2 Embedding in Kerr-Newman-(anti-)de Sitter spacetime

Kerr-Newman-(anti-)de Sitter spacetime (2.68) is an algebraically special solution of type D, depending on parameters m, a, Λ and e. The generic spacetime contains Killing horizons, that are isolated. For special values of its parameters, two or three of these horizons can overlap to create an extremal horizon. In this section we shall identify such extremal horizons with solutions of NHGE given by (3.124). We will show, that every solution obtained in the Section 3.4.1 is embeddable in the extremal KNadS spacetime.

By changing the coordinates in (2.104) with the transformation

$$x=-\cos\theta\,, \tag{3.147}$$

we express the metric of a section of an extremal horizon in KNadS spacetime as

$$\widetilde{q}=\frac{r_0^2+a^2}{\Xi}\left(\frac{\Xi\rho^2}{\Delta_\theta(r_0^2+a^2)}\frac{dx^2}{1-x^2}+\frac{\Delta_\theta(r_0^2+a^2)}{\Xi\rho^2}(1-x^2)d\varphi^2\right)\,. \tag{3.148}$$

By expressing an area of the horizon given by the metric (3.148) with the formula (3.24) we are able to read it radius

$$R^2=\frac{r_0^2+a^2}{\Xi}\,. \tag{3.149}$$

Meanwhile function

$$\widetilde{P}^2=\frac{\Delta_\theta(r_0^2+a^2)}{\Xi\rho^2}(1-x^2) \tag{3.150}$$

must correspond to the function (3.124). To identify the parameters further, we express (3.148) in the same form as (3.124), that is

$$\widetilde{P}^2=\left[\frac{\left(1+\frac{r_0^2}{a^2}\right)\left(1-\frac{\Lambda}{3}r_0^2\right)}{\left(1+\frac{\Lambda}{3}a^2\right)\left(x^2+\frac{r_0^2}{a^2}\right)}+\frac{\frac{\Lambda}{3}a^2\left(1+\frac{r_0^2}{a^2}\right)}{1+\frac{\Lambda}{3}a^2}\right](1-x^2)\,. \tag{3.151}$$

By comparing the coefficients, we see that there has to be

$$\Omega^2=\frac{r_0^2}{a^2}\,, \tag{3.152}$$

and the following equations must also be true:

$$\frac{\left(1+\frac{r_0^2}{a62}\right)\left(1-\frac{\Lambda}{3}r_0^2\right)}{1+\frac{\Lambda}{3}a^2} = (1+\Omega^2)\left(1-\frac{\Lambda}{3}R^2\right)\,, \tag{3.153}$$

$$\frac{\frac{\Lambda}{3}a^2\left(1+\frac{r_0^2}{a62}\right)}{1+\frac{\Lambda}{3}a^2} = \frac{\Lambda}{3}R^2\,. \tag{3.154}$$

They both can be solved, to yield

$$a^2 = \frac{R^2}{1+\Omega^2-\frac{1}{3}\Lambda R^2}\,. \tag{3.155}$$

To find the relationship between E^2 and e^2 we shall rewrite the Equation (3.123), using (3.149) and (3.152):

$$E^2 = \frac{R^2}{a^4}\left(r_0^2-\Lambda r_0^4-a^2\left(1+\frac{\Lambda}{3}r_0^2\right)\right)\,. \tag{3.156}$$

Comparing the result with (2.100), we obtain

$$E^2 = \frac{R^2}{a^4}e^2\,. \tag{3.157}$$

To summarize, the relation between (a, r_0, e) and (Ω, R, E) is

$$a^2 = \frac{R^2}{1+\Omega^2-\frac{1}{3}\Lambda R^2}\,, \tag{3.158}$$

$$r_0^2 = \frac{\Omega R^2}{1+\Omega^2-\frac{1}{3}\Lambda R^2}\,, \tag{3.159}$$

$$e = \frac{ER}{1+\Omega^2-\frac{1}{3}\Lambda R^2}\,; \tag{3.160}$$

while the inverse transformation is

$$\Omega = \frac{r_0}{a}\,, \tag{3.161}$$

$$R = \frac{r_0^2+a^2}{1+\frac{1}{3}\Lambda R^2}\,, \tag{3.162}$$

$$E = \frac{(r_0^2+a^2)e}{(1+\frac{1}{3}\Lambda R^2)a^2}\,. \tag{3.163}$$

Notice that

$$1+\Omega^2-\frac{1}{3}\Lambda R^2 > 0 \quad \text{for} \quad \Lambda R^2 < 1\,. \tag{3.164}$$

We can also express the mass m of KNadS spacetime in terms of R and Ω:

$$m = \Omega R\frac{(1+\Omega^2)\left(1-\frac{2}{3}\Lambda R^2\right)}{\left(1+\Omega^2-\frac{1}{3}\Lambda R^2\right)^{\frac{3}{2}}}\,. \tag{3.165}$$

To establish the correctness of our calculations, we can compare scalar Φ_1

$$|\Phi_1|^2 = \frac{|E_0|^2}{B_0^4\left(\Omega^2+x^2\right)^4} \tag{3.166}$$

from Equation (3.146) with the corresponding scalar in KNadS spacetime, given by

$$|\widetilde{\Phi}_1|^2 = \frac{e^2}{2\kappa_0(r_0^2+a^2x^2)^4}\,. \tag{3.167}$$

Indeed we recover (3.146), only if (3.152) and (3.157) apply.

We will now show, that every isolated horizon obtained in this chapter can be embedded in KNadS spacetime. We will achieve that by analysing relations (3.158) and (3.160), and comparing them with the allowed values of Λa^2 and Λe^2 given by (2.102). We will divide the calculation into two subcases: the one with positive cosmological constant Λ, and the one with the negative Λ. Let us firstly take $\Lambda > 0$, and look at the function $F_1 = \Lambda a^2$ as a function of $\xi = \Omega^2$ and $\eta = \Lambda R^2$:

$$F_1(\xi, \eta) = \frac{\eta}{1 + \xi - \frac{1}{3}\eta} \tag{3.168}$$

in the range (see (3.139)–(3.141))

$$\xi \in [1, \infty] , \qquad \eta \in [0, 1) \qquad \eta \leq \frac{\xi - 1}{\xi - \frac{1}{3}} . \tag{3.169}$$

One can see that F_1 is a continuous, non-negative function in the range that interests us, and that $F_1(\xi, 0) = 0$. In the interior of (3.169) function F_1 does not have a local extremum, and for a given η is a decreasing (to zero) function of ξ. Let us examine its value on a curve

$$\eta = \frac{\xi - 1}{\xi - \frac{1}{3}} . \tag{3.170}$$

In such a case, function F_1 on this curve

$$\widetilde{F}_1(\xi) = \frac{\xi - 1}{\xi(\xi + \frac{1}{3})} \tag{3.171}$$

has a maximum in

$$\xi_1 = 1 + \frac{2}{\sqrt{3}} > 1 , \tag{3.172}$$

which is equal to

$$\widetilde{F}_1(\xi_1) = 3(7 - 4\sqrt{3}) . \tag{3.173}$$

It means that the highest, allowable value of Λa^2 is $3(7 - 4\sqrt{3})$, which agrees with (2.102). The similar analysis can be conducted for a function $F_2 = \Lambda e^2$, which takes the form

$$F_2(\xi, \eta) = \frac{\eta(1 + \xi)\left(\xi - 1 - \eta\left(\xi - \frac{1}{3}\right)\right)}{\left(1 + \xi - \frac{1}{3}\eta\right)^2} . \tag{3.174}$$

Using (3.168) we can introduce Λa^2 instead of η, to get

$$F_2(\xi, \Lambda a^2) = -\Lambda a^2 \xi^2 + \left(1 - \frac{1}{3}\Lambda a^2\right)\xi - \Lambda a^2 . \tag{3.175}$$

By treating Λa^2 as a parameter in F_2, we can see that the maximum value of F_2 is achieved for

$$\xi_2 = \frac{1}{2\Lambda a^2} - \frac{1}{6} , \tag{3.176}$$

and is equal to

$$F_2\left(\xi_2, \Lambda a^2\right) = \frac{1}{36}\left(\Lambda^2 a^4 - 42\Lambda a^2 + 9\right) . \tag{3.177}$$

This result is in line with the permissible range of Λe^2 in (2.102) for KNadS spacetime. Let us now take $\Lambda < 0$, and analyse the function $F_1(\xi, \eta)$ in the range

$$\xi \in \left(\frac{1}{3}, 1\right) , \qquad \eta \in (-\infty, 0) , \qquad \eta \leq \frac{\xi - 1}{\xi - \frac{1}{3}} . \tag{3.178}$$

One is lead to the conclusion that now function F_1 is negative, and it achieves its minimum

$$F_1 \longrightarrow -3 \quad \text{for} \quad \begin{cases} \xi = \text{const.}\,, \\ \eta \longrightarrow -\infty\,. \end{cases} \tag{3.179}$$

Similarly on the curve (3.170) its limit is

$$F_1 \longrightarrow -3 \quad \text{for} \quad \xi \longrightarrow \frac{1}{3}^{+}\,. \tag{3.180}$$

It follows that

$$|\Lambda| a^2 \longrightarrow 3\,, \tag{3.181}$$

and we conclude that

$$a^2 \in \left[0, \frac{3}{|\Lambda|}\right)\,, \tag{3.182}$$

just like in (2.103). The analogous analysis for the function F_2 leads us to $e^2 \in [0, \infty)$, which is also in line with (2.103).

The investigation above proves the following theorem (the analogous one for a horizon without the electromagnetic field was proven in [81]):

Theorem 3.4.1 *Let $\mathcal{H}$ be an extremal, isolated horizon in four-dimensional spacetime, satisfying Einstein's equations with a cosmological constant and an energy-momentum tensor of an electromagnetic field. Suppose $\mathcal{H}$ admits two-dimensional, spacelike cross-section $\mathcal{S}$ diffeomorphic to $\mathbb{S}_2$. Then (q, ω), where q is metric tensor induced on $\mathcal{S}$ and ω is one-form pulled-back to $\mathcal{S}$, coincides with the data defined on a section of an extremal, isolated horizon in Kerr-Newman-(anti-)de Sitter spacetime, for certain values of constants m, a, Λ and e.*

In this section we were investigating the generic EIH. In the following sections we will analyse the particular cases of EIH with $E = 0$, $\Lambda = 0$ and $E = \Lambda = 0$.

3.4.3 Vanishing electromagnetic field ($E = 0$)

This particular solution to NHG equation was analysed in [81]. We will show that the results from an aforementioned article can be reproduced from the calculations in Section 3.4.1, with the assumption $E = 0$. When the electromagnetic field vanishes, the NHG equation (3.113) is

$$\partial_x^2 P^2 + \frac{2(x - x_0)}{(x - x_0)^2 + \Omega^2}\partial_x P^2 + \frac{4\Omega^2}{\left[(x - x_0)^2 + \Omega^2\right]^2} P^2 + 2\Lambda R^2 = 0\,, \tag{3.183}$$

with a solution [81, 85]

$$P^2 = \frac{c_1\left[(x - x_0)^2 - \Omega^2\right] - 2c_2\Omega(x - x_0) - \frac{1}{3}\Lambda R^2 (x - x_0)^2\left[(x - x_0)^2 + 3\Omega^2\right]}{(x - x_0)^2 + \Omega^2}\,. \tag{3.184}$$

To satisfy boundary conditions, we need to set

$$c_1 = 2 - \Lambda R^2\,, \quad c_2 = 0\,, \quad x_0 = 0\,, \quad \Omega^2 = \frac{\frac{1}{3}\Lambda R^2 - 1}{\Lambda R^2 - 1}\,; \tag{3.185}$$

which leads to

$$P^2 = \left(x^2 - 1\right)\frac{\Lambda R^2\left(\Lambda R^2 - x^2(\Lambda R^2 - 1) - 5\right) + 6}{\Lambda R^2 + 3x^2(\Lambda R^2 - 1) - 3}\,. \tag{3.186}$$

Notice, however that it is equivalent to the conditions (3.120)–(3.122), with $E = 0$. We can therefore write the metric as

$$P^2 = (1 - x^2)\left(\frac{\left(\Omega^2 + 1\right)\left(1 - \frac{1}{3}\Lambda R^2\right)}{x^2 + \Omega^2} + \frac{1}{3}\Lambda R^2\right)\,, \tag{3.187}$$

where

$$\Omega^2 = \frac{\frac{1}{3}\Lambda R^2 - 1}{\Lambda R^2 - 1}\,. \tag{3.188}$$

Positivity of both this metric and Ω^2 forces the following restriction:

$$\Lambda R^2 < 1 \implies \Omega^2 > \frac{1}{3}\,. \tag{3.189}$$

To retrieve (3.184) from (3.124) it is sufficient to set $E^2 = 0$, then (3.123) reduces to (3.188). There is, of course, also a degenerate solution

$$P^2 = 1 - x^2\,, \tag{3.190}$$

when we set

$$\Lambda R^2 = 3 \iff \Omega^2 = 0\,. \tag{3.191}$$

We can of course recover (3.190) straight from (3.184) by setting

$$c_1 = 1\,, \qquad x_0 = 0\,, \qquad \Lambda = \frac{3}{R^2}\,, \qquad \Omega = 0\,. \tag{3.192}$$

Interestingly, one retrieves a trivial solution for

$$c_1 = 2\,, \qquad c_2 = 0\,, \qquad x_0 = 0\,, \qquad R = 0\,, \qquad \Omega^2 = 1\,; \tag{3.193}$$

but a vanishing radius R would not make sense in the context of the whole metric.

We can embedd metric (3.186) into extremal Kerr-de Sitter spacetime by taking

$$r_0^2 = \frac{R^2}{2 - \Lambda R^2}\,, \tag{3.194}$$

$$a^2 = \frac{3R^2(1 - \Lambda R^2)}{(3 - \Lambda R^2)(2 - \Lambda R^2)}\,, \tag{3.195}$$

$$m = \frac{2}{3}\sqrt{\frac{R^2}{2 - \Lambda R^2}}\,\frac{(3 - 2\Lambda R^2)^2}{(2 - \Lambda R^2)(3 - \Lambda R^2)}\,. \tag{3.196}$$

We still retain the same range of permissible parameters:

$$\Lambda R^2 \in]-\infty, 1[\,. \tag{3.197}$$

Embedding into Kerr-(anti-)de Sitter spacetime is given by

$$x = -\cos\theta\,, \tag{3.198}$$

$$R^2 = \frac{r_0^2 + a^2}{1 + \frac{1}{3}\Lambda a^2}\,, \tag{3.199}$$

$$\Omega^2 = \frac{r_0^2}{a^2}\,. \tag{3.200}$$

3.4.4 Vanishing cosmological constant ($\Lambda = 0$)

By making cosmological constant vanish, (3.113) reduces to

$$\partial_x^2 P^2 + \frac{2(x - x_0)}{(x - x_0)^2 + \Omega^2}\partial_x P^2 + \frac{4\Omega^2}{\left[(x - x_0)^2 + \Omega^2\right]^2}P^2 + \frac{2E^2}{\left[(x - x_0)^2 + \Omega^2\right]^2} = 0 \tag{3.201}$$

with a solution [84]

$$P^2 = c_1\frac{\Omega^2 - (x - x_0)^2}{\Omega^2 + (x - x_0)^2} + c_2\frac{2\Omega(x - x_0)}{\Omega^2 + (x - x_0)^2} + \frac{E^2}{2\Omega^2}\,. \tag{3.202}$$

Notice that (3.201) it is equivalent to

$$\partial_x^2 Q^2 + \frac{2(x-x_0)}{(x-x_0)^2+\Omega^2}\partial_x Q^2 + \frac{4\Omega^2}{\left[(x-x_0)^2+\Omega^2\right]^2} Q^2 = 0\,, \tag{3.203}$$

with

$$Q^2 = P^2 + \frac{E^2}{2\Omega^2} = c_1 \frac{\Omega^2-(x-x_0)^2}{\Omega^2+(x-x_0)^2} + c_2 \frac{2\Omega(x-x_0)}{\Omega^2+(x-x_0)^2}\,. \tag{3.204}$$

Boundary conditions result in

$$c_1 = \frac{(1+\Omega^2)^2}{2\Omega^2}\,, \qquad c_2 = 0\,, \qquad x_0 = 0\,, \qquad E^2 = (1-\Omega^2)(1+\Omega^2)\,; \tag{3.205}$$

which reduces the solution to

$$P^2 = (1-x^2)\frac{1+\Omega^2}{x^2+\Omega^2}\,, \tag{3.206}$$

or

$$P^2 = (1-x^2)\frac{1+\sqrt{1-E^2}}{x^2+\sqrt{1-E^2}}\,. \tag{3.207}$$

The admissible range of parameters is

$$E^2 < 1 \Longleftrightarrow \Omega^2 < 1\,. \tag{3.208}$$

Embedding into Kerr-Newman spacetime is given by

$$x = -\cos\theta\,, \tag{3.209}$$

$$R^2 = r_0^2 + a^2\,, \tag{3.210}$$

$$\Omega^2 = \frac{r_0^2}{a^2}\,. \tag{3.211}$$

3.4.5 Vacuum ($\boldsymbol{E = \Lambda = 0}$)

For $E = \Lambda = 0$, the equation (3.34) reduces to

$$\partial_x^2 P^2 + \frac{2(x-x_0)}{(x-x_0)^2+\Omega^2}\partial_x P^2 + \frac{4\Omega^2}{\left[(x-x_0)^2+\Omega^2\right]^2} P^2 = 0\,, \tag{3.212}$$

which is also equivalent to (3.203). Solution is therefore of the form (3.204), that is

$$P^2 = c_1 \frac{\Omega^2-(x-x_0)^2}{\Omega^2+(x-x_0)^2} + c_2 \frac{2\Omega(x-x_0)}{\Omega^2+(x-x_0)^2}\,, \tag{3.213}$$

where c_1 and c_2 are integration constants. Boundary conditions (3.52)–(3.53) set

$$c_1 = 2\,, \qquad c_2 = 0\,, \qquad x_0 = 0\,, \qquad \Omega^2 = 1\,; \tag{3.214}$$

which results in

$$P^2 = 2\frac{1-x^2}{1+x^2}\,. \tag{3.215}$$

Retrieving this result from from (3.124) is straightforward. We need to take the limit of

$$E = \Lambda = 0 \tag{3.216}$$

in both (3.124) and (3.123).

One can easily embed this horizon into an extremal Kerr spacetime. The area of the Kerr black hole is

$$A = 8\pi\left(m^2 \pm m\sqrt{m^2-a^2}\right) = 8\pi r_0^2 \tag{3.217}$$

because

$$r_0 = m = |a|\,, \tag{3.218}$$

and we see that metric of an extremal horizon

$$q = 2r_0^2\left(\frac{1}{2}(1+\cos^2\theta)d\theta^2 + 2\frac{\sin^2\theta}{1+\cos^2\theta}d\varphi^2\right) \tag{3.219}$$

is equal to (3.215) by setting

$$x = -\cos\theta\,, \tag{3.220}$$

$$R^2 = r_0^2 + a^2 = 2r_0^2 = 2m^2 = 2a^2\,. \tag{3.221}$$

The extremality assumption results of course in

$$\Omega^2 = \frac{r_0^2}{a^2} = 1\,. \tag{3.222}$$

3.5. $\mathbb{R}\times\mathbb{T}_2$ topology

While we expect an event horizon to be topologically spherical (see e.g [86]), it is nevertheless possible for a toroidal horizon to emerge under special conditions [87, 88]. NHGE on compact two-dimensional manifold of any genus was analysed in [76, 89]. Therefore, this type of horizon stays in the area of our interest. As we have mentioned in Section 3.2, the axially symmetric metric can still be expressed as

$$q = R^2\left(\frac{1}{P^2}dx^2 + P^2d\varphi^2\right)\,, \quad P = P(x)\,, \quad R = \text{const.} \tag{3.223}$$

The NHG equations (3.32)–(3.34) lay on the assumption of axial symmetry generated by ∂_φ and are independent of topology of a horizon. It means we can still utilize them in $\mathbb{R}\times\mathbb{T}_2$ topology. On the other hand the boundary conditions for the funtion $P^2(x)$ will be

$$P^2\left(x=1\right) = P^2\left(x=-1\right)\,, \tag{3.224}$$

$$\partial_x^n P^2\left(x=1\right) = \partial_x^n P^2\left(x=-1\right) \qquad n = 1,2,3,\dots\,. \tag{3.225}$$

These constraints result of course from the assumptions on continuity of g and its derivatives. Similarly we require that components of ω and their derivatives are periodic

$$\omega_x(x=1) = \omega_x(x=-1)\,, \qquad \omega_\varphi(x=1) = \omega_\varphi(x=-1)\,; \tag{3.226}$$

$$\partial_x^n\omega_x(x=1) = \partial_x^n\omega_x(x=-1)\,, \quad \partial_x^n\omega_\varphi(x=1) = \partial_x^n\omega_\varphi(x=-1)\,, \qquad n = 1,2,3,\dots\,. \tag{3.227}$$

Now we can solve (3.32) as before, to get two branches of solution, depending on vanishing of Ω.

First we look at the case of $\Omega \neq 0$. The potentials are

$$U = \arctan\left(\frac{x-x_0}{\Omega}\right) + U_0\,, \tag{3.228}$$

$$B^2 = B_0^2\left[\Omega^2 + (x-x_0)^2\right]\,. \tag{3.229}$$

It follows from (3.226) that

$$\frac{1}{1+\left(\frac{1-x_0}{\Omega}\right)^2} = \frac{1}{1+\left(\frac{-1-x_0}{\Omega}\right)^2} \tag{3.230}$$

$$\frac{-1-x_0}{(x-x_0)^2+\Omega^2} = \frac{1-x_0}{(x-x_0)^2+\Omega^2} \tag{3.231}$$

These equations lead to contradiction, so the solution for $\Omega \neq 0$ does not exist.

The second branch results in

$$U = U_0\,, \tag{3.232}$$
$$B = B_1 x + B_2\,, \tag{3.233}$$

where again we must put $B_1 = 0$ because of (3.226). One-form ω vanishes, satisfying boundary conditions (3.226)–(3.227) automatically. The third NHG equation (2.24) is then reduced to

$$\partial_x^2 P^2 = -2\left(\Lambda R^2 + \frac{2\kappa_0|E_0|^2}{B_2^4}\right) \implies P^2 = -\left(\Lambda R^2 + \frac{2\kappa_0|E_0|^2}{B_2^4}\right)x^2 + c_1 x + c_2\,. \tag{3.234}$$

Boundary conditions for (3.224)–(3.225) result in

$$c_1 = 0\,, \qquad \Lambda R^2 + \frac{2\kappa_0|E_0|^2}{B_2^4} = 0\,, \tag{3.235}$$

so function P^2 is

$$P^2 = \text{const.} \tag{3.236}$$

To summarize EIH with the topology of $\mathbb{R}\times\mathbb{T}_2$ is trivial, that is its metric and one-from ω are

$$q = R^2\left(dx^2 + d\varphi^2\right)\,, \qquad \omega = 0\,. \tag{3.237}$$

The solution only exists for spacetimes with cosmological constant or electromagnetic field satisfying the following constraints:

$$\Lambda R^2 + \frac{2\kappa_0|E_0|^2}{B_2^4} = 0 \qquad \text{for} \qquad \Lambda < 0\,, \tag{3.238}$$
$$E_0 = 0 \qquad \text{for} \qquad \Lambda = 0\,. \tag{3.239}$$

In the case of $E_0 = 0$, the non-existence of solution for $\Lambda < 0$ were known [90].

The results above are in agreement with Section 3.1.1.

Table 3.1: The function $P^2(x)$ of topologically spherical EIH for particular values of different parameters, together with spacetimes they are embeddable in.

Ω	other parameters	form of $P^2(x)$	embeddable in
$\Omega = 0$	$\Lambda = \frac{1}{R^2}\,, \quad E_0 = 0$	$P^2 = 1 - x^2$	Schwarzschild-de Sitter
$\Omega = 0$	$\Lambda = 0\,, \quad E_0 = \text{const.}$	$P^2 = 1 - x^2$	Reissner-Nordström
$\Omega = 0$	$\Lambda R^2 + \frac{4\kappa_0 R^2 \lvert E_0\rvert^2}{B_2^4} = 1$	$P^2 = 1 - x^2$	Reissner-Nordström-(anti-)de Sitter
$\Omega = \pm 1$	$E = \Lambda = 0$	$P^2 = 2\frac{1-x^2}{1+x^2}$	Kerr
$\Omega \in \left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$	$\Lambda \in \left(-\infty, \frac{1}{R^2}\right)\,, \quad E = 0$	$P^2 = \left(x^2 - 1\right) \frac{\Lambda R^2 \left(\Lambda R^2 - x^2(\Lambda R^2 - 1) - 5\right) + 6}{\Lambda R^2 + 3x^2(\Lambda R^2 - 1) - 3}$	Kerr-(anti-)de Sitter
$\Omega \in (-1, 1)$	$\Lambda = 0\,, \qquad E^2 \in (0, 1)$	$P^2 = (1 - x^2)\frac{1 + \sqrt{1 - E^2}}{x^2 + \sqrt{1 - E^2}}$	Kerr-Newman
$\Omega^2 \geq \frac{1 - \frac{1}{3}\Lambda R^2}{1 - \Lambda R^2}$	$\begin{cases} \Lambda \in \left(-\infty, \frac{1}{R^2}\right) \\ E^2 = \left(\Omega^2 + 1\right)\left(\Omega^2 - 1 - \Lambda R^2\left(\Omega^2 - \frac{1}{3}\right)\right) \end{cases}$	$P^2 = (1 - x^2)\left(\frac{\left(\Omega^2 + 1\right)\left(1 - \frac{1}{3}\Lambda R^2\right)}{x^2 + \Omega^2} + \frac{1}{3}\Lambda R^2\right)$	Kerr-Newman-(anti-)de Sitter

Chapter 4

Topologically non-trivial isolated horizons

In this chapter we will consider extremal, isolated horizons with nontrivial topology. We assume the axial symmetry of the cross-section of EIH. We are also analysing embeddability of such horizons into Plebański-Demiański spacetimes.

4.1. Non-trivial horizon

In the chapter 3 we have exhausted all possible product topologies of type $\mathbb{R} \times \mathcal{S}$. Now we will engage in the study of exotic topologies – the ones without global section (see [41, 42, 91–95])

Similarly to trivial cases, we can consider topologically non-trivial horizons as principal fibre bundles. If the structure group is defined by the flow of the null vector field ℓ, then following the same argument as for trivial bundles, one can show that (q, ω) defined on $\mathcal{S}$ are regular everywhere. The solution to NHGE are thus in parallel to those from Chapter 3. The most interesting case however, is when the horizon null generators are not the fibres of the bundle [42]. We will now proceed with an explicit construction of such a non-trivial EIH. Let us consider an axisymmetric metric q on a (topological) sphere $\mathbb{S}_2$, defined everywhere, except for poles; together with and axisymmetric one-form ω, defined and regular everywhere on $\mathbb{S}_2$. The metric q shall have conical singularities at the poles (see Section 3.2.2). Such a metric is given by

$$q = \frac{R^2}{P^2(x)} dx^2 + R^2 P^2(x) d\varphi^2 \,, \tag{4.1}$$

but there exist constants β_- and β_+, defining auxillary metrics q_- and q_+, given by

$$q_\pm = \frac{R^2}{P^2(x)} dx^2 + \beta_\pm^2 R^2 P^2(x) d\varphi^2 \,. \tag{4.2}$$

These metrics are extendable in a differentiable way (that is, they are in the class C^k for some $k \in \mathbb{N} \setminus \{0\}$) to the pole $x = -1$, and $x = 1$ respectively. On such a manifold, we can define a C^k one-form ω

$$\omega = \omega_x dx + \omega_\varphi d\varphi \,. \tag{4.3}$$

We will now construct everywhere regular, extremal isolated horizon with $\mathbb{S}_3$ topology. Consider now two three-dimensional manifolds:

$$\mathcal{H}_+ = \mathbb{S}_2 \setminus \{p_+\} \times \mathbb{S}_1 \tag{4.4}$$

$$\mathcal{H}_- = \mathbb{S}_2 \setminus \{p_-\} \times \mathbb{S}_1 \tag{4.5}$$

where points p_-, and p_+ correspond to the poles $x = -1$, and $x = 1$ respectively. Submanifolds $\mathbb{S}_2$ are described by coordinates $(x_\pm, \varphi_\pm)$, while submanifolds $\mathbb{S}_1$ are equipped with a cyclic (range is $[0, 2\pi)$) coordinate $v_\pm$. On manifolds $\mathcal{H}_\pm$ we define degenerate metric tensors

$$q^\pm = \frac{R^2}{P^2(x_\pm)} dx_\pm^2 + R^2 P^2(x_\pm) \left[\beta_\mp d\varphi_\pm + (\beta_\pm - \beta_\mp) \, dv_\pm\right]^2 , \tag{4.6}$$

and rotation one-forms

$$\omega^{\pm} = \omega_x(x_{\pm})dx_{\pm} \pm \omega_{\varphi}\left[\beta_{\mp} d\varphi_{\pm} + (\beta_{\pm} - \beta_{\mp})\, dv_{\pm}\right] . \tag{4.7}$$

All metrics and one-forms are regular in their domain. We can now consider $\mathcal{H}_+$, and $\mathcal{H}_+$ as two charts of three-dimensional manifold $\mathcal{H}$, that is equipped with degenerate metric q, and one-from ω. The coordinate transformation between those two charts is

$$x_+ = x_- \,, \qquad \varphi_+ = -\varphi_- \,, \qquad v_+ = v_- - \varphi_- \,, \qquad \text{for} \qquad x_{\pm} \notin \{-1, 1\} \,. \tag{4.8}$$

There are two symmetry generators

$$\xi_L = \frac{\partial}{\partial v_+} = \frac{\partial}{\partial v_-} \,, \tag{4.9}$$

$$\xi_R = \frac{\partial}{\partial v_+} + 2\frac{\partial}{\partial \varphi_+} = -\frac{\partial}{\partial v_-} - 2\frac{\partial}{\partial \varphi_-} \,; \tag{4.10}$$

and both of them endow $\mathcal{H}$ with the structure of $U(1)$ principal bundle bundle over $\mathbb{S}_2$. Natural projection reduce metric $q_{\pm}$ to (4.2). The null generator of $\mathcal{H}$ is given by

$$\ell = \frac{1}{\beta_+}\frac{\partial}{\partial v_+} + \left(\frac{1}{\beta_+} - \frac{1}{\beta_-}\right)\frac{\partial}{\partial \varphi_+} = \frac{1}{\beta_-}\frac{\partial}{\partial v_-} + \left(\frac{1}{\beta_-} - \frac{1}{\beta_+}\right)\frac{\partial}{\partial \varphi_-} \,. \tag{4.11}$$

If

$$\beta_+ = \beta_- = \beta \,, \tag{4.12}$$

then

$$\ell = \xi_L = \xi_R \,; \tag{4.13}$$

and the space of null curves of $\mathcal{H}$ is $\mathbb{S}_2$. In the general case however the null generator is

$$\ell = \frac{1}{2(\beta_+ + \beta_-)}\left[(\beta_+ + \beta_-)\xi_L + (\beta_+ - \beta_-)\xi_R\right] \,, \tag{4.14}$$

and the space of null curves is not a differentiable manifold, even though $\mathcal{H}$ is regular. The genericity condition is

$$\frac{\beta_+ - \beta_-}{\beta_+ + \beta_-} \notin \mathbb{Q} \,. \tag{4.15}$$

Finally let us assume, that metric q, together with ω satisfies near-horizon geometry equation (2.24). Then the pull-backs of $g^{(3)}$ and $\omega^{(3)}$ to any local, spacelike section of $\mathcal{H}$ must also satisfy it. We can then call $\mathcal{H}$ an extremal, isolated horizon. As one can see, the whole construction sbove allows us to reduce the problem of solving NHGE on topologically non-trivial EIHs reduces to solving the same NHGE as for a topologically trivial EIH, for one metric on $\mathbb{S}_2$. The difference is lax boundary conditions. It is sufficient to demand, that

$$P(x = \pm 1) = 0 \,, \tag{4.16}$$

to make the metric regular outside of poles. We can quantify the degree of conical singularities, using the ratio of length to the radius of the infinitesimally small loop around the pole, just like in (3.57). By subtracting it from 2π, we get an angle deficit $\delta_{\pm}$

$$\delta_{\pm} = 2\pi\left(1 \pm \frac{1}{2}\partial_x P^2(\pm 1)\right) . \tag{4.17}$$

Setting $\partial_x P^2(\pm 1) = \mp 2$ would makes angle deficits vanish, trivializing the horizon. Hence we reject this boundary condition. As equations (3.32)–(3.33) do not contain cosmological constant or any components dependent on the electromagnetic field, we can integrate them, to once again get either (3.37) for $\Omega = 0$, or (3.44)–(3.45) for $\Omega \neq 0$.

Similarly as in the trivial case, one can analyze IHs and EIHs in the global context. Either as a whole spacetime or as degenerate Killing horizons [9, 90].

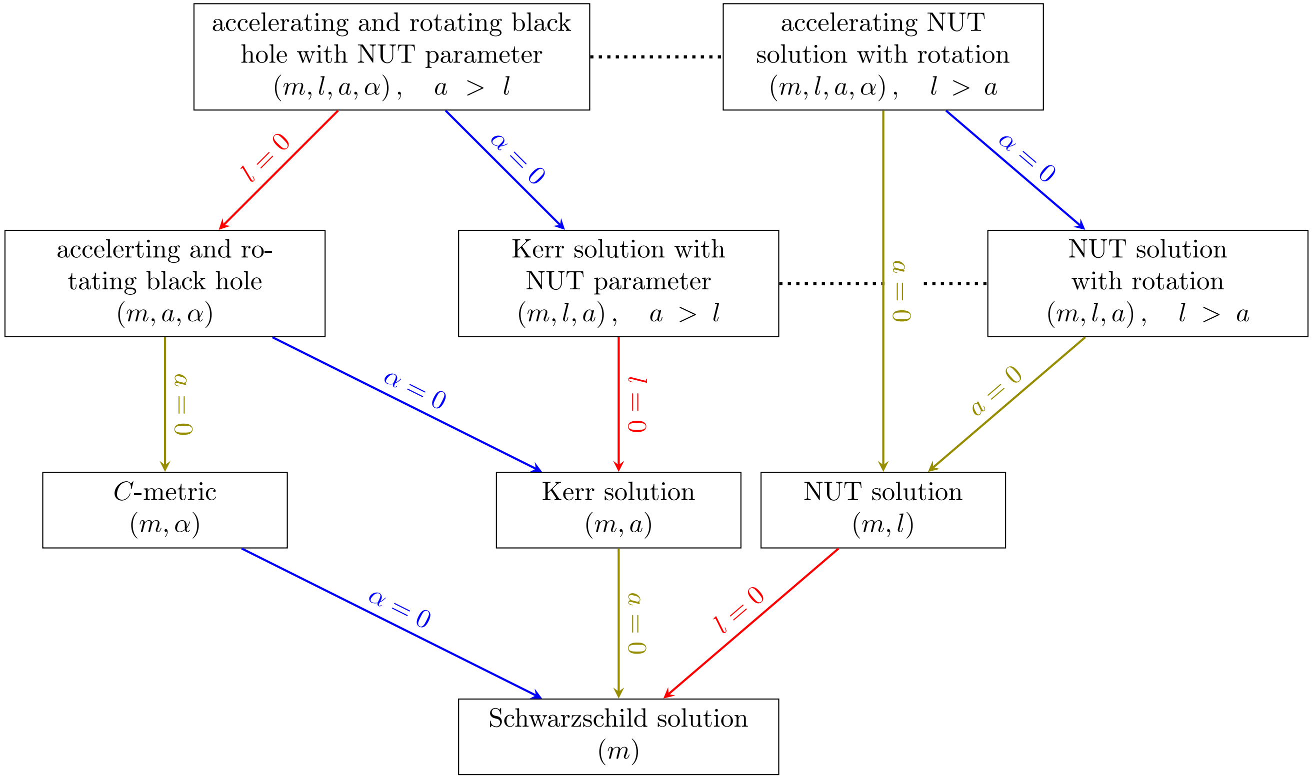

FIGURE 4.1: Relations between Plebański-Demianański spacetime and other known black hole solutions, for $e = \Lambda = 0$. The parameters in every metric are given in parentheses. It is trivial to generalize this diagram to the cases with non-vanishing charges or cosmological constant. Reproduced from [96] and [97]. See also [98]

4.2. Plebański-Demiański spacetimes

Plebański-Demiański spacetimes represent the complete class of type D (it has two principal, null directions) exact solutions to Einstein-Maxwell equations with double-aligned non-null electromagnetic field and cosmological constant. This family of solutions was derived by Plebański and Demiański [99], with earlier, less complete results by Carter [100], Kinnersley [101] and Debever [102] and others (see [97], and references therein). A more convenient form was later found by Griffiths and Podolsky [96], and then by Podolský, and Vrátný [103]. Our nomenclature is taken from the latter work. Plebański-Demiański spacetime has the general form

$$\begin{aligned} g = \frac{1}{\Xi^2}\Bigg(-\frac{Q}{\rho^2}\left[dt-\left(a\sin^2\theta+4l\sin^2\frac{1}{2}\theta\right)d\varphi\right]^2+\frac{\rho^2}{Q}dr^2+\frac{\rho^2}{T}d\theta^2 \\ +\frac{T}{\rho^2}\sin^2\theta\left[a dt-\left(r^2+(a+l)^2\right)d\varphi\right]^2\Bigg)\,, \end{aligned} \tag{4.18}$$

where

$$\Xi(r,\theta) = 1-\frac{a\alpha}{a^2+l^2}r(l+a\cos\theta)\,, \tag{4.19}$$

$$\rho^2(r,\theta) = r^2+(l+a\cos\theta)^2\,, \tag{4.20}$$

$$\begin{aligned} T(\theta) = 1-2\left(\frac{a\alpha}{a^2+l^2}m-\frac{\Lambda}{3}l\right)(l+a\cos\theta) \\ +\left(\frac{a^2\alpha^2}{(a^2+l^2)^2}(a^2-l^2+e^2)+\frac{\Lambda}{3}\right)(l+a\cos\theta)^2\,, \end{aligned} \tag{4.21}$$

$$\begin{aligned} Q(r) = \left[r^2-2mr+a^2-l^2+e^2\right]\left(1+a\alpha\frac{a-l}{a^2+l^2}r\right)\left(1-a\alpha\frac{a+l}{a^2+l^2}r\right) \\ -\frac{\Lambda}{3}r^2\left[r^2+2a\alpha l\frac{a^2-l^2}{a^2+l^2}r+a^2+3l^2\right]\,. \end{aligned} \tag{4.22}$$

There are six parameters characterizing this metric. By putting some of them to zero in an appropriate way, we can retrieve other, existing black hole solutions, as indicated in Figure 4.1. Parameters can be endowed with physical meaning, which can be seen in Table 4.1, but the physical meaning of the whole spacetime is still an open question [103]. In particular parameter l is connected to torison singularity in these spacetimes. We note however that its meaning in Taub–NUT spacetime was somewhat elucidated in [104] and [105]. The geodesics can be extended through singular (so-called) Misner string, making such spacetimes geodesically complete. Furthermore they do not contain causal timelike or null closed geodesics. This may allow one to adopt Taub-Nut spacetimes as physical, and gives hopes to applying analogous analysis to Plebański-Demiański spacetimes. Notice, that to retrieve for example

Table 4.1: Parameters of Demiański-Plebański spacetime, with their physical interpretation.

parameter	physical meaning
m	mass
a	Kerr-like rotation
e	electric charge
l	so-called NUT parameter; difference between the rotation of the axes $\theta = 0$ and $\theta = \pi$
α	acceleration of the black hole
Λ	cosmological constant

Kerr-Newman-(anti-)de Sitter metric in the form (2.68) we have to transform

$$dt \longrightarrow \frac{dt}{1+\frac{1}{3}\Lambda a^2}\,, \tag{4.23}$$

$$d\varphi \longrightarrow \frac{d\varphi}{1+\frac{1}{3}\Lambda a^2}\,, \tag{4.24}$$

and set $\alpha = l = 0$.

The electromagnetic field is described by the potential

$$A = -\frac{er}{\rho^2}dt + \frac{er\left(a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta\right)}{\rho^2}d\varphi\,, \tag{4.25}$$

so the solution without electromagnetic field indeed corresponds to $e = 0$.

We can choose the following NP tetrad

$$\ell = \frac{1}{\sqrt{2}}\frac{\Xi}{\rho}\left[\frac{1}{\sqrt{Q}}\left(\left(r^2+(a+l)^2\right)\partial_t + a\partial_\varphi\right) + \sqrt{Q}\partial_r\right]\,, \tag{4.26}$$

$$n = \frac{1}{\sqrt{2}}\frac{\Xi}{\rho}\left[\frac{1}{\sqrt{Q}}\left(\left(r^2+(a+l)^2\right)\partial_t + a\partial_\varphi\right) - \sqrt{Q}\partial_r\right]\,, \tag{4.27}$$

$$m = \frac{1}{\sqrt{2}}\frac{\Xi}{\rho}\left[\frac{1}{\sqrt{T}\sin\theta}\left(\partial_\varphi + \left(a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta\right)\partial_t\right) + i\sqrt{T}\partial_\theta\right]\,; \tag{4.28}$$

which results in the following spin coefficients

$$\kappa = \nu = \sigma = \lambda = 0\,, \tag{4.29}$$

$$\varrho = \mu = -\frac{\sqrt{Q}}{\sqrt{2}\rho^3}\left(1 + \mathrm{i}\frac{\alpha a}{a^2+l^2}(l + a\cos\theta)^2\right)\left(r - \mathrm{i}(l + a\cos\theta)\right)\,, \tag{4.30}$$

$$\tau = \pi = -\frac{a\sqrt{T}\sin\theta}{\sqrt{2}\rho^3}\left(1 - \mathrm{i}\frac{\alpha a}{a^2+l^2}r^2\right)\left(r - \mathrm{i}(l + a\cos\theta)\right)\,. \tag{4.31}$$

One can see, that for $Q = 0$ expansion and vorticity of congruence generated by ℓ vanish (see Appendix B). In NPF the non-vanishing scalar, describing Maxwell field is

$$\Phi_1 = \frac{1}{2}e\frac{(1 - ar\cos\theta)^2}{(r_0 + i(l + a\cos\theta))^2}\,, \tag{4.32}$$

so the electromagnetic field indeed vanishes for $e = 0$. The determinant of the whole metric g is

$$\det(g) = -\frac{\rho^2}{\Xi^6}Q\sin^2\theta\,, \tag{4.33}$$

so it indicates irregularity at $Q = 0$. Moreover, it can be seen, that radial coordinate r is timelike in the region $Q < 0$ and spacelike for $Q > 0$. This leads us to guess that the horizon exists for $r = r_0$, such that

$$\det\left(g|_{r=r_0}\right) = 0 \implies Q(r_0) = 0\,. \tag{4.34}$$

Indeed the linear combination of Killing vector fields ∂_t and ∂_φ

$$X = \partial_t + \frac{a}{r^2 + (a+l)^2}\partial_\varphi \tag{4.35}$$

is null at $r = r_0$, while Kretschmann scalar of g is not [103]. One can show that the horizon becomes extremal when the discriminant of Q also vanishes (and so does the surface gravity)

$$Q(r_0) = \mathrm{Disc}\left(Q(r_0)\right) = 0 \iff Q(r_0) = \partial_r Q(r_0) = 0\,. \tag{4.36}$$

Table 4.2: Limits of Demiański-Plebański spacetime, with coordinate r_0 of EIH, with acceleration $\alpha = 0$.

name	parameters	r_0
Kerr-NUT	$\Lambda = e = \alpha = 0$ $m^2 = a^2 - l^2$	$r_0 = m$
Kerr-NUT-(anti-)de Sitter	$e = \alpha = 0$; m is given by (4.41), with γ given by (4.43).	Given by Equation (4.39)–(4.40).
Kerr-NUT-Newman	$\Lambda = \alpha = 0$ $m^2 - l^2 = e^2$	$r_0 = m$
Kerr-NUT-Newman-$\tilde{(}$anti-)de Sitter	$\alpha = 0$; m is given by (4.41), with γ given by (4.42).	Given by Equation (4.39)–(4.40).

The above condition for derivative of Q is given explicitly by

$$\begin{aligned}\partial_r Q(r_0) = 0 = 2\left(r_0 - m\right)\left(1 + a\alpha\frac{a-l}{a^2+l^2}r_0\right)\left(1 - a\alpha\frac{a+l}{a^2+l^2}r_0\right)\\ - 2a\alpha\frac{r_0^2 - 2mr_0 + a^2 - l^2 + e^2}{a^2+l^2}\left(l + a\alpha r\frac{a^2-l^2}{a^2+l^2}\right)\\ - 2\frac{\Lambda}{3}r_0\left(2r_0^2 + a^2 + 3l^2 + 3a\alpha l\frac{a^2-l^2}{a^2+l^2}\right).\end{aligned} \tag{4.37}$$

This together with $Q(r_0) = 0$ results in constraints on our parameters. One could, for example, eliminate Λ and m from many further equations, by expressing them as functions of other parameters

$$\Lambda = \Lambda(\alpha, a, l, e)\,, \qquad m = m(\alpha, a, l, e)\,. \tag{4.38}$$

In the generic case, we have four horizons – two black hole horizons $\mathcal{H}_b^{\pm}$ and two cosmological horizons $\mathcal{H}_c^{\pm}$.

There are a few particular limits of metric (4.18) that we will be interested in. The roots of Q for spacetimes without acceleration have simple expressions and are summarized in Table (4.2). Horizon is located at [103]

$$r_0 = \sqrt{\frac{1}{2\Lambda}}\sqrt{X - \left(X^3 - 18\Lambda m^2 + 12\Lambda X\right)^{\frac{1}{3}}}\,, \tag{4.39}$$

where

$$X = \left(1 - \frac{\Lambda}{3}(a^2 + 3l^2)\right)\,, \tag{4.40}$$

and mass is expressed by other parameters as

$$m = \gamma X + \frac{\Lambda}{12} + \frac{\sqrt{64\gamma^3\Lambda^2 + \Lambda^2 X^6 + 12\gamma\Lambda^2 X^4 + 48\gamma^2\Lambda^2 X^2}}{12\Lambda^2}\,, \tag{4.41}$$

$$\gamma = a^2 - l^2 + e^2\,. \tag{4.42}$$

If we take the limit of $e = 0$, then the form of r_0 stays the same, but

$$\gamma = a^2 - l^2\,. \tag{4.43}$$

This defines the position of EIH on Kerr-NUT-(anti-)de Sitter.

As we have mentioned in the previous section, the Plebański-Demiański metric can be covered with two coordinate patches, one regular in 0, and one regular in π. If we regularize the metric in one pole, then there is angle deficit/excess δ_π, and δ_0 on the other respectively.

Deficits are given by [106]

$$\delta_0 = \frac{8\pi a\left[\alpha a\left[m\left(a^2+l^2\right)-\alpha a l\left(a^2-l^2+e^2\right)\right]-\frac{2}{3}\Lambda l\left(a^2+l^2\right)^2\right]}{\left\{\begin{array}{l}\left[1+\dfrac{1}{3}\Lambda(a-l)(a-3l)\right]\left(a^2+l^2\right)^2+2\alpha a m(a-l)\left(a^2+l^2\right)\\ \qquad +\alpha^2a^2(a-l)^2\left(a^2-l^2+e^2\right)\end{array}\right\}}\,, \tag{4.44}$$

$$\delta_\pi = -\frac{8\pi a\left[\alpha a\left[m\left(a^2+l^2\right)-\alpha a l\left(a^2-l^2+e^2\right)\right]-\frac{2}{3}\Lambda l\left(a^2+l^2\right)^2\right]}{\left\{\begin{array}{l}\left[1+\dfrac{1}{3}\Lambda(a+l)(a+3l)\right]\left(a^2+l^2\right)^2-2\alpha a m(a+l)\left(a^2+l^2\right)\\ \qquad +\alpha^2a^2(a+l)^2\left(a^2-l^2+e^2\right)\end{array}\right\}}\,; \tag{4.45}$$

for $\Lambda \neq 0$, and by

$$\delta_0 = \frac{8\pi\alpha a^2\left[m\left(a^2+l^2\right)-\alpha a l\left(a^2-l^2+e^2\right)\right]}{\left(a^2+l^2\right)^2+2\alpha a m(a-l)\left(a^2+l^2\right)+\alpha^2a^2(a-l)^2\left(a^2-l^2+e^2\right)}\,, \tag{4.46}$$

$$\delta_\pi = \frac{-8\pi\alpha a^2\left[m\left(a^2+l^2\right)-\alpha a l\left(a^2-l^2+e^2\right)\right]}{\left(a^2+l^2\right)^2-2\alpha a m(a+l)\left(a^2+l^2\right)+\alpha^2a^2(a+l)^2\left(a^2-l^2+e^2\right)}\,; \tag{4.47}$$

for $\Lambda = 0$. The usual interpretation of such singularities are related to the physical force along the axis that causes the acceleration, described by the parameter α (see [107] and references therein). One should note however, that it is possible to find such a class of observers, that perceive no conical singularity along the axis [107].

4.3. Topologically non-trivial EIH with $\mathbf{\Omega = 0}$

If the constant Ω vanishes, then the general solution of NHG equation (2.24) are functions U and B, given in (3.37), and the function P^2, satisfying boundary conditions (3.52). To analyse the positivity of the latter one, we shall write it as

$$P^2 = \frac{1-x^2}{(B_1x+B_2)^2}\left[\frac{2\kappa_0R^2|E_0|^2}{B_2^2-B_1^2}+\frac{\Lambda R^2}{3}(B_1x+3B_2)(B_1x+B_2)\right]\,, \tag{4.48}$$

remembering that

$$B = B_1x+B_2 > 0\,, \qquad B_2 > |B_1|\,. \tag{4.49}$$

We can see that

$$B_2^2-B_1^2 > 0\,, \qquad B_1x+3B_2 = B+2B_1 > 0\,. \tag{4.50}$$

It follows that for $\Lambda \geq 0$ the function P^2 is always positive. For $\Lambda < 0$ we must find the greatest value $v_{\max}$ of a polynomial

$$v(x) = (B_1x+3B_2)(B_1x+B_2) = (B+2B_2)B\,, \qquad x\in[-1,1]\,. \tag{4.51}$$

For generic B_2, constant B_1 can be negative, and we can show that

$$v_{\max} = \left(|B_1|x+3B_2\right)\left(|B_1|x+B_2\right)\,. \tag{4.52}$$

The positivity condition is therefore

$$\kappa_0|E_0|^2 > \frac{|\Lambda|}{6}(|B_1|x+3B_2)(|B_1|x+B_2)(B_2^2-B_1^2) \tag{4.53}$$

The solution (together with boundary conditions) with vanishing electromagnetic field has constants c_1 and c_2 given by (3.88), which results in

$$P^2 = \frac{\Lambda R^2}{3}\left(1-x^2\right)\frac{B_1x+3B_2}{B_1x+B_2}\,. \tag{4.54}$$

Using the same arguments as for general solution, we can see, that P^2 is positive only for $\Lambda > 0$.

For vanishing cosmological constant function P^2 is equal to (3.99)

$$P^2 = (1-x^2)\frac{2\kappa_0 |E_0|^2 R^2}{(B_2^2 - B_1^2)(B_1 x + B_2)^2}\,, \tag{4.55}$$

and it is well-defined for any value of E_0.

There is no solution for a complete vacuum, that would satisfy (3.52).

4.4. Topologically non-trivial EIH with $\Omega \neq 0$

4.4.1 General solution

The general solution to (2.24), satisfying boundary conditions (3.52) is a function given by (3.115), where c_1 and c_2 are given by (3.120)–(3.121). The details can be found in Appendix C. Just like in topologically trivial case we need to consider two possibilities. Firstly we can assume that $\Omega^2 + x_0^2 \neq 1$, then the function P^2 is given by

$$P^2 = \frac{1}{3}\frac{\Lambda R^2}{\Omega^2 + (x-x_0)^2}(1-x^2)\left[(x-x_0)(x-3x_0) + 3\Omega^2 + \frac{4\Omega^2 + \frac{3}{\Lambda R^2}E^2}{\Omega^2 + x_0^2 - 1}\right], \tag{4.56}$$

where we assume $\Lambda \neq 0$. The second case – $\Omega^2 + x_0^2 \neq 1$ – will be analysed later. In the generic case, non-negativity of the function P^2 puts constraints on parameters Λ, R^2, x_0 and Ω. Let us introduce the notation

$$w(x) = (x-x_0)(x-3x_0)\,, \tag{4.57}$$

and investigate the sign of the function

$$f(x) = w(x) + 3\Omega^2 + \frac{4\Omega^2 + \frac{3}{\Lambda R^2}E^2}{\Omega^2 + x_0^2 - 1}\,. \tag{4.58}$$

Notice that polynomial $w(x)$ (and the whole function $f(x)$) has minimum in

$$x = x_{\min} = 2x_0 \qquad \text{where} \qquad w(x_{\min}) = -x_0^2 \leq 0 \tag{4.59}$$

Let us firstly take cosmological constant Λ to be positive. In this case P^2 is well-defined if $f(x) > 0$ for $x \in (-1,1)$. One can now see, that positivity of $f(x)$ depends on parameter x_0 in the following way:

1. If $|x_0| > 1$, then the minimum of $w(x)$ is outside of the range $[-1,1]$. Moreover $w(-1) > 0$ for $x_0 < -1$ and $w(1) > 0$ for $x_0 > 1$. The denominator $\Omega^2 + x_0^2 - 1$ is also positive and it follows, that $f(x) > 0$. We conclude, that for $|x_0| > 1$ and any values of Ω and E, the function P^2 is well-defined.

2. If $|x_0| \leq \frac{1}{2}$, then the minimum of $w(x)$ is in the range $x \in [-1,1]$. The positivity of P^2 is then assured by the inequality

$$-x_0^2 + 3\Omega^2 + \frac{4\Omega^2 + \frac{3}{\Lambda R^2}E^2}{\Omega^2 + x_0^2 - 1} > 0\,, \tag{4.60}$$

that can be rewritten as

$$\frac{(\Omega^2 + x_0^2)(3\Omega^2 - x_0^2 + 1) + \frac{3}{\Lambda R^2}E^2}{\Omega^2 + x_0^2 - 1} > 0\,. \tag{4.61}$$

Notice, that

$$\frac{3}{4} \leq 1 - x_0^2 \leq 1 \implies 3\Omega^2 - x_0^2 + 1 > 0\,, \tag{4.62}$$

which means, that $f(x)$ is positive if $\Omega^2 + x_0^2 > 1$.

3. If $\frac{1}{2} < x_0 < 1$, then the minimum of the polynomial $w(x)$ is situated in $x > 1$. It follows, that $f(x) > 0$ if $f(x) \geq 0$, that is
$$\frac{\left(\Omega^2 + (x_0 - 1)^2\right)\left(\Omega^2 + \left(x_0 + \frac{1}{3}\right)^2 - \frac{4}{9}\right) + \frac{3}{\Lambda R^2}E^2}{\Omega^2 + x_0^2 - 1} \geq 0 \,. \tag{4.63}$$
Notice, that
$$\frac{1}{4} < \left(x_0 + \frac{1}{3}\right)^2 - \frac{4}{9} < \frac{4}{3} \,, \tag{4.64}$$
so the numerator in (4.63) is always positive. Therefore $f(x_0) > 0$ for $\Omega^2 + x_0^2 > 1$
4. The analysis for $-1 < x_0 < -\frac{1}{2}$ is analogous as the one for $\frac{1}{2} < x_0 < 1$.

To summarize, when $\Lambda > 0$, the function P^2 is well-defined only if
$$\Omega^2 + x_0^2 > 1 \,, \qquad E^2 > 0 \,. \tag{4.65}$$

Now we will now take Λ to be negative. Now the function P^2 is well-defined if $f(x) < 0$ for $x \in [-1, 1]$. We can break up our analysis into two parts:

1. Notice, that for $x_0 < 0$, the minimum of the function $f(x)$ is located in $x = 2x_0 < 0$, which means, that $f(-1) < f(1)$. Function $f(x)$ must then take its maximal value at $x = 1$. It follows that the sufficient condition for $f(x)$ to be negative in $x \in [-1, 1]$ is $f(1) \leq 0$. Ne have
$$f(1) = 3\frac{\left(\Omega^2 + (x_0 - 1)^2\right)\left(\Omega^2 + \left(x_0 + \frac{1}{3}\right)^2 - \frac{4}{9}\right) + \frac{3}{\Lambda R^2}E^2}{\Omega^2 + x_0^2 - 1} \leq 0 \,, \tag{4.66}$$
which imposes the following constraints on parameters:
$$\begin{aligned} &E^2 \geq |\Lambda| R^2 \left(\Omega^2 + (x_0 - 1)^2\right)\left(\Omega^2 + \left(x_0 + \frac{1}{3}\right)^2 - \frac{4}{9}\right) \\ &\qquad \text{for} \quad \Omega^2 + x_0^2 > 1 \,, \end{aligned} \tag{4.67}$$
$$\begin{aligned} &E^2 \leq |\Lambda| R^2 \left(\Omega^2 + (x_0 - 1)^2\right)\left(\Omega^2 + \left(x_0 + \frac{1}{3}\right)^2 - \frac{4}{9}\right) \\ &\qquad \text{for} \quad \Omega^2 + x_0^2 < 1 \quad \text{and} \quad \Omega^2 + \left(x_0 + \frac{1}{3}\right)^2 \geq \frac{4}{9} \,. \end{aligned} \tag{4.68}$$
2. If $x_0 \geq 0$, then function $f(x)$ attains its maximal value at $x = -1$, so we have to consider an inequality
$$f(1) = 3\frac{\left(\Omega^2 + (x_0 + 1)^2\right)\left(\Omega^2 + \left(x_0 - \frac{1}{3}\right)^2 - \frac{4}{9}\right) + \frac{3}{\Lambda R^2}E^2}{\Omega^2 + x_0^2 - 1} \leq 0 \,, \tag{4.69}$$
which implies
$$\begin{aligned} &E^2 \geq |\Lambda| R^2 \left(\Omega^2 + (x_0 + 1)^2\right)\left(\Omega^2 + \left(x_0 - \frac{1}{3}\right)^2 - \frac{4}{9}\right) \\ &\qquad \text{for} \quad \Omega^2 + x_0^2 > 1 \,, \end{aligned} \tag{4.70}$$
$$\begin{aligned} &E^2 \leq |\Lambda| R^2 \left(\Omega^2 + (x_0 + 1)^2\right)\left(\Omega^2 + \left(x_0 - \frac{1}{3}\right)^2 - \frac{4}{9}\right) \\ &\qquad \text{for} \quad \Omega^2 + x_0^2 < 1 \quad \text{and} \quad \Omega^2 + \left(x_0 - \frac{1}{3}\right)^2 \geq \frac{4}{9} \,. \end{aligned} \tag{4.71}$$

In the special case, when

$$\Omega^2 + x_0^2 = 1\,, \tag{4.72}$$

there has to be

$$E^2 = -\frac{4}{3}\Lambda R^2(1 - x_0^2)\,, \tag{4.73}$$

so it means, that cosmological constant has to be non-positive. The function P^2 is given by

$$P^2 = \frac{1}{3}\Lambda R^2(1 - x^2)\frac{x^2 - 4xx_0 + 4 + c}{x^2 - 2xx_0 + 1}\,, \tag{4.74}$$

where c is a constant.

4.4.2 Embedding into Plebański-Demiański spacetime

We will find the transformation between abstract, extremal isolated horizon given by the metric (4.56), and an isolated extremal horizon in Plebański-Demiański (PB) spacetime (4.18)–(4.22). The procedure of the construction of metric tensor of the Killing horizon in PB spacetime (4.18) leads to the following formula

$$\widetilde{q} = \frac{1}{\Xi^2(r_0,\theta)}\left(\frac{\rho^2(r_0,\theta)}{T(\theta)}d\theta^2 + \frac{T(\theta)}{\rho^2(r_0,\theta)}\sin^2\theta\left(r_0^2 + (a+l)^2\right)^2\beta^2 d\varphi^2\right)\,, \tag{4.75}$$

where $r = r_0 =$ const. defines the horizon, and the constant β assures, that coordinate φ varies in range $[0, 2\pi)$. The metric on an abstract horizon in coordiantes (x, φ) is

$$q = R^2\left(\frac{dx^2}{P^2(x)} + P^2(x)\beta^2 d\varphi^2\right)\,. \tag{4.76}$$

We will start with calculating the area of horizons, given by S and $\widetilde{S}$ for q and $\widetilde{q}$ respectively:

$$S = \int_{-1}^{1}dx\int_0^{2\pi}d\varphi\sqrt{\det q} = \int_{-1}^{1}dx\int_0^{2\pi}d\varphi R^2\beta = 4\pi R^2\beta\,, \tag{4.77}$$

$$\widetilde{S} = \int_0^{\pi}d\theta\int_0^{2\pi}d\varphi\sqrt{\det\widetilde{q}} = \int_0^{\pi}d\theta\int_0^{2\pi}d\varphi\frac{\sin\theta(r_0^2 + (a+l)^2)\beta}{\Xi^2} = \tag{4.78}$$

$$= 2\pi\beta(r_0^2 + (a+l)^2)\int_1^{-1}\frac{-d\cos\theta}{\left(1 - \frac{\alpha a r_0}{a^2+l^2}(l + a\cos\theta)\right)^2} = \tag{4.79}$$

$$= \frac{4\pi\beta\left[r_0^2 + (a+l)^2\right](a^2+l^2)^2}{\left[a^2 + l^2 + \alpha a r_0(a-l)\right]\left[a^2 + l^2 - \alpha a r_0(a+l)\right]}\,. \tag{4.80}$$

Equating S and $\widetilde{S}$ lets us express parameter R^2 as

$$R^2 = \frac{\left[r_0^2 + (a+l)^2\right](a^2+l^2)^2}{\left[a^2 + l^2 + \alpha a r_0(a-l)\right]\left[a^2 + l^2 - \alpha a r_0(a+l)\right]} = [r_0^2 + (a+l)^2]\xi\,. \tag{4.81}$$

Notice that for R^2 (and consequently area of a horizon) to be well-defined, there has to be

$$\left[a^2 + l^2 + \alpha a r_0(a-l)\right]\left[a^2 + l^2 - \alpha a r_0(a+l)\right] > 0\,, \tag{4.82}$$

which is equivalent to

$$\left(a^2 + l^2 - \alpha a l r_0\right)^2 - \left(\alpha a^2 r_0\right)^2 > 0 \implies \left|\frac{\alpha r_0 a^2}{a^2 + l^2 - \alpha l r_0 a}\right| < 1\,. \tag{4.83}$$

Now metric $\widetilde{q}$ can be rewritten as

$$\widetilde{q} = R^2\left(\frac{\rho^2}{\Xi^2 T R^2}d\theta^2 + \frac{TR^2}{\Xi^2\rho^2\xi^2}\sin^2\theta\beta^2 d\varphi^2\right)\,, \tag{4.84}$$

and comparing with q yields

$$P^2 = \frac{TR^2}{\Xi^2 \rho^2 \xi^2} \sin^2\theta \,, \qquad dx = \frac{\sin\theta}{\Xi^2 \xi} d\theta \tag{4.85}$$

We can integrate the latter equality to express x as a function of θ and vice versa:

$$x = \frac{\cos\theta - \frac{\alpha r_0 a^2}{a^2+l^2-\alpha l r_0 a}}{\frac{\alpha r_0 a^2}{a^2+l^2-\alpha l r_0 a}\cos\theta - 1} \,, \tag{4.86}$$

$$\cos\theta = \frac{\frac{\alpha r_0 a^2}{a^2+l^2-\alpha l r_0 a} - x}{1 - \frac{\alpha r_0 a^2}{a^2+l^2-\alpha l r_0 a} x} \,; \tag{4.87}$$

where the constants of integration has been chosen, so that

$$x\,(\theta = 0) = -1 \,, \qquad x\,(\theta = \pi) = 1 \,. \tag{4.88}$$

For this transformation to be well defined it enough for constants α, a, l and r_0 to satisfy (4.83). Notice, that for $\alpha = 0$ we recover $x = -\cos\theta$, just like in (3.147). The derivative of x

$$\frac{dx}{d\theta} = \frac{(a^2 + l^2 - \alpha l r_0 a)^2 - (\alpha r_0 a^2)^2}{\left[a^2 + l^2 - \alpha l r_0 a - \alpha r_0 a^2 \cos\theta\right]^2} \sin\theta \tag{4.89}$$

is either always non-negative in the range $\theta \in [0, \pi]$. It makes transformation (4.86) a bijection. The function P^2 from (4.85) can be brought to the form

$$P^2 = (1 - x)^2 \frac{a_1 x^2 + a_2 x + a_3}{x^2 + b_1 x + b_2} \,, \tag{4.90}$$

so that the comparison with (4.56) is an elementary albeit tedious exercise in algebra. It yields

$$x_0 = \frac{l(a^2 + l^2)^2 + \alpha a r_0 (a^2 + l^2)(a^2 - l^2 + r_0^2) - l\alpha^2 a^2 r_0^4}{a\left[\alpha^2 a^2 r_0^4 + (a^2 + l^2)^2\right]} \,, \tag{4.91}$$

$$\begin{aligned} \Omega^2 &= \frac{r_0^2}{a^2} \left[\frac{a^2 \alpha^2 r_0^2 \left(l^2 - a^2\right) - 2a\alpha l r_0 \left(a^2 + l^2\right) + \left(a^2 + l^2\right)^2}{a^2 \alpha^2 r_0^4 + \left(a^2 + l^2\right)^2} \right]^2 \\ &= \frac{r_0^2}{a^2} \frac{(a^2 + l^2)^4}{\left[(a^2 + l^2)^2 + \alpha^2 a^2 r_0^4\right]^2 \xi^2} \end{aligned} \tag{4.92}$$

where one can see, that the limit of $\alpha = 0$ is

$$x_0 = \frac{l}{a} \,, \qquad \Omega^2 = \frac{r_0^2}{a^2} \,. \tag{4.93}$$

Moreover this parametrization of Ω^2 is well defined for any value of parameters a, l, r_0 and α. Parameter E^2 can be now calculated as

$$E^2 = \left[a_3 - \Lambda R^2 (\Omega^2 + x_0^2)\right] (\Omega^2 + x_0^2 - 1) - \frac{4}{3} \Lambda R^2 \Omega^2 \,. \tag{4.94}$$

The full expression is rather cumbersome and is given in Equation (4.95), and so is its limit for $\alpha = 0$ and $\alpha = l = 0$. This limit also defines an embedding into non accelerated Kerr-Newman-NUT-(anti-)de Sitter spacetime. Now, as one can see from Equation (4.95), it is unfeasable to find ranges of parameters, in which constant E^2 is well defined. Notice that the latter agrees with (3.156).

In the following sections, we will consider the special cases of $E = 0$, $\Lambda = 0$ and $E = \Lambda = 0$, and their embeddings in non-accelerated spacetimes.

The parameter E^2 expressed in parameters α, a, l, m, e, Λ and $r_0 = r_0(\alpha, a, l, m, e, \Lambda)$:

$$
\begin{aligned}
E^2 = \frac{1}{3}R^2 &\left\{ \begin{array}{l}
- \dfrac{3\Lambda\left(a^2-l^2-r_0^2\right)\left[\alpha^2a^4r_0^2-(a^2+l^2-\alpha a l r_0)^2\right]\left[a^2\alpha^2r_0^2\left[\left(a^2-l^2\right)^2+l^2r_0^2\right]+2a\alpha l r_0\left(a^2+l^2\right)\left(a^2-l^2-r_0^2\right)+\left(a^2+l^2\right)^2\left(l^2+r_0^2\right)\right]}{a^4\left[\left(a^2+l^2\right)^2+a^2\alpha^2r_0^4\right]^2} \\
\qquad - \dfrac{\left(a^2+l^2\right)^4\left(a^2-l^2-r_0^2\right)}{a^4\left(a^2+l^2+\alpha a r_0(a-l)\right)^2\left(a^2+l^2-\alpha a r_0(a+l)\right)^2\left(\left(a^2+l^2\right)^2+a^2\alpha^2r_0^4\right)^2} \cdot \\
\qquad\quad \cdot \left(\begin{array}{l}
\alpha^2a^{10}r_0^2\left(3\alpha^2+\Lambda\right)+2\alpha a^9 l r_0\left(3\alpha^2+2\Lambda\right)+a^8\left(3\alpha^4e^2r_0^2+l^2\left(3\Lambda+\alpha^2\left(3-r_0^2\left(9\alpha^2+2\Lambda\right)\right)\right)-6\alpha^2mr_0+3\right) \\
\qquad -6\alpha a^7 l\left(\alpha^2r_0\left(l^2-e^2\right)-\Lambda l^2r_0-\alpha^2mr_0^2+m+r_0\right) \\
\qquad +a^6l^2\left(3\alpha^2\left(\alpha^2r_0^2\left(3l^2-2e^2\right)+e^2+l^2+r_0^2\right)-4\Lambda l^2\left(\alpha^2r_0^2-3\right)+12\right) \\
\qquad -6\alpha a^5l^3\left(r_0\left(l^2\left(\alpha^2+\Lambda\right)+3\right)+3m\right) \\
\qquad +a^4l^4\left(3\alpha^2\left(2e^2+\alpha^2r_0^2(e-l)(e+l)-l^2+2r_0(3m+r_0)\right)+2\Lambda l^2\left(\alpha^2r_0^2+9\right)+18\right) \\
\qquad -2\alpha a^3l^5\left(r_0\left(3\alpha^2(e-l)(e+l)+7\Lambda l^2+9\right)+3m\left(\alpha^2r^2+3\right)\right) \\
\qquad +3a^2l^6\left(\alpha^2\left(e^2-l^2+4mr_0+r_0^2\right)+\Lambda l^2\left(\alpha^2r_0^2+4\right)+4\right)-6\alpha a l^7\left(\Lambda l^2r_0+m+r_0\right)+3l^8\left(\Lambda l^2+1\right)
\end{array} \right) \\
\qquad - \dfrac{4\Lambda r^2\left(a^2+\alpha a r_0(a-l)+l^2\right)^2\left(a^2-\alpha a r_0(a+l)+l^2\right)^2}{a^2\left[a^2\alpha^2r_0^4+\left(a^2+l^2\right)^2\right]^2}
\end{array} \right\} \\
&\xrightarrow{\alpha=0} R^2\left\{\frac{\Lambda\left(a^2-l^2-r_0^2\right)\left(l^2+r_0^2\right)}{a^4}-\frac{\left(a^2-l^2-r_0^2\right)}{a^4}\left(\Lambda l^2+1\right)-\frac{4}{3}\frac{\Lambda r_0^2}{a^2}\right\} \xrightarrow{l=0} \frac{R^2}{a^4}\left[r_0^2\left(1-\Lambda r_0^2\right)-a^2\left(1+\frac{1}{3}\Lambda r_0^2\right)\right] .
\end{aligned}
\tag{4.95}
$$

4.4.3 Vanishing electromagnetic field ($\boldsymbol{E = 0}$)

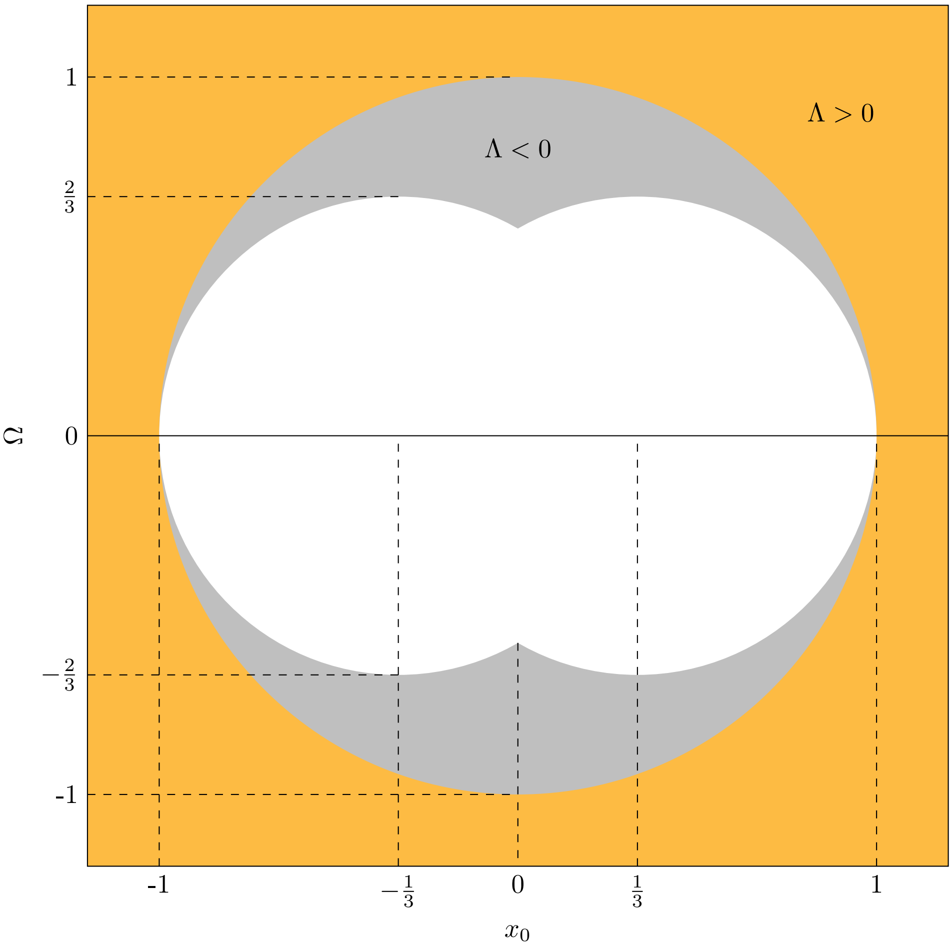

FIGURE 4.2: Possible values of Ω with regards to x_0 in the solution of the topologically non-trivial NHG equation, with vanishing electromagnetic field ($E = 0$).

If we set $E = 0$ in (4.56), we get

$$P^2 = \frac{1}{3}\Lambda R^2 \frac{1-x^2}{\Omega^2 + (x - x_0)^2}\left[(x - x_0)(x - 3x_0) + 3\Omega^2 + \frac{4\Omega^2}{\Omega^2 + x_0^2 - 1}\right], \tag{4.96}$$

which is the same as solution to (2.24) with boundary conditions (3.52). The positivity of the function P^2 can be analysed in the same way as in Section 4.4.1, with the assumption $E = 0$. Similarly to the generic case, for $\Lambda > 0$, the function P^2 is well-defined for

$$\Omega^2 + x_0^2 > 1\,. \tag{4.97}$$

For negative cosmological constant, inequalities (4.67) and (4.70) are contradictory. On the other hand inequalities (4.68) and (4.71) are always satisfied. To summarize, for $\Lambda < 0$ function (4.96) is positive if

$$\Omega^2 + x_0^2 < 1 \qquad \text{and} \qquad \Omega^2 + \left(|x_0| - \frac{1}{3}\right)^2 \geq \frac{4}{9}\,. \tag{4.98}$$

The results above are illustrated in Figure 4.2. Topologically non-trivial solution for $E = 0$ is analysed in detail in [91].

This solution can be embedded in Kerr-NUT-(anti-)de Sitter by taking

$$\Omega^2 = \frac{r_0^2}{a^2}\,, \qquad x_0 = \frac{l}{a}\,; \tag{4.99}$$

and constraining these parameters by

$$\Lambda r_0^2 \left(r_0^2 + l^2 + \frac{1}{3} a^2 \right) = r_0^2 + l^2 - a^2\,, \tag{4.100}$$

as per (4.95); which is just the extremality condition of Kerr-NUT-(anti-)de Sitter spacetime.

4.4.4 Vanishing cosmological constant ($\Lambda = 0$)

The general solution of (2.24) without cosmological constant is

$$P^2 = -\frac{E^2}{2\Omega^2} + c_1 \frac{\Omega^2 - (x - x_0)^2}{(x - x_0)^2 + \Omega^2} + c_2 \frac{2\Omega(x - x_0)}{(x - x_0)^2 + \Omega^2}\,. \tag{4.101}$$

Applying boundary conditions results in

$$c_1 = \frac{E^2 \left(\Omega^2 - x_0^2 + 1 \right)}{2\Omega^2 \left(\Omega^2 + x_0^2 - 1 \right)}\,, \qquad c_2 = -\frac{E^2 x_0}{\Omega \left(\Omega^2 + x_0^2 - 1 \right)}\,; \tag{4.102}$$

that is

$$P^2 = \left(1 - x^2 \right) \frac{E^2}{\left(\Omega^2 + x_0^2 - 1 \right) \left[(x - x_0)^2 + \Omega^2 \right]}\,. \tag{4.103}$$

This solution is viable for

$$\Omega^2 + x_0^2 > 1\,. \tag{4.104}$$

Angle deficit of (4.103) is

$$\delta_\pm = 2\pi \left(1 - \frac{E^2}{\left(x_0^2 + \Omega^2 - 1 \right) \left((x_0 \mp 1)^2 + \Omega^2 \right)} \right)\,. \tag{4.105}$$

This metric can be embedded into Kerr-NUT-Newman spacetime, by setting

$$x_0 = \frac{l}{a}\,, \qquad \Omega^2 = \frac{r_0^2}{a^2}\,, \qquad E^2 = \frac{e^2}{a^2}\,. \tag{4.106}$$

This transformation is valid for

$$a \in (-1, 1)\,, \tag{4.107}$$

which is consistent with (4.104).

4.4.5 Vacuum ($E = \Lambda = 0$)

In the limit of

$$E^2 = \Lambda = 0\,, \tag{4.108}$$

the solution to (2.24) is

$$P^2 = \frac{c_1 \left(\Omega^2 - (x - x_0)^2 \right) + 2 c_2 \Omega (x - x_0)}{(x - x_0)^2 + \Omega^2} \tag{4.109}$$

Boundary condition (4.16) makes us set

$$\Omega^2 + x_0^2 = 1\,, \qquad c_2 = -c_1 \frac{x_0}{\Omega}\,; \tag{4.110}$$

which leads to

$$P^2 = c_1 \frac{1 - x^2}{(x - x_0)^2 + 1 - x_0^2}\,, \tag{4.111}$$

that is well-defined for $c_1 > 0$. The admissible ranges of parameters Ω and x_0 are

$$\Omega \in (-1, 1)\,, \qquad x_0 \in (-1, 1)\,. \tag{4.112}$$

Angle deficit of (4.111) is

$$\delta_\pm = 2\pi \left(1 - \frac{1}{2} \frac{c_1}{1 \mp x_0} \right)\,. \tag{4.113}$$

Notice that for $\Omega = 0$, there is only the trivial solution of constant P^2 This solution is embeddable in Kerr-NUT spacetime, by setting

$$c_1 = \frac{1}{a^2}\,, \qquad \Omega^2 = \frac{r_0^2}{a^2}\,, \qquad x_0 = \frac{l}{a}\,. \tag{4.114}$$

Chapter 5

Conclusions

The results of this thesis have been described in Chapters 3 and 4. We have solved the near-horizon geometry equation in the presence of the electromagnetic field and cosmological constant.

In the chapter 3 we have found all extremal horizons of $\mathbb{R} \times \mathbb{S}_2$ topology. They are defined by metric q given by (3.21) on the cross-section $\mathcal{S} = \mathbb{S}_2$ of the horizon, and by the rotation one-form ω, given by (3.31). The non-rotating case has been described by formulas (3.72) and (3.74), together with the constraint (3.73). In the rotating case, the metric (3.21) and one-form (3.31) are given by functions (3.124) and (3.44)–(3.45), depending on parameters Ω^2, R^2 and E^2; constrained by an Equation (3.123). In Section 3.4.1 we have shown that the constant E^2 can take on any value, but the constants Ω^2 and R^2 cannot (see (3.139)–(3.141) and Figure 3.3). In Section 3.4.2 we have proven that every extremal horizon can be embedded in Kerr-Newman-(anti-)de Sitter spacetime. The explicit relations connecting an abstract horizon and the embedded one can be found in (3.158)–(3.160) and (3.161)–(3.163). This is one of the main results of this thesis and has been described in Theorem 3.4.1. It constitutes a generalization of the Theorem from [81], where horizons without the electromagnetic field were considered.

In the chapter 4 we have reviewed the basic properties of Plebański-Demiański spacetime, modelling black hole solutions. We have found and described abstract, axially symmetric extremal, isolated horizons, without rotation (4.48) and with rotation (4.56), in the presence of cosmological constant and electromagnetic field. In both cases we have analysed the admissible ranges of parameters (see (4.53) for $\Lambda < 0$ and (4.65) for $\Lambda > 0$; (4.67)–(4.68) and (4.70)–(4.71) for $\Lambda < 0$). The problem of embedding such horizons in Plebański-Demiański spacetime was considered in Section 4.4.2

Appendix A

Near-horizon Geometry

Near-horizon metric, and its inverse are given by

$$g = 2dv\left(\frac{1}{2}r^2 f dv + dr + rh_A dx^A\right) + \gamma_{AB}dx^A dx^B\,, \tag{A.1}$$

$$g^{-1} = 2\left(\frac{1}{2}r^2(h^2 - f)\partial_r + \partial_v - rh^A\partial_A\right)\partial_v + \gamma^{AB}\partial_A\partial_B\,; \tag{A.2}$$

where

$$f = f(r, x^A)\,, \qquad h_A = h_A(r, x^A)\,, \qquad \gamma_{AB} = \gamma_{AB}(r, x^A)\,, \qquad h^2 = \gamma^{AB}h_A h_B\,. \tag{A.3}$$

Energy-momentum tensor has the general form

$$T = 2dv\left[T_0 dv + r(\beta_A + T_0 h_A)dx^A + \frac{1}{2}r^2(\alpha + T_0 f)dv\right] + T_{AB}dx^A dx^B\,, \tag{A.4}$$

where

$$T_{AB} = T_{AB}(r, x^A)\,. \tag{A.5}$$

Null generator of a horizon is

$$\ell^\alpha\partial_\alpha = \partial_v\,, \tag{A.6}$$

$$\ell_\alpha dx^\alpha = r^2 f dv + dr + rh_A dx^A\,; \tag{A.7}$$

while the second, null vector is

$$n^\alpha\partial_\alpha = \partial_r\,, \tag{A.8}$$

$$n_\alpha dx^\alpha = -dv\,. \tag{A.9}$$

Out of all the non-vanishing Christoffel symbols, the only relevant one will be

$$\Gamma^A{}_{BC} = \frac{1}{2}\gamma^{AK}\left(\gamma_{KB,C} - \gamma_{BC,K} + \gamma_{CK,B}\right)\,, \tag{A.10}$$

$$\Gamma^v{}_{vv} = -\frac{1}{2}\left(r^2 f\right)_{,r}\,, \tag{A.11}$$

$$\Gamma^v{}_{vA} = -\frac{1}{2}\left(rh_A\right)_{,r}\,, \tag{A.12}$$

$$\Gamma^v{}_{AB} = -\frac{1}{2}\gamma_{AB,r}\,, \tag{A.13}$$

$$\Gamma^r{}_{vr} = \frac{1}{2}\left(r^2 f\right)_{,r} - \frac{1}{2}r\,h^A\left(rh_A\right)_{,r}\,, \tag{A.14}$$

$$\Gamma^r{}_{rA} = \frac{1}{2}\left(rh_A\right)_{,r} - \frac{1}{2}r\,h^B\gamma_{AB,r}\,, \tag{A.15}$$

$$\Gamma^r{}_{AB} = \frac{1}{2}r\left(h_{A,B} + h_{B,A}\right) - r\,h^C\Gamma_{CAB}\,, \tag{A.16}$$

$$\Gamma^A{}_{vr} = -\frac{1}{2}\gamma^{AB}\left(rh_B\right)_{,r}\,, \tag{A.17}$$

$$\Gamma^A{}_{vB} = \frac{1}{2} r h^A \left(r h_B\right)_{,r} + \frac{1}{2} r \gamma^{AC} \left(h_{C,B} - h_{B,C}\right) \,, \tag{A.18}$$

$$\Gamma^A{}_{rB} = \frac{1}{2} \gamma^{AC} \gamma_{BC,r} \,. \tag{A.19}$$

Others will either vanish after taking the limit $r \to 0$, or they will not be relevant in building Ricci tensor components, we are interested with. Let us now look at the R_{AB} component of EE:

$$R^{(4)}_{AB} - \Lambda g_{AB} = \kappa_0 T_{AB} \,. \tag{A.20}$$

This component is given by

$$R^{(4)}_{AB} = R^{(2)}_{AB} + R^v{}_{AvB} + R^r{}_{ArB} \,, \tag{A.21}$$

where we differentiate between four- and two-dimensional Ricci tensors with respective subscripts. The relevant Riemann tensor components are

$$\begin{aligned} R^v{}_{AvB} &= -\partial_B \Gamma^v{}_{Av} + \Gamma^v{}_{vC} \Gamma^C{}_{AB} - \Gamma^v{}_{Bv} \Gamma^v{}_{vA} && \text{(A.22)}\\ &= \frac{1}{2} \partial_B h_A - \frac{1}{2} \Gamma^C{}_{AB} h_C - \frac{1}{4} h_A h_B + \mathcal{O}(r) && \text{(A.23)}\\ &\xrightarrow{r\to 0} \frac{1}{2} \nabla_B \omega_A - \frac{1}{4} \omega_A \omega_B \,, && \text{(A.24)}\\ R^r{}_{ArB} &= \partial_r \Gamma^r{}_{AB} - \partial_B \Gamma^r{}_{Ar} + \Gamma^r{}_{rC} \Gamma^C{}_{AB} + \Gamma^r{}_{rv} \Gamma^v{}_{AB} - \Gamma^r{}_{Br} \Gamma^r{}_{Ar} - \Gamma^r{}_{BC} \Gamma^V{}_{Ar} && \text{(A.25)}\\ &= \partial_{(B} h_{A)} - \frac{1}{2} \Gamma^C{}_{AB} h_C - \frac{1}{4} h_A h_B + \mathcal{O}(r) && \text{(A.26)}\\ &\xrightarrow{r\to 0} \frac{1}{2} \nabla_A \omega^A - \frac{1}{4} \omega_A \omega_B \,; && \text{(A.27)} \end{aligned}$$

where

$$\lim_{r\to 0} h_A = \omega_A \,. \tag{A.28}$$

This leads us to the equation

$$\nabla_{(A} \omega_{B)} - \frac{1}{2} \omega_A \omega_B + R^{(2)}_{AB} - \Lambda g_{AB} = \kappa_0 T_{AB} \,, \tag{A.29}$$

which is consistent with the results from [9]. To retrieve the form we use through the whole these, one needs to transform

$$\omega^A \longrightarrow -2 \omega^A \,. \tag{A.30}$$

We are then left with

$$\nabla_{(A} \omega_{B)} + \omega_A \omega_B - \frac{1}{2} R^{(2)}_{AB} + \frac{1}{2} \Lambda g_{AB} + \frac{1}{2} \kappa_0 T_{AB} = 0 \tag{A.31}$$

as desired.

Appendix B

Newman-Penrose formalism for Isolated Horizons

B.1. Null complex basis in four dimensions

To see a more complete treatment of Newman-Penrose formalism, refer to the original articles [108, 109] or to the review in [110]. Our convension is compatible with the latter work, by the transformation

$$\begin{bmatrix} k \\ l \end{bmatrix}_{\text{in [110]}} \longrightarrow \begin{bmatrix} \ell \\ n \end{bmatrix}_{\text{this thesis}} . \tag{B.1}$$

In four dimensions we can choose a tetrad

$$\{e_k\} = \{\ell, n, m, \overline{m}\} = \{\ell^\mu \partial_\mu, n^\nu \partial_\nu, m^\rho \partial_\rho, \overline{m}^\sigma \partial_\sigma\} = \{e_k{}^\mu \partial_\mu\} , \quad k = 1, 2, 3, 4 ; \tag{B.2}$$

consisting of four null vectors, such that all the products of these basis vectors are zero, besides

$$\ell^\mu n_\mu = -1 , \qquad m^\nu \overline{m}_\nu = 1 . \tag{B.3}$$

Although it goes outside of the scope of the present work, it is possible to generalize this formalism to higher number of dimensions [111]. Naturally we can write

$$\{e^k\} = \{e^k{}_\mu dx^\mu\} = \{n_\mu dx^\mu, \ell_\nu dx^\nu, \overline{m}_\rho dx^\rho, m_\sigma dx^\sigma\} . \tag{B.4}$$

Such a complex tetrad can be transformed into a real orthonormal tetrad $\{E_k\}$ via transofrmation

$$E_1 = \frac{\overline{m} + m}{\sqrt{2}} , \qquad E_2 = \frac{\overline{m} - m}{i\sqrt{2}} , \qquad E_3 = \frac{\ell + n}{\sqrt{2}} , \qquad E_4 = \frac{\ell - n}{\sqrt{2}} . \tag{B.5}$$

Metric is given by

$$g_{\mu\nu} = 2 m_{(\mu} \overline{m}_{\nu)} - 2 \ell_{(\mu} n_{\nu)} , \tag{B.6}$$

while nonholonomic rotation/spin coefficients $\Gamma^a{}_{bc}$ (not to be confused with Christoffel symbols) are defined by

$$e_k{}^\mu \nabla_\mu e_l = e_k{}^\mu \nabla_\mu e_l = \Gamma^n{}_{kl} e_n . \tag{B.7}$$

It is customary to indicate spin coefficients by Greek minuscule (wee follow the same nomenclature as [110]):

$$-\kappa \equiv \Gamma_{144} = \ell_{\alpha;\beta} m^\alpha \ell^\beta = m^\alpha D \ell_\alpha , \tag{B.8}$$

$$-\rho \equiv \Gamma_{142} = \ell_{\alpha;\beta} m^\alpha \overline{m}^\beta = m^\alpha \delta \ell_\alpha , \tag{B.9}$$

$$-\sigma \equiv \Gamma_{141} = \ell_{\alpha;\beta} m^\alpha m^\beta = m^\alpha \delta \ell_\alpha , \tag{B.10}$$

$$-\tau \equiv \Gamma_{143} = \ell_{\alpha;\beta} m^\alpha n^\beta = m^\alpha \Delta \ell_\alpha , \tag{B.11}$$

$$\nu \equiv \Gamma_{233} = n_{\alpha;\beta} \overline{m}^\alpha n^\beta = \overline{m}^\alpha \Delta n_\alpha , \tag{B.12}$$

$$\mu \equiv \Gamma_{231} = n_{\alpha;\beta} \overline{m}^\alpha m^\beta = \overline{m}^\alpha \delta n_\alpha \tag{B.13}$$

$$\lambda \equiv \Gamma_{232} = n_{\alpha;\beta} \overline{m}^\alpha \overline{m}^\beta = \overline{m}^\alpha \overline{\delta} n_\alpha \tag{B.14}$$

$$\pi \equiv \Gamma_{234} = n_{\alpha;\beta}\overline{m}^{\alpha}\ell^{\beta} = \overline{m}^{\alpha} D n_{\alpha}\,, \tag{B.15}$$

$$-\epsilon \equiv \frac{1}{2}(\Gamma_{344} - \Gamma_{214}) = \frac{1}{2}(\ell_{\alpha;\beta} n^{\alpha}\ell^{\beta} - m_{\alpha;\beta}\overline{m}^{\alpha}\ell^{\beta}) = \frac{1}{2}(n^{\alpha} D\ell_{\alpha} - \overline{m}^{\alpha} D m_{\alpha})\,, \tag{B.16}$$

$$-\beta \equiv \frac{1}{2}(\Gamma_{341} - \Gamma_{211}) = \frac{1}{2}(\ell_{\alpha;\beta} n^{\alpha} m^{\beta} - m_{\alpha;\beta}\overline{m}^{\alpha} m^{\beta}) = \frac{1}{2}(n^{\alpha}\delta\ell_{\alpha} - \overline{m}^{\alpha}\delta m_{\alpha})\,, \tag{B.17}$$

$$\gamma \equiv \frac{1}{2}(\Gamma_{433} - \Gamma_{123}) = \frac{1}{2}(n_{\alpha;\beta}\ell^{\alpha} n^{\beta} - m_{\alpha;\beta}\overline{m}^{\alpha} n^{\beta}) = \frac{1}{2}(\ell^{\alpha}\Delta n_{\alpha} - m^{\alpha}\Delta\overline{m}_{\alpha})\,, \tag{B.18}$$

$$\alpha \equiv \frac{1}{2}(\Gamma_{432} - \Gamma_{122}) = \frac{1}{2}(n_{\alpha;\beta}\ell^{\alpha}\overline{m}^{\beta} - m_{\alpha;\beta}\overline{m}^{\alpha}\overline{m}^{\beta}) = \frac{1}{2}(\ell^{\alpha}\overline{\delta} n_{\alpha} - m^{\alpha}\overline{\delta}\overline{m}_{\alpha})\,; \tag{B.19}$$

where we have used the following differential operators:

$$D = \ell^{\alpha}\partial_{\alpha}\,, \qquad \Delta = n^{\alpha}\partial_{\alpha}\,, \qquad \delta = m^{\alpha}\partial_{\alpha}\,, \qquad \overline{\delta} = \overline{m}^{\alpha}\partial_{\alpha}\,. \tag{B.20}$$

One can now relate these operators by the commutators

$$[e_k, e_i] = -2\Gamma^{j}{}_{[ki} e_j\,. \tag{B.21}$$

Naturally, in the context of intristic geometry of a cross-section of a null hypersurface, we work in the basis

$$\{m, \overline{m}\}\,, \tag{B.22}$$

but all of the results in [110] can be easily reduced to it. To relate it to real basis from Section 3.2.2, let us express metric on $\mathcal{S}$ with two real vectors M and N:

$$q = \Sigma^2\left(d\theta^2 + \sin^2\theta d\varphi\right) = (M_A M_B + N_A N_B)\,dx^A dx^B\,, \tag{B.23}$$

where

$$M_A dx^A = \Sigma d\theta\,, \qquad N_A dx^A = \Sigma\sin\theta d\varphi\,. \tag{B.24}$$

Now we can write

$$M_A = \frac{m_A + \overline{m}_A}{\sqrt{2}}\,, \qquad N_A = \frac{m_A - \overline{m}_A}{i\sqrt{2}}\,, \tag{B.25}$$

or equivalently

$$m_A = \frac{M_A - iN_A}{\sqrt{2}}\,, \qquad \overline{m}_A = \frac{M_A + iN_A}{\sqrt{2}}\,, \tag{B.26}$$

to retrieve

$$q = (m_A\overline{m}_B + \overline{m}_A m_B)\,dx^A dx^B\,. \tag{B.27}$$

B.2. Connection and Curvature

To find curvature components in NP basis, we first and swcond Cartan equations calculate connection one-forms $\Gamma^{\alpha}{}_{\beta}$ from first Cartan equation

$$de^{\alpha} = -\Gamma^{\alpha}{}_{\beta}\wedge e^{\beta}\,, \qquad \Gamma_{\alpha\beta} = -\Gamma_{\beta\alpha}\,, \tag{B.28}$$

$$\Omega^{\alpha}{}_{\beta} = d\Gamma^{\alpha}{}_{\beta} + \Gamma^{\alpha}{}_{\gamma}\wedge\Gamma^{\gamma}{}_{\beta} = \frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}e^{\gamma}\wedge e^{\delta}\,; \tag{B.29}$$

where $\Omega^{\alpha}{}_{\beta}$ is curvature two-form.

The tetrad components of the Weyl tensor C, the traceless part of the Riemann tensor, are denoted as

$$\begin{aligned}
&\Psi_0 \equiv C_{\alpha\beta\gamma\delta}\ell^{\alpha}m^{\beta}\ell^{\gamma}m^{\delta}\,, \qquad &&\Psi_1 \equiv C_{\alpha\beta\gamma\delta}\ell^{\alpha}n^{\beta}\ell^{\gamma}m^{\delta}\,, \qquad &&\Psi_2 \equiv C_{\alpha\beta\gamma\delta}\ell^{\alpha}m^{\beta}n^{\gamma}m^{\delta}\,,\\
&\Psi_3 \equiv C_{\alpha\beta\gamma\delta}\ell^{\alpha}n^{\beta}\ell^{\gamma}n^{\delta}\,, \qquad &&\Psi_4 \equiv C_{\alpha\beta\gamma\delta}n^{\alpha}m^{\beta}n^{\gamma}m^{\delta}\,.
\end{aligned} \tag{B.30}$$

For an IH, optical scalars are given by

$$\theta^{(\ell)} = -\left(\rho + \overline{\rho}\right) , \tag{B.31}$$

$$\theta^{(n)} = \mu + \overline{\mu} , \tag{B.32}$$

$$\sigma_{ab} = -\sigma \overline{m}_a \overline{m}_b - \overline{\sigma} m_a m_b , \tag{B.33}$$

$$\omega_{ab} = \frac{1}{2} \left(\rho - \overline{\rho}\right) \left(m_a \overline{m}_b - \overline{m}_a m_b\right) . \tag{B.34}$$

The extremality of a horizon then translates to

$$\rho = \overline{\rho} = \sigma = \overline{\sigma} = 0 . \tag{B.35}$$

Symmetry of the metric (2.11) means, that

$$\mathcal{L}_\ell n = \mathcal{L}_\ell m = 0 , \tag{B.36}$$

moreover it lets us set [40, 112]

$$[\ell, m] = 0 . \tag{B.37}$$

Another consequence of (2.11) is

$$\Im(\varepsilon) = 0 \quad \text{and} \quad \pi = \alpha + \overline{\beta} . \tag{B.38}$$

Covariant derivative of ℓ is then given by

$$\ell_{\alpha;\beta} \ell^\beta = (\varepsilon + \overline{\varepsilon}) \ell_\alpha - \kappa \overline{m}_\alpha - \overline{\kappa} m_\alpha , \tag{B.39}$$

so, if ℓ is geodesic, then

$$\kappa = \overline{\kappa} = \Re(\varepsilon) = 0 . \tag{B.40}$$

Furthermore, due to condition (2.16), we have [40, 43, 112]

$$D\alpha = D\beta = D\lambda = D\mu = D\varepsilon = 0 , \tag{B.41}$$

and symmetry of equation (2.24) results in

$$\Im(\mu) = 0 . \tag{B.42}$$

Ricci components, relevant for an IH are

$$R_{m\overline{m}} = 2\kappa^{(\ell)} \mu - \mathrm{div}\omega - \omega^2 - K , \tag{B.43}$$

$$R_{\overline{mm}} = 2\kappa^{(\ell)} \lambda - 2\overline{\delta} - 4(\alpha - \overline{\beta})\pi - 2\pi^2 ; \tag{B.44}$$

where

$$R_{m\overline{m}} = \frac{1}{4} \left(R_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} R \right) m^\alpha \overline{m}^\beta , \tag{B.45}$$

$$R_{\overline{mm}} = \frac{1}{2} \left(R_{\alpha\beta} - \frac{1}{4} g_{\alpha\beta} R \right) \overline{m}^\alpha \overline{m}^\beta . \tag{B.46}$$

One-form ω is expressed as

$$\omega_\alpha = (\alpha + \overline{\beta}) m_\alpha + (\overline{\alpha} + \beta) \overline{m}_\alpha + \kappa^{(\ell)} n_\alpha , \tag{B.47}$$

while its divergence is

$$\nabla^\alpha \omega_\alpha = \delta\pi + \overline{\delta}\overline{\pi} - (\alpha - \overline{\beta})\overline{\pi} - (\overline{\alpha} - \beta)\pi . \tag{B.48}$$

Gaussian curvature of IH is

$$K = \delta(\alpha - \overline{\beta}) + \overline{\delta}(\overline{\alpha} - \beta) - 2(\alpha - \overline{\beta})(\overline{\alpha} - \beta) . \tag{B.49}$$

Constraints (B.45) can be combined to yield (2.24) in coordinate basis.

B.3. Energy-momentum tensor

Let

$$F = \frac{1}{2} F_{\alpha\beta} e^{\alpha} \wedge e^{\beta} \tag{B.50}$$

be a two-from describing electromagnetic field (Maxwell two-form) in the null basis. Using the standard notation, we can express it as

$$\begin{aligned} F = \Phi_0 e^1 \wedge e^4 + \Phi_1 \left(e^4 \wedge e^3 + e^2 \wedge e^1\right) + \Phi_2 e^2 \wedge e^3 \\ + \overline{\Phi}_0 e^2 \wedge e^4 + \overline{\Phi}_1 \left(e^4 \wedge e^3 + e^1 \wedge e^2\right) + \overline{\Phi}_2 e^1 \wedge e^3 \end{aligned} \tag{B.51}$$

Energy-momentum tensor of the electro-magnetic field is of the form

$$T_{\mu\nu} = F_{\mu\rho} F^{\rho}{}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \,, \tag{B.52}$$

and can be expressed in terms of Φ_0, Φ_1 and Φ_2. In our basis we have

$$T_{11} = -2\Phi_0 \overline{\Phi}_2 \,, \tag{B.53}$$

$$T_{12} = 2|\Phi_1|^2 \,, \tag{B.54}$$

$$T_{22} = -2\overline{\Phi}_0 \Phi_2 = \overline{T}_{11} \,. \tag{B.55}$$

Maxwell equations in NP formalism are

$$D\Phi_1 - \overline{\delta}\Phi_0 = (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 \,, \tag{B.56}$$

$$D\Phi_2 - \overline{\delta}\Phi_1 = -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\varepsilon)\Phi_2 \,, \tag{B.57}$$

$$\delta\Phi_1 - \Delta\Phi_0 = (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 \,, \tag{B.58}$$

$$\delta\Phi_2 - \Delta\Phi_1 = -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2 \,; \tag{B.59}$$

where

$$\Phi_0 \equiv F_{\alpha\beta} \ell^{\alpha} m^{\beta}, \tag{B.60}$$

$$\Phi_1 \equiv \tfrac{1}{2} F_{\alpha\beta} \left(\ell^{\alpha} n^{\beta} + \overline{m}^{\alpha} m^{\beta}\right) \,, \tag{B.61}$$

$$\Phi_2 \equiv F_{\alpha\beta} \overline{m}^{\alpha} n^{\beta} \,. \tag{B.62}$$

Appendix C

Solution to NHG equation

The near-horizon equation equation

$$\partial_x^2 P^2 + \frac{2(x-x_0)}{(x-x_0)^2+\Omega^2}\partial_x P^2 + \frac{4\Omega^2}{\left[(x-x_0)^2+\Omega^2\right]^2}P^2 + 2\Lambda R^2 + \frac{2E^2}{\left((x-x_0)^2+\Omega^2\right)^2} = 0\,. \tag{C.1}$$

is an inhomogeneous, linear, second order ordinary differential equation. Let us simplify it by introducing

$$t = x - x_0\,, \tag{C.2}$$

which results in

$$\partial_t^2 P^2 + \frac{2t}{t^2+\Omega^2}\partial_t P^2 + \frac{4\Omega^2}{\left[t^2+\Omega^2\right]^2}P^2 + 2\Lambda R^2 + \frac{2E^2}{\left(t^2+\Omega^2\right)^2} = 0\,. \tag{C.3}$$

We shall be using the following, handy notation:

$$f''(t) + T(t)f'(t) + Q(t)f(t) = S(t)\,, \tag{C.4}$$

where

$$f = P^2\,, \tag{C.5}$$

$$T = \frac{2t}{t^2+\Omega^2}\,, \tag{C.6}$$

$$Q = \frac{4\Omega^2}{\left[t^2+\Omega^2\right]^2}\,, \tag{C.7}$$

$$S = -2\Lambda R^2 - \frac{2E^2}{\left[t^2+\Omega^2\right]^2}\,. \tag{C.8}$$

The general solution to this second-order, non-homogenous ordinary differential equation is

$$f(t) = C_1 y_1(t) + C_2 y_2(t) + z_1(t) + z_2(t)\,, \tag{C.9}$$

where y_1 and y_2 are solution to homogenous equation, while functions z_1 and z_2 are particular solutions to non-homogenous problem, that can be expressed as

$$z_1(t) = -y_1(t)\int^t y_2(s)\frac{S(s)}{W(s)}ds\,, \tag{C.10}$$

$$z_2(t) = y_2(t)\int^t y_1(s)\frac{S(s)}{W(s)}ds\,; \tag{C.11}$$

where W is Wrońskian defined as

$$W_{(y_1,y_2)}(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)\,. \tag{C.12}$$

Homogenous problem can be brought to the simpler form by introducing coordinate θ as

$$t = \Omega\tan\theta \Longleftrightarrow \theta = \arctan\frac{t}{\Omega}\,. \tag{C.13}$$

We readily see, that

$$\frac{\partial \theta}{\partial t} = \frac{1}{\Omega} \frac{1}{1+\frac{t^2}{\Omega^2}} = \frac{\Omega}{t^2+\Omega^2} = \frac{1}{\Omega} \cos^2 \theta \,, \tag{C.14}$$

which lets us write derivatives of f as

$$\frac{\partial f}{\partial t} = \frac{1}{\Omega} \cos^2 \theta F' \,, \tag{C.15}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{1}{\Omega^2} \cos^2 \theta \left(F'' - 2\tan\theta F'\right) \; ; \tag{C.16}$$

where

$$F = F(\theta) = f(\theta) = f\left(t(\theta)\right) \,. \tag{C.17}$$

Now homogenous equation is simplified to a harmonic equation

$$F'' + 4F = 0 \,, \tag{C.18}$$

which means, that two independent solutions to the homogenous problem are (without the loss of generality)

$$y_1 = \cos(2\theta) = \cos\left(2 \arctan\left(\frac{t}{\Omega}\right)\right) = \frac{\Omega^2 - t^2}{\Omega^2 + t^2} \,, \tag{C.19}$$

$$y_2 = \sin(2\theta) = \sin\left(2 \arctan\left(\frac{t}{\Omega}\right)\right) = \frac{2t\Omega}{\Omega^2 + t^2} \,; \tag{C.20}$$

where we have used known trigonometric identities

$$\sin^2 x = \frac{\tan^2 x}{\tan^2 x + 1} \implies \sin\left(\arctan(x)\right) = \frac{x}{\sqrt{1+x^2}} \,, \tag{C.21}$$

$$\cos^2 x = 1 - \sin^2 x \implies \cos\left(\arctan(x)\right) = \frac{1}{\sqrt{1+x^2}} \,. \tag{C.22}$$

We now see that the Wrońskian is

$$W = \frac{2\Omega}{t^2+\Omega^2} \,, \tag{C.23}$$

which lets us calculate particular solutions, given by simple integrals

$$z_1(t) = \left(t^2 - \Omega^2\right) \left(\frac{E^2}{(t^2+\Omega^2)^2} - \Lambda R^2\right) \,, \tag{C.24}$$

$$z_2(t) = -\frac{2E^2 t^2}{(t^2+\Omega^2)^2} + \frac{2}{3} \frac{\Lambda R^2 t^2 \left(t^4 - 2t^2\Omega^2 - 3\Omega^4\right)}{\left(t^2+\Omega^2\right)^2} \,. \tag{C.25}$$

An alternative way to find a particular solutions is to assume that function

$$G = F + \frac{2E^2}{\Omega^2} \,, \tag{C.26}$$

which is a particular solution to

$$G'' + 4G = -2\Lambda R^2 \Omega^2 (t^2+\Omega^2)^2 \,, \tag{C.27}$$

is of the form

$$G = \frac{a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0}{(t^2+\Omega^2)^2} \,, \qquad a_1, \ldots a_4 = \text{const.} \tag{C.28}$$

The general solution to non-homogenous problem is therefore

$$\begin{aligned} P^2(x) = {} & c_1 \frac{\Omega^2 - (x-x_0)^2}{\Omega^2 + (x-x_0)^2} + c_2 \frac{2(x-x_0)\Omega}{(x-x_0)^2 + \Omega^2} \\ & - \frac{E^2 + \frac{1}{3}\Lambda R^2 \left[(x-x_0)^4 + 6(x-x_0)^2\Omega^2 - 3\Omega^4\right]}{(x-x_0)^2 + \Omega^2} \end{aligned} \tag{C.29}$$

which is equivalent to (3.115), after a suitable redefinition of constants c_1 and c_2, namely

$$c_1 \longrightarrow c_1 - \Lambda R^2 \left(\Omega^2 + x_0^2\right) , \tag{C.30}$$

$$c_2 \longrightarrow c_2 - \Lambda R^2 \frac{x_0}{\Omega} \left(\Omega^2 - \frac{1}{3} x_0^2\right) . \tag{C.31}$$

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