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Marsden–Meyer–Weinstein Reduction Theories and their Applications to Energy-Momentum Methods

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in
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Abstract

This PhD thesis presents a comprehensive review and a significant extension of the classical Marsden–Meyer–Weinstein reduction theorem to manifolds endowed with different geometric structures. Moreover, results are also applied to some relevant physical systems. In particular, this dissertation presents the k-polysymplectic Marsden–Meyer–Weinstein reduction theory by removing unnecessary technical assumptions, such as coadjoint equivariance of momentum maps, via a theory of affine Lie group actions, while correcting some misconceptions in the previous literature. The geometry of fibred k-polysymplectic manifolds is studied in depth, leading to a new k-polycosymplectic reduction theory, which is applied to systems with field symmetries, including coupled vibrating strings. A geometric reduction from k-cosymplectic to ℓ-cosymplectic structures is also developed and applied to vibrating membranes, giving one of the few geometric reduction theories of space-time variables in the literature. A Marsden–Meyer–Weinstein reduction for k-contact manifolds is also developed, and existing contact reduction results are revisited and clarified by solving some problems in the previous literature.

This dissertation develops new geometric methods for the analysis of non-autonomous Hamiltonian systems with symmetries, focusing on stability, reduction, and generalisations of classical energy-momentum techniques. A new cosymplectic energy-momentum method is formulated, providing a more general framework for analysing time-dependent Hamilton equations. Cosymplectic geometry enables the treatment of broader classes of Lie symmetries like Hamiltonian, gradient, and evolution vector fields. This also gives rise to the definition of new types of relative equilibria, such as gradient relative equilibria. These methods are applied to study physical systems like the restricted circular three-body problem. A cosymplectic-to-symplectic reduction method is introduced, extending results of C. Albert, and eigenfunctions of time-dependent Schrödinger equations are interpreted as relative equilibria in a cosymplectic setting.

Finally, a k-polysymplectic energy-momentum method is constructed, along with new stability analysis techniques for Hamiltonian systems on k-polysymplectic manifolds. Applications include integrable systems, polynomial dynamical systems, and quantum oscillators with dissipation.

Keywords:

symplectic geometry, Marsden–Meyer–Weinstein reduction, cosymplectic geometry, contact geometry, k-symplectic geometry, k-cosymplectic geometry, momentum maps, energy-momentum methods, Lyapunov stability, relative equilibrium points, Lie systems.

Streszczenie

Niniejsza rozprawa doktorska stanowi całościowy przegląd oraz istotne uogólnienie w zakresie klasycznego twierdzenia redukcji Marsdena–Meyera–Weinsteina. W szczególności, praca przedstawia teorię k-wielosymplektycznej redukcji Marsdena–Meyera–Weinsteina poprzez usunięcie zbędnych założeń technicznych, takich jak niezmienniczość odwzorowania momentu względem działania dołączonego, przy wykorzystaniu afinicznych działań grup Liego. Szczegółowo analizuje geometrię rozwłóknionych rozmaitości k-wielosymplektycznych, co prowadzi do nowej teorii redukcji k-wielokosymplektycznej, stosowanej następnie do układów z symetriami polowymi, w tym sprzężonych drgających strun. Opracowano również geometryczną redukcję struktur k-kosymplektycznych do ℓ-kosymplektycznych i zastosowano ją do analizy drgających membran. Rozwinięto także redukcję Marsdena–Meyera–Weinsteina dla struktur k-kontaktowych oraz poddano rewizji i uściśleniu istniejące wyniki dotyczące kontaktowej redukcji Marsdena–Meyera–Weinsteina.

W rozprawie rozwinięto również nowe metody geometryczne służące analizie nieautonomicznych układów hamiltonowskich z symetriami Liego, koncentrując się na problematyce stabilności, redukcji oraz uogólnieniach klasycznych metod energii-pędu. Sformułowano nową metodę energii-pędu w kontekście geometrii kosymplektycznej, co pozwala na analizę równań Hamiltona z czasowo zależnymi funkcjami Hamiltona w bardziej ogólnych ramach. Geometria kosymplektyczna umożliwia uwzględnienie szerszych klas symetrii Liego — w tym wektorów Hamiltonowskich, gradientowych i ewolucyjnych a także definicję nowych typów punktów równowagi względnej, takich jak równowagi względne gradientowe. Zastosowano te konstrukcje do badania układów takich jak ograniczony kołowy problem trzech ciał. Wprowadzono również metodę redukcji kosymplektycznej-symplektycznej oraz zinterpretowano funkcje własne równań Schrödingera zależnych od czasu jako punkty równowagi względnej w kontekście geometrii kosymplektycznej.

Ostatecznie, skonstruowano metodę energii-pędu dla geometrii k-wielosymplektycznej oraz opracowano nowe techniki analizy stabilności dla układów hamiltonowskich zdefiniowanych na rozmaitościach k-wielosymplektycznych. Pokazano również przykłady zastosowań k-wielosymplektycznej metody energii-pędu, obejmujące układy całkowalne, układy dynamiczne opisywane przez wielomiany oraz oscylatory kwantowe z dysypacją.

Słowa kluczowe

geometria symplektyczna, redukcja Marsdena–Meyera–Weinsteina, geometria kosymplektyczna, geometria kontaktowa, geometria k-symplektyczna, geometria k-cosymplektyczna, odwzorowania momentu, metody energii–pędu, stabilność Lyapunova, punkt względnej równowagi, system Liego.

Tytuł pracy w języku polskim

Teorie redukcji Marsdena–Meyera–Weinsteina i ich zastosowania w metodach energii-pędu

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Contents

Introduction

The Marsden–Meyer–Weinstein (MMW) reduction theory [4, 103, 109, 111] represents a pivotal result in the geometric formulation of classical mechanics, building upon foundational developments in symplectic geometry and Hamiltonian dynamics [128]. The MMW reduction not only clarified the geometric underpinnings of conserved quantities in classical Hamiltonian systems and their reduction to manifolds of smaller dimension, but also provides the foundation for powerful methods in dynamical systems, stability theory, and mathematical physics.

The roots of the MMW reduction can be traced back to Lie and Noether [115]. Nevertheless, significant advances towards our modern understanding can be found in the works by Souriau and Kostant [92, 143], who introduced the use of momentum maps to encode symmetries of Hamiltonian systems, as well as the ideas of symplectic quotients in the context of geometric quantisation [115]. After many other contributions [5, 6], they were Meyer in [120] and Marsden and Weinstein in [109] who unified these earlier ideas within the modern framework of symplectic geometry, showing that, under suitable regularity and symmetry conditions, one can construct a reduced phase space—obtained as a quotient of a level set of the momentum map—which inherits a natural symplectic structure.

Many physical systems of interest today, such as time-dependent mechanics and classical field theories, demand geometrical formalisms beyond the symplectic setting [128]. This dissertation extends the MMW reduction to k-polycosymplectic [50] and k-contact [51] frameworks. In addition, it corrects and clarifies some mistakes from the literature concerning contact [4] and k-polycosymplectic MMW reductions. At the same time, the already established results in the context of k-polysymplectic geometry are generalised to a broader class of k-polysymplectic momentum maps. This allows for broadening the applicability of reduction techniques to systems characterised by non-autonomous dynamics and multiple time-like parameters.

The core of this work consists of a rigorous development of momentum maps and reduction procedures within these new geometrical settings. In particular, novel reduction theorems for each of the above structures are proven and subtle aspects of the existing literature are clarified, especially by addressing issues such as the lack of equivariance in momentum maps and the role of affine Lie group actions. These generalisations are not merely of theoretical interest; they provide the mathematical underpinning necessary to apply energy-momentum methods to a broader class of systems. This is demonstrated through stability analysis of relative equilibria in reduced systems, incorporating both autonomous and time-dependent Hamiltonian dynamics. The results contribute to the refinement of modern geometric mechanics and offer tools applicable in both mathematical physics and engineering contexts.

The content of this Doctoral Dissertation is based on five research papers:

- J. de Lucas, B.M. Zawora, "*A time-dependent energy-momentum method*". Journal of Geometry and Physics **170**, 104364, 2021, 21 p.

  The article presents a generalisation of the energy-momentum method for studying the stability of non-autonomous Hamiltonian systems with a Lie group of Hamiltonian symmetries. A generalisation of the relative equilibrium point notion to a non-autonomous realm is provided and studied. Relative equilibrium points of a class of non-autonomous Hamiltonian systems are described via foliated Lie systems, which opens a new field of application of such systems of differential equations. Non-autonomous Hamiltonian systems are reduced via the MMW theorem, and conditions ensuring the stability of relative equilibrium points are given. Remarkably, the presented geometrical approach provides valuable insight, which is not common in the existing literature. As a byproduct, a geometrical extension of notions and results from Lyapunov stability theory on linear spaces to manifolds is provided. As an application, a class of mechanical systems, hereafter called *almost-rigid bodies* [113, 114, 138], which covers rigid bodies as a particular case, is analysed.

- J. de Lucas, X. Rivas, S. Vilariño, B.M. Zawora, "*On k-polycosymplectic Marsden–Weinstein reductions*". Journal of Geometry and Physics **191**, 104899, 2023, 36 p.

  This work reviews and slightly improves the known k-polysymplectic MMW reduction theory by removing some technical conditions on k-polysymplectic momentum maps by developing a theory of affine Lie group actions for k-polysymplectic momentum maps, avoiding the necessity of their co-adjoint equivariance. Then, a particular case of k-polysymplectic manifolds, the so-called fibred ones, is analysed, and their k-polysymplectic MMW reductions are studied. Previous results led to the development of a k-polycosymplectic MMW reduction theory, which is one of the main results of the article. Results are applied to study k-polycosymplectic Hamiltonian systems with field symmetries. In particular, two coupled vibrating strings are studied. Then, it is shown that k-polycosymplectic geometry can be understood as a particular type of k-polysymplectic geometry. Finally, a k-cosymplectic to ℓ-polycosymplectic geometric reduction theory is presented, which reduces, geometrically, the space-time variables in a k-cosymplectic framework. An application of this latter result to a vibrating membrane with symmetries is provided.

- J. de Lucas, A. Maskalaniec, B.M. Zawora, "*Cosymplectic geometry, reductions, and energy-momentum methods with applications*". Journal of Nonlinear Mathematical Physics **31**, 64, 2024, 58 p.

  This work devises a new cosymplectic energy-momentum method, providing a more general framework to study t-dependent Hamilton equations. Cosymplectic geometry allows for using more types of distinguished Lie symmetries (given by Hamiltonian, gradient, or evolution vector fields), relative equilibrium points, and reduction methods than symplectic techniques. Additionally, it reviews the cosymplectic formalism and the cosymplectic MMW reduction. Known and new types of relative equilibrium points are characterised and studied. Technical conditions used in previous energy-momentum methods, like the Ad^*-equivariance of momentum maps, are removed. Eigenfunctions of t-dependent Schrödinger equations are interpreted in terms of relative equilibrium points in cosymplectic manifolds. A new cosymplectic-to-symplectic reduction is developed, and a new associated type of relative equilibrium points, the so-called gradient relative equilibrium points, are introduced and applied to study the Lagrange points and Hill spheres of a restricted circular three-body system by means of a non-Hamiltonian Lie symmetry of the system.

- L. Colombo, J. de Lucas, X. Rivas and B.M. Zawora, "*An energy-momentum method for ordinary differential equations with an underlying k-polysymplectic manifold*". Journal of Nonlinear Science **35**, 42, 2025, 54 p.

  This work presents a comprehensive review of the k-polysymplectic MMW reduction theory, explaining previous errors and inaccuracies in the previous literature [62, 107] while introducing novel findings. It also emphasises the genuine practical significance of seemingly minor technical details. On this basis, a novel k-polysymplectic energy-momentum method and new related stability analysis techniques are introduced and applied to study Hamiltonian systems of ordinary differential equations relative to a k-polysymplectic manifold. Detailed and illustrative examples of both physical and mathematical significance are provided, including the study of complex Schwarz equations related to the Schwarz derivative, a series of isotropic oscillators, integrable Hamiltonian systems, quantum oscillators with dissipation, affine systems of differential equations, and polynomial dynamical systems.

- J. de Lucas, X. Rivas, S. Vilariño, B.M. Zawora, "*Marsden–Meyer–Weinstein reduction for k-contact field theories*". Preprint: arXiv:2503.03463, 2025, 50 p.

  This work devises a Marsden–Meyer–Weinstein k-contact reduction. Our techniques are illustrated with several examples of mathematical and physical relevance. As a byproduct, we review the previous contact reduction literature to clarify and to solve some inaccuracies.

These papers address the generalisation of the MMW theorem to the k-polysymplectic, k-cosymplectic, and k-contact settings, the formulation of energy-momentum methods in the cosymplectic framework, and applications to non-autonomous systems, including the restricted three-body problem and quantum quadratic Hamiltonians. The theoretical advances are motivated by significant references such as Ortega and Ratiu's foundational work on momentum maps and reduction [128], as well as modern treatments of polysymplectic and cosymplectic geometry by de León, Salgado, and Vilariño [43].

A primary goal of this research is to unify and extend the applicability of geometric reduction and stability analysis across different mathematical models of physical systems. This objective is achieved by systematically developing the theory of momentum maps in generalised geometrical contexts and establishing conditions under which reduction and stability theorems remain valid. An equally important aim is the derivation of criteria for Lyapunov and formal stability of equilibrium points in reduced Hamiltonian systems, particularly in those defined on cosymplectic or k-polysymplectic manifolds. These criteria are shown to be effective by application to important systems such as the circular restricted three-body problem and complex extensions of Lie systems such as the Schwarz equations.

There are several possibilities for further research that emerge from this dissertation. One natural direction involves relaxing some of the global assumptions made for simplicity, such as the existence of global Reeb vector fields or the connectedness of manifolds, which would make the results applicable to a wider class of models. Another promising extension is the adaptation of the methods developed in this PhD thesis to infinite-dimensional settings, such as those encountered in the study of classical field theories and quantum field theories, and in particular to deal with energy-Casimir methods [71, 84, 113]. The extension of these results to non-smooth or singular spaces is also of high interest, especially in connection with the reduction of singular Hamiltonian systems or systems with constraints. Finally, the interplay between generalised reduction and quantisation procedures remains an open field, where the methods introduced could provide a solid foundation for future explorations.

In summary, this PhD thesis presents geometric frameworks for the reduction and stability analysis of Hamiltonian systems in several contemporary geometrical settings. It addresses both theoretical foundations and practical applications, offering a substantial contribution to the fields of differential geometry, geometric mechanics, and mathematical physics.

There are other papers related, but not principal, to this doctoral dissertation:

1. C. Gonera, J. Gonera, J. de Lucas, W. Szczesek, B.M. Zawora, "*More on superintegrable models on spaces of constant curvature*". Regular and Chaotic Dynamics **27**, 561–571 (2022).

2. A.M. Grundland, J. de Lucas, B.M. Zawora, "*Stability analysis of the* $(1+1)$*-dimensional Nambu-Goto action gas models*". J. Phys. A **58**, 50LT01 (2025).

3. X. Rivas, N. Román-Roy, B.M. Zawora, "*Symmetries and Noether's theorem for action-dependent multicontact field theories*". Lett. Math. Phys. **115**, 108 (2025).

4. J. Lange, B.M. Zawora, "*Reduction of exact symplectic manifolds and energy hypersurfaces*". Proceedings of the 7th International Conference on Geometric Science of Information, LNCS, vol 16035:328- 336, Springer.

A brief description of all the above articles is as follows.

The article entitled "More on superintegrable models on spaces of constant curvature" is devoted to the previously less studied models characterised by the radial potential of the generalised Kepler type. Introduces a new two-parameter family of associated angular potentials expressed in terms of elementary functions. For a specific choice of parameters, this family reduces to the asymmetric spherical Higgs oscillator.

The main goal of the paper "Stability analysis of the $(1+1)$-dimensional Nambu-Goto action gas models" is to perform a nonlinear stability analysis of such action gas models. The study employs the energy-Casimir method, which is an extension of the energy-momentum method one [84], to examine in detail the Lyapunov stability of the Chaplygin and Born-Infeld models. Moreover, particular solutions are considered and their stability is studied to demonstrate the application of the obtained results. The paper deals with the stability of PDEs using infinite-dimensional geometry.

The article "Symmetries and Noether's theorem for action-dependent multicontact field theories" studies symmetries in action-dependent Lagrangian and Hamiltonian field theories together with the corresponding dissipation laws. In particular, the work introduces the notions of conserved and dissipated quantities, formulates the definitions of general symmetries of both the field equations and the underlying geometric structures, and studies their fundamental properties. These symmetries, referred to as Noether symmetries, give rise to a version of Noether's theorem adapted to this framework, which associates each such symmetry with the corresponding dissipated quantity and the related conservation law.

Finally, the article "Reduction of exact symplectic manifolds and energy hypersurfaces" presents two reduction schemes for Hamiltonian systems defined on exact symplectic manifolds endowed with Lie group symmetries. It is demonstrated that these reduction procedures are equivalent by employing a modified Marsden–Meyer–Weinstein reduction theorem for exact symplectic manifolds and contact manifolds arising as energy hypersurfaces. Each approach is illustrated through an example. Moreover, the Prize Committee recognised the work for its strong mathematical core, which established a connection between symplectic reduction and contact structures, as well as for the clarity and didactic quality of the presentation at GSI 2025 (selected among approximately 250 other participants).

This work concerns results previously obtained for this PhD dissertation and other results that appeared as a byproduct of my formation at the University of Warsaw.

The dissertation is divided into three chapters. Chapter 1, titled Fundamentals, establishes the necessary mathematical background, beginning with a generalisation of Lyapunov stability theory to differentiable manifolds. It proceeds with a detailed review of symplectic geometry and then introduces cosymplectic, contact, k-symplectic, and k-cosymplectic structures, focusing on their geometric properties. These tools are fundamental for the reduction and stability methods developed in the subsequent chapters.

Chapter 2 is the theoretical heart of this PhD thesis and is devoted to extending the Marsden–Meyer–Weinstein reduction to a wide range of geometric contexts. It begins by recalling the classical symplectic case, then moves through cosymplectic, k-polysymplectic, k-polycosymplectic, and k-contact reductions, providing rigorous proofs of new theorems and developing the theory of momentum maps suitable to each structure. It also discusses the particular case of k-contact MMW reduction when $k = 1$, explaining and correcting mistakes in the literature. The chapter highlights both mathematical and physical motivation, presenting examples such as the reduction of coupled vibrating strings and quantum systems.

The final chapter, Chapter 3, entitled "Energy-momentum methods", applies previous general reduction frameworks to the analysis of Lyapunov stability on reduced manifolds. The chapter is divided into sections for the symplectic, cosymplectic, and k-polysymplectic settings, each of which develops stability criteria and applies them to illustrative examples, including the restricted three-body problem, quantum

Hamiltonians, and affine Lie systems. The PhD thesis concludes by summarising the contributions: a unified theory of generalised MMW reduction, new criteria for stability of non-autonomous Hamiltonian systems, and corrections to earlier results in the literature. It also outlines possible directions for future research.

Chapter 1

Fundamentals

This chapter introduces concepts and geometric structures essential for this dissertation by establishing the foundations for the systematic study of reduced dynamics in the subsequent sections. It begins with an extension of Lyapunov stability theory [79, 90, 124, 147] to non-autonomous dynamical systems on manifolds, needed for the further analysis of stability of the reduced systems. Then, symplectic and cosymplectic geometries are briefly discussed [2, 25, 27, 95, 100, 101, 128], including the description of the properties of Hamiltonian systems. This chapter also reviews some geometric descriptions of field theories, with particular focus on the k-polysymplectic [7, 39, 41, 67, 78, 89, 117, 126, 134], k-polycosymplectic [43, 74, 125], and k-contact [130, 131, 132] frameworks. These structures generalise classical symplectic, cosymplectic, and contact geometries, respectively, and provide the appropriate setting for studying more general Hamiltonian systems. It presents the necessary mathematical formalism, including the theory of differential forms, vector fields, and Lie group actions. These tools are systematically developed, focusing on the application to reduction theory and the stability analysis of Hamiltonian systems.

Unless explicitly stated otherwise, several general assumptions hold throughout this work.

All geometric objects are smooth and real. Manifolds are assumed to be finite-dimensional, connected, paracompact, and Hausdorff. In particular, this ensures the existence of partitions of unity. Moreover, all geometric structures are globally defined. Of course, a more detailed treatment without previous assumptions is possible (especially since it is believed that most results could be extended to the complex case without much difficulty). These simplifications stress our key ideas and allow us to avoid minor or unnecessary technical problems. Moreover, $\Omega^k(M)$ and $\mathfrak{X}(M)$ stand for the spaces of differential k-forms and vector fields on a manifold M, respectively.

1.1 Fundamentals on the Lyapunov stability of non-autonomous systems

This section provides an adaptation of some fundamental results from the Lyapunov stability theory on $\mathbb{R}^n$ [79, 90, 124, 147] to the setting of manifolds. This generalisation enables the application of Lyapunov theory to study differential equations on manifolds. It is worth noting that there is not much work on Lyapunov stability on manifolds, and some results were presented in the work [122].

It is worth stressing that the extension of Lyapunov theory to manifolds is quite recent, with only a few published works on the subject (see [54, 122] and references therein). Remarkably, the presented generalisation retrieves the standard Lyapunov theory when restricted to problems on a Euclidean space $\mathbb{R}^n$. The main objective of the introduced techniques is to analyse the stability close to its equilibrium points of the Hamilton equations of various reduced Hamiltonian systems, including non-autonomous ones, obtained through the MMW theorems discussed in detail in Chapter 2.

To generalise Lyapunov theory from linear spaces to manifolds, it is necessary to find a substitute for the norm on linear spaces, which plays a fundamental role in the classical Lyapunov theory. This norm comes from a Euclidean metric on linear spaces. Its natural extension to manifolds is provided through Riemannian metrics. In particular, the existence of partitions of unity, which follows from the general assumptions in this work, ensures that any manifold P admits a Riemannian metric [11]. Assume that P is endowed with a Riemannian metric g. Then, the distance between two points $x_1, x_2 \in P$, denoted by $d(x_1, x_2)$, is defined as

$$d(x_1,x_2) := \inf_{\substack{\gamma\colon [0,1]\to P\\ \gamma(0)=x_1,\gamma(1)=x_2}} \operatorname{length}(\gamma),$$

where $\operatorname{length}(\gamma)$ is the length of a smooth curve γ in P relative to the Riemannian metric g (see [94]). Let B_{r,x_e} be the ball of radius r centred at $x_e \in P$ with respect to the distance induced by the Riemannian metric g, namely $B_{r,x_e} := \{x \in P \mid d(x, x_e) < r\}$ with $r > 0$. It can be proved that the topology induced by a Riemannian metric on P is equivalent to the topology of the manifold P [91, 95]. Consequently, for any point $x \in P$, every chart on P containing x gives a homomorphism to an open subset of $\mathbb{R}^n$. This implies that on an open coordinate neighbourhood of $x \in P$, the topology of the manifold is equivalent to the topology of an open subset in $\mathbb{R}^n$ induced by the standard norm in $\mathbb{R}^n$. Therefore, topological properties on an open coordinate neighbourhood of any $x \in P$ can be analysed using the norm on $\mathbb{R}^n$.

Throughout this section, t stands for the physical time. Consider a t-dependent vector field $X\colon (t,x) \in \mathbb{R}\times P \mapsto X(t,x) \in \mathrm{T}P$, which is a t-parametric family of vector fields $X_t\colon x \in P \mapsto X(t,x) \in \mathrm{T}P$ on P with $t \in \mathbb{R}$ (see [52, 151] for details). Consider the following non-autonomous dynamical system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(t,x)\,, \qquad \forall t \in \mathbb{R}\,, \qquad \forall x \in P\,. \tag{1.1.1}$$

where X is assumed to be smooth and (1.1.1) satisfies the conditions of the Theorem of existence and uniqueness of solutions [2, Theorem 2.1.2].

Define $\bar{\mathbb{R}} := \mathbb{R}_+ \cup \{0\}$ as the space of non-negative real numbers. Then, $I_{t'} := [t', \infty[$ for any $t' \in \mathbb{R}$ and $I_{-\infty} := \mathbb{R}$. A point $x_e \in P$ is an *equilibrium point* of (1.1.1) if $X(t, x_e) = 0$ for every $t \in \mathbb{R}$. An equilibrium point x_e is *stable* from $t^0 \in \mathbb{R}$ if, for every $t_0 \in I_{t^0}$ and any ball B_{ϵ,x_e} for $\epsilon > 0$, there exists a ball of radius $\delta(t_0,\epsilon)$, namely $B_{\delta(t_0,\epsilon),x_e}$, such that every solution $x(t)$ to (1.1.1) with $x(t_0) \in B_{\delta(t_0,\epsilon),x_e}$ satisfies that $x(t) \in B_{\epsilon,x_e}$ for all time $t \in I_{t_0}$. If t^0 is not hereafter explicitly detailed, it is assumed that $t^0 = -\infty$. An equilibrium point $x_e \in P$ is *uniformly stable* from $t^0 \in \mathbb{R}$ if for every $\epsilon > 0$, one can choose $\delta(t_0, \epsilon)$, with $t_0 \in I_{t^0}$, to be independent of t_0. An equilibrium point is *unstable* from t^0 if it is not stable from t^0.

The equilibrium point x_e is *asymptotically stable* from t^0 if x_e is stable and for every $t_0 \in I_{t^0}$ there exists an open neighbourhood $B_{r(t_0),x_e}$ of x_e such that every solution $x(t)$ to (1.1.1) with $x(t_0) \in B_{r(t_0),x_e}$ converges to x_e. Moreover, x_e is *uniformly asymptotically stable* from t^0 if it is asymptotically stable and $r(t_0)$ can be chosen to be independent of $t_0 \in I_{t^0}$ and the convergence to x_e is uniform relative to x in B_{r,x_e} and $t \in I_{t^0}$ (see [147, p. 140]).

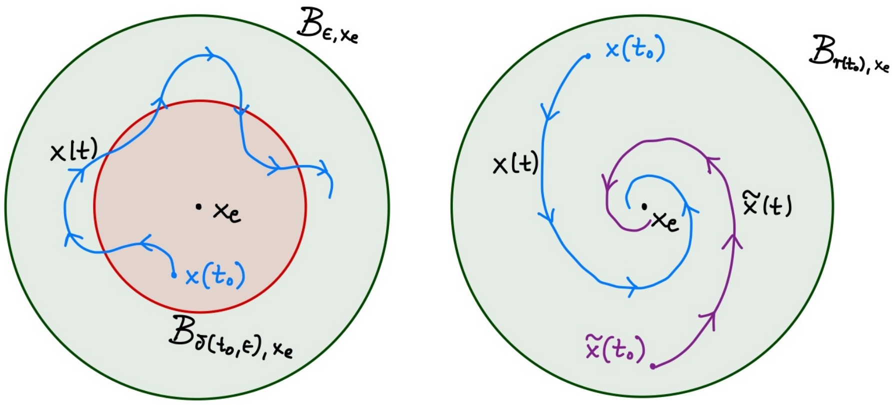
Definition 1.1.1. A continuous function $M\colon I_{t^0} \times P \to \mathbb{R}$ is a *locally positive definite function (lpdf) at an equilibrium point x_e from* $t^0 \in \mathbb{R}$ if, for some $r > 0$ and some continuous, strictly increasing function $\alpha\colon \bar{\mathbb{R}} \to \mathbb{R}$ with $\alpha(0) = 0$, one has that

$$M(t, x_e) = 0, \quad M(t,x) \geq \alpha(d(x, x_e)), \quad \forall t \in I_{t^0}, \quad \forall x \in B_{r,x_e}.$$

Definition 1.1.2. A continuous function $M\colon I_{t^0} \times P \to \mathbb{R}$ is *decrescent at an equilibrium point x_e* from $t^0 \in \mathbb{R}$ if, for some $s > 0$ and some continuous, strictly increasing function $\beta\colon \bar{\mathbb{R}} \to \mathbb{R}$ with $\beta(0) = 0$, is fulfilled that

$$M(t,x) \leq \beta(d(x,x_e)), \quad \forall t \in I_{t^0}, \quad \forall x \in B_{s,x_e}.$$

Although Definition 1.1.1 and Definition 1.1.2 concern a continuous function M, as in the literature [79, 90, 124, 147], for the present analysis, it is sufficient to assume that $M(t,x)$ is a $\mathscr{C}^1$ function. Hence, from now on, M is assumed to be $\mathscr{C}^1$. Note that Definition 1.1.1 could be reformulated without requiring



(a) (b)

Figure 1.1: An example of an equilibrium point $x_e \in \mathbb{R}^n$ that is stable (a) and asymptotically stable (b).

x_e to be an equilibrium point. Nevertheless, its current form is appropriate for further developments, which refer to the case for x_e being an equilibrium point.

Define $\dot{M}\colon I_{t^0} \times P \to \mathbb{R}$ as the function for which $\dot{M}(\hat{t},\hat{x})$, with $(\hat{t},\hat{x}) \in I_{t^0} \times P$, denotes the time derivative of $M(t, x(t))$ at $t = \hat{t}$ along the particular solution $x(t)$ of (1.1.1) with initial condition $x(\hat{t}) = \hat{x}$. Explicitly,

$$\dot{M}(\hat{t},\hat{x}) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=\hat{t}} M(t,x(t)) = \frac{\partial M}{\partial t}(\hat{t},\hat{x}) + \sum_{i=1}^{\dim P} \frac{\partial M}{\partial x^i}(\hat{t},\hat{x}) X^i(\hat{t},\hat{x}),$$

where $\{x^1,\ldots,x^{\dim P}\}$ is a local coordinate system on P around $\hat{x}$ and $X^1,\ldots,X^{\dim P}$ are the components of X in the basis of vector fields associated with the given local coordinates.

The above definitions play a crucial role in understanding Theorem 1.1.6, which characterises the stability of (1.1.1) by studying the properties of an appropriate associated function.

For completeness and clarity, the following theorems provide an extension to manifolds of several classical results for linear spaces presented in [147].

Theorem 1.1.3. *An equilibrium point $x_e \in P$ of the system (1.1.1) is stable from t^0 if there exists a lpdf $\mathscr{C}^1$-function $M\colon I_{t^0} \times P \to \mathbb{R}$ from $t^0 \in \mathbb{R}$ and a constant $r > 0$ such that*

$$\dot{M}(t,x) \leq 0\,, \qquad \forall t \in I_{t^0}\,, \quad \forall x \in B_{r,x_e}\,.$$

Proof. Since the function M is assumed to be lpdf from t^0, Definition 1.1.1 yields that there exists a continuous strictly increasing function $\alpha\colon \bar{\mathbb{R}} \to \mathbb{R}$ from t^0 and a constant $s > 0$ satisfying

$$\alpha(d(x,x_e)) \leq M(t,x)\,, \qquad \forall t \in I_{t^0}, \quad \forall x \in B_{s,x_e}\,.$$

The proof that x_e is stable from t^0 boils down to showing that for any $\epsilon > 0$, $t_0 \in I_{t^0}$, and $t \in I_{t_0}$, there exists $\delta(t_0,\epsilon) =: \delta$ such that, if $x(t)$ is the particular solution of (1.1.1) with initial condition $x_0 := x(t_0)$, then

$$d(x_0,x_e) < \delta \implies d(x(t),x_e) < \epsilon\,, \qquad \forall t \in I_{t_0}\,.$$

Fix ϵ, t_0, and set $\mu := \min(\epsilon, r, s)$. Then, there exists $\delta > 0$ so that

$$\sup_{d(x,x_e)<\delta} M(t_0,x) < \alpha(\mu)\,.$$

This holds since $\alpha(\mu) > 0$ and $\lim_{\delta\to 0^+} \sup_{d(x,x_e)<\delta} M(t_0,x) = 0$. To show that δ guarantees the stability of x_e, suppose that $d(x_0,x_e) < \delta$. Then, $M(t_0,x_0) \leq \sup_{d(x,x_e)<\delta} M(t_0,x) < \alpha(\mu)$.

Assume, for the moment, that $x(t)$ remains in B_{μ,x_e} for every $t \in I_{t_0}$. Then, $B_{\mu,x_e} \subset B_{r,x_e}$ and since $\dot{M}(t,x(t)) \leq 0$, together with the assumption that $M(t,x)$ is a $\mathscr{C}^1$-function, it follows that $M(t,x(t)) - M(t_0,x_0) \leq 0$. Thus,

$$M(t,x(t)) \leq M(t_0,x_0) < \alpha(\mu)\,, \qquad \forall t \in I_{t_0}\,.$$

Moreover, since $x(t) \in B_{\mu,x_e} \subset B_{s,x_e}$ for $t \in I_{t_0}$ by assumption, one also obtains

$$\alpha(d(x(t),x_e)) \leq M(t,x(t))\,, \qquad \forall t \in I_{t_0}\,.$$

Hence, combining the previous two inequalities yields

$$\alpha(d(x(t),x_e)) < \alpha(\mu)\,, \qquad \forall t \in I_{t_0}\,.$$

Since α is strictly increasing, it follows that

$$d(x(t),x_e) < \mu \leq \epsilon\,, \qquad \forall t \in I_{t_0}\,. \tag{1.1.2}$$

Consequently, x_e is a stable equilibrium point provided that $x(t)$ belongs to B_{μ,x_e} for every $t \in I_{t_0}$. It remains to show that this assumption is indeed satisfied.

Suppose that $T := \min\{t \in \mathbb{R} \mid d(x(t),x_e) \geq \mu\}$ (it is well defined, since $x(t)$ is continuous). By the definition of T, one has

$$d(x(t),x_e) < \mu\,, \qquad \forall t \in [t_0,T[\,,$$

and, by continuity, $d(x(T),x_e) = \mu$. Moreover, since $\mu \leq r$, it follows that

$$\dot{M}(t,x(t)) \leq 0\,, \qquad \forall t \in [t_0,T[\,.$$

Hence, since M is a $\mathscr{C}^1$-function, one obtains

$$M(T,x(T)) \leq M(t_0,x_0) < \alpha(\mu)\,. \tag{1.1.3}$$

On the other hand, as $\mu \leq s$, it follows that

$$M(T,x(T)) \geq \alpha(d(x(T),x_e)) = \alpha(\mu)\,. \tag{1.1.4}$$

However, Equations (1.1.3) and (1.1.4) contradict each other. Consequently, no such T exists, and therefore (1.1.2) holds. □

Theorem 1.1.4. *An equilibrium point x_e of system (1.1.1) is uniformly stable from t^0 if there exists a $\mathscr{C}^1$, lpdf and also decrescent function $M\colon I_{t^0} \times P \to \mathbb{R}$ from t^0 and a constant $r > 0$ such that*

$$\dot{M}(t,x) \leq 0\,, \qquad \forall t \in I_{t^0}\,, \quad \forall x \in B_{r,x_e}\,.$$

Proof. The proof of this theorem is only sketched, because it is essentially the same as the proof of Theorem 1.1.3.

As M is assumed to be decrescent from t^0, Definition 1.1.2 ensures the existence of a continuous, strictly increasing function $\beta\colon \bar{\mathbb{R}} \to \mathbb{R}$ satisfying $\beta(0) = 0$ and a constant $s > 0$ such that

$$M(t,x) \leq \beta(d(x,x_e))\,, \qquad \forall t \in I_{t^0}\,, \quad \forall x \in B_{s,x_e}\,.$$

Define

$$\omega(\delta) := \sup_{d(x,x_e)<\delta,\ t\in I_{t^0}} M(t,x)\,.$$

This function is well defined for $\delta < s$, since $M(t,x)$ is decrescent and $\omega(\delta) \leq \beta(\delta)$. Moreover, $\omega(\delta)$ is non-decreasing and

$$\lim_{\delta\to 0^+} \omega(\delta) = \lim_{\delta\to 0^+} \sup_{d(x,x_e)<\delta,\ t\in I_{t^0}} M(t,x) \leq \lim_{\delta\to 0^+} \beta(\delta) = 0\,.$$

Since M is a ldpf function, consider the function $\alpha\colon \bar{\mathbb{R}} \to \mathbb{R}$ and the constant $s_1 > 0$ satisfying

$$\alpha(d(x, x_e)) \leq M(t, x)\,, \qquad \forall t \in I_{t^0}\,, \quad \forall x \in B_{s_1, x_e}\,.$$

Fix $\epsilon > 0$ and define $\mu := \min(\epsilon, r, s, s_1)$. Choose δ such that $\beta(\delta) < \alpha(\mu)$.

The rest of the proof proceeds exactly as in Theorem 1.1.3, including the argument showing that $x(t)$ remains in B_{μ,x_e} for all $t \geq t_0 \geq t^0$ whenever $x(t_0) \in B_{\mu,x_e}$. □

Theorem 1.1.5. *The equilibrium point x_e of system (1.1.1) is uniformly asymptotically stable from t^0 if there exists a decrescent, lpdf, $\mathscr{C}^1$-function $M\colon I_{t^0} \times P \to \mathbb{R}$ from t^0 such that $-\dot{M}$ is a lpdf from t^0.*

Proof. Let $x(t)$ denote a solution of (1.1.1) with initial condition $x(t_0) = x_0$ for some $t_0 \geq t^0$. Since $-\dot{M}$ is a lpdf function, Definition 1.1.1 implies the existence of a continuous, strictly increasing function $\gamma\colon \bar{\mathbb{R}} \to \mathbb{R}$ with $\gamma(0) = 0$ and a constant $s > 0$ such that

$$\dot{M}(t, x) \leq -\gamma(d(x, x_e))\,, \qquad \forall t \in I_{t^0}\,, \quad \forall x \in B_{s,x_e}\,.$$

Since γ is a non-negative function, one has

$$\dot{M}(t, x) \leq 0\,, \qquad \forall t \in I_{t^0}\,, \quad \forall x \in B_{s,x_e}\,. \tag{1.1.5}$$

Thus, $\dot{M}$ satisfies the hypothesis of Theorem 1.1.4 and x_e is a uniformly stable equilibrium from t^0. Then, what remains to prove is that, for every $\epsilon > 0$ and $t_0 \geq t^0$, there exists $T := T(\epsilon)$ and $\delta > 0$ such that every solution $x(t)$ with $x(t_0) \in B_{\delta,x_e}$ satisfies

$$d(x(t), x_e) < \epsilon\,, \quad \forall t \geq T + t_0\,.$$

It is sufficient to show that such a constant δ exists. This condition can be equivalently written as

$$\forall \epsilon > 0\,, \quad \exists\, \delta > 0\,, \quad \exists T > 0\,, \quad d(x_0, x_e) < \delta \implies d(x(t), x_e) < \epsilon\,, \quad \forall t \geq T + t_0\,. \tag{1.1.6}$$

By the hypothesis, there exist functions $\alpha, \beta\colon \bar{\mathbb{R}} \to \mathbb{R}$ and constants $k, l > 0$ such that

$$\alpha(d(x, x_e)) \leq M(t, x)\,, \qquad \forall t \in I_{t^0}\,, \quad \forall x \in B_{k,x_e}\,, \tag{1.1.7}$$

$$M(t, x) \leq \beta(d(x, x_e))\,, \qquad \forall t \in I_{t^0}\,, \quad \forall x \in B_{l,x_e}\,. \tag{1.1.8}$$

Set $r := \min\{k, l, s, \epsilon\}$ and define positive constants κ_1, κ_2, T as follows

$$\kappa_1 < \beta^{-1}(\alpha(r))\,, \qquad \kappa_2 < \min\{\beta^{-1}(\alpha(\epsilon)), \kappa_1\}\,, \qquad T := \frac{\beta(\kappa_1)}{\gamma(\kappa_2)}\,.$$

To prove that fixing $\delta = \kappa_2$ and T satisfies (1.1.6), recall that every particular solution $x(t)$ to (1.1.1) with $x(t_0) =: x_0 \in B_{\kappa_2,x_e}$ remains inside the ball B_{r,x_e} for all $t \in I_{t_0}$ and κ_2 small enough. Indeed, the argument follows from the same reasoning as in the previous theorems. Therefore, one can assume that (1.1.7) and (1.1.8) apply to B_{κ_2,x_e}.

First, one needs to prove the following

$$d(x_0, x_e) < \kappa_1 \implies d(x(t_1), x_e) < \kappa_2\,, \qquad \exists\, t_1 \in [t_0, t_0 + T]\,. \tag{1.1.9}$$

The proof proceeds by contradiction. Suppose that

$$d(x_0, x_e) < \kappa_1 \qquad \wedge \qquad d(x(t), x_e) \geq \kappa_2\,, \qquad \forall t \in [t_0, t_0 + T]\,.$$

From (1.1.7), (1.1.8), and (1.1.5) it follows

$$\beta(d(x_0, x_e)) < \beta(\kappa_1)\,, \qquad \gamma(d(x(t), x_e)) \geq \gamma(\kappa_2)\,, \qquad \alpha(\kappa_2) \leq \alpha(d(x(t), x_e))\,,$$

for all $t_0 < t < t_0 + T$ and $x_0 \in B_{\kappa_2, x_e}$. Consequently,

$$0 < \alpha(\kappa_2) \leq M(t_0 + T, x(t_0 + T)) = M(t_0, x_0) + \int_{t_0}^{t_0+T} \dot{M}(\tau, x(\tau)) d\tau \leq$$

$$\beta(d(x_0, x_e)) - \int_{t_0}^{t_0+T} \gamma(d(x(\tau), x_e)) d\tau \leq \beta(\kappa_1) - T\gamma(\kappa_2) = 0\,,$$

which is a contradiction. Therefore, (1.1.9) holds.

To complete the proof, consider $t > t_0 + T$. Inequality (1.1.7) holds for all $t \in I_{t_0}$, and by (1.1.9) there exists $t_1 \in [t_0, t_0 + T]$ such that $\beta(d(x(t_1), x_e)) < \beta(\kappa_2)$. Then, by (1.1.5), it follows that

$$\alpha(d(x(t), x_e)) \leq M(t, x(t)) \leq M(t_1, x(t_1))$$

and

$$M(t_1, x(t_1)) \leq \beta(d(x(t_1), x_e)) < \beta(\kappa_2)\,.$$

Combining these two inequalities, one gets

$$\alpha(d(x(t), x_e)) < \beta(\kappa_2) \leq \alpha(\epsilon),$$

which establishes (1.1.6) for $\delta = \kappa_2$ and finishes the proof. □

The following theorem summarises the last three theorems in a single statement, referred to as the basic Lyapunov theorem on manifolds.

Theorem 1.1.6. (The basic Lyapunov theorem on manifolds [90, 147]) *Let* $M \colon I_{t^0} \times P \to \mathbb{R}$ *be a non-negative function, let* $x_e \in P$ *be an equilibrium point of (1.1.1), and let* $\dot{M}$ *stand for the function (1.1). Then, one has the following results:*

1. *If* M *is* $\mathscr{C}^1$ *and lpdf from* t^0 *and* $\dot{M}(t, x) \leq 0$ *for* x *locally around* x_e *and for all* $t \in I_{t^0}$*, then* x_e *is stable.*
2. *If* M *is* $\mathscr{C}^1$*, lpdf and decrescent from* t^0*, and* $\dot{M}(t, x) \leq 0$ *locally around* x_e *and for all* $t \in I_{t^0}$*, then* x_e *is uniformly stable.*
3. *If* M *is* $\mathscr{C}^1$*, lpdf and decrescent from* t^0*, and* $-\dot{M}(t, x)$ *is locally positive definite around* x_e *and* $t \in I_{t^0}$*, then* x_e *is uniformly asymptotically stable.*

1.2 Basics on symplectic geometry

This section presents fundamental notions and results in symplectic geometry while establishing the notation used hereafter [2, 25, 95, 128].

Definition 1.2.1. A *symplectic manifold* is a pair (P, ω), where P is a manifold and $\omega \in \Omega^2(P)$ is closed and *non-degenerate*, namely the map $\widehat{\omega} \colon \mathrm{T}P \mapsto \mathrm{T}^*P$, given by $\widehat{\omega}(v_p) := \omega_p(v_p, \cdot) \in \mathrm{T}^*_pP$ for every $p \in P$ and each $v_p \in \mathrm{T}_pP$, is a vector bundle isomorphism. The form ω is called a *symplectic form.*

Hereafter, (P, ω) stands for a symplectic manifold. For any subspace $V_p \subset \mathrm{T}_pP$, the *symplectic orthogonal* is defined as

$$V_p^{\perp_\omega} := \{\vartheta_p \in \mathrm{T}_pP \mid \omega_p(\vartheta_p, v_p) = 0, \forall v_p \in V_p\}.$$

Theorem 1.2.2. *Let* (P, ω) *be a symplectic manifold of dimension* $2n$*. Then, around any point* $p \in P$*, there exist an open neighbourhood* U *and a coordinate system* $\{q^i, p_i\}_{i=1,\ldots,n}$ *such that*

$$\omega|_U = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}p_i\,.$$

These coordinates are called the symplectic Darboux coordinates.

It is worth noting that symplectic Darboux coordinates are not unique.

Cotangent manifolds are naturally endowed with a symplectic form to be described next.

Definition 1.2.3. The *canonical one-form* or *Liouville form* on T^*Q is $\theta_Q \in \Omega^1(\mathrm{T}^*Q)$ on T^*Q defined by

$$(\theta_Q)_{\alpha_q}(v_{\alpha_q}) := \langle \alpha_q, \mathrm{T}_{\alpha_q}\tau(v_{\alpha_q})\rangle, \quad \forall q \in Q, \quad \forall \alpha_q \in \mathrm{T}^*_qQ, \quad \forall v_{\alpha_q} \in \mathrm{T}_{\alpha_q}\mathrm{T}^*_qQ$$

where Q is any manifold, $\tau\colon \mathrm{T}^*Q \to Q$ is the canonical cotangent bundle projection and $\langle\cdot,\cdot\rangle$ is the natural pairing between covectors and vectors. The *canonical two-form* $\omega_Q \in \Omega^2(\mathrm{T}^*Q)$ is the differential two-form on T^*Q given by

$$\omega_Q := -\mathrm{d}\theta_Q.$$

In local adapted coordinates $\{q^i, p_i\}_{i=1,\dots,n}$ to T^*Q, one has $\theta_Q = \sum_{i=1}^n p_i \mathrm{d}q^i$. Then, $\omega_Q = -\mathrm{d}\theta_Q = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}p_i$ becomes the *canonical symplectic form* [2, 25, 128]. The symplectic manifold $(\mathrm{T}^*Q, \omega_Q)$ plays a significant role in many physical applications [2].

Definition 1.2.4. A vector field $X \in \mathfrak{X}(P)$ is *Hamiltonian* if $\iota_X\omega = \mathrm{d}f$ for some $f \in \mathscr{C}^\infty(P)$. Then, f is called a *Hamiltonian function* associated with X.

Since ω is non-degenerate, every $f \in \mathscr{C}^\infty(P)$ corresponds to a unique Hamiltonian vector field X_f. The space of Hamiltonian vector fields on P relative to a symplectic form ω is denoted by $\mathrm{Ham}(P,\omega)$. Moreover, the *Cartan's magic formula* [2, p 194] yields

$$\mathscr{L}_{X_f}\omega = \iota_{X_f}\mathrm{d}\omega + \mathrm{d}\iota_{X_f}\omega = \iota_{X_f}\mathrm{d}\omega + \mathrm{d}^2 f = 0, \tag{1.2.1}$$

where $\mathscr{L}_{X_f}\omega$ is the *Lie derivative* of ω with respect to X_f.

Definition 1.2.5. A *Poisson bracket* is a bilinear map $\{\cdot,\cdot\}\colon \mathscr{C}^\infty(P)\times\mathscr{C}^\infty(P) \to \mathscr{C}^\infty(P)$ satisfying that $(\mathscr{C}^\infty(P), \{\cdot,\cdot\})$ is a Lie algebra and

$$\{f, gh\} = \{f,g\}h + g\{f,h\}\,, \qquad \forall f,g,h \in \mathscr{C}^\infty(P)\,.$$

Define a bracket

$$\{\cdot,\cdot\}\colon \mathscr{C}^\infty(P)\times\mathscr{C}^\infty(P) \ni (f,g) \mapsto \omega(X_f, X_g) \in \mathscr{C}^\infty(P).$$

This bracket is bilinear, antisymmetric, and, since $\mathrm{d}\omega = 0$, it satisfies the *Jacobi identity* [2, 93, 128], which makes $\{\cdot,\cdot\}$ into a *Lie bracket*. Furthermore, $\{\cdot,\cdot\}$ satisfies the *Leibniz rule*, i.e.

$$\{f, gh\} = \{f,g\}h + g\{f,h\}\,, \qquad \forall f,g,h \in \mathscr{C}^\infty(P)\,.$$

Due to all such properties, $\{\cdot,\cdot\}$ becomes a *Poisson bracket* according to Definition 1.2.5. Since $\iota_{[X_f,X_g]} = \mathscr{L}_{X_f}\iota_{X_g} - \iota_{X_g}\mathscr{L}_{X_f}$ for every $f,g \in \mathscr{C}^\infty(P)$ (see [2, p 121]), and by using (1.2.1), it follows that

$$\iota_{[X_f,X_g]}\omega = \mathscr{L}_{X_f}\iota_{X_g}\omega - \iota_{X_g}\mathscr{L}_{X_f}\omega = \mathscr{L}_{X_f}\iota_{X_g}\omega = \mathrm{d}X_f g = \mathrm{d}\{g,f\}$$

and $X_{\{g,f\}} = [X_f, X_g]$. In other words, the mapping $f \in \mathscr{C}^\infty(P) \mapsto -X_f \in \mathrm{Ham}(P,\omega)$ is a Lie algebra morphism relative to the Lie bracket $\{\cdot,\cdot\}$ in $\mathscr{C}^\infty(P)$ and the commutation of vector fields in $\mathfrak{X}(P)$.

The following definition plays a crucial role in reduction theory, as it serves as a fundamental tool for describing Lie symmetries. Assume G to be a Lie group with a Lie algebra $\mathfrak{g}$.

Definition 1.2.6. The *fundamental vector field* associated with a Lie group action $\Phi\colon G\times P \to P$ related to $\xi \in \mathfrak{g}$ is the vector field on P defined by

$$(\xi_P)_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(\exp(t\xi), p)\,, \quad \forall p \in P\,,$$

where $\exp\colon \mathfrak{g} \to G$ is the exponential map related to the Lie group G.

If the Lie group action Φ is known from context, the notation gp is used instead of $\Phi(g,p)$ for every $g \in G$ and each $p \in P$. The convention used in Definition 1.2.6 gives rise to an anti-morphism of Lie algebras $\xi \in \mathfrak{g} \mapsto \xi_P \in \mathfrak{X}(P)$ (see [2, 25, 30]). Define

$$\Phi_g \colon \tilde{p} \in P \mapsto g\tilde{p} \in P\,, \quad \Phi^z \colon \tilde{g} \in G \mapsto \tilde{g}p \in P\,, \qquad \forall g \in G, \quad \forall p \in P\,.$$

Each Φ_g, for $g \in G$, has an inverse $\Phi_{g^{-1}}$. Thus, Φ_g is a diffeomorphism for every $g \in G$.

Definition 1.2.7. The *isotropy subgroup* of Φ at $p \in P$ is the Lie subgroup of G defined by

$$G_p := \{g \in G \mid gp = p\} \subset G.$$

The orbit of a point $p \in P$ relative to Φ is given by

$$Gp := \{gp \mid g \in G\}\,.$$

The constant rank theorem yields that the orbits of a Lie group action Φ are immersed submanifolds of P [2, p 48]. Consequently, for each $\tilde{p} \in Gp$ one has

$$\mathrm{T}_{\tilde{p}}(Gp) = \{(\xi_P)_{\tilde{p}} \mid \xi \in \mathfrak{g}\}\,.$$

Recall that each $g \in G$ gives rise to the following diffeomorphisms on G

$$L_g \colon G \ni h \mapsto gh \in G, \qquad R_g \colon G \ni h \mapsto hg \in G,$$
$$I_g \colon G \ni h \mapsto ghg^{-1} \in G.$$

Then, the *adjoint action* of G is defined as

$$\mathrm{Ad} \colon (g,\xi) \in G \times \mathfrak{g} \mapsto \mathrm{Ad}_g \xi \in \mathfrak{g}\,, \tag{1.2.2}$$

where $\mathrm{Ad}_g\xi := (\mathrm{T}_e I_g)(\xi)$. The fundamental vector field associated with the adjoint action for a given $\xi \in \mathfrak{g}$ is given by

$$(\xi_{\mathfrak{g}})_v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\exp(t\xi)} v = [\xi, v] =: \mathrm{ad}_\xi v, \quad \forall v \in \mathfrak{g},$$

where $[\cdot,\cdot]$ denotes the Lie bracket in $\mathfrak{g}$. Note that $(\xi_{\mathfrak{g}})_v \in \mathrm{T}_v\mathfrak{g}$ and $\mathrm{ad}_\xi v \in \mathfrak{g}$. Although both elements belong to different spaces, $(\xi_{\mathfrak{g}})_v$ and $\mathrm{ad}_\xi v$ can be identified due to the existence of a natural isomorphism. Specifically, when $\dim V < \infty$, there exists an isomorphism $v \in V \simeq D_v \in \mathrm{T}_\vartheta V$, at each $\vartheta \in V$, identifying each $v \in V$ to the tangent vector at ϑ associated with the derivative at ϑ in the direction v. If $\mathcal{S}_\xi$ is the orbit of the adjoint action passing through $\xi \in \mathfrak{g}$. Then

$$\mathrm{T}_\nu \mathcal{S}_\xi = \{(\xi_{\mathfrak{g}})_\nu \mid \xi \in \mathfrak{g}\},$$

for every $\nu \in \mathcal{S}_\xi$.

The group G also acts on $\mathfrak{g}^*$ via the *co-adjoint action*, given by

$$\mathrm{Ad}^* \colon (g,\mu) \in G \times \mathfrak{g}^* \mapsto \mathrm{Ad}^*_{g^{-1}}\mu \in \mathfrak{g}^*,$$

where Ad^*_g is the dual map to Ad_g, namely $\langle \mathrm{Ad}^*_g \mu, \xi\rangle = \langle \mu, \mathrm{Ad}_g \xi\rangle$ for all $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$, where $\langle\cdot,\cdot\rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Then,

$$(\xi_{\mathfrak{g}^*})_\mu = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}^*_{\exp(-t\xi)}\mu = -\langle \mu, [\xi,\cdot]\rangle = -\mathrm{ad}^*_\xi \mu, \qquad \forall \mu \in \mathfrak{g}^*. \tag{1.2.3}$$

Consequently, ad^*_ξ is defined as $\langle \mathrm{ad}^*_\xi \vartheta, v\rangle := \langle v, \mathrm{ad}_\xi v\rangle$ for every $\vartheta \in \mathfrak{g}^*$ and $v, \xi \in \mathfrak{g}$. The co-adjoint orbit of $\mu \in \mathfrak{g}^*$ is given by

$$\mathcal{O}_\mu := \{\mathrm{Ad}^*_{g^{-1}}\mu \mid g \in G\}\,, \qquad \text{and} \qquad T_\vartheta \mathcal{O}_\mu = \{(\xi_{\mathfrak{g}^*})_\vartheta \mid \xi \in \mathfrak{g}\},$$

at every point $\vartheta \in \mathcal{O}_\mu$. Furthermore, the fundamental vector fields $\xi_\mathfrak{g}$ and $\xi_{\mathfrak{g}^*}$ are related by

$$\langle (\xi_{\mathfrak{g}^*})_\mu, v\rangle = \langle -\mathrm{ad}^*_\xi \mu, v\rangle = -\langle \mu, (\xi_\mathfrak{g})_v\rangle, \quad \forall v \in \mathfrak{g} \simeq \mathrm{T}^*_\mu \mathfrak{g}^*, \quad \forall \mu \in \mathfrak{g}^* \simeq \mathrm{T}^*_v \mathfrak{g}.$$

Finally, the following definition introduces Lie group actions that preserve the symplectic form. Such actions play a fundamental role in the symplectic Marsden–Meyer–Weinstein reduction [128].

Definition 1.2.8. A Lie group action $\Phi\colon G \times P \to P$ is a *symplectic* Lie group action relative to (P, ω) if $\Phi^*_g \omega = \omega$ for every $g \in G$. Equivalently, in terms of fundamental vector fields associated with Φ, one has

$$\mathscr{L}_{\xi_P} \omega = 0 \qquad \forall \xi \in \mathfrak{g}\,.$$

Furthermore, Φ is a *Hamiltonian Lie group action* if its fundamental vector fields are Hamiltonian relative to (P, ω).

1.3 Basics on cosymplectic geometry

This subsection establishes fundamental results on cosymplectic geometry [27, 100, 101] to be used hereafter.

Definition 1.3.1. A *cosymplectic manifold* is a triple (M, ω, τ), where M is a $(2n+1)$-dimensional manifold, $\omega \in \Omega^2(M)$ and $\tau \in \Omega^1(M)$ are closed forms satisfying that $\tau \wedge \omega^n$ does not vanish at any point of M.

Note that $\tau \wedge \omega^n$ does not vanish at any point of M if and only if $\tau \wedge \omega^n$ is a volume form. Hence, cosymplectic manifolds are always orientable and odd-dimensional.

The Darboux theorem for cosymplectic manifolds [4] states the following.

Theorem 1.3.2. *Let (M, ω, τ) be a cosymplectic manifold. Then, each point $x \in M$ admits a local coordinate system $\{t, q^1, \ldots, q^n, p_1, \ldots, p_n\}$ on an open neighbourhood U of x so that*

$$\omega|_U = \sum_{i=1}^n \mathrm{d}q^i \wedge \mathrm{d}p_i, \qquad \tau|_U = \mathrm{d}t.$$

Such local coordinates are referred to as cosymplectic Darboux coordinates.

As in the symplectic setting, cosymplectic Darboux coordinates are not unique. As shown in the following theorem, each cosymplectic manifold (M, ω, τ) admits a unique vector field R on M satisfying

$$\iota_R\, \omega = 0, \qquad \iota_R\, \tau = 1.$$

This vector field is called the *Reeb vector field* of (M, ω, τ). In cosymplectic Darboux coordinates $\{t, q^1, \ldots, q^n, p_1, \ldots, p_n\}$, it is given by $R = \frac{\partial}{\partial t}$.

Theorem 1.3.3. *Every cosymplectic manifold (M, ω, τ) admits a unique Reeb vector field.*

Proof. Since M is odd-dimensional, suppose that $\dim M = 2n + 1$. Since $\ker \tau_x \cap \ker \omega_x = 0$ for every $x \in M$ and $\ker \omega_x \neq 0$, it follows that τ does not vanish. Consequently, $\dim \ker \tau_x = 2n$ and $\dim \ker \omega_x = 1$. Indeed, if $\dim \ker \omega_x > 1$, then $\dim \ker(\tau_x \wedge \omega^n_x) > 0$, which leads to a contradiction. Therefore, $\ker \omega_x \oplus \ker \tau_x = T_x M$ for each $x \in M$. Let $D_x \in \ker \omega_x \setminus \{0\}$. A Reeb vector field is then defined by

$$R_x = \frac{D_x}{\iota_{D_x} \tau_x}, \qquad x \in M\,. \tag{1.3.1}$$

It satisfies $\iota_{R_x} \tau_x = 1$ and $\iota_{R_x} \omega_x = 0$. Moreover, if R_1 and R_2 are two Reeb vector fields, then

$$\iota_{R_1}(\tau \wedge \omega^n) = \iota_{R_2}(\tau \wedge \omega^n) \;\Rightarrow\; \iota_{R_1 - R_2}(\tau \wedge \omega^n) = 0 \;\Rightarrow\; R_1 = R_2.$$

Hence, the Reeb vector field is unique and is given by (1.3.1). □

Definition 1.3.4. A *cosymplectomorphism* is a map $\varphi\colon M_1 \to M_2$ between cosymplectic manifolds (M_1, ω_1, τ_1) and (M_2, ω_2, τ_2) such that $\varphi^*\omega_2 = \omega_1$ and $\varphi^*\tau_2 = \tau_1$.

Then, in terms of Lie group actions, one has the following definition.

Definition 1.3.5. A *cosymplectic Lie group action* relative to (M, ω, τ) is a Lie group action $\Phi\colon G \times M \to M$ such that, for every $g \in G$, the map $\Phi_g\colon M \to M$ is a cosymplectomorphism. In other words,

$$\Phi_g^*\omega = \omega, \qquad \Phi_g^*\tau = \tau, \qquad \forall g \in G.$$

Assuming G is connected, as assumed in this PhD thesis, $\Phi\colon G \times M \to M$ is a cosymplectomorphism if and only if

$$\mathscr{L}_{\xi_M}\omega = 0, \qquad \mathscr{L}_{\xi_M}\tau = 0, \qquad \forall \xi \in \mathfrak{g},$$

where $\mathscr{L}_{\xi_M}$ denotes the Lie derivative along the fundamental vector field ξ_M.

Additionally, since $\mathrm{d}\tau = 0$, the condition $\mathscr{L}_{\xi_M}\tau = 0$ implies that $\iota_{\xi_M}\tau$ is a constant function on M, which does not need to be one or zero. This property will be relevant in Section 3.2.7 when studying the restricted circular three-body problem [2, 60].

Proposition 1.3.6. *A triple (M, ω, τ) is a cosymplectic manifold if and only if the vector bundle homomorphism*

$$\flat\colon \mathrm{T}M \to \mathrm{T}^*M, \qquad v_x \in \mathrm{T}_xM \mapsto \flat(v_x) = \iota_{v_x}\omega_x + (\iota_{v_x}\tau_x)\tau_x, \qquad \forall x \in M,$$

is a vector bundle isomorphism.

Proof. Assume that $\flat(v_x) = 0$ for a certain $v_x \in \mathrm{T}_xM$. Then,

$$\iota_{v_x}\omega_x + (\iota_{v_x}\tau_x)\tau_x = 0. \tag{1.3.2}$$

Contracting both sides of (1.3.2) with the Reeb vector field at x, namely $R_x \in \mathrm{T}_xM$, one has

$$\iota_{R_x}\iota_{v_x}\omega_x + (\iota_{v_x}\tau_x)\iota_{R_x}\tau_x = \iota_{v_x}\tau_x = 0,$$

and $v_x \in \ker\tau_x$. Then,

$$0 = \flat(v_x) = \iota_{v_x}\omega_x$$

and $v_x \in \ker\omega_x$. Therefore, $v_x \in \ker\tau_x \cap \ker\omega_x = 0$. Hence, $\flat$ is an injective vector bundle morphism and becomes a vector bundle isomorphism since $\mathrm{T}M$ and T^*M are vector bundles of the same rank.

Conversely, by contradiction, if (M, ω, τ) is not a cosymplectic manifold, then there exists a non-zero $v_x \in \ker\tau_x \cap \ker\omega_x$, which exists by assumption. Then $\flat$ is not a vector bundle isomorphism. □

Definition 1.3.7. Given a cosymplectic manifold (M, ω, τ). Then, each $f \in \mathscr{C}^\infty(M)$ gives rise to three vector fields:

- A *gradient vector field*, namely

  $$\nabla f := \flat^{-1}(\mathrm{d}f)\,, \tag{1.3.3}$$

  which amounts to saying that $\iota_{\nabla f}\omega = \mathrm{d}f - (Rf)\tau$ and $\iota_{\nabla f}\tau = Rf$.

- A *Hamiltonian vector field*, X_f, given by

  $$X_f := \flat^{-1}(\mathrm{d}f - (Rf)\tau)\,, \tag{1.3.4}$$

  which is equivalent to $\iota_{X_f}\omega = \mathrm{d}f - (Rf)\tau$ and $\iota_{X_f}\tau = 0$.

- An *evolution vector field*

  $$E_f := R + X_f\,. \tag{1.3.5}$$

In cosymplectic Darboux coordinates for (M,ω,τ) around a point $x\in M$, the vector fields (1.3.3), (1.3.4), and (1.3.5) read

$$\nabla f=\frac{\partial f}{\partial t}\frac{\partial}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}-\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}\right),\qquad X_f=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}-\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}\right),$$

and

$$E_f=\frac{\partial}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i}-\frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}\right).$$

The integral curves of E_f are given, in cosymplectic Darboux coordinates, by the solutions of

$$\frac{\mathrm{d}t}{\mathrm{d}s}=1,\qquad \frac{\mathrm{d}q^i}{\mathrm{d}s}=\frac{\partial f}{\partial p_i}(t,q,p)\,,\qquad \frac{\mathrm{d}p_i}{\mathrm{d}s}=-\frac{\partial f}{\partial q^i}(t,q,p)\,,\qquad i=1,\ldots,n\,, \tag{1.3.6a}$$

where (t,q,p) stands for $(t,q^1,\ldots,q^n,p_1,\ldots,p_n)$.

Example 1.3.8. Let T be a one-dimensional manifold, and let (P,ω) be a symplectic manifold. Consider the product manifold $M=T\times P$ and the projection map $\pi_T\colon M\to T$ and $\pi_P\colon M\to P$. A symplectic form ω on P induces a closed differential two-form on M given by $\omega_P:=\pi_P^*\omega$. Similarly, a non-vanishing differential one-form τ on T gives rise to a closed differential one-form $\tau_T=\pi_T^*\tau$ on M. Consequently, $(T\times P,\omega_P,\tau_T)$ becomes a cosymplectic manifold.

Unless otherwise stated, cosymplectic Darboux coordinates on $(T\times P,\omega_P,\tau_T)$ are assumed to be of the form $\{t,q^1,\ldots,q^n,p_1,\ldots,p_n\}$, where t stands for the pull-back to M of a potential of τ, while $q^1,\ldots,q^n,p_1,\ldots,p_n$ are the pull-backs to M of symplectic Darboux coordinates for (P,ω). For clarity, the pull-backs of coordinate functions from T and P to M are denoted identically to their counterparts in T and P. △

If $M=\mathbb{R}\times\mathrm{T}^*Q$, with $\tau=\mathrm{d}t$ and $\omega=\sum_{i=1}^n\mathrm{d}q^i\wedge\mathrm{d}p_i$, then (1.3.6a) can be rewritten as

$$\frac{\mathrm{d}q^i}{\mathrm{d}t}=\frac{\partial f}{\partial p_i}(t,q,p),\qquad \frac{\mathrm{d}p_i}{\mathrm{d}t}=-\frac{\partial f}{\partial q^i}(t,q,p),\qquad i=1,\ldots,n. \tag{1.3.6b}$$

Thus, (1.3.6b) retrieves the Hamilton equations for a time-dependent symplectic Hamiltonian system on T^*Q (see Subsection 3.1.1 or [2, 54]).

More generally, one has the following definition.

Definition 1.3.9. Given a cosymplectic manifold (M,ω,τ), the *Hamilton equations* associated with $h\in\mathscr{C}^\infty(M)$ are defined as the system of differential equations which, locally on each coordinated open neighbourhood $U\subset M$ with cosymplectic Darboux coordinates $\{t,q^1,\ldots,q^n,p_1,\ldots,p_n\}$, is given by

$$\frac{\mathrm{d}q^i}{\mathrm{d}t}=\frac{\partial h}{\partial p_i}(t,q^1,\ldots,q^n,p_1,\ldots,p_n)\,,\quad \frac{\mathrm{d}p_i}{\mathrm{d}t}=-\frac{\partial h}{\partial q^i}(t,q^1,\ldots,q^n,p_1,\ldots,p_n)\,,\quad i=1,\ldots,n\,. \tag{1.3.7}$$

Roughly speaking, Equations (1.3.7) are the system of differential equations for the integral curves of E_h parametrised by points in T described by the coordinate t in the Darboux coordinates obtained from coordinates on T and P as indicated previously. Although the coordinate t is defined up to an additive constant, equations (1.3.7) are equivalent for any admissible choice of variable t within the cosymplectic Darboux coordinate class. Consequently, the Hamilton equations exhibit a geometrical meaning. This property remains valid even in the cases where $T=\mathbb{S}^1$, provided that the particular solutions are allowed to match every point of T with several points of P (see Figure 1.2).

The integral curves of X_f in M are given by the solutions of

$$\frac{\mathrm{d}t}{\mathrm{d}s}=0,\qquad \frac{\mathrm{d}q^i}{\mathrm{d}s}=\frac{\partial f}{\partial p_i}(t,q,p)\,,\qquad \frac{\mathrm{d}p_i}{\mathrm{d}s}=-\frac{\partial f}{\partial q^i}(t,q,p)\,,\qquad i=1,\ldots,n\,.$$

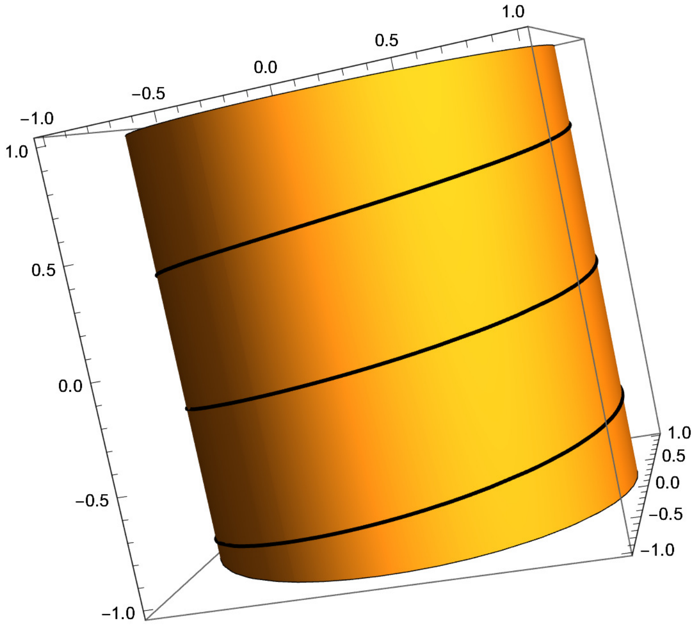


Figure 1.2: Example of solutions of Hamilton equations on a cosymplectic manifold $(\mathbb{S}^1 \times \mathrm{T}^*\mathbb{R}, \omega_{\mathrm{T}^*\mathbb{R}}, \tau_{\mathbb{S}^1})$ for $\mathbb{S}^1$ being the circle of radius one centred at zero. Only the coordinates of solutions in $\mathbb{S}^1 \times \mathbb{R}$ are represented.

It is worth noting that X_h on $M = \mathbb{R} \times P$ can also be considered as a *time-dependent vector field* [2, 54]. In this setting, its integral curves are $\mathbb{R} \ni t \mapsto (t, p(t)) \in \mathbb{R} \times P$, where $p(t)$ denotes a solution of (1.3.7). This provides the geometric interpretations of the solutions.

An analogous construction extends to any cosymplectic manifold of the form $(M := T \times P, \omega_P, \tau_T)$. However, in this setting, the solutions of the Hamilton equations can associate each point $t \in T$ with multiple distinct points in P. Nevertheless, locally in a neighbourhood of any point $t_0 \in T$, a solution can be considered as a union of local sections of $\pi_T \colon M \to T$, whose images do not intersect each other (see Figure 1.2).

The following result shows a useful property.

Proposition 1.3.10. *For any $f \in \mathscr{C}^\infty(M)$, the gradient vector field on (M, ω, τ) is given by $\nabla f = X_f + (Rf)R$. Moreover, if $Rf = 0$, then $[R, X_f] = 0$.*

Proof. From the definitions of the Hamiltonian and gradient vector fields, one has

$$\iota_{\nabla f}\, \omega = \mathrm{d}f - (Rf)\tau = \iota_{X_f}\omega \quad \Rightarrow \quad \iota_{\nabla f - X_f}\omega = 0\,.$$

Hence, $\nabla f = X_f + Y$ for some vector field Y on M such that $\iota_Y \omega = 0$. Furthermore,

$$\iota_{\nabla f}\tau = \iota_{X_f}\tau + \iota_Y \tau = Rf\,.$$

Since $\iota_{X_f}\tau = 0$ and $\ker \tau \oplus \ker \omega = \mathrm{T}M$, then $Y = (Rf)R$ and $\nabla f = X_f + (Rf)R$.

If, in addition, $Rf = 0$, then

$$\iota_{[X_f, R]}\omega = \mathscr{L}_{X_f}\iota_R \omega - \iota_R \mathscr{L}_{X_f}\omega = -\iota_R \mathrm{d}\iota_{X_f}\omega = \iota_R \mathrm{d}(\mathrm{d}f - (Rf)\tau) = 0,$$

and similarly

$$\iota_{[X_f, R]}\tau = \mathscr{L}_{X_f}\iota_R \tau - \iota_R \mathscr{L}_{X_f}\tau = -\iota_R \mathrm{d}\iota_{X_f}\tau = 0.$$

Since $\mathrm{T}M = \ker \tau \oplus \ker \omega$, it follows that $[X_f, R] = 0$. □

Any cosymplectic manifold (M, ω, τ) naturally induces a Poisson bracket $\{\cdot,\cdot\}_{\omega,\tau}\colon \mathscr{C}^\infty(M)\times\mathscr{C}^\infty(M)\to \mathscr{C}^\infty(M)$ of the form

$$\{f, g\}_{\omega,\tau} := \omega(\nabla f, \nabla g) = \omega(X_f, X_g), \qquad \forall f, g \in \mathscr{C}^\infty(M), \tag{1.3.8}$$

where the last equality is a consequence of Proposition 1.3.10 and the condition $\iota_R\omega = 0$. As in the symplectic case, the Poisson bracket satisfies

$$X_{\{f,g\}_{\omega,\tau}} = -[X_f, X_g], \qquad \forall f, g \in \mathscr{C}^\infty(M). \tag{1.3.9}$$

The Poisson bivector $\Lambda_{\omega,\tau}$ associated with the Poisson bracket $\{\cdot,\cdot\}_{\omega,\tau}$ is given by

$$\Lambda_{\omega,\tau}(x)(\alpha_x, \beta_x) = \{f, g\}_{\omega,\tau}(x) = \omega_x(X_f, X_g), \qquad \forall x \in M,$$

where $\mathrm{d}f(x) = \alpha_x$ and $\mathrm{d}g(x) = \beta_x$ are elements of T^*_xM for certain $f, g \in \mathscr{C}^\infty(M)$. In cosymplectic Darboux coordinates, the Poisson bivector $\Lambda_{\omega,\tau}$ reads

$$\Lambda_{\omega,\tau} = \sum_{i=1}^n \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}.$$

It is worth noting that $\mathscr{L}_R\Lambda_{\omega,\tau} = 0$.

The space of Hamiltonian vector fields relative to a cosymplectic manifold (M, ω, τ), denoted by $\mathrm{Ham}(M, \omega, \tau)$, forms a Lie subalgebra of $\mathfrak{X}(M)$. Moreover, the map

$$f \in \mathscr{C}^\infty(M) \mapsto -X_f \in \mathrm{Ham}(M, \omega, \tau)$$

is a Lie algebra homomorphism. Note that X_f is a Hamiltonian vector field relative to the Poisson bracket $\{\cdot,\cdot\}_{\omega,\tau}$, namely $X_f = \{\cdot, f\}_{\omega,\tau}$, whereas the corresponding evolution vector field E_f is never so and ∇f is not Hamiltonian in general either (cf. [146]).

1.3.1 Symplectic, cosymplectic, and Poisson geometries

In physical applications, particular attention is given to cosymplectic manifolds of the form $(T\times P, \omega_P, \tau_T)$, where T is a one-dimensional manifold representing a certain time interval and P is a symplectic manifold. Several choices of T and P are interesting in this context.

Let Q be the configuration manifold of a physical system. A standard choice for P is the cotangent bundle of Q, i.e. $P = \mathrm{T}^*Q$, equipped with its canonical symplectic structure, see Definition 1.2.3. Alternatively, one may consider $P = \mathrm{T}Q$ endowed with the symplectic structure induced by a *regular Lagrangian* function (see [112] for details).

Meanwhile, T can be chosen to be $\mathbb{R}$ with its natural variable t and the closed non-vanishing one-form $\tau = \mathrm{d}t$, which represents the flow of time over the real line. This choice is appropriate for systems evolving continuously over an unbounded time interval. Another relevant option is $T = \mathbb{S}^1$, the unit circle in $\mathbb{R}^2$, endowed with the closed non-degenerate one-form $\mathrm{d}\theta$, where θ denotes the angular coordinate relative to a reference point in it. This model is suited for the analysis of t-dependent Hamilton equations with a t-dependent periodic Hamiltonian.

Recall that cosymplectic Darboux coordinates for $(T \times P, \omega_P, \tau_T)$ are assumed to be of the form $\{t, q^1, \ldots, q^n, p_1, \ldots, p_n\}$, where the functions $\{q^1, \ldots, q^n, p_1, \ldots, p_n\}$ denote the pull-back to M of symplectic Darboux coordinates for a symplectic manifold P and the function t is the pull-back to M of a potential of a closed one-form on T.

Although a cosymplectic manifold (M, ω, τ) naturally induces both a symplectic manifold $(\mathbb{R} \times M, \widehat{\omega})$ and a Poisson manifold $(M, \{\cdot,\cdot\}_{\omega,\tau})$, it is shown that neither of these approaches provides an appropriate framework for extending the energy-momentum method from symplectic to cosymplectic setting.

The following lemma illustrates the natural relationship between symplectic and cosymplectic manifolds [44]. A complete proof is provided, as it is often omitted in the existing literature.

Lemma 1.3.11. *Let* $\omega \in \Omega^2(M), \tau \in \Omega^1(M)$ *and let* $\mathrm{pr}\colon \mathbb{R} \times M \to M$ *be the canonical projection onto* M*. Let* s *be the natural coordinate in* $\mathbb{R}$ *understood as a variable in* $\mathbb{R} \times M$ *in the natural manner. Then,* (M, ω, τ) *is a cosymplectic manifold if and only if* $(\mathbb{R} \times M, \mathrm{pr}^*\omega + \mathrm{d}s \wedge \mathrm{pr}^*\tau =: \widehat{\omega})$ *is a symplectic manifold. Moreover,* pr *is a Poisson morphism, i.e.*

$$\{f \circ \mathrm{pr}, k \circ \mathrm{pr}\}_{\widehat{\omega}} = \{f, k\}_{\omega,\tau} \circ \mathrm{pr}, \qquad \forall f, k \in \mathscr{C}^\infty(M)\,.$$

Proof. Recall that if (M, ω, τ) is a cosymplectic manifold and $\dim M = 2n + 1$, then $\omega^n \wedge \tau$ is a volume form. Since $\omega \in \Omega^2(M)$ and $\tau \in \Omega^1(M)$ are closed, then

$$\mathrm{d}\widehat{\omega} = \mathrm{d}(\mathrm{pr}^*\omega + \mathrm{d}s \wedge \mathrm{pr}^*\tau) = \mathrm{pr}^*\mathrm{d}\omega - \mathrm{d}s \wedge \mathrm{pr}^*\mathrm{d}\tau = 0, \tag{1.3.10}$$

and $\widehat{\omega} \in \Omega^2(\mathbb{R} \times M)$ is also closed. Since $\vartheta^{n+1} = 0$ for every differential two-form ϑ on M, one has that

$$\widehat{\omega}^{n+1} = (\mathrm{pr}^*\omega + \mathrm{d}s \wedge \mathrm{pr}^*\tau)^{n+1} = (n+1)(\mathrm{pr}^*\omega)^n \wedge \mathrm{d}s \wedge \mathrm{pr}^*\tau = (n+1)\mathrm{d}s \wedge \mathrm{pr}^*(\omega^n \wedge \tau), \tag{1.3.11}$$

is clearly a volume form on $\mathbb{R} \times M$ and it is non-zero. Thus, $\widehat{\omega}$ is non-degenerate.

Conversely, if $(\mathbb{R} \times M, \widehat{\omega})$ is a symplectic manifold, relation (1.3.11) shows that $\omega^n \wedge \tau \neq 0$. Moreover, (1.3.10) gives that $\mathrm{pr}^*\omega$ and $\mathrm{pr}^*\tau$ are closed forms. Since pr is a surjective submersion, $\mathrm{d}\omega = 0$ and $\mathrm{d}\tau = 0$. Therefore, (M, ω, τ) is a cosymplectic manifold.

Furthermore, if $\{\cdot, \cdot\}_{\widehat{\omega}}$ is the Poisson bracket induced by the symplectic form $\widehat{\omega}$, then

$$\begin{aligned}\mathrm{pr}^*\{f, k\}_{\omega,\tau} = -\mathrm{pr}^*(\iota_{X_f}\iota_{X_k}\omega) = -\mathrm{pr}^*(\iota_{X_f}\mathrm{d}k - (Rk)\iota_{X_f}\tau)&\\ = -\iota_{X_{\mathrm{pr}^*f}}\mathrm{pr}^*\mathrm{d}k = -\iota_{X_{\mathrm{pr}^*f}}\mathrm{d}\mathrm{pr}^*k = \{\mathrm{pr}^*f, \mathrm{pr}^*k\}_{\widehat{\omega}}&,\end{aligned}$$

for every $f, k \in \mathscr{C}^\infty(M)$, and $\mathrm{pr}\colon \mathbb{R} \times M \to M$ is a Poisson morphism. Note that X_{pr^*f} stands for the Hamiltonian vector field on $(\mathbb{R} \times M, \widehat{\omega})$ of the function $\mathrm{pr}^*f \in \mathscr{C}^\infty(\mathbb{R} \times M)$. □

This paragraph aims to show that the vector fields ∇f, X_f, and E_f on a cosymplectic manifold (M, ω, τ) cannot, in general, be considered as Hamiltonian vector fields relative to the symplectic manifold $(\mathbb{R} \times M, \widehat{\omega})$, where $\widehat{\omega}$ is a symplectic form induced by (M, ω, τ). To clarify this point, it is necessary to relate f, ∇f, X_f, and E_f to natural mathematical structures on $\mathbb{R} \times M$. Consider $\tilde{f} := \mathrm{pr}^*f \in \mathscr{C}^\infty(\mathbb{R} \times M)$, where $\mathrm{pr}\colon \mathbb{R} \times M \to M$ is the canonical projection. Define the vector fields $\tilde{F}_g$, $\tilde{X}_f$, and $\tilde{E}_f$ on $\mathbb{R} \times M$ to be the unique vector fields projecting onto ∇f, X_f, and E_f via pr_*, respectively. Then, taking into account the isomorphism $\mathrm{T}_{(s,x)}(\mathbb{R} \times M) \simeq \mathrm{T}_s\mathbb{R} \oplus \mathrm{T}_xM$ for every $s \in \mathbb{R}$ and $x \in M$, one gets that

$$\iota_{\tilde{F}_g}\mathrm{d}s = \iota_{\tilde{X}_f}\mathrm{d}s = \iota_{\tilde{E}_f}\mathrm{d}s = 0\,.$$

Furthermore,

$$\begin{aligned}\mathrm{d}\iota_{\tilde{F}_g}\hat{\omega} = d(\iota_{\tilde{F}_g}\mathrm{pr}^*\omega - (\iota_{\tilde{F}_g}\mathrm{pr}^*\tau)\mathrm{d}s) = \mathrm{d}(\mathrm{pr}^*(\iota_{\nabla f}\omega) - \mathrm{pr}^*(Rf)\mathrm{d}s)&\\ = \mathrm{pr}^*(\mathrm{d}(\mathrm{d}f - (Rf)\tau)) - \mathrm{pr}^*(\mathrm{d}(Rf)) \wedge \mathrm{d}s = -\mathrm{pr}^*(\mathrm{d}(Rf)) \wedge (\mathrm{d}s + \mathrm{pr}^*\tau)&\,.\end{aligned}$$

Consequently, $\tilde{F}_g$ is not, in general, a Hamiltonian vector field on $\mathbb{R} \times M$ relative to $\widehat{\omega}$. Similarly,

$$\begin{aligned}\mathrm{d}\iota_{\tilde{X}_f}\widehat{\omega} = \mathrm{d}(\iota_{\tilde{X}_f}\mathrm{pr}^*\omega - (\iota_{\tilde{X}_f}\mathrm{pr}^*\tau)\mathrm{d}s) = \mathrm{d}(\mathrm{pr}^*(\iota_{X_f}\omega) - \mathrm{pr}^*(\iota_{X_f}\tau)\mathrm{d}s)&\\ = \mathrm{pr}^*(\mathrm{d}\iota_{X_f}\omega) - \mathrm{pr}^*(\mathrm{d}\iota_{X_f}\tau) \wedge \mathrm{d}s = -\mathrm{pr}^*(\mathrm{d}(Rf) \wedge \tau)&,\end{aligned}$$

and $\tilde{X}_f$ is not, neither, a Hamiltonian vector field on $\mathbb{R} \times M$ in general. Finally,

$$\mathrm{d}\iota_{\tilde{E}_f}\widehat{\omega} = \mathrm{d}(\iota_{\tilde{R}}\hat{\omega} + \iota_{\tilde{X}_f}\widehat{\omega}) = \mathrm{d}\iota_{\tilde{X}_f}\hat{\omega} = -\mathrm{pr}^*(\mathrm{d}(Rf) \wedge \tau),$$

where $\tilde{R}$ is the unique vector field on $\mathbb{R} \times M$ that projects onto the Reeb vector field R on M via pr_* and satisfies $\iota_{\tilde{R}}\mathrm{d}s = 0$. Accordingly, $\tilde{E}_f$, in general, fails to be a Hamiltonian vector field with respect to $(\mathbb{R} \times M, \widehat{\omega})$.

Nevertheless, if $\mathrm{d}(Rf) = 0$, then ∇f, X_f, E_f naturally give rise to Hamiltonian vector fields $\tilde{F}_g$, $\tilde{X}_f$, and $\tilde{E}_f$, respectively, relative to the symplectic manifold $(\mathbb{R} \times M, \widehat{\omega})$. However, in general, the latter is not necessarily satisfied. The condition $Rf = 0$, which appears in a different form in cosymplectic theory [4], may be used to define, on cosymplectic manifolds, an analogue of the geometric structures and techniques appearing in symplectic manifolds [44].

It is worth noting that alternative approaches exist to consider some of the vector fields mentioned above in M as Hamiltonian vector fields on $\mathbb{R} \times M$ (see, for instance, [44] or the proof of Lemma 1.3.11). However, these methods are used to change the intrinsic properties of vector fields on M, potentially complicating their analysis. For example, certain methods can turn a vector field on M with equilibrium points into one without them in $\mathbb{R} \times M$, which can give rise to problems with studying the stability. Specifically, the vector field $(Rf)\frac{\partial}{\partial s} + X_f$ is Hamiltonian on $\mathbb{R} \times M$ with respect to $\widehat{\omega}$ and projecting onto X_f relative to pr_*. Nevertheless, the stability properties of $(Rf)\frac{\partial}{\partial s} + X_f$ significantly differ from those of X_f, e.g. it may admit no equilibrium points at all while X_f does, and thus introduces new difficulties in the stability analysis of the original dynamics on M.

Note that every cosymplectic Hamiltonian vector field is indeed Hamiltonian relative to the Poisson bracket associated with its underlying cosymplectic manifold. Nevertheless, gradient vector fields are not, in general, Hamiltonian, and evolution vector fields are never Hamiltonian with respect to such a Poisson bracket, as shown before. These facts, along with further results presented in the subsequent sections, indicate that the Poisson bracket associated with a cosymplectic structure is insufficient, by itself, for the analysis of the problems to be studied hereafter. Moreover, neither the classical energy-momentum method [113] nor the energy-Casimir method, developed for studying the stability of relative equilibrium points of Hamiltonian systems on Poisson manifolds [113], is applicable in the forthcoming analysis. In particular, the restricted circular three-body problem examined in Subsection 3.2.7 demonstrates the necessity of the new techniques introduced within this PhD thesis. It also highlights the limitations of existing techniques, such as the time-dependent energy-momentum method [54], which proves to be not enough to analyse certain types of problems addressed through the new methods proposed in Chapter 3.

1.4 Fundamentals on geometric field theory

This section reviews the geometric preliminaries required for the development of the geometric formulation of Hamiltonian field theories (see [7, 37, 38, 75, 78, 106, 107] for details on k-polysymplectic and k-polycosymplectic formalisms). Throughout this work, it is assumed that $\mathbb{R}^k$ has a fixed basis $\{e_1, \ldots, e_k\}$ giving rise to a dual basis $\{e^1, \ldots, e^k\}$ in $\mathbb{R}^{k*}$. Let $\boldsymbol{\theta} = \theta^\alpha \otimes e_\alpha \in \Omega^\ell(M, \mathbb{R}^k)$ be an $\mathbb{R}^k$-valued differential ℓ-form. The contraction of $\boldsymbol{\theta}$ with a vector field $X \in \mathfrak{X}(M)$ is defined as

$$\iota_X \boldsymbol{\theta} = \sum_{\alpha=1}^{k} (\iota_X \theta^\alpha) \otimes e_\alpha \in \Omega^{\ell-1}(M, \mathbb{R}^k)\,.$$

The contraction of $\boldsymbol{\theta}$ with a k-vector field $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{X}^k(M)$ is defined as

$$\iota_{\mathbf{X}} \boldsymbol{\theta} = \sum_{\alpha=1}^{k} \iota_{X_\alpha} \theta^\alpha \in \Omega^{\ell-1}(M)\,.$$

The exterior product of two $\mathbb{R}^k$-valued differential forms $\boldsymbol{\vartheta} = \vartheta^\alpha \otimes e_\alpha \in \Omega^{\ell_1}(M, \mathbb{R}^k)$ and $\boldsymbol{\mu} = \mu^\alpha \otimes e_\alpha \in \Omega^{\ell_2}(M, \mathbb{R}^k)$ is defined by

$$\boldsymbol{\vartheta} \,\bar{\wedge}\, \boldsymbol{\mu} = \sum_{\alpha=1}^{k} (\vartheta^\alpha \wedge \mu^\alpha) \otimes e_\alpha \in \Omega^{\ell_1+\ell_2}(M, \mathbb{R}^k)\,.$$

The above definitions are useful for simplifying the notation of the theory. Note that a point-wise analogue can be defined similarly. In particular, these definitions apply to the contraction of elements of E or $E \otimes \mathbb{R}^k$ with $E^* \otimes \mathbb{R}^k$ for a vector space E.

1.4.1 k-Vector fields and integral sections

This subsection reviews the theory of k-vector fields, which plays a crucial role in the geometric analysis of systems of partial differential equations [43].

First, the following fundamental definition is introduced.

Definition 1.4.1. *The Whitney sum of k copies of the tangent bundle to M* is defined as [1]

$$\bigoplus^k \mathrm{T}M = \mathrm{T}M \oplus_M \overset{(k)}{\cdots} \oplus_M \mathrm{T}M\,,$$

where the fibre product is taken over M. This construction admits the natural projections

$$\mathrm{pr}^\alpha \colon \bigoplus^k \mathrm{T}M \to \mathrm{T}M\,, \qquad \mathrm{pr}^1_M \colon \bigoplus^k \mathrm{T}M \to M, \qquad \alpha = 1, \dots, k.$$

Definition 1.4.2. A *k-vector field* on M is a section $\mathbf{X}\colon M \to \bigoplus^k \mathrm{T}M$ of the vector bundle $\mathrm{pr}_M \colon \bigoplus^k \mathrm{T}M \to M$. The space of k-vector fields on M is denoted by $\mathfrak{X}^k(M)$.

Each k-vector field $\mathbf{X} \in \mathfrak{X}^k(M)$ is equivalent to a family of vector fields $X_1, \dots, X_k \in \mathfrak{X}(M)$ defined by $X_\alpha = \mathrm{pr}^\alpha_M \circ \mathbf{X}$ for $\alpha = 1, \dots, k$. This fact justifies the notation $\mathbf{X} := (X_1, \dots, X_k)$.

Definition 1.4.3. Given a map $\phi\colon U \subset \mathbb{R}^k \to M$, its *first prolongation* is the map $\phi' \colon U \subset \mathbb{R}^k \to \bigoplus^k \mathrm{T}M$ defined as follows

$$\phi'(t) := \left(\phi(t); \mathrm{T}_t\phi\left(\frac{\partial}{\partial t^1}\bigg|_t\right), \dots, \mathrm{T}_t\phi\left(\frac{\partial}{\partial t^k}\bigg|_t\right)\right) =: (\phi(t); \phi'_\alpha(t))\,, \qquad t = (t^1, \dots, t^k) \in \mathbb{R}^k\,.$$

The integral sections of a k-vector field are defined in the following definition.

Definition 1.4.4. Let $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(M)$ be a k-vector field. An *integral section* of $\mathbf{X}$ is a map $\phi\colon U \subset \mathbb{R}^k \to M$ such that $\phi' = \mathbf{X} \circ \phi$, namely $\mathrm{T}\phi\left(\frac{\partial}{\partial t^\alpha}\right) = X_\alpha \circ \phi$ for $\alpha = 1, \dots, k$. A k-vector field $\mathbf{X} \in \mathfrak{X}^k(M)$ is *integrable* if $[X_\alpha, X_\beta] = 0$ for $1 \leq \alpha < \beta \leq k$.

Let $\mathbf{X} = (X_1, \dots, X_k)$ be a k-vector field on M with local expression $X_\alpha = X^i_\alpha \frac{\partial}{\partial x^i}$ for $\alpha = 1, \dots, k$. Then, $\phi\colon U \subset \mathbb{R}^k \to M$ is an integral section of $\mathbf{X}$ if and only if its coordinates satisfy the following system of PDEs

$$\frac{\partial \phi^i}{\partial t^\alpha} = X^i_\alpha \circ \phi\,, \qquad i = 1, \dots, n, \qquad \alpha = 1, \dots, k\,. \tag{1.4.1}$$

Indeed, (1.4.1) is integrable if and only if $[X_\alpha, X_\beta] = 0$ for $1 \leq \alpha < \beta \leq k$.

1.4.2 k-Polysymplectic geometry

Geometric covariant descriptions of first-order classical field theories can be performed by appropriate generalisations of some of the structures mentioned in the previous sections. One of the simplest among them is *k-symplectic geometry (also known as k-polysymplectic)*, originally introduced by A. Awane [7, 8] and subsequently used by M. de León et al. [39, 41, 42], and L.K. Norris [117, 126] to describe first-order classical field theories. This formalism coincides with the *polysymplectic structures* developed by G.C. Günther [78], although it differs from the polysymplectic frameworks introduced by G. Sardanashvily et al. [67, 134] and I.V. Kanatchikov [89]. As there are many k-symplectic-like definitions with related but mainly different and even contradictory meanings, it is relevant to fix the terminology properly, as accomplished in Definition 1.4.5.

k-Polysymplectic manifolds have been widely used in the analysis of physical systems described by partial differential equations. In particular, they provide a geometric framework for the Euler-Lagrange and Hamilton-de Donder-Weyl field equations, together with the dynamical systems described by them,

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[1] The subindex of the Whitney sum will be skipped if it is understood from context

notably including first-order regular autonomous field theories [37, 43, 46, 57]. Moreover, k-polysymplectic geometry offers a natural setting for the study of symmetries, conservation laws, and reduction procedures in field theories [7, 78, 107, 133]. Remarkably, k-polysymplectic geometry has also proved to be effective in the study of systems of ordinary differential equations, and their superposition rules [53]. It is also worth stressing that the analysis of ordinary differential equations within the framework of k-polysymplectic geometry differs significantly from the standard approach, which primarily addresses systems of partial differential equations, and therefore gives rise to new lines of research.

This subsection provides an overview of the theory of k-polysymplectic, polysymplectic, k-symplectic structures, and related concepts that appear in the literature. It clarifies the terminology adopted in this PhD thesis, as well as introduces the definitions to be used. Such clarification is particularly necessary due to the lack of terminological consistency in the literature, where the same term can refer to different, not equivalent, geometric concepts. Certain examples of this ambiguity can be found in the foundational works by Günther [78] and Awane [7].

Definition 1.4.5. A *k-polysymplectic form* on P is a closed non-degenerate $\mathbb{R}^k$-valued differential two-form

$$\boldsymbol{\omega} = \sum_{\alpha=1}^{k} \omega^\alpha \otimes e_\alpha \in \Omega^2(P, \mathbb{R}^k)\,.$$

The pair $(P, \boldsymbol{\omega})$ is called a *k-polysymplectic manifold*. In addition, if $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\theta}$ for some $\boldsymbol{\theta} \in \Omega^1(P, \mathbb{R}^k)$, then $(P, \boldsymbol{\theta})$ is an exact k-polysymplectic manifold.

In the literature, k-polysymplectic manifolds are often called, for simplicity, polysymplectic manifolds (see [107] for instance). However, the term ‘polysymplectic’ also refers to a different notion that is explained below. To prevent ambiguity, the terminology ‘k-polysymplectic manifold’ and other related ones in our work are not simplified, unless otherwise stated.

A manifold P admits a k-polysymplectic form $\boldsymbol{\omega}$ if and only if there exists a family of k closed two-forms $\omega^1, \ldots, \omega^k \in \Omega^2(P)$ satisfying the non-degeneracy condition

$$\ker \boldsymbol{\omega} = \ker\Big(\sum_{\alpha=1}^{k} \omega^\alpha \otimes e_\alpha\Big) = \bigcap_{\alpha=1}^{k} \ker \omega^\alpha = 0\,.$$

Hereafter, $\mathbb{R}^k$-valued differential forms are written in bold. Now, one can proceed to define polysymplectic manifolds as follows.

Definition 1.4.6. Let P be an $n(k+1)$-dimensional manifold. Then,

- A *polysymplectic form* on P is a differential $\mathbb{R}^k$-valued two-form of the form

  $$\boldsymbol{\omega} = \sum_{\alpha=1}^{k} \omega^\alpha \otimes e_\alpha \in \Omega^2(P, \mathbb{R}^k)\,,$$

  where $\omega^1, \ldots, \omega^k \in \Omega^2(P)$ are closed two-forms such that the non-degeneracy condition

  $$\ker \boldsymbol{\omega} = \bigcap_{\alpha=1}^{k} \ker \omega^\alpha = 0\,.$$

  holds. A manifold P equipped with a polysymplectic form $\boldsymbol{\omega}$ is referred to as a *polysymplectic manifold* and is denoted as a pair $(P, \boldsymbol{\omega})$.

- A *k-symplectic structure* on P is a pair $(\boldsymbol{\omega}, V)$, where $(P, \boldsymbol{\omega})$ is a polysymplectic manifold and $V \subset \mathrm{T}P$ is an integrable distribution on P of rank nk such that

  $$\boldsymbol{\omega}|_{V\times V} = 0\,.$$

  In this case, $(P, \boldsymbol{\omega}, V)$ is a *k-symplectic manifold* and the distribution V is called the *polarisation* of the k-symplectic manifold.

If the two-form $\boldsymbol{\omega}$ is exact, namely $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\theta}$ for some $\boldsymbol{\theta} \in \Omega^1(P, \mathbb{R}^k)$, then the corresponding structure, whether polysymplectic or k-symplectic as introduced in Definition 1.4.6, is said to be *exact*.

The definition of a k-symplectic manifold coincides with the one introduced by A. Awane [7, 9]. Additionally, it is locally equivalent to the concept of the *standard polysymplectic structure* introduced by C. Günther [78] (they are globally equivalent provided there exist compatible Darboux charts[2]) and globally equivalent to the *integrable p-almost cotangent structure* introduced by M. de León *et al* [41, 42]. In the special case when $k = 1$, Awane's definition reduces to the well-known notion of a *polarised symplectic manifold*, namely a symplectic manifold with a Lagrangian distribution [46].

In Günther's works, polysymplectic manifolds refer to the differential geometric structures obtained from the definition of a k-symplectic manifold by removing the existence of the distribution V. On the other hand, a *standard* polysymplectic manifold in Günther's terminology [78] is a polysymplectic manifold admitting local Darboux coordinates, which is equivalent to the definition of a k-symplectic manifold. The existence of the polarisation V in the definition of a k-symplectic structure is necessary to guarantee the existence of an atlas of compatible k-symplectic Darboux coordinates (see [7, 73] and [130, p 57]) and vice versa.

Theorem 1.4.7 (Darboux theorem for k-symplectic manifolds)**.** *Let $(P, \boldsymbol{\omega}, V)$ be a k-symplectic manifold. Then, on a neighbourhood U of any point $p \in P$, there exist local coordinates $\{q^i, p_i^\alpha\}$, with $i = 1, \ldots, n$ and $\alpha = 1, \ldots, k$, such that*

$$\boldsymbol{\omega} = \sum_{\alpha=1}^{k} \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i^\alpha \otimes e_\alpha \,, \qquad V = \left\langle \frac{\partial}{\partial p_i^\alpha} \right\rangle .$$

Such coordinates are called k-symplectic Darboux coordinates.

Before presenting the canonical example of a k-symplectic manifold, it is useful to recall the following example, which will be used extensively throughout this PhD thesis.

Example 1.4.8 (Canonical model for k-symplectic manifolds)**.** Let Q be an n-dimensional manifold and consider the Whitney sum

$$\bigoplus^k \mathrm{T}^*Q = \mathrm{T}^*Q \oplus_Q \overset{(k)}{\cdots} \oplus_Q \mathrm{T}^*Q \,,$$

with natural projections $\pi^\alpha \colon \bigoplus^k \mathrm{T}^*Q \to \mathrm{T}^*Q$, from the α-th component of $\bigoplus^k \mathrm{T}^*Q$ onto T^*Q, with $\alpha = 1, \ldots, k$, and $\pi_Q \colon \bigoplus^k \mathrm{T}^*Q \to Q$. A coordinate system $\{q^i\}$ in Q induces a natural coordinate system $\{q^i, p_i^\alpha\}$ in $\bigoplus^k \mathrm{T}^*Q$, where $\alpha = 1, \ldots, k$. Consider the canonical forms in the cotangent bundle T^*Q of Q given by $\theta \in \Omega^1(\mathrm{T}^*Q)$ and $\omega = -\mathrm{d}\theta \in \Omega^2(\mathrm{T}^*Q)$. Hence, the Whitney sum $\bigoplus^k \mathrm{T}^*Q$ has the canonical forms taking values in $\mathbb{R}^k$ given by

$$\boldsymbol{\theta}_k = \sum_{\alpha=1}^{k} (\pi^\alpha)^*\theta \otimes e_\alpha \,, \qquad \boldsymbol{\omega}_k = -\mathrm{d}\boldsymbol{\theta}_k \,,$$

which, in natural coordinates $\{q^i, p_i^\alpha\}$ in $\bigoplus^k \mathrm{T}^*Q$, read

$$\boldsymbol{\theta}_k = \sum_{\alpha=1}^{k} \sum_{i=1}^{n} p_i^\alpha \mathrm{d}q^i \otimes e_\alpha \;, \qquad \boldsymbol{\omega}_k = \sum_{\alpha=1}^{k} \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i^\alpha \otimes e_\alpha \,.$$

Taking all this into account, the triple $(\bigoplus^k \mathrm{T}^*Q, \boldsymbol{\omega}_k, V_k)$, with $V_k = \ker \mathrm{T}\pi_Q$, is a k-symplectic manifold. Notice that the natural coordinates $\{q^i, p_i^\alpha\}$ in $\bigoplus^k \mathrm{T}^*Q$ are the canonical example of k-symplectic Darboux coordinates. △

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[2] Note that it is not clear what Günther means by an atlas of canonical charts, namely, which is the equivalence between different pairs of Darboux charts.

Given a k-polysymplectic manifold $(P, \boldsymbol{\omega})$, the vector bundle morphism

$$\flat\colon (v_1, \dots, v_k) \in \bigoplus^k \mathrm{T}P \longmapsto \sum_{\alpha=1}^k \iota_{v_\alpha} \langle \boldsymbol{\omega}, e^\alpha \rangle \in \mathrm{T}^*P\,,$$

induces a morphism of $\mathscr{C}^\infty(P)$-modules $\flat\colon \mathfrak{X}^k(P) \to \Omega^1(P)$. The morphism $\flat$ is surjective because the annihilator of its image belongs to $\bigcap_{\alpha=1}^k \ker \omega^\alpha = \ker \boldsymbol{\omega} = 0$.

1.4.3 $\boldsymbol{\omega}$-Hamiltonian functions and vector fields

This subsection surveys the basic theory of k-polysymplectic vector fields and functions. These structures play a fundamental role in the k-polysymplectic energy-momentum method introduced in Section 3.3.

Definition 1.4.9. Let $(P, \boldsymbol{\omega} = \sum_{\alpha=1}^k \omega^\alpha \otimes e_\alpha)$ be a k-polysymplectic manifold. A vector field $Y \in \mathfrak{X}(P)$ is *$\boldsymbol{\omega}$-Hamiltonian* if it is Hamiltonian with respect to all the presymplectic forms $\omega^1, \dots, \omega^k \in \Omega^2(P)$, namely $\iota_Y \omega^\alpha$ is closed for $\alpha = 1, \dots, k$. The space of $\boldsymbol{\omega}$-Hamiltonian vector fields on a k-polysymplectic manifold $(P, \boldsymbol{\omega})$ is denoted by $\mathfrak{X}_{\boldsymbol{\omega}}(P)$.

Note that if $\iota_Y \omega^\alpha$ is closed, it generally admits a potential function only locally. Nevertheless, since the present PhD thesis is mainly focused on local aspects, the possible lack of a globally defined potential function does not affect what follows.

For the study of $\boldsymbol{\omega}$-Hamiltonian vector fields, it is useful to introduce a generalisation of the concept of the Hamiltonian function for presymplectic forms. This generalisation allows for dealing with all associated functions $h^1, \dots, h^k$ simultaneously (see [7, 53] for details).

Definition 1.4.10. Let $(P, \boldsymbol{\omega} = \sum_{\alpha=1}^k \omega^\alpha \otimes e_\alpha)$ be a k-polysymplectic manifold. An $\mathbb{R}^k$-valued function $\boldsymbol{h} = \sum_{\alpha=1}^k h^\alpha \otimes e_\alpha$ is an *$\boldsymbol{\omega}$-Hamiltonian function* if there exists a vector field $X_{\boldsymbol{h}}$ on P such that $\iota_{X_{\boldsymbol{h}}} \boldsymbol{\omega} = \mathrm{d}\boldsymbol{h}$, namely $\iota_{X_{\boldsymbol{h}}} \omega^\alpha = \mathrm{d}h^\alpha$ for $\alpha = 1, \dots, k$. In this case, $\boldsymbol{h} \in \mathscr{C}^\infty(P, \mathbb{R}^k)$ is an *$\boldsymbol{\omega}$-Hamiltonian function* associated with $X_{\boldsymbol{h}}$. The space of $\boldsymbol{\omega}$-Hamiltonian functions on $(P, \boldsymbol{\omega})$ is denoted by $\mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$.

An $\boldsymbol{\omega}$-Hamiltonian vector field (resp. function) is often called k-Hamiltonian at times if $\boldsymbol{\omega}$ is understood from context or its specific expression is not relevant. In [118], the author defined the k-Hamiltonian system associated with the $\mathbb{R}^k$-valued Hamiltonian function $\boldsymbol{h}$ as the vector field $X_{\boldsymbol{h}}$ from the previous definition. Additionally, A. Awane [7] called $\boldsymbol{h}$ a Hamiltonian map of X when X is additionally an infinitesimal automorphism of a certain distribution on which it is assumed that the presymplectic forms of the k-symplectic manifold vanish.

Example 1.4.11. Consider the two-polysymplectic manifold $(\mathbb{R}^3, \boldsymbol{\omega})$, where $\{u, v, w\}$ are linear coordinates on $\mathbb{R}^3$ and

$$\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2\,,$$

with

$$\omega^1 = -\frac{4w}{v^2}\mathrm{d}u \wedge \mathrm{d}w + \frac{1}{v}\mathrm{d}v \wedge \mathrm{d}w + \frac{4w^2}{v^3}\mathrm{d}u \wedge \mathrm{d}v\,, \qquad \omega^2 = -\frac{4}{v^2}\mathrm{d}u \wedge \mathrm{d}w + \frac{8w}{v^3}\mathrm{d}u \wedge \mathrm{d}v\,,$$

The vector fields

$$X_1 = 4u^2\frac{\partial}{\partial u} + 4uv\frac{\partial}{\partial v} + v^2\frac{\partial}{\partial w}\,, \qquad X_2 = \frac{\partial}{\partial u}\,,$$

are $\boldsymbol{\omega}$-Hamiltonian with $\boldsymbol{\omega}$-Hamiltonian functions

$$\boldsymbol{f} = \left(4uw - 8\frac{u^2w^2}{v^2} - \frac{v^2}{2}\right) \otimes e_1 + \left(4u - 16\frac{u^2w}{v^2}\right) \otimes e_2\,, \qquad \boldsymbol{g} = -2\frac{w^2}{v^2} \otimes e_1 - 4\frac{w}{v^2} \otimes e_2\,,$$

respectively, with respect to the two-polysymplectic form $\boldsymbol{\omega}$. △

The following propositions provide some properties of $\boldsymbol{\omega}$-Hamiltonian functions and vector fields. More details can be found in [53].

Proposition 1.4.12. *Each $\boldsymbol{\omega}$-Hamiltonian vector field is associated with at least one $\boldsymbol{\omega}$-Hamiltonian function. Conversely, any $\boldsymbol{\omega}$-Hamiltonian function uniquely determines a $\boldsymbol{\omega}$-Hamiltonian vector field.*

Proof. The non-trivial part of the proof is the converse. By the definition, each $\boldsymbol{\omega}$-Hamiltonian function $\boldsymbol{h} = h^1 \otimes e^1 + \cdots + h^k \otimes e_k$ is associated with a vector field $X_{\boldsymbol{h}}$. Suppose that there exist two $\boldsymbol{\omega}$-Hamiltonian vector fields $X^1_{\boldsymbol{h}}$ and $X^2_{\boldsymbol{h}}$ related to $\boldsymbol{h}$. Then,

$$\iota_{X^1_{\boldsymbol{h}}}\boldsymbol{\omega} = \iota_{X^2_{\boldsymbol{h}}}\boldsymbol{\omega} = \mathrm{d}\boldsymbol{h} \quad \Longrightarrow \quad \iota_{X^1_{\boldsymbol{h}} - X^2_{\boldsymbol{h}}}\boldsymbol{\omega} = 0 .$$

Hence $X^1_{\boldsymbol{h}} - X^2_{\boldsymbol{h}}$ takes values in $\ker \boldsymbol{\omega}$ yields that $X^1_{\boldsymbol{h}} = X^2_{\boldsymbol{h}}$. ☐

Proposition 1.4.13. *The space $\mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ relative to k-polysymplectic manifold $(P, \boldsymbol{\omega})$ becomes a Lie algebra when endowed with the natural operations*

$$\boldsymbol{h} + \boldsymbol{g} := (h^\alpha + g^\alpha) \otimes e_\alpha , \qquad \lambda \cdot \boldsymbol{h} := \lambda h^\alpha \otimes e_\alpha ,$$

where $\boldsymbol{h} = h^\alpha \otimes e_\alpha$, $\boldsymbol{g} = g^\alpha \otimes e_\alpha \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$, $\lambda \in \mathbb{R}$, and the Lie bracket $\{\cdot,\cdot\}_{\boldsymbol{\omega}} \colon \mathscr{C}^\infty_{\boldsymbol{\omega}}(P) \times \mathscr{C}^\infty_{\boldsymbol{\omega}}(P) \to \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ is of the form

$$\{\boldsymbol{h}, \boldsymbol{g}\}_{\boldsymbol{\omega}} = \{h^1, g^1\}_{\omega^1} \otimes e_1 + \cdots + \{h^k, g^k\}_{\omega^k} \otimes e_k ,$$

where $\{\cdot,\cdot\}_{\omega^\alpha}$ is the Poisson bracket naturally induced by the presymplectic form ω^α, with $\alpha = 1, \ldots, k$.

Proof. Let $X_{\boldsymbol{h}}$ and $X_{\boldsymbol{g}}$ be $\boldsymbol{\omega}$-Hamiltonian vector fields associated with $\boldsymbol{h}$ and $\boldsymbol{g}$, respectively. The linear combination $\lambda\boldsymbol{h} + \mu\boldsymbol{g}$, with $\lambda, \mu \in \mathbb{R}$, is also an $\boldsymbol{\omega}$-Hamiltonian function associated to the vector field $\lambda X_{\boldsymbol{h}} + \mu X_{\boldsymbol{g}}$ since

$$\iota_{\lambda X_{\boldsymbol{h}} + \mu X_{\boldsymbol{g}}} = \mathrm{d}(\lambda\boldsymbol{h} + \mu\boldsymbol{g}) .$$

Therefore, $\mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ becomes a vector space. Moreover,

$$\iota_{[X_{\boldsymbol{h}}, X_{\boldsymbol{g}}]}\boldsymbol{\omega} = \mathrm{d}\{\boldsymbol{g}, \boldsymbol{h}\}_{\boldsymbol{\omega}} .$$

Hence, $\{\boldsymbol{g}, \boldsymbol{h}\}_{\boldsymbol{\omega}}$ is an $\boldsymbol{\omega}$-Hamiltonian function with Hamiltonian vector field $[X_{\boldsymbol{h}}, X_{\boldsymbol{g}}]$. Thus, $\mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ is closed with respect to this bracket, which is trivially antisymmetric and satisfies the Jacobi identity, which turns $(\mathscr{C}^\infty_{\boldsymbol{\omega}}(P), \{\cdot,\cdot\}_{\boldsymbol{\omega}})$ into a Lie algebra. ☐

The product of $\boldsymbol{\omega}$-Hamiltonian functions, defined as

$$\boldsymbol{h} \star \boldsymbol{g} = (h^1 g^1) \otimes e_1 + \cdots + (h^k g^k) \otimes e_k ,$$

is not, in general, an $\boldsymbol{\omega}$-Hamiltonian function [53, p. 2239]. Therefore, $(\mathscr{C}^\infty_{\boldsymbol{\omega}}(P), \star, \{\cdot,\cdot\}_{\boldsymbol{\omega}})$ is not, in general, a Poisson algebra [53, p 2239]. Moreover, the map $\{\boldsymbol{h}, \cdot\}_{\boldsymbol{\omega}} \colon \boldsymbol{g} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P) \mapsto \{\boldsymbol{g}, \boldsymbol{h}\}_{\boldsymbol{\omega}} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$, with $\boldsymbol{h} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$, is not, in general, a derivation with respect to $\star$ either. Thus, k-polysymplectic geometry significantly differs from Poisson and presymplectic geometry. Nevertheless, $\{\boldsymbol{h}, \boldsymbol{g}\}_{\boldsymbol{\omega}}$ vanishes for every locally constant function $\boldsymbol{g} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ and any $\boldsymbol{h} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$. Additional properties of this Lie algebra are presented below.

Proposition 1.4.14. *Consider a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. Every $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ acts as a derivation on the Lie algebra $(\mathscr{C}^\infty_{\boldsymbol{\omega}}(P), \{\cdot,\cdot\}_{\boldsymbol{\omega}})$ in the form*

$$X_{\boldsymbol{h}}\boldsymbol{f} = \{\boldsymbol{f}, \boldsymbol{h}\}_{\boldsymbol{\omega}} , \qquad \forall \boldsymbol{f} \in \mathscr{C}^\infty_{\boldsymbol{\omega}}(P) ,$$

where $\boldsymbol{h}$ is an $\boldsymbol{\omega}$-Hamiltonian function related to $X_{\boldsymbol{h}}$.

Proof. Note that $\{\boldsymbol{f}, \boldsymbol{h}\}_{\boldsymbol{\omega}}$ does not depend on the chosen $\boldsymbol{\omega}$-Hamiltonian for X. Every two $\boldsymbol{\omega}$-Hamiltonian functions related to the same $\boldsymbol{\omega}$-Hamiltonian vector field differ by a constant (on each connected component of P). So, if $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ are $\boldsymbol{\omega}$-Hamiltonian functions for X. Then $\{\boldsymbol{f}, \boldsymbol{h}_1\}_{\boldsymbol{\omega}} = \{\boldsymbol{f}, \boldsymbol{h}_2\}_{\boldsymbol{\omega}}$ and $X_{\boldsymbol{f}}$ become well defined. Furthermore,

$$X_{\boldsymbol{h}}\{\boldsymbol{f}, \boldsymbol{g}\}_{\boldsymbol{\omega}} = \{\{\boldsymbol{f}, \boldsymbol{g}\}_{\boldsymbol{\omega}}, \boldsymbol{h}\}_{\boldsymbol{\omega}} = \{\{\boldsymbol{f}, \boldsymbol{h}\}_{\boldsymbol{\omega}}, \boldsymbol{g}\}_{\boldsymbol{\omega}} + \{\boldsymbol{f}, \{\boldsymbol{g}, \boldsymbol{h}\}_{\boldsymbol{\omega}}\}_{\boldsymbol{\omega}} = \{X_{\boldsymbol{h}}\boldsymbol{f}, \boldsymbol{g}\}_{\boldsymbol{\omega}} + \{\boldsymbol{f}, X_{\boldsymbol{h}}\boldsymbol{g}\}_{\boldsymbol{\omega}} .$$

Since $X_{\boldsymbol{h}}$ acts linearly on $\mathscr{C}^\infty_{\boldsymbol{\omega}}(P)$ the results follows. ☐

1.4.4 k-Polycosymplectic geometry

A natural generalisation of the k-polysymplectic structures is provided by *k-polycosymplectic manifolds*, which extends the cosymplectic framework for non-autonomous mechanical systems [2] to regular field theories whose Lagrangian and/or Hamiltonian functions, in the local description, depend on the space-time coordinates [38, 40]. Non-autonomous field theories can be effectively formulated within the framework of k-polycosymplectic [119] and k-cosymplectic [38] geometry. For further details on the k-symplectic and k-cosymplectic formalisms, see [43, 74, 125]. The relationships among k-symplectic, k-cosymplectic, and multisymplectic structures are systematically discussed in [129]. The basic definitions and properties associated with these geometric structures are introduced in this subsection.

Definition 1.4.15. A *k-polycosymplectic structure* on M is a pair $(\boldsymbol{\tau}, \boldsymbol{\omega})$, where $\boldsymbol{\tau} \in \Omega^1(M, \mathbb{R}^k)$ and $\boldsymbol{\omega} \in \Omega^2(M, \mathbb{R}^k)$ are closed differential one- and two-forms taking values in $\mathbb{R}^k$ such that

$$\operatorname{rk} \ker \boldsymbol{\omega} = \operatorname{rk}\left(\bigcap_{\alpha=1}^{k} \ker \omega^\alpha\right) = k\,, \qquad \ker \boldsymbol{\omega} \cap \ker \boldsymbol{\tau} = \bigcap_{\alpha=1}^{k} (\ker \tau^\alpha \cap \ker \omega^\alpha) = 0\,.$$

In this case, $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ is called a *k-polycosymplectic manifold*. If, in addition, $\dim M = k + n(k+1)$ for a certain $n \in \mathbb{N}$, it is said that $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ is a *polycosymplectic manifold* and $(\boldsymbol{\tau}, \boldsymbol{\omega})$ is a *polycosymplectic structure*.

Every k-polycosymplectic structure on a manifold M gives rise to two closed $\mathbb{R}^k$-valued differential forms $\boldsymbol{\omega} \in \Omega^2(M, \mathbb{R}^k)$ and $\boldsymbol{\tau} \in \Omega^1(M, \mathbb{R}^k)$ given by

$$\boldsymbol{\omega} = \sum_{\alpha=1}^{k} \omega^\alpha \otimes e_\alpha, \qquad \boldsymbol{\tau} = \sum_{\alpha=1}^{k} \tau^\alpha \otimes e_\alpha,$$

for a canonical basis $\{e_1, \ldots, e_k\}$ in $\mathbb{R}^k$ and some differential two- and one-forms on M given by ω^α and τ^α for $\alpha = 1, \ldots, k$, respectively.

Definition 1.4.16. A *k-cosymplectic structure* on M is a family $(\boldsymbol{\tau}, \boldsymbol{\omega}, V)$, where $(\boldsymbol{\tau}, \boldsymbol{\omega})$ is a polycosymplectic structure on M and $V \subset \mathrm{T}M$ is a distribution of rank nk on M such that

$$\boldsymbol{\tau}|_V = 0 \qquad \text{and} \qquad \boldsymbol{\omega}|_{V\times V} = 0\,.$$

Then, $(M, \boldsymbol{\tau}, \boldsymbol{\omega}, V)$ is *k-cosymplectic manifold*.

If $\boldsymbol{\omega}$ is exact, namely $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\theta}$ for some $\boldsymbol{\theta} \in \Omega^1(M, \mathbb{R}^k)$, the k-polycosymplectic (resp. polycosymplectic or k-cosymplectic) structure is said to be *exact*. Throughout this work, $M^{\boldsymbol{\omega}}_{\boldsymbol{\tau}}$ will be occasionally used to denote a k-polycosymplectic manifold $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$. This allows for shortening the notation.

Theorem 1.4.17 (Darboux theorem for k-cosymplectic manifolds)**.** *Let $(M^{\boldsymbol{\omega}}_{\boldsymbol{\tau}}, V)$ be a k-cosymplectic manifold. Then, on a neighbourhood U of any point $x \in M$, there exist local coordinates $\{t^\alpha, q^i, p^\alpha_i\}$, with $i = 1, \ldots, n$ and $\alpha = 1, \ldots, k$, such that*

$$\boldsymbol{\omega} = \sum_{\alpha=1}^{k}\sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p^\alpha_i \otimes e_\alpha\,, \quad \boldsymbol{\tau} = \sum_{\alpha=1}^{k} \mathrm{d}t^\alpha \otimes e_\alpha\,, \quad V = \left\langle \frac{\partial}{\partial p^\alpha_i} \right\rangle.$$

Such coordinates are called k-cosymplectic Darboux coordinates.

For the sake of clarity, it is convenient to establish the following result. Moreover, it will be necessary to relate the class of the considered k-polycosymplectic manifolds to a specific type of k-polysymplectic manifolds. It is a natural extension of Theorem 1.3.3.

Proposition 1.4.18. *Let $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ be a k-polycosymplectic manifold. There exists a unique family of vector fields $R_1, \ldots, R_k$ on M, called Reeb vector fields, such that*

$$\iota_{R_\alpha}\boldsymbol{\tau} = e_\alpha\,, \qquad \iota_{R_\alpha}\boldsymbol{\omega} = 0\,, \qquad \alpha = 1, \ldots, k\,. \tag{1.4.2}$$

Proof. By Definition 1.4.15, one has $\ker \boldsymbol{\tau} \cap \ker \boldsymbol{\omega} = 0$, which means that, if $\boldsymbol{\tau} = \sum_{\alpha=1}^{k} \tau^\alpha \otimes e_\alpha$, then $\tau^1 \wedge \cdots \wedge \tau^k$ does not vanish on $D = \ker \boldsymbol{\omega}$. The distribution D has rank k by the definition of a k-polycosymplectic manifold. Therefore, $\tau_x^1|_{D_x}, \ldots, \tau_x^k|_{D_x}$ are linearly independent at every $x \in M$ and the restrictions of $\tau^1, \ldots, \tau^k$ to D admit a unique dual basis $R_1, \ldots, R_k$ of vector fields on M taking values in D. Then, the vector fields $R_1, \ldots, R_k$ satisfy the conditions (1.4.2). □

1.4.5 k-Polycosymplectic Hamiltonian systems

This subsection presents the application of k-polycosymplectic geometry to the description of non-autonomous field theories.

Definition 1.4.19. Let $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ be a k-polycosymplectic manifold and let $h \in \mathscr{C}^\infty(M)$. Then, $(M, \boldsymbol{\tau}, \boldsymbol{\omega}, h)$ is said to be a *k-polycosymplectic Hamiltonian system*. A k-vector field $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{X}^k(M)$ on M is a *k-polycosymplectic Hamiltonian k-vector field* if it satisfies the system of equations

$$\begin{cases} \iota_{\mathbf{X}} \boldsymbol{\omega} = \mathrm{d}h - \displaystyle\sum_{\alpha=1}^{k} (R_\alpha h) \tau^\alpha \,, \\ \iota_{X_\beta} \boldsymbol{\tau} = e_\beta \,, \end{cases} \qquad \beta = 1, \ldots, k \,. \tag{1.4.3}$$

Then, the function h is a *Hamiltonian function* associated with $\mathbf{X}$. The space of all k-polycosymplectic Hamiltonian k-vector fields on M is denoted by $\mathfrak{X}^k_{\mathrm{Ham}}(M, \boldsymbol{\tau}, \boldsymbol{\omega})$.

It is worth noting that every function $f \in \mathscr{C}^\infty(M)$ corresponds to multiple distinct k-polycosymplectic Hamiltonian k-vector fields. Note that for $k = 1$, Definition 1.4.19 retrieves the evolution vector field in the cosymplectic setting from Definition 1.3.7.

Suppose that, in a neighbourhood of a point $x \in M$, there exist k-cosymplectic Darboux coordinates $\{t^\alpha, q^i, p_i^\alpha\}$. Consider a k-vector field $\mathbf{X} = (X_1, \ldots, X_k) \in \mathfrak{X}^k(M)$ which, in these k-cosymplectic Darboux coordinates, reads

$$X_\alpha = (X_\alpha)^\beta \frac{\partial}{\partial t^\beta} + (X_\alpha)^i \frac{\partial}{\partial q^i} + (X_\alpha)_i^\beta \frac{\partial}{\partial p_i^\beta} \,.$$

If $\mathbf{X}$ is a k-polycosymplectic Hamiltonian k-vector field, then conditions (1.4.3) imply the following

$$(X_\alpha)^\beta = \delta_\alpha^\beta \,, \qquad \frac{\partial h}{\partial p_i^\beta} = (X_\beta)^i \,, \qquad \frac{\partial h}{\partial q^i} = -\sum_{\alpha=1}^{k} (X_\alpha)_i^\alpha \,. \tag{1.4.4}$$

The Equations (1.4.4) imply that, for a given $h \in \mathscr{C}^\infty(M)$, there may exist multiple k-polycosymplectic Hamiltonian k-vector fields. Let $\psi \colon \mathbb{R}^k \to M$ be an integral section of a k-polycosymplectic Hamiltonian k-vector field $\mathbf{X}$, locally expressed as

$$\psi(s) = (t^\alpha(s), q^i(s), p_i^\alpha(s)) \,, \quad s \in \mathbb{R}^k \,.$$

Then, ψ satisfies the following system of partial differential equations

$$\frac{\partial t^\beta}{\partial s^\alpha} = \delta_\alpha^\beta \,, \qquad \frac{\partial q^i}{\partial s^\alpha} = \frac{\partial h}{\partial p_i^\alpha} \,, \qquad \sum_{\alpha=1}^{k} \frac{\partial p_i^\alpha}{\partial s^\alpha} = -\frac{\partial h}{\partial q^i} \,. \tag{1.4.5}$$

These equations are called *k-polycosymplectic Hamilton-De Donder-Weyl equations for a k-vector field* $\mathbf{X}$.

Example 1.4.20 (The vibrating membrane with external force)**.** Consider a horizontal vibrating membrane with coordinates $\{x, y\}$ subjected to a time-dependent external force given by a function $f(t, x, y)$. The phase space of this system is $M = \mathbb{R}^3 \times \bigoplus^3 \mathrm{T}^*\mathbb{R}$ and it admits global coordinates $\{t, x, y, \zeta, p^t, p^x, p^y\}$, where ζ stands for the distance of every point in the membrane with respect to its equilibrium position, and p^t, p^x, p^y are the corresponding momenta. This system is described by the Hamiltonian function $h \in \mathscr{C}^\infty(M)$ given by

$$h(t,x,y,\zeta,p^t,p^x,p^y) = \frac{1}{2}(p^t)^2 - \frac{1}{2c^2}(p^x)^2 - \frac{1}{2c^2}(p^y)^2 - \zeta f(t,x,y)\,,$$

where $c \in \mathbb{R}$ is a constant depending on the physical properties of the membrane, such as its tension. For a section

$$\psi\colon (t,x,y) \in \mathbb{R}^3 \mapsto (t,x,y,\zeta(t,x,y),p^t(t,x,y),p^x(t,x,y),p^y(t,x,y)) \in \mathbb{R}^3 \times \bigoplus_{\alpha=1}^3 \mathrm{T}^*\mathbb{R}\,,$$

Equations (1.4.5) yield

$$\begin{gathered}\frac{\partial p^t}{\partial t} + \frac{\partial p^x}{\partial x} + \frac{\partial p^y}{\partial y} = f(t,x,y)\,,\\ \frac{\partial \zeta}{\partial t} = p^t\,, \quad \frac{\partial \zeta}{\partial x} = -\frac{1}{c^2}p^x\,, \quad \frac{\partial \zeta}{\partial y} = -\frac{1}{c^2}p^y\,.\end{gathered}$$

The choice of $\{t,x,y\}$ as the coordinates of the domain of a section ψ is a slight abuse of notation that, however, is standard in the literature [130] and is adopted throughout this work.

Combining the above equations, one obtains the equation of a forced vibrating membrane, namely

$$\frac{\partial^2\zeta}{\partial t^2} - \frac{1}{c^2}\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right) = f(t,x,y)\,.$$

To rewrite this system in polar coordinates $\{r,\theta\}$, consider the Hamiltonian function

$$\widetilde{h}(t,r,\theta,\zeta,p^t,p^r,p^\theta) = \frac{1}{2r}\left((p^t)^2 - \frac{1}{c^2}(p^r)^2 - \frac{r^2}{c^2}(p^\theta)^2\right) - r\zeta f(t,r,\theta)\,.$$

Then, Equations (1.4.5) for a section

$$\psi\colon (t,r,\theta) \in \mathbb{R}^3 \mapsto (t,r,\theta,\zeta(t,r,\theta),p^t(t,r,\theta),p^x(t,r,\theta),p^y(t,r,\theta)) \in \mathbb{R}^3 \times \bigoplus_{\alpha=1}^3 \mathrm{T}^*\mathbb{R}$$

become

$$\begin{gathered}\frac{\partial p^t}{\partial t} + \frac{\partial p^r}{\partial r} + \frac{\partial p^\theta}{\partial \theta} = rf(t,r,\theta)\,,\\ \frac{\partial \zeta}{\partial t} = \frac{1}{r}p^t\,, \quad \frac{\partial \zeta}{\partial r} = -\frac{1}{rc^2}p^r\,, \quad \frac{\partial \zeta}{\partial \theta} = -\frac{r}{c^2}p^\theta\,.\end{gathered}$$

Combining the above equations yields the equation of a forced vibrating membrane in polar coordinates

$$\frac{\partial^2\zeta}{\partial t^2} - c^2\left(\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2}\right) = f(t,r,\theta)\,.$$

1.4.6 k-Contact geometry

k-Contact geometry arises as a natural generalisation of contact geometry, designed to describe non-autonomous Hamiltonian field theories [130, 131, 132]. Contact geometry dates back to 1872, when Sophus Lie introduced contact transformations to study differential equations [102]. Since then, it has evolved into rich geometric concept with numerous applications, including Gibbs' thermodynamics [21, 139], Huygens' geometric optics, non-autonomous Hamiltonian dynamics [45], control theory [144], Lie systems [47], and many others [22, 36, 59, 64]. The main concept of classical contact geometry is the contact distribution, defined as a maximally non-integrable distribution $\mathcal{C}$ on a manifold M.

The recent decades of growing mathematical and physical interest in contact geometry motivated its extension to field-theoretic settings [48, 49, 76, 85, 87, 131, 132, 142, 148]. In k-contact geometry, a manifold M is equipped with an $\mathbb{R}^k$-valued differential one-form $\boldsymbol{\eta}$ whose kernel is a non-zero regular distribution of corank k, satisfying $\ker \boldsymbol{\eta} \oplus \ker \mathrm{d}\boldsymbol{\eta} = \mathrm{T}M$ [7, 65, 97, 130, 131]. This definition extends the classical properties of contact forms. From a physical perspective, it provides a framework for studying Hamilton-De Donder-Weyl equations with dissipation through the use of k-vector fields and k-contact forms [78, 133]. It should be noted that most works on k-contact geometry focus on the co-orientable case [130]. More recently, a general formulation was introduced in [48], in which the main object of investigation is a distribution of corank k that is maximally non-integrable, and locally admits k commuting Lie symmetries - the k-contact distribution.

This section provides an overview of the theory of k-contact manifolds [63], as well as generalised subbundles and other related notions crucial for developing Marsden–Meyer–Weinstein reduction for k-contact manifolds [99].

Definition 1.4.21. Let $E \to M$ be a vector bundle over M. Then,

- A *generalised subbundle* on M is a subset $D \subset E$ such that $D_x = D \cap E_x$ is a vector subspace of the fibre E_x of the bundle E of every $x \in M$. The *rank* of D at $x \in M$ is the dimension of the subspace $D_x \subset E_x$.
- A generalised subbundle $D \subset E$ is *smooth* if it is locally spanned by a family of smooth sections of $E \to M$ taking values in D, namely, if for every $x \in M$, there exists a family of sections $e_1, \ldots, e_r \colon U \subset M \to E|_U$ defined in a neighbourhood U of x such that $D_{x'} = \langle e_1(x'), \ldots, e_r(x') \rangle$ for every $x' \in U$.
- A generalised subbundle D is *regular* if it is smooth and has constant rank.

A generalised subbundle in TM is called a generalised distribution. A generalised subbundle in T^*M is called a codistribution. For simplicity, the word generalised will be skipped.

Consider a differential one-form $\eta \in \Omega^1(M)$. Then, η spans a smooth co-distribution $\mathcal{C} = \langle \eta \rangle = \{\langle \eta_x \rangle \mid x \in M\} \subset \mathrm{T}^*M$. Then, $\mathcal{C}$ has rank one at every point where η does not vanish. The annihilator of $\mathcal{C}$ is the distribution $\ker \eta \subset \mathrm{T}M$. The distribution $\mathcal{C}^\circ$ has corank one at every point where η does not vanish, and zero otherwise.

Definition 1.4.22. A *k-contact form* on an open $U \subset M$ is a differential $\mathbb{R}^k$-valued one-form $\boldsymbol{\eta} = \sum_\alpha \eta^\alpha \otimes e_\alpha \in \Omega^1(U, \mathbb{R}^k)$ such that

(1) $\ker \boldsymbol{\eta} \subset \mathrm{T}U$ is a regular non-zero distribution of corank k,

(2) $\ker \mathrm{d}\boldsymbol{\eta} \subset \mathrm{T}U$ is a regular distribution of rank k,

(3) $\ker \boldsymbol{\eta} \cap \ker \mathrm{d}\boldsymbol{\eta} = 0$.

If $\boldsymbol{\eta} \in \Omega^1(M, \mathbb{R}^k)$ exists globally, the pair $(M, \boldsymbol{\eta})$ is a *co-oriented k-contact manifold.* Moreover, if $\dim M = n + nk + k$ for some $n, k \in \mathbb{N}$ and M is endowed with an integrable distribution $\mathcal{V} \subset \ker \boldsymbol{\eta}$ with $\operatorname{rk} \mathcal{V} = nk$, then $(M, \boldsymbol{\eta}, \mathcal{V})$ is a *polarised co-oriented k-contact manifold* and $\mathcal{V}$ is a *polarisation* of $(M, \boldsymbol{\eta})$.

It is worth noting that the case $k = 1$ recovers the classical notion of a co-oriented contact manifold [48]. Every co-orientable k-contact manifold admits a set of Reeb vector fields, which play a fundamental role in the k-contact geometry [43]. Note that Definition 1.4.22 does not cover one-dimensional contact manifolds to avoid considering $\ker \eta$ to be integrable.

From now on, $\psi^\alpha \in \Omega^s(M)$ denote the differential components of $\boldsymbol{\psi} = \sum_{\alpha=1}^k \psi^\alpha \otimes e_\alpha \in \Omega^s(M, \mathbb{R}^k)$.

Theorem 1.4.23. *Let $(M, \boldsymbol{\eta})$ be a k-contact manifold. Then, there exists a unique family of vector fields $R_1, \ldots, R_k \in \mathfrak{X}(M)$, called the Reeb vector fields of $(M, \boldsymbol{\eta})$, such that*

$$\iota_{R_\alpha} \eta^\beta = \delta^\beta_\alpha \,, \qquad \iota_{R_\alpha} \mathrm{d}\boldsymbol{\eta} = 0 \,,$$

for $\alpha, \beta = 1, \ldots, k$. Moreover, $[R_\alpha, R_\beta] = 0$ with $\alpha, \beta = 1, \ldots, k$, while $\ker \mathrm{d}\boldsymbol{\eta} = \langle R_1, \ldots, R_k \rangle$, and $R_1 \wedge \cdots \wedge R_k$ is non-vanishing.

Definition 1.4.24. Every $(M,\boldsymbol{\eta})$ defines a vector bundle morphism over M

$$\begin{array}{rccl} \flat_{\boldsymbol{\eta}}\colon & \bigoplus^k \mathrm{T}M & \longrightarrow & \mathrm{T}^*M\times\mathbb{R} \\ & \boldsymbol{v}=(v_1,\dots,v_k) & \longmapsto & \flat_{\boldsymbol{\eta}}(\boldsymbol{v})=\left(\sum_{\alpha=1}^k \iota_{v_\alpha}\mathrm{d}\eta^\alpha, \sum_{\alpha=1}^k \iota_{v_\alpha}\eta^\alpha\right). \end{array}$$

A *$\boldsymbol{\eta}$-gauge k-vector field* of $(M,\boldsymbol{\eta})$ is a k-vector field on M taking values in $\ker\flat_{\boldsymbol{\eta}}$.

The following example presents the canonical construction of a co-oriented polarised k-contact manifold.

Example 1.4.25. The manifold $M=(\bigoplus^k\mathrm{T}^*Q)\times\mathbb{R}^k$ carries a natural k-contact form given by

$$\boldsymbol{\eta}_Q=\sum_{\alpha=1}^k(\mathrm{d}z^\alpha-\theta^\alpha)\otimes e_\alpha\,,$$

where $\{z^1,\dots,z^k\}$ are the pull-back to M of standard linear coordinates in $\mathbb{R}^k$ and each θ^α is the pull-back of the Liouville one-form θ on T^*Q via the projection $\mathrm{pr}^\alpha\colon M\to\mathrm{T}^*Q$ onto the α-th component of $\bigoplus^k\mathrm{T}^*Q$. Furthermore, M admits a natural projection onto $Q\times\mathbb{R}^k$ and a related vertical distribution $\mathcal{V}$ of rank $k\cdot\dim Q$ contained in $\ker\boldsymbol{\eta}_Q$. Thus, $\left((\bigoplus^k\mathrm{T}^*Q)\times\mathbb{R}^k,\boldsymbol{\eta}_Q,\mathcal{V}\right)$ is a polarised co-oriented k-contact manifold.

Local coordinates $\{q^1,\dots,q^n\}$ on Q induce natural coordinates $\{q^i,p_i^\alpha\}$, for a fixed value of α, on the α-th component of $\bigoplus^k\mathrm{T}^*Q$ and $\{q^i,p_i^\alpha,z^\alpha\}$, with $\alpha=1,\dots,k$, on M. In these coordinates, one has

$$\boldsymbol{\eta}_Q=\sum_{\alpha=1}^k\left(\mathrm{d}z^\alpha-\sum_{i=1}^n p_i^\alpha\mathrm{d}q^i\right)\otimes e_\alpha\,,\qquad \ker\boldsymbol{\eta}_Q=\left\langle\frac{\partial}{\partial p_i^\alpha}\,,\ \frac{\partial}{\partial q^i}+\sum_{\alpha=1}^k p_i^\alpha\frac{\partial}{\partial z^\alpha}\right\rangle,$$

and $\mathrm{d}\boldsymbol{\eta}_Q=\sum_{i=1}^n\sum_{\alpha=1}^k(\mathrm{d}q^i\wedge\mathrm{d}p_i^\alpha)\otimes e_\alpha$. The associated Reeb vector fields are $R_\alpha=\partial/\partial z^\alpha$ for $\alpha=1,\dots,k$, and

$$\ker\mathrm{d}\boldsymbol{\eta}_Q=\left\langle\frac{\partial}{\partial z^1},\dots,\frac{\partial}{\partial z^k}\right\rangle.$$

△

Example 1.4.26 (Contactification of an exact k-symplectic manifold)**.** Let $(P,\boldsymbol{\omega}=\mathrm{d}\boldsymbol{\theta})$ be an exact k-symplectic manifold and consider the product manifold $M=P\times\mathbb{R}^k$. Let $\{z^1,\dots,z^k\}$ be the pull-back to M of some Cartesian coordinates in $\mathbb{R}^k$ and denote by θ_M^α the pull-back of θ^α to the product manifold M. Consider the $\mathbb{R}^k$-valued one-form $\boldsymbol{\eta}=\sum_{\alpha=1}^k(\mathrm{d}z^\alpha+\theta_M^\alpha)\otimes e_\alpha\in\Omega^1(M,\mathbb{R}^k)$. Then, $(M,\boldsymbol{\eta})$ is a co-oriented k-contact manifold, because $\ker\boldsymbol{\eta}\neq0$ has corank k, while $\mathrm{d}\boldsymbol{\eta}=\mathrm{d}\boldsymbol{\theta}_M$ and $\ker\mathrm{d}\boldsymbol{\eta}=\langle\partial/\partial z^1,\dots,\partial/\partial z^k\rangle$ has rank k since $\boldsymbol{\omega}$ is non-degenerate. It follows that $\boldsymbol{\eta}$ is a globally defined k-contact form.

Note that the so-called canonical k-contact form $\boldsymbol{\eta}_Q$ described in Example 1.4.25 is essentially a contactification of the k-symplectic manifold $(P=\bigoplus^k\mathrm{T}^*Q,\boldsymbol{\omega}_Q)$ described in Example 1.4.8. The only significant difference is that θ_M^α is minus the pull-back to M of the Liouville form to the α-copy of T^*Q in M. △

Theorem 1.4.27 (k-contact Darboux Theorem [63])**.** *Consider a polarised k-contact manifold $(M,\boldsymbol{\eta},\mathcal{V})$. Then, around every point of M, there exist local coordinates $\{q^i,p_i^\alpha,z^\alpha\}$, with $1\le\alpha\le k$ and $1\le i\le n=\dim M$, such that*

$$\boldsymbol{\eta}=\sum_{\alpha=1}^k\left(\mathrm{d}z^\alpha-\sum_{i=1}^n p_i^\alpha\mathrm{d}q^i\right)\otimes e_\alpha\,,\quad \ker\mathrm{d}\boldsymbol{\eta}=\left\langle\frac{\partial}{\partial z^\alpha}\right\rangle,\quad \mathcal{V}=\left\langle\frac{\partial}{\partial p_i^\alpha}\right\rangle.$$

These coordinates are called k-contact Darboux coordinates of the polarised k-contact manifold $(M,\boldsymbol{\eta},\mathcal{V})$.

Theorem 1.4.27 justifies treating the manifold introduced in Example 1.4.25 as the canonical model of polarised co-oriented k-contact manifolds. Furthermore, any polarised k-contact manifold arising as the contactification of a polarised k-symplectic manifold admits Darboux coordinates.

1.4.7 k-Contact Hamiltonian systems

This section presents the basics of the Hamiltonian formulation of classical field theories with dissipation in a co-orientable k-contact setting; for more details, see [49, 63, 130, 131].

Definition 1.4.28. A *k-contact Hamiltonian system* is a triple $(M, \boldsymbol{\eta}, h)$, where $(M, \boldsymbol{\eta})$ is a co-oriented k-contact manifold and $h \in \mathscr{C}^\infty(M)$ is called a *Hamiltonian function.* Let $\psi\colon U \subset \mathbb{R}^k \to M$, where U is an open subset of $\mathbb{R}^k$. The *k-contact Hamilton–De Donder–Weyl equations* associated with $(M, \boldsymbol{\eta}, h)$ are of the form

$$\sum_{\alpha=1}^{k} \iota_{\psi'_\alpha} \mathrm{d}\eta^\alpha = \left(\mathrm{d}h - \sum_{\alpha=1}^{k} (R_\alpha h)\eta^\alpha\right) \circ \psi\,, \qquad \sum_{\alpha=1}^{k} \iota_{\psi'_\alpha} \eta^\alpha = -h \circ \psi\,, \tag{1.4.6}$$

where $\psi'(t) = (\psi'_1(t), \dots, \psi'_k(t)) \in \bigoplus^k \mathrm{T}M$ denotes the first prolongation (see Definition 1.4.3).

In k-contact Darboux coordinates Equations (1.4.6) take the form

$$\frac{\partial q^i}{\partial t^\alpha} = \frac{\partial h}{\partial p_i^\alpha} \circ \psi\,, \qquad \sum_{\alpha=1}^{k} \frac{\partial p_i^\alpha}{\partial t^\alpha} = -\left(\frac{\partial h}{\partial q^i} + \sum_{\alpha=1}^{k} p_i^\alpha \frac{\partial h}{\partial z^\alpha}\right) \circ \psi\,, \qquad \sum_{\alpha=1}^{k} \frac{\partial z^\alpha}{\partial t^\alpha} = \left(\sum_{\alpha=1}^{k} \sum_{i=1}^{\dim M} p_i^\alpha \frac{\partial h}{\partial p_i^\alpha} - h\right) \circ \psi\,.$$

Definition 1.4.29. The *k-contact Hamilton–De Donder–Weyl equations*, associated with $(M, \boldsymbol{\eta}, h)$, for a k-vector field $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(M)$ are

$$\sum_{\alpha=1}^{k} \iota_{X_\alpha} \mathrm{d}\eta^\alpha = \mathrm{d}h - \sum_{\alpha=1}^{k} (R_\alpha h)\eta^\alpha\,, \qquad \sum_{\alpha=1}^{k} \iota_{X_\alpha} \eta^\alpha = -h\,. \tag{1.4.7}$$

Then, a k-vector field $\mathbf{X}$ that satisfies equations (1.4.7) is a *k-contact Hamiltonian k-vector field.*

Note that a k-contact Hamiltonian system admits a family of k-contact Hamiltonian vector fields. Indeed, if $\mathbf{X}$ is a k-contact hamiltonian k-vector field for $(M, \boldsymbol{\eta}, h)$, then so $\mathbf{X} + \mathbf{Y}$ for any $\boldsymbol{\eta}$-gauge k-vector field $\mathbf{Y}$ of $(M, \boldsymbol{\eta}, h)$.

Let $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(M)$ be a k-vector field expressed in k-contact Darboux coordinates as

$$X_\alpha = \sum_{i=1}^{\dim M} (X_\alpha)^i \frac{\partial}{\partial q^i} + \sum_{\beta=1}^{k} \sum_{i=1}^{\dim M} (X_\alpha)_i^\beta \frac{\partial}{\partial p_i^\beta} + \sum_{\beta=1}^{k} (X_\alpha)^\beta \frac{\partial}{\partial z^\beta}\,, \qquad \alpha = 1, \dots, k.$$

Then, equations (1.4.7) are equivalent to

$$(X_\alpha)^i = \frac{\partial h}{\partial p_i^\alpha}\,, \qquad \sum_{\alpha=1}^{k} (X_\alpha)_i^\alpha = -\left(\frac{\partial h}{\partial q^i} + \sum_{\alpha=1}^{k} p_i^\alpha \frac{\partial h}{\partial z^\alpha}\right)\,, \qquad \sum_{\alpha=1}^{k} (X_\alpha)^\alpha = \sum_{\alpha=1}^{k} \sum_{i=1}^{\dim M} p_i^\alpha \frac{\partial h}{\partial p_i^\alpha} - h\,.$$

Then, one immediately obtains the following propositions.

Proposition 1.4.30. *Let $\mathbf{X} \in \mathfrak{X}^k(M)$ be an integrable k-vector field. Then, every integral section $\psi\colon L \subset \mathbb{R}^k \to M$ of $\mathbf{X}$ satisfies the k-contact Hamilton–De Donder–Weyl equations* (1.4.6) *if, and only if, $\mathbf{X}$ is a solution to* (1.4.7)*.*

It is important to emphasise that the existence of a k-vector field satisfying equations (1.4.7) does not, in general, guarantee the existence of integral sections.

Corollary 1.4.31. *The k-contact Hamilton–De Donder–Weyl equations* (1.4.7) *are equivalent to the following conditions*

$$\begin{cases} \mathscr{L}_{\mathbf{X}} \boldsymbol{\eta} = \displaystyle\sum_{\alpha=1}^{k} \mathscr{L}_{X_\alpha} \eta^\alpha = \iota_{\mathbf{X}} \mathrm{d}\boldsymbol{\eta} + \mathrm{d}\iota_{\mathbf{X}} \boldsymbol{\eta} = -\sum_{\alpha=1}^{k} (R_\alpha h)\eta^\alpha\,, \\ \iota_{\mathbf{X}} \boldsymbol{\eta} = \displaystyle\sum_{\alpha=1}^{k} \iota_{X_\alpha} \eta^\alpha = -h\,, \end{cases}$$

where $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k(M)$.

Example 1.4.32 (The damped wave equation)**.** The dynamics of a vibrating string can be formulated within the k-contact Hamiltonian setting [63].

Let $\{t, x\}$ be coordinates on $\mathbb{R}^2$ and consider the configuration space $Q = \mathbb{R}$. The phase space becomes $\bigoplus^2 \mathrm{T}^*\mathbb{R} \times \mathbb{R}^2$, with coordinates (u, p^t, p^x, s^t, s^x), where u denotes the displacement of a point from its equilibrium point, while p^t and p^x are the momenta associated with u with respect to the independent variables t and x, respectively. The canonical two-contact on $\bigoplus^2 \mathrm{T}^*\mathbb{R} \times \mathbb{R}^2$ is given as in Example 1.4.25, namely

$$\eta^t = \mathrm{d}t - p^t \mathrm{d}u\,, \qquad \eta^x = \mathrm{d}x - p^x \mathrm{d}u\,.$$

A Hamiltonian function $h \in \mathscr{C}^\infty(\bigoplus^2 \mathrm{T}^*\mathbb{R} \times \mathbb{R}^2)$ describing the damped vibrating string reads

$$h(u, p^t, p^x, s^t, s^x) = \frac{1}{2\rho}(p^t)^2 - \frac{1}{2\tau}(p^x)^2 + ks^t\,,$$

where ρ is the linear mass density, τ is the tension, and $k > 0$ is the damping coefficient of the string, all of which are assumed to be constant.

A corresponding two-contact Hamiltonian two-vector field $\mathbf{X} = (X^1, X^2)$ has the form

$$\begin{aligned} X^1 &= \frac{p^t}{\rho}\frac{\partial}{\partial u} + A^1_t \frac{\partial}{\partial p^t} + A^1_x \frac{\partial}{\partial p^x} + B^1_t \frac{\partial}{\partial s^t} + B^1_x \frac{\partial}{\partial s^x}\,, \\ X^2 &= -\frac{p^x}{\tau}\frac{\partial}{\partial u} + A^2_t \frac{\partial}{\partial p^t} - (kp^t + A^1_t)\frac{\partial}{\partial p^x} + B^2_t \frac{\partial}{\partial s^t} + \left(\frac{(p^t)^2}{2\rho} - \frac{(p^x)^2}{2\tau} - ks^t - B^1_t\right)\frac{\partial}{\partial s^x}\,, \end{aligned}$$

where $A^1_t, A^1_x, B^1_t, B^1_x, A^2_t, B^2_t$ are arbitrary functions on M. For instance, by choosing $B^1_x = \tau k (p^t)^2 u/(\rho p^x)$ and $B^2_t = B^2_t(s^t, p^x)$, and setting $A^2_t, B^1_t, A^1_x = 0$ and $A^1_t = -kp^t$, it follows that $\mathbf{X}$ becomes an integrable k-vector field. Thus, the Hamilton–De Donder–Weyl equations are

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{\rho}p^t\,, \\ \dfrac{\partial u}{\partial x} = -\dfrac{1}{\tau}p^x\,, \\ \dfrac{\partial p^t}{\partial t} + \dfrac{\partial p^x}{\partial x} = -kp^t\,, \\ \dfrac{\partial s^t}{\partial t} + \dfrac{\partial s^x}{\partial x} = \dfrac{1}{2\rho}(p^t)^2 - \dfrac{1}{2\tau}(p^x)^2 - ks^t\,. \end{cases}$$

Substituting the first two equations into the third yields the damped wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + k\frac{\partial u}{\partial t} = 0\,,$$

where $c^2 = \dfrac{\tau}{\rho}$. △

Chapter 2

Marsden–Meyer–Weinstein reduction theorems

The reduction problem for systems with symmetry has, for decades, drawn significant attention from both mathematicians and theoretical physicists, motivated by the objective of reducing the number of equations describing the behaviour of the dynamics of such systems through the first integrals or conservation laws [115, 128]. The general procedure of the symplectic reduction can be traced back to E. Cartan, and it goes as follows (see [2, p 298] or [115, 128] and references therein):

> "Suppose that P is a manifold and ω is a closed two-form on P; let $\ker\omega = \{v \in \mathrm{T}P \mid \iota_v\omega = 0\}$ be the *characteristic distribution* of ω and call ω *regular* if $\ker\omega$ is a subbundle of $\mathrm{T}P$. In the regular case, $\ker\omega$ is an involutive distribution. By the Fröbenius theorem, $\ker\omega$ is integrable and it defines a foliation $\mathcal{F}$ on P. This gives rise to a quotient space $P/\mathcal{F}$ by identification of all points on the same leaf of $\mathcal{F}$. Assume now that $P/\mathcal{F}$ is a manifold, the canonical projection $x \in P \mapsto [x] \in P/\mathcal{F}$ being a submersion. Then, the tangent space at a point $[x]$ is isomorphic to $\mathrm{T}_xP/\ker\omega_x$ and ω projects onto a well-defined closed, nondegenerate two-form on $P/\mathcal{F}$; that is, $P/\mathcal{F}$ is a symplectic manifold: a so-called *reduced space.*"

The application of geometric methods has proven to be a particularly powerful tool in the analysis of this problem. A breakthrough was achieved by Marsden and Weinstein in their work on the reduction of autonomous Hamiltonian systems on symplectic manifolds admitting the action of a Lie group of symmetries, under the assumption that the momentum map takes regular values [109]. One year before, Meyer had obtained related results [120], although not as detailed and comprehensive as those later presented by Marsden and Weinstein [115]. More generally, the results in [120] and [109] were indeed the culmination of many other previous achievements by Smale, Sternberg, Kostant, Robbin, and many others, who had provided partial but significant approaches to the reduction procedure (see [115] for a more detailed history review of the reduction procedure).

In the famous work [109], Marsden and Weinstein applied a very powerful version of the previous reduction scheme to submanifolds defined by the level sets of an Ad^*-equivariant symplectic momentum map $\mathbf{J}^\Phi\colon P \to \mathfrak{g}^*$ associated with a certain Lie group action on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ and a Hamiltonian Lie group action Φ on a symplectic manifold P leaving invariant a Hamiltonian function on P. The resulting reduced space carries a natural symplectic structure and inherits the Hamiltonian dynamics induced by the original system. Nowadays, this procedure is very well known as the *Marsden–Meyer–Weinstein reduction.*

After Marsden and Weinstein's foundational work, the Marsden–Meyer–Weinstein reduction technique was subsequently extended and applied to a wide range of different settings. For instance, the reduction of Hamiltonian systems with singular values of the momentum map was studied in several articles, including [141] for the autonomous case, where the resulting reduced spaces are stratified manifolds endowed with symplectic structures [128]. Furthermore, the orbifolds naturally arise in Marsden–Meyer–Weinstein reductions, leading to separate research topics with both physical and mathematical applications [68, 81, 97]. The reduction of time-dependent regular Hamiltonian systems with regular values is developed in the framework of cosymplectic geometry in [4, 44], where the corresponding reduced phase spaces inherit cosymplectic structure. Autonomous systems arising from certain classes of singular Lagrangians were analysed in [26], where conditions ensuring that the reduced phase space carries an almost-tangent structure were established.

Moreover, numerous generalisations of the Marsden–Meyer–Weinstein reduction have been proposed in order to cope with different geometric structures. In particular, Marsden and Ratiu extended the theory to Poisson manifolds in [111], the case of locally conformally symplectic manifolds was developed in [80], and the reduction of Dirac structures was devised and further analysed in several papers [23, 24, 34]. Finally, the Marsden–Meyer–Weinstein reduction of Jacobi manifolds was studied in [86].

Almost fifty years after the foundational work [109], the development of the Marsden–Meyer–Weinstein reduction for various geometric settings and reduction schemes remains an active and evolving research field. The theory admits numerous modifications and generalisations, for instance, to the singular cases [44, 88], and has found a wide range of applications, as illustrated by the growing list of literature using these techniques [10, 12, 19, 68, 81, 97, 135, 136]. It is worth noting that the multisymplectic analogue of the Marsden–Meyer–Weinstein reduction has persisted as an open problem for decades now [58], and even partial advances toward this achievement attract attention [13, 14, 35, 121].

One such generalisation is the Marsden–Meyer–Weinstein reduction theorem for k-polysymplectic manifolds. The first attempt to develop a k-polysymplectic reduction was due to Günther [78]. Unfortunately, his approach contained fundamental flaws, arising from an improper analysis of the double orthogonal relative to a k-polysymplectic form. More precisely, [78, Lemma 7.5 and Theorem 7.7] are the main source of mistakes in Günther's work, while [107, Section 2.2] provides an interesting counterexample explicitly demonstrating Günther's error. Another similarly flawed attempt to develop a k-polysymplectic reduction was accomplished in [123]. These problems were subsequently corrected in [107], where sufficient conditions to accomplish a k-polysymplectic reduction were formulated. Nevertheless, [107, Lemma 3.4] implicitly suggests that the assumption of the k-polysymplectic momentum map being a submersion is justified by Sard's Theorem. While this assumption indeed works very well in the classical symplectic Marsden–Meyer–Weinstein reduction theory and Sard's Theorem can be used to justify it [17], the authors of [50] demonstrate that this condition is very restrictive in the k-polysymplectic geometry realm and clarifies why Sard's Theorem cannot be used in this context. Explicit examples illustrating that it is convenient to assume that the momentum maps in k-polysymplectic geometry are not submersions are provided. Consequently, it is appropriate to adopt the formalism introduced in [50], where k-polysymplectic momentum maps admit only weak regular values. This approach offers a practical generalisation of the k-polysymplectic Marsden–Meyer–Weinstein reduction and completes the analysis initiated in [12, 62, 78, 107].

Necessary and sufficient conditions for the k-polysymplectic Marsden–Meyer–Weinstein reduction were formulated implicitly in [107, p 12] and subsequently presented in detail in [12]. Unfortunately, one of the Blacker's main results, namely [12, Theorem 3.22], contains a minor but potentially misleading typo in the statement of the conditions (as observed in [62]). Moreover, the proof of that theorem admits other minor technical issues concerning the existence of certain submanifold structures. These issues are clarified and rigorously analysed in [33, 50]. It is also worth noting that Blacker analyses the presence of orbifolds in k-polysymplectic Marsden–Meyer–Weinstein reductions corresponding to regular values of k-polysymplectic momentum maps related to pathological Lie group actions.

The requirement for k-polysymplectic momentum maps to be Ad^{*k}-equivariant in the k-polysymplectic Marsden–Meyer–Weinstein reductions was removed in [50] by extending the classical theory of affine Lie group actions on symplectic manifolds [128] to the k-polysymplectic setting. Then, García-Toraño and Mestdag [62] re-examined the sufficient conditions for the k-polysymplectic Marsden–Meyer–Weinstein reduction established in [107], claiming that the single condition, namely [107, Theorem 3.17, condition (3.6)], suffices to guarantee the existence of a k-polysymplectic Marsden–Meyer–Weinstein reduction. However, the proof of the main result in [62], used to justify the previous claim, contains a fundamental mistake. Indeed, [62, Lemma 3.1] is shown to be false by presenting an explicit counterexample, and the general independence of the conditions in [107, Theorem 3.17] is established in [33]. Furthermore, additional properties concerning these sufficient conditions are clarified.

Another important generalisation of the Marsden–Meyer–Weinstein reduction is its extension to the setting of k-polycosymplectic manifolds [10, 43]. The search for a k-polycosymplectic reduction can be traced back to [15], where a particular case was analysed. Next, certain ideas regarding the possible scheme of a k-polycosymplectic reduction were outlined in [108], although no proofs were provided. Some of these ideas led to [106], where no k-polycosymplectic reduction was studied, but the k-polysymplectic reduction was developed instead. Nowadays, more than a decade after [108], the Marsden–Meyer–Weinstein k-polycosymplectic reduction has been drawing some attention until the present day and was finally proven in [50].

$$\begin{tikzcd}
(\mathbb{R}^k \times M, \widetilde{\boldsymbol{\omega}} = \mathrm{pr}_M^* \boldsymbol{\omega} + \mathrm{d}\boldsymbol{u} \mathbin{\bar\wedge} \mathrm{pr}_M^* \boldsymbol{\tau}) \arrow[r, "\mathrm{pr}_M"] & (M, \boldsymbol{\tau}, \boldsymbol{\omega}) \\
(\widetilde{\mathbf{J}}^{\Phi-1}(\boldsymbol{\mu}) = \mathbb{R}^k \times \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}), \widetilde{\jmath}_{\boldsymbol{\mu}}^* \widetilde{\boldsymbol{\omega}}) \arrow[u, "\widetilde{\jmath}_{\boldsymbol{\mu}}"] \arrow[d, "\widetilde{\pi}_{\boldsymbol{\mu}} = \mathrm{Id}_{\mathbb{R}^k} \otimes \pi_{\boldsymbol{\mu}}"'] & (\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}), \jmath_{\boldsymbol{\mu}}^* \boldsymbol{\tau}, \jmath_{\boldsymbol{\mu}}^* \boldsymbol{\omega}) \arrow[l, "\iota_{\boldsymbol{u}}^{\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})}"] \arrow[u, "\jmath_{\boldsymbol{\mu}}"] \arrow[d, "\pi_{\boldsymbol{\mu}}"'] \\
(\mathbb{R}^k \times M_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}, \widetilde{\boldsymbol{\omega}}_{\boldsymbol{\mu}} = \mathrm{pr}_{M_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}}^* \boldsymbol{\omega}_{\boldsymbol{\mu}} + \mathrm{d}\boldsymbol{u} \mathbin{\bar\wedge} \mathrm{pr}_{M_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}}^* \boldsymbol{\tau}_{\boldsymbol{\mu}}) \arrow[r, bend right=10, "\mathrm{pr}_{M_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}}"'] & (M_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}, \boldsymbol{\tau}_{\boldsymbol{\mu}}, \boldsymbol{\omega}_{\boldsymbol{\mu}}) \arrow[l, bend right=10, "\iota_{\boldsymbol{u}}^{M_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}}"']
\end{tikzcd}$$

Figure 2.1: Scheme of the different structures involved in the k-polycosymplectic reduction through k-polysymplectic fibred manifolds. It is worth noting that k-polysymplectic manifolds in the above diagram admit a series of vector fields satisfying properties extending the ones for Reeb vector fields in k-polycosymplectic geometry.

The authors of [50] used the ideas of [108], along with other ones in [107] and new ones to be presented hereafter in this chapter, to devise a k-polycosymplectic Marsden–Meyer–Weinstein reduction. In particular, a k-polycosymplectic manifold can be associated with a k-polysymplectic manifold of a larger dimension and a specific type, referred to as *k-polysymplectic fibred manifold*. This result is very relevant, as it shows that k-polycosymplectic geometry is a particular case of k-polysymplectic geometry, and it allows us to use the techniques of k-polysymplectic geometry to study k-polycosymplectic manifolds. Such k-polysymplectic fibred manifolds possess, among other properties, a distinguished family of vector fields called *k-polysymplectic Reeb vector fields*. Then, a slight generalisation of the k-polysymplectic Marsden–Meyer–Weinstein reduction developed in [107] is applied to k-polysymplectic fibred manifolds, constructed from k-polycosymplectic manifolds, thereby producing reduced k-polysymplectic fibred manifolds. These reduced manifolds are related to k-polycosymplectic manifolds that arise as Marsden–Meyer–Weinstein reductions of the original k-polycosymplectic manifolds. The general scheme illustrating the k-polycosymplectic reduction developed in [50] is presented in Figure 2.1.

It is convenient to stress that Theorem 2.4.7, which establishes the important connection between a k-polycosymplectic structure on M and a k-polysymplectic structure on $\mathbb{R}^k \times M$, may lead to potential complications. For example, Subsection 2.4.3 shows that Hamiltonian k-vector fields in the k-polycosymplectic setting correspond to k-polysymplectic Hamiltonian k-vector fields with different equilibrium points, which may potentially introduce difficulties to study certain problems. In particular, Subsection 1.3.1 shows that the extension from cosymplectic Hamiltonian vector fields with equilibrium points may lead to Hamiltonian vector fields in the associated symplectic manifolds without them, which gives rise to problems, for instance, in the study of relative equilibrium points. In summary, although these extension techniques provide valuable insights for studying geometric structures, their applicability to the analysis of the corresponding dynamical systems remains limited.

Chapter 2 presents the theoretical contributions of the PhD thesis by developing the generalisations of the Marsden–Meyer–Weinstein (MMW) reduction theorems. Beginning with the classical symplectic setting, the chapter advances through the cosymplectic, k-polysymplectic, k-polycosymplectic, and k-contact frameworks. For each geometric structure, a suitable notion of a momentum map is introduced and analysed in detail, including non-Ad*-equivariant cases.

In addition to formulating new reduction theorems, the chapter compares and corrects certain previous results in the literature, ensuring that the presented methods are mathematically correct and applicable. Illustrative examples are included to highlight the geometric intuition and demonstrate the practical applicability of the theory. These applications range from mechanical systems with time-dependent symmetries to field-theoretic models, such as the vibrating string.

2.1 Symplectic Marsden–Meyer–Weinstein reduction

This section recalls the basic notions and results required to obtain the classical Marsden–Meyer–Weinstein reduction necessary for the time-dependent symplectic energy-momentum method presented in Section 3.1. It begins with the definition of a symplectic momentum map and then presents a proof of the reduction theorem. In addition, it explains how the Ad*-equivariance assumption may be omitted.

Definition 2.1.1. Let (P,ω) be a symplectic manifold and let $\Phi\colon G\times P\to P$ be a Hamiltonian action of the Lie group G on P. A map $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^*$ is a *symplectic momentum map* for the action Φ if

$$\mathrm{d}J^\Phi_\xi=\iota_{\xi_P}\omega,\qquad\forall\xi\in\mathfrak{g}\,,$$

where $J^\Phi_\xi\colon P\to\mathbb{R}$ is defined by

$$J^\Phi_\xi(p):=\langle\mathbf{J}^\Phi(p),\xi\rangle,\quad\forall p\in P,\quad\forall\xi\in\mathfrak{g}\,,$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Additionally, a symplectic momentum map $\mathbf{J}^\Phi$ is said to be Ad*-*equivariant* if it satisfies

$$\mathbf{J}^\Phi(gp)=\mathrm{Ad}^*_{g^{-1}}(\mathbf{J}^\Phi(p)),\qquad\forall g\in G,\qquad\forall p\in P,$$

or, equivalently, if the following diagram commutes

$$\begin{tikzcd}
P \arrow[r, "\Phi_g"] \arrow[d, "\mathbf{J}^\Phi"'] & P \arrow[d, "\mathbf{J}^\Phi"] \\
\mathfrak{g}^* \arrow[r, "\mathrm{Ad}^*_{g^{-1}}"] & \mathfrak{g}^*
\end{tikzcd}\quad.$$

In other words, a map $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^*$ is a symplectic momentum map associated with the Lie group action $\Phi\colon G\times P\to P$ if and only if

$$X_{J^\Phi_\xi}=\xi_P,\qquad\forall\,\xi\in\mathfrak{g}\,,$$

where $X_{J^\Phi_\xi}$ denotes the Hamiltonian vector field associated with $J^\Phi_\xi\colon P\to\mathbb{R}$.

Under the assumption of Ad*-equivariance property of $\mathbf{J}^\Phi$, the following holds

$$(\xi_P J^\Phi_\nu)(p)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle\mathbf{J}^\Phi(\exp(t\xi)p),\nu\rangle=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\langle\mathrm{Ad}^*_{\exp(-t\xi)}\mathbf{J}^\Phi(p),\nu\rangle=J^\Phi_{[\nu,\xi]}(p),\ \forall\xi,\nu\in\mathfrak{g},\ \forall p\in P.\quad(2.1.1)$$

Consequently, $\{J^\Phi_\nu, J^\Phi_\xi\} = J^\Phi_{[\nu,\xi]}$, so that $\mathbf{J}^\Phi$ gives rise to a Lie algebra morphism $\nu \in \mathfrak{g} \mapsto J^\Phi_\nu \in \mathscr{C}^\infty(P)$, where the bracket on $\mathfrak{g}$ is the Lie algebra bracket and the bracket on $\mathscr{C}^\infty(P)$ is the Poisson bracket from Definition 1.2.5.

A Lie group action $\Psi\colon G \times Q \to Q$ induces a natural Lie group action

$$\Phi\colon G \times \mathrm{T}^*Q \ni (g, \alpha_q) \mapsto \Phi_g(\alpha_q) \in \mathrm{T}^*Q$$

of the form

$$\langle \Phi_g(\alpha_q),\ v_{gq}\rangle = \langle \alpha_q, \mathrm{T}_{gq}\Psi_{g^{-1}}(v_{gq})\rangle, \quad \forall q \in Q, \quad \forall v_{gq} \in \mathrm{T}_{gq}Q\,. \tag{2.1.2}$$

This construction is known as the *cotangent lift* of the Lie group action $\Psi\colon G \times Q \to Q$. It plays a fundamental role in geometric mechanics, as it yields canonical momentum maps [2, p 283]. Further details are provided in Proposition 2.1.3 below (see also [2, p 283]).

Before, however, it is useful to prove the following identity.

Lemma 2.1.2. *Let $\Phi\colon G \times P \to P$ be a Lie group action. Then,*

$$(\mathrm{Ad}_g\xi)_P = \Phi_{g*}\xi_P, \quad \forall g \in G, \quad \forall \xi \in \mathfrak{g}\,.$$

Proof. Recall that $\exp(\mathrm{Ad}_g\xi) = I_g \exp(\xi)$ for all $g \in G$ and every $\xi \in \mathfrak{g}$. Therefore,

$$\begin{aligned}(\mathrm{Ad}_g\xi)_P(p) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\exp(t\mathrm{Ad}_g\xi)}(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{I_g \exp(t\xi)}(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{g \exp(t\xi) g^{-1}}(p)\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{g\exp(t\xi)}(\Phi_{g^{-1}}(p)) = \mathrm{T}_{\Phi_{g^{-1}}(p)}\Phi_g \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\exp(t\xi)} \circ \Phi_{g^{-1}}\right)(p)\\ &= \mathrm{T}_{\Phi_{g^{-1}}(p)}\Phi_g(\xi_P)_{\Phi_{g^{-1}}(p)} = (\Phi_{g*}\xi_P)(p)\,.\end{aligned}$$

Thus, $(\mathrm{Ad}_g\xi)_P = \Phi_{g*}\xi_P$, for every $g \in G$ and $\xi \in \mathfrak{g}$. □

Proposition 2.1.3. *Every Lie group action $\Psi\colon G \times Q \to Q$ induces a cotangent lift $\Phi\colon G \times \mathrm{T}^*Q \to \mathrm{T}^*Q$ that admits an Ad^*-equivariant symplectic momentum map $\mathbf{J}^\Phi\colon \mathrm{T}^*Q \to \mathfrak{g}^*$, defined by*

$$J^\Phi_\xi(\alpha_q) := \langle \alpha_q, (\xi_Q)_q\rangle\,, \quad \forall q \in Q\,, \quad \forall \alpha_q \in \mathrm{T}^*_qQ\,, \quad \forall \xi \in \mathfrak{g}.$$

Proof. Recall that the canonical symplectic form on T^*Q is given by $\omega_Q = -\mathrm{d}\theta_Q$, where θ_Q is the Liouville one-form on T^*Q, see Definition 1.2.3. One has to show that for each $\xi \in \mathfrak{g}$, a function J^Φ_ξ corresponds to the Hamiltonian vector field ξ_{T^*Q}, i.e. $\iota_{\xi_{\mathrm{T}^*Q}}\omega_Q = \mathrm{d}J^\Phi_\xi$. Using Cartan's formula, it follows that

$$-\iota_Y \iota_{\xi_{\mathrm{T}^*Q}}\mathrm{d}\theta_Q = -\xi_{\mathrm{T}^*Q}\iota_Y\theta_Q + Y\iota_{\xi_{\mathrm{T}^*Q}}\theta_Q + \iota_{[\xi_{\mathrm{T}^*Q}, Y]}\theta_Q\,. \tag{2.1.3}$$

From the definition of θ_Q, one has

$$(\iota_Y\theta_Q)(\alpha_q) = \langle \alpha_q, \mathrm{T}_{\alpha_q}\tau(Y_{\alpha_q})\rangle\,, \qquad \forall q \in Q\,, \quad \forall \alpha_q \in \mathrm{T}^*_qQ\,, \quad \forall Y_{\alpha_q} \in \mathrm{T}_{\alpha_q}\mathrm{T}^*Q\,,$$

where $\tau\colon \mathrm{T}^*Q \to Q$ is the canonical projection onto Q. Let $g_t := \exp(t\xi)$ and define $\alpha_t := \alpha_q \circ \mathrm{T}_{g_tq}\Psi_{g_{-t}} \in \mathrm{T}^*_{g_tq}Q$, which gives the integral curve through α_q of ξ_{T^*Q}.

Then, note that $\Psi_{g_{-t}} \circ \tau = \tau \circ \Phi_{g_{-t}}$. Furthermore,

$$\begin{aligned}(\xi_{\mathrm{T}^*Q}\iota_Y\theta_Q)(\alpha_q) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\iota_Y\theta_Q)(\alpha_t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \alpha_q, \mathrm{T}_{\alpha_t}(\Psi_{g_{-t}} \circ \tau)(Y_{\alpha_t})\rangle\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \alpha_q, \mathrm{T}_{\alpha_q}\tau \circ \mathrm{T}_{\alpha_t}\Phi_{g_{-t}}(Y_{\alpha_t})\rangle = \langle \alpha_q, \mathrm{T}_{\alpha_q}\tau(\mathscr{L}_{\xi_{\mathrm{T}^*Q}}Y)_{\alpha_q}\rangle = (\iota_{[\xi_{\mathrm{T}^*Q}, Y]}\theta_Q)(\alpha_q).\end{aligned}$$

Let $\{\exp(tY)\}_{t\in\mathbb{R}}$ denote the one-parameter group of diffeomorphisms of Y. Substituting this into (2.1.3) and using (2.1.2), it follows that

$$
\begin{aligned}
(\iota_Y \iota_{\xi_{\mathrm{T}^*Q}}\omega)(\alpha_q) = (Y \iota_{\xi_{\mathrm{T}^*Q}}\theta_Q)(\alpha_q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \exp(tY)\alpha_q, \mathrm{T}\tau(\xi_{\mathrm{T}^*Q})_{\exp(tY)\alpha_q}\rangle \\
= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle \exp(tY)\alpha_q, (\xi_Q)_{\tau(\exp(tY)\alpha_q)}\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} J^\Phi_\xi(\exp(tY)\alpha_q) = (Y J^\Phi_\xi)(\alpha_q) = (\iota_Y \mathrm{d}J^\Phi_\xi)(\alpha_q)
\end{aligned}
$$

for all $Y \in \mathfrak{X}(\mathrm{T}^*Q)$, $\alpha_q \in \mathrm{T}^*_qQ$ and $q \in Q$. Consequently, $\iota_{\xi_{\mathrm{T}^*Q}}\omega = \mathrm{d}J^\Phi_\xi$, as claimed.

To prove Ad*-equivariance, using 2.1.2, one has

$$
J^\Phi_\xi(g\alpha_q) \quad = \quad J^\Phi_\xi(\alpha_q \circ \mathrm{T}_{gq}\Psi_{g^{-1}}) \quad = \quad \langle \alpha_q, \mathrm{T}_{gq}\Psi_{g^{-1}}(\xi_Q)_{gq}\rangle \quad = \quad \langle \alpha_q, ((\mathrm{Ad}_{g^{-1}}\xi)_Q)_q\rangle \quad = \quad J^\Phi_{\mathrm{Ad}_{g^{-1}}\xi}(\alpha_q).
$$

Thus, $\mathbf{J}^\Phi(g\alpha_q) = \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi(\alpha_q)$ for every $g \in G$, $\alpha_q \in \mathrm{T}^*_qQ$, and $q \in Q$. □

The following definition introduces the essential notion of regular values, which plays a fundamental role in reduction theorems.

Definition 2.1.4. A *weak regular value* of a map $F\colon \to N$ is a point $x_0 \in N$ such that $F^{-1}(x_0)$ is a submanifold of M and $\ker \mathrm{T}_pF = \mathrm{T}_p[F^{-1}(x_0)]$ for every $p \in F^{-1}(x_0)$. Moreover, if F is a submersion, then x_0 is a regular value of F. In particular, any regular value of F is also a weak regular value.

To clarify the concept of a weak regular value, which is essential in this work, the following elementary example of a point that is neither a regular value nor a weak regular value is presented.

Example 2.1.5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^2$. Consider the vector field $X = \frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Then, $(\iota_X \mathrm{d}f)(x,y) = 0$ if $x = 0$. However, X is not tangent to $f^{-1}(0) = \{(x,y) \in \mathbb{R}^2 \,|\, x = 0\}$, since $\mathrm{T}_{(0,y)}f^{-1}(0) = \langle \frac{\partial}{\partial y}\rangle$ for every $y \in \mathbb{R}$. Therefore, as $\ker \mathrm{T}_{(0,y)}f \neq \mathrm{T}_{(0,y)}f^{-1}(0)$, it follows that $0 \in \mathbb{R}$ is not a weak regular value of f. Indeed, it is not a regular value either since $\mathrm{T}f = 0$ at points of $f^{-1}(0)$.

More generally, for any function $f\colon M \to N$, a point $\lambda \in N$ is not a weak regular value of f if $\mathrm{T}_pf(v_p) = 0$ for some $v_p \in \mathrm{T}_pM$ with $p \in f^{-1}(\lambda)$ that is not tangent to the submanifold $f^{-1}(\lambda)$.

A Lie group action $\Phi\colon G \times M \to M$ is *quotientable* [4] if the orbit space M/G is a manifold and the canonical projection $\pi\colon M \to M/G$ is a submersion. This condition is automatically satisfied when Φ is free and proper.

If $\mu \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}^\Phi$. By the Implicit Function Theorem (see [2, p 29]), $\mathbf{J}^{\Phi-1}(\mu)$ is a submanifold of P and $\mathrm{T}_z(\mathbf{J}^{\Phi-1}(\mu)) = \ker(\mathrm{T}_p\mathbf{J}^\Phi)$ for every $p \in \mathbf{J}^{\Phi-1}(\mu)$.

It is hereafter assumed that $\mu \in \mathfrak{g}^*$ is a weak regular value of a symplectic momentum map $\mathbf{J}^\Phi : P \to \mathfrak{g}^*$.

Theorem 2.1.6. *If $\mu \in \mathfrak{g}^*$ is a regular value for the symplectic momentum map $\mathbf{J}^\Phi$, then every μ' belonging to the coadjoint orbit, $\mathcal{O}_\mu$, of μ is also a regular value. If G_μ acts properly and freely in $\mathbf{J}^{\Phi-1}(\mu)$, then $G_{\mu'}$ acts also freely and properly on $\mathbf{J}^{\Phi-1}(\mu')$ for every $\mu' \in \mathcal{O}_\mu$. Finally, $\mathbf{J}^{\Phi-1}(\mathcal{O}_\mu)$ is a submanifold of P.*

Proof. If μ is a regular point of $\mathbf{J}^\Phi$, then $\mathrm{T}\mathbf{J}^\Phi$ is a surjection at every point of $\mathbf{J}^{\Phi-1}(\mu)$.

Let $\mu' := \mathrm{Ad}^*_{g^{-1}}\mu$. If $p \in \mathbf{J}^{\Phi-1}(\mu)$, then $gp \in \mathbf{J}^{\Phi-1}(\mu')$ since $\mathbf{J}^\Phi$ is Ad*-equivariant. Moreover, from the fact that Φ_g is a diffeomorphism, it follows that

$$
\mathbf{J}^{\Phi-1}(\mathrm{Ad}^*_{g^{-1}}\mu) = \Phi_g(\mathbf{J}^{\Phi-1}(\mu)), \qquad \forall g \in G, \quad \forall \mu \in \mathbf{J}^\Phi(P).
$$

Furthermore, $\mathrm{T}_{gp}\mathbf{J}^\Phi = \mathrm{Ad}^*_{g^{-1}}\mathrm{T}_p\mathbf{J}^\Phi$ for every $p \in \mathbf{J}^{\Phi-1}(\mu)$ and $g \in G$. Thus, $\mathrm{T}\mathbf{J}^\Phi$ is a surjection on $\mathbf{J}^{\Phi-1}(\mathrm{Ad}^*_{g^{-1}}\mu)$ for every $g \in G$.

Note that $G_{\mathrm{Ad}^*_{g^{-1}}\mu} = I_gG_\mu$ for every $g \in G$ and $\mu \in \mathbf{J}^\Phi(P)$. Moreover, if $\Phi\colon G_\mu \times \mathbf{J}^{\Phi-1}(\mu) \to \mathbf{J}^{\Phi-1}(\mu)$ is free and proper, by the equivariance of Φ, it follows that $\Phi\colon G_{\mu'} \times \mathbf{J}^{\Phi-1}(\mu') \to \mathbf{J}^{\Phi-1}(\mu')$ is free and proper also for $\mu' \in \mathcal{O}_\mu$.

To prove that $\mathbf{J}^{\Phi-1}(\mathcal{O}_\mu)$ is a submanifold of P, recall that if $f\colon M \to N$, $S \subset N$ is a submanifold of the manifold N and $\operatorname{Im} T_pf + T_sS = T_sN$ for every $s \in S$ and $p \in f^{-1}(s)$, then f is *transversal* to S and hence $f^{-1}(S)$ is a submanifold of M (see [2, p 49]).

Since μ is a regular point of $\mathbf{J}^\Phi$, one has that $\operatorname{Im} \mathrm{T}_p\mathbf{J} = \mathrm{T}_{\mathbf{J}^\Phi(p)}\mathfrak{g}^*$ for every $p \in \mathbf{J}^{\Phi-1}(\mu)$. Consequently,

$$\operatorname{Im} \mathrm{T}_p\mathbf{J}^\Phi + \mathrm{T}_{\mathbf{J}^\Phi(p)}\mathcal{O}_\mu = \mathrm{T}_{\mathbf{J}(p)}\mathfrak{g}^*$$

for every $p \in \mathbf{J}^{\Phi-1}(\mathcal{O}_\mu)$. Therefore, $\mathbf{J}^\phi$ is transversal to $\mathcal{O}_\mu$ and $\mathbf{J}^{\Phi-1}(\mathcal{O}_\mu)$ is a submanifold of P. □

Lemma 2.1.7. *Let $p \in \mathbf{J}^{\Phi-1}(\mu)$ and assume that symplectic momentum map $\mathbf{J}^\Phi$ is* Ad**-equivariant. Let $G_\mu \subset G$ be the isotropy group at $\mu \in \mathfrak{g}^*$ of the coadjoint action of G. Then*

a) $\mathrm{T}_p(G_\mu p) = \mathrm{T}_p(Gp) \cap \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\mu))$,

b) $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu) = (\mathrm{T}_p(Gp))^{\perp_\omega}$.

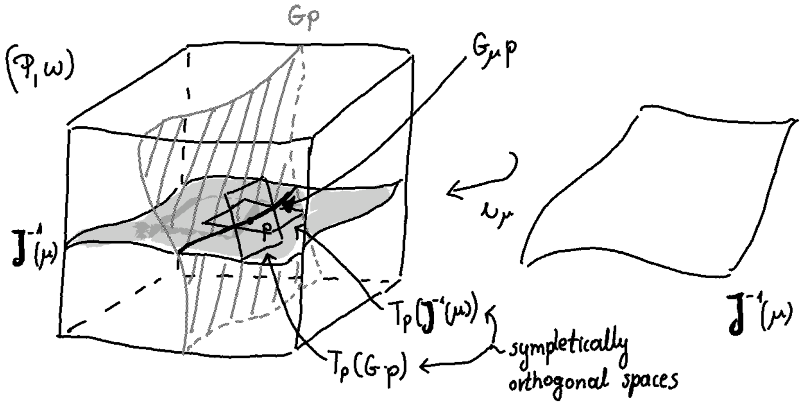

Figure 2.2: Symplectically orthogonal spaces $\mathrm{T}_p(Gp)$ and $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)$.

Proof. Recall that

$$\mathrm{T}_p(Gp) = \{(\xi_P)_p \mid \xi \in \mathfrak{g}\}, \qquad \mathrm{T}_p(G_\mu p) = \{(\xi_P)_p \mid \xi \in \mathfrak{g}_\mu\},$$

where $\mathfrak{g}_\mu$ is the Lie algebra of G_μ.

The proof of a) amounts to proving that $(\xi_P)_p \in \mathrm{T}_p(\mathbf{J}^{\Phi-1}(\mu))$ if and only if $\xi \in \mathfrak{g}_\mu$. Since $\mathbf{J}^\Phi$ is Ad*-equivariant and by (1.2.3), it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathbf{J}^\Phi \circ \Phi_{\exp(t\xi)}(p) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \mathrm{Ad}^*_{\exp(-t\xi)}\mathbf{J}^\Phi(p) \Rightarrow \mathrm{T}_p\mathbf{J}^\Phi((\xi_P)_p) = (\xi_{\mathfrak{g}^*})_\mu,$$

for all $\xi \in \mathfrak{g}$ and $p \in \mathbf{J}^{\Phi-1}(\mu)$. Therefore, $(\xi_P)_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu) = \ker(\mathrm{T}_p\mathbf{J}^\Phi)$ implies that $(\xi_{\mathfrak{g}^*})_\mu = 0$. Then, for $(\xi_{\mathfrak{g}^*})_\mu = 0$, one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} \mathrm{Ad}^*_{\exp(-t\xi)}\mu &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=s} \mathrm{Ad}^*_{\exp(-s\xi)}\mathrm{Ad}^*_{\exp(-(t-s)\xi)}(\mu) \\ &= \mathrm{T}_\mu\mathrm{Ad}^*_{\exp(-s\xi)}\left(\frac{\mathrm{d}}{\mathrm{d}(t-s)}\bigg|_{t=s} \mathrm{Ad}^*_{\exp((s-t)\xi)}\mu\right) = \mathrm{T}_\mu\mathrm{Ad}^*_{\exp(-s\xi)}(\xi_{\mathfrak{g}^*})_\mu = 0\end{aligned}$$

and $(\xi_{\mathfrak{g}^*})_\mu = 0$ boils down to $\exp(t\xi) \in G_\mu$ for all $t \in \mathbb{R}$. Thus, $(\xi_{\mathfrak{g}^*})_\mu = 0$ whenever $\xi \in \mathfrak{g}_\mu$. This finishes the proof of a).

To prove b), note that the definition of a symplectic momentum map $\mathbf{J}^\Phi$ yields

$$\omega_p((\xi_P)_p, v_p) = (\mathrm{d}J^\Phi_\xi)_p(v_p) = \langle \mathrm{T}_p\mathbf{J}^\Phi(v_p), \xi\rangle, \qquad \forall v_p \in \mathrm{T}_pP, \qquad \forall \xi \in \mathfrak{g}, \qquad \forall p \in P.$$

Thus, $v_p \in \ker \mathrm{T}_p\mathbf{J}^\Phi = \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)$ if and only if $\langle \mathrm{T}_p\mathbf{J}^\Phi(v_p), \xi\rangle = 0$ for all $\xi \in \mathfrak{g}$. Therefore,

$$(\mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu))^{\perp_\omega} = \mathrm{T}_p(Gp)\,,$$

for all $p \in \mathbf{J}^{\Phi-1}(\mu)$. □

The following theorem presents the classical symplectic Marsden–Meyer–Weinstein reduction [109].

Theorem 2.1.8. *Let $\Phi\colon G\times P \to P$ be a Hamiltonian Lie group action of G on the symplectic manifold (P,ω) admitting an Ad^*-equivariant symplectic momentum map $\mathbf{J}^\Phi\colon P \to \mathfrak{g}^*$. Assume that $\mu \in \mathfrak{g}^*$ is a weak regular value of $\mathbf{J}^\Phi$ and G_μ acts freely and properly on $\mathbf{J}^{\Phi-1}(\mu)$. Let $i_\mu\colon \mathbf{J}^{\Phi-1}(\mu) \hookrightarrow P$ be the natural embedding of $\mathbf{J}^{\Phi-1}(\mu)$ into P and let $\pi_\mu\colon \mathbf{J}^{\Phi-1}(\mu) \to \mathbf{J}^{\Phi-1}(\mu)/G_\mu =: P_\mu$ be the canonical projection. Then, there exists a unique symplectic structure ω_μ on P_μ such that*

$$\pi^*_\mu\omega_\mu = i^*_\mu\omega\,.$$

Proof. Since μ is a weak regular value of $\mathbf{J}^\Phi$, it follows that $\mathbf{J}^{\Phi-1}(\mu) \subset P$ is a submanifold of P. Additionally, as G_μ acts freely and properly on $\mathbf{J}^{\Phi-1}(\mu)$, the quotient $P_\mu = \mathbf{J}^{\Phi-1}(\mu)/G_\mu$ inherits a manifold structure. For any $v_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)$, the equivalence class of v_p within $\mathrm{T}_{\pi_\mu(p)}P_\mu \simeq \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)/\mathrm{T}_p(G_\mu p)$ is denoted by $[v_p] := \mathrm{T}_p\pi_\mu(v_p) \in T_{\pi_\mu(p)}P_\mu$. Then,

$$\pi^*_\mu\omega_\mu = i^*_\mu\omega \Longleftrightarrow (\omega_\mu)_{\pi_\mu(p)}([v_p],[\vartheta_p]) = \omega_p(v_p,\vartheta_p), \quad \forall v_p,\vartheta_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu), \quad \forall p \in \mathbf{J}^{\Phi-1}(\mu). \tag{2.1.4}$$

To verify that ω_μ is well-defined, it is necessery to prove that

$$\omega_p(v_p,\vartheta_p) = \omega_p(\tilde{v}_p,\tilde{\vartheta}_p)\,,$$

for all $\tilde{v}_p \in [v_p]$, $\tilde{\vartheta}_p \in [\vartheta_p]$, and $p \in \mathbf{J}^{\Phi-1}(\mu)$. Since π_μ is a submersion, $\pi_\mu^{-1}(\pi_\mu(p)) = G_\mu p$ is a submanifold and $\mathrm{T}_p(G_\mu p) = \ker \mathrm{T}_p\pi_\mu$. Moreover, $\tilde{v}_p - v_p \in \ker \mathrm{T}_p\pi_\mu = \mathrm{T}_p(G_\mu p) \subset \mathrm{T}_p(Gp)$ and from the part b) of Lemma 2.1.7, i.e. $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu) = \mathrm{T}_p(Gp)^{\perp_\omega}$, it follows that $\omega_p(\tilde{v}_p - v_p, \vartheta_p) = 0$ for all $\vartheta_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)$. Therefore,

$$\begin{aligned}\omega_p(\tilde{v}_p,\tilde{\vartheta}_p) &= \omega_p(\tilde{v}_p - v_p + v_p, \tilde{\vartheta}_p - \vartheta_p + \vartheta_p)\\ &= \omega_p(\tilde{v}_p - v_p, \tilde{\vartheta}_p - \vartheta_p) + \omega_p(\tilde{v}_p - v_p, \vartheta_p) + \omega_p(v_p, \tilde{\vartheta}_p - \vartheta_p) + \omega_p(v_p,\vartheta_p)\\ &= \omega_p(v_p,\vartheta_p).\end{aligned}$$

Let $p' = \Phi(g,p)$ for some $g \in G_\mu$. Now the goal is to prove that $(\omega_\mu)_{\pi_\mu(p)} = (\omega_\mu)_{\pi_\mu(p')}$. Since Φ_g is symplectic by assumption, one has $\Phi^*_g\omega = \omega$ for every $g \in G$. Moreover, $\mathrm{T}_p\Phi_g$ is an isomorphism and, if $g \in G_\mu$, it follows that $\mathrm{T}_p\Phi_g\mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu) = \mathrm{T}_{p'}\mathbf{J}^{\Phi-1}(\mu)$. Consequently, for all $v_{p'},\vartheta_{p'} \in \mathrm{T}_{p'}\mathbf{J}^{\Phi-1}(\mu)$, there exist $v_p,\vartheta_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)$ such that

$$\omega_{p'}(v_{p'},\vartheta_{p'}) = \omega_{p'}(\mathrm{T}_p\Phi_g(v_p), \mathrm{T}_p\Phi_g(\vartheta_p)) = [\mathrm{T}^*_{p'}\Phi_g(\omega_{p'})](v_p,\vartheta_p) = \omega_p(v_p,\vartheta_p),$$

and $(\omega_\mu)_{\pi_\mu(x')}$ becomes a well-defined two-form, namely

$$(\omega_\mu)_{\pi_\mu(p')}([v_{p'}],[\vartheta_{p'}]) = (\omega_\mu)_{\pi_\mu(p)}([v_p],[\vartheta_p])$$

for all $v_p,\vartheta_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)$, $v_{p'},\vartheta_{p'} \in \mathrm{T}_{p'}\mathbf{J}^{\Phi-1}(\mu)$ and for any $p,p' \in \mathbf{J}^{\Phi-1}(\mu)$ such that $\pi_\mu(p) = \pi_\mu(p')$. Thus, ω_μ is well-defined on $\mathrm{Im}\,\pi_\mu$. Since π_μ is surjective, it follows that ω_μ is well-defined on the entire P_μ.

To prove that ω_μ is uniquely defined, assume that there exists another two-form $\tilde{\omega}_\mu \in \Omega^2(P_\mu)$ such that $\pi_\mu^* \tilde{\omega}_\mu = i_\mu^* \omega$. Then,

$$(\omega_\mu - \tilde{\omega}_\mu)_{\pi_\mu(p)}(\mathrm{T}_p\pi_\mu(v_p), \mathrm{T}_p\pi_\mu(\vartheta_p)) = 0$$

for all $v_p, \vartheta_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)$ and $p \in \mathbf{J}^{\Phi-1}(\mu)$. Since π_μ is a surjective submersion, $\tilde{\omega}_\mu = \omega_\mu$ and ω_μ is unique.

To verify that ω_μ is smooth, by the fact that π_μ is a submersion, the local section theorem [94, Theorem 4.26] states that there exists a smooth section $\varsigma\colon P_\mu \to \mathbf{J}^{\Phi-1}(\mu)$, i.e. $\pi_\mu \circ \varsigma = \mathrm{Id}_{P_\mu}$. Then,

$$\omega_\mu = (\pi_\mu \circ \varsigma)^* \omega_\mu = \varsigma^*(\pi_\mu^* \omega_\mu) = \varsigma^*(i_\mu^* \omega) = (i_\mu \circ \varsigma)^* \omega.$$

Thus, ω_μ is locally the pull-back of a smooth differential form relative to a smooth map. Consequently, ω_μ is smooth.

To establish that ω_μ is non-degenerate, consider $v_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)$. Then, using (2.1.4), one has

$$(\omega_\mu)_{\pi_\mu(p)}([v_p], [\vartheta_p]) = 0, \quad \forall \vartheta_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu) \implies \omega_p(v_p, \vartheta_p) = 0, \quad \forall \vartheta_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu).$$

Therefore, $v_p \in \mathrm{T}_p(Gp)$. By Lemma 2.1.7 it follows that $v_p \in \mathrm{T}_p(G_\mu p)$. Thus, $[v_p] = \mathrm{T}_p\pi_\mu(v_p) = 0$, and therefore ω_μ is non-degenerate on P_μ.

To show that ω_μ is closed, note that

$$\mathrm{d}\omega = 0 \implies 0 = i_\mu^* \mathrm{d}\omega = \mathrm{d}(i_\mu^* \omega) = \mathrm{d}(\pi_\mu^* \omega_\mu) = \pi_\mu^* \mathrm{d}\omega_\mu \implies \mathrm{d}\omega_\mu = 0,$$

where the last step stems from the fact that π_μ is a surjective submersion following the same argument given for the uniqueness of ω_μ. Therefore, $\omega_\mu \in \Omega^2(P_\mu)$ is closed, and thus it is a symplectic form on P_μ. □

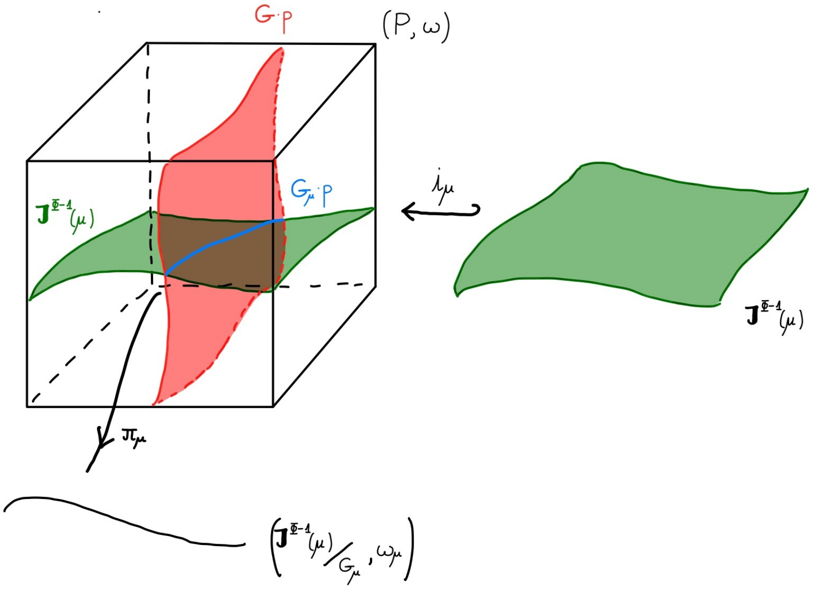

Figure 2.3: Scheme of the Marsden–Meyer–Weinstein symplectic reduction.

To simplify the notation, the following definition is introduced.

Definition 2.1.9. The five-tuple $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$ is called a *G-invariant Hamiltonian system*, where (P, ω) is a symplectic manifold, Φ is a symplectic action leaving invariant the Hamiltonian function $h \in \mathscr{C}^\infty(P)$, i.e. $h(\Phi(g,p)) = h(p)$ for every $g \in G$ and $p \in P$, and $\mathbf{J}^\Phi$ is a symplectic momentum map. An Ad^*-*equivariant G-invariant Hamiltonian system* is a G-invariant Hamiltonian system with an Ad^*-equivariant symplectic momentum map.

Proposition 2.1.10 analyses the evolution of $\mathbf{J}^\Phi \colon P \to \mathfrak{g}^*$ under the dynamics of X_h associated with an Ad^**-equivariant G-invariant Hamiltonian system* $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$.

Proposition 2.1.10. *Let* $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$ *be an* Ad^**-equivariant* G*-invariant Hamiltonian system. Then, the symplectic momentum map* $\mathbf{J}^\Phi \colon P \to \mathfrak{g}^*$ *is invariant relative to the evolution of* h*, i.e. if* $F \colon (t,p) \in \mathbb{R} \times P \mapsto F_t(p) := F(t,p) \in P$ *is the flow of* X_h *on* P*, then* $\mathbf{J}^\Phi(F(t,p)) = \mathbf{J}^\Phi(p)$ *for all* $p \in P$ *and* $t \in \mathbb{R}$.

Proof. Since h is invariant under the action of Φ by assumption, it follows that $\xi_P h = 0$ for every $\xi \in \mathfrak{g}$. Additionally,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} J_\xi^\Phi(F_t) = X_h J_\xi^\Phi = \{J_\xi^\Phi, h\} = -\xi_P h = 0, \qquad \forall \xi \in \mathfrak{g}.$$

Hence, $J_\xi^\Phi(F(t,p)) = J_\xi^\Phi(p)$, for every $\xi \in \mathfrak{g}$, every $t \in \mathbb{R}$, and $p \in P$. Therefore, $\mathbf{J}^\Phi(F(t,p)) = \mathbf{J}^\Phi(p)$ for every $t \in \mathbb{R}$ and $p \in P$. □

The consequence of Theorem 2.1.8 and Proposition 2.1.10 is the following proposition that ensures the reduction of the dynamics given by a symplectic Hamiltonian vector field X_h.

Proposition 2.1.11. *Assume the assumptions of Theorem 2.1.8 hold. Then, the symplectic Hamiltonian vector field* X_h *is projectable onto* $P_\mu = \mathbf{J}^{\Phi-1}(\mu)/G_\mu$ *and* $\pi_{\mu *}(X_h) = X_{k_\mu}$*, where* k_μ *is the unique function on* P_μ *satisfying* $\pi_\mu^* k_\mu = i_\mu^* h$.

Proof. By Proposition 2.1.10, the vector field X_h is tangent to $\mathbf{J}^{\Phi-1}(\mu)$. Moreover, since Φ is a symplectic Lie group action that leaves h invariant, it follows that

$$\iota_{[\xi_P, X_h]}\omega = \mathscr{L}_{\xi_P}\iota_{X_h}\omega - \iota_{X_h}\mathscr{L}_{\xi_P}\omega = \mathscr{L}_{\xi_P}\mathrm{d}h = 0\,.$$

Consequently, X_h is projectable onto P_μ, i.e. $\pi_{\mu *}X_h = Y$ for some vector field Y on P_μ. Moreover, there exists a unique function $k_\mu \in \mathscr{C}^\infty(P_\mu)$ such that $\pi_\mu^* k_\mu = i_\mu^* h$. Then, Theorem 2.1.8 yields that there exists $\omega_\mu \in \Omega^2(P_\mu)$ satisfying $\pi_\mu^*\omega_\mu = i_\mu^*\omega$. Furthermore,

$$\pi_\mu^*(\iota_Y \omega_\mu) = \pi_\mu^* * (\iota_{\pi_{\mu *}(X_h)}\omega_\mu) = \iota_{X_h}\pi_\mu^*\omega_\mu = \iota_{X_h} i_\mu^*\omega = i_\mu^*(\iota_{X_h}\omega) = i_\mu^*\mathrm{d}h \; = \pi_\mu^*\mathrm{d}k_\mu\,,$$

where X_h denotes both the vector field on P and its restriction to $\mathbf{J}^{\Phi-1}(\mu)$. Therefore, $Y = \pi_{\mu *}X_h = X_{k_\mu}$ is the reduced symplectic Hamiltonian vector field. □

2.1.1 General symplectic momentum maps

In this section, it is shown that the Ad^*-equivariance condition imposed in Definition 2.1.1 can be relaxed in certain cases. In particular, the existence of a non-Ad^*-equivariant symplectic momentum map still allows for the definition of some equivariance of the symplectic momentum map associated with the same Lie group action. A general symplectic momentum map $\mathbf{J}^\Phi$ may not induce a Poisson algebra homomorphism between P and $\mathfrak{g}^*$. The following results provide fundamental properties of such symplectic momentum maps. More details can be found in [54, 128, 151].

Proposition 2.1.12. *Let* $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$ *be a* G*-invariant Hamiltonian system. Define the functions on* P *of the form*

$$\psi_{g,\xi} \colon p \in P \longmapsto J_\xi(\Phi_g(p)) - J_{\mathrm{Ad}_{g^{-1}}\xi}(p) \in \mathbb{R}\,, \qquad g \in G\,, \qquad \xi \in \mathfrak{g}\,.$$

Then, $\psi_{g,\xi}$ *is a constant function on* P *for every* $g \in G$ *and* $\xi \in \mathfrak{g}$.

Proof. To verify that $\psi_{g,\xi}$ is constant on P, compute the following

$$\begin{aligned} \mathrm{d}\psi_{g,\xi}(p) &= \mathrm{d}[J^\Phi_\xi \circ \Phi_g](p) - \mathrm{d}J^\Phi_{\mathrm{Ad}_{g^{-1}}\xi}(p) = (\Phi_g^*(\iota_{\xi_P}\omega))_p - (\iota_{(\mathrm{Ad}_{g^{-1}}\xi)_P}\omega)_p \\ &= (\Phi_g^*(\iota_{\xi_P}\omega))_p - (\iota_{\Phi_{g^{-1}*}\xi_P}\omega)_p = (\iota_{\Phi_{g^{-1}*}\xi_P}\Phi_g^*\omega)_p - (\iota_{\Phi_{g^{-1}*}\xi_P}\omega)_p \\ &= (\iota_{\Phi_{g^{-1}*}\xi_P}\omega)_p - (\iota_{\Phi_{g^{-1}*}\xi_P}\omega)_p = 0, \end{aligned}$$

where the assumption that Φ is a symplectic Lie group action and Lemma 2.1.2 have been used. Hence, $\psi_{g,\xi}$ is constant for every $g \in G$ and $\xi \in \mathfrak{g}$. □

To capture all such function $\psi_{g,\xi}$ simultaneously with $g \in G$ and $\xi \in \mathfrak{g}$, consider

$$\begin{aligned} \psi_{g,\xi}(p) &= J_\xi(\Phi_g(p)) - J^\Phi_{\mathrm{Ad}_{g^{-1}}\xi}(p) = \langle \mathbf{J}^\Phi(\Phi_g(p)), \xi\rangle - \langle \mathbf{J}^\Phi(p), \mathrm{Ad}_{g^{-1}}\xi\rangle \\ &= \langle \mathbf{J}^\Phi(\Phi_g(p)), \xi\rangle - \langle \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi(p), \xi\rangle = \langle \mathbf{J}^\Phi(\Phi_g(p)) - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi(p), \xi\rangle \end{aligned}$$

and define $\sigma\colon G \ni g \mapsto \mathbf{J}^\Phi(\Phi_g(p)) - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi(p) \in \mathfrak{g}^*$. Then, $\langle \sigma(g), \xi\rangle = \psi_{g,\xi}(p)$ for every $g \in G$ and $\xi \in \mathfrak{g}$. Moreover,

$$\begin{aligned} \sigma(gh) &= \mathbf{J}(\Phi_{gh}(p)) - \mathrm{Ad}^*_{gh^{-1}}\mathbf{J}(p) = \mathbf{J}(\Phi_g(\Phi_h(p))) - \mathrm{Ad}^*_{g^{-1}}\mathrm{Ad}^*_{h^{-1}}\mathbf{J}(p) \\ &= \mathbf{J}(\Phi_g(\Phi_h(p))) - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}(\Phi_h(p)) + \mathrm{Ad}^*_{g^{-1}}\mathbf{J}(\Phi_h(p)) - \mathrm{Ad}^*_{g^{-1}}\mathrm{Ad}^*_{h^{-1}}\mathbf{J}(p) \\ &= \mathbf{J}(\Phi_g(p)) - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}(p) + \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(\Phi_h(p)) - \mathrm{Ad}^*_{h^{-1}}\mathbf{J}(p)) = \sigma(g) + \mathrm{Ad}^*_{g^{-1}}\sigma(h). \end{aligned}$$

Definition 2.1.13. A map $\sigma\colon G \to \mathfrak{g}^*$ satisfying $\sigma(gh) = \sigma(g) + \mathrm{Ad}^*_{g^{-1}}\sigma(h)$ for every $g, h \in G$ is called a *cocycle* and a map $\sigma\colon G \to \mathfrak{g}^*$ is the *co-adjoint cocycle* associated with the symplectic momentum map $\mathbf{J}^\Phi$ on P, if

$$\sigma(g) := \mathbf{J}^\Phi \circ \Phi_g - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi(p)\,, \qquad p \in P\,.$$

A map $\sigma\colon G \to \mathfrak{g}^*$ is a *coboundary* if there exists $\mu \in \mathfrak{g}^*$ such that

$$\sigma(g) = \mu - \mathrm{Ad}^*_{g^{-1}}\mu, \qquad \forall g \in G. \tag{2.1.5}$$

Then, for a coboundary σ, condition (2.1.5) implies that, for any $g, h \in G$, one has

$$\sigma(gh) = \mu - \mathrm{Ad}^*_{(gh)^{-1}}\mu = \mu - \mathrm{Ad}^*_{g^{-1}}\mu + \mathrm{Ad}^*_{g^{-1}}\mu - \mathrm{Ad}^*_{g^{-1}}\mathrm{Ad}^*_{h^{-1}}\mu = \sigma(g) + \mathrm{Ad}^*_{g^{-1}}\sigma(h)\,.$$

Consequently, any coboundary is a cocycle.

Note that $\sigma(e) = 0$ for the neutral element $e \in G$. Moreover, if $\mathbf{J}^\Phi$ is an Ad^*-equivariant symplectic momentum map, then $\sigma = 0$. In other words, the cocycle σ measures the deviation from the Ad^*-equivariance of a symplectic momentum map.

The set of cocycles of G, viewed as functions on G taking values in a vector space, in this case, $\mathfrak{g}^*$, forms a vector space relative to the pointwise addition of functions and scalar multiplication. The subset of coboundaries is a linear subspace of this vector space. Denote by $[\sigma]$ the equivalence classes of cocycles differing from a cocycle σ by a coboundary. Explicitly, for two cocycles σ_1 and σ_2, one has $[\sigma_1] = [\sigma_2]$ if and only if there exists $\mu \in \mathfrak{g}^*$ such that

$$\sigma_1 - \sigma_2 = \mu - \mathrm{Ad}^*_{g^{-1}}\mu.$$

The following proposition establishes the existence of a well-defined cohomology class, $[\sigma]$, associated with any symplectic action that admits a symplectic momentum map.

Proposition 2.1.14. *Let $\Phi\colon G \times P \to P$ be a symplectic Lie group action and let $\mathbf{J}^\Phi_1, \mathbf{J}^\Phi_2\colon P \to \mathfrak{g}^*$ be two symplectic momentum maps corresponding to the same Lie group action, with associated co-adjoint cocycles σ_1 and σ_2, respectively. Then $[\sigma_1] = [\sigma_2]$.*

Proof. By the definition of the cocycles associated with the symplectic momentum maps $\mathbf{J}_1^\Phi, \mathbf{J}_2^\Phi$, one obtains

$$\langle \sigma_1(g) - \sigma_2(g), \xi \rangle = \langle \mathbf{J}_1^\Phi(\Phi_g(p)) - \mathbf{J}_2^\Phi(\Phi_g(p)), \xi \rangle - \langle \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}_1^\Phi(p) - \mathbf{J}_2^\Phi(p)), \xi \rangle, \quad \forall g \in G, \quad \forall \xi \in \mathfrak{g}.$$

Then,

$$\mathrm{d}\langle \mathbf{J}_1^\Phi - \mathbf{J}_2^\Phi, \xi \rangle = \mathrm{d}J^\Phi_{1,\xi} - \mathrm{d}J^\Phi_{2,\xi} = \iota_{\xi_P}\omega - \iota_{\xi_P}\omega = 0$$

and it follows that $\mathbf{J}_1^\Phi - \mathbf{J}_2^\Phi$ has a constant value in $\mathfrak{g}^*$, say μ. Hence, $(\mathbf{J}_1^\Phi - \mathbf{J}_2^\Phi) \circ \Phi_g = \mathbf{J}_1^\Phi - \mathbf{J}_2^\Phi$ for every $g \in G$. Consequently,

$$\sigma_1(g) - \sigma_2(g) = \mu - \mathrm{Ad}^*_{g^{-1}}\mu\,, \qquad \forall g \in G,$$

which implies $[\sigma_1] = [\sigma_2]$. □

The above proposition implies that a necessary condition for the existence of an Ad*-equivariant symplectic momentum map is that the associated cocycle must be a coboundary. Indeed, if a Hamiltonian Lie group action admits an Ad*-equivariant symplectic momentum map $\mathbf{J}_2^\Phi$, then $\sigma_2 = 0$ and hence any other symplectic momentum map $\mathbf{J}_1^\Phi$ for the same action must satisfy $[\sigma_1] = [\sigma_2] = 0$, making the cocycle of $\mathbf{J}_1^\Phi$ a coboundary. Conversely, if σ_1 is a coboundary, then the symplectic momentum map

$$\mathbf{J}^\Phi(p) := \mathbf{J}_1^\Phi(p) - \mu, \quad \forall p \in P,$$

is an Ad*-equivariant symplectic momentum map for the same Hamiltonian Lie group action of $\mathbf{J}_1^\Phi$, where $\mu \in \mathfrak{g}^*$ satisfies that $\sigma_1(g) = \mu - \mathrm{Ad}^*_{g^{-1}}\mu$ for every $g \in G$. In fact,

$$\langle \mathbf{J}^\Phi, \xi \rangle = \langle \mathbf{J}_1^\Phi, \xi \rangle - \langle \mu, \xi \rangle = J^\Phi_{1,\xi} - \langle \mu, \xi \rangle, \quad \forall \xi \in \mathfrak{g}$$

and

$$\mathbf{J}^\Phi(\Phi(g,p)) = \mathbf{J}_1^\Phi(\Phi(g,p)) - \mu = \sigma_1(g) + \mathrm{Ad}^*_{g^{-1}}\mathbf{J}_1^\Phi(p) - \mu = \sigma_1(g) + \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi(p) + \mathrm{Ad}^*_{g^{-1}}\mu - \mu\,,$$

for any $p \in P$ and $g \in G$. Therefore,

$$\mathbf{J}^\Phi(\Phi(g,p)) - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi(p) = \sigma_1(g) + \mathrm{Ad}^*_{g^{-1}}\mu - \mu = 0\,, \quad \forall p \in P, \quad \forall g \in G\,.$$

Proposition 2.1.15. *Let $\mathbf{J}^\Phi \colon P \to \mathfrak{g}^*$ be a symplectic momentum map for the symplectic Lie group action $\Phi \colon G \times P \to P$ with associated cocycle $\sigma \colon G \to \mathfrak{g}^*$. Then*

1. *the map $\Delta \colon G \times \mathfrak{g}^* \to \mathfrak{g}^* \mid (g, \mu) \mapsto \mathrm{Ad}^*_{g^{-1}}\mu + \sigma(g)$ is an action of G on $\mathfrak{g}^*$, the so-called symplectic affine Lie group action,*
2. *the symplectic momentum map $\mathbf{J}^\Phi$ is equivariant with respect to Δ, in other words, the following diagram commutes*

$$\begin{tikzcd} P \arrow[r, "\Phi_g"] \arrow[d, "\mathbf{J}^\Phi"] & P \arrow[d, "\mathbf{J}^\Phi"] \\ \mathfrak{g}^* \arrow[r, "\Delta_g"] & \mathfrak{g}^* \end{tikzcd}$$

where $\Delta_g(\mu) \colon p \in P \mapsto \Delta(g, \mu) \in \mathfrak{g}^*$.

Proof. First, it is verified that $\Delta(e, \mu) = \mu$ and $\Delta(g, \Delta(h, \mu)) = \Delta(gh, \mu)$ for the neutral element $e \in G$, every $g, h \in G$, and all $\mu \in \mathfrak{g}^*$. Indeed,

$$\Delta(e, \mu) = \mathrm{Ad}^*_{e^{-1}}\mu + \sigma(e) = \mu$$

and

$$\begin{aligned}\Delta(g,\Delta(h,\mu)) &= \mathrm{Ad}^*_{g^{-1}}(\mathrm{Ad}^*_{h^{-1}}\mu+\sigma(h))+\sigma(g) = \mathrm{Ad}^*_{g^{-1}}\mathrm{Ad}^*_{h^{-1}}\mu+\mathrm{Ad}^*_{g^{-1}}\sigma(h)+\sigma(g)\\ &= \mathrm{Ad}^*_{(gh)^{-1}}\mu+\mathrm{Ad}^*_{g^{-1}}\sigma(h)+\sigma(g) = \mathrm{Ad}^*_{(gh)^{-1}}\mu+\sigma(gh) = \Delta(gh,\mu).\end{aligned}$$

Thus, Δ is a Lie group action on $\mathfrak{g}^*$.

Second, from the definition of Δ and σ, one gets

$$\Delta_g(\mathbf{J}^\Phi(p)) = \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi(p)+\sigma(g) = \mathbf{J}^\Phi(\Phi_g(p)), \qquad \forall g\in G, \quad \forall p\in P.$$

□

This result implies that even when a symplectic momentum map is not Ad*-equivariant, it becomes an equivariant relative to the symplectic affine Lie group action on $\mathfrak{g}^*$ given by Δ. The following theorem studies the commutation relations associated with a symplectic momentum map $\mathbf{J}^\Phi$.

Theorem 2.1.16. *Let $\Phi\colon G\times P\to P$ be a symplectic Lie group action with a symplectic momentum map $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^*$ and let $\sigma\colon G\to\mathfrak{g}^*$ be an associated cocycle of $\mathbf{J}^\Phi$. Define*

$$\sigma_\eta\colon g\in G\mapsto\langle\sigma(g),\eta\rangle\in\mathbb{R}, \qquad \Sigma\colon(\xi,\eta)\in\mathfrak{g}\times\mathfrak{g}\mapsto \mathrm{T}_e\sigma_\eta(\xi)\in\mathbb{R}.$$

Then:

1. *Σ is a skew-symmetric bilinear form on $\mathfrak{g}$ satisfying the Jacobi identity,*
2. *$\Sigma(\xi,\nu)=\{J^\Phi_\nu,J^\Phi_\xi\}-J^\Phi_{[\nu,\xi]}$, for all $\xi,\nu\in\mathfrak{g}$.*

Proof. The derivative of $\sigma_\eta(g)$ at $g=e$, is given by

$$\begin{aligned}\Sigma(\xi,\nu)=\mathrm{T}_e\sigma_\nu(\xi) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\Big(\langle\mathbf{J}^\Phi(\Phi_{\exp(s\xi)}p),\nu\rangle-\langle\mathrm{Ad}^*_{\exp(-s\xi)}\mathbf{J}^\Phi(p),\nu\rangle\Big)\\ &= \mathrm{d}J^\Phi_\nu(\xi_P)_p-\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\langle\mathbf{J}^\Phi(p),\mathrm{Ad}_{\exp(-s\xi)}\nu\rangle=(\iota_\nu\iota_{\xi_P}\omega)_p-\langle\mathbf{J}^\Phi(p),[\nu,\xi]\rangle=\{J^\Phi_\nu,J^\Phi_\xi\}(p)-J^\Phi_{[\nu,\xi]}(p),\end{aligned}$$

which establishes point 2.

Point 1. follows from the fact that $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$ both are skew-symmetric, bilinear, and satisfy the Jacobi identity. □

Recall that if the symplectic momentum map $\mathbf{J}^\Phi$ is Ad*-equivariant, then the associated cocycle vanishes, i.e. $\sigma(g)=0$ for any $g\in G$. Consequently, $\Sigma(\xi,\eta)=0$ for all $\xi,\eta\in\mathfrak{g}$. Therefore, the following corollary is an immediate consequence of the previous theorem.

Corollary 2.1.17. *Assume that $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^*$ is an Ad*-equivariant symplectic momentum map. Then,*

$$\{J^\Phi_\xi,J^\Phi_\nu\}=J_{[\xi,\nu]}, \quad \forall\xi,\nu\in\mathfrak{g}.$$

In other words, $\lambda\colon\xi\in\mathfrak{g}\mapsto J^\Phi_\xi\in\mathscr{C}^\infty(P)$ is a Lie algebra homomorphism.

This result recovers, as a special case, the general identity for Ad*-equivariant symplectic momentum maps that there exists a Lie algebra morphism $\mathfrak{g}\ni\xi\mapsto J^\Phi_\xi\in\mathscr{C}^\infty(P)$. It was previously obtained in (2.1.1) through direct computation.

The following lemma provides the generalisation of Lemma 2.1.7 for a general symplectic momentum map.

Lemma 2.1.18. *Let $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^*$ be a symplectic momentum map associated with a Lie group action $\Phi\colon G\times P\to P$ and cocycle $\sigma\colon G\to\mathfrak{g}^*$. Let G^Δ_μ be the isotropy group at $\mu\in\mathfrak{g}^*$ of the action $\Delta\colon G\times P\to P$*

- *a)* $\mathrm{T}_p(G_\mu^\Delta p) = \mathrm{T}_p(Gp) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)$,
- *b)* $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu) = \mathrm{T}_p(Gp)^{\perp_\omega}$.

Proof. As in the proof of Lemma 2.1.7, part a) follows by showing that $(\xi_P)_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu)$ if and only if $\xi \in \mathfrak{g}_\mu^\Delta$, where $\mathfrak{g}_\mu^\Delta$ denotes the Lie algebra of G_μ^Δ. By Proposition 2.1.15, the symplectic momentum map $\mathbf{J}^\Phi$ is equivariant with respect to the symplectic affine action Δ. Hence,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{J}^\Phi \circ \Phi_{\exp(t\xi)}(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Delta_{\exp(t\xi)}\mathbf{J}^\Phi(p) \Rightarrow \mathrm{T}_p\mathbf{J}^\Phi((\xi_P)_p) = (\xi_{\mathfrak{g}^*}^\Delta)_\mu,$$

where $\xi_{\mathfrak{g}^*}^\Delta$ is the fundamental vector field on $\mathfrak{g}^*$ induced by the symplectic affine Lie group action Δ corresponding to $\xi \in \mathfrak{g}$, for all $\xi \in \mathfrak{g}$ and $p \in \mathbf{J}^{\Phi-1}(\mu)$. Therefore, $(\xi_P)_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mu) = \ker(\mathrm{T}_p\mathbf{J})$ if and only if $(\xi_{\mathfrak{g}^*}^\Delta)_\mu = 0$. Suppose that $(\xi_{\mathfrak{g}^*}^\Delta)_\mu = 0$, then

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} \Delta_{\exp(t\xi)}(\mu) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} \Delta_{\exp(s\xi)}\Delta_{\exp((t-s)\xi)}(\mu) \\ &= \mathrm{T}_\mu\Delta_{\exp(s\xi)}\left(\left.\frac{\mathrm{d}}{\mathrm{d}(t-s)}\right|_{t=s} \Delta_{\exp((t-s)\xi)}\mu\right) = \mathrm{T}_\mu\Delta_{\exp(s\xi)}(\xi_{\mathfrak{g}^*}^\Delta)_\mu = 0\end{aligned}$$

and thus $(\xi_{\mathfrak{g}^*}^\Delta)_\mu = 0$ yields that $\exp(t\xi) \in G_\mu^\Delta$ for all $t \in \mathbb{R}$. Therefore, $(\xi_{\mathfrak{g}^*}^\Delta)_\mu = 0$ implies that $\xi \in \mathfrak{g}_\mu^\Delta$. This completes the proof of a).

Part b) follows identically to the corresponding statement in Lemma 2.1.7. ☐

Note that the previous lemma establishes the conditions necessary to formulate the Marsden–Meyer–Weinstein reduction for a Hamiltonian system $(P, \omega, \Phi, h, \mathbf{J}^\Phi)$ in the case where the symplectic momentum map $\mathbf{J}^\Phi$ is not Ad^*-equivariant. Consequently, the following generalisation of Theorem 2.1.8 is immediate.

Theorem 2.1.19. *Let $\Phi\colon G \times P \to P$ be a Hamiltonian Lie group action of G on the symplectic manifold (P, ω) admitting a (not necessarily* Ad^**-equivariant) symplectic momentum map $\mathbf{J}^\Phi\colon P \to \mathfrak{g}^*$. Assume that $\mu \in \mathfrak{g}^*$ is a weak regular value of $\mathbf{J}^\Phi$ and G_μ^Δ acts freely and properly on $\mathbf{J}^{\Phi-1}(\mu)$. Let $i_\mu\colon \mathbf{J}^{\Phi-1}(\mu) \hookrightarrow P$ be the natural embedding of $\mathbf{J}^{\Phi-1}(\mu)$ into P and let $\pi_\mu\colon \mathbf{J}^{\Phi-1}(\mu) \to \mathbf{J}^{\Phi-1}(\mu)/G_\mu^\Delta =: P_\mu^\Delta$ be the canonical projection. Then, there exists a unique symplectic structure ω_μ on P_μ^Δ such that*

$$\pi_\mu^*\omega_\mu = i_\mu^*\omega\,.$$

2.2 Cosymplectic Marsden–Meyer–Weinstein reduction

The main objective of this section is to present the cosymplectic Marsden–Meyer–Weinstein reduction [4]. This reduction plays a fundamental role in the cosymplectic energy-momentum method, devised in Subsection 3.2, since the associated relative equilibrium points, introduced in Subsection 3.2.1, are, roughly speaking, points projecting onto equilibrium points of a reduced Hamiltonian obtained through the cosymplectic Marsden–Meyer–Weinstein reduction.

Although some of the following results can be found in the literature, full proofs are included here to maintain a self-contained exposition, as such proofs are generally not available. In particular, the classical works [4, 101] are written in French, and the online versions of [4] are partially illegible in essential sections. Furthermore, several results presented in this section constitute natural generalisations of well-known theorems in symplectic geometry to the cosymplectic setting. The discussion begins by introducing momentum maps for cosymplectic structures.

2.2.1 Cosymplectic momentum maps

This subsection introduces the notion of Ad^*-equivariant momentum maps in a framework of cosymplectic geometry.

Definition 2.2.1. Let $\Phi\colon G \times M \to M$ be a Lie group action on a cosymplectic manifold (M, ω, η) such that $\iota_{\xi_M}\eta = 0$. A *cosymplectic momentum map* associated with a Lie group action $\Phi\colon G \times M \to M$ is a map $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^*$ defined by

$$\iota_{\xi_M}\omega = \mathrm{d}\langle \mathbf{J}^\Phi, \xi\rangle := \mathrm{d}J^\Phi_\xi, \qquad RJ^\Phi_\xi = 0, \qquad \forall \xi \in \mathfrak{g}, \tag{2.2.1}$$

where R denotes the Reeb vector field associated with (M, ω, η).

In the existing literature, it is often assumed that the cosymplectic momentum map is Ad^*-equivariant. Hence, one has the following definition that is analogous to Definition 2.1.1 in a symplectic setting.

Definition 2.2.2. A momentum map $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^*$ is Ad^**-equivariant* if

$$\mathbf{J}^\Phi \circ \Phi_g = \mathrm{Ad}^*_{g^{-1}} \circ \mathbf{J}^\Phi, \qquad \forall g \in G.$$

In other words, the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\mathbf{J}^\Phi"] \arrow[d, "\Phi_g"] & \mathfrak{g}^* \arrow[d, "\mathrm{Ad}^*_{g^{-1}}"] \\ M \arrow[r, "\mathbf{J}^\Phi"] & \mathfrak{g}^* \end{tikzcd},$$

for every $g \in G$ and $\mathrm{Ad}^*_{g^{-1}}$ being the transpose of $\mathrm{Ad}_{g^{-1}}$.

The condition $RJ^\Phi_\xi = 0$ in (2.2.1) is necessary in order to apply the cosymplectic Marsden–Meyer–Weinstein reduction theorem to be introduced in Section 2.2.3. Although this condition may appear restrictive, it is satisfied by many physically relevant systems, including certain classes of time-dependent or dissipative Hamiltonian systems. These cases will be studied in detail using the cosymplectic energy-momentum method developed in Section 3.2.

Let $\Phi\colon G \times M \to M$ be a cosymplectic Lie group action, see Definition 1.3.5. Since $\mathrm{d}\tau = 0$, the condition $\mathscr{L}_{\xi_M}\tau = 0$ implies that $\iota_{\xi_M}\tau$ takes a constant value, not necessarily zero, on M. This apparently minor detail has relevant applications in the reduction of cosymplectic manifolds to symplectic manifolds and its applications to circular restricted three-body problems (cf. [4, 105]). As shown in Section 2.4.6, the fact that $\iota_{\xi_M}\tau$ may not be zero will play a relevant role in the description of radically new types of reductions.

Note that if a Lie group action $\Phi\colon G \times M \to M$ admits a cosymplectic momentum map relative to (M, ω, η), then Φ is a cosymplectic Lie group action. However, not every cosymplectic Lie group action on M admits a momentum map. A counterexample is provided by the flow of the Reeb vector field R, which is a cosymplectic Lie group action, but it does not admit a cosymplectic momentum map relative to its associated cosymplectic manifold since $\iota_R\eta = 1 \neq 0$.

In view of (2.2.1), the Reeb vector field R is always tangent to the level sets of a momentum map $\mathbf{J}^\Phi$ relative to (M, ω, η). Nonetheless, R can not be tangent to the orbits of Φ, as that would imply $\iota_R\,\eta = 0$, contradicting the definition of R.

The following example introduces the canonical cosymplectic structure on $T \times \mathrm{T}^*Q$, where T is a one-dimensional manifold. It also shows the construction of a cosymplectic momentum map and cosymplectic Lie group action.

Example 2.2.3. Let θ_{T^*Q} be the canonical Liouville one-form and $\omega_{\mathrm{T}^*Q} = -\mathrm{d}\theta_{\mathrm{T}^*Q}$ the associated symplectic form on T^*Q. One can define canonical one- and two-forms on $T \times \mathrm{T}^*Q$ in the following way

$$\theta_{T\times\mathrm{T}^*Q} := \pi^*_{\mathrm{T}^*Q}\theta_{\mathrm{T}^*Q}, \qquad \omega_{T\times\mathrm{T}^*Q} := \pi^*_{\mathrm{T}^*Q}\omega_{\mathrm{T}^*Q},$$

where $\pi_{\mathrm{T}^*Q}\colon T \times \mathrm{T}^*Q \to \mathrm{T}^*Q$ is the canonical projection onto T^*Q. Then, $\omega_{T\times\mathrm{T}^*Q} = -\mathrm{d}\theta_{T\times\mathrm{T}^*Q}$ is closed and satisfies that $\ker\omega_{T\times\mathrm{T}^*Q}$ is a distribution of rank 1. Consequently, one has a canonical cosymplectic manifold

$$\left(T \times \mathrm{T}^*Q, \omega_{T\times\mathrm{T}^*Q}, \eta_{T\times\mathrm{T}^*Q}\right), \tag{2.2.2}$$

where $\eta_{T\times \mathrm{T}^*Q}$ is the pull-back to $T\times \mathrm{T}^*Q$ of a closed non-vanishing one-form η on T. In the particular case when $T=\mathbb{R}$, it is natural to take $\eta=\mathrm{d}t$, where t is the standard coordinate on $\mathbb{R}$.

Recall that according to Proposition 2.1.3 every Lie group action $\Phi\colon G\times Q\to Q$ induces a canonical lift $\widehat{\Phi}\colon G\times \mathrm{T}^*Q\to \mathrm{T}^*Q$ given by

$$\langle \widehat{\Phi}_g(\alpha_q), v_{\Phi_g(q)}\rangle := \langle \alpha_q, \mathrm{T}_{\Phi_g(q)}\Phi_{g^{-1}}(v_{\Phi_g(q)})\rangle,\ \forall g\in G,\ \forall q\in Q,\ \forall \alpha_q\in \mathrm{T}^*_qQ,\ \forall v_{\Phi_g(q)}\in \mathrm{T}_{\Phi_g(q)}Q.$$

This construction allows one to define a natural Lie group action $\Psi\colon G\times T\times Q\to T\times Q$ and $\widehat{\Psi}\colon G\times T\times \mathrm{T}^*Q\to T\times \mathrm{T}^*Q$ in the following manner

$$\Psi\colon G\times T\times Q\ni (g,t,q)\mapsto (t,\Phi_g(q))\in T\times Q$$

and

$$\widehat{\Psi}\colon G\times T\times \mathrm{T}^*Q\ni (g,t,\alpha_q)\mapsto (t,\widehat{\Phi}_g(\alpha_q))\in T\times \mathrm{T}^*Q\,. \tag{2.2.3}$$

If the lifted Lie group action $\widehat{\Phi}$ is symplectic relative to the canonical symplectic structure on T^*Q, then the lifted Lie group action $\widehat{\Psi}$ is a cosymplectic Lie group action relative to the cosymplectic manifold (2.2.2).

The following proposition is equivalent to Proposition 2.1.3 but in a cosymplectic setting.

Proposition 2.2.4. *Let $\Phi\colon G\times Q\to Q$ be a Lie group action. Then, the lifted Lie group action $\widehat{\Psi}\colon G\times T\times \mathrm{T}^*Q\to T\times \mathrm{T}^*Q$, defined in (2.2.3), admits an* Ad^**-equivariant cosymplectic momentum map $\mathbf{J}^{\widehat{\Psi}}\colon T\times \mathrm{T}^*Q\to \mathfrak{g}^*$ satisfying*

$$\langle \mathbf{J}^{\widehat{\Psi}},\xi\rangle = \iota_{\xi_{T\times \mathrm{T}^*Q}}\theta_{T\times \mathrm{T}^*Q},\qquad \forall \xi\in\mathfrak{g}, \tag{2.2.4}$$

with respect to the canonical cosymplectic structure $(T\times \mathrm{T}^*Q,\omega_{T\times \mathrm{T}^*Q},\eta_{T\times \mathrm{T}^*Q})$.

Proof. Since $\mathbf{J}^{\widehat{\Psi}}$, $\xi_{T\times \mathrm{T}^*Q}$, and $\theta_{T\times \mathrm{T}^*Q}$ are invariant relative to the Lie derivative with respect to the Reeb vector field R, it follows that (2.2.4) amounts to the pull-back via $\pi_{\mathrm{T}^*Q}\colon T\times \mathrm{T}^*Q\to \mathrm{T}^*Q$ of

$$\langle \mathbf{J}^{\widehat{\Phi}},\xi\rangle = \iota_{\xi_{\mathrm{T}^*Q}}\theta_{\mathrm{T}^*Q},\qquad \forall \xi\in\mathfrak{g},$$

which is a well-defined Ad^*-equivariant momentum map on T^*Q (see [2]). □

To simplify the notation, the cosymplectic manifold (M,ω,η) is frequently denoted by M^ω_η.

Definition 2.2.5. A triple $(M^\omega_\eta,h,\mathbf{J}^\Phi)$ is said to be a *G-invariant cosymplectic Hamiltonian system* if it consists of a cosymplectic manifold (M,ω,η), an associated cosymplectic Lie group action $\Phi\colon G\times M\to M$ such that $\Phi^*_g h=h$ for every $g\in G$, and a cosymplectic momentum map $\mathbf{J}^\Phi\colon M\to\mathfrak{g}^*$ related to Φ. If, additionally, the cosymplectic momentum map $\mathbf{J}^\Phi$ is Ad^*-equivariant, a G-invariant cosymplectic Hamiltonian system $(M^\omega_\eta,h,\mathbf{J}^\Phi)$ is referred to as an Ad^**-equivariant G-invariant cosymplectic Hamiltonian system.*

2.2.2 General cosymplectic momentum maps

This section develops the theory of non-Ad^*-equivariant cosymplectic momentum maps. The results presented in this subsection constitute a straightforward extension of the corresponding theory for general momentum maps on symplectic manifolds, as detailed in Subsection 2.1.1. It also presents a slight adaptation of the results by Albert in [4].

Proposition 2.2.6. *Let $(M^\omega_\eta,h,\mathbf{J}^\Phi)$ be a G-invariant cosymplectic Hamiltonian system. Define the functions*

$$\psi_{g,\xi}\colon M\ni x\mapsto J^\Phi_\xi(\Phi_g(x))-J^\Phi_{\mathrm{Ad}_{g^{-1}}\xi}(x)\in\mathbb{R},\qquad \forall g\in G,\qquad \forall\xi\in\mathfrak{g}.$$

Then, for every $g\in G$ and $\xi\in\mathfrak{g}$, the function $\psi_{g,\xi}$ is constant on M. Moreover, $\sigma\colon G\ni g\mapsto\sigma(g)\in\mathfrak{g}^*$, defined by $\langle\sigma(g),\xi\rangle:=\psi_{g,\xi}$ for all $\xi\in\mathfrak{g}$ satisfies the cocycle condition

$$\sigma(gg')=\sigma(g)+\mathrm{Ad}^*_{g^{-1}}\sigma(g'),\qquad \forall g,g'\in G. \tag{2.2.5}$$

Proof. To show that each $\psi_{g,\xi}$ is constant on M, note that

$$\begin{aligned}\mathrm{d}\psi_{g,\xi} &= \mathrm{d}[J^\Phi_\xi \circ \Phi_g] - dJ^\Phi_{\mathrm{Ad}_{g^{-1}}\xi} = \Phi_g^*(\iota_{\xi_M}\omega) - \iota_{(\mathrm{Ad}_{g^{-1}}\xi)_M}\omega \\ &= \Phi_g^*(\iota_{\xi_M}\omega) - \iota_{\Phi_{g^{-1}*}\xi_M}\omega = \iota_{\Phi_{g^{-1}*}\xi_M}\Phi_g^*\omega - \iota_{\Phi_{g^{-1}*}\xi_M}\omega = \iota_{\Phi_{g^{-1}*}\xi_M}\omega - \iota_{\Phi_{g^{-1}*}\xi_M}\omega = 0,\end{aligned}$$

where the fact that Φ is a cosymplectic Lie group action and $(\mathrm{Ad}_g\xi)_M = \Phi_{g*}\xi_M$ for every $g \in G$ and every $\xi \in \mathfrak{g}$ were used, see Lemma 2.1.2. Since each $\psi_{g,\xi}$ is constant on every connected component of M. Since M is connected by assumption, it follows that $\psi_{g,\xi}$ is constant on M for all $g \in G$ and $\xi \in \mathfrak{g}$.

Similarly to the symplectic setting, to analyse the family of mappings $\{\psi_{g,\xi}\}_{g\in G,\xi\in\mathfrak{g}}$, note that each $\psi_{g,\xi}$ can be rewritten as

$$\begin{aligned}\psi_{g,\xi}(x) &= J^\Phi_\xi(\Phi_g(x)) - J^\Phi_{\mathrm{Ad}_{g^{-1}}\xi}(x) = \langle \mathbf{J}^\Phi(\Phi_g(x)), \xi\rangle - \langle \mathbf{J}^\Phi(x), \mathrm{Ad}_{g^{-1}}\xi\rangle \\ &= \langle \mathbf{J}^\Phi(\Phi_g(x)), \xi\rangle - \langle \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi(x), \xi\rangle = \langle \mathbf{J}^\Phi(\Phi_g(x)) - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi(x), \xi\rangle,\end{aligned}$$

for all $x \in M$. Since $\langle\sigma(g), \xi\rangle = \psi_{g,\xi}$ is constant on M for every $g \in G$ and $\xi \in \mathfrak{g}$, the map σ can be expressed as

$$\sigma\colon G \ni g \mapsto \mathbf{J}^\Phi \circ \Phi_g - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi \in \mathfrak{g}^*,$$

and each $\sigma(g)$, with $g \in G$, is constant on M.

A straightforward computation, using that $\psi_{g,\xi}$ are constant, yields that

$$\begin{aligned}\sigma(gg') =& \mathbf{J}^\Phi \circ \Phi_{gg'} - \mathrm{Ad}^*_{(gg')^{-1}}\mathbf{J}^\Phi = \mathbf{J}^\Phi \circ \Phi_g \circ \Phi_{g'} - \mathrm{Ad}^*_{g^{-1}}\mathrm{Ad}^*_{g'^{-1}}\mathbf{J}^\Phi \\ =& \mathbf{J}^\Phi \circ \Phi_g \circ \Phi_{g'} - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi \circ \Phi_{g'} + \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi \circ \Phi_{g'} - \mathrm{Ad}^*_{g^{-1}}\mathrm{Ad}^*_{g'^{-1}}\mathbf{J}^\Phi \\ =& \mathbf{J}^\Phi \circ \Phi_g - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi + \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}^\Phi \circ \Phi_{g'} - \mathrm{Ad}^*_{g'^{-1}}\mathbf{J}^\Phi) = \sigma(g) + \mathrm{Ad}^*_{g^{-1}}\sigma(g')\end{aligned}$$

for every $g, g' \in G$, which proves (2.2.5). ☐

As in the symplectic case, the map (2.2.2) is called the *co-adjoint cocycle* associated with the cosymplectic momentum map $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^*$. Again, analogously to the symplectic setting, a cosymplectic momentum map $\mathbf{J}^\Phi$ is Ad^*-equivariant if and only if $\sigma = 0$. Therefore, σ measures the lack of Ad^*-equivariance of a cosymplectic momentum map.

The following terminology is the same as in the symplectic setting. A map $\sigma\colon G \to \mathfrak{g}^*$ is a *coboundary* if there exists $\mu \in \mathfrak{g}^*$ such that

$$\sigma(g) = \mu - \mathrm{Ad}^*_{g^{-1}}\mu, \qquad \forall g \in G\,.$$

Every coboundary satisfies (2.2.5) and is therefore a *co-adjoint cocycle*. The space of co-adjoint cocycles admits an equivalence relation, whose equivalence classes are called *cohomology classes*, given by setting that two co-adjoint cocycles belong to the same cohomology class if their difference is a coboundary. The following proposition shows that any two cosymplectic momentum maps associated with a given cosymplectic Lie group action induce a well-defined cohomology class $[\sigma]$.

Proposition 2.2.7. *Let $\Phi\colon G \times M \to M$ be a cosymplectic Lie group action relative to (M, ω, η). If $\mathbf{J}^\Phi_1$ and $\mathbf{J}^\Phi_2$ are two cosymplectic momentum maps related to Φ with co-adjoint cocycles σ_1 and σ_2, respectively, then $[\sigma_1] = [\sigma_2]$.*

Proof. From the definition of the co-adjoint cocycles corresponding to $\mathbf{J}^\Phi_1$ and $\mathbf{J}^\Phi_2$, one has

$$\langle\sigma_1(g) - \sigma_2(g), \xi\rangle = \langle \mathbf{J}^\Phi_1 \circ \Phi_g - \mathbf{J}^\Phi_2 \circ \Phi_g, \xi\rangle - \langle \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}^\Phi_1 - \mathbf{J}^\Phi_2), \xi\rangle,$$

for all $g \in G$ and $\xi \in \mathfrak{g}$. Since $\mathbf{J}^\Phi_1$ and $\mathbf{J}^\Phi_2$ are both momentum maps related to the same cosymplectic Lie group action Φ, the difference $\mathbf{J}^\Phi_1 - \mathbf{J}^\Phi_2$ is a constant map with value $\mu \in \mathfrak{g}^*$. Indeed,

$$\mathrm{d}\langle \mathbf{J}^\Phi_1 - \mathbf{J}^\Phi_2, \xi\rangle = \mathrm{d}J^\Phi_{1,\xi} - \mathrm{d}J^\Phi_{2,\xi} = \iota_{\xi_M}\omega - \iota_{\xi_M}\omega = 0, \qquad \forall \xi \in \mathfrak{g}.$$

Consequently, $(\mathbf{J}^\Phi_1 - \mathbf{J}^\Phi_2) \circ \Phi_g = \mathbf{J}^\Phi_1 - \mathbf{J}^\Phi_2$ for every $g \in G$ and then

$$\sigma_1(g) - \sigma_2(g) = \mu - \mathrm{Ad}^*_{g^{-1}}\mu, \qquad \forall g \in G\,.$$

□

Proposition 2.2.7 implies that a cosymplectic Lie group action admits an Ad*-equivariant momentum map $\mathbf{J}^\Phi$ if and only if its associated coadjoint cocycle is a coboundary. Indeed, if a cosymplectic Lie group action has an Ad*-equivariant momentum map $\mathbf{J}^\Phi_2$ relative to (M, ω, η), then its associated co-adjoint cocycle satisfies $\sigma_2 = 0$, and any other momentum map $\mathbf{J}^\Phi_1$ for the same action is such that its co-adjoint cocycle, σ_1, satisfies $[\sigma_1] = [\sigma_2] = 0$, and σ_1 becomes a coboundary.

Conversely, if a momentum map $\mathbf{J}^\Phi_1$ admits a coboundary σ_1 associated with $\mu \in \mathfrak{g}^*$, then the momentum map

$$\mathbf{J}^\Phi := \mathbf{J}^\Phi_1 - \mu,$$

is an Ad*-equivariant momentum map for the same cosymplectic Lie group action of $\mathbf{J}^\Phi_1$, where $\mu \in \mathfrak{g}^*$ satisfies that $\sigma_1(g) = \mu - \mathrm{Ad}^*_{g^{-1}}\mu$ for every $g \in G$. Indeed,

$$\langle \mathbf{J}^\Phi, \xi \rangle = \langle \mathbf{J}^\Phi_1, \xi \rangle - \langle \mu, \xi \rangle = J^\Phi_{1,\xi} - \langle \mu, \xi \rangle, \qquad \forall \xi \in \mathfrak{g},$$

and

$$\sigma(g) = \mathbf{J}^\Phi \circ \Phi_g - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi = \sigma_1(g) + \mathrm{Ad}^*_{g^{-1}}\mu - \mu = 0,$$

for every $g \in G$.

To summarise, if a co-adjoint cocycle associated with a given momentum map is a coboundary, then it is possible to construct an Ad*-equivariant cosymplectic momentum map. The following proposition shows, however, that for any momentum map, there exists a Lie group action $\Delta\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$ such that the momentum map becomes Δ-*equivariant*. That is, for every $g \in G$, the following diagram is commutative

$$\begin{tikzcd} M \arrow[r, "\Phi_g"] \arrow[d, "\mathbf{J}^\Phi"] & M \arrow[d, "\mathbf{J}^\Phi"] \\ \mathfrak{g}^* \arrow[r, "\Delta_g"] & \mathfrak{g}^*. \end{tikzcd}$$

One sees that the result is analogous to Proposition 2.1.15 and follows from the same techniques.

Proposition 2.2.8. *Let $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^*$ be a cosymplectic momentum map for a cosymplectic Lie group action $\Phi\colon G \times M \to M$ with associated co-adjoint cocycle σ. Then,*

1. *the map $\Delta\colon G \times \mathfrak{g}^* \ni (g, \mu) \mapsto \Delta_g(\mu) := \mathrm{Ad}^*_{g^{-1}}\mu + \sigma(g) \in \mathfrak{g}^*$ is a Lie group action,*
2. *the momentum map $\mathbf{J}^\Phi$ is Δ-equivariant.*

Proof. First, since $\sigma(e) = 0$, one has $\Delta(e, \mu) = \mathrm{Ad}^*_{e^{-1}}\mu + \sigma(e) = \mu$ and,

$$\begin{aligned} \Delta(g, \Delta(g', \mu)) &= \mathrm{Ad}^*_{g^{-1}}(\mathrm{Ad}^*_{g'^{-1}}\mu + \sigma(g')) + \sigma(g) = \mathrm{Ad}^*_{g^{-1}}\mathrm{Ad}^*_{g'^{-1}}\mu + \mathrm{Ad}^*_{g^{-1}}\sigma(g') + \sigma(g) \\ &= \mathrm{Ad}^*_{(gg')^{-1}}\mu + \mathrm{Ad}^*_{g^{-1}}\sigma(g') + \sigma(g) = \mathrm{Ad}^*_{(gg')^{-1}}\mu + \sigma(gg') = \Delta(gg', \mu). \end{aligned}$$

Thus, Δ is a Lie group action on $\mathfrak{g}^*$, which proves 1. Second, from the definition of Δ and σ, one gets

$$\Delta_g \circ \mathbf{J}^\Phi = \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi + \sigma(g) = \mathbf{J}^\Phi \circ \Phi_g, \qquad \forall g \in G,$$

which shows that $\mathbf{J}^\Phi$ is Δ-equivariant. □

Proposition 2.2.8 ensures that a general cosymplectic momentum map $\mathbf{J}^\Phi$ gives rise to an equivariant momentum map relative to a new action $\Delta\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$, called a *cosymplectic affine action.* In a particular case where $\mathbf{J}^\Phi$ is Ad^*-invariant, it follows that $\sigma = 0$ and $\Delta = \mathrm{Ad}^*$ reduces the co-adjoint action of G.

The next result, which is a generalisation of Theorem 2.1.16 to a cosymplectic setting, characterises the Poisson bracket of functions $\{J_\xi\}^\Phi_{\xi\in\mathfrak{g}}$ associated with a cosymplectic momentum map $\mathbf{J}^\Phi$.

Theorem 2.2.9. *Let $\Phi\colon G \times M \to M$ be a cosymplectic Lie group action on (M, ω, η), and let $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^*$ be a corresponding cosymplectic momentum map with coadjoint cocycle $\sigma\colon G \to \mathfrak{g}^*$. Define*

$$\sigma_\nu\colon G \ni g \mapsto \langle \sigma(g), \nu\rangle \in \mathbb{R}, \qquad \Sigma\colon \mathfrak{g} \times \mathfrak{g} \ni (\xi_1, \xi_2) \mapsto \mathrm{T}_e\sigma_{\xi_2}(\xi_1) \in \mathbb{R}, \qquad \forall \nu \in \mathfrak{g}.$$

Then,

1. *the map Σ is a skew-symmetric bilinear form on $\mathfrak{g}$ satisfying the following identity*

$$\Sigma(\xi, [\zeta, \nu]) + \Sigma(\nu, [\xi, \zeta]) + \Sigma(\zeta, [\nu, \xi]) = 0, \qquad \forall \xi, \zeta, \nu \in \mathfrak{g},$$

2. *For all $\xi, \nu \in \mathfrak{g}$ the relation $\Sigma(\xi, \nu) = \{J_\nu, J_\xi\}_{\omega,\eta} - J_{[\nu,\xi]}$ holds.*

Proof. To establish 2, consider the tangent map of σ_ν at e. Then,

$$\begin{aligned}\Sigma(\xi, \nu) = \mathrm{T}_e\sigma_\nu(\xi) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} & \left(\langle \mathbf{J}^\Phi(\Phi_{\exp(s\xi)}(x)), \nu\rangle - \langle \mathrm{Ad}^*_{\exp(-s\xi)}\mathbf{J}^\Phi(x), \nu\rangle\right) \\ &= \mathrm{d}J_\nu(\xi_M)_x - \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \langle \mathbf{J}^\Phi(x), \mathrm{Ad}_{\exp(-s\xi)}\nu\rangle \\ &\qquad = -(\iota_{\nu_M}\iota_{\xi_M}\omega)_x - \langle \mathbf{J}^\Phi(x), [\nu, \xi]\rangle = \{J_\nu, J_\xi\}_{\omega,\eta}(x) - J_{[\nu,\xi]}(x),\end{aligned}$$

where the last equality stems from (1.3.8).

Since $X_{\{J^\Phi_\nu, J^\Phi_\xi\}_{\omega,\eta}} = -[X_{J^\Phi_\nu}, X_{J^\Phi_\xi}] = -[\nu_M, \xi_M] = [\nu, \xi]_M$. Hence, $X_{\{J^\Phi_\nu, J^\Phi_\xi\}_{\omega,\eta}}$ coincides with $X_{J^\Phi_{[\nu,\xi]}}$, implying that the functions $\{J^\Phi_\nu, J^\Phi_\xi\}_{\omega,\eta}$ and $J^\Phi_{[\nu,\xi]}$ differ by a constant. Consequently, Σ does not depend on $x \in M$, which proves 2.

To prove 1., consider

$$-\Sigma(\xi, [\zeta, \nu]) = \{J_\xi, J_{[\zeta,\nu]}\}_{\omega,\eta} - J_{[\xi,[\zeta,\nu]]} = \{J_\xi, \{J_\zeta, J_\nu\}_{\omega,\eta} - \Sigma(\nu, \zeta)\}_{\omega,\eta} - J_{[\xi,[\zeta,\nu]]}\,.$$

The desired identity follows from the fact that $\{\cdot, \cdot\}_{\omega,\eta}$ and $[\cdot, \cdot]$ are anti-symmetric, bilinear, and satisfy the Jacobi identity. □

If a cosymplectic momentum map $\mathbf{J}^\Phi$ is Ad^*-equivariant, then $\sigma(g) = 0$ for every $g \in G$, which in turn implies that $\Sigma(\xi, \nu) = 0$ for all $\xi, \nu \in \mathfrak{g}$. In this case, part 2. of Theorem 2.2.9 recovers the standard result that there exists a Lie algebra homomorphism $\mathfrak{g} \ni \xi \mapsto J^\Phi_\xi \in \mathscr{C}^\infty(M)$.

2.2.3 Cosymplectic Marsden–Meyer–Weinstein reduction theorem

This subsection is devoted to the formulation of the Marsden–Meyer–Weinstein reduction theorem in the cosymplectic setting [4]. Although the results were originally established by C. Albert in [4], the present exposition broadens its applicability by incorporating cosymplectic momentum maps that are not necessarily Ad^*-equivariant. This generalisation extends the range of potential applications beyond those considered in the original paper [4]. Moreover, since [4] is written in French, and the only online version that is accessible is scanned in poor quality, a complete and self-contained presentation is provided here. The result presented in this subsection also plays a fundamental role in the cosymplectic energy-momentum method developed in Section 3.2.

The following results are natural extensions to the cosymplectic setting of their symplectic analogues presented in Section 2.1. Although some Proofs of the following results are available in [4], that reference is not widely accessible (some of them contain typographical issues). For the sake of clarity and completeness, full proofs are included below.

The next proposition shows that the cosymplectic momentum map $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^*$ associated with $(M^\omega_\tau, h, \mathbf{J}^\Phi)$ is preserved under the flows generated by the vector fields ∇h, X_h, and E_h. It is worth noting that Proposition 2.2.10 appears as a novel contribution, although it extends a partial result presented in [82].

Proposition 2.2.10. *Let $(M^\omega_\tau, h, \mathbf{J}^\Phi)$ be a G-invariant cosymplectic Hamiltonian system and let a map $F\colon (s,x) \in \mathbb{R}\times M \mapsto F_s(x) = F(s,x) \in M$ denote the flow of ∇h. Then, $\mathbf{J}^\Phi \circ F_s = \mathbf{J}^\Phi$ for all $s \in \mathbb{R}$. Analogous statement holds for the flows of E_h and X_h.*

Proof. The G-invariance of h implies that $\xi_M h = 0$ for every $\xi \in \mathfrak{g}$. Consequently,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} J^\Phi_\xi \circ F_s = \iota_{\nabla h}\mathrm{d}J^\Phi_\xi = \iota_{X_h+(Rh)R}\mathrm{d}J^\Phi_\xi = \iota_{X_h}\mathrm{d}J^\Phi_\xi = \iota_{X_h}\iota_{\xi_M}\omega = \iota_{\xi_M}((Rh)\eta - \mathrm{d}h) = 0,$$

for every $\xi \in \mathfrak{g}$. Thus, $J^\Phi_\xi \circ F_s = J^\Phi_\xi$, for every $\xi \in \mathfrak{g}$ and every $s \in \mathbb{R}$, which yields $\mathbf{J}^\Phi \circ F_s = \mathbf{J}^\Phi$ for all $s \in \mathbb{R}$.

Similarly, let L be the flow of E_h. Then,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} J^\Phi_\xi \circ L_s = \iota_{E_h}\mathrm{d}J^\Phi_\xi = \iota_{X_h+R}\,\mathrm{d}J^\Phi_\xi = \iota_{X_h}\mathrm{d}J^\Phi_\xi = \iota_{X_h}\iota_{\xi_M}\omega = \iota_{\xi_M}((Rh)\eta - \mathrm{d}h) = 0,$$

for every $\xi \in \mathfrak{g}$ and thus $\mathbf{J}^\Phi \circ L_s = \mathbf{J}^\Phi$ for every $s \in \mathbb{R}$. Since $\iota_{X_h+R}\mathrm{d}J^\Phi_\xi = \iota_{X_h}\mathrm{d}J^\Phi_\xi$, it follows that $\mathbf{J}^\Phi \circ K_s = \mathbf{J}^\Phi$ for the diffeomorphisms K_s of the one-parametric group of diffeomorphisms of X_h. ☐

Remark 2.2.11. Recall that, for $(M := \mathbb{R}\times P, \omega_P, \tau)$, the vector field X_h on M may be viewed as a time-dependent vector field on P. The integral curves of X_h, considered in this context, coincide with the integral curves of E_h of the form $t \mapsto (t, x(t))$. Consequently, Proposition 2.2.10 also applies to the time-dependent vector field X_h, a fact already established in [54].

The next lemma generalises a standard result in symplectic geometry and is fundamental in formulating the cosymplectic version of the Marsden–Meyer–Weinstein reduction theorem.

Recall that according to Definition 2.1.4 a *weak regular value* of $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^*$ is a point $\mu \in \mathfrak{g}^*$ such that $\mathbf{J}^{\Phi-1}(\mu)$ is a submanifold in M and $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\mu) = \ker \mathrm{T}_x\mathbf{J}^\Phi$ for every $x \in \mathbf{J}^{\Phi-1}(\mu)$. It is hereafter assumed that $\mu \in \mathfrak{g}^*$ is a weak regular value of a cosymplectic momentum map $\mathbf{J}^\Phi$. Additionally, it is also assumed that the isotropy subgroup G^Δ_μ of $\mu \in \mathfrak{g}^*$ relative to the cosymplectic affine Lie group action $\Delta\colon G\times\mathfrak{g}^* \to \mathfrak{g}^*$ acts via Φ on $\mathbf{J}^{\Phi-1}(\mu)$ in a quotientable manner, namely $\mathbf{J}^{\Phi-1}(\mu)/G^\Delta_\mu$ is a manifold and the projection $\pi_\mu\colon \mathbf{J}^{\Phi-1}(\mu) \to \mathbf{J}^{\Phi-1}(\mu)/G^\Delta_\mu$ is a submersion. A sufficient condition for $\mathbf{J}^{\Phi-1}(\mu)/G^\Delta_\mu$ to be a manifold is that G^Δ_μ acts freely and properly on $\mathbf{J}^{\Phi-1}(\mu)$ (see [2, 4, 105] for details).

Lemma 2.2.12. *Let $\mu \in \mathfrak{g}^*$ be a weak regular value of a cosymplectic momentum map $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^*$ and let G^Δ_μ be the isotropy group at $\mu \in \mathfrak{g}^*$ of the cosymplectic affine Lie group action $\Delta\colon G\times\mathfrak{g}^* \to \mathfrak{g}^*$ relative to the co-adjoint cocycle $\sigma\colon G \to \mathfrak{g}^*$ of $\mathbf{J}^\Phi$. Then, for any $x \in \mathbf{J}^{\Phi-1}(\mu)$, the following hold*

1. $\mathrm{T}_x(G^\Delta_\mu x) = \mathrm{T}_x(Gx) \cap \mathrm{T}_x\mathbf{J}^{\Phi-1}(\mu)$,
2. $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\mu) = \mathrm{T}_x\,(Gx)^{\perp_\omega}$,
3. $\left(\mathrm{T}_x\mathbf{J}^{\Phi-1}(\mu)\right)^{\perp_\omega} = \mathrm{T}_x(Gx) \oplus \langle R_x\rangle$.

Proof. To prove 1., let $(\xi_M)_x \in \mathrm{T}_x\mathbf{J}^{\Phi-1}(\mu)$. Since $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\mu) = \ker \mathrm{T}_x\mathbf{J}^\Phi$, then,

$$(\iota_{\xi_M}\mathrm{d}J^\Phi_\nu)_x = \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} J^\Phi_\nu(\Phi(\exp(u\xi), x)) = \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} \mathbf{J}^\Phi(\Phi(\exp(u\xi), x)), \nu\right\rangle = \left\langle \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_{u=0} \Delta_{\exp(u\xi)}\mathbf{J}^\Phi(x), \nu\right\rangle = 0,$$

for every $\nu \in \mathfrak{g}$ if and only if $\xi \in \mathfrak{g}_\mu^\Delta$, where $\mathfrak{g}_\mu^\Delta$ is the Lie algebra of G_μ^Δ.

To prove 2., recall that

$$\omega_x((\xi_M)_x, v_x) = (\mathrm{d}J_\xi^\Phi)_x(v_x) = \left\langle \mathrm{T}_x\mathbf{J}^\Phi(v_x), \xi \right\rangle, \qquad \forall x \in M, \quad \forall v_x \in \mathrm{T}_xM, \quad \forall \xi \in \mathfrak{g}.$$

Therefore, $v_x \in \ker \mathrm{T}_x\mathbf{J}^\Phi = \mathrm{T}_x\mathbf{J}^{\Phi-1}(\mu)$ if and only if $\langle \mathrm{T}_x\mathbf{J}^\Phi(v_x), \xi\rangle = 0$ for all $\xi \in \mathfrak{g}$, and consequently $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\mu) = (\mathrm{T}_x(Gx))^{\perp_\omega}$ for all $x \in \mathbf{J}^{\Phi-1}(\mu)$, as claimed.

For 3., let $X = \xi_M + \lambda R$ for any $\lambda \in \mathbb{R}$. Then, for any $v_x \in \ker T_x\mathbf{J}^\Phi$, it follows that

$$\omega_x(X_x, v_x) = (\mathrm{d}J_\xi^\Phi)_x(v_x) = 0, \qquad \forall \xi \in \mathfrak{g},$$

and $\mathrm{T}_x(Gx) \oplus \langle R_x \rangle \subset \left(\mathrm{T}_x\mathbf{J}^{\Phi-1}(\mu)\right)^{\perp_\omega}$. On the other hand, for every $x \in \mathbf{J}^{\Phi-1}(\mu)$, since $R_x \in T_x\mathbf{J}^{\Phi-1}(\mu)$ but is not tangent to Gx, it follows that

$$\left(\mathrm{T}_x\mathbf{J}^{\Phi-1}(\mu)\right)^{\perp_\omega} = (\mathrm{T}_x(Gx))^{\perp_\omega\perp_\omega} = \mathrm{T}_x(Gx) \oplus \langle R_x \rangle, \qquad \forall x \in \mathbf{J}^{\Phi-1}(\mu).$$

This completes the proof. □

The following theorem extends the classical Marsden–Meyer–Weinstein reduction theorem to the cosymplectic framework. It follows the ideas of the proof given in [4].

Theorem 2.2.13. *Let $\Phi\colon G \times M \to M$ be a cosymplectic Lie group action on a cosymplectic manifold (M, ω, τ) associated with a cosymplectic momentum map $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^*$. Assume that $\mu \in \mathfrak{g}^*$ is a weak regular value of $\mathbf{J}^\Phi$ and let $\mathbf{J}^{\Phi-1}(\mu)$ be quotientable, i.e. $M_\mu^\Delta := \mathbf{J}^{\Phi-1}(\mu)/G_\mu^\Delta$ is a manifold and $\pi_\mu\colon \mathbf{J}^{\Phi-1}(\mu) \to M_\mu^\Delta$ is a submersion. Let $i_\mu\colon \mathbf{J}^{\Phi-1}(\mu) \hookrightarrow M$ be the natural immersion and let $\pi_\mu\colon \mathbf{J}^{\Phi-1}(\mu) \to M_\mu^\Delta$ be the canonical projection. Then, there exists a unique cosymplectic manifold $(M_\mu^\Delta, \omega_\mu, \tau\mu)$ such that*

$$i_\mu^*\omega = \pi_\mu^*\omega_\mu, \qquad i_\mu^*\tau = \pi_\mu^*\tau_\mu. \tag{2.2.6}$$

Proof. The quotient space $M_\mu^\Delta = \mathbf{J}^{\Phi-1}(\mu)/G_\mu^\Delta$ is a manifold, since $\mathbf{J}^{\Phi-1}(\mu)$ is assumed to be quotientable. Meanwhile, $\pi_\mu\colon \mathbf{J}^{\Phi-1}(\mu) \to M_\mu^\Delta$ is a surjective submersion by assumption. Consequently, $\ker \mathrm{T}\pi_\mu$ is a subbundle of $\mathrm{T}\mathbf{J}^{\Phi-1}(\mu)$.

Given that Φ_g is a cosymplectic Lie group action for every $g \in G$, it follows that $\mathscr{L}_{\xi_M}\omega = 0$ and $\mathscr{L}_{\xi_M}\tau = 0$ for every $\xi \in \mathfrak{g}$. Thus, $\mathscr{L}_{\xi_{\mathbf{J}^{\Phi-1}(\mu)}} i_\mu^*\omega = 0$ and $\mathscr{L}_{\xi_{\mathbf{J}^{\Phi-1}(\mu)}} i_\mu^*\tau = 0$ for every $\xi \in \mathfrak{g}_\mu^\Delta$, where $\mathfrak{g}_\mu^\Delta$ is the Lie algebra of G_μ^Δ and $\xi_{\mathbf{J}^{\Phi-1}(\mu)}$ is the fundamental vector field of the restriction of the action of G_μ^Δ to $\mathbf{J}^{\Phi-1}(\mu)$ via Φ.

Moreover, for any vector field $Y_{\mathbf{J}^{\Phi-1}(\mu)}$ on $\mathbf{J}^{\Phi-1}(\mu)$ and tangent to $\mathbf{J}^{\Phi-1}(\mu)$, one can consider some vector field Y on M coinciding with $Y_{\mathbf{J}^{\Phi-1}(\mu)}$ on $\mathbf{J}^{\Phi-1}(\mu)$. Then,

$$\iota_{Y_{\mathbf{J}^{\Phi-1}(\mu)}} \iota_{\xi_{\mathbf{J}^{\Phi-1}(\mu)}} i_\mu^*\omega = i_\mu^*(\iota_Y \iota_{\xi_M}\omega) = i_\mu^*(\iota_Y \mathrm{d}J_\xi) = 0\,,$$

and

$$\iota_{\xi_{\mathbf{J}^{\Phi-1}(\mu)}} i_\mu^*\tau = i_\mu^*(\iota_{\xi_M}\tau) = 0\,.$$

These conditions guarantee the existence of $\omega_\mu \in \Omega^2(M_\mu^\Delta)$ and $\tau\mu \in \Omega^1(M_\mu^\Delta)$ satisfying (2.2.6). Since π_μ^* is injective, the forms ω_μ and $\tau\mu$ are uniquely determined. The fact that both ω and τ are closed ensures that ω_μ and $\tau\mu$ are also closed.

By Definition 2.2.1, one has $\iota_R \mathrm{d}J_\xi^\Phi = 0$ for every $\xi \in \mathfrak{g}$, implying that the Reeb vector field R is tangent to $\mathbf{J}^{\Phi-1}(\mu)$. Consequently, there exists a vector field $\widetilde{R}$ on $\mathbf{J}^{\Phi-1}(\mu)$ such that $\widetilde{R} = R|_{\mathbf{J}^{\Phi-1}(\mu)}$. Since $\Phi_{g*}\widetilde{R} = \widetilde{R}$ and $\mathscr{L}_{\xi_{\mathbf{J}^{\Phi-1}(\mu)}}\widetilde{R} = 0$ for every $g \in G_\mu^\Delta$ and $\xi \in \mathfrak{g}_\mu^\Delta$, it follows that there exists a well-defined vector field R_μ on M_μ^Δ such that $R_\mu = \pi_{\mu*}\widetilde{R}$. Moreover,

$$\pi_\mu^*(\iota_{R_\mu}\tau_\mu) = \iota_{\widetilde{R}}\pi_\mu^*\tau_\mu = i_\mu^*(\iota_R\tau) = 1\,,$$

and

$$\pi_\mu^*(\iota_{R_\mu}\omega_\mu) = \iota_{\widetilde{R}}\pi_\mu^*\omega_\mu = i_\mu^*(\iota_R\omega) = 0\,.$$

Thus, $\iota_{R_\mu}\tau_\mu = 1$ and $\iota_{R_\mu}\omega_\mu = 0$. To prove that $\ker\omega_\mu \oplus \ker\tau_\mu = \mathrm{T}M_\mu^\Delta$, it is sufficient to show that the map

$$\flat_\mu\colon X_\mu \in \mathrm{T}M_\mu^\Delta \mapsto \iota_{X_\mu}\omega_\mu + (\iota_{X_\mu}\tau_\mu)\tau_\mu \in \mathrm{T}^*M_\mu^\Delta$$

is an isomorphism. To prove injectivity, suppose that X_μ takes values in $\ker\flat_\mu$. Since $\iota_{R_\mu}\flat_\mu(X_\mu) = 0$, then $\iota_{X_\mu}\tau_\mu = 0$ and $\iota_{X_\mu}\omega_\mu = 0$. Consequently, there exist $\widetilde{X} \in \mathfrak{X}(\mathbf{J}^{\Phi-1}(\mu))$ and $X \in \mathfrak{X}(M)$, such that $\pi_{\mu*}\widetilde{X} = X_\mu$ and $\widetilde{X} = X|_{\mathbf{J}^{\Phi-1}(\mu)}$. Then, $\pi_\mu^*(\iota_{X_\mu}\omega_\mu) = \iota_X\omega|_{\mathbf{J}^{\Phi-1}(\mu)} = 0$ implies that X takes values in $\left(\mathrm{T}_x(\mathbf{J}^{\Phi-1}(\mu))\right)^{\perp_\omega} = \mathrm{T}_x(Gx)\oplus\langle R_x\rangle$ for all $x \in \mathbf{J}^{\Phi-1}(\mu)$. Therefore, $X_x = (\xi_M)_x + \lambda R_x$ for some $\xi \in \mathfrak{g}_\mu^\Delta$ and $\lambda \in \mathbb{R}$ depending on $x \in \mathbf{J}^{\Phi-1}(\mu)$. Since $\tau_\mu(\mathrm{T}_x\pi_\mu X_x) = 0$, one gets $\lambda = 0$. Then, $(X_\mu)_{\pi_\mu(x)} = \mathrm{T}_x\pi_\mu X_x = \mathrm{T}_x\pi_\mu(\xi_M)_x = 0$. Thus, $\ker\flat_\mu = 0$, proving that $\flat_\mu$ is injective. Moreover, $\flat_\mu$ as an injective bundle morphism between vector bundles of equal rank, it must also be surjective. Consequently, $(M_\mu^\Delta, \omega_\mu, \tau_\mu)$ is a cosymplectic manifold. □

The following result is of particular relevance for the physical applications of the cosymplectic energy-momentum method to be developed in the following chapter. Note that the existence of a cosymplectic momentum map associated with $\Phi\colon G\times M \to M$ relative to $(M := T\times P, \omega_P, \tau_T)$ implies that Φ can be restricted to a well-defined Lie group action of G on P. This follows from the fact that the fundamental vector fields of Φ are required to take values in $\ker\tau$.

Corollary 2.2.14. *Assume the hypothesis of Theorem 2.2.13 hold. Furthermore, suppose that the cosymplectic manifold is given by $(T\times P, \omega_P, \tau_T)$. Then,*

$$\mathbf{J}^{\Phi-1}(\mu) \simeq T\times\pi_P(\mathbf{J}^{\Phi-1}(\mu)), \qquad M_\mu^\Delta \simeq T\times P_\mu^\Delta,$$

where $P_\mu^\Delta := \pi_P(\mathbf{J}^{\Phi-1}(\mu))/G_\mu^\Delta$.

Proof. By Definition 2.2.1, one has $\iota_R\mathrm{d}J_\xi^\Phi = 0$ and $\mathscr{L}_R\mathrm{d}J_\xi^\Phi = 0$ for every $\xi\in\mathfrak{g}$. Hence, $\mathrm{d}J_\xi^\Phi$ is a basic one-form with respect to the projection $\pi_P\colon \mathbf{J}^{\Phi-1}(\mu) \to P_\mu^\Delta$. Therefore, for each $\xi\in\mathfrak{g}$, there exists $\widetilde{J}_\xi \in \mathscr{C}^\infty(P)$ such that $\pi_P^*\widetilde{J}_\xi^\Phi = J_\xi^\Phi$. Consequently, there exists $\widetilde{\mathbf{J}}^\Phi\colon P\to\mathfrak{g}^*$ satisfying $\mathbf{J}^\Phi = \widetilde{\mathbf{J}}^\Phi\circ\pi_P$, and hence $\mathbf{J}^{\Phi-1}(\mu) = T\times\widetilde{\mathbf{J}}^{\Phi-1}(\mu) = T\times\pi_P(\mathbf{J}^{\Phi-1}(\mu))$.

Let $\widetilde{R}$ denote the restriction of R to $\mathbf{J}^{\Phi-1}(\mu)$. Note that $\Phi^\mu\colon G_\mu^\Delta\times T\times\mathbf{J}^{\Phi-1}(\mu)\to T\times\mathbf{J}^{\Phi-1}(\mu)$ is a well-defined Lie group action obtained by restricting the action Φ of G_μ^Δ on $T\times P$ to $T\times\mathbf{J}^{\Phi-1}(\mu)$. Since $\iota_{\xi_{\mathbf{J}^{\Phi-1}(\mu)}}i_\mu^*\tau_T = 0$ for every $\xi\in\mathfrak{g}_\mu^\Delta$, there exists a Lie group action $\widetilde{\Phi}^\mu\colon G_\mu^\Delta\times\widetilde{\mathbf{J}}^{\Phi-1}(\mu)\to\widetilde{\mathbf{J}}^{\Phi-1}(\mu)$ such that $\widetilde{\Phi}_g^\mu\circ\pi_P = \pi_P\circ\Phi_g^\mu$ for every $g\in G_\mu^\Delta$. Therefore, $\Phi_g^\mu(t,p) = (t,\widetilde{\Phi}_g^\mu(p))$ for every $t\in T$ and $p\in\widetilde{\mathbf{J}}^{\Phi-1}(\mu)$. Thus, $\mathbf{J}^{\Phi-1}(\mu)/G_\mu^\Delta = (T\times\widetilde{\mathbf{J}}^{\Phi-1}(\mu))/G_\mu^\Delta = T\times(\widetilde{\mathbf{J}}^{\Phi-1}(\mu)/G_\mu^\Delta)$. □

Proposition 2.2.15. *Let the assumptions of Theorem 2.2.13 hold for $(M_\tau^\omega, h, \mathbf{J}^\Phi)$. Then, the restriction of E_h to $\mathbf{J}^{\Phi-1}(\mu)$ is projectable onto $M_\mu^\Delta = \mathbf{J}^{\Phi-1}(\mu)/G_\mu^\Delta$ and $\pi_{\mu*}(E_h|_{\mathbf{J}^{\Phi-1}(\mu)}) = E_{k_\mu}$, where k_μ is the only function on M_μ^Δ such that $\pi_\mu^*k_\mu = \iota_\mu^*h$.*

Proof. By Proposition 2.2.10, the vector field E_h is tangent to $\mathbf{J}^{\Phi-1}(\mu)$. Moreover, for every $\xi\in\mathfrak{g}_\mu^\Delta$, one has $\xi_M = X_{J_\xi^\Phi}$. Since $RJ_\xi^\Phi = 0$, Proposition 1.3.10 yields that $[\xi_M, R] = 0$. Therefore,

$$[\xi_M, E_h] = [\xi_M, R + X_h] = [\xi_M, X_h]\,, \qquad \forall\xi\in\mathfrak{g}\,.$$

Using (1.3.9), one gets

$$[\xi_M, X_h] = X_{\{h, J_\xi^\Phi\}} = X_{\xi_M h} = 0\,.$$

Hence, $E_h|_{\mathbf{J}^{\Phi-1}(\mu)}$ is projectable onto M_μ^Δ. By Theorem 2.2.13, the differential forms $i_\mu^*\omega$ and $i_\mu^*\tau$ are also projectable, and from the proof of that theorem, it follows

$$\iota_{\pi_{\mu*}(E_h|_{\mathbf{J}^{\Phi-1}(\mu)})}\omega_\mu = \mathrm{d}k_\mu - (R_\mu k_\mu)\tau_\mu\,, \qquad \iota_{\pi_{\mu*}(E_h|_{\mathbf{J}^{\Phi-1}(\mu)})}\tau_\mu = 1\,,$$

where the Hamiltonian function $k_\mu \in \mathscr{C}^\infty(M_\mu^\Delta)$ is determined uniquely by the condition $\pi_\mu^* k_\mu = i_\mu^* h$, which holds since h is invariant relative to G_μ^Δ. It follows that $\pi_{\mu *} E_h|_{\mathbf{J}^{\Phi-1}(\mu)}$ is an evolutionary vector field with respect to the reduced cosymplectic manifold $(M_\mu^\Delta, \omega_\mu, \tau_\mu)$. $\square$

2.2.4 Relation between cosymplectic and symplectic Marsden–Meyer–Weinstein reductions

This subsection presents how the cosymplectic Marsden–Meyer–Weinstein reduction may be reformulated by using the classical symplectic Marsden–Meyer–Weinstein reduction theorem (see [44]). Recall that Lemma 1.3.11 establishes that every cosymplectic manifold naturally induces a symplectic form on a manifold of a larger dimension [105]. The following construction will be generalised to devise a k-polycosymplectic Marsden–Meyer–Weinstein reduction derived from a k-polysymplectic one in Subsection 2.4.3.

Let $\Psi\colon G \times M \to M$ be a cosymplectic Lie group action with an associated cosymplectic momentum map $\mathbf{J}^\Psi\colon M \to \mathfrak{g}^*$ relative to (M, τ, ω). Then, Ψ and $\mathbf{J}^\Psi$ naturally extend to $\mathbb{R} \times M$ in the following way, respectively,

$$\widetilde{\Psi}\colon (g, u, x) \in G \times \mathbb{R} \times M \mapsto (u, \Psi_g(x)) \in \mathbb{R} \times M$$

and

$$\mathbf{J}^{\widetilde{\Psi}}\colon (u, x) \in \mathbb{R} \times M \mapsto \mathbf{J}^\Psi(x) \in \mathfrak{g}^* \,.$$

The fundamental vector fields ξ_M, with $\xi \in \mathfrak{g}$ corresponding to Ψ are understood as vector fields on $\mathbb{R} \times M$ via the isomorphisms $\mathrm{T}_{(u,x)}(\mathbb{R} \times M) \simeq \mathrm{T}_u\mathbb{R} \times \mathrm{T}_x M$ for every $(u, x) \in \mathbb{R} \times M$. These vector fields are locally Hamiltonian with respect to $\widetilde{\omega}$ if and only if $\mathrm{d}(RJ_\xi^\Phi) = 0$ (see discussion in Subsection 1.3.1). Thus, condition $RJ_\xi^\Phi = 0$ ensures that $\widetilde{\Psi}$ admits a momentum map $\mathbf{J}^{\widetilde{\Psi}}$ relative to the symplectic manifold $(\mathbb{R} \times M, \widetilde{\omega})$, namely

$$\iota_{\xi_{\mathbb{R}\times M}} \widetilde{\omega} = \mathrm{d}\langle \mathbf{J}^{\widetilde{\Psi}}, \xi \rangle \,, \qquad \forall \xi \in \mathfrak{g} \,.$$

Moreover, if $\mathbf{J}^\Psi$ is Δ-equivariant with respect to Ψ, then $\mathbf{J}^{\widetilde{\Psi}}$ is also Δ-equivariant with respect to $\widetilde{\Psi}$. Further, by Corollary 2.2.14, since $\mathbf{J}^{\widetilde{\Psi}-1}(\mu) \simeq \mathbb{R} \times \mathbf{J}^{\Psi-1}(\mu)$ for every $\mu \in \mathfrak{g}^*$ and $\mathrm{pr} \circ \widetilde{\Psi}_g = \Psi_g \circ \mathrm{pr}$ for every $g \in G$, i.e. $\widetilde{\Psi}$ does not change the first component of $\mathbb{R} \times M$, then $\mathbf{J}^{\widetilde{\Psi}-1}(\mu)$ is quotientable by G_μ^Δ if and only if $\mathbf{J}^{\Psi-1}(\mu)$ is so. Moreover, $\mu \in \mathfrak{g}^*$ is a (resp. weak) regular value of $\mathbf{J}^\Psi$ if and only if μ is a (resp. weak) regular value of $\mathbf{J}^{\widetilde{\Psi}}$.

Consequently, the classical symplectic Marsden–Meyer–Weinstein Reduction Theorem 2.1.8 can be applied to the symplectic manifold $(\mathbb{R}\times M, \widetilde{\omega})$ to obtain the reduced symplectic manifold $\widetilde{M_\mu^\Delta} = \mathbf{J}^{\widetilde{\Psi}-1}(\mu)/G_\mu^\Delta$ endowed with the reduced symplectic form, $\widetilde{\omega}_\mu$, determined univocally by the condition

$$\widetilde{\jmath}_\mu^* \widetilde{\omega} = \widetilde{\pi}_\mu^* \widetilde{\omega}_\mu \,,$$

where $\widetilde{\jmath}_\mu\colon \mathbf{J}^{\widetilde{\Psi}-1}(\mu) \hookrightarrow \mathbb{R} \times M$ is the natural immersion and $\widetilde{\pi}_\mu\colon \mathbf{J}^{\widetilde{\Psi}-1}(\mu) \to \mathbf{J}^{\widetilde{\Psi}-1}(\mu)/G_\mu^\Delta$ is the canonical projection [2]. Since $\widetilde{M_\mu^\Delta} \simeq (\mathbb{R} \times \mathbf{J}^{\Psi-1}(\mu))/G_\mu^\Delta \simeq \mathbb{R} \times M_\mu^\Delta$, where $M_\mu^\Delta = \mathbf{J}^{\Psi-1}(\mu)/G_\mu^\Delta$, one can retrieve the reduced cosymplectic manifold $(M_\mu^\Delta, \omega_\mu, \tau_\mu)$ from $\widetilde{\omega}_\mu$, in the following way

$$\tau_\mu = i_u^* \left(\iota_{\partial/\partial u} \widetilde{\omega}_\mu \right) \,, \qquad \omega_\mu = i_u^* \widetilde{\omega}_\mu \,,$$

where $i_u\colon M_\mu^\Delta \ni [x] \mapsto (u, [x]) \in \mathbb{R} \times M_\mu^\Delta$ and $[x]$ stands for the orbit of $x \in \mathbf{J}^{\Psi-1}(\mu)$ relative to G_μ^Δ. In particular,

$$\mathrm{d}\omega_\mu = \mathrm{d}i_u^* \widetilde{\omega}_\mu = i_u^* \mathrm{d}\widetilde{\omega}_\mu = 0 \,, \qquad \mathrm{d}\tau_\mu = \mathrm{d}i_u^* \left(\iota_{\partial/\partial u} \widetilde{\omega}_\mu \right) = i_u^* \left(\mathscr{L}_{\partial/\partial u} \widetilde{\omega}_\mu \right) = 0 \,,$$

where the last equality follows from $\mathscr{L}_{\partial/\partial u} \widetilde{\omega}_\mu = 0$. Moreover, if $X \in \mathfrak{X}(M_\mu^\Delta)$, then $i_{u*}X$ takes values in $\ker \mathrm{d}u$. If, in addition, $\iota_X \omega_\mu = 0$ and $\iota_X \tau_\mu = 0$, then $\iota_{i_{u*}X} \widetilde{\omega}_\mu = 0$ and $X = 0$ because $\widetilde{\omega}_\mu$ is symplectic. Therefore, $\ker \omega_\mu \cap \ker \tau_\mu = 0$.

A Reeb vector field R on M gives rise to a unique vector field $\widetilde{R}$ on $\mathbb{R}\times M$ projecting onto M via pr and taking values in $\ker \mathrm{d}u$. Since $\widetilde{R}$ is tangent to $\mathbf{J}^{\widetilde{\Psi}-1}(\mu)$ and projectable onto a vector field $\widetilde{R}_\mu$ on $\widetilde{M}_\mu^\Delta$, one has

$$\iota_{R_\mu}\tau_\mu = \iota_{\widetilde{R}_\mu}\iota_{\partial/\partial u}\widetilde{\omega}_\mu = \iota_R\iota_{\partial/\partial u}\omega = 1\,.$$

Hence, τ_μ is different from zero, and $(M_\mu^\Delta,\omega_\mu,\tau_\mu)$ becomes a cosymplectic manifold.

The above approach demonstrates that the cosymplectic Marsden–Meyer–Weinstein Reduction Theorem 2.2.13 arises as a particular case of a symplectic reduction of $(\mathbb{R}\times M,\widetilde{\omega})$. Although the above construction uses $T=\mathbb{R}$, one may similarly consider $T=\mathbb{S}^1$, equipped with $\mathrm{d}\theta$, where θ is a locally defined angular coordinate on $\mathbb{S}^1$ giving rise to a global closed differential one-form. The entire procedure is analogous in the latter case.

It is worth noting that, and it has not been stressed so far in the literature, the above discussion also implies that cosymplectic geometry can be understood as a particular type of symplectic geometry in a larger manifold. Although conceptually appealing, such a perspective can introduce additional complexity. In fact, the extension of mathematical entities on a cosymplectic manifold to a symplectic one of a larger dimension may change the properties of such entities in such a way that it complicates their analysis, as observed, for example, in the study of cosymplectic energy-momentum methods in Section 3.2.

2.3 k-Polysymplectic Marsden–Meyer–Weinstein reduction

This section reviews existing results in the literature on the k-polysymplectic Marsden–Meyer–Weinstein reduction in order to address and correct certain inaccuracies and errors previously presented in the literature. Furthermore, it introduces the reduction of the dynamical system governed by an $\boldsymbol{\omega}$-Hamiltonian vector field. This concept was studied for the first time in [33], as prior research has primarily focused on dynamical systems given by Hamiltonian k-vector fields [12, 107]. In particular, this section first reviews the previous k-polysymplectic Marsden–Meyer–Weinstein reduction theory and explains some minor, yet conceptually relevant, inaccuracies. After that, a mistake in one of the main results in [62] regarding the conditions for the existence of a k-polysymplectic reduction is addressed. Finally, in Subsection 2.3.6, the assumptions required for the k-polysymplectic reduction as formulated in [107] are analysed and compared.

It is important to emphasise that the k-polysymplectic reduction outlined in [107] does not rely on any relationship between the dimension of the manifold and the number k of the k-polysymplectic form $\boldsymbol{\omega}$ on it. Furthermore, the term *polysymplectic manifold* in [107] is just a simplification of the term k-polysymplectic manifold, which is defined in this work. Additionally, Definition 1.4.6 leads to a linear analogue definition by assuming $\boldsymbol{\omega}$ to be restricted to a point $x\in M$. This allows us to define *k-polysymplectic structures on linear spaces*, *k-polysymplectic spaces*, and so on. In such cases, one assumes $\boldsymbol{\omega}\in\Lambda^2E^*\otimes\mathbb{R}^k$, where E is a vector space and Λ^2E^* stands for the space of two-covectors on E.

2.3.1 k-Polysymplectic momentum maps

This subsection introduces the notion of momentum maps on k-polysymplectic manifolds, with particular attention given to the case of Ad^{*k}-equivariant momentum maps.

The following definition introduces a natural class of Lie group actions preserving a k-polysymplectic structure. As presented in the following, such actions are fundamental to the formulation of the k-polysymplectic Marsden–Meyer–Weinstein reduction.

Definition 2.3.1. Let $(P,\boldsymbol{\omega})$ be a k-polysymplectic manifold. A Lie group action $\Phi\colon G\times P\to P$ of a Lie group G is a *k-polysymplectic Lie group action* if $\Phi_g^*\boldsymbol{\omega}=\boldsymbol{\omega}$ for each $g\in G$.

Definition 2.3.2. A *k-polysymplectic momentum map* for a Lie group action $\Phi\colon G\times P\to P$ relative to a k-polysymplectic manifold $(P,\boldsymbol{\omega})$ is a map $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^{*k}$ such that

$$\iota_{\xi_P}\boldsymbol{\omega}=(\iota_{\xi_P}\omega^\alpha)\otimes e_\alpha=\mathrm{d}\left\langle\mathbf{J}^\Phi,\xi\right\rangle,\qquad\forall\xi\in\mathfrak{g}\,,\tag{2.3.1}$$

where $\mathfrak{g}^{*k}=\mathfrak{g}^*\times\overset{(k)}{\cdots}\times\mathfrak{g}^*$.

From (2.3.1), it follows that $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^{*k}$ satisfies

$$\iota_{\boldsymbol{\xi}_P}\boldsymbol{\omega}=\mathrm{d}\left\langle\mathbf{J}^\Phi,\boldsymbol{\xi}\right\rangle=:\mathrm{d}\mathbf{J}^\Phi_{\boldsymbol{\xi}},\qquad\forall\boldsymbol{\xi}\in\mathfrak{g}^k\,,\tag{2.3.2}$$

where $\boldsymbol{\xi}_P$ is the k-vector field on P whose k vector field components are the fundamental vector fields of Φ related to the k-components of $\boldsymbol{\xi}\in\mathfrak{g}^k$ and $\mathbf{J}^\Phi_{\boldsymbol{\xi}}\colon P\to\mathbb{R}$. Writing $\boldsymbol{\xi}=(0,\ldots,\overset{(\alpha)}{\xi},\ldots,0)\in\mathfrak{g}^k$ for any $\xi\in\mathfrak{g}$ and $\alpha=1,\ldots,k$ and impose (2.3.2) to hold for a basis $\{\xi_1,\ldots,\xi_r\}$ for $\mathfrak{g}$ yields kr conditions, which uniquely determine the value of the kr coordinates of $\mathbf{J}^\Phi$. Conversely, the equation (2.3.1) evaluated on the basis of $\mathfrak{g}$ imposes r conditions for each one of the k components of $\mathbf{J}^\Phi$, giving rise to kr conditions and showing that both formulations are equivalent.

The following definition, while standard in the literature [107], adopts the more concise notation Ad^{*k} instead of Coad^k. Nevertheless, it is later demonstrated that the Ad^{*k}-equivariance condition is not essential and can be omitted.

Definition 2.3.3. A k-polysymplectic momentum map $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^{*k}$ is Ad^{*k}*-equivariant* if

$$\mathbf{J}^\Phi\circ\Phi_g=\mathrm{Ad}^{*k}_{g^{-1}}\circ\ \mathbf{J}^\Phi\,,\quad\forall g\in G\,,$$

where $\mathrm{Ad}^{*k}_{g^{-1}}=\mathrm{Ad}^*_{g^{-1}}\otimes\overset{(k)}{\cdots}\otimes\mathrm{Ad}^*_{g^{-1}}$ and

$$\begin{array}{rccc}\mathrm{Ad}^{*k} & :G\times\mathfrak{g}^{*k} & \longrightarrow & \mathfrak{g}^{*k}\\ & (g,\boldsymbol{\mu}) & \longmapsto & \mathrm{Ad}^{*k}_{g^{-1}}\boldsymbol{\mu}\,.\end{array}$$

That is, for every $g\in G$, the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "\mathbf{J}^\Phi"] \arrow[d, "\Phi_g"] & \mathfrak{g}^{*k} \arrow[d, "\mathrm{Ad}^{*k}_{g^{-1}}"] \\ P \arrow[r, "\mathbf{J}^\Phi"] & \mathfrak{g}^{*k}. \end{tikzcd}$$

To simplify the notation, the following definition is introduced.

Definition 2.3.4. The four-tuple $(P,\boldsymbol{\omega},h,\mathbf{J}^\Phi)$ is a *G-invariant k-polysymplectic Hamiltonian system* if it consists of a k-polysymplectic manifold $(P,\boldsymbol{\omega})$, a k-polysymplectic Lie group action $\Phi\colon G\times P\to P$ such that $\Phi_g^*h=h$ for every $g\in G$, and a k-polysymplectic momentum map $\mathbf{J}^\Phi$ related to Φ. An Ad^{*k}*-equivariant G-invariant k-polysymplectic Hamiltonian system* is a G-invariant k-polysymplectic Hamiltonian system whose k-polysymplectic momentum map is Ad^{*k}-equivariant.

2.3.2 General k-polysymplectic momentum maps

This subsection develops the theory of k-polysymplectic momentum maps that are not necessarily Ad^{*k}-equivariant. In particular, it is shown that every k-polysymplectic momentum map $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^{*k}$ determines a Lie group action on $\mathfrak{g}^{*k}$ with respect to which it is equivariant. The techniques introduced here, although technically more intricate, are analogous to those used in previous sections devoted to non-Ad^*-equivariant symplectic momentum maps in Subsection 2.1.1.

Proposition 2.3.5. *Let* $(P,\boldsymbol{\omega},h,\mathbf{J}^\Phi)$ *be a* G*-invariant* k*-polysymplectic Hamiltonian system. Define*

$$\boldsymbol{\psi}_{g,\boldsymbol{\xi}}\colon x\in P\longmapsto \mathbf{J}^\Phi_{\boldsymbol{\xi}}(\Phi_g(x)) - \mathbf{J}^\Phi_{\mathrm{Ad}^k_{g^{-1}}\boldsymbol{\xi}}(x)\in\mathbb{R}\,,\quad \forall g\in G\,,\quad \forall\boldsymbol{\xi}\in\mathfrak{g}^k\,.$$

Then, $\boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ *is constant on* P *for every* $g\in G$ *and* $\boldsymbol{\xi}\in\mathfrak{g}^k$*. Moreover, the map* $\boldsymbol{\sigma}\colon G\ni g\mapsto\boldsymbol{\sigma}(g)\in\mathfrak{g}^{*k}$ *determined by the condition* $\langle\boldsymbol{\sigma}(g),\boldsymbol{\xi}\rangle=\boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ *satisfies the cocycle property*

$$\boldsymbol{\sigma}(g_1g_2)=\boldsymbol{\sigma}(g_1)+\mathrm{Ad}^{*k}_{g_1^{-1}}\boldsymbol{\sigma}(g_2)\,,\qquad\forall g_1,g_2\in G\,.$$

Proof. Note that

$$\begin{aligned}\mathrm{d}\boldsymbol{\psi}_{g,\boldsymbol{\xi}}&=\mathrm{d}(\mathbf{J}^\Phi_{\boldsymbol{\xi}}\circ\Phi_g)-\mathrm{d}\mathbf{J}^\Phi_{\mathrm{Ad}^k_{g^{-1}}\boldsymbol{\xi}}=\Phi_g^*(\iota_{\boldsymbol{\xi}_P}\boldsymbol{\omega})-\iota_{(\mathrm{Ad}^k_{g^{-1}}\boldsymbol{\xi})_P}\boldsymbol{\omega}\\&=\iota_{\Phi_{g^{-1}*}\boldsymbol{\xi}_P}\Phi_g^*\boldsymbol{\omega}-\iota_{\Phi_{g^{-1}*}\boldsymbol{\xi}_P}\boldsymbol{\omega}=\iota_{\Phi_{g^{-1}*}\boldsymbol{\xi}_P}\boldsymbol{\omega}-\iota_{\Phi_{g^{-1}*}\boldsymbol{\xi}_P}\boldsymbol{\omega}=0\,,\end{aligned}$$

where it was used that Φ is a k-polysymplectic Lie group action and Lemma 2.1.2, i.e. $(\mathrm{Ad}_g\xi)_P=\Phi_{g*}\xi_P$ for every $g\in G$ and each $\xi\in\mathfrak{g}$. Therefore, $(\mathrm{Ad}^k_{g^{-1}}\boldsymbol{\xi})_P=\Phi_{g^{-1}*}\boldsymbol{\xi}_P$ for every $\boldsymbol{\xi}\in\mathfrak{g}^k$, so that $\boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ is constant on P for all $g\in G$ and any $\boldsymbol{\xi}\in\mathfrak{g}^k$.

Notice that $\boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ can be rewritten in the following way

$$\begin{aligned}\boldsymbol{\psi}_{g,\boldsymbol{\xi}}=\mathbf{J}^\Phi_{\boldsymbol{\xi}}\circ\Phi_g-\mathbf{J}^\Phi_{\mathrm{Ad}^k_{g^{-1}}\boldsymbol{\xi}}&=\left\langle\mathbf{J}^\Phi\circ\Phi_g,\boldsymbol{\xi}\right\rangle-\left\langle\mathbf{J}^\Phi,\mathrm{Ad}^k_{g^{-1}}\boldsymbol{\xi}\right\rangle\\&=\left\langle\mathbf{J}^\Phi\circ\Phi_g,\boldsymbol{\xi}\right\rangle-\left\langle\mathrm{Ad}^{*k}_{g^{-1}}\mathbf{J}^\Phi,\boldsymbol{\xi}\right\rangle=\left\langle\mathbf{J}^\Phi\circ\Phi_g-\mathrm{Ad}^{*k}_{g^{-1}}\mathbf{J}^\Phi,\boldsymbol{\xi}\right\rangle\,,\end{aligned}$$

where $\mathrm{Ad}^k_{g^{-1}}\colon\mathfrak{g}^k\to\mathfrak{g}^k$ is the transpose to $\mathrm{Ad}^{*k}_{g^{-1}}$. Hence,

$$\boldsymbol{\sigma}\colon G\ni g\longmapsto\mathbf{J}^\Phi\circ\Phi_g-\mathrm{Ad}^{*k}_{g^{-1}}\mathbf{J}^\Phi=\boldsymbol{\sigma}(g)\in\mathfrak{g}^{*k}\,.$$

Thus, $\boldsymbol{\sigma}(g)$ is constant on P for every $g\in G$ and $\langle\boldsymbol{\sigma}(g),\boldsymbol{\xi}\rangle=\boldsymbol{\psi}_{g,\boldsymbol{\xi}}$, for every $g\in G$ and $\boldsymbol{\xi}\in\mathfrak{g}^k$. To prove the cocycle identity, one has

$$\begin{aligned}\boldsymbol{\sigma}(g_1g_2)&=\left(\mathbf{J}^\Phi\circ\Phi_{g_1g_2}-\mathrm{Ad}^{*k}_{(g_1g_2)^{-1}}\mathbf{J}^\Phi\right)=\left(\mathbf{J}^\Phi\circ\Phi_{g_1}\circ\Phi_{g_2}-\mathrm{Ad}^{*k}_{g_1^{-1}}\mathrm{Ad}^{*k}_{g_2^{-1}}\mathbf{J}^\Phi\right)\\&=\left(\mathbf{J}^\Phi\circ\Phi_{g_1}\circ\Phi_{g_2}-\mathrm{Ad}^{*k}_{g_1^{-1}}(\mathbf{J}^\Phi\circ\Phi_{g_2})+\mathrm{Ad}^{*k}_{g_1^{-1}}(\mathbf{J}^\Phi\circ\Phi_{g_2})-\mathrm{Ad}^{*k}_{g_1^{-1}}\mathrm{Ad}^{*k}_{g_2^{-1}}\mathbf{J}^\Phi\right)\\&=\left(\mathbf{J}^\Phi\circ\Phi_{g_1}-\mathrm{Ad}^{*k}_{g_1^{-1}}\mathbf{J}^\Phi+\mathrm{Ad}^{*k}_{g_1^{-1}}(\mathbf{J}^\Phi\circ\Phi_{g_2}-\mathrm{Ad}^{*k}_{g_2^{-1}}\mathbf{J}^\Phi)\right)=\boldsymbol{\sigma}(g_1)+\mathrm{Ad}^{*k}_{g_1^{-1}}\boldsymbol{\sigma}(g_2)\end{aligned}$$

for any $g_1,g_2\in G$. □

The map $\boldsymbol{\sigma}\colon G\to\mathfrak{g}^{*k}$ of the form

$$\boldsymbol{\sigma}(g)=\mathbf{J}^\Phi\circ\Phi_g-\mathrm{Ad}^{*k}_{g^{-1}}\mathbf{J}^\Phi,\qquad g\in G,$$

is called the *co-adjoint cocycle* associated with the k-polysymplectic momentum map $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^{*k}$. A map $\boldsymbol{\sigma}\colon G\to\mathfrak{g}^{*k}$ is a *coboundary* if there exists $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$ such that

$$\boldsymbol{\sigma}(g)=\boldsymbol{\mu}-\mathrm{Ad}^{*k}_{g^{-1}}\boldsymbol{\mu},\qquad\forall g\in G\,.$$

In particular, if $\mathbf{J}^\Phi$ is an Ad^{*k}-equivariant k-polysymplectic momentum map, then $\boldsymbol{\sigma}=0$.

The next result shows that the cohomology class $[\boldsymbol{\sigma}]$ associated with the cocycle $\boldsymbol{\sigma}$ depends only on the k-polysymplectic Lie group action and not on the particular choice of a momentum map. The proof is analogous to the corresponding result for symplectic momentum maps introduced in Section 2.1.1.

Proposition 2.3.6. *Let* $\Phi\colon G\times P\to P$ *be a* k*-polysymplectic Lie group action. If* $\mathbf{J}^{1\Phi}$ *and* $\mathbf{J}^{2\Phi}$ *are two associated* k*-polysymplectic momentum maps with co-adjoint cocycles* $\boldsymbol{\sigma}_1$ *and* $\boldsymbol{\sigma}_2$*, respectively, then* $[\boldsymbol{\sigma}_1]=[\boldsymbol{\sigma}_2]$*.*

Proposition 2.3.7. *Let $\mathbf{J}^\Phi\colon P \to \mathfrak{g}^{*k}$ be a k-polysymplectic momentum map associated with a k-polysymplectic Lie group action $\Phi\colon G\times P\to P$ with co-adjoint cocycle $\boldsymbol{\sigma}$. Then,*

(1) *the map*

$$\boldsymbol{\Delta}\colon G\times\mathfrak{g}^{*k}\ni(g,\boldsymbol{\mu})\mapsto\boldsymbol{\sigma}(g)+\mathrm{Ad}^{*k}_{g^{-1}}\boldsymbol{\mu}=\boldsymbol{\Delta}_g(\boldsymbol{\mu})\in\mathfrak{g}^{*k},$$

is a Lie group action of G on $\mathfrak{g}^{*k}$,

(2) *the k-polysymplectic momentum map $\mathbf{J}^\Phi$ is equivariant with respect to $\boldsymbol{\Delta}$, in other words, for every $g\in G$, the following diagram commutes*

$$\begin{tikzcd} P \arrow[r, "\mathbf{J}^\Phi"] \arrow[d, "\Phi_g"] & \mathfrak{g}^{*k} \arrow[d, "\boldsymbol{\Delta}_g"] \\ P \arrow[r, "\mathbf{J}^\Phi"] & \mathfrak{g}^{*k}. \end{tikzcd}$$

Proof. First, since $\boldsymbol{\sigma}(e)=0$, it follows that

$$\boldsymbol{\Delta}(e,\boldsymbol{\mu})=\boldsymbol{\mu}+\boldsymbol{\sigma}(e)=\boldsymbol{\mu}\,,$$

Thus, $\boldsymbol{\Delta}(e,\boldsymbol{\mu})=\boldsymbol{\mu}$. Then, Proposition 2.3.5 yields

$$\begin{aligned}\boldsymbol{\Delta}(g_1,\boldsymbol{\Delta}(g_2,\boldsymbol{\mu}))&=\mathrm{Ad}^{*k}_{g_1^{-1}}(\mathrm{Ad}^{*k}_{g_2^{-1}}\boldsymbol{\mu}+\boldsymbol{\sigma}(g_2))+\boldsymbol{\sigma}(g_1)=\mathrm{Ad}^{*k}_{g_1^{-1}}\mathrm{Ad}^{*k}_{g_2^{-1}}\boldsymbol{\mu}+\mathrm{Ad}^{*k}_{g_1^{-1}}\boldsymbol{\sigma}(g_2)+\boldsymbol{\sigma}(g_1)\\&=\mathrm{Ad}^{*k}_{(g_1g_2)^{-1}}\boldsymbol{\mu}+\mathrm{Ad}^{*k}_{g_1^{-1}}\boldsymbol{\sigma}(g_2)+\boldsymbol{\sigma}(g_1)=\mathrm{Ad}^{*k}_{(g_1g_2)^{-1}}\boldsymbol{\mu}+\boldsymbol{\sigma}(g_1g_2)=\boldsymbol{\Delta}(g_1g_2,\boldsymbol{\mu})\,,\end{aligned}$$

for every $g_1,g_2\in G$ and $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$. Hence, $\boldsymbol{\Delta}$ is a Lie group action of G on $\mathfrak{g}^{*k}$.

Second, by the definition of $\boldsymbol{\Delta}$ and $\boldsymbol{\sigma}$, one has

$$\boldsymbol{\Delta}_g\circ\mathbf{J}^\Phi=\mathrm{Ad}^{*k}_{g^{-1}}\mathbf{J}^\Phi+\boldsymbol{\sigma}(g)=\mathbf{J}^\Phi\circ\Phi_g\,,\qquad\forall g\in G\,,$$

which shows that $\mathbf{J}^\Phi$ is $\boldsymbol{\Delta}$-equivariant, as required. □

Proposition 2.3.7 implies that any (not necessarily Ad^{*k}-equivariant) k-polysymplectic momentum map $\mathbf{J}^\Phi$ becomes equivariant with respect to a new Lie group action $\boldsymbol{\Delta}\colon G\times\mathfrak{g}^{*k}\to\mathfrak{g}^{*k}$, called a *$k$-polysymplectic affine Lie group action.*

A k-polysymplectic affine Lie group action can be equivalently described component-wise. Let $\boldsymbol{\mu}=(\mu^1,\dots,\mu^k)\in\mathfrak{g}^{*k}$, then $\boldsymbol{\Delta}(g,\boldsymbol{\mu})=(\Delta^1_g\mu^1,\dots,\Delta^k_g\mu^k)\in\mathfrak{g}^{*k}$, where the maps $\Delta^1,\dots,\Delta^k$ take the form $\Delta^\alpha\colon G\times\mathfrak{g}^*\ni(g,\vartheta)\mapsto\mathrm{Ad}^*_{g^{-1}}\vartheta+\sigma^\alpha(g)=\Delta^\alpha_g(\vartheta)\in\mathfrak{g}^*$ and $\boldsymbol{\sigma}(g)=(\sigma^1(g),\dots,\sigma^k(g))$, where $\sigma^\alpha(g)=\mathbf{J}^\Phi_\alpha\circ\Phi_g-\mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi_\alpha$ for $\alpha=1,\dots,k$.

2.3.3 k-Polysymplectic reduction by a submanifold

This section surveys the well-known k-polysymplectic reduction by a submanifold [107]. To study k-polysymplectic reduction by a submanifold, it is important to consider the notion of k-polysymplectic orthogonal complement (see [46] for details).

Definition 2.3.8. Let $(E,\boldsymbol{\omega})$ be a k-polysymplectic vector space and let W be a linear subspace of E. Then, the *k-polysymplectic orthogonal complement* of W relative to $(E,\boldsymbol{\omega})$ is the linear subspace defined by

$$W^{\perp,k}=\{v\in E\mid\iota_w\iota_v\boldsymbol{\omega}=0\,,\forall w\in W\}\,.$$

In the k-polysymplectic framework, attention is restricted to a weak regular value $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$ of a k-polysymplectic momentum map $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^{*k}$. Nevertheless, the assumption that $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$ is a weak regular value is insufficient, since it is essential that, for each component of the k-polysymplectic momentum map, the level set $\mathbf{J}^{\Phi-1}_\alpha(\mu^\alpha)$ is a submanifold of P. This motivates the introduction of the following definition.

Definition 2.3.9. A *weak regular k-value* of $\mathbf{J}^\Phi$ is $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ such that each $\mu^\alpha \in \mathfrak{g}^*$ is a weak regular value of $\mathbf{J}^\Phi_\alpha \colon P \to \mathfrak{g}^*$ for $\alpha = 1, \ldots, k$. This implies that $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is a submanifold of P

However, it should be noted that the converse does not necessarily hold, since in general, one has the inclusion

$$\mathbf{J}^{\Phi-1}_\alpha(\mu^\alpha) \supset \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}), \qquad \alpha = 1, \ldots, k\,.$$

The following theorem plays an important role in the reduction theory (see [46, 107] for a proof).

Theorem 2.3.10. *(k-Polysymplectic reduction by a submanifold.) Let $(P, \boldsymbol{\omega})$ be a k-polysymplectic manifold and let S be a submanifold of P with an injective immersion $\jmath \colon S \hookrightarrow P$. Assume that* $\ker \jmath^*\boldsymbol{\omega}$ *has a constant rank for $(P, \boldsymbol{\omega})$, the quotient space $S/\mathcal{F}_S$ is a manifold, where $\mathcal{F}_S$ is a foliation on S given by* $\ker \jmath^*\boldsymbol{\omega}$*, and assume that the canonical projection $\pi \colon S \to S/\mathcal{F}_S$ is a submersion. Then, $(S/\mathcal{F}_S, \boldsymbol{\omega}_S)$ is a k-polysymplectic manifold defined univocally by*

$$\jmath^*\boldsymbol{\omega} = \pi^*\boldsymbol{\omega}_S\,,$$

and $\ker \jmath^*\boldsymbol{\omega}_p = \mathrm{T}_pS \cap (\mathrm{T}_pS)^{\perp_k}$ *for any $p \in S$.*

Recall the following lemma, the proof of which can be found in [107].

Lemma 2.3.11. *Let $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathfrak{g}^{*k}$. Then,*

$$G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} = \bigcap_{\alpha=1}^{k} G^{\Delta^\alpha}_{\mu^\alpha}\,, \qquad \mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} = \bigcap_{\alpha=1}^{k} \mathfrak{g}^{\Delta^\alpha}_{\mu^\alpha}\,,$$

where $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ is the isotropy group of $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ under the k-polysymplectic affine Lie group action Δ and $\mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ is its Lie algebra.

2.3.4 k-Polysymplectic Marsden–Meyer–Weinstein reduction theorem

This subsection extends the k-polysymplectic reduction theory to the setting where the associated k-polysymplectic momentum map $\mathbf{J}^\Phi \colon P \to \mathfrak{g}^{*k}$ is not necessarily Ad^{*k}-equivariant. As Proposition 2.3.7 shows, every k-polysymplectic momentum map $\mathbf{J}^\Phi$ admits a k-polysymplectic affine Lie group action $\boldsymbol{\Delta} \colon G \times \mathfrak{g}^{*k} \to \mathfrak{g}^{*k}$ with respect to which $\mathbf{J}^\Phi$ is $\boldsymbol{\Delta}$-equivariant. It is important to note that the isotropy subgroup, $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$, of $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ relative to $\boldsymbol{\Delta}$ may differ from the isotropy group of $\boldsymbol{\mu}$ with respect to Ad^{*k} since the it is not necessarily the same (cf. [128, Theorem 6.1.1]). This distinction plays a crucial role in generalising the k-polysymplectic Marsden–Meyer–Weinstein reduction theorem [107] to the case of non-Ad^{*k}-equivariant momentum maps. This subsection also presents a generalisation of the reduction theorems from [107] to this broader class of k-polysymplectic momentum maps.

The following results provide a slight generalisation of some foundational constructions in the framework of k-polysymplectic geometry as originally developed in [107]. Specifically, the extension addresses the case in which the k-polysymplectic momentum map $\mathbf{J}^\Phi \colon P \to \mathfrak{g}^{*k}$ is not required to be Ad^{*k}-equivariant. In fact, the whole work [107] may be reformulated by replacing the use of the coadjoint representation with that of k-polysymplectic affine Lie group actions. The necessary changes involve substituting G_{μ^α} by $G^{\Delta^\alpha}_{\mu^\alpha}$, $\mathfrak{g}_{\mu^\alpha}$ by $\mathfrak{g}^{\Delta^\alpha}_{\mu^\alpha}$, $G_{\boldsymbol{\mu}}$ by $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$, along with other minor modifications of this type. Although these substitutions are conceptually straightforward, identifying all instances in which such corrections must be applied requires considerable effort and attention to detail.

The following lemma is a slight generalisation of a standard result in a k-polysymplectic Marsden–Meyer–Weinstein reduction.

Lemma 2.3.12. *Let $(P, \boldsymbol{\omega}, h, \mathbf{J}^\Phi)$ be a G-invariant k-polysymplectic Hamiltonian system and let $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ be a weak regular k-value of a k-polysymplectic momentum map $\mathbf{J}^\Phi \colon P \to \mathfrak{g}^{*k}$ associated with Φ. Then, for every $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, one has*

(1) $\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}p) = \mathrm{T}_p(Gp) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$,

(2) $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \mathrm{T}_p(Gp)^{\perp,k}$.

Proof. Let $\xi_P(p) \in \mathrm{T}_p(Gp)$ for some $\xi \in \mathfrak{g}$. Then, $\xi_P(p) \in \mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}p)$ if and only if $\xi_P(p) \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, or equivalently $\xi \in \mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ if and only if $\xi_P(p) \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, where $\mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ is the Lie algebra of $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$.

The proof of (2) follows essentially the same as in [107] and is analogous to the symplectic case in Lemma 2.1.18. □

A non-necessarily Ad^{*k}-equivariant analogue of the essential technical results from [107, Lemmas 3.15 and 3.16] is provided below. These results constitute a foundation for the generalised k-polysymplectic Marsden–Meyer–Weinstein reduction theory.

Lemma 2.3.13. *The linear map*

$$\widetilde{\pi}^\alpha_p\colon \frac{\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})}{\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}p)} \longrightarrow \frac{\left(\frac{\mathrm{T}_p\mathbf{J}^\Phi_\alpha}{\ker\omega^\alpha_p}\right)}{\{\mathrm{T}_p\pi_{\boldsymbol{\mu}}(\xi_P)_p \mid \xi\in\mathfrak{g}^{\Delta^\alpha}_{\mu^\alpha}\}}\,, \qquad p\in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\subset P\,,$$

for some α belonging to $\{1,\ldots,k\}$ is a surjection if and only if

$$\ker \mathrm{T}_p\mathbf{J}^\Phi_\alpha = \mathrm{T}_p(\mathbf{J}^{\Phi-1}\boldsymbol{\mu}) + \ker\omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha}p)\,.$$

Furthermore, $\bigcap_{\alpha=1}^k \ker\widetilde{\pi}^\alpha_p = 0$ holds, if and only if

$$\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}p) = \bigcap_{\alpha=1}^{k}\Big(\ker\omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha}p)\Big)\cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,.$$

The following results establish the main statements in the slightly generalised k-polysymplectic Marsden–Meyer–Weinstein reduction theory. It must be emphasised that there is no restriction on the dimension of the manifold P for Theorem 2.3.14 and Theorem 2.3.15 to remain valid.

Theorem 2.3.14 (The general k-polysymplectic Marsden–Meyer–Weinstein reduction theorem)**.** *Let $(P,\boldsymbol{\omega},h,\mathbf{J}^\Phi)$ be a G-invariant k-polysymplectic Hamiltonian system. Suppose that $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$ is a weak regular k-value of a k-polysymplectic momentum map $\mathbf{J}^\Phi$ and $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ acts in a quotientable manner on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Let $G^{\Delta^\alpha}_{\mu^\alpha}$ denote the isotropy group at μ^α of the Lie group action $\Delta^\alpha\colon (g,\vartheta)\in G\times\mathfrak{g}^*\mapsto \Delta^\alpha(g,\vartheta)\in\mathfrak{g}^*$ for $\alpha=1,\ldots,k$. Moreover, assume that the following conditions hold*

$$\ker(\mathrm{T}_p\mathbf{J}^\Phi_\alpha) = \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) + \ker\omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha}p)\,, \tag{2.3.3}$$

$$\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}p) = \bigcap_{\alpha=1}^{k}\Big(\ker\omega^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha}p)\Big)\cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,, \tag{2.3.4}$$

for every $p\in\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and all $\alpha=1,\ldots,k$. Then, $(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}},\boldsymbol{\omega}_{\boldsymbol{\mu}})$ is a k-polysymplectic manifold, where $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ is univocally determined by

$$\pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega}\,,$$

where $\jmath_{\boldsymbol{\mu}}\colon \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\hookrightarrow P$ is the natural immersion and $\pi_{\boldsymbol{\mu}}\colon\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\to\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ is the canonical projection.

This theorem presents several improvements over the original k-polysymplectic Marsden–Meyer–Weinstein reduction theorem developed in [107]. In particular, the requirement that the k-polysymplectic momentum map $\mathbf{J}^\Phi$ be Ad^{*k}-equivariant is eliminated. This generalisation is achieved by employing the k-polysymplectic affine Lie group action $\boldsymbol{\Delta}$.

The next result characterises the conditions under which a k-polysymplectic Hamiltonian system $(P,\boldsymbol{\omega},h,\mathbf{J}^\Phi)$ induces a reduced k-polysymplectic Hamiltonian system on the quotient $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ obtained by Theorem 2.3.14.

Theorem 2.3.15. *Let the assumptions of Theorem 2.3.14 hold. Let* $h \in \mathscr{C}^\infty(P)$ *be a Hamiltonian function that is* G*-invariant relative to* Φ *and let* $\mathbf{X}^h = (X_1^h, \dots, X_k^h)$ *be a* k*-vector field associated with* h*. Assume that* $\Phi_{g*}\mathbf{X}^h = \mathbf{X}^h$ *for every* $g \in G$ *and* $\mathbf{X}^h$ *is tangent to* $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$*. Then, the flows* F_t^α *of* X_α^h *leave* $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ *invariant and induce unique flows* $\mathcal{F}_t^\alpha$ *on* $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}$ *such that* $\pi_{\boldsymbol{\mu}} \circ F_t^\alpha = \mathcal{F}_t^\alpha \circ \pi_{\boldsymbol{\mu}}$ *for every* $\alpha = 1, \dots, k$*.*

The proof of Theorem 2.3.15 follows directly from Theorem 2.3.14 and is essentially the same as in [107]. It should be noted that the k-polysymplectic Hamiltonian k-vector field $\mathbf{X}^h$ need not be invariant under the G-action, even in cases where its corresponding Hamilton-de Donder-Weyl (HDW) equations are. In addition, $\mathbf{X}^h$ is not tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ in general. This is illustrated by the examples in Section 2.4.

2.3.5 Comments on the k-polysymplectic Marsden–Meyer–Weinstein reduction

To improve the applicability of the k-polysymplectic Marsden–Meyer–Weinstein reductions, this subsection comments on some technical conditions imposed in the literature [107, 62]. The notion of a regular value is particularly subtle in the context of k-polysymplectic momentum maps. The codomain of a k-polysymplectic momentum map $\mathbf{J}^\Phi\colon P \to \mathfrak{g}^{*k}$ often has a dimension larger than that of P, especially for large k, due to the k-fold product of $\mathfrak{g}^*$. As a result, $\mathbf{J}^\Phi$ may fail to be a submersion. While being a submersion is a standard assumption in many classical Marsden–Meyer–Weinstein reduction frameworks (see, e.g., [62, 107]), this property becomes very restrictive in the k-polysymplectic setting. It is sometimes assumed in the literature that Sard's Theorem ensures that $\mathbf{J}^\Phi$ is frequently a submersion because the set of singular points in P of $\mathbf{J}^\Phi$, i.e. the set of points where $\mathbf{J}^\Phi$ is not a submersion, has an image with zero measure (see [108, Lemma 3.4] or [17, p 212]). Nevertheless, the whole image of $\mathbf{J}^\Phi$ may also be a zero measure subset and, in this case, it may happen that $\mathbf{J}^\Phi$ is not a submersion at points in a dense subset of P. Indeed, $\mathbf{J}^\Phi$ is not a submersion at any point in P whenever $k \dim \mathfrak{g}^* > \dim P$. In such a case, $\mathbf{J}^\Phi$ does not admit regular values in $\mathfrak{g}^{*k}$. This fact highlights the importance of the analysis of weak regular values for k-polysymplectic momentum maps in [50]. It also explains why, in the symplectic case, when $k = 1$, the assumption of $\mathbf{J}^\Phi$ being a submersion is not so problematic and justifies the notion of weak regular k-values introduced in Definition 2.3.9.

It is also worth stressing that Blacker, in [12, Theorem 3.22], does not provide any explicit assumptions regarding the structure of $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, although it is implicitly assumed that $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is a manifold. In general, Blacker's work [12] does not study in detail the technical conditions required to guarantee that $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is a submanifold. Nevertheless, the structure of the quotient space $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is studied.

Lemma 2.3.12 plays a significant role in characterising the so-called k-polysymplectic relative equilibrium points of G-invariant $\boldsymbol{\omega}$-Hamiltonian systems studied in Section 3.3. More importantly, it is essential for establishing the k-polysymplectic Marsden–Meyer–Weinstein reduction theorem.

Interestingly, the incorrect formulation of Lemma 2.3.12 in Günther's article [78, Lemma 7.5] implies that the main reduction result presented is not valid. In particular, [78, Lemma 7.5] contains an incorrect expression for item (1) in Lemma 2.3.12. In this work, the claim is made that, similarly to the symplectic setting, the following identity holds

$$\ker \jmath_{\boldsymbol{\mu}}^*\boldsymbol{\omega} = \mathrm{T}_p\left(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\right)^{\perp,k} \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \mathrm{T}_p(Gp) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \mathrm{T}_p(G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}p),$$

However, the identification between the second and third expressions is, in general, only an inclusion $\supset$, rather than an equality, as clarified in [107, p 12]. Moreover, the justification presented for Lemma 7.5 in [78, p 48] is limited to a brief remark indicating that the proof is analogous to that in the symplectic setting, which is insufficient in the k-polysymplectic context.

Additionally, a related mistake appears in [123], where similar reasoning is applied without appropriate justification. Further discussion of these issues and their implications may be found in [107, Sections 1 and 2.2].

The conditions for the k-polysymplectic Marsden–Meyer–Weinstein reduction theorem are now reviewed, as they are crucial in the formulation of the k-polysymplectic energy-momentum method, particularly in addressing a mistake present in one of the principal results of [62], namely [62, Proposition 1], giving the name to the paper.

The first correct version of the k-polysymplectic Marsden–Meyer–Weinstein reduction theorem was formulated in [107]. Subsequently, necessary and sufficient conditions for the reduction procedure were established by C. Blacker in [12], although a typographical error in the statement of the main theorem is noted and discussed in [62]. In [107], the k-polysymplectic Marsden–Meyer–Weinstein reduction theorem was proved under the assumption that the k-polysymplectic momentum map $\mathbf{J}^\Phi \colon P \to \mathfrak{g}^{*k}$ is Ad^{*k}-equivariant. A more general version, which removes the assumption of Ad^{*k}-equivariance, was devised in [50, Theorem 5.10] and is presented in the most modern form in Theorem 2.3.14.

The next theorem presents the reduction of the dynamics given by $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ on P, as a consequence of Theorem 2.3.14. This result is crucial in the construction of the k-polysymplectic energy-momentum method introduced in Section 3.3. In previous formulations of the k-polysymplectic Marsden–Meyer–Weinstein reduction theorem, the reduction was applied to the dynamics generated by a k-polysymplectic Hamiltonian k-vector field; see, for instance, [107, Theorem 4.4]. In contrast, the version of the theorem stated above applies directly to $\boldsymbol{\omega}$-Hamiltonian vector fields, thereby simplifying the required assumptions.

Theorem 2.3.16. *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system and let $\Phi_{g*}\boldsymbol{h} = \boldsymbol{h}$ for each $g \in G$. Then, the one-parametric group of diffeomorphisms F_t of the vector field $X_{\boldsymbol{h}}$ induces the one-parametric group of diffeomorphisms $\mathcal{F}_t$ of the vector field $X_{\boldsymbol{f}_{\boldsymbol{\mu}}}$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ such that*

$$\iota_{X_{\boldsymbol{f}_{\boldsymbol{\mu}}}}\boldsymbol{\omega}_{\boldsymbol{\mu}} = \mathrm{d}\boldsymbol{f}_{\boldsymbol{\mu}} \qquad \text{and} \qquad \jmath^*_{\boldsymbol{\mu}}\boldsymbol{h} = \pi^*_{\boldsymbol{\mu}}\boldsymbol{f}_{\boldsymbol{\mu}}\,.$$

Proof. The G-invariance of $\boldsymbol{h}$, together with the assumption $\Phi_g^*\boldsymbol{\omega} = \boldsymbol{\omega}$, yields $\Phi_g X_{\boldsymbol{h}} = X_{\boldsymbol{h}}$ for each $g \in G$. Consequently,

$$\iota_{X_{\boldsymbol{h}}}\mathrm{d}\langle \mathbf{J}^\Phi, \xi\rangle = -\iota_{\xi_P}\iota_{X_{\boldsymbol{h}}}\boldsymbol{\omega} = -\iota_{\xi_P}\mathrm{d}\boldsymbol{h} = 0\,, \qquad \forall \xi \in \mathfrak{g}\,,$$

which implies that $X_{\boldsymbol{h}}$ is tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$.

Furthermore, for every $\xi \in \mathfrak{g}$, one has

$$\iota_{[\xi_P, X_{\boldsymbol{h}}]}\boldsymbol{\omega} = \mathscr{L}_{\xi_P}\iota_{X_{\boldsymbol{h}}}\boldsymbol{\omega} - \iota_{\xi_P}\mathscr{L}_{X_{\boldsymbol{h}}}\boldsymbol{\omega} = 0\,,$$

and since $\ker \boldsymbol{\omega} = 0$, it follows that $[\xi_P, X_{\boldsymbol{h}}] = 0$. Hence, $X_{\boldsymbol{h}}$ projects onto a vector field Y on the reduced space $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$. The one-parameter group of diffeomorphisms F_t generated by $X_{\boldsymbol{h}}$ then descends to a one-parameter group $\mathcal{F}_t$ of diffeomorphisms of Y such that $\pi_{\boldsymbol{\mu}} \circ F_t = \mathcal{F}_t \circ \pi_{\boldsymbol{\mu}}$ for every $t \in \mathbb{R}$.

Then, by Theorem 2.3.14, one obtains

$$\jmath^*_{\boldsymbol{\mu}}\mathrm{d}\boldsymbol{h} = \jmath^*_{\boldsymbol{\mu}}(\iota_{X_{\boldsymbol{h}}}\boldsymbol{\omega}) = \iota_{X_h}\jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega} = \iota_{X_h}\pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}}, = \pi^*_{\boldsymbol{\mu}}(\iota_Y\boldsymbol{\omega}_{\boldsymbol{\mu}})\,, \tag{2.3.5}$$

where $X_{\boldsymbol{h}}$ denotes both the vector field on P and its restriction to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. The same slight abuse of notation is used hereafter to simplify the notation.

Since $\boldsymbol{h}$ is $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$-invariant, there exists a reduced $\mathbb{R}^k$-valued function $\boldsymbol{f}_{\boldsymbol{\mu}}$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\Delta}_{\boldsymbol{\mu}}$ such that

$$\jmath^*_{\boldsymbol{\mu}}\boldsymbol{h} = \pi^*_{\boldsymbol{\mu}}\boldsymbol{f}_{\boldsymbol{\mu}}\,.$$

Substituting this into (2.3.5) yields

$$\pi^*_{\boldsymbol{\mu}}\mathrm{d}\boldsymbol{f}_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}}\mathrm{d}\boldsymbol{h} = \pi^*_{\boldsymbol{\mu}}\iota_Y\boldsymbol{\omega}_{\boldsymbol{\mu}}$$

which shows that $Y = X_{\boldsymbol{f}_{\boldsymbol{\mu}}}$ is an $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian vector field and $\boldsymbol{f}_{\boldsymbol{\mu}}$ is an $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian function associated with $X_{\boldsymbol{f}_{\boldsymbol{\mu}}}$. □

The sufficient and necessary conditions for k-polysymplectic reduction given by Blacker in equation (2.3.6) are now recalled. The main result is reformulated in Theorem 2.3.17 using the present notation. The formulation also corrects a typographical error in [12, Theorem 3.22], which appears both in the statement and the proof, as revealed by application of [12, Theorem 2.14] to $\boldsymbol{\omega}_x$. Although these assumptions were remarked by Blacker, they appear to have been overlooked by Mestdag and García-Toraño in [62]. Furthermore, condition (2.3.6) had already appeared implicitly in [107, p 12]. For related results, see also [116], which discusses the reduction of poly-Poisson structures.

Theorem 2.3.17. *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be an Ad^{*k}-equivariant G-invariant k-polysymplectic Hamiltonian system and let $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ be a fixed regular k-value of $\mathbf{J}^\Phi$. If the stabiliser subgroup $G_{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ under the Ad^{*k} action is connected, and $P_{\boldsymbol{\mu}} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is a smooth manifold, then there is a unique $\mathbb{R}^k$-valued two-form $\boldsymbol{\omega}_{\boldsymbol{\mu}} \in \Omega^2(P_{\boldsymbol{\mu}}, \mathbb{R}^k)$ such that $\pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega}$ where $\jmath_{\boldsymbol{\mu}} \colon \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \hookrightarrow P$ is the inclusion and $\pi_{\boldsymbol{\mu}} \colon \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \to P_{\boldsymbol{\mu}}$ is the canonical projection. The form $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ is closed and nondegenerate if and only if*

$$\mathrm{T}_p(G_{\boldsymbol{\mu}}p) = (\mathrm{T}_p(Gp)^{\perp,k})^{\perp,k} \cap \mathrm{T}_p(Gp)^{\perp,k}\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,. \tag{2.3.6}$$

For the sake of completeness, the reason why assuming $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ to be a regular value of a k-polysymplectic momentum map is a very restrictive condition is now presented. Consider an example of a k-polysymplectic Marsden–Meyer–Weinstein reduction associated with a non-regular k-value of a k-polysymplectic momentum map. Further examples with potential applications are presented in Section 3.4. Consider the completely integrable and separable system on $\mathbb{R}^{2k}$ given by

$$\frac{\mathrm{d}I_\alpha}{\mathrm{d}t} = 0\,, \qquad \frac{\mathrm{d}\theta_\alpha}{\mathrm{d}t} = F_\alpha(I_\alpha)\,, \qquad \alpha = 1, \ldots, k > 1\,, \tag{2.3.7}$$

for some arbitrary functions $F_1, \ldots, F_k \colon \mathbb{R} \to \mathbb{R}$.

This system defines a k-polysymplectic Hamiltonian system on $\mathbb{R}^{2k}$ with respect to the k-polysymplectic form $\boldsymbol{\omega} = \omega^\alpha \otimes e_\alpha$, where

$$\omega^\alpha = \mathrm{d}\theta^\alpha \wedge \mathrm{d}I^\alpha\,, \qquad \alpha = 1, \ldots, k\,,$$

where it is important to stress that the right-hand side is not summed over the indices $\alpha = 1, \ldots, k$.

The Lie group action

$$\Phi \colon (\lambda_1, \ldots, \lambda_{k-1}; \theta_1, \ldots, \theta_k, I) \in \mathbb{R}^{k-1} \times \mathbb{R}^{2k} \mapsto (\lambda_1 + \theta_1, \ldots, \lambda_{k-1} + \theta_{k-1}, \theta_k, I) \in \mathbb{R}^{2k}\,,$$

induces fundamental vector fields $\partial/\partial\theta^1, \ldots, \partial/\partial\theta^{k-1}$, with $I = (I_1, \ldots, I_k) \in \mathbb{R}^k$.

Note that the functions $F_1, \ldots, F_k$ are chosen to be of the form $F_\alpha = F_\alpha(I_\alpha)$, with $\alpha = 1, \ldots, k$, to ensure that (2.3.7) is $\boldsymbol{\omega}$-Hamiltonian and separable.

Consider a k-polysymplectic momentum map

$$\mathbf{J}^\Phi \colon (\theta, I) \in \mathbb{R}^{2k} \longmapsto (I_1, \ldots, 0) \otimes e_1 + \cdots + (0, \ldots, I_{k-1}) \otimes e_{k-1} + (0, \ldots, 0) \otimes e_k \in \left(\mathbb{R}^{(k-1)*}\right)^k\,,$$

which is Ad^{*k}-equivariant and has no regular values, as its codomain has dimension greater than its domain for $k > 3$. Note that (2.3.7) gives rise to an $\mathbb{R}^{k-1}$-invariant $\boldsymbol{\omega}$-Hamiltonian system.

The reduction of $\boldsymbol{\omega}$ and (2.3.7) may be considered for any value of the form

$$\boldsymbol{\mu} = (\mu^1, \ldots, 0) \otimes e_1 + \cdots + (0, \ldots, \mu^{k-1}) \otimes e_{k-1} \in \left(\mathbb{R}^{(k-1)*}\right)^k\,.$$

Then,

$$\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \{(\theta_\alpha, I_\alpha) \in \mathbb{R}^{2k} \mid I_1 = \mu^1,\ \ldots,\ I_{k-1} = \mu^{k-1},\ \theta_1, \ldots, \theta_k, I_k \in \mathbb{R}\} \simeq \mathbb{R}^k \times \mathbb{R}\,.$$

The isotropy subgroup $\mathbb{R}^{k-1}_{\boldsymbol{\mu}} \simeq \mathbb{R}^{k-1}$ acts on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ via Φ, and the reduced manifold is diffeomorphic to $\mathbb{R}^2$. The presymplectic forms $\omega^1, \ldots, \omega^{k-1}$ vanish, whereas the reduction of ω^k gives a symplectic form. Consequently, $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ is a k-polysymplectic form in which only one component is non-zero and symplectic.

Since the $\boldsymbol{\omega}$-Hamiltonian function of the original system is a first integral of $\theta_1, \ldots, \theta_{k-1}$, one can project the initial system onto

$$\frac{\mathrm{d}I_k}{\mathrm{d}t} = 0\,, \qquad \frac{\mathrm{d}\theta_k}{\mathrm{d}t} = F_k(I_k)\,,$$

which is Hamiltonian with respect to the symplectic form $\mathrm{d}\theta_k \wedge \mathrm{d}I_k$, where θ_k, I_k are treated as coordinates on $\mathbb{R}^2$ in the natural manner.

2.3.6 On the conditions for the k-polysymplectic Marsden–Meyer–Weinstein reduction

This subsection focuses on the conditions established in [107] and stated in Theorem 2.3.14 that guarantee the existence of the k-polysymplectic Marsden–Meyer–Weinstein reduction theorem. It was asserted in [62, Proposition 1] that condition (2.3.4) suffices to guarantee the existence of a k-polysymplectic Marsden–Meyer–Weinstein reduction. This subsection demonstrates that such a claim is incorrect. The error is first identified in the proof of [62, Proposition 1], and a counterexample is subsequently provided in which condition (2.3.4) is satisfied, yet no k-polysymplectic Marsden–Meyer–Weinstein reduction exists, and, in fact, condition (2.3.3) fails to hold. Finally, an example of a possible k-polysymplectic reduction is provided where (2.3.3) and (2.3.4) are not simultaneously satisfied. To keep the exposition simple and highlight the main ideas, in this subsection all k-polysymplectic momentum maps are assumed to be Ad^{*k}-equivariant, as in [62, 107]. Furthermore, an example is presented where condition (2.3.3) is satisfied, but (2.3.4) is not. Finally, a case is given in which a k-polysymplectic reduction is possible, even though neither condition (2.3.3) nor (2.3.4) is simultaneously satisfied.

The proof of [62, Proposition 1] contains a crucial error: an inclusion is written in the opposite direction. Specifically, since $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \subset \mathrm{T}_p\mathbf{J}^{\Phi-1}_{\alpha}(\mu^\alpha)$ for $\alpha = 1, \ldots, k$ and every $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ for a regular k-value $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$, one has

$$\begin{aligned}\big\{v \in \mathrm{T}_pP \mid \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_1(\mu^1)) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_k(\mu^k)) = 0\big\}&\\ \subset \big\{v \in \mathrm{T}_pP \mid \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = 0\big\}&\,,\end{aligned}$$

contrary to what is stated at the end of page 8 in the proof of [62, Proposition 1], where the opposite inclusion is claimed

$$\begin{aligned}\big\{v \in \mathrm{T}_pP \mid \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_1(\mu^1)) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}_k(\mu^k)) = 0\big\}&\\ \supset \big\{v \in \mathrm{T}_pP \mid \omega^1(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = \cdots = \omega^k(v, \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})) = 0\big\}&\,.\end{aligned}$$

In other words, if v is perpendicular to $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ relative to each ω^α, it does not imply that v is perpendicular to each $\mathrm{T}_p\mathbf{J}^{\Phi-1}_\alpha(\mu^\alpha)$ relative to ω^α for $\alpha = 1, \ldots, k$, since the latter conditions impose a stronger constraint. Consequently, the proof of [62, Proposition 1] yields only the inclusion

$$\bigcap_{\alpha=1}^{k} (\ker \jmath^*_{\mu^\alpha}\omega^\alpha|_p) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \subset (\mathrm{T}_p(Gp)^{\perp,k})^{\perp,k} \cap \mathrm{T}_p(Gp)^{\perp,k}\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,,$$

instead of the claimed

$$\bigcap_{\alpha=1}^{k} (\ker \jmath^*_{\mu^\alpha}\omega^\alpha|_p) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \supset (\mathrm{T}_p(Gp)^{\perp,k})^{\perp,k} \cap \mathrm{T}_p(Gp)^{\perp,k}\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,.$$

As a result, the proof of Proposition 1 fails to establish the validity of condition (2.3.6), which is the necessary and sufficient condition for the existence of a k-polysymplectic Marsden–Meyer–Weinstein reduction theorem. Therefore, the statement of [62, Proposition 1] cannot be considered correct. The origin of this failure lies in the incorrectness of [62, Proposition 1] itself. Moreover, several of the related remarks following that proposition in [62] also contain inaccuracies.

A counterexample is now presented to demonstrate the invalidity of [62, Proposition 1]. Specifically, an $\mathbb{R}$-invariant k-polysymplectic Hamiltonian system associated with a two-symplectic form is constructed, which satisfies condition (2.3.4), yet does not yield a k-polysymplectic Marsden–Meyer–Weinstein reduction. Before that, it is convenient to recall some results from [107].

It was proved in [107] that $\ker \omega^\alpha_p \subset \ker \mathrm{T}_p\mathbf{J}^\Phi_\alpha$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, which allows one to define the following commutative diagram (see [107, p 12])

$$\begin{tikzcd}
\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \arrow[r, "\jmath"] \arrow[rr, bend left, "\pi^\alpha_p"] & \ker \mathrm{T}_p\mathbf{J}^\Phi_\alpha \arrow[r, "\pi"] & \dfrac{\ker \mathrm{T}_p\mathbf{J}^\Phi_\alpha}{\ker \omega^\alpha_p}
\end{tikzcd}$$

for all $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, where $\jmath$ and π are the canonical immersion and projection, respectively. For simplicity, the equivalence class of an element v in a quotient is denoted by $[v]$. To avoid making the notation too complicated, the specific meaning of $[v]$ is understood from the context.

According to Proposition 3.12 in [107], the above diagram induces the maps

$$\widetilde{\pi}^\alpha_p \colon \frac{\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})}{\mathrm{T}_p(G_{\boldsymbol{\mu}}p)} \longrightarrow \frac{\frac{\ker \mathrm{T}_p\mathbf{J}^\Phi_\alpha}{\ker \omega^\alpha_p}}{\{[(\xi_P)_p] \mid \xi \in \mathfrak{g}_{\mu^\alpha}\}}\,, \qquad \alpha = 1, \ldots, k\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,,$$

where $\mathfrak{g}_{\mu^\alpha}$ is the Lie algebra of G_{μ^α} and $\{[(\xi_P)_p] \mid \xi \in \mathfrak{g}_{\mu^\alpha}\} = \mathrm{pr}^P_\alpha(\{(\xi_P)_p \mid \xi \in \mathfrak{g}_{\mu^\alpha}\})$ and $\mathrm{pr}^P_\alpha \colon \mathrm{T}_pP \to \mathrm{T}_pP/\ker \omega^\alpha_p$ is the canonical projection onto the quotient.

The conditions (2.3.3) at $p \in P$ are equivalent to each $\widetilde{\pi}^\alpha_p$ being surjective, respectively [107, Lemma 3.15], while (2.3.4) amounts to $\bigcap_{\alpha=1}^k \ker \widetilde{\pi}^\alpha_p = 0$ (see [107, Lemma 3.16]).

Consider $P = \mathbb{R}^4$ with linear coordinates $\{x, y, z, t\}$ and the presymplectic forms

$$\omega^1 = \mathrm{d}x \wedge \mathrm{d}y\,, \qquad \omega^2 = \mathrm{d}x \wedge \mathrm{d}t + \mathrm{d}y \wedge \mathrm{d}z\,,$$

which give rise to a two-polysymplectic form $\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2$, because ω^2 is a symplectic form and $\ker \omega^1 \cap \ker \omega^2 = 0$. Consider the Lie group action

$$\Phi \colon (\lambda; x, y, z, t) \in \mathbb{R} \times \mathbb{R}^4 \mapsto (x + \lambda, y, z, t) \in \mathbb{R}^4\,.$$

The Lie algebra of fundamental vector fields of Φ is $V = \langle \partial_x \rangle \simeq \mathbb{R}$. Moreover, Φ admits a two-polysymplectic momentum map relative to $(\mathbb{R}^4, \boldsymbol{\omega})$ given by

$$\mathbf{J}^\Phi \colon (x, y, z, t) \in \mathbb{R}^4 \longmapsto \boldsymbol{\mu} = (y, t) \in \mathbb{R}^{*2},$$

which is clearly Ad^{*2}-equivariant. Additionally, any $\boldsymbol{\mu} \in \mathbb{R}^{*2}$ is a regular k-value of $\mathbf{J}^\Phi$. Consequently,

$$\mathbf{J}^{\Phi-1}(y, t) = \{(x, y, z, t) \in \mathbb{R}^4 \mid x, z \in \mathbb{R}\} \simeq \mathbb{R}^2$$

is a submanifold for every $(y, t) \in \mathbb{R}^{*2}$ and

$$\mathrm{T}_p\mathbf{J}^{\Phi-1}(y, t) = \langle \partial_x, \partial_z \rangle\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(y, t)\,.$$

Moreover, $G_{\boldsymbol{\mu}} = \mathbb{R}$ for each $\boldsymbol{\mu} = (y, t) \in \mathbb{R}^{*2}$ and $G_{\boldsymbol{\mu}}$ acts freely and properly on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$.

It is now shown that condition (2.3.4) does not imply either the reduction condition (2.3.6) for the two-polysymplectic form $\boldsymbol{\omega}$ nor (2.3.3).

In this example, $\boldsymbol{\mu} = (y, t)$ with $\mu^1 = y$ and $\mu^2 = t$, while

$$\ker \mathrm{T}_p\mathbf{J}^\Phi_1 = \langle \partial_x, \partial_z, \partial_t \rangle\,, \quad \ker \omega^1 = \langle \partial_t, \partial_z \rangle\,, \quad \ker \mathrm{T}_p\mathbf{J}^\Phi_2 = \langle \partial_x, \partial_y, \partial_z \rangle\,, \quad \ker \omega^2 = 0\,,$$

and

$$\{[(\xi_P)_p] \mid \xi \in \mathfrak{g}_{\mu^1}\} = \langle [\partial_x] \rangle\,, \qquad \{[(\xi_P)_p] \mid \xi \in \mathfrak{g}_{\mu^2}\} = \langle [\partial_x] \rangle\,,$$

for any $p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. The associated maps read

$$\widetilde{\pi}^1_p \colon \langle[\partial_z]\rangle = \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \longmapsto \langle 0\rangle = (\ker \mathrm{T}_p\mathbf{J}^\Phi_1 / \ker\omega^1_p)/\langle[\partial_x]\rangle$$

and

$$\widetilde{\pi}^2_p \colon \langle[\partial_z]\rangle = \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \longmapsto \langle[\partial_y],[\partial_z]\rangle = (\ker \mathrm{T}_p\mathbf{J}^\Phi_2 / \ker\omega^2_p)/\langle[\partial_x]\rangle\,.$$

Since $\widetilde{\pi}^2_p([\partial_z]) = [\partial_z]$, one obtain

$$\ker\widetilde{\pi}^1_p = \langle[\partial_z]\rangle\,, \qquad \ker\widetilde{\pi}^2_p = \langle 0\rangle\,.$$

Thus, $\ker\widetilde{\pi}^1_p \cap \ker\widetilde{\pi}^2_p = 0$, so that condition (2.3.4) is satisfied. However, $\operatorname{Im}\widetilde{\pi}^2_p = \langle[\partial_z]\rangle$ and $\widetilde{\pi}^2_p$ is not surjective. Therefore, condition (2.3.3) does not hold for $\alpha = 2$.

Furthermore, both forms ω^1 and ω^2 become isotropic when restricted to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, and their induce the zero two-forms on the reduced manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$, which is one-dimensional. Hence, no two-polysymplectic form is induced on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$, despite the fact that condition (2.3.4) holds.

An explicit computation confirms that condition (2.3.4) is satisfied in the above example, while condition (2.3.3) is not. This directly shows that Proposition 1 from [62] fails and that condition (2.3.4) does not imply condition (2.3.3). The construction also illustrates the method by which the counterexample was obtained.

The failure of condition (2.3.3) in this setting is due to the non-surjectivity of the map $\widetilde{\pi}^2_p$. Recall that

$$\ker \mathrm{T}_p\mathbf{J}^\Phi_2 = \langle\partial_x,\partial_y,\partial_z\rangle\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,,$$

while

$$\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) + \ker\omega^2_p + \mathrm{T}_p(G_{\mu^2}p) = \langle\partial_x,\partial_z\rangle + \{0\} + \langle\partial_x\rangle = \langle\partial_x,\partial_z\rangle\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,.$$

On the other hand, condition (2.3.4) is satisfied since

$$\mathrm{T}_p(G_{\boldsymbol{\mu}}p) = \langle\partial_x\rangle$$

and

$$(\ker\omega^1_p + \mathrm{T}_p(G_{\mu^1}p)) \cap (\ker\omega^2_p + \mathrm{T}_p(G_{\mu^2}p)) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \langle\partial_x\rangle\,,$$

reads

$$(\langle\partial_t,\partial_z\rangle + \langle\partial_x\rangle) \cap (\langle 0\rangle + \langle\partial_x\rangle) \cap \langle\partial_x,\partial_z\rangle = \langle\partial_x\rangle\,.$$

The example is constructed so that the reduced space $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is one-dimensional. Therefore, the reduction of $\boldsymbol{\omega}$ does not give a two-polysymplectic form on the quotient manifold.

The following examples illustrate some relations between the conditions (2.3.3), (2.3.4) and the existence of k-polysymplectic Marsden–Meyer–Weinstein reductions.

Example 2.3.18. This example shows that if condition (2.3.3) is satisfied, then condition (2.3.4) does not need to hold. Consider a two-polysymplectic manifold $(\mathbb{R}^6, \boldsymbol{\omega})$. Let $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ be global linear coordinates on $\mathbb{R}^6$ and define

$$\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 = (\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_5 \wedge \mathrm{d}x_6) \otimes e_1 + (\mathrm{d}x_3 \wedge \mathrm{d}x_4 + \mathrm{d}x_5 \wedge \mathrm{d}x_6) \otimes e_2\,.$$

Then, $\ker\omega^1_p = \langle\partial_3,\partial_4\rangle$, $\ker\omega^2_p = \langle\partial_1,\partial_2\rangle$, and $\ker\omega^1_p \cap \ker\omega^2_p = 0$ for every $p \in \mathbb{R}^6$. This guarantees that $\boldsymbol{\omega}$ is a two-polysymplectic form.

Consider the Lie group action

$$\Phi\colon (\lambda; x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R} \times \mathbb{R}^6 \mapsto (x_1 + \lambda, x_2, x_3 + \lambda, x_4, x_5, x_6) \in \mathbb{R}^6\,,$$

whose the fundamental vector field reads $\langle \partial_1+\partial_3\rangle$. The associated two-polysymplectic momentum map is given by

$$\mathbf{J}^\Phi\colon (x_1,x_2,x_3,x_4,x_5,x_6)\in\mathbb{R}^6\longmapsto (x_2,x_4)=\boldsymbol{\mu}\in\mathbb{R}^{*2},$$

which is Ad^{*2}-equivariant and every $\boldsymbol{\mu}=(x_2,x_4)\in\mathbb{R}^{*2}$ is a regular two-value of $\mathbf{J}^\Phi$. Therefore,

$$\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})=\{(x_1,x_2,x_3,x_4,x_5,x_6)\in\mathbb{R}^6\mid x_1,x_3,x_5,x_6\in\mathbb{R}\}\simeq\mathbb{R}^4$$

is a submanifold of $\mathbb{R}^6$ for every $\boldsymbol{\mu}\in\mathbb{R}^{*2}$. Furthermore,

$$\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})=\langle\partial_1,\partial_3,\partial_5,\partial_6\rangle\,,\qquad \forall p\in\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,.$$

Hence,

$$\ker \mathrm{T}_p\mathbf{J}^\Phi_1=\langle\partial_1,\partial_3,\partial_4,\partial_5,\partial_6\rangle,\qquad \ker\mathrm{T}_p\mathbf{J}^\Phi_2=\langle\partial_1,\partial_2,\partial_3,\partial_5,\partial_6\rangle\,.$$

Condition (2.3.3) is satisfied since both sides of the condition are equal to

$$\begin{aligned}\langle\partial_1,\partial_3,\partial_4,\partial_5,\partial_6\rangle&=\langle\partial_1,\partial_3,\partial_5,\partial_6\rangle+\langle\partial_3,\partial_4\rangle+\langle\partial_1+\partial_3\rangle\,,\\ \langle\partial_1,\partial_2,\partial_3,\partial_5,\partial_6\rangle&=\langle\partial_1,\partial_3,\partial_5,\partial_6\rangle+\langle\partial_1,\partial_2\rangle+\langle\partial_1+\partial_3\rangle\,,\end{aligned}$$

for $\mathbf{J}^\Phi_1$ and $\mathbf{J}^\Phi_2$, respectively. However, condition (2.3.4) is not fulfilled. In particular,

$$\begin{aligned}&\bigcap_{\alpha=1}^{2}\left(\ker\omega^\alpha_p+\mathrm{T}_p(G_{\mu^\alpha}p)\right)\cap\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\\ &\qquad=(\langle\partial_3,\partial_4\rangle+\langle\partial_1+\partial_3\rangle)\cap(\langle\partial_1,\partial_2\rangle+\langle\partial_1+\partial_3\rangle)\cap\langle\partial_1,\partial_3,\partial_5,\partial_6\rangle\\ &\qquad\qquad=\langle\partial_1,\partial_3\rangle\neq\langle\partial_1+\partial_3\rangle=\mathrm{T}_p(G_{\boldsymbol{\mu}}p)\,,\end{aligned}$$

for any $p\in\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. By [107, Lemmas 3.15 and 3.16], the maps $\widetilde{\pi}^1_p$ and $\widetilde{\pi}^2_p$ are surjective but $\ker\widetilde{\pi}^1_p\cap\ker\widetilde{\pi}^2_p\neq 0$.

This can be explicitly verified by computing $\widetilde{\pi}^\alpha_p$ for $\alpha=1,2$. Indeed, for each $p\in\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, one has

$$\begin{aligned}\widetilde{\pi}^1_p&\colon\langle[\partial_1],[\partial_5],[\partial_6]\rangle\in\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p)\longmapsto\langle[\partial_5],[\partial_6]\rangle=(\ker\mathrm{T}_p\mathbf{J}^\Phi_1/\ker\omega^1_p)/\langle[\partial_1+\partial_3]\rangle\,.\\ \widetilde{\pi}^2_p&\colon\langle[\partial_1],[\partial_5],[\partial_6]\rangle\in\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p)\longmapsto\langle[\partial_5],[\partial_6]\rangle=(\ker\mathrm{T}_p\mathbf{J}^\Phi_2/\ker\omega^2_p)/\langle[\partial_1+\partial_3]\rangle\,.\end{aligned}$$

Note that $[\partial_1+\partial_3]=[\partial_1]$ in the first line, while in the second $[\partial_1+\partial_3]=[\partial_3]$. △

Example 2.3.19. This example illustrates that the conditions stated in [107] for the k-polysymplectic Marsden–Meyer–Weinstein reduction are sufficient but not necessary. In this respect, there are cases where reduction is possible, condition (2.3.4) holds, while condition (2.3.3) does not. Consider a two-polysymplectic manifold $(\mathbb{R}^7,\boldsymbol{\omega})$, where $\{x_1,\ldots,x_7\}$ are global linear coordinates and

$$\begin{aligned}\boldsymbol{\omega}&=\omega^1\otimes e_1+\omega^2\otimes e_2\\ &=(\mathrm{d}x_1\wedge\mathrm{d}x_2+\mathrm{d}x_5\wedge\mathrm{d}x_7+\mathrm{d}x_3\wedge\mathrm{d}x_6)\otimes e_1+(\mathrm{d}x_3\wedge\mathrm{d}x_4+\mathrm{d}x_5\wedge\mathrm{d}x_6)\otimes e_2\,.\end{aligned}$$

This give rise to a two-polysymplectic structure on $\mathbb{R}^7$ since $\ker\omega^1=\langle\partial_4\rangle$, $\ker\omega^2=\langle\partial_1,\partial_2,\partial_7\rangle$ and $\ker\omega^1\cap\ker\omega^2=0$.

Consider the Lie group action $\Phi\colon\mathbb{R}\times\mathbb{R}^7\to\mathbb{R}^7$ corresponding to translations along the x_5 coordinate. The associated Lie algebra of the fundamental vector fields is $\langle\partial_5\rangle$. A two-polysymplectic momentum map corresponding to this action is given by

$$\mathbf{J}^\Phi\colon(x_1,x_2,x_3,x_4,x_5,x_6,x_7)\in\mathbb{R}^7\longmapsto(x_7,x_6)=\boldsymbol{\mu}\in\mathbb{R}^{2*}\,.$$

This momentum map is Ad^{*2}-equivariant, and every $\boldsymbol{\mu}\in\mathbb{R}^{2*}$ is a regular two-value. Therefore,

$$\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})=\{(x_1,x_2,x_3,x_4,x_5,x_6,x_7)\in\mathbb{R}^7\mid x_1,x_2,x_3,x_4,x_5\in\mathbb{R}\}\simeq\mathbb{R}^5$$

is a submanifold of $\mathbb{R}^7$ for every $\boldsymbol{\mu} = (x_7, x_6) \in \mathbb{R}^2$, and

$$\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \langle \partial_1, \partial_2, \partial_3, \partial_4, \partial_5 \rangle \,, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,.$$

Condition (2.3.4) is satisfied, while (2.3.3) for $\mathbf{J}^\Phi_1$ is not since

$$\widetilde{\pi}^1_p \colon \langle [\partial_1], [\partial_2], [\partial_3], [\partial_4] \rangle \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \mapsto \langle [\partial_1], [\partial_2], [\partial_3], [\partial_6] \rangle = (\ker \mathrm{T}_p\mathbf{J}^\Phi_1 / \ker \omega^1_p)/\langle [\partial_5] \rangle\,.$$

Therefore, $\widetilde{\pi}^1_p$ is not surjective. Nevertheless, the reduced manifold $P_{\boldsymbol{\mu}} = \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_p(G_{\boldsymbol{\mu}}p) \simeq \mathbb{R}^4$ inherits a two-polysymplectic structure given by

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \otimes e_1 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 \otimes e_2\,,$$

in the variables x_1, x_2, x_3, x_4 naturally defined in $P_{\boldsymbol{\mu}}$.

In summary, this example shows that although both conditions (2.3.3) and (2.3.4) suffice to ensure the validity of a k-polysymplectic Marsden–Meyer–Weinstein reduction, they are not necessary. △

2.3.7 Example: The product of k symplectic manifolds

An illustrative example of a k-polysymplectic manifold is now presented, along with the application of Theorem 2.3.14, as discussed in [107]. This construction demonstrates the k-polysymplectic Marsden–Meyer–Weinstein reduction. Numerous applications are formulated in this k-polysymplectic framework, including one of the physical examples analysed in Section 3.4.

Let $P = P_1 \times \cdots \times P_k$ for some symplectic manifolds $(P_\alpha, \omega^\alpha)$ with $\alpha = 1, \ldots, k$. If $\mathrm{pr}_\alpha \colon P \to P_\alpha$ is the canonical projection onto the α-th component, P_α, in P. Then $(P, \boldsymbol{\omega} = \sum_{\alpha=1}^k \mathrm{pr}^*_\alpha \omega^\alpha \otimes e_\alpha)$ is a k-polysymplectic manifold. To simplify the notation, $\mathrm{pr}^*_\alpha \omega^\alpha$ is simply denoted as ω^α. Assume that a Lie group action $\Phi^\alpha \colon G_\alpha \times P_\alpha \to P_\alpha$ admits a symplectic momentum map $\mathbf{J}^{\Phi^\alpha} \colon P_\alpha \to \mathfrak{g}^*_\alpha$ and each Φ^α acts in a quotientable manner on the level sets given by weak regular values of $\mathbf{J}^{\Phi^\alpha}$ for each $\alpha = 1, \ldots, k$.

Define the Lie group action

$$\Phi \colon (g_1, \ldots, g_k, x_1, \ldots, x_k) \in G \times P \longmapsto (\Phi^1_{g_1}(x_1), \ldots, \Phi^k_{g_k}(x_k)) \in P\,, \tag{2.3.8}$$

where $G = G_1 \times \cdots \times G_k$. Then, $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ is the Lie algebra of G. The k-polysymplectic momentum map associated with Φ reads

$$\mathbf{J} \colon (x_1, \ldots, x_k) \in P \longmapsto (0, \ldots, \mathbf{J}^\alpha, \ldots, 0) \otimes e_\alpha \in \mathfrak{g}^{*k}\,,$$

where $\mathbf{J}^\alpha(x_1, \ldots, x_k) = \mathbf{J}^{\Phi^\alpha}(x_\alpha)$ for $\alpha = 1, \ldots, k$ and $\mathfrak{g}^* = \mathfrak{g}^*_1 \times \cdots \times \mathfrak{g}^*_k$ is the dual space to $\mathfrak{g}$.

Let $\mu^\alpha \in \mathfrak{g}^*_\alpha$ be a weak regular value of $\mathbf{J}^{\Phi^\alpha} \colon P_\alpha \to \mathfrak{g}^*_\alpha$ for each $\alpha = 1, \ldots, k$. Then, $\boldsymbol{\mu} = (0, \ldots, \mu^\alpha, \ldots, 0) \otimes e_\alpha \in \mathfrak{g}^{*k}$ is a weak regular k-value of $\mathbf{J}$, and Φ acts in a quotientable on the level sets of $\mathbf{J}$.

Therefore, if $p = (x_1, \ldots, x_k) \in \mathbf{J}^{-1}(\boldsymbol{\mu})$, it follows that

$$\begin{aligned}
\ker \mathrm{T}_p\mathbf{J}^{\Phi^\alpha} &= \mathrm{T}_{x_1}P_1 \oplus \cdots \oplus \ker \mathrm{T}_{x_\alpha}\mathbf{J}^{\Phi^\alpha} \oplus \cdots \oplus \mathrm{T}_{x_k}P_k\,,\\
\mathrm{T}_p\mathbf{J}^{-1}(\boldsymbol{\mu}) &= \ker \mathrm{T}_{x_1}\mathbf{J}^{\Phi^1} \oplus \cdots \oplus \ker \mathrm{T}_{x_k}\mathbf{J}^{\Phi^k}\,,\\
\ker \omega^\alpha_p &= \mathrm{T}_{x_1}P_1 \oplus \cdots \oplus \mathrm{T}_{x_{\alpha-1}}P_{\alpha-1} \oplus \{0\} \oplus \mathrm{T}_{x_{\alpha+1}}P_{\alpha+1} \oplus \cdots \oplus \mathrm{T}_{x_k}P_k\,,\\
\mathrm{T}_p\left(G^{\Delta^\alpha}_{\mu^\alpha}p\right) &= \mathrm{T}_{x_1}\left(G_1x_1\right) \oplus \cdots \oplus \mathrm{T}_{x_\alpha}\left(G^{\Delta^\alpha}_{\alpha\mu^\alpha}x_\alpha\right) \oplus \cdots \oplus \mathrm{T}_{x_k}\left(G_kx_k\right)\,,\\
\mathrm{T}_p\left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}p\right) &= \mathrm{T}_{x_1}\left(G^{\Delta^1}_{1\mu^1}x_1\right) \oplus \cdots \oplus \mathrm{T}_{x_k}\left(G^{\Delta^k}_{k\mu^k}x_k\right)\,.
\end{aligned}$$

Then, it follows immediately that

$$\ker \mathrm{T}_p\mathbf{J}^{\Phi^\alpha} = \mathrm{T}_p\mathbf{J}^{-1}(\mu) + \ker \omega^\alpha_p + \mathrm{T}_p\left(G^{\Delta^\alpha}_{\mu^\alpha}p\right)\,, \qquad \alpha = 1, \ldots, k\,,$$

and

$$\mathrm{T}_p\left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}p\right) = \bigcap_{\beta=1}^{k}\left(\ker\omega^{\beta}_p + \mathrm{T}_p\left(G^{\Delta^\beta}_{\mu^\beta}p\right)\right) \cap \mathrm{T}_p\mathbf{J}^{-1}(\boldsymbol{\mu}\,,$$

for every weak regular k-value $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ and $p \in \mathbf{J}^{-1}(\boldsymbol{\mu})$.

According to Theorem 2.3.14, these conditions guarantee that the reduced space $\mathbf{J}^{-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ carries a natural k-polysymplectic structure. Furthermore,

$$\mathbf{J}^{-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} \simeq \mathbf{J}^{\Phi^1 -1}(\mu^1)/G^{\Delta^1}_{1\mu^1} \times \cdots \times \mathbf{J}^{\Phi^k -1}(\mu^k)/G^{\Delta^k}_{k\mu^k}\,.$$

△

2.4 k-Polycosymplectic Marsden–Meyer–Weinstein reduction

This section presents several fundamental results, including a k-polycosymplectic Marsden–Meyer–Weinstein reduction theorem that does not require the Ad^{*k}-equivariance of the corresponding k-polycosymplectic momentum map. As a consequence, k-polycosymplectic geometry arises as a particular case of k-polysymplectic geometry. It is established by constructing a k-polysymplectic structure on a manifold of higher dimension derived from a given k-polycosymplectic structure. Finally, a new Marsden–Meyer–Weinstein reduction is introduced, namely the reduction from a k-polycosymplectic structure to an ℓ-polysymplectic structure for $k \geq \ell$. All of the techniques presented are illustrated through an example.

2.4.1 k-Polycosymplectic momentum maps

The aim of this section is to develop the notion of a k-polycosymplectic momentum map by extending the construction used in the cosymplectic setting to the k-polycosymplectic framework introduced in Subsection 2.2.2. The definition of a k-polycosymplectic momentum map that is Ad^{*k}-equivariant is provided.

Definition 2.4.1. A Lie group action $\Phi\colon G \times M \to M$ is said to be a *k-polycosymplectic Lie group action* relative to the k-polycosymplectic manifold $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ if, for each $g \in G$, the diffeomorphism $\Phi_g\colon M \to M$ satisfies $\Phi_g^*\boldsymbol{\omega} = \boldsymbol{\omega}$ and $\Phi_g^*\boldsymbol{\tau} = \boldsymbol{\tau}$.

Definition 2.4.2. A *k-polycosymplectic momentum map* for a Lie group action $\Phi\colon G \times M \to M$ relative to a k-polycosymplectic manifold $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ such that ξ_M takes values in $\ker\boldsymbol{\tau}$ for every $\xi \in \mathfrak{g}$, is a map $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^{*k}$ satisfying that

$$\iota_{\xi_M}\boldsymbol{\omega} = \mathrm{d}\left\langle \mathbf{J}^\Phi, \xi\right\rangle = \mathrm{d}J^\Phi_\xi\,, \quad \iota_{\xi_M}\boldsymbol{\tau} = 0\,, \quad \mathscr{L}_{R_\alpha}J^\Phi_\xi = 0\,, \quad \forall \xi \in \mathfrak{g}\,, \quad \alpha = 1, \ldots, k\,.$$

In this context, similarly to k-polysymplectic setting, for each fixed $\xi \in \mathfrak{g}$, the function J^Φ_ξ takes values in $\mathbb{R}^k$. In terms of the notation introduced in Definition 2.3.2, the first and second conditions can be rewritten as

$$\iota_{\boldsymbol{\xi}_M}\boldsymbol{\omega} = \mathrm{d}\left\langle \mathbf{J}^\Phi, \boldsymbol{\xi}\right\rangle \quad \text{and} \quad \iota_{\boldsymbol{\xi}_M}\boldsymbol{\tau} = 0, \qquad \forall \boldsymbol{\xi} \in \mathfrak{g}^k,$$

where $\boldsymbol{\xi} = (0, \ldots, \overset{(\alpha)}{\xi}, \ldots, 0) \in \mathfrak{g}^k$ for any $\xi \in \mathfrak{g}$ and $\alpha = 1, \ldots, k$. The Reeb vector fields $R_1, \ldots, R_k$ corresponding to $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ are tangent to the level sets of $\mathbf{J}^\Phi$. However, $R_1, \ldots, R_k$ are not tangent to the orbits of Φ since $\boldsymbol{\tau}$ is required to vanish when restricted to the tangent space to the orbits of Φ.

The following definition introduces the standard notion of Ad^{*k}-equivariance of a k-polycosymplectic momentum map, which is common in the literature. However, it showed in the next subsection that this assumption is not essential for the reduction theory developed later on.

Definition 2.4.3. A k-polycosymplectic momentum map $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^{*k}$ is Ad^{*k}*-equivariant* if it satisfies

$$\mathbf{J}^\Phi \circ \Phi_g = \mathrm{Ad}^{*k}_{g^{-1}} \circ\ \mathbf{J}^\Phi, \qquad \forall g \in G\,,$$

where

$$\begin{aligned} \mathrm{Ad}^{*k}\colon G\times\mathfrak{g}^{*k} &\longrightarrow \mathfrak{g}^{*k}\\ (g,\boldsymbol{\mu}) &\longmapsto \overbrace{(\mathrm{Ad}^*_{g^{-1}}\otimes\cdots\otimes\mathrm{Ad}^*_{g^{-1}})}^{k-\text{times}}(\boldsymbol{\mu}). \end{aligned}$$

In other words, for every $g\in G$, the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "\mathbf{J}^\Phi"] \arrow[d, "\Phi_g"] & \mathfrak{g}^{*k} \arrow[d, "\mathrm{Ad}^{*k}_{g^{-1}}"] \\ M \arrow[r, "\mathbf{J}^\Phi"] & \mathfrak{g}^{*k} \end{tikzcd} .$$

Note that k-polycosymplectic Lie group actions are the analogue of the k-polysymplectic Lie group actions in the k-polycosymplectic setting.

To simplify the notation, the following definition is introduced.

Definition 2.4.4. The four-tuple $(M^{\boldsymbol{\omega}}_{\boldsymbol{\tau}}, h, \mathbf{J}^\Phi)$ is called a *G-invariant k-polycosymplectic Hamiltonian system* if it consists of a k-polycosymplectic manifold $(M,\boldsymbol{\tau},\boldsymbol{\omega})$, a k-polycosymplectic Lie group action $\Phi\colon G\times M\to M$ such that $\Phi_g^*h=h$ for every $g\in G$, and the k-polycosymplectic momentum map $\mathbf{J}^\Phi\colon M\to\mathfrak{g}^{*k}$ related to Φ. An *Ad^{*k}-equivariant G-invariant k-polycosymplectic Hamiltonian system* is a G-invariant k-polycosymplectic Hamiltonian whose k-polycosymplectic momentum map is Ad^{*k}-equivariant.

2.4.2 General k-polycosymplectic momentum maps

This section establishes that the standard requirement of Ad^{*k}-equivariance for a k-polycosymplectic momentum map $\mathbf{J}^\Phi\colon M\to\mathfrak{g}^{*k}$ can be replaced by a more general form of equivariance, similar to in the k-polysymplectic setting introduced in Subsection 2.3.2. Analogously, it is demonstrated that the momentum map $\mathbf{J}^\Phi$ is $\boldsymbol{\Delta}$-equivariant with respect to a k-polycosymplectic affine Lie group action on $\mathfrak{g}^{*k}$. The proofs of the statements presented in this section follow analogously to their counterparts in the k-polysymplectic setting in Subsection 2.3.2 and are therefore omitted. The underlying techniques are analogous to those presented in Subsection 2.2.4, where the concept of momentum maps on symplectic manifolds was extended to the cosymplectic manifolds. However, the k-polycosymplectic case is much more technically involved.

Proposition 2.4.5. *Let $(M^{\boldsymbol{\omega}}_{\boldsymbol{\tau}}, h, \mathbf{J}^\Phi)$ be a G-invariant k-polycosymplectic Hamiltonian system. Consider the functions on M of the form*

$$\boldsymbol{\psi}_{g,\boldsymbol{\xi}} = \mathbf{J}^\Phi_{\boldsymbol{\xi}}\circ\Phi_g - \mathbf{J}^\Phi_{\mathrm{Ad}^k_{g^{-1}}\boldsymbol{\xi}}\colon M\to\mathbb{R}\,,\quad \forall g\in G\,,\quad \forall\boldsymbol{\xi}\in\mathfrak{g}^k\,.$$

Then, each function $\boldsymbol{\psi}_{g,\boldsymbol{\xi}}$ is constant on M for all $g\in G$ and $\boldsymbol{\xi}\in\mathfrak{g}^k$. Furthermore, the map $\boldsymbol{\sigma}\colon G\ni g\mapsto\boldsymbol{\sigma}(g)\in\mathfrak{g}^{*k}$, defined by the relation

$$\langle\boldsymbol{\sigma}(g),\boldsymbol{\xi}\rangle = \boldsymbol{\psi}_{g,\boldsymbol{\xi}}$$

satisfies the cocycle condition

$$\boldsymbol{\sigma}(g_1g_2) = \boldsymbol{\sigma}(g_1) + \mathrm{Ad}^{*k}_{g_1^{-1}}\boldsymbol{\sigma}(g_2)\,,\qquad \forall g_1,g_2\in G\,.$$

The proof of Proposition 2.4.5 is essentially the same as the proof of Proposition 2.3.5.

Note that the map $\boldsymbol{\sigma}$ introduced in Proposition 2.4.5 also can be brought into the form

$$\boldsymbol{\sigma}(g) = \mathbf{J}^\Phi\circ\Phi_g - \mathrm{Ad}^{*k}_{g^{-1}}\mathbf{J}^\Phi = (\sigma^1(g),\dots,\sigma^k(g))\in\mathfrak{g}^{*k}\,,$$

where $\sigma^\alpha(g) = \mathbf{J}^\Phi_\alpha \circ \Phi_g - \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^\Phi_\alpha$ for each $\alpha = 1, \ldots, k$. This map $\boldsymbol{\sigma}$ is called the *co-adjoint cocycle associated with* $\mathbf{J}^\Phi$. The vanishing of $\boldsymbol{\sigma}$ characterises the Ad^{*k}-equivariance of $\mathbf{J}^\Phi$, that is, $\mathbf{J}^\Phi$ is Ad^k-equivariant if and only if $\boldsymbol{\sigma} = 0$. Furthermore, any k-polycosymplectic Lie group action that admits a k-polycosymplectic momentum map induces a well-defined cohomology class $[\boldsymbol{\sigma}]$.

An analogue of Proposition 2.3.7 is now introduced to show that a k-polycosymplectic momentum map $\mathbf{J}^\Phi$ determines a k-polycosymplectic affine Lie group action $\boldsymbol{\Delta}$ of G on $\mathfrak{g}^{*k}$ satisfying the relation

$$\mathbf{J}^\Phi \circ \Phi_g = \boldsymbol{\Delta}_g \circ \mathbf{J}^\Phi \qquad \forall g \in G.$$

Proposition 2.4.6. *Let $\mathbf{J}^\Phi \colon M \to \mathfrak{g}^{*k}$ be a k-polycosymplectic momentum map associated with a k-polycosymplectic Lie group action Φ, and let $\boldsymbol{\sigma}$ denote the corresponding coadjoint cocycle. Then,*

(1) *the map*

$$\boldsymbol{\Delta} \colon G \times \mathfrak{g}^{*k} \ni (g, \boldsymbol{\mu}) \longmapsto \mathrm{Ad}^{*k}_{g^{-1}}\boldsymbol{\mu} + \boldsymbol{\sigma}(g) = \boldsymbol{\Delta}_g\boldsymbol{\mu} \in \mathfrak{g}^{*k},$$

is a Lie group action of G on $\mathfrak{g}^{*k}$,

(2) *the k-polycosymplectic momentum map $\mathbf{J}^\Phi$ is equivariant with respect to $\boldsymbol{\Delta}$, in other words, every $g \in G$ gives rise to a commutative diagram*

$$\begin{tikzcd} M \arrow[r, "\mathbf{J}^\Phi"] \arrow[d, "\Phi_g"] & \mathfrak{g}^{*k} \arrow[d, "\boldsymbol{\Delta}_g"] \\ M \arrow[r, "\mathbf{J}^\Phi"] & \mathfrak{g}^{*k} \end{tikzcd}$$

As in the k-polysymplectic case (cf. Proposition 2.3.7), the proof of Proposition 2.4.6 follows analogously and is therefore omitted.

The action $\boldsymbol{\Delta}$ can be equivalently written in componentwise form as

$$\begin{aligned} \boldsymbol{\Delta}(g, \mu^1, \ldots, \mu^k) &= (\mathrm{Ad}^*_{g^{-1}}(\mu^1) + \sigma^1(g), \ldots, \mathrm{Ad}^*_{g^{-1}}(\mu^k) + \sigma^k(g)) \\ &= (\Delta^1(g, \mu^1), \ldots, \Delta^k(g, \mu^k)) \in \mathfrak{g}^{*k}, \end{aligned}$$

which gives rise to defining k affine Lie group actions

$$\Delta^\alpha \colon (g, \vartheta) \in G \times \mathfrak{g}^* \mapsto \mathrm{Ad}^*_{g^{-1}}(\vartheta) + \sigma^\alpha(g) \in \mathfrak{g}^*, \qquad \alpha = 1, \ldots, k.$$

2.4.3 k-Polycosymplectic Marsden–Meyer–Weinstein reduction theorem

This section presents a k-polycosymplectic Marsden–Meyer–Weinstein reduction procedure by means of a particular type of k-polysymplectic Marsden–Meyer–Weinstein reduction. Analogously to the cosymplectic case discussed in Section 2.2.4, any k-polycosymplectic manifold can be extended to a k-polysymplectic manifold of a particular kind, referred to as a fibred k-polysymplectic manifold [50]. Furthermore, the k-polycosymplectic momentum map $\mathbf{J}^\Phi \colon M \to \mathfrak{g}^{*k}$ associated with a Lie group action $\Phi \colon G \times M \to M$ determines an extended momentum map for an extended Lie group action on the product manifold $\mathbb{R}^k \times M$, endowed with a fibred k-polysymplectic structure. Consequently, the k-polycosymplectic Marsden–Meyer–Weinstein reduction boils down to the Marsden–Meyer–Weinstein reduction for fibred k-polysymplectic manifolds, developed within this subsection.

Theorem 2.4.7 shows how a k-polycosymplectic manifold $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ induces a fibred k-polysymplectic manifold $(\mathbb{R}^k \times M, \widetilde{\boldsymbol{\omega}})$ equipped with certain vector fields, called k-polysymplectic Reeb vector fields, and vice versa. In particular, the fibred k-polysymplectic manifold admits a global symmetry structure.

Recall that the exterior product of two $\mathbb{R}^k$-valued differential forms $\boldsymbol{\vartheta} = \vartheta^\alpha \otimes e_\alpha \in \Omega^{\ell_1}(M, \mathbb{R}^k)$ and $\boldsymbol{\mu} = \mu^\alpha \otimes e_\alpha \in \Omega^{\ell_2}(M, \mathbb{R}^k)$ is given by

$$\boldsymbol{\vartheta} \barwedge \boldsymbol{\mu} = \sum_{\alpha=1}^{k} (\vartheta^\alpha \wedge \mu^\alpha) \otimes e_\alpha \in \Omega^{\ell_1+\ell_2}(M, \mathbb{R}^k).$$

Theorem 2.4.7. *Let $\boldsymbol{\omega} \in \Omega^2(M, \mathbb{R}^k)$, $\boldsymbol{\tau} \in \Omega^1(M, \mathbb{R}^k)$, and let $\mathrm{pr}_M \colon \mathbb{R}^k \times M \to M$ be the canonical projection onto M. Let $\boldsymbol{u} = (u^1, \ldots, u^k)$ be a natural global coordinate system in $\mathbb{R}^k$. Then, $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ is a k-polycosymplectic manifold if and only if $(\mathbb{R}^k \times M, \mathrm{pr}_M^*\boldsymbol{\omega} + \mathrm{d}\boldsymbol{u} \,\overline{\wedge}\, \mathrm{pr}_M^*\boldsymbol{\tau} = \widetilde{\boldsymbol{\omega}})$ is a k-polysymplectic manifold, where $\mathrm{d}\boldsymbol{u} = \sum_{\alpha=1}^n \mathrm{d}u^\alpha \otimes e_\alpha$, admitting some vector fields $\widetilde{R}_1, \ldots, \widetilde{R}_k$ on $\mathbb{R}^k \times M$, so-called k-polysymplectic Reeb vector fields, such that $\iota_{\widetilde{R}_\alpha}\widetilde{\omega}^\beta = -\delta^\beta_\alpha \mathrm{d}u^\alpha$ and $\widetilde{R}_\alpha u^\beta = 0$ for $\alpha, \beta = 1, \ldots, k$.*

Proof. Note that the form $\widetilde{\boldsymbol{\omega}}$ decomposes into k components. By the argument presented in the proof of Lemma 1.3.11, it follows that $\widetilde{\boldsymbol{\omega}}$ is closed if and only if both $\boldsymbol{\omega}$ and $\boldsymbol{\tau}$ are closed.

First aim is to show that if $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ is a k-polycosymplectic manifold, then $\widetilde{\boldsymbol{\omega}}$ is non-degenerate and it possesses k-polysymplectic Reeb vector fields.

By Proposition 1.4.18, there exists a family of Reeb vector fields $R_1, \ldots, R_k$ on M associated with $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$. These vector fields can be uniquely lifted to vector fields $\widetilde{R}_1, \ldots, \widetilde{R}_k \in \mathfrak{X}(\mathbb{R}^k \times M)$ satisfying $\widetilde{R}_\beta u^\alpha = 0$ for $\alpha, \beta = 1, \ldots, k$ and they project onto $R_1, \ldots, R_k$ via pr_M. By the construction of $\widetilde{\boldsymbol{\omega}}$, the vector fields $\widetilde{R}_1, \ldots, \widetilde{R}_k$ satisfy $\iota_{\widetilde{R}_\alpha}\widetilde{\omega}^\beta = -\delta^{\widehat{\beta}}_\alpha \mathrm{d}u^{\widehat{\beta}}$ for $\alpha, \beta = 1, \ldots, k$ and therefore become k-polysymplectic Reeb vector fields related to $\widetilde{\boldsymbol{\omega}}$. Note that in $\iota_{\widetilde{R}_\alpha}\widetilde{\omega}^\beta = -\delta^{\widehat{\beta}}_\alpha \mathrm{d}u^{\widehat{\beta}}$ there is no summation over β.

Suppose that $X \in \mathfrak{X}(\mathbb{R}^k \times M)$ takes values in $\ker(\mathrm{pr}_M^*\boldsymbol{\omega} + \mathrm{d}\boldsymbol{u} \,\overline{\wedge}\, \mathrm{pr}_M^*\boldsymbol{\tau})$ at some point. Then, at that point

$$\iota_{\partial/\partial u^\alpha}\iota_X \widetilde{\boldsymbol{\omega}} = 0 \quad \Longrightarrow \quad \iota_X \mathrm{pr}_M^*\tau^\alpha = 0\,, \qquad \alpha = 1, \ldots, k\,.$$

Consequently, X takes values in $\ker \mathrm{pr}_M^*\boldsymbol{\tau}$. Moreover,

$$\iota_{\widetilde{R}_\alpha}\iota_X \widetilde{\boldsymbol{\omega}} = 0 \quad \Longrightarrow \quad X u^\alpha = 0\,, \qquad \alpha = 1, \ldots, k\,.$$

Therefore,

$$\iota_X \mathrm{pr}_M^*\boldsymbol{\omega} = 0$$

and hence $X = 0$, since $\ker \mathrm{d}\boldsymbol{u} \cap \ker \mathrm{pr}_M^*\boldsymbol{\tau} \cap \ker \mathrm{pr}_M^*\boldsymbol{\omega} = 0$. Thus, $\widetilde{\boldsymbol{\omega}}$ is non-degenerate.

Conversely, assume that $(\mathbb{R}^k \times M, \widetilde{\boldsymbol{\omega}})$ is a k-polysymplectic manifold endowed with k-polysymplectic Reeb vector fields. If a vector field X takes values in $\ker \boldsymbol{\omega} \cap \ker \boldsymbol{\tau}$, then X uniquely lifts to a vector field $\widetilde{X}$ on $\mathbb{R}^k \times M$ so that $\mathrm{pr}_{M*}\widetilde{X} = X$ and $\iota_{\widetilde{X}}\mathrm{d}\boldsymbol{u} = 0$. Since $\widetilde{\boldsymbol{\omega}}$ is assumed to be non-degenerate, it follows that

$$\iota_{\widetilde{X}}\widetilde{\boldsymbol{\omega}} = 0 \quad \Longrightarrow \quad \widetilde{X} = 0\,.$$

Therefore, $X = 0$, and consequently $\ker \boldsymbol{\omega} \cap \ker \boldsymbol{\tau} = 0$.

To prove that the k-polysymplectic Reeb vector fields $\widetilde{R}_1, \ldots, \widetilde{R}_k$ project onto vector fields on M spanning a distribution of rank k equal to the kernel of $\boldsymbol{\omega}$, observe that $\iota_{\widetilde{R}_\alpha}\widetilde{\omega}^\beta = -\delta^\beta_{\widehat{\alpha}}\mathrm{d}u^{\widehat{\alpha}}$ for $\alpha, \beta = 1, \ldots, k$. Since $\widetilde{\boldsymbol{\omega}}$ is a k-polysymplectic form invariant relative to the Lie derivatives with respect to $\partial/\partial u^1, \ldots, \partial/\partial u^k$, the definition of $\widetilde{R}_1, \ldots, \widetilde{R}_k$ implies that

$$\mathscr{L}_{\partial/\partial u^\alpha}\iota_{\widetilde{R}_\beta}\widetilde{\boldsymbol{\omega}} = 0\,.$$

Consequently, $\iota_{[\partial/\partial u^\alpha, \widetilde{R}_\beta]}\widetilde{\boldsymbol{\omega}} = 0$ for every $\alpha, \beta = 1, \ldots, k$ yields that $\widetilde{R}_1, \ldots, \widetilde{R}_k$ project onto M.

Moreover, for every $\alpha, \beta = 1, \ldots, k$, one has that

$$\iota_{\widetilde{R}_\alpha}\widetilde{\omega}^\beta = \iota_{\widetilde{R}_\alpha}\mathrm{pr}_M^*\omega^\beta + (\iota_{\widetilde{R}_\alpha}\mathrm{d}u^{\widehat{\beta}})\mathrm{pr}_M^*\tau^{\widehat{\beta}} - (\iota_{\widetilde{R}_\alpha}\mathrm{pr}_M^*\tau^{\widehat{\beta}})\mathrm{d}u^{\widehat{\beta}} = -\mathrm{d}u^{\widehat{\alpha}}\delta^\beta_{\widehat{\alpha}}\,,$$

where there is no sum over the possible values of β or α as indicated by the hatted indices. Hence, again without summing over β,

$$\iota_{\partial/\partial u^{\widehat{\beta}}}\iota_{\widetilde{R}_\alpha}\widetilde{\omega}^{\widehat{\beta}} = -\iota_{\widetilde{R}_\alpha}\mathrm{pr}_M^*\tau^\beta = -\delta^\alpha_\beta \quad \Longrightarrow \quad \langle \tau^\beta, \mathrm{pr}_{M*}\widetilde{R}_\alpha \rangle = \delta^\alpha_\beta\,, \qquad \forall \alpha, \beta = 1, \ldots, k\,, \tag{2.4.1}$$

and

$$\iota_{\widetilde{R}_\alpha}\mathrm{pr}_M^*\boldsymbol{\omega} = 0\,.$$

Condition (2.4.1) yields that the vector fields $\mathrm{pr}_{M*}\widetilde{R}_\alpha = R_\alpha$ with $\alpha = 1, \ldots, k$, span a distribution on M of rank k taking values in $\ker \boldsymbol{\omega}$. The rank of $D = \ker \boldsymbol{\omega}$ cannot exceed k. Otherwise, there would exist a non-zero tangent vector $v_x \in \ker \boldsymbol{\omega}_x \cap \ker \boldsymbol{\tau}_x$, for some $x \in M$, since the annihilator of $\langle \tau^1|_D, \ldots, \tau^k|_D \rangle$ in D would be non-zero which contradicts already established fact that $\ker \boldsymbol{\omega} \cap \ker \boldsymbol{\tau} = 0$. □

Definition 2.4.8. A k-polysymplectic manifold satisfying the conditions of Theorem 2.4.7 is referred to as a *k-polysymplectic fibred manifold.* In particular, the manifold $(\mathbb{R}^k \times M, \widetilde{\boldsymbol{\omega}})$ with

$$\widetilde{\boldsymbol{\omega}} := \mathrm{pr}_M^* \boldsymbol{\omega} + \mathrm{d}\boldsymbol{u} \,\bar{\wedge}\, \mathrm{pr}_M^* \boldsymbol{\tau} \,,$$

is called the *k-polysymplectic fibred manifold associated with the k-polycosymplectic manifold* $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$.

Remark 2.4.9. The condition on the existence of the k-polysymplectic Reeb vector fields $\widetilde{R}_\alpha$ in Theorem 2.4.7 is essential to ensure that a k-polysymplectic structure on $\mathbb{R}^k \times M$ gives rise to a k-polycosymplectic one on M. The necessity of this condition is illustrated by the following example.

Example 2.4.10. Consider $M = \mathbb{R}^4$ equipped with standard linear coordinates $\{x, y, w, v\}$, and define the closed differential forms

$$\tau^1 = \mathrm{d}y\,, \qquad \tau^2 = \mathrm{d}x\,, \qquad \omega^1 = \mathrm{d}x \wedge \mathrm{d}w\,, \qquad \omega^2 = \mathrm{d}y \wedge \mathrm{d}v\,.$$

These differential forms give rise to

$$\boldsymbol{\tau} = \tau^1 \otimes e_1 + \tau^2 \otimes e_2, \qquad \boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2\,.$$

Consider now the closed two-forms $\widetilde{\omega}^1, \widetilde{\omega}^2 \in \Omega^2(\mathbb{R}^2 \times M)$ given by

$$\widetilde{\omega}^1 = \omega^1 + \mathrm{d}u^1 \wedge \tau^1 = \mathrm{d}x \wedge \mathrm{d}w + \mathrm{d}u^1 \wedge \mathrm{d}y\,,$$

$$\widetilde{\omega}^2 = \omega^2 + \mathrm{d}u^2 \wedge \tau^2 = \mathrm{d}y \wedge \mathrm{d}v + \mathrm{d}u^2 \wedge \mathrm{d}x\,.$$

These define the $\mathbb{R}^2$-valued two-form $\widetilde{\boldsymbol{\omega}} = \widetilde{\omega}^1 \otimes e_1 + \widetilde{\omega}^2 \otimes e_2 \in \omega^2(\mathbb{R}^2 \times M, \mathbb{R}^2)$. Then,

$$\ker \widetilde{\omega}^1 = \left\langle \partial/\partial u^2, \partial/\partial v \right\rangle, \qquad \ker \widetilde{\omega}^2 = \left\langle \partial/\partial u^1, \partial/\partial w \right\rangle\,.$$

Consequently,

$$\ker \widetilde{\boldsymbol{\omega}} = \ker \left(\widetilde{\omega}^1 \otimes e_1 + \widetilde{\omega}^2 \otimes e_2 \right) = \ker \widetilde{\omega}^1 \cap \ker \widetilde{\omega}^2 = 0\,,$$

which implies that $(\mathbb{R}^2 \times M, \widetilde{\boldsymbol{\omega}})$ is a two-polysymplectic manifold.

However, a direct computation shows that $\widetilde{\boldsymbol{\omega}}$ has no two-polysymplectic Reeb vector fields. Although

$$\ker \boldsymbol{\tau} \cap \ker \boldsymbol{\omega} = \ker \tau^1 \cap \ker \tau^2 \cap \ker \omega^1 \cap \ker \omega^2 = 0\,,$$

the rank of $\ker \omega^1 \cap \ker \omega^2$ is not 2, and thus $(\boldsymbol{\tau}, \boldsymbol{\omega})$ cannot be a two-polycosymplectic structure on M. △

Let $(M^{\boldsymbol{\omega}}_{\boldsymbol{\tau}}, h, \mathbf{J}^\Phi)$ be a G-invariant k-polycosymplectic Hamiltonian system. Then, a Lie group action $\Phi\colon G \times M \to M$ and a k-polycosymplectic momentum map $\mathbf{J}^\Phi\colon M \to \mathfrak{g}^{*k}$ associated with Φ admit the following extensions to the product manifold $\mathbb{R}^k \times M$ as follows

$$\widetilde{\Phi}\colon G \times \mathbb{R}^k \times M \ni (g, \boldsymbol{u}, x) \longmapsto (\boldsymbol{u}, \Phi(g, x)) \in \mathbb{R}^k \times M\,, \tag{2.4.2}$$

and

$$\mathbf{J}^{\widetilde{\Phi}}\colon \mathbb{R}^k \times M \ni (\boldsymbol{u}, x) \longmapsto \mathbf{J}^\Phi(x) \in \mathfrak{g}^{*k}\,, \tag{2.4.3}$$

for every $(\boldsymbol{u}, x) \in \mathbb{R}^k \times M$ and every $g \in G$. In particular, $k = 1$ recovers the extension of a cosymplectic manifold to a symplectic manifold of the form $\mathbb{R} \times M$ presented in Lemma 1.3.11.

The map $\mathbf{J}^{\widetilde{\Phi}}$ gives rise to a k-polysymplectic momentum map associated with a Lie group action $\widetilde{\Phi}$ with respect to the fibred k-polysymplectic manifold $(\mathbb{R}^k \times M, \widetilde{\boldsymbol{\omega}})$, which arises as an extension of the k-polycosymplectic manifold $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$.

Lemma 2.4.11 and Lemma 2.4.12 follow immediately as a straightforward consequence of previous facts and the relation $\mathrm{pr}_{\mathbb{R}^k} \circ \widetilde{\Phi}_g = \mathrm{pr}_{\mathbb{R}^k}$ for every $g \in G$, where $\mathrm{pr}_{\mathbb{R}^k} \colon \mathbb{R}^k \times M \to \mathbb{R}^k$, is the natural projection onto $\mathbb{R}^k$.

Lemma 2.4.11. *Let* $(\mathbb{R}^k \times M, \widetilde{\boldsymbol{\omega}})$ *be a k-polysymplectic fibred manifold. Then,*

$$\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu}) \simeq \mathbb{R}^k \times \mathrm{pr}_M\big(\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})\big) \simeq \mathbb{R}^k \times \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,, \quad \mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})/G^{\Delta}_{\boldsymbol{\mu}} \simeq \mathbb{R}^k \times \big(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}\big)$$

for every weak regular k-value $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$, where the quotients $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})/G^{\Delta}_{\boldsymbol{\mu}}$ and $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ are relative to the actions of $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ on $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})$ and $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, respectively.

Lemma 2.4.12. *A k-polycosymplectic momentum map $\mathbf{J}^{\Phi} \colon M \to \mathfrak{g}^{*k}$ is $\boldsymbol{\Delta}$-equivariant with respect to a Lie group action $\Phi \colon G \times M \to M$ if and only if the associated k-polysymplectic momentum map $\mathbf{J}^{\widetilde{\Phi}} \colon \mathbb{R}^k \times M \to \mathfrak{g}^{*k}$ is $\boldsymbol{\Delta}$-equivariant relative to $\widetilde{\Phi} \colon G \times \mathbb{R}^k \times M \to \mathbb{R}^k \times M$. Additionally, $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ is a (resp. weak) regular k-value of $\mathbf{J}^{\Phi}$ if and only if $\boldsymbol{\mu}$ is a (resp. weak) regular k-value of $\mathbf{J}^{\widetilde{\Phi}}$. Moreover, $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu}) = \mathbb{R}^k \times \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})$ is quotientable by $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ if and only if $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is so.*

The k-polysymplectic Marsden–Meyer–Weinstein reduction Theorem 2.3.14, provides the conditions (2.3.3) and (2.3.4) ensuring the existence of a k-polysymplectic structure on the reduced manifold $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$.

Lemmas 2.4.11, 2.4.12, and 2.4.13 imply that the k-polysymplectic fibred manifold induced from a k-polycosymplectic one satisfies the hypotheses required for applying the reduction procedure.

Moreover, recall that the k-polysymplectic Reeb vector fields on $\mathbb{R}^k \times M$ are tangent to the level set $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})$, as the original Reeb vector fields $R_1, \ldots, R_k$ are tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and remain invariant under the extended group action $\widetilde{\Phi}$. Consequently, they project to the reduced manifold.

Theorem 2.4.14 further analyses the structure of the reduced space, asserting that the reduced k-polysymplectic form defined on $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} \simeq \mathbb{R}^k \times M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ is fibred, and thus corresponds to a k-polycosymplectic structure on $M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$. This completes the geometric construction of a k-polycosymplectic reduction manifold. The reduction of Hamiltonian dynamics on such structures is addressed subsequently.

Lemma 2.4.13. *Let $(M, \boldsymbol{\tau}, \boldsymbol{\omega})$ be a k-polycosymplectic manifold and let $(\mathbb{R}^k \times M, \widetilde{\boldsymbol{\omega}})$ be its associated k-polysymplectic fibred manifold. Then,*

$$\mathrm{T}_x(G^{\Delta}_{\boldsymbol{\mu}} x) = \bigcap_{\alpha=1}^{k} \Big((\ker \omega^\alpha_x \cap \ker \tau^\alpha_x) + \mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha} x) \Big) \cap \mathrm{T}_x \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \tag{2.4.4}$$

and

$$\ker \mathrm{T}_x \mathbf{J}^{\Phi}_\alpha = \ker \omega^\alpha_x \cap \ker \tau^\alpha_x + \mathrm{T}_x \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) + \mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha} x) \tag{2.4.5}$$

hold for every $x \in \mathbf{J}^{\Phi-1}(\mu)$ and $\alpha = 1, \ldots, k$ if and only if the expressions (2.3.3) and (2.3.4) concerning the extensions $\mathbf{J}^{\widetilde{\Phi}}$ and $\widetilde{\Phi}$ to $\mathbb{R}^k \times M$ of $\mathbf{J}^{\Phi}$ and Φ, namely

$$\mathrm{T}_p(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} p) = \bigcap_{\alpha=1}^{k} \Big(\ker \widetilde{\omega}^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha} p) \Big) \cap \mathrm{T}_p \mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})\,, \tag{2.4.6}$$

and

$$\ker(\mathrm{T}_p \mathbf{J}^{\widetilde{\Phi}}_\alpha) = \mathrm{T}_p \mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu}) + \ker \widetilde{\omega}^\alpha_p + \mathrm{T}_p(G^{\Delta^\alpha}_{\mu^\alpha} p) \tag{2.4.7}$$

are satisfied for every $p = (\boldsymbol{u}, x) \in \mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})$ and all $\alpha = 1, \ldots, k$.

Proof. Given the canonical projection $\mathrm{pr}_M\colon \mathbb{R}^k\times M\to M$ and the natural isomorphisms $\mathrm{T}_{(\boldsymbol{u},x)}(\mathbb{R}^k\times M)\simeq \mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus \mathrm{T}_xM$ for every $(\boldsymbol{u},x)\in\mathbb{R}^k\times M$, it yields that, for $\alpha=1,\ldots,k$, the following hold

$$\begin{gathered}(\ker \mathrm{pr}_M^*\omega^\alpha)_{(\boldsymbol{u},x)}=\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\ker\omega^\alpha_x\,,\qquad (\ker \mathrm{pr}_M^*\tau^\alpha)_{(\boldsymbol{u},x)}=\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\ker\tau^\alpha_x\,,\\ (\ker \mathrm{d}u^\alpha)_{(\boldsymbol{u},x)}=A^\alpha_{\boldsymbol{u}}\oplus\mathrm{T}_xM\,,\end{gathered}\tag{2.4.8}$$

where $A^\alpha_{\boldsymbol{u}}=\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\cap(\ker\mathrm{d}u^\alpha)_{(\boldsymbol{u},x)}$ and $\ker\boldsymbol{\omega}_x,\ker\boldsymbol{\tau}_x\subset\mathrm{T}_xM$ for every $(\mathbf{u},x)\in\mathbb{R}^k\times M$. The contraction of $\widetilde{\omega}^\alpha$ with $\partial/\partial u^\alpha$ and the extended Reeb vector fields $\widetilde{R}_1,\ldots,\widetilde{R}_k$ on $\mathbb{R}^k\times M$ yield, along with (2.4.8), that

$$\begin{aligned}(\ker\widetilde{\omega}^\alpha)_{(\boldsymbol{u},x)}&=(\ker\mathrm{pr}_M^*\omega^\alpha\cap\ker\mathrm{d}u^\alpha\cap\ker\mathrm{pr}_M^*\tau^\alpha)_{(\boldsymbol{u},x)}\\&=\big(\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\ker\omega^\alpha_x\big)\cap(A^\alpha_{\boldsymbol{u}}\oplus\mathrm{T}_xM)\cap\big(\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\ker\tau^\alpha_x\big)=A^\alpha_{\boldsymbol{u}}\oplus(\ker\omega^\alpha_x\cap\ker\tau^\alpha_x)\,,\end{aligned}\tag{2.4.9}$$

for every $(\boldsymbol{u},x)\in\mathbb{R}^k\times M$.

Moreover, from (2.4.2) and (2.4.3), it follows that $\mathscr{L}_{\frac{\partial}{\partial u^\alpha}}\mathbf{J}^{\widetilde{\Phi}}=0$ and $\iota_{\xi_{\mathbb{R}^k\times M}}\mathrm{d}\boldsymbol{u}=0$ hold for every $\xi\in\mathfrak{g}$ and $\alpha=1,\ldots,k$. Therefore,

$$\begin{gathered}\mathrm{T}_{(\boldsymbol{u},x)}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}(\boldsymbol{u},x))=\{0\}\oplus\mathrm{T}_x(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}x)\,,\qquad \mathrm{T}_{(\boldsymbol{u},x)}\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})=\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,,\\ \mathrm{T}_{(\boldsymbol{u},x)}(G^{\Delta^\alpha}_{\mu^\alpha}(\boldsymbol{u},x))=\{0\}\oplus\mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha}x)\,,\qquad \ker(\mathrm{T}_{(\boldsymbol{u},x)}\mathbf{J}^{\widetilde{\Phi}}_\alpha)=\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\ker\mathrm{T}_x\mathbf{J}^{\Phi}_\alpha\,,\end{gathered}\tag{2.4.10}$$

for $\alpha=1,\ldots,k$ and arbitrary $\boldsymbol{u}\in\mathbb{R}^k$ and $x\in\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Suppose that (2.4.4) and (2.4.5) hold. Consequently, the condition (2.4.4) gives

$$\begin{aligned}\mathrm{T}_{(\boldsymbol{u},x)}\left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}(\boldsymbol{u},x)\right)&=\{0\}\oplus\mathrm{T}_x(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}x)=\{0\}\oplus\bigcap_{\alpha=1}^k\Big(\ker\omega^\alpha_x\cap\ker\tau^\alpha_x+\mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha}x)\Big)\cap\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\\&=\bigcap_{\alpha=1}^k\Big(A^\alpha_{\boldsymbol{u}}\oplus(\ker\omega^\alpha_x\cap\ker\tau^\alpha_x)+\{0\}\oplus\mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha}x)\Big)\cap\big(\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\big)\\&=\bigcap_{\alpha=1}^k\Big((\ker\widetilde{\omega}^\alpha)_{(\boldsymbol{u},x)}+\mathrm{T}_{(\boldsymbol{u},x)}(G^{\Delta^\alpha}_{\mu^\alpha}(\boldsymbol{u},x))\Big)\cap\mathrm{T}_{(\boldsymbol{u},x)}\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})\,,\end{aligned}$$

and (2.3.3) amounts to

$$\begin{aligned}\ker\mathrm{T}_{(\boldsymbol{u},x)}\mathbf{J}^{\widetilde{\Phi}}_\alpha&=\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\ker\mathrm{T}_x\mathbf{J}^\Phi_\alpha=\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\Big(\ker\omega^\alpha_x\cap\ker\tau^\alpha_x+\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})+\mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha}x)\Big)\\&=A^\alpha_{\boldsymbol{u}}\oplus(\ker\omega^\alpha_x\cap\ker\tau^\alpha_x)+\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})+\{0\}\oplus\mathrm{T}_x\left(G^{\Delta^\alpha}_{\mu^\alpha}x\right)\\&=\mathrm{T}_{(\boldsymbol{u},x)}\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})+\ker\widetilde{\omega}^\alpha_{(\boldsymbol{u},x)}+\mathrm{T}_{(\boldsymbol{u},x)}\left(G^{\Delta^\alpha}_{\mu^\alpha}(\boldsymbol{u},x)\right)\,,\end{aligned}$$

where (2.4.8), (2.4.9), and (2.4.10) have been used, for every $(\boldsymbol{u},x)\in\mathbb{R}^k\times\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, every $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$, and $\alpha=1,\ldots,k$. Thus, (2.4.4) and (2.4.5) imply the conditions (2.4.6) and (2.4.7), respectively.

Conversely, assume that (2.4.6) and (2.4.7) are satisfied. Then, condition (2.4.6) implies

$$\begin{aligned}\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\ker\mathrm{T}_x\mathbf{J}^\Phi_\alpha&=\ker\mathrm{T}_{(\boldsymbol{u},x)}\mathbf{J}^{\widetilde{\Phi}}_\alpha=\mathrm{T}_{(\boldsymbol{u},x)}\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})+\ker\widetilde{\omega}^\alpha_{(\boldsymbol{u},x)}+\mathrm{T}_{(\boldsymbol{u},x)}\left(G^{\Delta^\alpha}_{\mu^\alpha}(\boldsymbol{u},x)\right)\\&=\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})+A^\alpha_{\boldsymbol{u}}\oplus(\ker\omega^\alpha_x\cap\ker\tau^\alpha_x)+\{0\}\oplus\mathrm{T}_x\left(G^{\Delta^\alpha}_{\mu^\alpha}x\right)\\&=\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\Big(\ker\omega^\alpha_x\cap\ker\tau^\alpha_x+\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})+\mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha}x)\Big)\,,\end{aligned}$$

and (2.4.6) boils down to

$$\{0\}\oplus \mathrm{T}_x(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}x)=\mathrm{T}_{(\boldsymbol{u},x)}\left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}(\boldsymbol{u},x)\right)=\bigcap_{\alpha=1}^{k}\left((\ker\widetilde{\omega}^{\alpha})_{(\boldsymbol{u},x)}+\{0\}\oplus\mathrm{T}_x\left(G^{\Delta^\alpha}_{\mu^\alpha}x\right)\right)\cap\mathrm{T}_{(\boldsymbol{u},x)}\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})$$
$$=\bigcap_{\alpha=1}^{k}\left(A^{\alpha}_{\boldsymbol{u}}\oplus(\ker\omega^{\alpha}_x\cap\ker\tau^{\alpha}_x)+\mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha}x)\right)\cap\left(\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k\oplus\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\right)$$
$$=\{0\}\oplus\bigcap_{\alpha=1}^{k}\left[(\ker\omega^{\alpha}_x\oplus\ker\tau^{\alpha}_x)+\mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha}x)\right]\cap\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,,$$

where, again the identities (2.4.8), (2.4.9), and (2.4.10) have been used, for every $(\boldsymbol{u},x)\in\mathbb{R}^k\times\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, every $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$, and $\alpha=1,\ldots,k$. Therefore, conditions (2.4.4) and (2.4.5) are equivalent with (2.4.6) and (2.4.7), respectively. This completes the proof. □

Note that, according to Lemma 2.4.13, if conditions (2.4.4) and (2.4.5) are satisfied, then Theorem 2.3.14 implies that the k-polycosymplectic manifold $(M,\boldsymbol{\tau},\boldsymbol{\omega})$ gives rise to a reduced k-polysymplectic manifold $(\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}},\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\mu}})$ associated with the extended fibred structure $(\mathbb{R}^k\times M,\widetilde{\boldsymbol{\omega}})$.

It remains to verify that the latter structure corresponds, via Theorem 2.4.7, to the reduced k-polycosymplectic manifold $(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}},\boldsymbol{\tau}_{\boldsymbol{\mu}},\boldsymbol{\omega}_{\boldsymbol{\mu}})$ obtained by applying k-polycosymplectic Marsden–Meyer–Weinstein reduction, introduced below, to the initial k-polycosymplectic manifold $(M,\boldsymbol{\tau},\boldsymbol{\omega})$.

Theorem 2.4.14. *Let $(M,\boldsymbol{\tau},\boldsymbol{\omega})$ be a k-polycosymplectic manifold and let $\mathbf{J}^\Phi\colon M\to\mathfrak{g}^{*k}$ be a k-polycosymplectic momentum map associated with a k-polycosymplectic Lie group action $\Phi\colon G\times M\to M$. Let $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$ be a weak regular k-value of $\mathbf{J}^\Phi$ and let $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ be quotientable by $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$. In addition, assume that*

$$\mathrm{T}_x(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}x)=\bigcap_{\alpha=1}^{k}\left(\ker\omega^{\alpha}_x\cap\ker\tau^{\alpha}_x+\mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha}x)\right)\cap\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,,\tag{2.4.11}$$

and

$$\ker\mathrm{T}_x\mathbf{J}^{\Phi}_{\alpha}=\ker\omega^{\alpha}_x\cap\ker\tau^{\alpha}_x+\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})+\mathrm{T}_x(G^{\Delta^\alpha}_{\mu^\alpha}x)\,,\tag{2.4.12}$$

for every $x\in\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and $\alpha=1,\ldots,k$. Then, $(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}},\boldsymbol{\tau}_{\boldsymbol{\mu}},\boldsymbol{\omega}_{\boldsymbol{\mu}})$ is a k-polycosymplectic manifold such that $\boldsymbol{\tau}_{\boldsymbol{\mu}}$ and $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ are defined univocally by

$$\pi^*_{\boldsymbol{\mu}}\boldsymbol{\tau}_{\boldsymbol{\mu}}=\jmath^*_{\boldsymbol{\mu}}\boldsymbol{\tau}\,,\qquad \pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}}=\jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega}\,,$$

where $\jmath_{\boldsymbol{\mu}}\colon\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\hookrightarrow M$ is the natural immersion and $\pi_{\boldsymbol{\mu}}\colon\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\to\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ is the canonical projection.

Proof. Consider the fibred k-polysymplectic manifold $(\mathbb{R}^k\times M,\widetilde{\boldsymbol{\omega}})$ associated with $(M,\boldsymbol{\tau},\boldsymbol{\omega})$, the extended action $\widetilde{\Phi}\colon G\times\mathbb{R}^k\times M\to\mathbb{R}^k\times M$, and extended k-polysymplectic momentum map $\mathbf{J}^{\widetilde{\Phi}}\colon\mathbb{R}^k\times M\to\mathfrak{g}^{*k}$. Denote by $\{u^1,\ldots,u^k\}$ the standard coordinate system on $\mathbb{R}^k$ which gives rise, in the standard way, to k coordinates on $\mathbb{R}^k\times M$ that, for simplicity, will be denoted in the same manner.

According to Lemma 2.4.12, if $\boldsymbol{\mu}$ is a weak regular k-value for $\mathbf{J}^\Phi$, then $\boldsymbol{\mu}$ is also a weak regular k-value for $\mathbf{J}^{\widetilde{\Phi}}$. Furthermore, $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})$ is quotientable by the restriction of $\widetilde{\Phi}$ to $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ if and only if $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is so relative to the restriction of Φ to $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$.

Moreover, Lemma 2.4.13 ensures that the conditions (2.4.11) and (2.4.12) imply that the conditions (2.3.3) and (2.3.4) for the k-polysymplectic Marsden–Meyer–Weinstein reduction on $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})$ hold. As a consequence, a k-polysymplectic Marsden–Meyer–Weinstein reduction can be performed on $\mathbb{R}^k\times\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$.

The k-polysymplectic manifold $(\mathbb{R}^k\times M,\widetilde{\boldsymbol{\omega}})$ admits, under the assumptions of the present theorem and Theorem 2.4.7, a collection of k-polysymplectic Reeb vector fields $\widetilde{R}_1,\ldots,\widetilde{R}_k$ that are tangent to the level set $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})$. This follows from the fact that the Reeb vector fields $R_1,\dots,R_k$ of the k-polycosymplectic manifold $(M,\boldsymbol{\tau},\boldsymbol{\omega})$ are tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and that $\mathrm{pr}_{M*}\widetilde{R}_\alpha = R_\alpha$ for $\alpha = 1,\dots,k$. Furthermore, the Reeb vector fields $R_1,\dots,R_k$ are invariant under the action of $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ via Φ. Consequently, the extensions $\widetilde{R}_1,\dots,\widetilde{R}_k$ are invariant under the extended action $\widetilde{\Phi}$ of $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ and satisfy $\widetilde{R}_\alpha u^\beta = 0$ for all $\alpha,\beta = 1,\dots,k$. The projections of the restrictions of these vector fields to $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})$ onto the quotient $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ define k-polysymplectic Reeb vector fields $\widetilde{R}_{1\boldsymbol{\mu}},\dots,\widetilde{R}_{k\boldsymbol{\mu}}$ on the reduced k-polysymplectic manifold $(\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}},\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\mu}})$.

In addition, the vector fields $\partial/\partial u^1,\dots,\partial/\partial u^k$ project onto the quotient $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$, which is diffeomorphic to $\mathbb{R}^k \times (\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}})$ by Lemma 2.4.11, and the corresponding projections are linearly independent. The contractions $\iota_{\partial/\partial u^\beta}\iota_{\widetilde{R}_\alpha}\widetilde{\boldsymbol{\omega}}$ are projectable from $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})$ to $\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ and are proportional, up to a non-zero constant, to δ^β_α

It is left to show how the reduced k-polysymplectic manifold $\left(\widetilde{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}} = \mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}},\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\mu}}\right)$ gives rise to a k-polycosymplectic structure on $M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$.

Consider the embedding $i_{\boldsymbol{u}}\colon x \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \ni x \mapsto (\boldsymbol{u},x) \in \mathbb{R}^k \times \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ for any $\boldsymbol{u} \in \mathbb{R}^k$. Using Lemma 2.4.11, one defines the k-polycosymplectic structure on $M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ as follows

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = i^*_{\boldsymbol{u}}\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\mu}}\,, \qquad \boldsymbol{\tau}_{\boldsymbol{\mu}} = \sum_{\alpha=1}^{k} i^*_{\boldsymbol{u}}\left(\iota_{\frac{\partial}{\partial u^\alpha}}\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\mu}}\right)\,. \tag{2.4.13}$$

Since the reduced k-polysymplectic form $\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\mu}}$ is closed and $\mathscr{L}_{\partial/\partial u^\alpha}\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\mu}} = 0$ for $\alpha = 1,\dots,k$, it follows that both $\boldsymbol{\omega}_\mu$ and $\boldsymbol{\tau}_\mu$ are closed.

Let $\mathrm{pr}_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}}\colon \mathbb{R}^k \times M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} \mapsto M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ and $\widetilde{\pi}_{\boldsymbol{\mu}}\colon \mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu}) \to \widetilde{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}}$ be the canonical projections. Then, the reduced k-polysymplectic form can be expressed as

$$\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\mu}} = \mathrm{pr}^*_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}}\boldsymbol{\omega}_{\boldsymbol{\mu}} + \mathrm{d}\boldsymbol{u} \,\bar{\wedge}\, \mathrm{pr}^*_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}}\boldsymbol{\tau}_{\boldsymbol{\mu}}.$$

Indeed, this satisfies the relations defining $\boldsymbol{\omega}_{\boldsymbol{\mu}}$ and $\boldsymbol{\tau}_{\boldsymbol{\mu}}$, and, more importantly,

$$\widetilde{\pi}^*_{\boldsymbol{\mu}}(\mathrm{pr}^*_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}}\boldsymbol{\omega}_{\boldsymbol{\mu}} + \mathrm{d}\boldsymbol{u} \,\bar{\wedge}\, \mathrm{pr}^*_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}}\boldsymbol{\tau}_{\boldsymbol{\mu}}) = \widetilde{\jmath}^*_{\boldsymbol{\mu}}\widetilde{\boldsymbol{\omega}}\,, \tag{2.4.14}$$

which determines uniquely the reduced k-polycosymplectic structure on $M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$. To prove (2.4.14), note that both sides vanish on pairs of tangent vectors belonging to $\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k$ understood as a subspace of $\mathrm{T}_{(\boldsymbol{u},x)}\mathbf{J}^{\widetilde{\Phi}-1}(\boldsymbol{\mu}) \simeq \mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k \oplus \mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Furthermore, due to the first expression in (2.4.13), both sides of equality (2.4.14) take the same values on pairs of tangent vectors of the space $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Finally, for any pair of tangent vectors belonging to $\mathrm{T}_{\boldsymbol{u}}\mathbb{R}^k$ and $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, respectively, a direct computation shows that both sides of (2.4.14) coincide. This, along with previous facts, yields that (2.4.14) holds.

Since $(\mathbb{R}^k \times \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}, \mathrm{pr}^*_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}}\boldsymbol{\omega}_{\boldsymbol{\mu}} + \mathrm{d}\boldsymbol{u}\bar{\wedge}\mathrm{pr}^*_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}}\boldsymbol{\tau}_{\boldsymbol{\mu}})$ is a k-polysymplectic manifold, by Theorem 2.3.14, and admits k-polysymplectic Reeb vector fields, then $(\widetilde{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}},\widetilde{\boldsymbol{\omega}}_{\boldsymbol{\mu}})$ is a k-polysymplectic fibred manifold and Theorem 2.4.7 yields that $(M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}},\boldsymbol{\tau}_{\boldsymbol{\mu}},\boldsymbol{\omega}_{\boldsymbol{\mu}})$ is a k-polycosymplectic manifold. To prove the following

$$\jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega} = \pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}}\,, \qquad \jmath^*_{\boldsymbol{\mu}}\boldsymbol{\tau} = \pi^*_{\boldsymbol{\mu}}\boldsymbol{\tau}_{\boldsymbol{\mu}}\,. \tag{2.4.15}$$

observe that (2.4.14) yields

$$\widetilde{\pi}^*_{\boldsymbol{\mu}}\left(\mathrm{pr}^*_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}}\boldsymbol{\omega}_{\boldsymbol{\mu}} + \mathrm{d}\boldsymbol{u} \,\bar{\wedge}\, \mathrm{pr}^*_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}}\boldsymbol{\tau}_{\boldsymbol{\mu}}\right) = \widetilde{\jmath}^*_{\boldsymbol{\mu}}\left(\mathrm{pr}^*_M\boldsymbol{\omega} + \mathrm{d}\boldsymbol{u} \,\bar{\wedge}\, \mathrm{pr}^*_M\boldsymbol{\tau}\right),$$

which amounts to

$$(\mathrm{pr}_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}} \circ \widetilde{\pi}_{\boldsymbol{\mu}})^*\boldsymbol{\omega}_{\boldsymbol{\mu}} + \mathrm{d}\boldsymbol{u} \,\bar{\wedge}\, (\mathrm{pr}_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}} \circ \widetilde{\pi}_{\boldsymbol{\mu}})^*\boldsymbol{\tau}_{\boldsymbol{\mu}} = (\mathrm{pr}_M \circ \widetilde{\jmath}_{\boldsymbol{\mu}})^*\boldsymbol{\omega} + \mathrm{d}\boldsymbol{u} \,\bar{\wedge}\, (\mathrm{pr}_M \circ \widetilde{\jmath}_{\boldsymbol{\mu}})^*\boldsymbol{\tau}. \tag{2.4.16}$$

Composing on both sides of the last equality by $i^*_{\boldsymbol{u}}$, one gets

$$(\mathrm{pr}_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}} \circ \widetilde{\pi}_{\boldsymbol{\mu}} \circ i_{\boldsymbol{u}})^*\boldsymbol{\omega}_{\boldsymbol{\mu}} = (\mathrm{pr}_M \circ \widetilde{\jmath}_{\boldsymbol{\mu}} \circ i_{\boldsymbol{u}})^*\boldsymbol{\omega}\,.$$

Since $\jmath_{\boldsymbol{\mu}} = \mathrm{pr}_M \circ \widetilde{\jmath}_{\boldsymbol{\mu}} \circ i_{\boldsymbol{u}}$ and $\mathrm{pr}_{M^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}} \circ \widetilde{\pi}_{\boldsymbol{\mu}} \circ i_{\boldsymbol{u}} = \pi_{\boldsymbol{\mu}}$, it follows that

$$\pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega},$$

which proves the first equality in (2.4.15). By contracting both sides of (2.4.16) with $\partial/\partial u^1, \ldots, \partial/\partial u^k$ and repeating the above procedure, the second equality in (2.4.15) follows. ☐

Theorem 2.4.15. *Let the assumptions of Theorem 2.4.14 be satisfied. Let* $\mathbf{X}^h = (X^h_1, \ldots, X^h_k)$ *be a* k*-polycosymplectic Hamiltonian* k*-vector field associated with a* G*-invariant Hamiltonian function* $h \in \mathscr{C}^\infty(M)$ *relative to the Lie group action* Φ*. Assume that* $\Phi_{g*}\mathbf{X}^h = \mathbf{X}^h$ *for every* $g \in G$ *and* $\mathbf{X}_h$ *is tangent to* $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$*. Then, for every* $\alpha = 1, \ldots, k$*, the flow* F^α_s *of* X^h_α *leave* $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ *invariant and induces a unique flow* K^α_s *on* $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ *satisfying* $\pi_{\boldsymbol{\mu}} \circ F^\alpha_s = K^\alpha_s \circ \pi_{\boldsymbol{\mu}}$*.*

Proof. Given the assumption that a k-polycosymplectic Hamiltonian k-vector field $\mathbf{X}^h$ is tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, it follows that each integral curve F^α_s of X^h_α with initial condition within $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is contained in $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ for all the values of its parameter $s \in \mathbb{R}$ and $\alpha = 1, \ldots, k$. Since $\Phi_{g*}\mathbf{X}^h = \mathbf{X}^h$ for every $g \in G$, one has that $\mathscr{L}_{\xi_M} X^h_\alpha = 0$ for $\alpha = 1, \ldots, k$. This gives rise to a reduced k-vector field $\mathbf{Y} = (Y_1, \ldots, Y_k)$ defined on the quotient manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$, such that $\pi_{\boldsymbol{\mu}} \circ F^\alpha_s = K^\alpha_s \circ \pi_{\boldsymbol{\mu}}$, where K^α_s is the flow of Y_α, for $\alpha = 1, \ldots, k$. Furthermore, the G-invariance of $h \in \mathscr{C}^\infty(M)$ implies that there exists a reduced Hamiltonian function $h_{\boldsymbol{\mu}} \in \mathscr{C}^\infty(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}})$ such that $\pi^*_{\boldsymbol{\mu}} h_{\boldsymbol{\mu}} = \jmath^*_{\boldsymbol{\mu}} h$.

It remains to verify that $\mathbf{Y}$ is a reduced k-polycosymplectic Hamiltonian k-vector field associated with $h_{\boldsymbol{\mu}}$. Indeed, the Reeb vector fields $R_1, \ldots, R_k$ are tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and give rise to linearly independent vector fields $R^{\mathbf{J}^\Phi}_1, \ldots, R^{\mathbf{J}^\Phi}_k$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Due to this fact and Theorem 2.4.14, it follows that

$$\begin{aligned}\mathrm{d}\pi^*_{\boldsymbol{\mu}} h_{\boldsymbol{\mu}} = \mathrm{d}\jmath^*_{\boldsymbol{\mu}} h = \jmath^*_{\boldsymbol{\mu}}\Big(\iota_{\mathbf{X}^h}\boldsymbol{\omega} + \sum_{\alpha=1}^{k}(R_\alpha h)\tau^\alpha\Big) &= \iota_{\mathbf{X}^h}\jmath^*_{\boldsymbol{\mu}}\boldsymbol{\omega} + \sum_{\alpha=1}^{k}(R^{\mathbf{J}^\Phi}_\alpha \jmath^*_{\boldsymbol{\mu}} h)\jmath^*_{\boldsymbol{\mu}}\tau^\alpha\Big) \\ &= \iota_{\mathbf{X}^h}\pi^*_{\boldsymbol{\mu}}\boldsymbol{\omega}_{\boldsymbol{\mu}} + \sum_{\alpha=1}^{k}(R^{\mathbf{J}^\Phi}_\alpha \pi^*_{\boldsymbol{\mu}} h_{\boldsymbol{\mu}})\pi^*_{\boldsymbol{\mu}}\tau^\alpha_{\boldsymbol{\mu}}\Big) = \pi^*_{\boldsymbol{\mu}}\Big(\iota_{\mathbf{Y}}\boldsymbol{\omega}_{\boldsymbol{\mu}} + \sum_{\alpha=1}^{k}(R_{\alpha\boldsymbol{\mu}} h_{\boldsymbol{\mu}})\tau^\alpha_{\boldsymbol{\mu}}\Big),\end{aligned}$$

where $\mathbf{X}^h$ denotes both a k-polycosymplectic Hamiltonian k-vector field on M and its restriction to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. Moreover,

$$\pi^*_{\boldsymbol{\mu}}(\iota_{Y_\alpha}\tau^\beta_{\boldsymbol{\mu}}) = \iota_{X^h_\alpha}(\pi^*_{\boldsymbol{\mu}}\tau^\beta_{\boldsymbol{\mu}}) = \jmath^*_{\boldsymbol{\mu}}(\iota_{X^h_\alpha}\tau^\beta) = \delta^\beta_\alpha.$$

Therefore, $\mathbf{Y}$ is a reduced k-polycosymplectic Hamiltonian k-vector field such that $\pi_{\boldsymbol{\mu}*}\mathbf{X}^h = \mathbf{Y} = \mathbf{X}^{h_{\boldsymbol{\mu}}}$ and $\pi_{\boldsymbol{\mu}} \circ F^\alpha_s = K^\alpha_s \circ \pi_{\boldsymbol{\mu}}$ holds for $\alpha = 1, \ldots, k$ and $s \in \mathbb{R}$. ☐

Additionally, Theorem 2.4.15 could be established via k-polysymplectic reduction, as stated in Theorem 2.3.14, by extending the Hamiltonian function $h \in \mathscr{C}^\infty(M)$ to $\mathbb{R}^k \times M$. This approach is examined in detail below.

Define the extended Hamiltonian function $\widetilde{h} \in \mathscr{C}^\infty(\mathbb{R}^k \times M)$ as

$$\widetilde{h}(\boldsymbol{u}, x) = h(x) - \sum_{\alpha=1}^{k} u^\alpha\,, \qquad \forall x \in M\,, \qquad \forall \boldsymbol{u} = (u_1, \ldots, u_k) \in \mathbb{R}^k\,.$$

Recall that a k-polycosymplectic Hamiltonian k-vector field $\mathbf{X}^h$ associated with h satisfies the following equations

$$\iota_{\mathbf{X}^h}\boldsymbol{\omega} = \mathrm{d}h - \sum_{\beta=1}^{k}(R_\beta h)\tau^\beta\,, \qquad \iota_{X^\alpha_h}\boldsymbol{\tau} = e^\alpha\,, \qquad \alpha = 1, \ldots, k\,.$$

The aim is to extend $\mathbf{X}^h$ to a k-polysymplectic Hamiltonian k-vector field $\mathbf{X}^{\widetilde{h}}$ associated with $\widetilde{h}$. It can be checked that $\mathbf{X}^{\widetilde{h}}$ of the form

$$\mathbf{X}^{\widetilde{h}} = \mathbf{X}^h + \sum_{\alpha=1}^{k}(R_\alpha h)\frac{\partial}{\partial u^\alpha}\,,$$

satisfies the required conditions for a k-polysymplectic Hamiltonian k-vector field on $\mathbb{R}^k \times M$ relative to $\widetilde{\boldsymbol{\omega}}$, namely

$$\iota_{\mathbf{X}^{\widetilde{h}}}\widetilde{\boldsymbol{\omega}} = \iota_{\mathbf{X}^h}\boldsymbol{\omega} + (R_\beta h)\tau^\beta - \sum_{\alpha=1}^{k}(\iota_{X_h^\alpha}\tau^\alpha)\mathrm{d}u^\alpha = \mathrm{d}h - \sum_{\alpha=1}^{k}\mathrm{d}u^\alpha = \mathrm{d}\widetilde{h}\,.$$

Therefore, $\mathbf{X}^{\widetilde{h}}$ is a k-polysymplectic Hamiltonian k-vector field on $\mathbb{R}^k \times M$ corresponding to the Hamiltonian function $\widetilde{h} \in \mathscr{C}^\infty(\mathbb{R}^k \times M)$ with respect to $\widetilde{\boldsymbol{\omega}}$. From Theorem 2.3.15 and Lemma 2.4.12, it follows that the reduced k-polysymplectic Hamiltonian k-vector field $\mathbf{X}^{h_{\boldsymbol{\mu}}}$ is given by

$$\mathbf{X}^{\widetilde{h_{\boldsymbol{\mu}}}} = \mathbf{X}^{h_{\boldsymbol{\mu}}} + \sum_{\alpha=1}^{k}(R_{\alpha\boldsymbol{\mu}}\, h_{\boldsymbol{\mu}})\frac{\partial}{\partial u^\alpha}\,.$$

Consequently, $\mathbf{X}^{\widetilde{h_{\boldsymbol{\mu}}}}$ projects onto $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}$ and its projection is $\mathbf{X}^{h_{\boldsymbol{\mu}}}$. This construction yields the desired reduction.

It is also worth stressing that the discussion regarding conditions 2.4.11 and 2.4.12 that guarantee the k-polycosymplectic Marsden–Meyer–Weinstein reduction in the context of k-polycosymplectic geometry is essentially identical. In fact, these conditions are sufficient, but not necessary. Indeed, analogous examples to those presented in Subsection 2.3.6 can be constructed.

2.4.4 Example: The product of k cosymplectic manifolds

This section presents an illustrative example of the k-polycosymplectic reduction of a product of k cosymplectic manifolds. Let

$$M = \bigtimes_{\alpha=1}^{k} M_\alpha$$

for some k cosymplectic manifolds $(M_\alpha, \tau_M^\alpha, \omega_M^\alpha)$ for $\alpha = 1, \dots, k$. If $\mathrm{pr}_\alpha \colon M \to M_\alpha$ is the canonical projection onto the α-component, then

$$(M, \sum_{\alpha=1}^{k}\mathrm{pr}_\alpha^*\tau_M^\alpha \otimes e_\alpha, \sum_{\alpha=1}^{k}\mathrm{pr}_\alpha^*\omega_M^\alpha \otimes e_\alpha)$$

is a k-polycosymplectic manifold. Assume further that the Lie group action $\Phi^\alpha \colon G_\alpha \times M_\alpha \to M_\alpha$ admits a cosymplectic momentum map $\mathbf{J}^{\Phi^\alpha} \colon M_\alpha \to \mathfrak{g}_\alpha^*$ for each $\alpha = 1, \dots, k$, and each Φ^α acts in a quotientable manner on the level sets given by regular values of $\mathbf{J}^{\Phi^\alpha}$.

Define the Lie group action of $G = G_1 \times \cdots \times G_k$ on M as

$$\Phi \colon G \times M \ni (g_1, \dots, g_k, x_1, \dots, x_k) \longmapsto (\Phi_{g_1}^1(x_1), \dots, \Phi_{g_k}^k(x_k)) \in M\,.$$

Let $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ be the Lie algebra of G. Then, the corresponding k-polycosymplectic momentum map is given by

$$\mathbf{J} \colon M \ni (x_1, \dots, x_k) \longmapsto (\mathbf{J}^{\Phi_1}(x_1), \dots, \mathbf{J}^{\Phi_k}(x_k)) \in \mathfrak{g}^{*k}\,,$$

where $\mathfrak{g}^* = \mathfrak{g}_1^* \times \cdots \times \mathfrak{g}_k^*$ is dual space to $\mathfrak{g}$.

If $\mu^\alpha \in \mathfrak{g}_\alpha^*$ is a regular value of $\mathbf{J}^{\Phi_\alpha} \colon M_\alpha \to \mathfrak{g}_\alpha^*$ for each $\alpha = 1, \dots, k$, then $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k) \in \mathfrak{g}^{*k}$ is a regular k-value of $\mathbf{J}$. Consequently, Φ acts in a quotientable manner on the associated level sets of $\mathbf{J}$.

Therefore, for $x = (x_1, \dots, x_k) \in \mathbf{J}^{-1}(\boldsymbol{\mu})$, it follows that

$$\begin{aligned}
\ker \mathrm{T}_x\mathbf{J}_\alpha &= \mathrm{T}_{x_1}M_1 \oplus \cdots \oplus \ker \mathrm{T}_{x_\alpha}\mathbf{J}^{\Phi_\alpha} \oplus \cdots \oplus \mathrm{T}_{x_k}M_k,\\
\mathrm{T}_x\mathbf{J}^{-1}(\boldsymbol{\mu}) &= \ker \mathrm{T}_{x_1}\mathbf{J}^{\Phi_1} \oplus \cdots \oplus \ker \mathrm{T}_{x_k}\mathbf{J}^{\Phi_k},\\
\ker \omega_x^\alpha \cap \ker \tau_x^\alpha &= \mathrm{T}_{x_1}M_1 \oplus \cdots \oplus \mathrm{T}_{x_{\alpha-1}}M_{\alpha-1} \oplus \{0\} \oplus \mathrm{T}_{x_{\alpha+1}}M_{\alpha+1} \oplus \cdots \oplus \mathrm{T}_{x_k}M_k\,,\\
\mathrm{T}_x\left(G_{\mu^\alpha}^{\Delta^\alpha}x\right) &= \mathrm{T}_{x_1}\left(G_1x_1\right) \oplus \cdots \oplus \mathrm{T}_{x_\alpha}\left(G_{\alpha\mu^\alpha}^{\Delta^\alpha}x_\alpha\right) \oplus \cdots \oplus \mathrm{T}_{x_k}\left(G_kx_k\right),\\
\mathrm{T}_x\left(G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}x\right) &= \mathrm{T}_{x_1}\left(G_{1\mu^1}^{\Delta^1}x_1\right) \oplus \cdots \oplus \mathrm{T}_{x_k}\left(G_{k\mu^k}^{\Delta^k}x_k\right).
\end{aligned}$$

Then,

$$\begin{aligned}\ker \mathrm{T}_x \mathbf{J}_\alpha &= \mathrm{T}_x \mathbf{J}^{-1}(\mu) + \ker \omega^\alpha_x \cap \ker \tau^\alpha_x ,\\ \mathrm{T}_x\left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} x\right) &= \bigcap_{\beta=1}^{k}\left(\ker \omega^\beta_x \cap \ker \tau^\beta_x + \mathrm{T}_x\left(G^{\Delta^\beta}_{\mu^\beta} x\right)\right),\end{aligned}$$

for $\alpha = 1,\dots,k$ and any $x \in \mathbf{J}^{-1}(\boldsymbol{\mu})$. Recall that, by Theorem 2.4.14, these equations guarantee that the reduced space $\mathbf{J}^{-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ can be endowed with a k-polycosymplectic structure, while

$$\mathbf{J}^{-1}(\boldsymbol{\mu})/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}} \simeq \mathbf{J}^{\Phi_1 -1}(\mu^1)/G^{\Delta^1}_{1\mu^1} \times \cdots \times \mathbf{J}^{\Phi_k -1}(\mu^k)/G^{\Delta^k}_{k\mu^k} .$$

2.4.5 Example: Two coupled vibrating strings

This subsection presents an example of a two-polycosymplectic reduction of two coupled vibrating strings. Moreover, the dynamics of the considered system is also reduced via Theorem 2.4.15.

Consider the manifold $M = \mathbb{R}^2 \times \bigoplus^2 \mathrm{T}^*\mathbb{R}^2$ with adapted coordinates $\{t, x; q^1, q^2, p^t_1, p^x_1, p^t_2, p^x_2\}$ and the standard associated two-polycosymplectic structure

$$\boldsymbol{\tau} = \mathrm{d}t \otimes e_1 + \mathrm{d}x \otimes e_2 , \qquad \boldsymbol{\omega} = (\mathrm{d}q^1 \wedge \mathrm{d}p^t_1 + \mathrm{d}q^2 \wedge \mathrm{d}p^t_2) \otimes e_1 + (\mathrm{d}q^1 \wedge \mathrm{d}p^x_1 + \mathrm{d}q^2 \wedge \mathrm{d}p^x_2) \otimes e_2 .$$

Consider the Hamiltonian function $h \in \mathscr{C}^\infty(M)$ of the form

$$h(t, x, q^1, q^2, p^t_1, p^t_2, p^x_1, p^x_2) = \frac{1}{2}\left((p^t_1)^2 + (p^t_2)^2 - (p^x_1)^2 - (p^x_2)^2\right) + C(t, x, q^1 - q^2) ,$$

where $C(t, x, q^1 - q^2)$ is a coupling function between the two strings. This system admits a Lie symmetry given by

$$\xi_M = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} ,$$

that is associated with the Lie group action $\Phi\colon \mathbb{R} \times M \to M$ acting by translations along the $q^1 + q^2$ direction, namely

$$\Phi\colon (\lambda; t, x, q^1, q^2, p^t_1, p^t_2, p^x_1, p^x_2) \ni \mathbb{R} \times M \mapsto (t, x, q^1 + \lambda, q^2 + \lambda, p^t_1, p^t_2, p^x_1, p^x_2) \in M .$$

The Lie group action Φ gives rise to a two-polycosymplectic momentum map $\mathbf{J}^\Phi$ given by

$$\mathbf{J}^\Phi\colon (t, x, q^1, q^2, p^t_1, p^t_2, p^x_1, p^x_2) \in \mathbb{R}^2 \times \bigoplus^2 \mathrm{T}^*\mathbb{R}^2 \longmapsto (p^t_1 + p^t_2, p^x_1 + p^x_2) =: (\mu^1, \mu^2) = \boldsymbol{\mu} \in \mathbb{R}^{*2} .$$

Consequently, the level set of the two-polycosymplectic momentum map $\mathbf{J}^\Phi$ is as follows

$$\mathbf{J}^{\Phi -1}(\boldsymbol{\mu}) = \{(t, x, q^1, q^2, p^t_1, \mu^1 - p^t_1, p^x_1, \mu^2 - p^x_1) \in M \mid (t, x, q^1, q^2, p^t_1, p^x_1) \in \mathbb{R}^6\}.$$

It is immediate that $\boldsymbol{\mu} = (\mu^1, \mu^2)$ is a weak regular k-value of $\mathbf{J}^\Phi$ and $\mathbf{J}^\Phi$ is Ad^{*2}-equivariant. Note that $\mathbf{J}^{\Phi -1}(\boldsymbol{\mu}) \simeq \mathbb{R}^6$ and $\mathbb{R} = G_{\boldsymbol{\mu}} = G_{\mu^\alpha}$ for $\alpha = 1, 2$. Then, for any $m \in \mathbf{J}^{\Phi -1}(\boldsymbol{\mu})$, one has

$$\begin{gathered}\mathrm{T}_m\left(G_{\boldsymbol{\mu}} m\right) = \mathrm{T}_m\left(G_{\mu^\alpha} m\right) = \left\langle \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} \right\rangle_m ,\\ (\ker \omega^1 \cap \ker \tau^1)_m = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial p^x_1}, \frac{\partial}{\partial p^x_2} \right\rangle_m , \qquad (\ker \omega^2 \cap \ker \tau^2)_m = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial p^t_1}, \frac{\partial}{\partial p^t_2} \right\rangle_m ,\\ \mathrm{T}_m \mathbf{J}^{\Phi -1}(\boldsymbol{\mu}) = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p^t_1} - \frac{\partial}{\partial p^t_2}, \frac{\partial}{\partial p^x_1} - \frac{\partial}{\partial p^x_2} \right\rangle_m ,\\ \ker \mathrm{T}_m \mathbf{J}^\Phi_1 = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p^x_1}, \frac{\partial}{\partial p^t_1} - \frac{\partial}{\partial p^t_2}, \frac{\partial}{\partial p^x_2} \right\rangle_m ,\\ \ker \mathrm{T}_m \mathbf{J}^\Phi_2 = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p^x_1} - \frac{\partial}{\partial p^x_2}, \frac{\partial}{\partial p^t_1}, \frac{\partial}{\partial p^t_2} \right\rangle_m\end{gathered}$$

and, indeed, the conditions (2.4.11) and (2.4.12) hold.

Recall that the dynamics on M is given by a two-polycosymplectic Hamiltonian two-vector field. Therefore, a general two-vector field $\mathbf{X}^h = (X_1^h, X_2^h) \in \mathfrak{X}^2(M)$ with local expression reads

$$X_\alpha^h = A_\alpha^t \frac{\partial}{\partial t} + A_\alpha^x \frac{\partial}{\partial x} + B_\alpha^1 \frac{\partial}{\partial q^1} + B_\alpha^2 \frac{\partial}{\partial q^2} + C_{\alpha 1}^t \frac{\partial}{\partial p_1^t} + C_{\alpha 1}^x \frac{\partial}{\partial p_1^x} + C_{\alpha 2}^t \frac{\partial}{\partial p_2^t} + C_{\alpha 2}^x \frac{\partial}{\partial p_2^x} .$$

Imposing the two-polycosymplectic Hamiltonian equations (1.4.3), the previous two-polycosymplectic Hamiltonian two-vector field $\mathbf{X}^h = (X_1^h, X_2^h)$ must be of the form

$$X_1^h = \frac{\partial}{\partial t} + p_1^t \frac{\partial}{\partial q^1} + p_2^t \frac{\partial}{\partial q^2} + C_{11}^t \frac{\partial}{\partial p_1^t} + C_{11}^x \frac{\partial}{\partial p_1^x} + C_{12}^t \frac{\partial}{\partial p_2^t} + C_{12}^x \frac{\partial}{\partial p_2^x} ,$$
$$X_2^h = \frac{\partial}{\partial x} - p_1^x \frac{\partial}{\partial q^1} - p_2^x \frac{\partial}{\partial q^2} + C_{21}^t \frac{\partial}{\partial p_1^t} - \left(C_{11}^t + \frac{\partial C}{\partial q} \right) \frac{\partial}{\partial p_1^x} + C_{22}^t \frac{\partial}{\partial p_2^t} + \left(\frac{\partial C}{\partial q} - C_{12}^t \right) \frac{\partial}{\partial p_2^x} ,$$

where $q = q^1 - q^2$ and $C_{11}^t, C_{11}^x, C_{12}^t, C_{12}^x, C_{21}^t, C_{22}^t \in \mathscr{C}^\infty(M)$ are, in principle, arbitrary functions.

Its integral sections, with t, x being the coordinates in its domain, satisfy

$$C_{11}^t = \frac{\partial p_1^t}{\partial t} , \qquad C_{11}^x = \frac{\partial p_1^x}{\partial t} , \qquad C_{12}^x = \frac{\partial p_2^x}{\partial t} , \qquad C_{21}^t = \frac{\partial p_1^t}{\partial x} , \qquad C_{22}^t = \frac{\partial p_2^t}{\partial x} ,$$
$$-C_{11}^t - \frac{\partial C}{\partial q} = \frac{\partial p_1^x}{\partial x} , \qquad \frac{\partial C}{\partial q} - C_{12}^t = \frac{\partial p_2^x}{\partial x} .$$

This system of PDEs is integrable when $[X_1^h, X_2^h] = 0$, for instance, if $C = qF(x) + \widehat{F}(t,x)$ for arbitrary functions $\widehat{F}(t,x)$, $F(x)$, while $C_{11}^t, C_{11}^x, C_{12}^t, C_{12}^x, C_{21}^t, C_{22}^t$ vanish. Then, to apply Theorem 2.4.15, we require $\mathbf{X}^h$ to be tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and $\mathscr{L}_{\xi_M} X_\alpha = 0$ for $\alpha = 1, 2$. Thus, $C_{12}^t + C_{11}^t = 0$, $C_{11}^x + C_{12}^x = 0$, and $C_{21}^t + C_{22}^t = 0$ and C_{ij}^t, C_{ij}^x must be first-integrals of ξ_M for $i, j = 1, 2$. A two-polycosymplectic Hamiltonian two-vector field gives rise to the following Hamilton–De Donder–Weyl equations

$$\frac{\partial q^1}{\partial t} = p_1^t , \qquad \frac{\partial q^1}{\partial x} = -p_1^x , \qquad \frac{\partial q^2}{\partial t} = p_2^t , \qquad \frac{\partial q^2}{\partial x} = -p_2^x ,$$
$$\frac{\partial p_1^t}{\partial t} + \frac{\partial p_1^x}{\partial x} = -\frac{\partial C}{\partial q} , \qquad \frac{\partial p_2^t}{\partial t} + \frac{\partial p_2^x}{\partial x} = \frac{\partial C}{\partial q} .$$

Since $G = \mathbb{R}$ acts on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ by translations along the $q^1 + q^2$ direction, the Lie group action Φ is free and proper. Therefore, $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ is a smooth manifold and

$$\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}} \simeq \mathbb{R}^2 \times \mathrm{T}^*\mathbb{R}^2/\mathbb{R} \simeq \mathbb{R}^2 \times \mathbb{R}^2/\mathbb{R} \times \mathbb{R} \simeq \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 .$$

Then, on the reduced manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathbb{R}$, the reduced two-polycosymplectic structure reads

$$\boldsymbol{\tau}_{\boldsymbol{\mu}} = \mathrm{d}t \otimes e_1 + \mathrm{d}x \otimes e_2 , \qquad \boldsymbol{\omega}_{\boldsymbol{\mu}} = \mathrm{d}q \wedge \mathrm{d}p_1^t \otimes e_1 + \mathrm{d}q \wedge \mathrm{d}p_1^x \otimes e_2 .$$

Indeed, it becomes a two-cosymplectic structure since

$$\ker \boldsymbol{\omega}_{\boldsymbol{\mu}} = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right\rangle , \qquad \ker \boldsymbol{\tau}_{\boldsymbol{\mu}} \cap \ker \boldsymbol{\omega}_{\boldsymbol{\mu}} = 0 .$$

The reduced dynamics on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathbb{R}$ is given by the reduced two-polycosymplectic Hamiltonian two-vector field $\mathbf{X}^{h_{\boldsymbol{\mu}}} = (X_1^{h_{\boldsymbol{\mu}}}, X_2^{h_{\boldsymbol{\mu}}})$ of the form

$$X_1^{h_{\boldsymbol{\mu}}} = \frac{\partial}{\partial t} + p^t \frac{\partial}{\partial q} + 2C_{11}^t \frac{\partial}{\partial p^t} + 2C_{11}^x \frac{\partial}{\partial p^x} ,$$
$$X_2^{h_{\boldsymbol{\mu}}} = \frac{\partial}{\partial x} - p^x \frac{\partial}{\partial q} - 2C_{22}^t \frac{\partial}{\partial p^t} - 2 \left(C_{11}^t + \frac{\partial C}{\partial q} \right) \frac{\partial}{\partial p^x} ,$$

where $h_{\boldsymbol{\mu}} = \frac{1}{4}\left((p^t)^2 + (p^x)^2 + (\mu^1)^2 - (\mu^2)^2\right) + C(t,x,q)$ is the reduced Hamiltonian function, where $p^t = p_1^t - p_2^t$ and $p^x = p_1^x - p_2^x$. A reduced two-polycosymplectic Hamiltonian two-vector field induces the following Hamilton–De Donder–Weyl equations

$$\frac{\partial q}{\partial x} = -p^x\,, \qquad \frac{\partial q}{\partial t} = p^t\,,$$
$$\frac{\partial p^t}{\partial t} + \frac{\partial p^x}{\partial x} = -2\frac{\partial C}{\partial q}\,.$$

It is relevant to stress that this procedure does not allow for a reduction involving the variables of $\mathbb{R}^k$. From a physical perspective, the reduction acts on the variables in $\bigoplus_{\alpha=1}^k \mathrm{T}^*Q$, corresponding to fields and their associated momenta. In contrast, the variables in $\mathbb{R}^k$, which are typically interpreted as space-time coordinates or parameters on the base manifold where the physical system is defined, remain unaffected by this reduction process. To do so, a new method is introduced in the next section to address the reduction of these $\mathbb{R}^k$-variables.

2.4.6 A k-cosymplectic to ℓ-cosymplectic reduction

This section presents a Marsden–Meyer–Weinstein type reduction from a k-cosymplectic manifold to an ℓ-cosymplectic one. Under additional conditions, this procedure may be followed by further reduction steps using the methods previously established, yielding a reduced ℓ-polycosymplectic manifold. To simplify the exposition and avoid the discussion of trivial cases, only the situation in which $\ell < k$ is considered.

From the perspective of physical applications, this approach provides a method for reducing field theories by eliminating space-time variables. Such a reduction is not possible within the framework of earlier sections, since the fundamental vector fields of the considered k-polycosymplectic Lie group actions were required to take values in $\ker \boldsymbol{\tau}$. The reduction method introduced in this subsection is of particular interest, as it constitutes a relatively novel and non-standard approach in the existing literature, which is frequently based on other methods involving principal bundles or Lie group actions that preserve the base manifold [31, 32].

Furthermore, the conditions are provided to enable the reduction of the Hamilton-De Donder-Weyl equations from the original k-cosymplectic setting to the resulting ℓ-cosymplectic manifold, and possibly to a further ℓ-polycosymplectic reduction.

The setting is restricted to the canonical k-cosymplectic manifold $(M_k = \mathbb{R}^k \times \bigoplus_{\alpha=1}^k \mathrm{T}^*Q, \boldsymbol{\tau}_k, \boldsymbol{\omega}_k)$, equipped with its natural polarisation V_k. According to the Darboux theorem for k-cosymplectic manifolds, see Theorem 1.4.17, every k-cosymplectic manifold is locally diffeomorphic to $(M_k, \boldsymbol{\tau}_k, \boldsymbol{\omega}_k)$. Therefore, the results obtained here apply locally to an arbitrary k-cosymplectic manifold.

As in previous sections, the basis $e_1, \ldots, e_k$ of $\mathbb{R}^k$ is used to define the k-cosymplectic structure through $\boldsymbol{\tau}_k = \tau^\alpha \otimes e_\alpha \in \Omega^1(M_k, \mathbb{R}^k)$ and $\boldsymbol{\omega}_k = \omega^\alpha \otimes e_\alpha \in \Omega^2(M_k, \mathbb{R}^k)$. Summation conventions are employed throughout repeated crossed indices over their natural ranges, e.g. $\alpha = 1, \ldots, k$, unless otherwise specified.

Theorem 2.4.16. *Let $(M_k, \boldsymbol{\tau}_k, \boldsymbol{\omega}_k, V_k)$ be a canonical k-cosymplectic manifold, let $\Phi\colon G \times M_k \to M_k$ be an associated k-cosymplectic Lie group action, and let $\{\overline{\tau}^1, \ldots, \overline{\tau}^\ell\}$ be a basis of the linear subspace (over the real numbers) of $\langle \tau^1, \ldots, \tau^k\rangle$ vanishing on the space W of fundamental vector fields of Φ. Without loss of generality, it may be assumed that the last $k-\ell$ forms in the original basis $\tau^1, \ldots, \tau^k$ are linearly independent when restricted to W. e.g. set $\overline{\tau}^\beta = c_\alpha^\beta\, \tau^\alpha$ for certain unique constants c_α^β with $\alpha = 1, \ldots, k$ and $\beta = 1, \ldots, \ell$. Define $\overline{\boldsymbol{\tau}} = \sum_{\beta=1}^\ell c_\alpha^\beta\, \tau^\alpha \otimes e_\beta$ and $\overline{\boldsymbol{\omega}} = \sum_{\beta=1}^\ell c_\alpha^\beta\, \omega^\alpha \otimes e_\beta$. If the map*

$$\pi\colon (\bar{x}^1, \ldots, \bar{x}^k, q, \bar{p}^1, \ldots, \bar{p}^k) \in \mathbb{R}^k \times \bigoplus_{\alpha=1}^k \mathrm{T}^*Q \longmapsto (\bar{x}^1, \ldots, \bar{x}^\ell, q, \bar{p}^1, \ldots, \bar{p}^\ell) \in \mathbb{R}^\ell \times \bigoplus_{\alpha=1}^\ell \mathrm{T}^*Q$$

is the canonical projection, then $(M_\ell = \mathbb{R}^\ell \times \bigoplus_{\alpha=1}^\ell \mathrm{T}^*Q, \boldsymbol{\tau}_\ell, \boldsymbol{\omega}_\ell)$ is an ℓ-cosymplectic manifold with

$$\pi^*\boldsymbol{\omega}_\ell = \overline{\boldsymbol{\omega}}\,, \qquad \pi^*\boldsymbol{\tau}_\ell = \overline{\boldsymbol{\tau}}\,.$$

Furthermore, there exists a Lie group action $\Phi_\ell\colon G \times M_\ell \to M_\ell$ that is equivariant relative to π.

Proof. Let $\{x^\alpha, y^j, p^\alpha_j\}$ denote locally adapted coordinates to $(M_k, \boldsymbol{\tau}_k, \boldsymbol{\omega}_k)$, namely

$$\boldsymbol{\omega}_k = \mathrm{d}y^j \wedge \mathrm{d}p^\alpha_j \otimes e_\alpha \,, \qquad \boldsymbol{\tau}_k = \mathrm{d}x^\alpha \otimes e_\alpha \,, \qquad V_k = \left\langle \frac{\partial}{\partial p^\alpha_j} \right\rangle_{\alpha=1,\dots,k,\ j=1,\dots,\dim Q} .$$

The condition $\Phi^*_g \boldsymbol{\omega}_k = \boldsymbol{\omega}_k$ for every $g \in G$ implies that $\ker \boldsymbol{\omega}_k = \langle \partial/\partial x^1, \dots, \partial/\partial x^k \rangle$ is G-invariant with respect to the lifted Lie group action of Φ to $\mathrm{T}M_k$. Furthermore, since $\Phi^*_g \boldsymbol{\tau}_k = \boldsymbol{\tau}_k$ for every $g \in G$, there exists a local linear Lie group action $\Phi^k\colon G \times \mathbb{R}^k \to \mathbb{R}^k$ whose space of orbits, around a point of $\mathbb{R}^k$, is a quotient space, $\mathbb{R}^k/E$, for some linear subspace $E \subset \mathbb{R}^k$. Consequently, $\mathbb{R}^k/E$ inherits a natural structure of ℓ-dimensional linear space.

Since $\boldsymbol{\tau}_k$ is closed and satisfies $\mathscr{L}_{\xi_{M_k}} \boldsymbol{\tau}_k = 0$ for every $\xi \in \mathfrak{g}$, it follows that $\iota_{\xi_{M_k}} \boldsymbol{\tau}_k$ is constant for each $\xi \in \mathfrak{g}$. This implies that the subspace $\mathfrak{W} = \langle \tau^1, \dots, \tau^k \rangle$ can be considered as a linear subspace of the dual, W^*, to the linear (over the reals) space W of fundamental vector fields of the Lie group action Φ. Hence, there exists a linear subspace $\mathfrak{A} \subset \mathfrak{W}$ consisting of the elements of $\mathfrak{W}$ vanishing on W. Let $\{\bar\tau^1, \dots, \bar\tau^\ell\}$ be a basis of $\mathfrak{A}$. Then, one can define $\bar\tau^\beta = c^\beta_\alpha\, \tau^\alpha$ and $\bar\omega^\beta = c^\beta_\alpha \omega^\alpha$ for some unique constants c^β_α, where $\alpha = 1, \dots, k$ and $\beta = 1, \dots, \ell$. Note that $\bar{\boldsymbol{\tau}} \in \Omega^1(\mathbb{R}^k \times \bigoplus_{\alpha=1}^k \mathrm{T}^*Q, \mathbb{R}^\ell)$ and $\bar{\boldsymbol{\omega}} \in \Omega^2(\mathbb{R}^k \times \bigoplus_{\alpha=1}^k \mathrm{T}^*Q, \mathbb{R}^\ell)$. It follows that $\bar{\boldsymbol{\tau}}$ and $\bar{\boldsymbol{\omega}}$ are closed, since $\boldsymbol{\tau}_k$ and $\boldsymbol{\omega}_k$ are closed and the coefficients c^β_α, with $\alpha = 1, \dots, k$ and $\beta = 1, \dots \ell$, are constants. There exist new local adapted coordinates to M_k obtained linearly from the previous ones, namely $\{\bar x^\alpha = A^\alpha_\beta x^\beta, y^j, \bar p^\alpha_j = A^\alpha_\beta p^\beta_j\}$ for a certain constant $(k \times k)$-matrix A^α_β, such that

$$\bar{\boldsymbol{\omega}} = \sum_{\beta=1}^{\ell} \mathrm{d}y^j \wedge \mathrm{d}\bar p^\beta_j \otimes e_\beta \,, \qquad \bar{\boldsymbol{\tau}} = \sum_{\beta=1}^{\ell} \mathrm{d}\bar x^\beta \otimes e_\beta \,.$$

Note that $\ker \bar{\boldsymbol{\tau}} \cap \ker \bar{\boldsymbol{\omega}}$ is an integrable regular distribution on M_k given by

$$\ker \bar{\boldsymbol{\tau}} \cap \ker \bar{\boldsymbol{\omega}} = \left\langle \frac{\partial}{\partial \bar p^\alpha_j}, \frac{\partial}{\partial \bar x^\alpha} \right\rangle_{\substack{\alpha=\ell+1,\dots,k,\\ j=1,\dots,\dim Q}} .$$

The pair $(\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{\omega}})$ is not a k-cosymplectic structure on M_k, but a k-precosymplectic one (see [75] for details).

The space $\mathrm{T}_x M_k / (\ker \bar{\boldsymbol{\tau}} \cap \ker \bar{\boldsymbol{\omega}})_x$ is diffeomorphic to $\mathrm{T}_{\pi(x)} M_\ell$ for $x \in M_k$, where $M_\ell = \mathbb{R}^\ell \times \bigoplus_{\alpha=1}^\ell \mathrm{T}^*Q$. Since $\bar{\boldsymbol{\tau}}$ and $\bar{\boldsymbol{\omega}}$ vanish on the fundamental vector field in $\ker \bar{\boldsymbol{\omega}} \cap \ker \bar{\boldsymbol{\tau}}$ and are closed, they are projectable via the canonical projection $\pi\colon M_k \to M_\ell$ onto M_ℓ giving rise to an ℓ-cosymplectic manifold $(M_\ell, \boldsymbol{\tau}_\ell, \boldsymbol{\omega}_\ell)$, where $\boldsymbol{\tau}_\ell$ and $\boldsymbol{\omega}_\ell$ are the unique $\mathbb{R}^k$-valued differential forms on M_ℓ so that

$$\pi^* \boldsymbol{\tau}_\ell = \bar{\boldsymbol{\tau}} \,, \qquad \pi^* \boldsymbol{\omega}_\ell = \bar{\boldsymbol{\omega}} \,.$$

Moreover, $\iota_{\xi_{M_k}} \bar{\boldsymbol{\tau}} = 0$ for every $\xi \in \mathfrak{g}$. Then, for every vector field X on M_k taking values in $\ker \bar{\boldsymbol{\tau}} \cap \ker \bar{\boldsymbol{\omega}}$, one has

$$\iota_{[\xi_{M_k}, X]} \bar{\boldsymbol{\omega}} = \mathscr{L}_{\xi_{M_k}} \iota_X \bar{\boldsymbol{\omega}} - \iota_X \mathscr{L}_{\xi_{M_k}} \bar{\boldsymbol{\omega}} = 0$$

and, similarly, $\iota_{[\xi_{M_k}, X]} \bar{\boldsymbol{\tau}} = 0$. This implies that the fundamental vector fields of Φ project onto M_ℓ and give rise to a new Lie group action $\Phi^\ell\colon G \times M_\ell \to M_\ell$ equivariant to Φ relative to the canonical projection $\pi\colon M_k \to M_\ell$. ☐

The previous procedure can, in principle, be continued by using an ℓ-polycosymplectic momentum map $\mathbf{J}_\ell$ to perform an ℓ-polycosymplectic Marsden–Meyer–Weinstein reduction according to Theorem 2.4.14. Note that the fundamental vector fields of Φ^ℓ leave invariant $\boldsymbol{\omega}_\ell$ and $\iota_{\xi_{M_\ell}} \boldsymbol{\omega}_\ell = \mathrm{d}\boldsymbol{h}_\xi$ for each $\xi \in \mathfrak{g}$ and certain $\boldsymbol{h}_\xi$. Furthermore, by construction, it follows that $\iota_{\xi_{M_\ell}} \boldsymbol{\tau}_\ell = 0$. Nevertheless, it is necessary to impose the additional condition that $R_{\ell\alpha} \boldsymbol{h}_\xi = 0$ for $\alpha = 1, \dots, \ell$. Note that this condition is not automatically satisfied. Indeed, since the initial Φ is assumed to be k-polycosymplectic, the existence of a corresponding k-polycosymplectic momentum map is not guaranteed.

The aim is now to establish that Theorem 2.4.16 induces a well-defined dynamics on the reduced space M_ℓ, starting from a dynamical system on M_k which satisfies certain conditions. Recall that, in the reductions of k-polycosymplectic structures considered in this section, the final ℓ components of $\boldsymbol{\tau}$ are, without loss of generality, assumed to form a linearly independent set when evaluated on the fundamental vector fields associated with the Lie group action Φ.

Before continuing, it is convenient to introduce and recall certain notational conventions that are used hereafter.

Original k-cosymplectic manifold and Hamiltonian k-cosymplectic k-vector field.

New k-cosymplectic manifold and Hamiltonian k-vector field obtained by making linear combinations of the components in $\mathbb{R}^k$ of $\boldsymbol{\omega}$ and $\boldsymbol{\tau}$ to ensure that the first ℓ components of $\widehat{\boldsymbol{\tau}}$ vanish on the fundamental vector fields of a k-cosymplectic Lie group action.

An ℓ-precosymplectic manifold and its k-vector field obtained by cutting the last $k-\ell$ components of the previous k-cosymplectic manifold and k-cosymplectic Hamiltonian k-vector field.

Projected ℓ-cosymplectic manifold and its ℓ-cosymplectic Hamiltonian ℓ-vector field.

$$\begin{tikzcd}
\big((M_k,\boldsymbol{\tau}_k,\boldsymbol{\omega}_k),\ \boldsymbol{X}^h\big) \arrow[d] \\
\big((M_k,\widehat{\boldsymbol{\tau}},\widehat{\boldsymbol{\omega}}),\ \widehat{\boldsymbol{X}^h}\big) \arrow[d] \\
\big((M_k,\overline{\boldsymbol{\tau}},\overline{\boldsymbol{\omega}}),\ \overline{\boldsymbol{X}}^h\big) \arrow[d, "\pi"] \\
\big((M_\ell,\boldsymbol{\tau}_\ell,\boldsymbol{\omega}_\ell),\ \boldsymbol{X}^{h_\ell}\big)
\end{tikzcd}$$

Theorem 2.4.17. *Let $(M_k,\boldsymbol{\tau}_k,\boldsymbol{\omega}_k,V_k)$ be a k-cosymplectic manifold and let $\Phi\colon G\times M_k\to M_k$ be an associated k-cosymplectic Lie group action. Assume that $h\in\mathscr{C}^\infty(M_k)$ and $\mathbf{X}^h$, its associated k-cosymplectic Hamiltonian k-vector field, are invariant relative to Φ. Suppose that h is also invariant relative to the vector fields taking values in $\ker\overline{\boldsymbol{\tau}}\cap\mathrm{T}\mathbb{R}^k$, and that the Lie bracket of any component of $\mathbf{X}^h$ with any vector field taking values in $\ker\overline{\boldsymbol{\tau}}\cap\ker\overline{\boldsymbol{\omega}}$ takes values in the kernel of $\mathrm{T}\pi$. Additionally, assume that*

$$\sum_{\alpha=\ell+1}^{k}[\widehat{X}^h_\alpha]^\alpha_i=0,\qquad i=1,\dots,\dim Q\,.$$

Then, there exists a function $h_\ell\in\mathscr{C}^\infty(M_\ell)$ such that $\mathbf{X}^{h_\ell}$ is the projection of $(\widehat{X}^h_1,\dots,\widehat{X}^h_\ell)$ onto M_ℓ and $\pi^*h_\ell=h$ on a submanifold of constant values of the momenta p^α_i with $\alpha=\ell+1,\dots,k$ and $i=1,\dots,\dim Q$. The ℓ-vector field $\mathbf{X}^{h_\ell}$ is Hamiltonian relative to $(M_\ell,\boldsymbol{\tau}_\ell,\boldsymbol{\omega}_\ell)$ and the solutions for the HDW equations of h_ℓ are solutions of the original HDW equations for constant associated momenta with $\alpha=\ell+1,\dots,k$ for $\overline{\boldsymbol{\tau}},\overline{\boldsymbol{\omega}}$.

Proof. Let $\widehat{c}^\beta_\alpha$ be the matrix of the change of bases mapping $\{\tau^1,\dots,\tau^k\}$ into the new basis

$$\widehat{\tau}^1=\bar{\tau}^1,\dots,\widehat{\tau}^\ell=\bar{\tau}^\ell,\widehat{\tau}^{\ell+1}=\tau^{\ell+1},\dots,\widehat{\tau}^k=\tau^k\,,$$

and let $\widehat{d}^\beta_\alpha$ be the inverse matrix, namely $\tau^\alpha=\widehat{d}^\alpha_\beta\widehat{\tau}^\beta$, for $\alpha,\beta=1,\dots,k$. Define a new Hamiltonian k-cosymplectic k-vector field $\widehat{\mathbf{X}}^h$ on M_k relative to $(M_k,\widehat{\boldsymbol{\tau}}=\widehat{c}^\beta_\alpha\tau^\alpha\otimes e_\beta,\widehat{\boldsymbol{\omega}}=\widehat{c}^\beta_\alpha\omega^\alpha\otimes e_\beta)$ of the form

$$\widehat{X}^h_\alpha=\widehat{d}^\beta_\alpha\,X^h_\beta\,,\qquad \alpha,\beta=1,\dots,k\,.$$

Since $\widehat{c}^\beta_\alpha$ is such that $\widehat{c}^\beta_\alpha=\delta^\beta_\alpha$ for $\beta=\ell+1,\dots,k$ and $\alpha=1,\dots,k$ by construction of $\widehat{\boldsymbol{\tau}}$, then $\widehat{d}^\beta_\alpha=\delta^\beta_\alpha$ for $\beta=\ell+1,\dots,k$ and $\alpha=1,\dots,k$. The relations between the new canonical coordinates in $\mathbb{R}^k\times\bigoplus_{\alpha=1}^k\mathrm{T}^*Q$

and the previous ones are given by

$$\widehat{x}^{\beta} = \widehat{c}^{\beta}_{\alpha} x^{\alpha}\,, \qquad \widehat{p}^{\beta}_{i} = \widehat{c}^{\beta}_{\alpha} p^{\alpha}_{i}\,, \qquad \alpha,\beta = 1,\ldots,k\,, \quad i = 1,\ldots,\dim Q\,,$$

while $q^1,\ldots,q^{\dim Q}$ are the same in the new and the old coordinate systems.

If $\widehat{R}_\alpha = \widehat{d}^{\beta}_{\alpha} R_\beta$, it follows that

$$\iota_{\widehat{X}^h_\alpha}\widehat{\omega}^\alpha = \mathrm{d}h - (\widehat{R}_\alpha h)\widehat{\tau}^\alpha, \qquad \iota_{\widehat{X}^h_\alpha}\widehat{\tau}^\beta = \delta^\beta_\alpha. \tag{2.4.17}$$

It is worth noting that if $\psi\colon s = (s^1,\ldots,s^k) \in \mathbb{R}^k \mapsto (x^\alpha(s), q^i(s), p^\alpha_i(s)) \in \mathbb{R}^k \times \bigoplus_{\alpha=1}^k \mathrm{T}^*Q$ is a solution to the HDW equations of the original $\mathbf{X}^h$, then the same ψ is a solution for the HDW equations for $\widehat{\mathbf{X}}^h$ in the new coordinates $\psi\colon \widehat{s} = (\widehat{s}^1,\ldots,\widehat{s}^k) \in \mathbb{R}^k \mapsto (\widehat{x}^\alpha(\widehat{s}), q^i(\widehat{s}), \widehat{p}^\alpha_i(\widehat{s})) \in \mathbb{R}^k \times \bigoplus_{\alpha=1}^k \mathrm{T}^*Q$ with $\widehat{s}^\beta = \widehat{c}^\beta_\alpha s^\alpha$ for $\alpha,\beta = 1,\ldots,k$, namely

$$\frac{\partial \widehat{x}^\beta}{\partial \widehat{s}^\alpha} = \delta^\beta_\alpha\,, \quad \frac{\partial q^i}{\partial \widehat{s}^\alpha} = \frac{\partial h}{\partial \widehat{p}^\alpha_i}\,, \quad \sum_{\alpha=1}^k \frac{\partial \widehat{p}^\alpha_i}{\partial \widehat{s}^\alpha} = -\frac{\partial h}{\partial q^i}\,, \qquad \alpha,\beta = 1,\ldots,k\,, \quad i = 1,\ldots,\dim Q\,. \tag{2.4.18}$$

Then, the aim is to show that there is a new ℓ-vector field $\overline{\mathbf{X}}^h$ on $\mathbb{R}^k \times \bigoplus_{\alpha=1}^k \mathrm{T}^*Q$ related to $(\mathbb{R}^k \times \bigoplus_{\alpha=1}^k \mathrm{T}^*Q, \overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\omega}})$ of the form

$$\overline{\mathbf{X}}^h = \sum_{\alpha=1}^{\ell} \widehat{X}^h_\alpha \otimes e_\alpha\,,$$

satisfying

$$\iota_{\overline{\mathbf{X}}^h}\overline{\boldsymbol{\omega}} = \mathrm{d}_\ell h - \sum_{\alpha=1}^{\ell} (\overline{R}_\alpha h)\overline{\tau}^\alpha\,, \qquad \overline{R}_\alpha = \widehat{R}_\alpha\,, \qquad \alpha = 1,\ldots,\ell\,,$$

where d_ℓ is the differential taking into account all canonical coordinates apart from $\widehat{x}^\alpha$ and $\widehat{p}^\alpha_i$ for $\alpha = \ell+1,\ldots,k$ and $i = 1,\ldots,\dim Q$. From (2.4.17) it follows that

$$\sum_{\beta=1}^{\ell} \iota_{\overline{X}^h_\beta}\overline{\omega}^\beta + \sum_{\beta=\ell+1}^{k} \frac{\partial h}{\partial \widehat{p}^\beta_i}\mathrm{d}\widehat{p}^\beta_i - \sum_{\beta=\ell+1}^{k} [\widehat{X}^h_\beta]^\beta_i \mathrm{d}q^i = \frac{\partial h}{\partial q^i}\mathrm{d}q^i + \sum_{\alpha=1}^{\ell} \frac{\partial h}{\partial \widehat{p}^\alpha_i}\mathrm{d}\widehat{p}^\alpha_i + \sum_{\alpha=\ell+1}^{k} \frac{\partial h}{\partial \widehat{p}^\alpha_i}\mathrm{d}\widehat{p}^\alpha_i\,.$$

The assumption $\sum_{\beta=\ell+1}^k [\widehat{X}^h_\beta]^\beta_i = 0$ for $i = 1,\ldots,\dim Q$, yields

$$\sum_{\beta=1}^{\ell} \iota_{\overline{X}^h_\beta}\overline{\omega}^\beta = \frac{\partial h}{\partial q^i}\mathrm{d}q^i + \sum_{\alpha=1}^{\ell} \frac{\partial h}{\partial \widehat{p}^\alpha_i}\mathrm{d}\widehat{p}^\alpha_i = \mathrm{d}_\ell h - \sum_{\alpha=1}^{\ell} (\overline{R}_\alpha h)\overline{\tau}_\alpha\,. \tag{2.4.19}$$

In particular, the previous expression holds on the submanifold S_λ for $\widehat{p}^\alpha_i = \lambda^\alpha_i$ for certain constants λ^α_i, with $\alpha = \ell{+}1,\ldots,k$ and $i = 1,\ldots,\dim Q$. The projection of this submanifold relative to $\pi\colon M_k \to M_\ell$ is surjective and open. By the given assumptions, the restriction of h to S_λ is projectable onto a function h_ℓ on M_ℓ. Since the Lie bracket of $\mathbf{X}^h$ with any vector field in $\ker \overline{\boldsymbol{\tau}} \cap \ker \overline{\boldsymbol{\omega}}$ belongs to the kernel of $\mathrm{T}\pi$, it follows that the same applies to $\widehat{\mathbf{X}}^h$ and $(\overline{X}^h_1,\ldots,\overline{X}^h_\ell)$ is projectable onto M_ℓ, which implies that the Lie derivatives of the $\overline{X}^h_1,\ldots,\overline{X}^h_\ell$ with $\partial/\partial\bar{x}^{\ell+1},\ldots,\partial/\partial\bar{x}^k$ and their associated momenta belong to the kernel of $\mathrm{T}\pi$. Then, (2.4.19) projects onto M_ℓ. In addition, $\iota_{\overline{X}_\alpha}\overline{\tau}^\beta = \delta^\beta_\alpha$ for $\alpha,\beta = 1,\ldots,\ell$. These facts show that the projection of $(\overline{X}^h_1,\ldots,\overline{X}^h_\ell)$ is Hamiltonian relative to the induced ℓ-cosymplectic manifold $(M_\ell, \boldsymbol{\tau}_\ell, \boldsymbol{\omega}_\ell)$. The new local canonical variables are given by

$$\bar{x}^\alpha = \widehat{x}^\alpha\,, \quad \bar{p}^\alpha_i = \widehat{p}^\alpha_i \qquad i = 1,\ldots,\dim Q\,, \qquad \alpha = 1,\ldots,\ell\,.$$

The HDW equations for the ℓ-cosymplectic structure take the form

$$\frac{\partial \bar{x}^\beta}{\partial \bar{s}^\alpha} = \delta^\beta_\alpha\,, \quad \frac{\partial q^i}{\partial s^\alpha} = \frac{\partial h_\ell}{\partial \bar{p}^\alpha_i}\,, \quad \sum_{\alpha=1}^{\ell} \frac{\partial \bar{p}^\alpha_i}{\partial \bar{s}^\alpha} = -\frac{\partial h_\ell}{\partial q^i}\,, \qquad \alpha,\beta = 1,\ldots,\ell\,, \qquad i = 1,\ldots,\dim Q\,.$$

These are indeed the equations for the solutions to (2.4.18) with constant $\bar{p}^\alpha_i$ with $\alpha = \ell+1,\ldots,k$ and such that h does not depend on $\bar{x}_{\ell+1},\ldots,\bar{x}_k$. □

It is worth noting that if the fundamental vector fields associated with Φ are tangent to the submanifolds S_λ, then the function h_ℓ remains invariant relative to Φ and a standard ℓ-polycosymplectic reduction can be applied, provided that the reduced action satisfies certain additional requirements, such as the existence of a ℓ-polycosymplectic momentum map.

Theorem 2.4.17 gives sufficient conditions under which the k-cosymplectic Hamiltonian k-vector field $\widehat{\mathbf{X}}^h$ admits a projection onto the manifold M_ℓ. However, these conditions are not necessary. The theorem holds under weaker assumptions. For instance, it is sufficient to require that only the first ℓ components of the k-vector field $\widehat{\mathbf{X}}^h$ are projectable.

The general framework described above may be applied to a concrete physical example, such as a vibrating membrane subject to an external force depending only on the radial distance.

Example 2.4.18. The system is given by the Hamiltonian function $\widetilde{h} \in \mathscr{C}^\infty(\mathbb{R}^3 \times \bigoplus_{\alpha=1}^3 \mathrm{T}^*\mathbb{R})$ of the form

$$\widetilde{h}(t,r,\theta,\zeta,p^t,p^r,p^\theta) = \frac{1}{2r}\left((p^t)^2 - \frac{1}{c^2}(p^r)^2 - \frac{r^2}{c^2}(p^\theta)^2\right) - r\zeta f(r)\,,$$

and the canonical three-cosymplectic structure on $\mathbb{R}^3 \times \bigoplus_{\alpha=1}^3 \mathrm{T}^*\mathbb{R}$ given by

$$\boldsymbol{\tau} = \mathrm{d}t \otimes e_1 + \mathrm{d}r \otimes e_2 + \mathrm{d}\theta \otimes e_3\,, \qquad \boldsymbol{\omega} = \mathrm{d}\zeta \wedge \mathrm{d}p^t \otimes e_1 + \mathrm{d}\zeta \wedge \mathrm{d}p^r \otimes e_2 + \mathrm{d}\zeta \wedge \mathrm{d}p^\theta \otimes e_3\,.$$

A section

$$\psi\colon (t,r,\theta) \in \mathbb{R}^3 \mapsto (t,r,\theta,\zeta(t,r,\theta),p^t(t,r,\theta),p^x(t,r,\theta),p^y(t,r,\theta)) \in \mathbb{R}^3 \times \bigoplus_{\alpha=1}^3 \mathrm{T}^*\mathbb{R} =: M_3^v\,,$$

is a solution of the HDW equations of the three-cosymplectic Hamiltonian three-vector field $\mathbf{X}^{\widetilde{h}} = (X_1^{\widetilde{h}}, X_2^{\widetilde{h}}, X_3^{\widetilde{h}})$ on M_3^v, where

$$X_1^{\widetilde{h}} = \frac{\partial}{\partial t} + \frac{p^t}{r}\frac{\partial}{\partial \zeta}, \quad X_2^{\widetilde{h}} = \frac{\partial}{\partial r} - \frac{p^r}{c^2 r}\frac{\partial}{\partial \zeta} + rf(r)\frac{\partial}{\partial p^r}, \quad X_3^{\widetilde{h}} = \frac{\partial}{\partial \theta} - \frac{rp^\theta}{c^2}\frac{\partial}{\partial \zeta}$$

if the following conditions are satisfied

$$\begin{gathered}\frac{\partial p^t}{\partial t} + \frac{\partial p^r}{\partial r} + \frac{\partial p^\theta}{\partial \theta} = rf(r)\,,\\ \frac{\partial \zeta}{\partial t} = \frac{1}{r}p^t\,, \qquad \frac{\partial \zeta}{\partial r} = -\frac{1}{rc^2}p^r\,, \qquad \frac{\partial \zeta}{\partial \theta} = -\frac{r}{c^2}p^\theta\,.\end{gathered}$$

Combining the above equations yields the standard wave equation in polar coordinates with a radially forced vibrating membrane

$$\frac{\partial^2 \zeta}{\partial t^2} - c^2\left(\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \zeta}{\partial \theta^2}\right) = f(r)\,.$$

Define the Lie group action

$$\Phi\colon \mathbb{R}^2 \times M_3^v \ni (\lambda_1,\lambda_2;t,r,\theta,\zeta,p^t,p^r,p^\theta) \mapsto (t+\lambda_1, r, (\theta+\lambda_2) \bmod 2\pi, \zeta, p^t, p^r, p^\theta) \in M_3^v\,,$$

which describes symmetries of $\widetilde{h}$ and defines a three-cosymplectic Lie group action, as it leaves invariant $\boldsymbol{\tau}$, $\boldsymbol{\omega}$, and their polarisation V. The restriction of Φ to $\mathbb{R}^3$ reads

$$\Phi_3\colon \mathbb{R}^2 \times \mathbb{R}^3 \ni (\lambda_1,\lambda_2;t,r,\theta) \mapsto (\lambda_1 + t, r, (\theta+\lambda_2) \bmod 2\pi) \in \mathbb{R}^3\,.$$

Note that the space of orbits of Φ is diffeomorphic to $\mathbb{R} \times \mathbb{S}^1$. In fact, the existence of such a Lie group action is guaranteed by Theorem 2.4.17. The space of fundamental vector fields is

$$D = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta} \right\rangle$$

and the one-forms of $\langle \mathrm{d}r, \mathrm{d}t, \mathrm{d}\theta\rangle$ vanishing on D are $\langle \mathrm{d}r\rangle$. Therefore,

$$\overline{\boldsymbol{\tau}} = \mathrm{d}r, \qquad \overline{\boldsymbol{\omega}} = \mathrm{d}\zeta \wedge \mathrm{d}p^r .$$

Note that $\widehat{\mathbf{X}}^{\widetilde{h}} = (\mathbf{X}_2^{\widetilde{h}}, \mathbf{X}_1^{\widetilde{h}}, \mathbf{X}_3^{\widetilde{h}})$ and the HDW equations for $\widehat{\mathbf{X}}^{\widetilde{h}}$ are the same as before (up to a reparametrization of the indexes of the variables in $\mathbb{R}^3$).

Consider the submanifold S_λ in $\mathbb{R}^3 \times \bigoplus_{\alpha=1}^3 \mathrm{T}^*\mathbb{R}$ given by

$$p^t = \lambda_t, \quad p^\theta = \lambda_\theta, \qquad \lambda = (\lambda_t, \lambda_\theta) \in \mathbb{R}^2 , \qquad \lambda_t, \lambda_\theta \in \mathbb{R} ,$$

which projects onto $\mathbb{R}^6$ diffeomorphically. Note that there exists a function

$$k(r, \zeta, p^r) = \frac{1}{2r}\left(\lambda_t^2 - \frac{1}{c^2}(p^r)^2 - \frac{r^2}{c^2}\lambda_\theta^2\right) - r\zeta f(r) ,$$

whose pull-back to $\mathbb{R}^3 \times \bigoplus_{\alpha=1}^3 \mathrm{T}^*\mathbb{R}$ coincides the value of $\widetilde{h}$ on S_λ.

The distribution D spanned by the fundamental vector fields of Φ, and

$$\ker \overline{\boldsymbol{\tau}} \cap \ker \overline{\boldsymbol{\omega}} = \langle \partial/\partial t, \partial/\partial_\theta, \partial/\partial p^t, \partial/\partial p^\theta\rangle$$

are integrable. Moreover, $\widetilde{h}$ is a first integral of any vector field taking value in $\ker \overline{\boldsymbol{\tau}} \cap \mathrm{T}\mathbb{R}^k$. Additionally, one has

$$X_2^{\widetilde{h2}} + X_3^{\widetilde{h3}} = \widehat{X}_2^{\widetilde{h2}} + \widehat{X}_3^{\widetilde{h}} = 0.$$

Then, the space of orbits of Φ_3 is diffeomorphic to $\mathbb{R} \times \mathrm{T}^*\mathbb{R}$ and one has the canonical projection

$$\pi \colon (t, r, \theta, \zeta, p^t, p^r, p^\theta) \in \mathbb{R}^3 \times \bigoplus_{\alpha=1}^{3} \mathrm{T}^*\mathbb{R} \longmapsto (r, \zeta, p^r) \in \mathbb{R} \times \mathrm{T}^*\mathbb{R} .$$

The Lie brackets of each of the components of $\mathbf{X}^h$ with vector fields of $\ker \overline{\boldsymbol{\tau}} \cap \ker \overline{\boldsymbol{\omega}}$ also belong to the kernel of $\mathrm{T}\pi$, ensuring that there exists induced one-cosymplectic manifold

$$(\mathbb{R} \times \mathrm{T}^*\mathbb{R}, \mathrm{d}r, \mathrm{d}\zeta \wedge \mathrm{d}p^r).$$

In fact, this is the canonical cosymplectic structure in the reduced manifold. Consequently, the Hamiltonian three-vector field with components

$$X_1^{\widetilde{h}} = \frac{\partial}{\partial t} + \frac{\lambda_t}{r}\frac{\partial}{\partial \zeta}, \quad X_2^{\widetilde{h}} = \frac{\partial}{\partial r} - \frac{p^r}{c^2 r}\frac{\partial}{\partial \zeta} + r f(r)\frac{\partial}{\partial p^r}, \quad X_3^{\widetilde{h}} = \frac{\partial}{\partial \theta} - \frac{r\lambda_\theta}{c^2}\frac{\partial}{\partial \zeta} ,$$

projects onto the quotient. The resulting HDW equations read

$$\frac{\partial p^r}{\partial r} = r f(r) , \qquad \frac{\partial \zeta}{\partial r} = -\frac{p^r}{rc^2} ,$$

which, when combined, yield the reduced wave equation

$$c^2\left(\frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r}\frac{\partial \zeta}{\partial r}\right) = -f(r) .$$

These are the HDW equations obtained by assuming p^θ and p^t to be constant in the initial HDW equations.

2.5 k-Contact Marsden–Meyer–Weinstein reduction

This Section presents the modified k-polysymplectic Marsden–Meyer–Weinstein reduction for exact k-polysymplectic manifolds. As demonstrated in subsequent subsections, the reduction is adjusted to retrieve the reduced k-contact geometric structure from the reduced exact k-polysymplectic one. The section focuses on exact k-polysymplectic manifolds, which are a specific case of k-polysymplectic manifolds. Hence, the standard definitions are recalled in a form adapted to the exact setting. However, since several of the subsequent results remain valid in a broader context, the definitions for general k-polysymplectic manifolds are also included.

2.5.1 Exact k-polysymplectic momentum maps

This subsection introduces the definition of an exact k-polysymplectic momentum map. First, recall that, as previously, $\mathfrak{g}^k = \mathfrak{g}\times \overset{k}{\cdots} \times\mathfrak{g}$, and analogously, $\mathfrak{g}^{*k} = \mathfrak{g}^*\times \overset{k}{\cdots} \times\mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space to the Lie algebra $\mathfrak{g}$. Moreover, recall that $(P, \boldsymbol{\theta})$ stands for an exact k-polysymplectic manifold, see Definition 1.4.5.

Definition 2.5.1. A Lie group action $\Phi\colon G \times P \to P$ is an *exact k-polysymplectic Lie group action* relative to $(P, \boldsymbol{\theta})$ if $\Phi_g^*\boldsymbol{\theta} = \boldsymbol{\theta}$ for each $g \in G$, where $\Phi_g\colon P \ni p \mapsto \Phi_g(p) := \Phi(g, p) \in P$. In other words,

$$\mathscr{L}_{\xi_P}\boldsymbol{\theta} = 0\,, \qquad \forall\xi \in \mathfrak{g}\,.$$

A k-polysymplectic momentum map for an exact k-polysymplectic manifold $(P, \boldsymbol{\theta})$ is naturally defined using the property of exactness.

Definition 2.5.2. An *exact k-polysymplectic momentum map* associated with $(P, \boldsymbol{\theta})$ and a Lie group action $\Phi\colon G \times P \to P$ is a map $\mathbf{J}^\Phi_{\boldsymbol{\theta}}\colon P \to \mathfrak{g}^{*k}$ such that

$$\iota_{\xi_P}\boldsymbol{\theta} = \langle \mathbf{J}^\Phi_{\boldsymbol{\theta}}, \xi\rangle, \qquad \forall\xi \in \mathfrak{g}\,.$$

If $\Phi\colon G \times P \to P$ is an exact k-polysymplectic Lie group action, then

$$\mathrm{d}\iota_{\xi_P}\boldsymbol{\theta} = -\iota_{\xi_P}\boldsymbol{\omega}, \qquad \forall\xi \in \mathfrak{g}.$$

Thus, one can retrieve a k-polysymplectic momentum map defined in the standard way, as in the following definition.

Proposition 2.5.3. *An exact k-polysymplectic momentum map $\mathbf{J}^\Phi_{\boldsymbol{\theta}}\colon P \to \mathfrak{g}^{*k}$ associated with a Lie group action $\Phi\colon G \times P \to P$ related to an exact k-polysymplectic manifold $(P, \boldsymbol{\theta})$ is Ad^{*k}-equivariant.*

The proof amounts to showing that $\mathbf{J}^\Phi_{\boldsymbol{\theta}}(\Phi_g(p)) = \mathrm{Ad}^{*k}_{g^{-1}}\mathbf{J}^\Phi_{\boldsymbol{\theta}}(p)$ for each $p \in P$ and every $g \in G$, and it follows from Lemma 2.1.2.

2.5.2 Marsden–Meyer–Weinstein reduction theorem for exact k-polysymplectic manifolds

This subsection is devoted to the proof of the modified k-polysymplectic Marsden–Meyer–Weinstein reduction (Theorem 2.3.14) via an extension of Lemma 2.3.12 in the symplectic setting [2, Lemma 4.3.2].

The idea to prove the k-contact Marsden–Meyer–Weinstein repeats the procedure devised in [107] and presented in Subsection 2.3.6, which leads to Theorem 2.3.14 but is carried out in a slightly different setting. Specifically, the construction is performed with $\widetilde{N} = \mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ instead of taking $N = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, where $\mathbb{R}^\times = \mathbb{R}\setminus\{0\}$ and $\mathbb{R}^{\times k}\boldsymbol{\mu}$ is an invariant set with respect to dilations in each component of $\mu^\alpha \in \mathfrak{g}^*$ for $\alpha = 1, \dots, k$, namely

$$\mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu}) = \{p \in P \mid \exists\lambda_1, \dots, \lambda_k \in \mathbb{R}^\times,\ \mathbf{J}^\Phi_\alpha(p) = \lambda_\alpha\mu^\alpha,\ \alpha = 1, \dots, k\}\,.$$

Equivalently, $\mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ is the pre-image of the orbit of $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ relative to the natural action defined by

$$\mathbb{R}^{\times k} \times \mathfrak{g}^{*k} \ni (\lambda_1, \dots, \lambda_k; \mu^1, \dots, \mu^k) \longmapsto (\lambda_1\mu^1, \dots, \lambda_k\mu^k) \in \mathfrak{g}^{*k}\,.$$

Note that if $\boldsymbol{\mu}$ is a weak regular k-value, the spaces $\mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ and $\mathbf{J}^{\Phi-1}_\alpha(\mathbb{R}^\times\mu^\alpha)$ are submanifolds of P.

The following proposition is a starting point for the modified k-polysymplectic Marsden–Meyer–Weinstein reduction.

It is natural to expect that the Lie subgroup of G acting on $\mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ may be required to satisfy some additional conditions compared to the subgroup appearing in the k-polysymplectic case of Theorem 2.3.14. It arises from dimensional considerations, as explained later in Subsection 2.6.2. In this context, the following proposition is introduced.

Proposition 2.5.4. *Let* $\mathfrak{k}_{[\mu]} := \ker \mu \cap \mathfrak{g}_{[\mu]}$*, where* $\ker \mu = \langle \mu \rangle^\circ$ *and* $\mathfrak{g}_{[\mu]} = \{\xi \in \mathfrak{g} \mid \mathrm{ad}^*_\xi \mu \wedge \mu = 0\}$ *for some* $\mu \in \mathfrak{g}*$*. Then,* $\mathfrak{k}_{[\mu]}$ *is a Lie subalgebra of* $\mathfrak{g}$*.*

Proof. Note that $\mathfrak{k}_{[\mu]}$ is a linear space. If $\xi, \nu \in \mathfrak{k}_{[\mu]}$, then $\mathrm{ad}^*_\xi \mu = \lambda \mu$ and $\mathrm{ad}^*_\nu \mu = \kappa \mu$ for some $\lambda, \kappa \in \mathbb{R}$. Thus,

$$\mathrm{ad}^*_{[\xi,\nu]}\mu = \mathrm{ad}^*_\nu \mathrm{ad}^*_\xi \mu - \mathrm{ad}^*_\nu \mathrm{ad}^*_\xi \mu = (\lambda\kappa - \kappa\lambda)\mu = 0, \qquad \langle \mathrm{ad}^*_\xi \mu, \nu \rangle = \langle \mu, [\xi, \nu] \rangle = \langle \lambda \mu, \nu \rangle = 0.$$

The first equality shows that $[\xi, \nu] \in \mathfrak{g}_{[\mu]}$ and the second that $[\xi, \nu]$ belongs to $\ker \mu$. Therefore, $[\xi, \nu] \in \mathfrak{k}_{[\mu]}$ and $\mathfrak{k}_{[\mu]}$ is a Lie subalgebra of $\mathfrak{g}$. □

Proposition 2.5.4 implies that there exists a unique and simply connected Lie subgroup of G, denoted as $K_{[\mu]}$, whose Lie algebra is $\mathfrak{k}_{[\mu]}$. The following lemma presents the properties of $\mathfrak{k}_{[\boldsymbol{\mu}]}$ and $K_{[\boldsymbol{\mu}]}$ in the k-polysymplectic setting, where $\ker \boldsymbol{\mu} = \cap_{\alpha=1}^k \ker \mu^\alpha$ and $\mathfrak{k}_{[\boldsymbol{\mu}]} = \ker \boldsymbol{\mu} \cap \mathfrak{g}_{[\boldsymbol{\mu}]}$.

Lemma 2.5.5. *Let* $(P, \boldsymbol{\omega})$ *be a* k*-polysymplectic manifold and let* $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ *be a weak regular* k*-value of a* k*-polysymplectic momentum map* $\mathbf{J}^\Phi \colon P \to \mathfrak{g}^{*k}$ *associated with a Lie group action* $\Phi \colon G \times P \to P$*. Then,*

$$K_{[\boldsymbol{\mu}]} = \bigcap_{\alpha=1}^k K_{[\mu^\alpha]}, \qquad \mathfrak{k}_{[\boldsymbol{\mu}]} = \bigcap_{\alpha=1}^k \mathfrak{k}_{[\mu^\alpha]}. \tag{2.5.1}$$

Moreover, $\mathrm{T}(K_{[\boldsymbol{\mu}]}p) \subseteq \ker \jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega}$ *for the natural embedding* $\jmath_{[\boldsymbol{\mu}]} \colon \mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu}) \hookrightarrow P$*.*

Proof. The first statement in (2.5.1) follows from Definition 2.5.2 and the fact that if $g \in K_{[\boldsymbol{\mu}]}$, then $g \in K_{[\mu^\alpha]}$ for $\alpha = 1, \ldots, k$. The identity for the Lie algebra then follows. Second, for any $\xi \in \mathfrak{k}_{[\boldsymbol{\mu}]}$, one has

$$\mathrm{T}\mathbf{J}^\Phi(\xi_P) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{J}^\Phi \circ \Phi_{\exp(t\xi)} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}^{*k}_{\exp(-t\xi)}\mathbf{J}^\Phi = -\mathrm{ad}^{*k}_\xi \mathbf{J}^\Phi = -\lambda \mathbf{J}^\Phi,$$

for some $\lambda \in \mathbb{R}$. This yields that $\mathrm{T}_p(K_{[\boldsymbol{\mu}]}p) \subseteq \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ for any $p \in \mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Next, for any $v_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})$, one has

$$(\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega})(p)(v_p, \xi_{\mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})}(p)) = \boldsymbol{\omega}(p)(\mathrm{T}_p\jmath_{[\boldsymbol{\mu}]}v_p, \xi_P(p)) = -\langle \mathrm{T}_p\mathbf{J}^\Phi(v_p), \xi \rangle = -\langle \lambda_\alpha \mu^\alpha \oplus e_\alpha, \xi \rangle = 0,$$

for any $\xi \in \mathfrak{k}_{[\boldsymbol{\mu}]}$ and $p \in \mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})$, where we denoted by v_p both a vector $v_p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mathbb{R}^\times \boldsymbol{\mu})$ and its induced $v_p \in \mathrm{T}_pP$. Therefore, $\mathrm{T}_p(K_{[\boldsymbol{\mu}]}p) \subseteq \ker(\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega})(p)$ for every $p \in \mathbf{J}^{\Phi-1}(\mathbb{R}^\times \boldsymbol{\mu})$.

□

Therefore, Lemma 2.5.5 and Theorem 2.3.10 show that

$$\mathrm{T}_p(K_{[\boldsymbol{\mu}]}p) \subseteq \ker(\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega})(p) = \mathrm{T}_p\mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu}) \cap \left(\mathrm{T}_p\mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})\right)^{\perp_k}, \qquad \forall p \in \mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu}).$$

In general, the converse does not hold, as detailed in [107]. Consequently, analogously to [107], one may ask under which conditions $\mathrm{T}_p(K_{[\boldsymbol{\mu}]}p) = \ker(\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega})(p)$ holds for any $p \in \mathbf{J}^{\Phi-1}(\mathbb{R}^\times \boldsymbol{\mu})$. The answer follows essentially as in [107] but with a significant difference given in Lemma 2.5.6.

Recall that, if $k = 1$, then (P, ω) is a symplectic manifold. Now, assume that $(P, \omega = \mathrm{d}\theta)$ is an exact symplectic manifold.

Lemma 2.5.6. *Let* (P, θ) *be an exact symplectic manifold and let* $G_{[\mu]} = \{g \in G \mid \mathrm{Ad}^*_{g^{-1}}\mu \wedge \mu = 0\}$*. Assume that* $J^\Phi(p) = \mu \in \mathfrak{g}^*$ *is a weak regular value of an exact symplectic momentum map* $J^\Phi \colon P \to \mathfrak{g}^*$ *associated with a Lie group action* $\Phi \colon G \times P \to P$*. Then, for any* $p \in J^{\Phi-1}(\mathbb{R}^\times \mu)$*, one has*

(1) $\mathrm{T}_p(G_{[\mu]}p) = \mathrm{T}_p(Gp) \cap \mathrm{T}_pJ^{\Phi-1}(\mathbb{R}^\times \mu)$,

(2) $\mathrm{T}_pJ^{\Phi-1}(\mathbb{R}^\times \mu) = (\mathrm{T}_p(Gp) \cap \ker \theta_p)^{\perp_\omega}$,

where $^{\perp_\omega}$ *denotes the symplectic orthogonal and* θ *is the Liouville one-form.*

Proof.

(1) First, for any $\xi \in \mathfrak{g}$, one has that

$$\mathrm{T}_pJ^\Phi(\xi_P(p)) = (\iota_{\xi_P}\mathrm{d}J^\Phi)_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} J^\Phi \circ \Phi_{\exp(t\xi)}(p) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(\mathrm{Ad}^*_{\exp(-t\xi)}J^\Phi\right)(p) = -\mathrm{ad}^*_\xi\mu. \tag{2.5.2}$$

To prove that $\mathrm{T}_p(G_{[\mu]}p) \subset \mathrm{T}_pJ^{\Phi-1}(\mathbb{R}^\times\mu)$. consider $\xi_P(p) \in \mathrm{T}_p\left(G_{[\mu]}p\right)$, i.e. $\xi \in \mathfrak{g}_{[\mu]}$. Then, by (2.5.2) it follows that

$$0 = \mu \wedge \mathrm{ad}^*_\xi\mu = -\mu \wedge \mathrm{T}_pJ^\Phi(\xi_P(p)),$$

and hence $\xi_P(p) \in \mathrm{T}_pJ^{\Phi-1}(\mathbb{R}^\times\mu)$.

To prove reverse inclusion $\mathrm{T}_p(G_{[\mu]}p) \supset \mathrm{T}_pJ^{\Phi-1}(\mathbb{R}^\times\mu) \cap \mathrm{T}_p(Gp)$, let $v \in \mathrm{T}_p(Gp) \cap \mathrm{T}_pJ^{\Phi-1}(\mathbb{R}^\times\mu)$. Then, $v = \xi_P(p) \in \mathrm{T}_pJ^{\Phi-1}(\mathbb{R}^\times\mu)$. Applying (2.5.2), one gets

$$0 = \mu \wedge \mathrm{T}_pJ^\Phi(\xi_P(p)) = -\mu \wedge \mathrm{ad}^*_\xi\mu.$$

Therefore, $\xi_P(p) \in \mathrm{T}_p(G_{[\mu]}p)$. This proves (1).

(2) Recall that, by Definition 2.5.2, if $\xi_P(p) \in \ker\theta_p$ for some $p \in J^{\Phi-1}(\mathbb{R}^\times\mu)$, then $\xi \in \ker\mu$. Let $v \in (\mathrm{T}_p(Gp) \cap \ker\theta_p)^{\perp_\omega}$. Then,

$$0 = (\iota_v\iota_{\xi_P}\omega)_p = -\iota_v\mathrm{d}\langle J^\Phi, \xi\rangle(p) = \langle \mathrm{T}_pJ^\Phi(v), \xi\rangle, \qquad \forall \xi \in \ker\mu\,.$$

Hence, $v \in \mathrm{T}_pJ^{\Phi-1}(\mathbb{R}^\times\mu)$.

Conversely, let $v \in \mathrm{T}_pJ^{\Phi-1}(\mathbb{R}^\times\mu)$. Then, for $\xi \in \ker\mu$, one has

$$0 = \langle \mathrm{T}_pJ^\Phi(v), \xi\rangle = (\iota_{\xi_P}\iota_v\omega)_p$$

and $v \in (\mathrm{T}_p(Gp) \cap \ker\theta_p)^{\perp_\omega}$.

□

Let $(P, \boldsymbol{\theta})$ be an exact k-polysymplectic manifold and let $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ be a weak regular k-value of an exact k-symplectic momentum $\mathbf{J}^\Phi_{\boldsymbol{\theta}}\colon P \to \mathfrak{g}^{*k}$ associated with an exact k-polysymplectic Lie group action $\Phi\colon G \times P \to P$ that acts in a quotientable manner on $\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. In what follows, using Lemma 2.5.6 the conditions under which equality $\mathrm{T}_p(K_{[\boldsymbol{\mu}]}p) = \ker(\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega})(p)$ holds are provided. The proof consists of two steps.

(1) The vector space

$$V^p_\alpha := \frac{\left(\frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}_\alpha}(\mathbb{R}^\times\mu^\alpha))}{\ker\omega^\alpha_p}\right)}{\left\{[\xi_P(p)] \mid \xi \in \mathfrak{k}_{[\mu^\alpha]}\right\}},$$

is a symplectic vector space, where $\mathrm{pr}^P_\alpha\colon \mathrm{T}P \to \frac{\mathrm{T}P}{\ker\omega^\alpha}$ is the canonical vector bundle projection (over the base P) and $[\xi_P(p)] := \mathrm{pr}^P_\alpha(\xi_P(p))$.

(2) The linear surjective morphisms

$$\Pi^\alpha_p\colon \mathrm{T}_{\pi_{[\mu]}(p)}\left(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}\right) \longrightarrow \frac{\left(\frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}_\alpha}(\mathbb{R}^\times\mu^\alpha))}{\ker\omega^\alpha_p}\right)}{\left\{[\xi_P(p)] \mid \xi \in \mathfrak{k}_{[\mu^\alpha]}\right\}}\,, \qquad \alpha = 1, \ldots, k\,,$$

satisfy $\bigcap_{\alpha=1}^k \ker\Pi^\alpha_p = 0$, where $\pi_{[\boldsymbol{\mu}]}\colon \mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) \to \mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$ is the canonical projection.

Assuming that the above steps are satisfied, Lemma 2.5.7 implies that $\ker(\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega})(p) = \mathrm{T}_p(K_{[\boldsymbol{\mu}]}p)$ and that $\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$ is a k-polysymplectic manifold.

$$\begin{tikzcd}
\left(\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times\mu^\alpha)), \omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)\right) \arrow[r, hook, "j^\alpha_p"] \arrow[d, "\mathrm{pr}^{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}"] & (\mathrm{T}_pP, \omega^\alpha(p)) \arrow[d, "\mathrm{pr}^P_\alpha"] \\
\left(\frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times\mu^\alpha))}{\ker\omega^\alpha_p}, \widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)}\right) \arrow[r, hook, "\widetilde{j^\alpha_p}"] \arrow[d, "\widetilde{\mathrm{pr}^\alpha}"] & \left(\frac{\mathrm{T}_pP}{\ker\omega^\alpha_p}, \widetilde{\omega^\alpha(p)}\right) \\
\left(\frac{\left(\frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times\mu^\alpha))}{\ker\omega^\alpha_p}\right)}{\left\{[\xi_P(p)]\mid\xi\in\mathfrak{k}_{[\mu^\alpha]}\right\}}, \omega_{[\mu^\alpha]}(p)\right).
\end{tikzcd}$$

Figure 2.4: Diagram illustrating part of the processes to accomplish a k-polysymplectic MMW reduction. Note that $\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p) := (j^\alpha_p)^*\omega^\alpha(p)$ and $\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)} = \widetilde{(j^\alpha_p)}^*\widetilde{\omega^\alpha(p)}$, while $\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p) = \widetilde{\mathrm{pr}^\alpha}^*\omega_{[\mu^\alpha]}(p)$ and $\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p) = (\mathrm{pr}^{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}})^*\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)}$

Lemma 2.5.7. *Let $\pi_\alpha\colon V_p \to V^p_\alpha$ be surjective linear morphisms and let $(V^p_\alpha, \mathrm{d}\theta^\alpha(p))$ be symplectic vector spaces for $\alpha = 1, \dots, k$. If $\cap^k_{\alpha=1}\ker\pi_\alpha = 0$, then $(V_p, \mathrm{d}\boldsymbol{\theta}(p) = \sum^k_{\alpha=1}(\mathrm{d}\pi^*_\alpha\theta^\alpha)(p)\otimes e_\alpha)$ is a k-polysymplectic vector space.*

The proofs of the aforementioned steps are now presented. They essentially follow from the same techniques as those applied in [107]. To clarify the relations between spaces and morphisms, the diagram in Figure 2.4 is provided.

The following lemma is immediate.

Lemma 2.5.8. *Let $(P, \boldsymbol{\theta})$ be an exact k-polysymplectic manifold, then there exists a unique symplectic linear form $\widetilde{\omega^\alpha(p)} = \widetilde{\mathrm{d}\theta^\alpha(p)}$ on $\frac{\mathrm{T}_pP}{\ker\omega^\alpha(p)}$ satisfying*

$$\left(\mathrm{pr}^P_\alpha\right)^*\widetilde{\omega^\alpha(p)} = \left(\mathrm{pr}^P_\alpha\right)^*\widetilde{\mathrm{d}\theta^\alpha(p)} = \mathrm{d}\theta^\alpha(p) = \omega^\alpha(p), \qquad \forall p \in P\,.$$

Moreover, there exists $\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)} \in \Omega^2\left(\frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times\mu^\alpha))}{\ker\omega^\alpha_p}\right)$ such that

$$\left(\mathrm{pr}^{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}\right)^*\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)} = \omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)\,, \qquad \widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)} = \left(\widetilde{j^\alpha_p}\right)^*\widetilde{\omega^\alpha(p)}\,, \qquad \forall p \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\,.$$

Proof. The first part of the Lemma is immediate. Now, by definition

$$\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p) := (j^{\alpha*}_p\omega^\alpha)(p) \in \Omega^2(\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times\mu^\alpha))\,,$$

for $p \in \mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times\mu^\alpha)$. Note that $\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)$ is exact because $\widetilde{\omega^\alpha(p)}$ is exact. Since

$$\ker\omega^\alpha(p) \subseteq \mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times\mu^\alpha)\,,$$

it follows that $\ker\omega^\alpha(p) \subseteq \ker\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)$ and there exists, therefore, a unique $\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)} \in \Omega^2\left(\frac{\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times\mu^\alpha)}{\ker\omega^\alpha_p}\right)$ satisfying $(\mathrm{pr}^{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}})^*\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)} = \omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)$. Additionally,

$$\begin{aligned}
\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p) &= (j^{\alpha*}_p\omega^\alpha)(p) = j^{\alpha*}_p\,\mathrm{pr}^{P*}_\alpha\,\widetilde{\omega^\alpha(p)} = \left(\mathrm{pr}^P_\alpha \circ j^\alpha_p\right)^*\widetilde{\omega^\alpha(p)} \\
&= \left(\widetilde{j^\alpha_p} \circ \mathrm{pr}^{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}\right)^*\widetilde{\omega^\alpha(p)} = \mathrm{pr}^{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}*}\,\widetilde{j^\alpha_p}^*\,\widetilde{\omega^\alpha(p)}\,.
\end{aligned}$$

Therefore, $\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)} = (\widetilde{j^\alpha_p})^*\widetilde{\omega^\alpha(p)}$. □

The next lemma is a direct consequence of Lemma 2.5.6, Lemma 2.5.8, and, in the case of (2), (3), (4), the fact that ω^α and $\mathbf{J}^\Phi_{\boldsymbol{\theta}}$ are exact.

Lemma 2.5.9. *For $\alpha = 1, \ldots, k$ and $p \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha)$, one has*

(1) $\left\{[\xi_P(p)] \mid \xi \in \mathfrak{g}_{[\mu^\alpha]}\right\} = \{[\xi_P(p)] \mid \xi \in \mathfrak{g}\} \cap \frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha))}{\ker \omega^\alpha(p)}$,

and if $\mathbf{J}^\Phi_{\boldsymbol{\theta}} \colon P \to \mathfrak{g}^{*k}$ is an exact k-symplectic momentum map relative to $(P, \boldsymbol{\theta})$, the following conditions hold

(2) $\frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha))}{\ker \omega^\alpha(p)} = \left(\{[\xi_P(p)] \mid \xi \in \mathfrak{g}\} \cap \ker \widetilde{\theta^\alpha(p)}\right)^{\perp_\alpha}$,

(3) $\left(\frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha))}{\ker \omega^\alpha(p)}\right)^{\perp_\alpha} = \{[\xi_P(p)] \mid \xi \in \mathfrak{g}\} \cap \ker \widetilde{\theta^\alpha(p)}$,

(4) $\left\{[\xi_P(p)] \mid \xi \in \mathfrak{g}_{[\mu^\alpha]}\right\} = \frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha))}{\ker \omega^\alpha(p)} \cap \left(\frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha))}{\ker \omega^\alpha(p)}\right)^{\perp_\alpha}$,

where $\perp_\alpha$ denotes the symplectic orthogonal in $\frac{\mathrm{T}_p P}{\ker \omega^\alpha_p}$ with respect to $\widetilde{\omega^\alpha(p)}$.

The proof of point (1) in Lemma 2.5.9 does not require $(P, \boldsymbol{\theta})$ to be an exact k-polysymplectic manifold. The following proposition establishes the first step of the Marsden–Meyer–Weinstein reduction theorem for exact k-polysymplectic manifolds.

Proposition 2.5.10. *The vector space*

$$V^p_\alpha := \frac{\left(\frac{\mathrm{T}_p(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha))}{\ker \omega^\alpha_p}\right)}{\left\{[\xi_P(p)] \mid \xi \in \mathfrak{k}_{[\mu^\alpha]}\right\}}$$

is a symplectic vector space for $\alpha = 1, \ldots, k$.

Proof. Since $\left\{[\xi_P(p)] \mid \xi \in \mathfrak{k}_{[\mu^\alpha]}\right\} \subseteq \frac{\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha)}{\ker \omega^\alpha_p}$, the quotient space V^p_α is well-defined and there is the canonical projection

$$\widetilde{\mathrm{pr}_\alpha} \colon \frac{\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha)}{\ker \omega^\alpha_p} \longrightarrow V^p_\alpha .$$

By Lemma 2.5.9 points (2), (3), (4) and Lemma 2.5.8, $\{[\xi_P(p)] \mid \xi \in \mathfrak{k}_{[\mu^\alpha]}\}$ belongs to $\ker \widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)}$ and there exists a symplectic form $\omega_{[\mu^\alpha]}(p) \in \Omega^2\left(V^p_\alpha\right)$ satisfying $\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)} = \widetilde{\mathrm{pr}_\alpha}^* \omega_{[\mu^\alpha]}(p)$ for $\alpha = 1, \ldots, k$. ☐

This concludes the first part of the proof. The second part establishes that the quotient manifold $\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$ admits the structure of an exact k-polysymplectic manifold under certain assumptions. Furthermore, the technical conditions ensuring the validity of these assumptions are formulated.

Proposition 2.5.11. *The map*

$$\Pi^\alpha_p := \mathrm{pr}^{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}} \circ \jmath^\alpha \colon \mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) \longrightarrow \frac{\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha)}{\ker \omega^\alpha_p} ,$$

where $\jmath^\alpha \colon \mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) \hookrightarrow \mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^\times \mu^\alpha)$ is the natural embedding, induces the map

$$\widetilde{\Pi}^\alpha_p \colon \mathrm{T}_{\pi_{[\boldsymbol{\mu}]}(p)}\left(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}\right) \longrightarrow V^p_\alpha , \qquad \alpha = 1, \ldots, k .$$

The proof follows from part (4) in Lemma 2.5.9 and is analogous to the one in [107].

Lemma 2.5.12. *Let $(P, \boldsymbol{\theta})$ be an exact k-polysymplectic manifold and let $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ be a weak regular k-value of an exact k-polysymplectic momentum map $\mathbf{J}^\Phi_{\boldsymbol{\theta}} \colon P \to \mathfrak{g}^{*k}$ associated with an exact k-polysymplectic*

Lie group action $\Phi\colon G\times P\to P$ *that acts in a quotientable manner on* $\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$*. Then, there exists an exact* $\mathbb{R}^k$*-valued differential two-form* $\boldsymbol{\omega}_{[\boldsymbol{\mu}]}\in\Omega^2(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]},\mathbb{R}^k)$ *satisfying*

$$\pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega}_{[\boldsymbol{\mu}]}=\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega},$$

and

$$\widetilde{\Pi}^{\alpha *}_p\omega_{[\mu^\alpha]}=\omega^\alpha_{[\boldsymbol{\mu}]},\qquad \alpha=1,\dots,k,$$

where $\jmath_{[\boldsymbol{\mu}]}\colon\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\hookrightarrow P$ *and* $\pi_{[\boldsymbol{\mu}]}\colon\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\to\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$ *are the canonical embedding and the canonical projection, respectively.*

Proof. First, recall that $\mathrm{T}_p(K_{[\boldsymbol{\mu}]}p)\subseteq\ker\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega}(p)$ for every $p\in\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ by Lemma 2.5.5. Then, $\iota_{\xi_P}\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega}=0$ for any $\xi\in\mathfrak{k}_{[\boldsymbol{\mu}]}$. Since $\jmath^*_{[\boldsymbol{\mu}]}\omega^\alpha$ is also closed, it gives rise to a unique closed $\mathbb{R}^k$-valued two-form $\boldsymbol{\omega}_{[\boldsymbol{\mu}]}\in\Omega^2\left(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]},\mathbb{R}^k\right)$ satisfying $\pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega}_{[\boldsymbol{\mu}]}=\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega}$. According to the following commutative diagram

$$\begin{tikzcd}
\left(\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}),\,\jmath^*_{[\boldsymbol{\mu}]}\omega^\alpha_p\right) \arrow[r, hook, "\jmath^\alpha_{\mathbf{J}^\Phi_{\boldsymbol{\theta}}}"] \arrow[rr, hook, bend left, "\jmath_{[\boldsymbol{\mu}]}"] \arrow[d, "\mathrm{T}_p\pi_{[\boldsymbol{\mu}]}"] \arrow[dr, "\Pi^\alpha_p"] & \left(\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^{\times}\boldsymbol{\mu}),\,\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)\right) \arrow[r, hook, "\jmath^\alpha_p"] \arrow[d, "\mathrm{pr}^{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}"] & (\mathrm{T}_pP,\,\omega^\alpha(p)) \arrow[d, "\mathrm{pr}^P_\alpha"] \\
\left(\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/\mathrm{T}_p\left(K_{[\boldsymbol{\mu}]}p\right),\,\omega^\alpha_{[\boldsymbol{\mu}]}(p)\right) \arrow[r, hook] \arrow[dr, "\widetilde{\Pi}^\alpha_p"] & \left(\frac{\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^{\times}\mu^\alpha)}{\ker\omega^\alpha_p},\,\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)}\right) \arrow[r, hook, "\widetilde{\jmath}^\alpha_p"] \arrow[d, "\widetilde{\mathrm{pr}}_\alpha"] & \left(\frac{\mathrm{T}_pP}{\ker\omega^\alpha_p},\,\widetilde{\omega^\alpha(p)}\right) \\
 & \left(\frac{\left(\frac{\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^{\times}\mu^\alpha)}{\ker\omega^\alpha_p}\right)}{\left\{[\xi_P(p)]\mid\xi\in\mathfrak{k}_{[\mu^\alpha]}\right\}},\,\omega_{[\mu^\alpha]}(p)\right) &
\end{tikzcd}$$

and the fact that $\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\subseteq\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^{\times}\mu^\alpha)$ for $\alpha=1,\dots,k$, it follows $\Pi^{\alpha*}_p\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}(p)}=\jmath^*_{[\boldsymbol{\mu}]}\omega^\alpha_p$ and $\widetilde{\Pi}^\alpha_p\circ\mathrm{T}_p\pi_{[\boldsymbol{\mu}]}=\widetilde{\mathrm{pr}}_\alpha\circ\Pi^\alpha_p$. Then, using Proposition 2.5.10 for any $v_p,w_p\in\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$, one gets

$$\begin{aligned}
\pi^*_{[\boldsymbol{\mu}]}\widetilde{\Pi}^{\alpha*}_p\omega_{[\mu^\alpha]}(p)(v_p,w_p)&=\omega_{[\mu^\alpha]}(p)\left(\widetilde{\Pi}^\alpha_p\circ\mathrm{T}_p\pi_{[\boldsymbol{\mu}]}(v_p),\widetilde{\Pi}^\alpha_p\circ\mathrm{T}_p\pi_{[\boldsymbol{\mu}]}(w_p)\right)\\
&=\omega_{[\mu^\alpha]}(p)\left(\widetilde{\mathrm{pr}}_\alpha\circ\Pi^\alpha_p(v_p),\widetilde{\mathrm{pr}}_\alpha\circ\Pi^\alpha_p(w_p)\right)=\widetilde{\mathrm{pr}}_\alpha{}^*\omega_{[\mu^\alpha]}(p)\left(\Pi^\alpha_p(v_p),\Pi^\alpha_p(w_p)\right)\\
&=\widetilde{\omega_{\mathbf{J}^\Phi_{\boldsymbol{\theta}\alpha}}}(p)\left(\Pi^\alpha_p(v_p),\Pi^\alpha_p(w_p)\right)=\jmath^*_{[\boldsymbol{\mu}]}\omega^\alpha(p)\left(v_p,w_p\right)=\pi^*_{[\boldsymbol{\mu}]}\omega^\alpha_{[\boldsymbol{\mu}]}(p).
\end{aligned}$$

Thus, $\widetilde{\Pi}^{\alpha*}_p\omega_{[\mu^\alpha]}(p)=\omega^\alpha_{[\boldsymbol{\mu}]}(p)$. ☐

The immediate consequence of Lemma 2.5.7 and Lemma 2.5.12 is the following proposition.

Proposition 2.5.13. *Assume that* $\bigcap_{\alpha=1}^k\ker\Pi^\alpha_p=0$ *and* Π^α_p *is a surjective morphism for every* $p\in\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ *and* $\alpha=1,\dots,k$*. Then,* $(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]},\boldsymbol{\theta}_{[\boldsymbol{\mu}]})$ *is an exact k-polysymplectic manifold, where*

$$\mathrm{d}\pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\theta}_{[\boldsymbol{\mu}]}=\pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega}_{[\boldsymbol{\mu}]}=\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega}=\mathrm{d}\jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\theta}\,.$$

The following lemmas provide the necessary, but not sufficient, conditions to ensure that Π^α_p is a surjective morphism and $\bigcap_{\alpha=1}^k\ker\Pi^\alpha_p=0$ for every $p\in\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ and $\alpha=1,\dots,k$. On the discussion on these conditions, see Subsection 2.3.6.

Lemma 2.5.14. *The map*

$$\Pi^\alpha_p\colon\mathrm{T}_{\pi_{[\boldsymbol{\mu}]}(p)}\left(\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}\right)\longrightarrow\frac{\left(\frac{\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^{\times}\mu^\alpha)}{\ker\omega^\alpha_p}\right)}{\left\{[\xi_P(p)]\ \mid\ \xi\in\mathfrak{k}_{[\mu^\alpha]}\right\}}$$

is a surjection if and only if

$$\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^{\times}\mu^\alpha) = \mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) + \ker\omega^\alpha(p) + \mathrm{T}_p\left(K_{[\mu^\alpha]}p\right) .$$

Additionally, the condition $\bigcap_{\alpha=1}^k \ker\Pi^\alpha_p = 0$ *is satisfied if and only if*

$$\mathrm{T}_p\left(K_{[\boldsymbol{\mu}]}p\right) = \bigcap_{\alpha}^{k}\left(\ker\omega^\alpha(p) + \mathrm{T}_p\left(K_{[\mu^\alpha]}p\right)\right) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) .$$

The proofs of the previous lemmas are analogous to the standard k-polysymplectic setting described in [107]. The following theorem summarises the above results.

Theorem 2.5.15. *Let* $(P,\boldsymbol{\theta})$ *be an exact* k*-polysymplectic manifold, let* $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$ *be a regular* k*-value of an exact* k*-polysymplectic momentum map* $\mathbf{J}^\Phi_{\boldsymbol{\theta}}\colon P\to\mathfrak{g}^{*k}$ *associated with an exact* k*-polysymplectic Lie group action* $\Phi\colon G\times P\to P$ *that acts in a quotientable manner on* $\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$*. Assume that for every* $p\in\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ *the following conditions hold*

$$\mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}\alpha}(\mathbb{R}^{\times}\mu^\alpha) = \mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) + \ker\omega^\alpha(p) + \mathrm{T}_p\left(K_{[\mu^\alpha]}p\right) , \qquad \forall\alpha=1,\ldots,k , \tag{2.5.3}$$

and

$$\mathrm{T}_p\left(K_{[\boldsymbol{\mu}]}p\right) = \bigcap_{\alpha}^{k}\left(\ker\omega^\alpha(p) + \mathrm{T}_p\left(K_{[\mu^\alpha]}p\right)\right) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) . \tag{2.5.4}$$

Then, $(P_{[\boldsymbol{\mu}]} := \mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]},\boldsymbol{\theta}_{[\boldsymbol{\mu}]})$ *is an exact* k*-polysymplectic manifold, such that*

$$\pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\theta}_{[\boldsymbol{\mu}]} = \jmath^*_{[\boldsymbol{\mu}]}\boldsymbol{\theta} ,$$

where $\pi_{[\boldsymbol{\mu}]}\colon\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\to P_{[\boldsymbol{\mu}]}$ *is the canonical projection and* $\jmath_{[\boldsymbol{\mu}]}\colon\mathbf{J}^{\Phi-1}_{\boldsymbol{\theta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\hookrightarrow P$ *is the canonical inclusion.*

2.5.3 k-Contact momentum maps

This subsection presents the definition of a k-contact momentum map and its properties. Additionally, it establishes the notation used hereafter.

Definition 2.5.16. Let $(M,\boldsymbol{\eta})$ be a k-contact manifold. A Lie group action $\Phi\colon G\times M\to M$ is a *k-contact Lie group action* if $\Phi^*_g\boldsymbol{\eta}=\boldsymbol{\eta}$ for each $g\in G$. A *k-contact momentum map* associated with $\Phi\colon G\times M\to M$ is a map $\mathbf{J}^\Phi_{\boldsymbol{\eta}}=(\mathbf{J}^\Phi_1,\ldots,\mathbf{J}^\Phi_k)\colon M\to\mathfrak{g}^{*k}$ such that

$$\left\langle\mathbf{J}^\Phi_{\boldsymbol{\eta}},\xi\right\rangle := \iota_{\xi_M}\boldsymbol{\eta} = (\iota_{\xi_M}\eta^\alpha)\otimes e_\alpha , \qquad \forall\xi\in\mathfrak{g} . \tag{2.5.5}$$

Note that if $\Phi\colon G\times M\to M$ is a k-contact Lie group action, then equation (2.5.5) implies that $\iota_{\xi_M}\mathrm{d}\boldsymbol{\eta} = -\mathrm{d}\iota_{\xi_M}\boldsymbol{\eta} = -\mathrm{d}\langle\mathbf{J}^\Phi_{\boldsymbol{\eta}},\xi\rangle$ and

$$\mathrm{d}\left\langle\mathbf{J}^\Phi_{\boldsymbol{\eta}},\xi\right\rangle = -\iota_{\xi_M}\mathrm{d}\boldsymbol{\eta} , \qquad \forall\xi\in\mathfrak{g} .$$

Then,

$$\iota_{R_\beta}\iota_{\xi_M}\mathrm{d}\boldsymbol{\eta} = -R_\beta\langle\mathbf{J}^\Phi_{\boldsymbol{\eta}},\xi\rangle = 0 , \qquad \forall\beta=1,\ldots,k, \qquad \forall\xi\in\mathfrak{g} .$$

Next, the definition of Ad^{*k}-equivariance in k-contact setting is presented.

Definition 2.5.17. A k-contact momentum map $\mathbf{J}^\Phi_{\boldsymbol{\eta}}\colon M\to\mathfrak{g}^{*k}$ is Ad^{*k}-*equivariant* if

$$\mathbf{J}^\Phi_{\boldsymbol{\eta}}\circ\Phi_g = \mathrm{Ad}^{*k}_{g^{-1}}\circ\,\mathbf{J}^\Phi_{\boldsymbol{\eta}} , \qquad \forall g\in G ,$$

where

$$\begin{array}{rcll} \mathrm{Ad}^{*k} & : G\times\mathfrak{g}^{*k} & \longrightarrow & \mathfrak{g}^{*k} \\ & (g,\boldsymbol{\mu}) & \longmapsto & \mathrm{Ad}^{*k}_{g^{-1}}\boldsymbol{\mu} \end{array} .$$

In other words, the diagram aside commutes for every $g\in G$.

$$\begin{tikzcd} M \arrow[r, "\mathbf{J}^\Phi_{\boldsymbol{\eta}}"] \arrow[d, "\Phi_g"] & \mathfrak{g}^{*k} \arrow[d, "\mathrm{Ad}^{*k}_{g^{-1}}"] \\ M \arrow[r, "\mathbf{J}^\Phi_{\boldsymbol{\eta}}"] & \mathfrak{g}^{*k}. \end{tikzcd}$$

Proposition 2.5.18. *A k-contact momentum map $\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}\colon M \to \mathfrak{g}^{*k}$ associated with a Lie group action $\Phi\colon G\times M \to M$ related to a k-contact manifold $(M,\boldsymbol{\eta})$ is Ad^{*k}-equivariant.*

Similarly as in Proposition 2.5.3, it is sufficient to show that $\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}(\Phi_g(x)) = \mathrm{Ad}^{*k}_{g^{-1}}\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}(x)$ for each $x\in M$ and every $g\in G$, and it follows from the identity $\Phi_{g*}\xi_M = (\mathrm{Ad}^*_{g^{-1}}\xi)_M$ in Lemma 2.1.2.

Analogously, the following definition simplifies the notation.

Definition 2.5.19. A *k-contact Hamiltonian system* is a triple $(M,\boldsymbol{\eta},\mathbf{J}^{\Phi})$, where $(M,\boldsymbol{\eta})$ is a k-contact manifold and $\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}\colon M\to\mathfrak{g}^{*k}$ is a k-contact momentum map associated with a k-contact Lie group action $\Phi\colon G\times M\to M$. A *$k$-contact G-invariant Hamiltonian system* is a tuple $(M,\boldsymbol{\eta},\mathbf{J}^{\Phi}_{\boldsymbol{\eta}},h)$, where $(M,\boldsymbol{\eta},\mathbf{J}^{\Phi}_{\boldsymbol{\eta}})$ is a k-contact Hamiltonian system, $h\in\mathscr{C}^\infty(M)$ is a Hamiltonian function associated with a k-contact Hamiltonian k-vector field $\mathbf{X}^h$, and the map $\Phi\colon G\times M\to M$ is a k-contact Lie group action satisfying $\Phi^*_g h = h$ for every $g\in G$.

2.5.4 k-Contact reduction by a submanifold

This subsection establishes a general k-contact reduction theorem by submanifold and provides the necessary and sufficient conditions for performing the reduction.

Definition 2.5.20. The *k-contact orthogonal* of $W_x\subset \mathrm{T}_xM$ at some $x\in M$ with respect to $(M,\boldsymbol{\eta})$ is

$$W_x^{\perp_{\mathrm{d}\boldsymbol{\eta}}} := \{v_x\in \mathrm{T}_xM \mid \mathrm{d}\boldsymbol{\eta}(v_x,w_x)=0\,,\ \forall w_x\in W_x\}\,.$$

Theorem 2.5.21. *(k-contact reduction by a submanifold.) Let N be a submanifold of M with an injective immersion $\jmath\colon N\hookrightarrow M$. Suppose that $\ker\jmath^*\boldsymbol{\eta}$ and $\ker\jmath^*\mathrm{d}\boldsymbol{\eta}$ have constant ranks for $(M,\boldsymbol{\eta})$. Let $N/\mathcal{F}_N$ be a manifold, where $\mathcal{F}_N$ is a foliation on N given by $\mathcal{D}:=\ker\jmath^*\boldsymbol{\eta}\cap\ker\jmath^*\mathrm{d}\boldsymbol{\eta}$ and let the canonical projection $\pi\colon N\to N/\mathcal{F}_N$ be a submersion. Moreover, assume that Reeb vector fields associated with $(M,\boldsymbol{\eta})$ are tangent to N. Then, $(N/\mathcal{F}_N,\boldsymbol{\eta}_N)$ is a k-contact manifold defined uniquely by*

$$\jmath^*\boldsymbol{\eta} = \pi^*\boldsymbol{\eta}_N,$$

and $\ker\jmath^*\boldsymbol{\eta}_x\cap\ker\jmath^*\mathrm{d}\boldsymbol{\eta}_x = \mathrm{T}_xN\cap(\mathrm{T}_xN)^{\perp_{\mathrm{d}\boldsymbol{\eta}}}\cap\ker\boldsymbol{\eta}_x$ for any $x\in N$.

Proof. For any $X,Y\in\mathfrak{X}(N)$ taking values in $\mathcal{D}$, one has

$$\jmath^*(\iota_{[X,Y]}\boldsymbol{\eta})=0\,,\qquad \jmath^*(\iota_{[X,Y]}\mathrm{d}\boldsymbol{\eta})=0\,.$$

Hence, $[X,Y]$ takes values in $\mathcal{D}$. By the Fröbenius theorem and the fact that $\ker\jmath^*\boldsymbol{\eta}\cap\ker\jmath^*\mathrm{d}\boldsymbol{\eta}$ has constant rank on N, the distribution $\mathcal{D}$ defines a foliation $\mathcal{F}_N$ on N.

Then, by definition of $\mathcal{D}$, it follows that $\jmath^*\boldsymbol{\eta}$ is basic with respect to $\mathcal{F}_N$. Therefore, there exists a unique $\boldsymbol{\eta}_N\in\Omega^1(N/F_N,\mathbb{R}^k)$, such that

$$\jmath^*\boldsymbol{\eta}=\pi^*\boldsymbol{\eta}_N\,.$$

Now, it must be verified that $\ker\boldsymbol{\eta}_N\cap\mathbf{d}\eta_N=0$, $\operatorname{cork}\ker\boldsymbol{\eta}_N=k$, and $\operatorname{rk}\ker\mathrm{d}\boldsymbol{\eta}=k$. Let $X_N=\mathrm{T}\pi(X)$ takes values in $\ker\boldsymbol{\eta}_N\cap\ker\mathrm{d}\boldsymbol{\eta}_N$. Then,

$$\iota_X\jmath^*\boldsymbol{\eta}=\iota_X\pi^*\boldsymbol{\eta}_N=\pi^*(\iota_{X_N}\boldsymbol{\eta}_N)=0$$

and

$$\iota_X\iota_Y\jmath^*\mathrm{d}\boldsymbol{\eta}=\iota_X\iota_Y\pi^*\mathrm{d}\boldsymbol{\eta}_N=\pi^*(\iota_{X_N}\iota_{Y_N}\mathrm{d}\boldsymbol{\eta}_N)=0,$$

for any $Y_N:=\mathrm{T}\pi(Y)$. Thus, X is tangent to $\mathcal{F}_N$ and $\mathrm{T}\pi(X)=0$. Then, $\ker\boldsymbol{\eta}_N\cap\ker\mathrm{d}\boldsymbol{\eta}_N=0$.

Note that $R_1,\dots,R_k$ are tangent to N and $[R_\alpha,X]$ takes values in $\mathcal{D}$ for every X tangent to $\mathcal{F}_N$ and $\alpha=1,\dots,k$. Thus, the Reeb vector fields project via π onto $\langle R^N_1,\dots,R^N_k\rangle\in\mathfrak{X}(N/\mathcal{F}_N)$. Additionally,

$$\pi^*(\iota_{R^N_\alpha}\eta^\beta_N)=\jmath^*(\iota_{R_\alpha}\eta^\beta)=\delta^\beta_\alpha,\qquad \pi^*(\iota_{R^N_\alpha}\mathrm{d}\boldsymbol{\eta}_N)=\jmath^*(\iota_{R_\alpha}\mathrm{d}\boldsymbol{\eta})=0,\qquad \alpha,\beta=1,\dots,k.$$

Therefore, $R_1^N, \dots, R_k^N$ are the Reeb vector fields associated with $(N/\mathcal{F}_N, \boldsymbol{\eta}_N)$ and $\operatorname{rk}\ker \boldsymbol{\eta}_N = k$.

Let $n := \dim N$, and let $\langle X_1, \dots, X_n\rangle = \mathrm{T}_x N$ for any $x \in N$. One can choose a family of vectors $\langle Y_1, \dots, Y_{n-k}\rangle \subset \ker \boldsymbol{\eta}_x$ such that $\langle Y_1, \dots, Y_{n-k}\rangle \oplus \langle R_1, \dots, R_k\rangle = \mathrm{T}_x N$. Moreover, among $\langle Y_1, \dots, Y_{n-k}\rangle$ there are vectors $\langle Z_1, \dots, Z_\ell\rangle = \ker \jmath^*\boldsymbol{\eta}_x \cap \ker \mathrm{d}\jmath^*\boldsymbol{\eta}_x$. Thus,

$$\mathrm{T}_x N = \langle Y_1, \dots, Y_{n-k-\ell}\rangle \oplus \langle Z_1, \dots, Z_\ell\rangle \oplus \langle R_1, \dots, R_k\rangle,$$

for any $x \in N$. Since, Z_i projects to zero for $\alpha = 1, \dots, \ell$ it follows that $\ker \mathrm{d}\boldsymbol{\eta}_N$ is a corank k distribution on $N/\mathcal{F}_N$ and a pair $(N/\mathcal{F}_N, \boldsymbol{\eta}_N)$ is a k-contact manifold.

Additionally, $\ker \jmath^*\boldsymbol{\eta}_x = \mathrm{T}_x N \cap \ker \boldsymbol{\eta}_x$ and $\ker \jmath^*\mathrm{d}\boldsymbol{\eta}_x = \mathrm{T}_x N \cap (\mathrm{T}_x N)^{\perp_{\mathrm{d}\boldsymbol{\eta}}}$ yields that

$$\ker \jmath^*\boldsymbol{\eta}_x \cap \ker \jmath^*\mathrm{d}\boldsymbol{\eta}_x = \mathrm{T}_x N \cap (\mathrm{T}_x N)^{\perp_{\mathrm{d}\boldsymbol{\eta}}} \cap \ker \boldsymbol{\eta}_x,$$

for any $x \in N$. □

The analogue of Lemma 2.3.12 is established next.

Lemma 2.5.22. *Let $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ be a weak regular k-value of a k-contact momentum map $\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}\colon M \to \mathfrak{g}^{*k}$ associated with a Lie group action $\Phi\colon G \times M \to M$ and $(M, \boldsymbol{\eta})$. Then, for any $x \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$, one has*

(1) $\mathrm{T}_x(G_{[\boldsymbol{\mu}]}x) = \mathrm{T}_x(Gx) \cap \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$,

(2) $\mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) = (\mathrm{T}_x(Gx) \cap \ker \boldsymbol{\eta}_x)^{\perp_{\mathrm{d}\boldsymbol{\eta}}}$.

Proof. (1) Note that for any $\xi \in \mathfrak{g}$, it follows that

$$\mathrm{T}_x\mathbf{J}^{\Phi}_{\boldsymbol{\eta}\,\alpha}(\xi_M(x)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathbf{J}^{\Phi}_{\boldsymbol{\eta}} \circ \Phi_{\exp(t\xi)}\right)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathrm{Ad}^{*k}_{\exp(-t\xi)}\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}\right)(x) = -\mathrm{ad}^{*k}_{\xi}\boldsymbol{\mu}.$$

First, let $\xi \in \mathfrak{g}_{[\boldsymbol{\mu}]}$, then

$$0 = \boldsymbol{\mu} \wedge \mathrm{ad}^{*k}_{\xi}\boldsymbol{\mu} = -\boldsymbol{\mu} \wedge \mathrm{T}_x\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}(\xi_M(x)),$$

for any $x \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Hence, $\xi_M(x) \in \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$.

Conversely, $v \in \mathrm{T}_x(Gx) \cap \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ yields that $v = \xi_M(x) \in \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ and

$$0 = \boldsymbol{\mu} \wedge \mathrm{T}_x\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}(\xi_M(x)) = -\boldsymbol{\mu} \wedge \mathrm{ad}^{*k}_{\xi}\boldsymbol{\mu},$$

for any $x \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Therefore, $\xi_M(x) \in \mathrm{T}_x(G_{[\boldsymbol{\mu}]}x)$. This proves (1).

(2) Definition 2.5.16 yields that if $\xi_M(x) \in \ker \boldsymbol{\eta}_x$ for some $x \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$, then $\xi \in \ker \boldsymbol{\mu}$. Let $v \in (\mathrm{T}_x(Gx) \cap \ker \boldsymbol{\eta}_x)^{\perp_{\mathrm{d}\boldsymbol{\eta}}}$. Thus,

$$0 = (\iota_v\iota_{\xi_M}\mathrm{d}\boldsymbol{\eta})_x = -\iota_v\mathrm{d}\left\langle \mathbf{J}^{\Phi}_{\boldsymbol{\eta}}, \xi\right\rangle = \left\langle \mathrm{T}_x\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}(v), \xi\right\rangle, \qquad \forall \xi \in \ker \boldsymbol{\mu}.$$

Thus, $v \in \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$.

Conversely, let $v \in \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Then,

$$0 = \left\langle \mathrm{T}_x\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}(v), \xi\right\rangle = (\iota_v\iota_{\xi_M}\mathrm{d}\boldsymbol{\eta})_x, \qquad \forall \xi \in \ker \boldsymbol{\mu}.$$

Hence, $v \in (\mathrm{T}_x(Gx) \cap \ker \boldsymbol{\eta}_x)^{\perp_{\mathrm{d}\boldsymbol{\eta}}}$. This proves (2).

□

Blacker, in [12], provides sufficient and necessary conditions for performing the Marsden–Meyer–Weinstein reduction on k-polysymplectic manifolds, see Theorem 2.3.17. Remarkably, Theorem 2.5.21 yields the analogue theorem in [12, Theorem 2.14] but in k-contact setting. Taking $\mathrm{T}_x N := (\mathrm{T}_x W)^{\perp_{\mathrm{d}\boldsymbol{\eta}}}$ for some $W \subseteq M$ leads to the following theorem.

Theorem 2.5.23. *Let $W \subset M$ be a submanifold of $(M, \boldsymbol{\eta})$. Then,*

$$\frac{(\mathrm{T}_x W)^{\perp_{\mathrm{d}\boldsymbol{\eta}}}}{((\mathrm{T}_x W)^{\perp_{\mathrm{d}\boldsymbol{\eta}}})^{\perp_{\mathrm{d}\boldsymbol{\eta}}} \cap (\mathrm{T}_x W)^{\perp_{\mathrm{d}\boldsymbol{\eta}}} \cap \ker \boldsymbol{\eta}_x}$$

is a k-contact vector space.

By combining Theorem 2.5.21 and Lemma 2.5.22, the following theorem is obtained by setting $\mathrm{T}_x W := \mathrm{T}_x(Gx) \cap \ker \boldsymbol{\eta}_x$ in Theorem 2.5.23, which is the k-contact analogue of the k-polysymplectic result presented in Theorem 2.3.17 in [12, Theorem 3.22].

Theorem 2.5.24. *Let $(M, \boldsymbol{\eta}, \mathbf{J}^{\Phi}_{\boldsymbol{\eta}})$ be a k-contact Hamiltonian system and let Φ be a k-contact Lie group action. Assume that $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ is a weak regular k-value of k-contact momentum map $\mathbf{J}^{\Phi}_{\boldsymbol{\eta}} \colon M \to \mathfrak{g}^{*k}$ and $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ is a quotientable by $K_{[\boldsymbol{\mu}]}$. Moreover, assume that*

$$\mathrm{T}_x(K_{[\boldsymbol{\mu}]}x) = \left((\mathrm{T}_x(Gx) \cap \ker \boldsymbol{\eta}_x)^{\perp_{\mathrm{d}\boldsymbol{\eta}}}\right)^{\perp_{\mathrm{d}\boldsymbol{\eta}}} \cap (\mathrm{T}_x(Gx) \cap \ker \boldsymbol{\eta}_x)^{\perp_{\mathrm{d}\boldsymbol{\eta}}} \cap \ker \boldsymbol{\eta}_x, \tag{2.5.6}$$

for any $x \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Then, $(M_{[\boldsymbol{\mu}]} = \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}, \boldsymbol{\eta}_{[\boldsymbol{\mu}]})$ is a k-contact manifold, where $\boldsymbol{\eta}_{[\boldsymbol{\mu}]}$ is uniquely defined by

$$\pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}_{[\boldsymbol{\mu}]} = i^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta},$$

where $i_{[\boldsymbol{\mu}]} \colon \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) \hookrightarrow M$ is the natural immersion and $\pi_{[\boldsymbol{\mu}]} \colon \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) \to \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$ is the canonical projection.

The following subsections provide conditions that ensure that Equation (2.5.6) is satisfied.

2.5.5 k-Contact Marsden–Meyer–Weinstein reduction theorem

This subsection develops the Marsden–Meyer–Weinstein reduction for k-contact manifolds. The theorem is obtained by extending the framework of k-contact manifolds to the setting of k-polysymplectic manifolds and by employing the modified k-polysymplectic Marsden–Meyer–Weinstein reduction theorem introduced in Theorem 2.5.15. A crucial distinction from the k-polysymplectic and k-polycosymplectic reduction procedures considered in Section 2.4 lies in the fact that the preimage of the momentum map is taken with respect to $\mathbb{R}^{\times k}\boldsymbol{\mu}$, where $\mathbb{R}^{\times k} = \mathbb{R}^k \setminus 0$, rather than with respect to $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$. Additionally, a simple example of a product of k different contact manifolds is provided to illustrate the applicability of the results.

The following theorem establishes how a k-contact manifold can be extended to a k-polysymplectic manifold, and vice versa.

Theorem 2.5.25. *Let $\boldsymbol{\eta} \in \Omega^1(M, \mathbb{R}^k)$, let $\mathrm{pr}_M \colon \mathbb{R}^\times \times M \to M$ be the canonical projection onto M, and let $s \in \mathbb{R}^\times$ be a natural coordinate on $\mathbb{R}^\times$. Then, $(M, \boldsymbol{\eta})$ is a k-contact manifold if and only if $(\mathbb{R}^\times \times M, \mathrm{d}(s \cdot \mathrm{pr}^*_M \boldsymbol{\eta}) =: \boldsymbol{\omega})$ is a k-polysymplectic manifold with some vector fields $\widetilde{R}_1, \ldots, \widetilde{R}_k$ on $\mathbb{R}^\times \times M$ such that $\iota_{\widetilde{R}_\alpha}\omega^\beta = -\delta^\beta_\alpha \mathrm{d}s$ and $\widetilde{R}^\alpha s = 0$ for $\alpha, \beta = 1, \ldots, k$.*

Proof. Assume that $(M, \boldsymbol{\eta})$ is a k-contact manifold and let $X \in \mathfrak{X}(\mathbb{R}^\times \times M)$. Note that $\mathrm{d}\boldsymbol{\omega} = \mathrm{d}^2(s \cdot \mathrm{pr}^*_M \boldsymbol{\eta}) = 0$ and

$$\boldsymbol{\omega} = \mathrm{d}(s \cdot \mathrm{pr}^*_M \boldsymbol{\eta}) = \mathrm{d}s \wedge \mathrm{pr}^*_M \boldsymbol{\eta} + s\, \mathrm{d}\, \mathrm{pr}^*_M \boldsymbol{\eta}\,.$$

By Theorem 1.4.23, there exists a family of Reeb vector fields $R_1, \ldots, R_k$ on M that can be uniquely lifted to $\widetilde{R}_1, \ldots, \widetilde{R}_k$ on $\mathbb{R}^\times \times M$ so that $\widetilde{R}_\alpha s = 0$ and $\mathrm{pr}_{M*} \widetilde{R}_\alpha = R_\alpha$ for $\alpha = 1, \ldots, k$. In addition, $\widetilde{R}_\alpha$ satisfies that $\iota_{\widetilde{R}_\alpha}\omega^\beta = -\delta^\beta_\alpha \mathrm{d}s$. Thus, $\widetilde{R}_1, \ldots, \widetilde{R}_k$ span a rank k distribution on $\mathbb{R}^\times \times M$ given by $\ker \mathrm{pr}^*_M \mathrm{d}\boldsymbol{\eta} \cap \ker \mathrm{d}s$.

To prove that $\boldsymbol{\omega}$ is nondegenerate, suppose that X takes values in $\ker \boldsymbol{\omega} = \ker(s\, \mathrm{pr}^*_M \mathrm{d}\boldsymbol{\eta} + \mathrm{d}s \wedge \mathrm{pr}^*_M \boldsymbol{\eta})$ at least at one point $x \in M$. Then,

$$0 = (\iota_{\frac{\partial}{\partial s}} \iota_X \boldsymbol{\omega})_x \quad \Longrightarrow \quad (\iota_X\, \mathrm{pr}^*_M \boldsymbol{\eta})_x = 0,$$

and X_x takes values in $\ker(\mathrm{pr}_M^*\,\boldsymbol{\eta})_x$. Moreover,

$$0 = (\iota_{\widetilde{R}_\alpha}\iota_X\boldsymbol{\omega})_x \quad\Longrightarrow\quad (Xs)(x) = 0.$$

Therefore, $(\iota_X\mathrm{d}\,\mathrm{pr}_M^*\,\boldsymbol{\eta})_x = 0$ and $X_x = 0$, since $\ker\mathrm{d}s \cap \ker\mathrm{pr}_M^*\,\boldsymbol{\eta} \cap \ker\mathrm{d}\,\mathrm{pr}_M^*\,\boldsymbol{\eta} = 0$. Consequently, $(\mathbb{R}^\times \times M, \boldsymbol{\omega})$ is a k-polysymplectic manifold.

Conversely, let $(\mathbb{R}^\times \times M, \boldsymbol{\omega} = \mathrm{d}(s\,\mathrm{pr}_M^*\,\boldsymbol{\eta}))$ be a k-symplectic manifold with vector fields $\widetilde{R}_\alpha$ spanning a rank k distribution given by $\ker\mathrm{d}\,\mathrm{pr}_M^*\,\boldsymbol{\eta} \cap \ker\mathrm{d}s$.

Suppose that $X \in \mathfrak{X}(M)$ takes values in $\ker\boldsymbol{\eta} \cap \ker\mathrm{d}\boldsymbol{\eta}$ at least at one point $x \in M$. Then, X can be lifted uniquely to a vector field $\widetilde{X}$ such that $\mathrm{pr}_{M*}\widetilde{X} = X$ and $\widetilde{X}s = 0$. Then, $\iota_{\widetilde{X}}\boldsymbol{\omega} = 0$ at any point $(s, x) \in \mathbb{R}^\times \times \{x\}$ yields that $\widetilde{X} = 0$ on such points since $\boldsymbol{\omega}$ is nondegenerate. Therefore, $\ker\boldsymbol{\eta}_x \cap \ker\mathrm{d}\boldsymbol{\eta}_x = 0$ and, in general $\ker\boldsymbol{\eta} \cap \ker\mathrm{d}\boldsymbol{\eta} = 0$.

Next, since $\widetilde{R}_\alpha s = 0$ and $\iota_{\widetilde{R}_\alpha}\boldsymbol{\omega}^\beta = -\delta_\alpha^\beta\mathrm{d}s$, it follows that

$$\iota_{\widetilde{R}_\alpha}\mathrm{pr}_M^*\eta^\beta = \delta_\alpha^\beta, \qquad \iota_{\widetilde{R}_\alpha}\mathrm{pr}_M^*\mathrm{d}\boldsymbol{\eta} = 0.$$

Then,

$$\iota_{[\widetilde{R}_\alpha, \frac{\partial}{\partial s}]}\boldsymbol{\omega} = \mathscr{L}_{\widetilde{R}_\alpha}\iota_{\frac{\partial}{\partial s}}\boldsymbol{\omega} - \iota_{\frac{\partial}{\partial s}}\mathscr{L}_{\widetilde{R}_\alpha}\boldsymbol{\omega} = \mathscr{L}_{\widetilde{R}_\alpha}\mathrm{pr}_M^*\,\boldsymbol{\eta} = 0\,.$$

Therefore, $\widetilde{R}_1, \ldots, \widetilde{R}_k$ project onto the family of vector fields $R_1 := \mathrm{pr}_{M*}\widetilde{R}_1, \ldots, R_k := \mathrm{pr}_{M*}\widetilde{R}_k$ on M satisfying

$$\mathrm{pr}_M^*(\iota_{R_\alpha}\eta^\beta) = \mathrm{pr}_M^*(\iota_{\mathrm{pr}_{M*}\widetilde{R}_\alpha}\eta^\beta) = \iota_{\widetilde{R}_\alpha}\mathrm{pr}_M^*\,\eta^\beta = \delta_\alpha^\beta \quad\Longrightarrow\quad \iota_{R_\alpha}\eta^\beta = \delta_\alpha^\beta, \qquad \alpha, \beta = 1, \ldots, k\,,$$

and

$$\iota_{R_\alpha}\mathrm{d}\boldsymbol{\eta} = 0\,, \qquad \alpha = 1, \ldots, k\,.$$

Hence, $\mathrm{pr}_{M*}\widetilde{R}_1 = R_1, \ldots, \mathrm{pr}_{M*}\widetilde{R}_1 = R_k$ span a distribution of rank k given by $\ker\mathrm{d}\boldsymbol{\eta}$. Moreover, $\ker\boldsymbol{\eta}$ must have corank k, as otherwise there would be a non-zero vector $v_x \in \mathrm{T}_xM$ such that $v_x \in \ker\boldsymbol{\eta}_x \cap \ker\mathrm{d}\boldsymbol{\eta}_x$, which leads to a contradiction. Therefore, $(M, \boldsymbol{\eta})$ is a k-contact manifold. ☐

From now on, a k-polysymplectic manifold $(\mathbb{R}^\times \times M, \boldsymbol{\omega})$ that comes from the extension of a k-contact manifold $(M, \boldsymbol{\eta})$ is referred to as *a k-polysymplectic fibred manifold associated with* $\mathrm{pr}_M\colon \mathbb{R}^\times \times M \to M$, or simply a *$k$-polysymplectic fibred manifold*. The immediate conclusion from Theorem 2.6.1 is that a k-polysymplectic fibred manifold $(\mathbb{R}^\times \times M, \mathrm{d}(s\,\mathrm{pr}_M^*\,\boldsymbol{\eta}))$ is a one-homogeneous k-polysymplectic manifold related to the natural action $\phi\colon (\lambda; s, x) \in \mathbb{R}^\times \times \mathbb{R}^\times \times M \mapsto (\lambda s, x) \in \mathbb{R}^\times \times M$.

Every k-contact Lie group action $\Phi\colon G \times M \to M$ that leaves the k-contact form $\boldsymbol{\eta}$ invariant admits a k-contact momentum map $\mathbf{J}^\Phi_{\boldsymbol{\eta}}\colon M \to \mathfrak{g}^{*k}$. Since $\mathscr{L}_{R_\beta}\mathbf{J}^\Phi_{\boldsymbol{\eta}} = 0$ for $\beta = 1, \ldots, k$, the Lie group action Φ can be lifted to the *extended Lie group action*

$$\widetilde{\Phi}\colon (g; s, x) \in G \times \mathbb{R}^\times \times M \mapsto (s, \Phi_g(x)) \in \mathbb{R}^\times \times M\,,$$

admitting an *extended momentum map*

$$\mathbf{J}^{\widetilde{\Phi}}_{\boldsymbol{\eta}}\colon (s, x) \in \mathbb{R}^\times \times M \mapsto s\,\mathbf{J}^\Phi_{\boldsymbol{\eta}}(x) \in \mathfrak{g}^{*k}$$

relative to the k-polysymplectic fibred manifold $(\mathbb{R}^\times \times M, \boldsymbol{\omega} = \mathrm{d}(s\,\mathrm{pr}_M^*\,\boldsymbol{\eta}))$. Moreover, $\mathbf{J}^{\widetilde{\Phi}}_{\boldsymbol{\eta}}\colon \mathbb{R}^\times \times M \to \mathfrak{g}^{*k}$ is an exact k-polysymplectic momentum map associated with an exact k-polysymplectic Lie group action $\widetilde{\Phi}\colon G \times \mathbb{R}^\times \times M \to \mathbb{R}^\times \times M$.

It is worth noting that the alternative extension of k-contact manifold to k-polysymplectic manifolds is through $\mathbb{R}^{\times k}$ [48], similarly as in Theorem 2.4.7 for k-polycosymplectic manifolds [50, 62]. However, for the k-contact Marsden–Meyer–Weinstein reduction, the extension via $\mathbb{R}^\times$ is considered.

Example 2.5.26. The assumption of the existence of vector fields $\widetilde{R}_1, \ldots, \widetilde{R}_k$ in Theorem 2.5.25 is necessary as illustrated in this example. Consider a manifold $(\mathbb{R}^\times \times \mathbb{R}^4, \boldsymbol{\omega})$ with

$$\boldsymbol{\omega} = \mathrm{d}(s\, \mathrm{pr}_M^* \boldsymbol{\eta}) = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 = \mathrm{d}(s\, \mathrm{pr}_M^* \eta^1) \otimes e_1 + \mathrm{d}(s\, \mathrm{pr}_M^* \eta^2) \otimes e_2 ,$$

where $\{s; x_1, x_2, x_3, x_4\} \in \mathbb{R}^\times \times \mathbb{R}^4$ are local linear coordinates and

$$\mathrm{pr}_M^* \boldsymbol{\eta} = \mathrm{pr}_M^* \eta^1 \otimes e_1 + \mathrm{pr}_M^* \eta^2 \otimes e_2 = (\mathrm{d}x_1 + x_3 \mathrm{d}x_4) \otimes e_1 + (\mathrm{d}x_2 + x_2 \mathrm{d}x_1) \otimes e_2 .$$

Then, $\ker \omega^1 = \langle \frac{\partial}{\partial x_2}, \rangle$ and $\ker \omega^2 = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle$, leading to $\ker \boldsymbol{\omega} = 0$. Consequently, $(\mathbb{R}^\times \times \mathbb{R}^4, \boldsymbol{\omega})$ is a two-polysymplectic manifold. However, $\ker \mathrm{pr}_M^* \mathrm{d}\boldsymbol{\eta} = 0$ is not a rank two distribution on $\mathbb{R}^\times \times \mathbb{R}^4$ and $(\mathbb{R}^4, \boldsymbol{\eta})$ fails to be a two-contact manifold. △

Lemma 2.5.27. *Let $\mathbf{J}^\Phi_{\boldsymbol{\eta}} \colon M \to \mathfrak{g}^{*k}$ be a k-contact momentum map. Then, $\mu^\alpha \in \mathfrak{g}^*$ is a weak regular value of $\mathbf{J}^\Phi_{\boldsymbol{\eta}\,\alpha}$ for $\alpha = 1, \ldots, k$ if and only if $\mu^\alpha \in \mathfrak{g}^*$ is a weak regular value of $\mathbf{J}^{\widetilde{\Phi}}_{\boldsymbol{\eta}\,\alpha}$. Moreover, if $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ is a weak regular k-value of $\mathbf{J}^\Phi_{\boldsymbol{\eta}}$, then, for a k-polysymplectic momentum map $\mathbf{J}^{\widetilde{\Phi}}_{\boldsymbol{\eta}} \colon \mathbb{R}^\times \times M \ni (s, x) \mapsto s\mathbf{J}^\Phi_{\boldsymbol{\eta}}(x) \in \mathfrak{g}^{*k}$ associated with $\widetilde{\Phi} \colon G \times \mathbb{R}^\times \times M \to \mathbb{R}^\times \times M$, one has*

$$\mathrm{T}_{(s,x)} \mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu}) = \left\langle \frac{\partial}{\partial s} \right\rangle \oplus \mathrm{T}_x \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu}) , \qquad \forall (s, x) \in \mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu}) .$$

Proof. The proof of the lemma follows immediately from the construction of $\mathbf{J}^\Phi_{\boldsymbol{\eta}}$ and $\mathbf{J}^{\widetilde{\Phi}}_{\boldsymbol{\eta}}$ and the fact that $\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu}) = \mathbb{R}^\times \times \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu})$. However, there is a slight abuse of notation, as $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu})$ is denoted as a submanifold of P and M at the same time. ☐

Theorem 2.5.15 provides sufficient conditions for the existence of an exact k-polysymplectic form on the quotient manifold $\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu}) / K_{[\boldsymbol{\mu}]}$.

The following lemma translates conditions (2.5.3) and (2.5.4) in Theorem 2.5.15 into the k-contact setting on M. This allows for establishing sufficient conditions for the k-contact Marsden–Meyer–Weinstein reduction on M. The discussion on conditions (2.5.3) and (2.5.4) to be sufficient is analogous to one presented in Subsection 2.3.6.

Lemma 2.5.28. *Let $(M, \boldsymbol{\eta})$ be a k-contact manifold and let $(\mathbb{R}^\times \times M, s\, \mathrm{pr}_M^* \boldsymbol{\eta})$ be its associated k-polysymplectic fibred manifold with the canonical projection $\mathrm{pr}_M \colon \mathbb{R}^\times \times M \to M$. Then,*

$$\mathrm{T}_x \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}\,\alpha}(\mathbb{R}^\times \mu^\alpha) = \ker \eta^\alpha_x \cap \ker \mathrm{d}\eta^\alpha_x + \mathrm{T}_x \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu}) + \mathrm{T}_x(K_{[\mu^\alpha]} x) \tag{2.5.7}$$

and

$$\mathrm{T}_x(K_{[\boldsymbol{\mu}]} x) = \bigcap_{\alpha=1}^{k} \big((\ker \eta^\alpha_x \cap \ker \mathrm{d}\eta^\alpha_x) + \mathrm{T}_x(K_{[\mu^\alpha]} x) \big) \cap \mathrm{T}_x \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu}) , \tag{2.5.8}$$

hold for every $x \in \mathbf{J}^{\Phi-1}(\mathbb{R}^{\times k} \boldsymbol{\mu})$ and $\alpha = 1, \ldots, k$, if and only if

$$\mathrm{T}_p \mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}\,\alpha}(\mathbb{R}^\times \mu^\alpha) = \mathrm{T}_p \mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu}) + \ker \omega^\alpha_p + \mathrm{T}_p \left(K_{[\mu^\alpha]} p \right) \tag{2.5.9}$$

and

$$\mathrm{T}_p(K_{[\boldsymbol{\mu}]} p) = \bigcap_{\alpha=1}^{k} \left(\ker \omega^\alpha_p + \mathrm{T}_p(K_{[\mu^\alpha]} p) \right) \cap \mathrm{T}_p \mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu}) , \tag{2.5.10}$$

hold for every $p = (s, x) \in \mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k} \boldsymbol{\mu})$ and $\alpha = 1, \ldots, k$.

Proof. Taking into account the canonical projection $\mathrm{pr}_M \colon \mathbb{R}^\times \times M \to M$ and the natural isomorphisms $\mathrm{T}_{(s,x)}(\mathbb{R}^\times \times M) \simeq \mathrm{T}_x \mathbb{R}^\times \oplus \mathrm{T}_x M$, for every $(s, x) \in \mathbb{R}^\times \times M$ yields

$$(\ker \mathrm{pr}_M^* \eta^\alpha)_{(s,x)} = \mathrm{T}_s \mathbb{R}^\times \oplus \ker \eta^\alpha_x ,$$
$$(\ker \mathrm{pr}_M^* \mathrm{d}\eta^\alpha)_{(s,x)} = \mathrm{T}_s \mathbb{R}^\times \oplus \ker \mathrm{d}\eta^\alpha_x ,$$
$$(\ker \mathrm{d}s)_{(s,x)} = \{0\} \oplus \mathrm{T}_x M ,$$

for every $(s,x)\in\mathbb{R}^\times\times M$ and $\alpha=1,\dots,k$. Then,

$$\begin{aligned}(\ker\omega^\alpha)_{(s,x)} &= (\ker \mathrm{d}s\cap\ker\mathrm{pr}_M^*\,\eta^\alpha\cap\ker\mathrm{pr}_M^*\,\mathrm{d}\eta^\alpha)_{(s,x)}\\ &= (\{0\}\oplus\mathrm{T}_xM)\cap\left(\mathrm{T}_s\mathbb{R}^\times\oplus\ker\eta^\alpha_x\right)\cap\left(\mathrm{T}_s\mathbb{R}^\times\oplus\ker\mathrm{d}\eta^\alpha_x\right)\\ &= \{0\}\oplus(\ker\eta^\alpha_x\cap\ker\mathrm{d}\eta^\alpha_x)\,,\end{aligned}$$

for every $(s,x)\in\mathbb{R}^\times\times M$ and $\alpha=1,\dots,k$. Moreover, the definition of the extended momentum map and the extended Lie group action imply

$$\begin{aligned}&\mathrm{T}_{(s,x)}\left(K_{[\boldsymbol{\mu}]}(s,x)\right)=\{0\}\oplus\mathrm{T}_x\left(K_{[\boldsymbol{\mu}]}x\right), &&\mathrm{T}_{(s,x)}\mathbf{J}_{\boldsymbol{\eta}}^{\widetilde{\Phi}-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})=\mathrm{T}_s\mathbb{R}^\times\oplus\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu}),\\ &\mathrm{T}_{(s,x)}\left(K_{[\mu^\alpha]}(s,x)\right)=\{0\}\oplus\mathrm{T}_x\left(K_{[\mu^\alpha]}x\right), &&\mathrm{T}_{(s,x)}\mathbf{J}_{\boldsymbol{\eta}\,\alpha}^{\widetilde{\Phi}-1}(\mathbb{R}^{\times}\boldsymbol{\mu})=\mathrm{T}_s\mathbb{R}^\times\oplus\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}\,\alpha}^{\Phi-1}(\mathbb{R}^{\times}\boldsymbol{\mu})\,,\end{aligned}$$

for $\alpha=1,\dots,k$ and any $(s,x)\in\mathbf{J}_{\boldsymbol{\eta}}^{\widetilde{\Phi}-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. First, suppose that conditions (2.5.7) and (2.5.8) are satisfied. Then, condition (2.5.7) yields that

$$\begin{aligned}\mathrm{T}_p\mathbf{J}_{\boldsymbol{\eta}\,\alpha}^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu^\alpha)&=\mathrm{T}_s\mathbb{R}^\times\oplus\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}\,\alpha}^{\Phi-1}(\mathbb{R}^\times\mu^\alpha)\\ &=\mathrm{T}_s\mathbb{R}^\times\oplus\left(\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})+(\ker\eta^\alpha_x\cap\ker\mathrm{d}\eta^\alpha_x)+\mathrm{T}_x\left(K_{[\mu^\alpha]}x\right)\right)\\ &=\mathrm{T}_s\mathbb{R}^\times\oplus\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})+\{0\}\oplus(\ker\eta^\alpha_x\cap\ker\mathrm{d}\eta^\alpha_x)+\{0\}\oplus\mathrm{T}_x\left(K_{[\mu^\alpha]}x\right)\\ &=\mathrm{T}_p\mathbf{J}_{\boldsymbol{\eta}}^{\widetilde{\Phi}-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})+\ker\omega^\alpha_p+\mathrm{T}_p\left(K_{[\mu^\alpha]}p\right)\ ,\end{aligned}$$

and condition (2.5.8), gives

$$\begin{aligned}\mathrm{T}_p\left(K_{[\boldsymbol{\mu}]}p\right)&=\{0\}\oplus\mathrm{T}_x\left(K_{[\boldsymbol{\mu}]}x\right)\\ &=\{0\}\oplus\bigcap_{\alpha=1}^k\left(\ker\eta^\alpha_x\cap\ker\mathrm{d}\eta^\alpha_x+\mathrm{T}_x(K_{[\mu^\alpha]}x)\right)\cap\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})\\ &=\bigcap_{\alpha=1}^k\left(\{0\}\oplus(\ker\eta^\alpha_x\cap\ker\mathrm{d}\eta^\alpha_x)+\{0\}\oplus\mathrm{T}_x(K_{[\mu^\alpha]}x)\right)\cap\left(\mathrm{T}_x\mathbb{R}^\times\oplus\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})\right)\\ &=\bigcap_{\alpha=1}^k\left(\ker\omega^\alpha_p+\mathrm{T}_p\left(K_{[\mu^\alpha]}p\right)\right)\cap\mathrm{T}_p\mathbf{J}_{\boldsymbol{\eta}}^{\widetilde{\Phi}-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})\,,\end{aligned}$$

for every $p=(s,x)\in\mathbb{R}^\times\times M$, every $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$, and $\alpha=1,\dots,k$. Hence, (2.5.9) and (2.5.10) are satisfied.

Conversely, assume that (2.5.9) and (2.5.10) hold. Then, condition (2.5.9) can be rewritten as follows

$$\begin{aligned}\mathrm{T}_s\mathbb{R}^\times\oplus\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}\,\alpha}^{\Phi-1}(\mathbb{R}^\times\mu^\alpha)&=\mathrm{T}_p\mathbf{J}_{\boldsymbol{\eta}\,\alpha}^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu^\alpha)\\ &=\mathrm{T}_p\mathbf{J}_{\boldsymbol{\eta}}^{\widetilde{\Phi}-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})+\ker\omega^\alpha_p+\mathrm{T}_p\left(K_{[\mu^\alpha]}p\right)\\ &=\mathrm{T}_s\mathbb{R}^\times\oplus\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})+\{0\}\oplus(\ker\eta^\alpha_x\cap\ker\mathrm{d}\eta^\alpha_x)+\{0\}\oplus\mathrm{T}_x\left(K_{[\mu^\alpha]}x\right)\\ &=\mathrm{T}_s\mathbb{R}^\times\oplus\left(\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})+(\ker\eta^\alpha_x\cap\ker\mathrm{d}\eta^\alpha_x)+\mathrm{T}_x\left(K_{[\mu^\alpha]}x\right)\right)\ ,\end{aligned}$$

and (2.5.10) amounts to

$$\begin{aligned}\{0\}\oplus\mathrm{T}_x\left(K_{[\boldsymbol{\mu}]}x\right)&=\mathrm{T}_p\left(K_{[\boldsymbol{\mu}]}p\right)\\ &=\bigcap_{\alpha=1}^k\left(\ker\omega^\alpha_p+\mathrm{T}_p\left(K_{[\mu^\alpha]}p\right)\right)\cap\mathrm{T}_p\mathbf{J}_{\boldsymbol{\eta}}^{\widetilde{\Phi}-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})\\ &=\bigcap_{\alpha=1}^k\left(\{0\}\oplus(\ker\eta^\alpha_x\cap\ker\mathrm{d}\eta^\alpha_x)+\{0\}\oplus\mathrm{T}_x(K_{[\mu^\alpha]}x)\right)\cap\left(\mathrm{T}_x\mathbb{R}^\times\oplus\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})\right)\\ &=\{0\}\oplus\bigcap_{\alpha=1}^k\left(\ker\eta^\alpha_x\cap\ker\mathrm{d}\eta^\alpha_x+\mathrm{T}_x(K_{[\mu^\alpha]}x)\right)\cap\mathrm{T}_x\mathbf{J}_{\boldsymbol{\eta}}^{\Phi-1}(\mathbb{R}^{\times k}\boldsymbol{\mu})\,,\end{aligned}$$

for every $p=(s,x)\in\mathbb{R}^\times\times M$, every $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$, and $\alpha=1,\dots,k$. Therefore, conditions (2.5.9) and (2.5.10) are equivalent with the conditions (2.5.7) and (2.5.8), respectively. □

Theorem 2.5.29. *Let $(M,\boldsymbol{\eta},\mathbf{J}^{\Phi}_{\boldsymbol{\eta}})$ be a k-contact Hamiltonian system and let Φ be a k-contact Lie group action. Assume that $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$ is a weak regular k-value of $\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}$ and $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ is quotientable by $K_{[\boldsymbol{\mu}]}$. Moreover, let the following conditions hold*

$$\mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}\,\alpha}(\mathbb{R}^{\times}\mu^{\alpha}) = \ker\eta^{\alpha}_x\cap\ker\mathrm{d}\eta^{\alpha}_x + \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) + \mathrm{T}_x(K_{[\mu^{\alpha}]}x) \tag{2.5.11}$$

and

$$\mathrm{T}_x(K_{[\boldsymbol{\mu}]}x) = \bigcap_{\alpha=1}^{k}\left(\ker\eta^{\alpha}_x\cap\ker\mathrm{d}\eta^{\alpha}_x + \mathrm{T}_x(K_{[\mu^{\alpha}]}x)\right)\cap\mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\,, \tag{2.5.12}$$

for every $x\in\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ and $\alpha=1,\dots,k$. Then, $(M_{[\boldsymbol{\mu}]}=\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]},\boldsymbol{\eta}_{[\boldsymbol{\mu}]})$ is a k-contact manifold, while $\boldsymbol{\eta}_{[\boldsymbol{\mu}]}$ is uniquely defined by

$$\pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}_{[\boldsymbol{\mu}]} = i^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}\,,$$

where $i_{[\boldsymbol{\mu}]}\colon\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\hookrightarrow M$ and $\pi_{[\boldsymbol{\mu}]}\colon\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\to\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$ are the natural immersion and the canonical projection, respectively.

Proof. Theorem 2.5.25 guarantees that $(\mathbb{R}^{\times}\times M, s\,\mathrm{pr}^*_M\,\boldsymbol{\eta})$ is a k-polysymplectic fibred manifold associated with $(M,\boldsymbol{\eta})$. Consider the extended k-polysymplectic momentum map $\mathbf{J}^{\widetilde{\Phi}}_{\boldsymbol{\eta}}\colon\mathbb{R}^{\times}\times M\to\mathfrak{g}^{*k}$ associated with the extended k-polysymplectic Lie group action $\widetilde{\Phi}\colon G\times\mathbb{R}^{\times}\times M\to\mathbb{R}^{\times}\times M$ as defined before. Then, Lemma 2.5.27 implies that $\boldsymbol{\mu}\in\mathfrak{g}^{*k}$ is a weak regular k-value of $\mathbf{J}^{\widetilde{\Phi}}_{\boldsymbol{\eta}}$ while the conditions (2.5.11) and (2.5.12) imply that

$$\mathrm{T}_{(s,x)}\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}\,\alpha}(\mathbb{R}^{\times}\mu^{\alpha}) = \mathrm{T}_{(s,x)}\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) + \ker\omega^{\alpha}_{(s,x)} + \mathrm{T}_{(s,x)}\left(K_{[\mu^{\alpha}]}(s,x)\right)$$

and

$$\mathrm{T}_{(s,x)}(K_{[\boldsymbol{\mu}]}(s,x)) = \bigcap_{\alpha=1}^{k}\Big(\ker\omega^{\alpha}_{(s,x)} + \mathrm{T}_{(s,x)}(K_{[\mu^{\alpha}]}(s,x))\Big)\cap\mathrm{T}_{(s,x)}\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\,,$$

for every $(s,x)\in\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Hence, Theorem 2.5.15 gives that $(\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]},\boldsymbol{\omega}_{[\boldsymbol{\mu}]})$ is an exact k-polysymplectic manifold, with

$$\widetilde{i}^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega} = \widetilde{\pi}^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega}_{[\boldsymbol{\mu}]}\,,$$

where $\widetilde{i}_{[\boldsymbol{\mu}]}\colon\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\hookrightarrow P$ is the natural immersion and $\widetilde{\pi}_{[\boldsymbol{\mu}]}\colon\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})\to\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$ is the canonical projection. Additionally, from the definition of $\widetilde{\Phi}\colon G\times P\to P$ and Lemma 2.5.27, it follows that $\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]} = \mathbb{R}^{\times}\times\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]} =: \mathbb{R}^{\times}\times M_{[\boldsymbol{\mu}]}$. Note that $\boldsymbol{\vartheta}\in\Omega^1(\mathbb{R}^{\times}\times M_{[\boldsymbol{\mu}]},\mathbb{R}^k)$ defined as $\boldsymbol{\vartheta}=\iota_{\frac{\partial}{\partial s}}\boldsymbol{\omega}_{[\boldsymbol{\mu}]}$ is projectable with respect to the natural projection $\mathrm{pr}_{M_{[\boldsymbol{\mu}]}}\colon\mathbb{R}^{\times}\times M_{[\boldsymbol{\mu}]}\to M_{[\boldsymbol{\mu}]}$. Therefore, $\boldsymbol{\vartheta}=\mathrm{pr}^*_{M_{[\boldsymbol{\mu}]}}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}$ for some $\boldsymbol{\eta}_{[\boldsymbol{\mu}]}\in\Omega^1(M_{[\boldsymbol{\mu}]},\mathbb{R}^k)$ and $\widetilde{i}^*_{[\boldsymbol{\mu}]}\,\mathrm{pr}^*_M\,\boldsymbol{\eta} = \widetilde{\pi}^*_{[\boldsymbol{\mu}]}\,\mathrm{pr}^*_{M_{[\boldsymbol{\mu}]}}\,\boldsymbol{\eta}_{[\boldsymbol{\mu}]}$. It is worth noting that the following diagram commutes

$$\begin{tikzcd}
(\mathbb{R}^{\times}\times M,\boldsymbol{\omega}) \arrow[rr, "\mathrm{pr}_M"] & & (M,\boldsymbol{\eta}) \\
(\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}),\widetilde{i}^*_{[\boldsymbol{\mu}]}\boldsymbol{\omega}) \arrow[u, "\widetilde{i}_{[\boldsymbol{\mu}]}"] \arrow[rr, "\mathrm{pr}_M\big|_{\mathbf{J}^{\widetilde{\Phi}-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})}"] \arrow[d, "\widetilde{\pi}_{[\boldsymbol{\mu}]}"] & & (\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}),i^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}) \arrow[u, "i_{[\boldsymbol{\mu}]}"'] \arrow[d, "\pi_{[\boldsymbol{\mu}]}"] \\
(\mathbb{R}^{\times}\times M_{[\boldsymbol{\mu}]},\boldsymbol{\omega}_{[\boldsymbol{\mu}]}) \arrow[rr, "\mathrm{pr}_{M_{[\boldsymbol{\mu}]}}"] & & (M_{[\boldsymbol{\mu}]},\boldsymbol{\eta}_{[\boldsymbol{\mu}]}).
\end{tikzcd}$$

Figure 2.5: Note that the geometric structures on the left are covers of the structures of the right in the sense that are of the form $\Omega=\mathrm{d}(s\,\mathrm{pr}^*\,\eta)$.

Thus,

$$\widetilde{i}^*_{[\boldsymbol{\mu}]}\,\mathrm{pr}^*_M\,\boldsymbol{\eta} = \mathrm{pr}^*_M\,i^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta} \qquad \text{and} \qquad \widetilde{\pi}^*_{[\boldsymbol{\mu}]}\,\mathrm{pr}^*_{M_{[\boldsymbol{\mu}]}}\,\boldsymbol{\eta}_{[\boldsymbol{\mu}]} = \mathrm{pr}^*_M\,\pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}\,,$$

and it yields $i^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta} = \pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}$.

Recall that for $(M_{[\boldsymbol{\mu}]}, \boldsymbol{\eta}_{[\boldsymbol{\mu}]})$ to be a k-contact manifold, it is required that $\ker \boldsymbol{\eta}_{[\boldsymbol{\mu}]} \cap \ker \mathrm{d}\boldsymbol{\eta}_{[\boldsymbol{\mu}]} = 0$. Additionally, $\ker \boldsymbol{\eta}_{[\boldsymbol{\mu}]}$ and $\ker \mathrm{d}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}$ must be a corank k and rank k distributions, respectively. By Theorem 1.4.23, there exists a unique family of vector fields $R_1, \ldots, R_k \in \mathfrak{X}(M)$ such that $\iota_{R_\alpha}\eta^\beta = \delta^\beta_\alpha$ and $\iota_{R_\alpha}\mathrm{d}\boldsymbol{\eta} = 0$, for $\alpha, \beta = 1, \ldots, k$. Moreover,

$$\iota_{R_\alpha}\mathrm{d}\left\langle \mathbf{J}^\Phi_{\boldsymbol{\eta}}, \xi \right\rangle = \iota_{R_\alpha}\mathrm{d}\iota_{\xi_M}\boldsymbol{\eta} = \iota_{\xi_M}\iota_{R_\alpha}\mathrm{d}\boldsymbol{\eta} = 0\,, \qquad \forall \xi \in \mathfrak{g}\,, \qquad \alpha = 1, \ldots, k\,,$$

yields that $R_1, \ldots, R_k$ are tangent to $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Since $\Phi\colon G \times M \to M$ is a k-contact Lie group action, one has

$$\iota_{[\xi_M, R_\alpha]}\boldsymbol{\eta} = 0\,, \qquad \text{and} \qquad \iota_{[\xi_M, R_\alpha]}\mathrm{d}\boldsymbol{\eta} = 0\,, \qquad \forall \xi \in \mathfrak{g}\,, \qquad \alpha = 1, \ldots, k\,.$$

Therefore, $[R_\alpha, \xi_M] = 0$ for any $\xi \in \mathfrak{g}$ and $\alpha = 1, \ldots, k$. Thus, $R_1, \ldots, R_k$ project via $\pi_{[\boldsymbol{\mu}]}\colon \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) \to M_{[\boldsymbol{\mu}]}$ onto $R_{[\boldsymbol{\mu}]\,1}, \ldots, R_{[\boldsymbol{\mu}]\,k} \in \mathfrak{X}(M_{[\boldsymbol{\mu}]})$. Additionally,

$$\pi^*_{[\boldsymbol{\mu}]}(\iota_{R_{[\boldsymbol{\mu}]\,\alpha}}\eta^\beta_{[\boldsymbol{\mu}]}) = \iota_{R_\alpha}i^*_{[\boldsymbol{\mu}]}\eta^\beta = i^*_{[\boldsymbol{\mu}]}(\iota_{R_\alpha}\eta^\beta) = \delta^\beta_\alpha\,, \qquad \alpha, \beta = 1, \ldots, k\,,$$

and

$$\pi^*_{[\boldsymbol{\mu}]}(\iota_{R_{[\boldsymbol{\mu}]\,\alpha}}\mathrm{d}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}) = \iota_{R_\alpha}i^*_{[\boldsymbol{\mu}]}\mathrm{d}\boldsymbol{\eta} = i^*_{[\boldsymbol{\mu}]}(\iota_{R_\alpha}\mathrm{d}\boldsymbol{\eta}) = 0\,, \qquad \alpha = 1, \ldots, k\,,$$

where R_α denotes both the vector field R_α on M itself and its restriction to $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Hence, $R_{[\boldsymbol{\mu}]\,1}, \ldots, R_{[\boldsymbol{\mu}]\,k} \in \mathfrak{X}(M_{[\boldsymbol{\mu}]})$ are Reeb vector fields related to $(M_{[\boldsymbol{\mu}]}, \boldsymbol{\eta}_{[\boldsymbol{\mu}]})$, namely they give rise to a basis of the distribution given by $\ker \mathrm{d}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}$.

Let $\ell := \dim \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$, and let $\langle X_1, \ldots, X_\ell\rangle = \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ for any $x \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Since $\langle R_1, \ldots, R_k\rangle \subset \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$, within $\langle X_1, \ldots, X_\ell\rangle$, one can always choose a family of vector fields $\langle Y_1, \ldots, Y_{\ell-k}\rangle \subset \ker \boldsymbol{\eta}_x$ such that $\langle Y_1, \ldots, Y_{\ell-k}\rangle \oplus \langle R_1, \ldots, R_k\rangle = \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Taking into account, that $K_{[\boldsymbol{\mu}]}x \subset \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$, it follows that within $\langle Y_1, \ldots, Y_{\ell-k}\rangle$ there are vector fields $\xi^j_M(x)$, where $\xi^j \in \mathfrak{k}_{[\boldsymbol{\mu}]}$ and $j = 1, \ldots, \dim K_{[\boldsymbol{\mu}]}$. Consequently,

$$\mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) = \left\langle Y_1, \ldots, Y_{\ell-\dim K_{[\boldsymbol{\mu}]}-k}\right\rangle \oplus \left\langle \xi^1_M(x), \ldots, \xi^{\dim K_{[\boldsymbol{\mu}]}}_M(x)\right\rangle \oplus \langle R_1, \ldots, R_k\rangle\,,$$

for any $x \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Moreover, $\left\langle Y_1, \ldots, Y_{\ell-\dim K_{[\boldsymbol{\mu}]}-k}\right\rangle$ is a family of vector fields that project onto $M_{[\boldsymbol{\mu}]}$ and take values in $\ker \boldsymbol{\eta}_{[\boldsymbol{\mu}]}$. Since the Reeb vector fields $R_1, \ldots, R_k$ project onto $R_{[\boldsymbol{\mu}]1}, \ldots, R_{[\boldsymbol{\mu}]k}$, the vector fields $\xi^1_M(x), \ldots, \xi^{\dim K_{[\boldsymbol{\mu}]}}_M(x)$ project to zero, and $\ker i^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}_x \cap \ker \mathrm{d}i^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}_x = \mathrm{T}_x(K_{[\boldsymbol{\mu}]}x)$ by (2.5.11) and (2.5.12), it follows that the pair $(M_{[\boldsymbol{\mu}]}, \boldsymbol{\eta}_{[\boldsymbol{\mu}]})$ is indeed a k-contact manifold. □

The following example demonstrates the application of the k-contact Marsden–Meyer–Weinstein reduction theorem. Remarkably, numerous practical examples admit a related k-contact structure similar to the one presented below.

Example 2.5.30. (Product of contact manifolds) Let $M = M_1 \times \cdots \times M_k$ for some one-contact manifolds (co-oriented contact manifolds) (M_α, η^α) with $\alpha = 1, \ldots, k$. Let $\mathrm{pr}_\alpha\colon M \to M_\alpha$ be the canonical projection onto the α-th component M_α in M. Then, $(M, \boldsymbol{\eta} = \sum_{\alpha=1}^k \mathrm{pr}^*_\alpha\,\eta^\alpha \otimes e_\alpha)$ is a k-contact manifold since $\mathrm{rk}(\ker \mathrm{d}\boldsymbol{\eta}) = k$, $\mathrm{corank}(\ker \boldsymbol{\eta}) = k$, and $\ker \boldsymbol{\eta} \cap \ker \mathrm{d}\boldsymbol{\eta} = 0$.

For simplicity, denote $\mathrm{pr}^*_\alpha\,\eta^\alpha$ as η^α. Additionally, suppose that a contact Lie group action $\Phi^\alpha\colon G_\alpha \times M_\alpha \to M_\alpha$ admits a contact momentum map $\mathbf{J}^{\Phi^\alpha}_{\eta^\alpha}\colon M_\alpha \to \mathfrak{g}^*_\alpha$ and each Φ^α acts in a quotientable manner on $\mathbf{J}^{\Phi^\alpha-1}_{\eta^\alpha}(\mathbb{R}^\times\mu^\alpha)$ for each $\alpha = 1, \ldots, k$.

Define the k-contact Lie group action $G = G_1 \times \cdots \times G_k$ on M in the following way

$$\Phi\colon G \times M \ni (g_1, \ldots, g_k, x_1, \ldots, x_k) \longmapsto (\Phi^1_{g_1}(x_1), \ldots, \Phi^k_{g_k}(x_k)) \in M\,.$$

Then, $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_k$ is the Lie algebra of G and the associated k-contact momentum mapis given by

$$\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}\colon M \ni (x_1,\dots,x_k) \longmapsto \sum_{\alpha=1}^{k} (0,\dots,\mathbf{J}^{\alpha},\dots,0)\otimes e_\alpha \in \mathfrak{g}^{*k}\,,$$

where $\mathbf{J}^{\alpha}(x_1,\dots,x_k) = \mathbf{J}^{\Phi^\alpha}_{\eta^\alpha}(x_\alpha)$ for $\alpha = 1,\dots,k$ and $\mathfrak{g}^* = \mathfrak{g}^*_1 \times \cdots \times \mathfrak{g}^*_k$ is the dual space to $\mathfrak{g}$. Assume that $\mu^\alpha \in \mathfrak{g}^*_\alpha$ is a weak regular value of $\mathbf{J}^{\Phi^\alpha}_{\eta^\alpha}\colon M_\alpha \to \mathfrak{g}^*_\alpha$ for each $\alpha = 1,\dots,k$. Hence, $\boldsymbol{\mu} = \sum_{\alpha=1}^{k}(0,\dots,\mu^\alpha,\dots,0)\otimes e_\alpha \in \mathfrak{g}^{*k}$ is a weak regular k-value of $\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}$ and Φ acts in a quotientable manner on $\mathbf{J}^{\Phi}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Therefore, $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ is a submanifold of M, where $\mathbb{R}^{\times k}\boldsymbol{\mu} = (\mathbb{R}^{\times}\mu^1,0,\dots,0)\otimes e_1 + \cdots + (0,\dots,0,\mathbb{R}^{\times}\mu^k)\otimes e_k \subset \mathfrak{g}^{*k}$.

By Theorem 2.5.4 there exists a unique and simply connected Lie group $K_{[\boldsymbol{\mu}]} \subset G$, whose Lie algebra is $\mathfrak{k}_{[\boldsymbol{\mu}]} = \ker \boldsymbol{\mu} \cap \mathfrak{g}_{[\boldsymbol{\mu}]}$, where $\mathfrak{k}_{[\boldsymbol{\mu}]} = \mathfrak{k}_{[\mu^1]} \cap \cdots \cap \mathfrak{k}_{[\mu^k]}$, $\ker \boldsymbol{\mu} = \ker \mu^1 \cap \cdots \cap \ker \mu^k$, and $\mathfrak{g}_{[\boldsymbol{\mu}]} = \mathfrak{g}_{[\mu^1]} \cap \cdots \cap \mathfrak{g}_{[\mu^k]}$. Therefore, for $x = (x_1,\dots,x_k) \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$, the following relations hold

$$\begin{aligned}
\mathrm{T}_x\mathbf{J}^{\alpha-1}(\mathbb{R}^{\times}\mu^\alpha) &= \mathrm{T}_{x_1}M_1 \oplus \cdots \oplus \mathrm{T}_{x_\alpha}\mathbf{J}^{\Phi^\alpha-1}_{\eta^\alpha}(\mathbb{R}^{\times}\mu^\alpha) \oplus \cdots \oplus \mathrm{T}_{x_k}M_k\,,\\
\mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) &= \mathrm{T}_{x_1}\mathbf{J}^{\Phi^1-1}_{\eta^1}(\mathbb{R}^{\times}\mu^1) \oplus \cdots \oplus \mathrm{T}_{x_k}\mathbf{J}^{\Phi^k-1}_{\eta^k}(\mathbb{R}^{\times}\mu^k)\,,\\
\ker \eta^\alpha_x \cap \ker \mathrm{d}\eta^\alpha_x &= \mathrm{T}_{x_1}M_1 \oplus \cdots \oplus \mathrm{T}_{x_{\alpha-1}}M_{\alpha-1} \oplus \{0\} \oplus \mathrm{T}_{x_{\alpha+1}}M_{\alpha+1} \oplus \cdots \oplus \mathrm{T}_{x_k}M_k\,,\\
\mathrm{T}_x\left(K_{[\mu^\alpha]}x\right) &= \mathrm{T}_{x_1}\left(G_1x_1\right) \oplus \cdots \oplus \mathrm{T}_{x_\alpha}\left(K_{\alpha[\mu^\alpha]}x_\alpha\right) \oplus \cdots \oplus \mathrm{T}_{x_k}\left(G_kx_k\right)\,,\\
\mathrm{T}_x\left(K_{[\boldsymbol{\mu}]}x\right) &= \mathrm{T}_{x_1}\left(K_{1[\mu^1]}x_1\right) \oplus \cdots \oplus \mathrm{T}_{x_k}\left(K_{k[\mu^k]}x_k\right)\,.
\end{aligned}$$

Then, immediately follows that

$$\mathrm{T}_x\mathbf{J}^{\Phi^\alpha}_{\eta^\alpha}(\mathbb{R}^{\times}\mu^\alpha) = \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu}) + \ker \eta^\alpha_x \cap \ker \mathrm{d}\eta^\alpha_x + \mathrm{T}_x\left(K_{[\mu^\alpha]}x\right)\,,\qquad \alpha = 1,\dots,k\,,$$

and

$$\mathrm{T}_x\left(K_{[\boldsymbol{\mu}]}x\right) = \bigcap_{\beta=1}^{k}\left(\ker \eta^\beta_x \cap \ker \mathrm{d}\eta^\beta_x + \mathrm{T}_x\left(K_{[\mu^\beta]}x\right)\right) \cap \mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times}\boldsymbol{\mu})\,,$$

for every weak regular k-value $\boldsymbol{\mu} \in \mathfrak{g}^{*k}$ and $x \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$. Recall that, according to Theorem 2.5.29, these equations guarantee that the reduced space $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$ inherits a k-contact structure, while

$$\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]} \simeq \mathbf{J}^{\Phi^1-1}_{\eta^1}(\mathbb{R}^{\times}\mu^1)/K_{1[\mu^1]} \times \cdots \times \mathbf{J}^{\Phi^k-1}_{\eta^k}(\mathbb{R}^{\times}\mu^k)/K_{k[\mu^k]}\,.$$

△

Based on Theorem 2.5.29, the following theorem presents the reduction of the dynamics given by k-contact Hamiltonian k-vector fields.

Theorem 2.5.31. *Let assumptions of the Theorem 2.5.29 hold. Let* $(M,\boldsymbol{\eta},\mathbf{J}^{\Phi}_{\boldsymbol{\eta}},h)$ *be a* G*-invariant* k*-contact Hamiltonian system. Assume that* $\Phi_{g*}\mathbf{X}^h = \mathbf{X}^h$ *for every* $g \in K_{[\boldsymbol{\mu}]}$*, and* $\mathbf{X}^h$ *is tangent to* $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$*. Then, the flow* $\mathcal{F}^\alpha_t$ *of* X^h_α *leave* $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ *invariant and induces a unique flow* $\mathcal{K}^\alpha_t$ *on* $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$ *satisfying*

$$\pi_{[\boldsymbol{\mu}]} \circ \mathcal{F}^\alpha_t = \mathcal{K}^\alpha_t \circ \pi_{[\boldsymbol{\mu}]}\,,$$

for $\alpha = 1,\dots,k$*.*

Proof. Since $\mathbf{X}^h$ is tangent to $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ it follows that each integral curve $\mathcal{F}^\alpha_t$ of X^h_α is contained within $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ for all $t \in \mathbb{R}$ and $\alpha = 1,\dots,k$. The assumption that $\Phi_{g*}\mathbf{X}^h = \mathbf{X}^h$, for every $g \in K_{[\boldsymbol{\mu}]}$, implies that $\mathbf{X}^h = (X^h_1,\dots,X^h_k)$ projects onto a k-vector field $\mathbf{Y} = (Y_1,\dots,Y_k)$ on $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$, namely $\pi_{[\boldsymbol{\mu}]*}X^h_\alpha = Y_\alpha$ for $\alpha = 1,\dots,k$. Since $h \in \mathscr{C}^\infty(M)$ is G-invariant, it gives rise to a function $h_{[\boldsymbol{\mu}]} \in \mathscr{C}^\infty(\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]})$ satisfying $\pi^*_{[\boldsymbol{\mu}]}h_{[\boldsymbol{\mu}]} = i^*_{[\boldsymbol{\mu}]}h$. By Theorem 2.5.29, it follows that $(M_{[\boldsymbol{\mu}]},\boldsymbol{\eta}_{[\boldsymbol{\mu}]})$ is a k-contact manifold, while $\pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}_{[\boldsymbol{\mu}]} = i^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}$. The Reeb vector fields, $R_1,\dots,R_k$, are tangent to

$\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times k}\boldsymbol{\mu})$ and they project onto Reeb vector fields, $R_{[\boldsymbol{\mu}]1},\ldots,R_{[\boldsymbol{\mu}]k}$, on $M_{[\boldsymbol{\mu}]}$. Then, for $\alpha=1,\ldots,k$, one has

$$\begin{aligned}\pi^*_{[\boldsymbol{\mu}]}\mathrm{d}h_{[\boldsymbol{\mu}]} &= \mathrm{d}i^*_{[\boldsymbol{\mu}]}h = i^*_{[\boldsymbol{\mu}]}\left(\iota_{\mathbf{X}^h}\mathrm{d}\boldsymbol{\eta}+\sum_{\alpha=1}^{k}(R_\alpha h)\eta^\alpha\right) = \iota_{\mathbf{X}^h}i^*_{[\boldsymbol{\mu}]}\mathrm{d}\boldsymbol{\eta}+\sum_{\alpha=1}^{k}\left(R_\alpha\left(i^*_{[\boldsymbol{\mu}]}h\right)\right)i^*_{[\boldsymbol{\mu}]}\eta^\alpha\\
&= \iota_{\mathbf{X}^h}\pi^*_{[\boldsymbol{\mu}]}\mathrm{d}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}+\sum_{\alpha=1}^{k}\left(R_\alpha\left(\pi^*_{[\boldsymbol{\mu}]}h_{[\boldsymbol{\mu}]}\right)\right)\pi^*_{[\boldsymbol{\mu}]}\eta^\alpha_{[\boldsymbol{\mu}]}\\
&= \pi^*_{[\boldsymbol{\mu}]}\left(\iota_{\pi_{[\boldsymbol{\mu}]*}\mathbf{X}^h}\mathrm{d}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}\right)+\pi^*_{[\boldsymbol{\mu}]}\left(\sum_{\alpha=1}^{k}\left(\left(\pi_{[\boldsymbol{\mu}]*}R_\alpha\right)h_{[\boldsymbol{\mu}]}\right)\eta^\alpha_{[\boldsymbol{\mu}]}\right)\\
&= \pi^*_{[\boldsymbol{\mu}]}\left(\iota_{\mathbf{Y}}\mathrm{d}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}\right)+\pi^*_{[\boldsymbol{\mu}]}\left(\sum_{\alpha=1}^{k}\left(R_{[\boldsymbol{\mu}]\alpha}h_{[\boldsymbol{\mu}]}\right)\eta^\alpha_{[\boldsymbol{\mu}]}\right) = \pi^*_{[\boldsymbol{\mu}]}\left(\iota_{\mathbf{Y}}\mathrm{d}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}+\sum_{\alpha=1}^{k}(R_{[\boldsymbol{\mu}]\alpha}h_{[\boldsymbol{\mu}]})\eta^\alpha_{[\boldsymbol{\mu}]}\right),\end{aligned}$$

and

$$-\pi^*_{[\boldsymbol{\mu}]}h_{[\boldsymbol{\mu}]} = -i^*_{[\boldsymbol{\mu}]}h = i^*_{[\boldsymbol{\mu}]}\left(\iota_{\mathbf{X}^h}\boldsymbol{\eta}\right) = \iota_{\mathbf{X}^h}i^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta} = \iota_{\mathbf{X}^h}\pi^*_{[\boldsymbol{\mu}]}\boldsymbol{\eta}_{[\boldsymbol{\mu}]} = \pi^*_{[\boldsymbol{\mu}]}\left(\iota_{\pi_{[\boldsymbol{\mu}]*}\mathbf{X}^h}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}\right) = \pi^*_{[\boldsymbol{\mu}]}\left(\iota_{\mathbf{Y}}\boldsymbol{\eta}_{[\boldsymbol{\mu}]}\right).$$

Therefore, $\mathbf{Y}$ is a k-contact Hamiltonian k-vector field with respect to $h_{[\boldsymbol{\mu}]}\in\mathscr{C}^\infty(M_{[\boldsymbol{\mu}]})$, namely $\mathbf{Y}=\mathbf{X}^{h_{[\boldsymbol{\mu}]}}$. This completes the proof. □

Example 2.5.32 (Coupled strings with damping)**.** Consider the manifold $M=\oplus^2\mathrm{T}^*\mathbb{R}^2\times\mathbb{R}^2$ with coordinates $(q^1,q^2,p^t_1,p^t_2,p^x_1,p^x_2,s^t,s^x)$. The pair $(M,\boldsymbol{\eta})$ is a two-contact manifold, where the two-contact form is defined as

$$\boldsymbol{\eta}=\eta^t\otimes e_1+\eta^x\otimes e_2=(\mathrm{d}s^t-p^t_1\mathrm{d}q^1-p^t_2\mathrm{d}q^2)\otimes e_1+(\mathrm{d}s^x-p^x_1\mathrm{d}q^1-p^x_2\mathrm{d}q^2)\otimes e_2\,.$$

The Reeb vector fields associated with η^t and η^x are $R_t=\partial/\partial s^t$ and $R_x=\partial/\partial s^x$, respectively. Define the Lie group action

$$\Phi\colon\mathbb{R}^2\times M\ni(\lambda_1,\lambda_2;q^1,q^2,p^t_1,p^t_2,p^x_1,p^x_2,s^t,s^x)\longmapsto(q^1+\lambda_2,q^2+\lambda_2,p^t_1,p^t_2,p^x_1,p^x_2,s^t,s^x+\lambda_1)\in M\,.$$

This action is a two-contact, free, and proper Lie group action. Its fundamental vector fields are

$$\xi^1_M=\frac{\partial}{\partial s^x}\,,\qquad \xi^2_M=\frac{\partial}{\partial q^1}+\frac{\partial}{\partial q^2}\,.$$

The two-contact momentum map $\mathbf{J}^\Phi_{\boldsymbol{\eta}}\colon\ \oplus^2\,\mathrm{T}^*\mathbb{R}\times\mathbb{R}^2\to(\mathbb{R}^2)^{*2}$ is then

$$\mathbf{J}^\Phi_{\boldsymbol{\eta}}\colon M\ni y\longmapsto\boldsymbol{\mu}=\mu^1\otimes e_1+\mu^2\otimes e_2=(0,-p^t_1-p^t_2)\otimes e_1+(1,-p^x_1-p^x_2)\otimes e_2\in(\mathbb{R}^2)^{*2}\,.$$

Recall that for any $y\in\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})$, one has

$$\mathrm{T}_y\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})=\{v_y\in\mathrm{T}_yM\ \mid\ \mathrm{T}_y\mathbf{J}^{\Phi}_{\boldsymbol{\eta}\,\alpha}(v_y)=\lambda_\alpha\mu^\alpha,\quad\lambda_\alpha\in\mathbb{R}^\times,\quad\alpha=1,\ldots,k\}\,.$$

Fixing $\boldsymbol{\mu}=(0,0)\otimes e_1+(1,0)\otimes e_2\in\mathfrak{g}^{*2}$, it follows that $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})$ is a submanifold of M given by

$$\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})=\{y\in M\ \mid\ p^t_1=-p^t_2\,,\quad p^x_1=-p^x_2\}$$

with

$$\mathrm{T}_y\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})=\left\langle\frac{\partial}{\partial s^x},\frac{\partial}{\partial s^t},\frac{\partial}{\partial q^1},\frac{\partial}{\partial q^2},\frac{\partial}{\partial p^x_1}-\frac{\partial}{\partial p^x_2},\frac{\partial}{\partial p^t_1}-\frac{\partial}{\partial p^t_2}\right\rangle\,.$$

The element $\boldsymbol{\mu}\in(\mathbb{R}^2)^{*2}$ is a weak regular 2-value of $\mathbf{J}^\Phi_{\boldsymbol{\eta}}$ but not a regular 2-value. Since $\ker\boldsymbol{\mu}=\langle\xi^2\rangle$ and $\mathfrak{g}_{[\boldsymbol{\mu}]}=\langle\xi^1,\xi^2\rangle$, it follows that $\mathfrak{k}_{[\boldsymbol{\mu}]}=\ker\boldsymbol{\mu}\cap\mathfrak{g}_{[\boldsymbol{\mu}]}=\langle\xi^2\rangle$, and thus

$$\mathrm{T}_y(K_{[\boldsymbol{\mu}]}y)=\left\langle\frac{\partial}{\partial q^1}+\frac{\partial}{\partial q^2}\right\rangle\,.$$

Moreover, one has

$$\mathfrak{k}_{\mu^1} = \ker \mu^1 \cap \mathfrak{g}_{\mu^1} = \langle \xi^1, \xi^2 \rangle \qquad \mathfrak{k}_{\mu^2} = \ker \mu^2 \cap \mathfrak{g}_{\mu^2} = \langle \xi^2 \rangle$$

and

$$\begin{aligned}
\mathrm{T}_y \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}\,1}(\mathbb{R}^\times \mu^1) &= \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial s^t}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p_1^x}, \frac{\partial}{\partial p_2^x}, \frac{\partial}{\partial p_1^t} - \frac{\partial}{\partial p_2^t} \right\rangle , \\
\mathrm{T}_y \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}\,2}(\mathbb{R}^\times \mu^2) &= \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial s^t}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p_1^x} - \frac{\partial}{\partial p_2^x}, \frac{\partial}{\partial p_1^t}, \frac{\partial}{\partial p_2^t} \right\rangle , \\
\mathrm{T}_y \left(K_{[\mu^1]} y\right) = \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} \right\rangle , &\qquad \mathrm{T}_y \left(K_{[\mu^2]} y\right) = \left\langle \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} \right\rangle , \\
\ker \eta^t_y \cap \ker \mathrm{d}\eta^t_y = \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial p_1^x}, \frac{\partial}{\partial p_2^x} \right\rangle , &\qquad \ker \eta^x_y \cap \ker \mathrm{d}\eta^x_y = \left\langle \frac{\partial}{\partial s^t}, \frac{\partial}{\partial p_1^t}, \frac{\partial}{\partial p_2^t} \right\rangle .
\end{aligned}$$

Finally, for each $y \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})$, condition (2.5.11) holds since

$$\begin{aligned}
&\ker \eta^t_y \cap \ker \mathrm{d}\eta^t_y + \mathrm{T}_y \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu}) + \mathrm{T}_y \left(K_{[\mu^1]} y\right)\big) \\
&= \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial p_1^x}, \frac{\partial}{\partial p_2^x} \right\rangle + \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial s^t}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p_1^x} - \frac{\partial}{\partial p_2^x}, \frac{\partial}{\partial p_1^t} - \frac{\partial}{\partial p_2^t} \right\rangle + \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} \right\rangle \\
&= \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial s^t}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p_1^x}, \frac{\partial}{\partial p_2^x}, \frac{\partial}{\partial p_1^t} - \frac{\partial}{\partial p_2^t} \right\rangle = \mathrm{T}_y \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}\,1}(\mathbb{R}^\times \mu^1),
\end{aligned}$$

and

$$\begin{aligned}
&\ker \eta^x_y \cap \ker \mathrm{d}\eta^x_y + \mathrm{T}_y \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu}) + \mathrm{T}_y \left(K_{[\mu^2]} y\right)\big) \\
&= \left\langle \frac{\partial}{\partial s^t}, \frac{\partial}{\partial p_1^t}, \frac{\partial}{\partial p_2^t} \right\rangle + \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial s^t}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p_1^x} - \frac{\partial}{\partial p_2^x}, \frac{\partial}{\partial p_1^t} - \frac{\partial}{\partial p_2^t} \right\rangle + \left\langle \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} \right\rangle \\
&= \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial s^t}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p_1^x} - \frac{\partial}{\partial p_2^x}, \frac{\partial}{\partial p_1^t}, \frac{\partial}{\partial p_2^t} \right\rangle = \mathrm{T}_y \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}\,2}(\mathbb{R}^{\times 2} \mu^2).
\end{aligned}$$

Similarly, condition (2.5.12) holds because

$$\begin{aligned}
&\big(\ker \eta^t_y \cap \ker \mathrm{d}\eta^t_y + \mathrm{T}_y \left(K_{[\mu^1]} y\right)\big) \cap \big(\ker \eta^x_y \cap \ker \mathrm{d}\eta^x_y + \mathrm{T}_y \left(K_{[\mu^2]} y\right)\big) \cap \mathrm{T}_y \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu}) = \\
&= \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial p_1^x}, \frac{\partial}{\partial p_2^x}, \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} \right\rangle \cap \left\langle \frac{\partial}{\partial s^t}, \frac{\partial}{\partial p_1^t}, \frac{\partial}{\partial p_2^t}, \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} \right\rangle \\
&\qquad \cap \left\langle \frac{\partial}{\partial s^x}, \frac{\partial}{\partial s^t}, \frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^2}, \frac{\partial}{\partial p_1^x} - \frac{\partial}{\partial p_2^x}, \frac{\partial}{\partial p_1^t} - \frac{\partial}{\partial p_2^t} \right\rangle = \mathrm{T}_y \left(K_{[\boldsymbol{\mu}]} y\right) ,
\end{aligned}$$

for any $y \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})$. Consequently, Theorem 2.5.29 ensures that the quotient manifold $M_{[\boldsymbol{\mu}]} = (\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}, \boldsymbol{\eta}_{[\boldsymbol{\mu}]})$ is a two-contact manifold with

$$\boldsymbol{\eta}_{[\boldsymbol{\mu}]} = \eta^t_{[\boldsymbol{\mu}]} \otimes e_1 + \eta^x_{[\boldsymbol{\mu}]} \otimes e_2 = \left(\mathrm{d}s^t - \frac{1}{2} p^t \mathrm{d}q\right) \otimes e_1 + \left(\mathrm{d}s^x - \frac{1}{2} p^x \mathrm{d}q\right) \otimes e_2 \,,$$

where $(q := q^1 - q^2, p^t := p_1^t - p_2^t, p^x := p_1^x - p_2^x, s^t, s^x)$ are local coordinates on $M_{[\boldsymbol{\mu}]} \simeq \mathbb{R}^5$.

Consider now a system of coupled damped strings with a Hamiltonian function $h \colon M \to \mathbb{R}$ of the form

$$h(q^1, q^2, p_1^t, p_2^t, p_1^x, p_2^x, s^t, s^x) = \frac{1}{2}\left((p_1^t)^2 + (p_2^t)^2 - (p_1^x)^2 - (p_1^x)^2\right) + C(q^1 - q^2) + \gamma s^t \,,$$

where $C(q^1 - q^2)$ is a coupling function between the two strings. The dynamics on $(M, \boldsymbol{\eta})$ is given by the two-contact Hamiltonian two-vector field $\mathbf{X}^h = (X^h_t, X^h_s) \in \mathfrak{X}^2(M)$, whose local expression is

$$\begin{aligned}
X^h_t &= p_1^t \frac{\partial}{\partial q^1} + p_2^t \frac{\partial}{\partial q^2} + \left(-\frac{\partial C}{\partial q} - \gamma p_1^t - G^x_{x1}\right) \frac{\partial}{\partial p_1^t} + \left(\frac{\partial C}{\partial q} - \gamma p_2^t - G^x_{x2}\right) \frac{\partial}{\partial p_2^t} + G^x_{t1} \frac{\partial}{\partial p_1^x} \\
&\quad + G^x_{t2} \frac{\partial}{\partial p_2^x} + \left(\frac{1}{2}((p_1^t)^2 + (p_2^t)^2 - (p_1^x)^2 - (p_2^x)^2) - C(q) - \gamma s^t - g^x_x\right) \frac{\partial}{\partial s^t} + g^x_t \frac{\partial}{\partial s^x} \,, \\
X^h_x &= -p_1^x \frac{\partial}{\partial q^1} - p_2^x \frac{\partial}{\partial q^2} + G^t_{x1} \frac{\partial}{\partial p_1^t} + G^t_{x2} \frac{\partial}{\partial p_2^t} + G^x_{x1} \frac{\partial}{\partial p_1^x} + G^x_{x2} \frac{\partial}{\partial p_2^x} + g^t_x \frac{\partial}{\partial s^t} + g^x_x \frac{\partial}{\partial s^x} \,,
\end{aligned}$$

with arbitrary functions $G^x_{x1}, G^x_{x2}, G^t_{x1}, G^t_{x2}, G^x_{t1}, G^x_{t2}, g^x_x, g^x_t, g^t_x$ on M.

According to Theorem 2.5.31, reduction onto $M_{[\boldsymbol{\mu}]} = \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]}$ requires $\mathbf{X}^h$ to be tangent to $\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})$ and G-invariant, i.e. $\Phi_{g*}\mathbf{X}^h = \mathbf{X}^h$ for every $g \in \mathbb{R}^2$. Therefore, assume that all these arbitrary functions are $\mathbb{R}^2$-invariant and satisfy

$$G^x_{x1} = -G^x_{x2}, \quad G^x_{t1} = -G^x_{t2}, \quad G^t_{x1} = -G^t_{x2}.$$

Then, $\mathbf{X}^h$ takes the form

$$\begin{aligned} X^h_t &= p^t_1\left(\frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}\right) - \left(\frac{\partial C}{\partial q} + \gamma p^t_1 + G^x_{x1}\right)\left(\frac{\partial}{\partial p^t_1} - \frac{\partial}{\partial p^t_2}\right) + G^x_{t1}\left(\frac{\partial}{\partial p^x_1} - \frac{\partial}{\partial p^x_2}\right) \\ &\quad + \left((p^t_1)^2 - (p^x_1)^2 - C(q) - \gamma s^t - g^x_x\right)\frac{\partial}{\partial s^t} + g^x_t\frac{\partial}{\partial s^x}\,, \\ X^h_x &= -p^x_1\left(\frac{\partial}{\partial q^1} - \frac{\partial}{\partial q^2}\right) + G^t_{x1}\left(\frac{\partial}{\partial p^t_1} - \frac{\partial}{\partial p^t_2}\right) + G^x_{x1}\left(\frac{\partial}{\partial p^x_1} - \frac{\partial}{\partial p^x_2}\right) + g^t_x\frac{\partial}{\partial s^t} + g^x_x\frac{\partial}{\partial s^x}\,, \end{aligned}$$

for any point $y \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})$. Applying Theorem 2.5.31, the reduced two-contact Hamiltonian two-vector field $\mathbf{X}_{[\boldsymbol{\mu}]} = (X^{h_{[\boldsymbol{\mu}]}}_t, X^{h_{[\boldsymbol{\mu}]}}_x)$ on $M_{[\boldsymbol{\mu}]}$ reads

$$\begin{aligned} X^{h_{[\boldsymbol{\mu}]}}_t &= p^t\frac{\partial}{\partial q} - \left(2\frac{\partial C}{\partial q} + \gamma p^t + 2\widetilde{G^x_{x1}}\right)\frac{\partial}{\partial p^t} + \left(\frac{1}{4}(p^2_t - p^2_x) - C(q) - \gamma s^t - \widetilde{g^x_x}\right)\frac{\partial}{\partial s^t} + 2\widetilde{G^x_{t1}}\frac{\partial}{\partial p^x} + \widetilde{g^x_t}\frac{\partial}{\partial s^x}\,, \\ X^{h_{[\boldsymbol{\mu}]}}_x &= -p^x\frac{\partial}{\partial q} + 2\widetilde{G^t_{x1}}\frac{\partial}{\partial p^t} + 2\widetilde{G^x_{x1}}\frac{\partial}{\partial p^x} + \tilde{g}^t_x\frac{\partial}{\partial s^t} + \widetilde{g^x_x}\frac{\partial}{\partial s^x}\,, \end{aligned}$$

where $\widetilde{G^x_{x1}}$, $\widetilde{G^t_{x1}}$, $\widetilde{G^x_{t1}}$, $\widetilde{g^x_x}$, $\widetilde{g^x_t}$, and $\tilde{g}^t_x$ are functions on $M_{[\boldsymbol{\mu}]}$ coming from the G-invariant functions without tildes on M and $h_{[\boldsymbol{\mu}]}$ is the reduced Hamiltonian function on $M_{[\boldsymbol{\mu}]}$ given by

$$h_{[\boldsymbol{\mu}]} = \frac{1}{4}\left((p^t)^2 + (p^x)^2\right) + C(q) + \gamma s^t\,.$$

The two-vector field $\mathbf{X}^h$ is integrable when $[X^h_t, X^h_x] = 0$. To guarantee the integrability, consider the restriction to the submanifold $N := \{y \in M \mid p^x_1 = 0 = p^x_2\} \subset \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})$. Additionally, assume that the functions $G^x_{x1}, G^x_{x2}, g^x_x$ are constant, while $G^t_{x1}, G^x_{x2}, G^x_{t1}, G^x_{t2}, g^x_t, g^t_x$ vanish. Under these assumptions, the two-contact Hamiltonian two-vector field $\mathbf{X}^h$ on N gives rise to the following Hamilton–De Donder–Weyl equations on N (note that these are not exactly the Hamilton–De Donder–Weyl equations since N is not a k-contact manifold)

$$\begin{gathered} \frac{\partial q^1}{\partial t} = p^t_1\,, \qquad \frac{\partial q^2}{\partial t} = p^t_2\,, \qquad \frac{\partial q^1}{\partial x} = -p^x_1 = 0\,, \qquad \frac{\partial q^2}{\partial x} = -p^x_2 = 0\,, \\ \frac{\partial p^t_1}{\partial t} = -\frac{\partial C}{\partial q} - \gamma p^t_1 - G^x_{x1} = -\frac{\partial C}{\partial q} - \gamma p^t_1 - \frac{\partial p^x_1}{\partial x} = -\frac{\partial C}{\partial q} - \gamma p^t_1\,, \\ \frac{\partial p^t_2}{\partial t} = \frac{\partial C}{\partial q} - \gamma p^t_2 - G^x_{x2} = \frac{\partial C}{\partial q} - \gamma p^t_2 - \frac{\partial p^x_2}{\partial x} = \frac{\partial C}{\partial q} - \gamma p^t_2\,. \end{gathered}$$

By combining the above equations, one obtains the following system of PDEs

$$\frac{\partial^2 q^1}{\partial t^2} = -\gamma\frac{\partial q^1}{\partial t} - \frac{\partial C}{\partial q}\,, \qquad \frac{\partial^2 q^2}{\partial t^2} = -\gamma\frac{\partial q^2}{\partial t} + \frac{\partial C}{\partial q}\,.$$

This system describes two coupled, damped, vibrating strings constrained to the submanifold N.

Furthermore, the integral sections of the reduced two-contact Hamiltonian two-vector field $\mathbf{X}^{h_{[\boldsymbol{\mu}]}}$, restricted to $\pi_{[\boldsymbol{\mu}]}(N) = \{\pi_{[\boldsymbol{\mu}]}(y) \in M_{[\boldsymbol{\mu}]} \mid p^x = 0\}$, lead to the following system of PDEs

$$\begin{gathered} \frac{\partial q}{\partial t} = p^t\,, \qquad \frac{\partial q}{\partial x} = -p^x = 0\,, \\ \frac{\partial p^t}{\partial t} = -2\frac{\partial C}{\partial q} + \gamma p^t - 2\widetilde{G^x_{x1}} = -2\frac{\partial C}{\partial q} + \gamma p^t - \frac{\partial p^x}{\partial x} = -2\frac{\partial C}{\partial q} + \gamma p^t\,. \end{gathered}$$

Consequently, this system reduces to

$$\frac{\partial p^t}{\partial t} = \gamma p^t - 2\frac{\partial C}{\partial q} \quad \Rightarrow \quad \frac{\partial^2 q}{\partial t^2} = \gamma \frac{\partial q}{\partial t} - 2\frac{\partial C}{\partial q}\,,$$

which represents the equation of a single damped vibrating string with an external force acting on it, constrained to $\pi_{[\boldsymbol{\mu}]}(N)$. △

Example 2.5.33. Consider the manifold $M = \mathbb{R}^5 \times \mathbb{R}^5$ with

$$\boldsymbol{\eta} = \eta^1 \otimes e_1 + \eta^2 \otimes e_2 = (\mathrm{d}s_1 - x_2\mathrm{d}x_1 - x_4\mathrm{d}x_3) \otimes e_1 + (\mathrm{d}s_2 - y_2\mathrm{d}y_1 - y_4\mathrm{d}y_3)\,,$$

where $(x_1,\dots,x_4,s_1,y_1,\dots,y_4,s_2)$ are linear coordinates on $\mathbb{R}^{10}$. Since, each pair $(\mathbb{R}^5,\eta^\alpha)$ is a one-contact manifold with local coordinates $(x_1,\dots,x_4,s_1)$ and $(y_1,\dots,y_4,s_2)$, it follows from Example 2.5.30 that $(M,\boldsymbol{\eta})$ is a two-contact manifold.

Consider the vector fields on M of the form

$$\xi^1_M = \frac{\partial}{\partial s_2}\,, \qquad \xi^2_M = \frac{\partial}{\partial x_3}\,, \qquad \xi^3_M = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3}\,.$$

These vector fields are generated by an abelian three-dimensional Lie group acting via translations on M and leaving $\boldsymbol{\eta}$ invariant. This Lie group action acts in a quotientable manner on M. The corresponding two-contact momentum map $\mathbf{J}^\Phi_{\boldsymbol{\eta}}\colon M \to (\mathbb{R}^3)^{*2}$ reads

$$\begin{aligned}\mathbf{J}^\Phi_{\boldsymbol{\eta}}\colon M \ni &(x_1,\dots,x_4,s_1,y_1,\dots,y_4,s_2)\\ &\longmapsto \mu^1 \otimes e_1 + \mu^2 \otimes e_2 = (0,-x_4,0)\otimes e_1 + (1,0,-y_2-y_4)\otimes e_2 \in (\mathbb{R}^3)^{*2}\,.\end{aligned}$$

Choosing $\boldsymbol{\mu} = (1,0,-1)\otimes e_2$, it follows that

$$\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu}) = \{x \in M \mid x_4 = 0\,, \quad y_2 + y_4 = 1\}$$

and

$$\mathrm{T}_x\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu}) = \left\langle \frac{\partial}{\partial s_1}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial s_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_4}, \frac{\partial}{\partial y_3} \right\rangle.$$

Moreover, $\mathfrak{k}_{[\boldsymbol{\mu}]} = \langle \xi^2, \xi^1 + \xi^3\rangle$. By Example 2.5.30, both reduction conditions (2.5.7) and (2.5.8) hold. Introducing the following change of coordinates

$$\begin{gathered}\alpha = \frac{1}{3}(y_1 + y_3 + s_2)\,, \qquad \beta = \frac{1}{3}(y_1 + y_3 - 2s_2)\,,\\ z_2 = y_2 + y_4\,, \qquad z_3 = y_1 - y_3\,, \qquad z_4 = y_2 - y_4\,,\end{gathered}$$

while (s_1,x_2,x_2,x_3,x_4) remain unchanged, one has

$$\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu}) = \{x \in M \mid x_4 = 0, \quad z_2 = 1\}$$

and

$$\mathrm{T}_x\left(K_{[\boldsymbol{\mu}]}x\right) = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial \alpha} \right\rangle$$

for any $x \in \mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})$. Hence, Theorem 2.5.29 implies that $(\mathbf{J}^{\Phi-1}_{\boldsymbol{\eta}}(\mathbb{R}^{\times 2}\boldsymbol{\mu})/K_{[\boldsymbol{\mu}]} \simeq \mathbb{R}^6, \boldsymbol{\eta}_{[\boldsymbol{\mu}]})$ is a two-contact manifold with

$$\boldsymbol{\eta}_{[\boldsymbol{\mu}]} = \eta^1_{[\boldsymbol{\mu}]} \otimes e_1 + \eta^2_{[\boldsymbol{\mu}]} \otimes e_2 = (\mathrm{d}s_1 - x_2\mathrm{d}x_1)\otimes e_1 + \left(-\frac{3}{2}\mathrm{d}\beta - \frac{1}{2}z_4\mathrm{d}z_3\right) \oplus e_2.$$

The corresponding reduced Reeb vector fields are given by

$$R_{[\boldsymbol{\mu}]\,1} = \frac{\partial}{\partial s_1}\,, \qquad R_{[\boldsymbol{\mu}]\,2} = -\frac{2}{3}\frac{\partial}{\partial \beta}\,.$$

Example 2.5.34. The following example presents the one-contact reduction for the spherical cotangent bundle of a Riemannian manifold. Let (Q, g) be an n-dimensional Riemannian manifold and let 0_Q denote the zero section of the cotangent bundle $\pi_Q\colon \mathrm{T}^*Q \to Q$. Consider the action of $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ on $\mathrm{T}^*Q - 0_Q$ defined by

$$\phi\colon (s, \alpha) \in \mathbb{R}^+ \times (\mathrm{T}^*Q - 0_Q) \longmapsto \phi_s(\alpha) := s\alpha \in (\mathrm{T}^*Q - 0_Q)\,.$$

This action gives rise to an $\mathbb{R}^+$-symplectic principal bundle $\tau\colon (\mathrm{T}^*Q - 0_Q) \to (\mathrm{T}^*Q - 0_Q)/\mathbb{R}^+$. The canonical symplectic form on $(\mathrm{T}^*Q - 0_Q)$ is one-homogeneous with respect to ϕ. Hence, the quotient manifold $(\mathrm{T}^*Q - 0_Q)/\mathbb{R}^+$ is diffeomorphic to the spherical cotangent bundle given by

$$\mathbb{S}(\mathrm{T}^*Q) = \left\{\alpha \in \mathrm{T}^*Q \mid \sqrt{g(\alpha, \alpha)} = 1\right\},$$

where g also denotes the corresponding metric on T^*Q. Furthermore, $(\mathbb{S}(\mathrm{T}^*Q), \eta = i^*\theta_{\mathrm{T}^*Q})$ is a one-contact (co-orientable contact) manifold, where $i\colon \mathbb{S}(\mathrm{T}^*Q) \hookrightarrow \mathrm{T}^*Q$ is the inclusion and θ_{T^*Q} is the Liouville form on T^*Q, for more details see [16, 66].

Consider now the case $Q = G$, where G is a finite-dimensional Lie group. A Riemannian metric on T^*G can be defined using the Killing form κ on $\mathfrak{g}$, which can be extended to a Riemannian metric on G via left multiplication. Once having the Riemannian metric on G, this metric induces a Riemannian metric on T^*G through the canonical isomorphism between $\mathrm{T}G$ and T^*G. Recall that the left multiplication $L\colon (g, h) \in G \times G \mapsto L_g(h) = gh \in G$ gives rise to the trivialisation of T^*G in the following manner $\lambda\colon \alpha_g \in \mathrm{T}^*G \mapsto (g, \mathrm{T}_e^*L_g(\alpha_g)) \in G \times \mathfrak{g}^*$. Therefore, the lift of the Lie group action L to $G \times \mathfrak{g}^*$ is given by

$$\Psi\colon (h; g, \vartheta) \in G \times (G \times \mathfrak{g}^*) \mapsto (hg, \vartheta) \in G \times \mathfrak{g}^*\,.$$

Then, $\phi\colon (s, g, \vartheta) \in \mathbb{R}^+ \times G \times (\mathfrak{g}^* - 0_{\mathfrak{g}^*}) \mapsto (g, s\vartheta) \in G \times (\mathfrak{g}^* - 0_{\mathfrak{g}^*})$ and $(\mathbb{S}(\mathrm{T}^*G) \simeq G \times \mathbb{S}\mathfrak{g}^*, \eta = i^*\theta_{\mathrm{T}^*G})$ is a co-orientable contact manifold. Since Ψ_g is fibrewise linear, it follows that $\phi_s \circ \Psi_g = \Psi_g \circ \phi_s$ for any $s \in \mathbb{R}^+$ and $g \in G$. Consequently, Ψ induces the Lie group action $\Phi\colon G \times (G \times \mathbb{S}\mathfrak{g}^*) \to G \times \mathbb{S}\mathfrak{g}^*$. Additionally, $\Phi_g^*\eta = \eta$ since $\Psi_g^*\theta_{\mathrm{T}^*G} = \theta_{\mathrm{T}^*G}$ for every $g \in G$.

Then, the contact momentum map

$$J^\Phi_\eta\colon (g, [\vartheta]) \in G \times \mathbb{S}\mathfrak{g}^* \mapsto \mathrm{Ad}^*_{g^{-1}}[\vartheta] \in \mathfrak{g}^*\,,$$

induces the map

$$\widetilde{J}^\Phi_\eta\colon (g, [\vartheta]) \in G \times \mathbb{S}\mathfrak{g}^* \mapsto [\mathrm{Ad}^*_{g^{-1}}\vartheta] \in \mathbb{S}\mathfrak{g}^*\,.$$

Then, for some $\mu \in (\mathfrak{g}^* - 0_{\mathfrak{g}^*})$, one has $\widetilde{J}^{\Phi-1}_\eta([\mu]) = J^{\Phi-1}_\eta(\mathbb{R}^+\mu)$[1], where

$$\widetilde{J}^{\Phi-1}_\eta([\mu]) = \{(g, [\vartheta]) \in G \times \mathbb{S}\mathfrak{g}^* \mid [\mathrm{Ad}^*_{g^{-1}}\vartheta] = [\mu]\}\,.$$

By Example 2.5.30, conditions (2.5.11) and (2.5.12) hold automatically when $k = 1$. Consequently, by Theorem 2.5.29, the pair $(\widetilde{J}^{\Phi-1}_\eta([\mu])/K_{[\mu]}, \eta_{[\mu]})$ becomes a contact manifold, where $\eta_{[\mu]}$ satisfies

$$i^*_{[\mu]}\eta = \pi^*_{[\mu]}\eta_{[\mu]}\,,$$

with $i_{[\mu]}\colon \widetilde{J}^{\Phi-1}_\eta([\mu]) \hookrightarrow G \times \mathbb{S}\mathfrak{g}^*$ being the natural immersion and $\pi_{[\mu]}\colon \widetilde{J}^{\Phi-1}_\eta([\mu]) \to \widetilde{J}^{\Phi-1}_\eta([\mu])/K_{[\mu]}$ being the canonical projection.

It is worth noting that this construction recovers the contact Marsden–Meyer–Weinstein reduction for spherical cotangent bundles. Previous studies [55, 56] employed Willett's reduction, which requires the technical assumption $\ker\mu + \mathfrak{g}_\mu = \mathfrak{g}$ [150]. In contrast, Theorem 2.5.29, for $k = 1$, provides a more general framework for the Marsden–Meyer–Weinstein reduction of co-orientable contact manifolds, including spherical cotangent bundles.

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[1] Note that it is considered $\mathbb{R}^\times$ instead of $\mathbb{R}^+$. However, in the contact co-oriented case, this distinction is irrelevant, for details see [70]

2.6 Comparison with previous contact reductions

This section analyses the relations between several previous contact reduction theories [4, 70, 150] and the one-contact reduction introduced in the previous section. The contact Marsden–Meyer–Weinstein reduction has been extensively studied for many years [4, 70, 150].

First, the correspondence between line symplectic principal bundles and contact manifolds is recalled [70]. Then, after commenting on Albert's contact reduction [4], Willett's approach is analysed (see [150] for a comparison between Willett's and Albert's reductions). This section focuses on the contact reductions developed by Willett [150] on the one hand, and K. Grabowska and J. Grabowski [70] on the other. In particular, the reduction subgroup in the main contact reduction theorem in [70] is revisited and corrected.

Some fundamental notions from contact geometry are recalled, as they are required to understand the relationship between contact and symplectic manifolds, as well as the reduction procedure itself.

A *contact manifold* is a pair $(M, \mathcal{C})$, where M is a $(2n+1)$-dimensional manifold and $\mathcal{C}$ is the *contact distribution*, i.e. a *maximally non-integrable distribution* with corank one on M. In other words, a contact distribution is a distribution $\mathcal{C}$ on M defined around any point $x \in M$ by $\mathcal{C}|_U = \ker \eta$, for some $\eta \in \Omega^1(U)$ and an open neighbourhood $U \ni x$ so that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form on U. Then, η is called a *(local) contact form*. A contact manifold $(M, \mathcal{C})$ is *co-oriented* if it admits an associated contact form $\eta \in \Omega^1(M)$ defined globally on M. A co-oriented contact manifold is denoted as (M, η). According to Definition 1.4.22, a k-contact manifold retrieves a co-oriented contact manifold for $k = 1$. A diffeomorphism on M that preserves $\mathcal{C}$ is called a *contactomorphism*. A *symplectic $\mathbb{R}^\times$-principal bundle* is a triple (P, ϕ, ω), where P is an $\mathbb{R}^\times$-principle bundle $\tau\colon P \to M$ relative to the Lie group action $\phi\colon \mathbb{R}^\times \times P \to P$ and $\omega \in \Omega^2(P)$ is a one-homogeneous symplectic form.

Within this section, pairs (P, ω) and (M, η) denote a symplectic and contact manifold, respectively. The following theorem (see [69, Theorem 3.8]) shows the relation between contact distributions $\mathcal{C}$ and symplectic $\mathbb{R}^\times$-principal bundles.

Theorem 2.6.1. *There is a one-to-one correspondence between contact distributions $\mathcal{C}$ on M and symplectic $\mathbb{R}^\times$-principal bundles over M. In this correspondence, the symplectic $\mathbb{R}^\times$-principal bundle associated with $\mathcal{C}$ is $(\mathcal{C}^\circ)^\times \subset \mathrm{T}^*M$, where $\mathcal{C}^\circ$ denotes the annihilator of $\mathcal{C}$.*

2.6.1 Previous contact reductions

The first contact Marsden–Meyer–Weinstein reduction, restricted to only co-oriented contact manifolds, was introduced by C. Albert in [4]. In his construction, the reduced manifold depends on the choice of a contact form within its conformal class. Indeed, the contact distribution $\mathcal{C} = \ker \eta$ remains unchanged when η is multiplied by a non-vanishing function. The problem of the dependence on η was solved by C. Willett in [150]. However, Willett's reduction requires the assumption $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$, where $\mathfrak{g}_\mu$ is the Lie algebra of $G_\mu = \{g \in G \mid \mathrm{Ad}^*_{g^{-1}}\mu = \mu\}$. The condition is not always satisfied, and there exist many cases when it fails (see [150] and the example in forthcoming Subsection 2.6.2).

More recently, a contact Marsden–Meyer–Weinstein reduction for general contact manifolds was devised in [70]. The new approach relies on the one-to-one correspondence between contact manifolds and one-homogeneous symplectic line bundles, while the contact quotient remains essentially the same as in [150]. Indeed, both contact reductions [70, 150] rely on the same Lie subgroup $K_\mu \subset G$ with Lie algebra $\mathfrak{k}_\mu := \ker \mu \cap \mathfrak{g}_\mu$ and the reduced contact manifold is of the form $J_\eta^{\Phi-1}(\mathbb{R}^\times \mu)/K_\mu$. However, the approach in [70] is claimed to work when $\ker \mu + \mathfrak{g}_\mu \neq \mathfrak{g}$. As explained in Subsection 2.6.2, the contact reduction theorem presented in [70] requires the modification of the reduction group. This necessity becomes evident in the case $\ker \mu + \mathfrak{g}_\mu \neq \mathfrak{g}$ and arises from a common mistake in Marsden–Meyer–Weinstein reduction theories, namely, the incorrect determination of the orthogonal complement.

Willett's results originally concerned rather reduced *contact orbifolds*, where an *orbifold* is a generalisation of manifolds obtained as a quotient by discrete groups [3, 145]. In the present setting, attention is restricted to manifolds by assuming that $J^{\Phi-1}_\eta(\mathbb{R}^\times\mu)$ is a submanifold and the Lie group action $\Phi\colon K_\mu\times M\to M$ acts in a quotientable manner on $J^{\Phi-1}_\eta(\mathbb{R}^\times\mu)$ ensuring that $J^{\Phi-1}_\eta(\mathbb{R}^\times\mu)/K_\mu$ is a manifold. In particular, Willett guarantees that $J^{\Phi-1}_\eta(\mathbb{R}^\times\mu)$ is a submanifold of M by assuming that J^Φ_η is transverse to $\mathbb{R}^\times\mu$ [150]. However, it is sufficient to assume that $\mu\in\mathfrak{g}^*$ is a weak regular value of J^Φ_η. The proof of the following theorem can be found in [150, Theorem 1].

Theorem 2.6.2. *Let (M,η,J^Φ_η) be a co-oriented contact Hamiltonian system, let $\mu\in\mathfrak{g}^*$ be a weak regular value of J^Φ_η, and assume that $\Phi\colon G\times M\to M$ acts in a quotientable manner on $J^{\Phi-1}_\eta(\mathbb{R}^\times\mu)$ with* $\ker\mu+\mathfrak{g}_\mu=\mathfrak{g}$*. Then, $(J^{\Phi-1}_\eta(\mathbb{R}^\times\mu)/K_\mu,\eta_\mu)$ such that $\pi^*_\mu\eta_\mu=i^*_\mu\eta$, is a co-oriented contact manifold, where $i_\mu\colon J^{\Phi-1}_\eta(\mathbb{R}^\times\mu)\hookrightarrow M$ is the natural immersion and $\pi_\mu\colon J^{\Phi-1}_\eta(\mathbb{R}^\times\mu)\to J^{\Phi-1}_\eta(\mathbb{R}^\times\mu)/K_\mu$ is the canonical projection by the Lie group K_μ with Lie algebra $\mathfrak{k}_\mu$.*

Meanwhile, the contact Marsden–Meyer–Weinstein reduction introduced in [70] allows one to reduce every contact manifold. Theorem 2.6.3 presents the Marsden–Meyer–Weinstein contact reduction from [70, Theorem 1.1]. The definition of transversality of a submanifold N to a contact distribution $\mathcal{C}$ used in [70] amounts to $\mathrm{T}_xN\not\subset\mathcal{C}_x$ for every $x\in N$.

Theorem 2.6.3. *Let $(M,\mathcal{C})$ be a contact manifold with a symplectic cover $\tau\colon P\to M$, let $\Phi\colon G\times M\to M$ be a contact Lie group, and let $J^{\widetilde{\Phi}}\colon P\to\mathfrak{g}^*$ be an exact symplectic momentum map associated with the lifted Lie group action $\widetilde{\Phi}\colon G\times P\to P$. Let $\mu\in\mathfrak{g}^*$ be a weak regular value of $J^{\widetilde{\Phi}}$ so that the simply connected Lie subgroup K_μ of G, corresponding to the Lie subalgebra*

$$\mathfrak{k}_\mu=\{\xi\in\ker\mu\mid\mathrm{ad}^*_\xi\mu=0\}$$

of $\mathfrak{g}$, acts in a quotientable manner on the submanifold $\tau(J^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu))$ of M. Additionally, suppose that $\mathrm{T}\left(\tau(J^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu))\right)$ *is transversal to $\mathcal{C}$. Then, one has a canonical submersion*

$$\pi\colon\tau(J^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu))\longrightarrow\tau(J^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu))/K_\mu\,,$$

where $\left(\tau(J^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu))/K_\mu,\mathcal{C}_\mu\right)$ is canonically a contact manifold equipped with the contact distribution $\mathcal{C}_\mu:=\mathrm{T}\pi\left(\mathcal{C}\cap\mathrm{T}\left(\tau(J^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu))/K_\mu\right)\right)$.

2.6.2 Correcting previous literature

Theorem 1.1 in [70] does not require the condition $\ker\mu+\mathfrak{g}_\mu=\mathfrak{g}$. However, as shown in [150, Example 3.7], if $\ker\mu+\mathfrak{g}_\mu\neq\mathfrak{g}$, the reduced manifold $J^{\Phi-1}_\eta(\mathbb{R}^\times\mu)/K_\mu$ may fail to be a contact manifold, which indicates a potential mistake in [70, Theorem 1.1]. The problem in [70, Theorem 1.1] is that the Lie group performing the reduction is not properly calculated, and should coincide with the expression (2) in Lemma 2.5.6. More precisely, in the proof of [70, Theorem 1.1], the expression for the kernel of the reduced homogeneous symplectic form ω on the submanifold $J^{\Phi-1}_\theta(\mathbb{R}^\times\mu)$ contains a mistake in the calculation of the symplectic orthogonal. The issue is clarified in detail through a relevant example adapted from Willett's construction in [150].

Consider the contact manifold $(M:=\mathrm{T}^*\mathrm{SL}_2\times\mathbb{R},\eta=\mathrm{d}t-\theta)$, where t is the canonical variable on $\mathbb{R}$ and θ is the pull-back of the Liouville form on $\mathrm{T}^*\mathrm{SL}_2$ to M.

The canonical identification is of the form

$$\vartheta_g\in\mathrm{T}^*\mathrm{SL}_2\mapsto(g,L^*_g\vartheta_g)\in\mathrm{SL}_2\times\mathfrak{sl}^*_2\,,$$

where L_g is the left multiplication in SL_2 by g. The Lie group action $R\colon(g,h)\in\mathrm{SL}_2\times\mathrm{SL}_2\mapsto hg^{-1}\in\mathrm{SL}_2$ lifts naturally to a Lie group action of SL_2 on $\mathrm{T}^*\mathrm{SL}_2\times\mathbb{R}\simeq\mathrm{SL}_2\times\mathfrak{sl}^*_2\times\mathbb{R}$ given by

$$\Phi\colon\mathrm{SL}_2\times(\mathrm{SL}_2\times\mathfrak{sl}^*_2\times\mathbb{R})\ni(g;h,\vartheta,t)\longmapsto(hg^{-1},\mathrm{Ad}^*_{g^{-1}}\vartheta,t)\in\mathrm{SL}_2\times\mathfrak{sl}^*_2\times\mathbb{R}\,.$$

Since R is a free Lie group action, the fundamental vector fields associated with Φ span a distribution of rank three on $\mathrm{SL}_2 \times \mathfrak{sl}_2^* \times \mathbb{R}$. Moreover, since Φ is a lift of R, it leaves invariant both the tautological one-form and the canonical symplectic forms on $\mathrm{T}^*\mathrm{SL}_2$, as well as their lifts to M. Consequently, Φ is a contact Lie group action of (M, η). The canonical isomorphisms of $\mathfrak{sl}_2^* \simeq \mathrm{T}_\mu \mathfrak{sl}_2^*$ for every $\mu \in \mathfrak{sl}_2^*$, together with the trivialisations $\mathrm{T}^*\mathrm{SL}_2 \simeq \mathrm{SL}_2 \times \mathfrak{sl}_2^*$ and $\mathrm{TSL}_2 \simeq \mathrm{SL}_2 \times \mathfrak{sl}_2$ via left group multiplications give the following decompositions

$$\mathrm{T}_{(g,\mu)}\mathrm{T}^*\mathrm{SL}_2 \simeq \mathrm{T}_g\mathrm{SL}_2 \oplus \mathrm{T}_\mu\mathfrak{sl}_2^* \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2^*, \qquad \mathrm{T}^*_{(g,\mu)}\mathrm{T}^*\mathrm{SL}_2 \simeq \mathrm{T}^*_g\mathrm{SL}_2 \oplus \mathrm{T}^*_\mu\mathfrak{sl}_2^* \simeq \mathfrak{sl}_2^* \oplus \mathfrak{sl}_2 \,.$$

Then, the tautological one-form is of the form $\theta_{(g,\mu)}(v, \vartheta) = \langle \mu, v\rangle$, for $(g, \mu) \in \mathrm{SL}_2 \times \mathfrak{sl}_2^*$ and every $(v, \vartheta) \in \mathrm{T}_{(g,\mu)}\mathrm{T}^*\mathrm{SL}_2$. Alternatively, $\theta = \sum_{i=1}^3 \lambda_i \tilde{\eta}_L^i$, where $\lambda_1, \lambda_2, \lambda_3$ are the coordinates of $\mathfrak{sl}_2^*$ lifted to $\mathrm{T}^*\mathrm{SL}_2$ according to the diffeomorphism $\mathrm{T}^*\mathrm{SL}_2 \simeq \mathrm{SL}_2 \times \mathfrak{sl}_2^*$, while $\tilde{\eta}_L^i$ are the pull-back to $\mathrm{T}^*\mathrm{SL}_2$ of the left-invariant one-forms η_L^i on SL_2 whose values at $\mathrm{Id} \in \mathrm{SL}_2$ coincide with the corresponding coordinates of $\mathfrak{sl}_2^*$. This yields an Ad^*-equivariant contact momentum map associated with Φ given by

$$J^\Phi_\eta \colon M \simeq \mathrm{SL}_2 \times \mathfrak{sl}_2^* \times \mathbb{R} \ni (g, \vartheta, t) \longmapsto \vartheta \in \mathfrak{sl}_2^* \,.$$

Consider the standard basis of $\mathfrak{sl}_2$ of the form

$$\mathfrak{sl}_2 = \left\langle \xi_1 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right), \quad \xi_2 = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right), \quad \xi_3 = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \right\rangle ,$$

with commutation relations

$$[\xi_1, \xi_2] = 2\xi_2 \,, \qquad [\xi_1, \xi_3] = -2\xi_3 \,, \qquad [\xi_2, \xi_3] = \xi_1 \,.$$

Let $\mathfrak{sl}_2^* = \langle \mu^1, \mu^2, \mu^3 \rangle$ be the dual basis to $\{\xi_1, \xi_2, \xi_3\}$. Then,

$$J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3) = \{(g, \vartheta, t) \in M \mid J^\Phi_\eta(g, \vartheta, t) = \lambda\, \mu^3, \lambda \in \mathbb{R}^\times\} \simeq \mathrm{SL}_2 \times \mathbb{R}^\times \mu^3 \times \mathbb{R}$$

is a five-dimensional submanifold of M. Since the vector field ∂_t is tangent to $J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3)$ but it does not takes values in $\mathcal{C}$ at any point, it follows that $J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3)$ is transverse to $\mathcal{C}_{(g,\mu,t)}$ for any $(g, \mu, t) \in J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3)$. Furthermore, one has

$$\begin{aligned}
&\mathrm{ad}^*_{\xi_1}\mu^1 = 0 \,, && \mathrm{ad}^*_{\xi_1}\mu^2 = 2\mu^2 \,, && \mathrm{ad}^*_{\xi_1}\mu^3 = -2\mu^3 \,, \\
&\mathrm{ad}^*_{\xi_2}\mu^1 = \mu^3 \,, && \mathrm{ad}^*_{\xi_2}\mu^2 = -2\mu^1 \,, && \mathrm{ad}^*_{\xi_2}\mu^3 = 0 \,, \\
&\mathrm{ad}^*_{\xi_3}\mu^1 = -\mu^2 \,, && \mathrm{ad}^*_{\xi_3}\mu^2 = 0 \,, && \mathrm{ad}^*_{\xi_3}\mu^3 = 2\mu^1 \,.
\end{aligned}$$

Then, the above relations yield

$$\ker \mu^3 = \langle \xi_1, \xi_2 \rangle \,, \qquad \mathfrak{g}_{\mu^3} = \langle \xi_2 \rangle \,, \qquad \mathfrak{k}_{\mu^3} = \mathfrak{g}_{\mu^3} \cap \ker \mu^3 = \langle \xi_2 \rangle \,.$$

The restriction of Φ to the action of K_{μ^3} on $J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3)$ is free. To verify that Φ is proper, consider the Bourbaki definition of properness using the shear map. Let $A \subset J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3) \times J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3)$ be a compact subset. Since $J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3)$ is metrizable (every smooth manifold is), A is sequentially compact.

The idea is to prove that $\widehat{\Phi}^{-1}(A)$ is sequentially compact relative to the induced shear map $\widehat{\Phi} \colon K_{\mu^3} \times J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3) \to J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3) \times J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3)$.

Take a sequence (k_i, x_i) in $\widehat{\Phi}^{-1}(A)$. Then, the sequence $(x_i, \Phi(k_i, x_i))$, which lies in A, admits a subsequence of points $(x_\alpha, \Phi(k_\alpha, x_\alpha))$ that is convergent in A. Therefore, (x_α) and $(\Phi(k_\alpha, x_\alpha))$ converge to some $x, y \in J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3)$, respectively. Moreover, $(x, y) \in A \subset J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3) \times J^{\Phi-1}_\eta(\mathbb{R}^\times \mu^3)$. Using the diffeomorphism $M = \mathrm{T}^*\mathrm{SL}_2 \times \mathbb{R} \simeq \mathrm{SL}_2 \times \mathfrak{sl}_2^* \times \mathbb{R}$, it follows that $(x_\alpha) = (g_\alpha, \vartheta_\alpha, t_\alpha)$, $y = (g_y, \vartheta_y, t_y)$ and $x = (g_x, \vartheta_x, t_x)$ in a unique manner. Then, $(g_\alpha k_\alpha^{-1})$ and $(g_x k_\alpha^{-1})$ tend to g_y. Since K_{μ^3} is of the form

$$K_{\mu^3} = \left\{ k_\alpha = \begin{pmatrix} 1 & \lambda_\alpha \\ 0 & 1 \end{pmatrix} : \lambda_\alpha \in \mathbb{R} \right\} ,$$

and acts freely, it yields that (k_α) must tend to an element $\delta = g_x^{-1}g_y$ in K_{μ^3}. Thus, $(x, \Phi(\delta, x)) \in A$ and (k_α, x_α) converges to a point in $\widehat{\Phi}^{-1}(A)$, which makes the restriction of Φ to $K_{\mu^3} \times J_\eta^{\Phi-1}(\mathbb{R}^\times\mu^3)$ to be proper.

To be in a setting of Theorem 2.6.3, consider the trivial symplectic $\mathbb{R}^\times$-principal bundle $\tau\colon \mathbb{R}^\times \times M \to M$ with $P = \mathbb{R}^\times \times M$. Then, $\omega \in \Omega^2(P)$ is of the form $\omega = \mathrm{d}(s\tau^*\eta) = \mathrm{d}\theta$, with $s \in \mathbb{R}^\times$. By construction, ω is a one-homogeneous symplectic form relative to

$$\phi\colon \mathbb{R}^\times \times P \ni (\lambda; s, x) \longmapsto (\lambda s, x) \in P\,.$$

Since the Lie group action $\Phi\colon SL_2 \times M \to M$ leaves the contact form η invariant, its lifted Lie group action $\widetilde{\Phi}\colon (g; s, x) \in \mathrm{SL}_2 \times P \mapsto (s, \Phi(g, x)) \in P$, leaves the s coordinate invariant, is free, proper, and exact symplectic relative to $\theta = s\tau^*\eta$. Moreover, $\widetilde{\Phi}$ gives rise to an Ad^*-equivariant exact symplectic momentum map $J_\theta^{\widetilde{\Phi}}\colon P \to \mathfrak{sl}_2^*$, defined as in Definition 2.5.2 for $k = 1$, of the form

$$J_\theta^{\widetilde{\Phi}}\colon P \ni (s, g, \vartheta, t) \longmapsto -s\vartheta \in \mathfrak{sl}_2^*\,.$$

Then,

$$J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu^3) = \{(s, g, \vartheta, t) \in P \mid J_\theta^{\widetilde{\Phi}}(s, g, \vartheta, t) = \kappa\mu^3, \kappa \in \mathbb{R}^\times\} \simeq \mathbb{R}^\times \times \mathrm{SL}_2 \times \mathbb{R}^\times\mu^3 \times \mathbb{R}\,,$$

is a six-dimensional submanifold of P. Therefore, all the assumptions of Theorem 2.6.3 are satisfied. However, $J_\eta^{\Phi-1}(\mathbb{R}^\times\mu^3)/K_{\mu^3}$ can not be a contact manifold since it is four-dimensional. Likewise, $J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu^3)/K_{\mu^3}$ can not be a symplectic manifold since it is a five-dimensional manifold. Consequently, Theorem 2.6.3 fails.

One of the problems of the Marsden–Meyer–Weinstein contact reduction in [70] is the expression $\chi(\omega_{[\mu]}) = \widehat{\mathfrak{g}}_\mu^0 + \chi(\omega)$ [70, p 2831]. In the present notation, this expression boils down to

$$\ker \jmath_{[\mu]}^*\omega = \mathrm{T}_p(K_\mu p)\,, \qquad \forall p \in J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu)\,, \tag{2.6.1}$$

where $\jmath_{[\mu]}\colon J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu) \hookrightarrow P$ is the natural embedding, $J_\theta^{\widetilde{\Phi}}$ is the exact symplectic momentum map associated with the Lie group action $\widetilde{\Phi}\colon G \times P \to P$ induced by the initial contact Lie group action $\Phi\colon G \times M \to M$ on the contact manifold $(M, \mathcal{C})$, and K_μ is the Lie subgroup of G with Lie algebra $\mathfrak{g}_\mu^0 = \ker\mu \cap \mathfrak{g}_\mu$, where $\mathfrak{g}_\mu$ is the Lie algebra of the isotropy subgroup of the coadjoint action of G on $\mathfrak{g}^*$ at $\mu \in \mathfrak{g}^*$. Note also that $\widehat{\mathfrak{g}}_\mu^0$ represents the fundamental vector fields on P related to the Lie algebra $\mathfrak{g}_\mu^0$, which is denoted by $\mathfrak{k}_\mu$ in the introduced notation. It is worth recalling now that since the $\mathbb{R}^\times$-bundle action commutes with $J_\theta^{\widetilde{\Phi}}$, one has that $K_{\mu'}$ is the same for every $\mu' \in \mathbb{R}^\times\mu$.

To justify why (2.6.1) does not hold in general, observe that

$$\ker \jmath_{[\mu]}^*\omega = \mathrm{T}J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu) \cap \left(\mathrm{T}J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu)\right)^{\perp_\omega}\,.$$

Assume for simplicity that $\mu' = J_\theta^{\widetilde{\Phi}}(p) \neq 0$. Then,

$$\mathrm{T}_pJ_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu) = \mathrm{T}_pJ_\theta^{\widetilde{\Phi}-1}(\mu') \oplus \langle\nabla_p\rangle\,,$$

where ∇ denotes the Euler vector field of the $\mathbb{R}^\times$-principal bundle $\tau\colon \mathbb{R}^\times \times M \to M$. Since $\iota_\nabla\omega = \theta$, it follows that

$$\mathrm{T}_pJ_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu) \cap \left(\mathrm{T}_pJ_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu)\right)^{\perp_\omega} = (\mathrm{T}_pJ_\theta^{\widetilde{\Phi}-1}(\mu') \oplus \langle\nabla_p\rangle) \cap \left(\mathrm{T}_pJ_\theta^{\widetilde{\Phi}-1}(\mu')\right)^{\perp_\omega} \cap \ker\theta_p\,.$$

From standard Marsden–Meyer–Weinstein symplectic reduction theory presented in Section 2.1, one has

$$\mathrm{T}_pJ_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu) \cap \left(\mathrm{T}_pJ_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times\mu)\right)^{\perp_\omega} = \left(\mathrm{T}_pJ_\theta^{\widetilde{\Phi}-1}(\mu') \oplus \langle\nabla_p\rangle\right) \cap \mathrm{T}_p(Gp) \cap \ker\theta_p\,. \tag{2.6.2}$$

Assuming that $\mathrm{T}(\tau(J_\theta^{\widetilde{\Phi}-1}(\mu')))$ is transversal to the contact distribution $\mathcal{C}$, as done in [70], essentially ensures that $\nabla_p \notin \ker \jmath^*_{[\mu]}\omega_p$. Indeed, if $\nabla_p \in \ker \jmath^*_{[\mu]}\omega_p$, it follows that $\iota_\nabla \omega = \theta$ vanishes on the tangent bundle to $J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu)$, and hence $\mathrm{T}\tau(J_\theta^{\widetilde{\Phi}-1}(\mu')) \subset \mathcal{C}$, contradicting transversality.

This observation, however, does not justify the identity $\chi(\omega_\mu) = \widehat{\mathfrak{g}}^0_\mu$ as in [70], where $\chi(\omega_\mu)$ represents the kernel of the restriction of ω to $J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu)$. In other words, according to [70], it follows that

$$\mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu) \cap \left(\mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu)\right)^{\perp_\omega} = \mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mu') \cap \mathrm{T}_p(Gp) \cap \ker \theta_p \ = \mathrm{T}_p(K_{\mu'}p)\,,$$

which is not correct in general. The intersection with a direct sum is not the direct sum of the intersections in general, and the transversality condition on $\tau(J_\theta^{\widetilde{\Phi}-1}(\mu'))$ assumed in [70] does not change that fact. Moreover, the Lemma 2.5.6 shows that there may be fundamental vector fields of $\widetilde{\Phi}$ tangent to $\mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu)$ that are symplectically orthogonal to it, but are not tangent to any $\mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mu')$, e.g. the fundamental vector fields related to $\ker \mu \cap \mathfrak{g}_{[\mu]}$ not belonging to $\ker \mu \cap \mathfrak{g}_\mu$.

To clarify the above arguments and illustrate the mistake, consider the following counterexample.

Note that both ∂_s and ∂_t are tangent to $J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3)$. Taking into account the form of ω, it turns out that

$$\mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3) \cap \left(\mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3)\right)^{\perp_\omega} \subset \ker \tau^*\eta \cap \ker \mathrm{d}s\,.$$

The fundamental vector fields ξ_{1P} and ξ_{2P} are tangent to $J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3)$, satisfy $\xi_{1P} \wedge \xi_{2P} \neq 0$, and $\omega(\xi_{1P}, \xi_{2P}) = 0$. Hence, in view of (2.6.2), it follows that

$$\mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3) \cap \left(\mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3)\right)^{\perp_\omega} \supset \langle \xi_{1P}(p), \xi_{2P}(p)\rangle \neq \mathrm{T}_p(K_{\mu^3}p) = \langle \xi_{2P}(p)\rangle\,,$$

for any $p \in J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3)$. Therefore, the claim (2.6.1) is incorrect, and Theorem 2.6.3 from [70] fails in this case. In contrast, Lemma 2.5.22 implies that

$$\mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3) \cap \left(\mathrm{T}_p J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3)\right)^{\perp_\omega} = \langle \xi_{1P}(p), \xi_{2P}(p)\rangle = \mathrm{T}_p(K_{[\mu^3]}p)\,,$$

for any $p \in J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3)$, where $K_{[\mu^3]}$ as defined in Proposition 2.5.4. Since [70] assumes that the symplectic orthogonal to $\mathrm{T}J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu^3)$ must be included in the isotropy group G_{μ^3}, the authors did not notice that ξ_{1P} is missing in (2.6.1).

A corrected version of Theorem 2.6.3 could be stated as follows, and the proof is analogous to the original provided in [70] once the required symplectic orthogonal is corrected.

Theorem 2.6.4. *Let* $(M, \mathcal{C})$ *be a contact manifold with a symplectic cover* (P, θ) *and* $\tau\colon P \to M$*, let* $\Phi\colon G \times M \to M$ *be a contact Lie group action, and let* $J_\theta^{\widetilde{\Phi}}\colon P \to \mathfrak{g}^*$ *be an exact symplectic momentum map associated with the lifted Lie group action* $\widetilde{\Phi}\colon G \times P \to P$*. Let* $\mu \in \mathfrak{g}^*$ *be a weak regular value of* $J_\theta^{\widetilde{\Phi}}$ *so that the connected Lie subgroup* $K_{[\mu]}$ *of* G*, corresponding to the Lie subalgebra of* $\mathfrak{g}$ *given by*

$$\mathfrak{k}_{[\mu]} = \{\xi \in \ker \mu \mid \mathrm{ad}^*_\xi \mu \wedge \mu = 0\},$$

acts in a quotientable manner on the submanifold $\tau(J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu))$ *of* M*. Additionally, suppose that* $\mathrm{T}[\tau(J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu))]$ *is transversal to* $\mathcal{C}$*. Then, one has a canonical submersion* $\pi\colon \tau(J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu)) \to \tau(J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu))/K_{[\mu]}$*, where* $\left(\tau(J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu))/K_{[\mu]}, \mathcal{C}_{[\mu]}\right)$ *is a contact manifold with*

$$\mathcal{C}_{[\mu]} := \mathrm{T}\pi\left(\mathcal{C} \cap \mathrm{T}\left(\tau(J_\theta^{\widetilde{\Phi}-1}(\mathbb{R}^\times \mu))/K_{[\mu]}\right)\right)\,.$$

The authors of [70] also distinguish the specific case when $\mu = 0$. This case is included as a special case of the present framework. If $\mu = 0$ then $\ker \mu = \mathfrak{g}$ and $K_{[\mu]} = G$, so the reduced manifold is

$\tau(J^{\widetilde{\Phi}-1}(0))/G$. In the co-oriented setting, this was already known to be a reduced co-orientable contact manifold as shown in [98, 104].

There is another reason that illustrates that the reduction Lie group in [70] should be changed. Suppose that $\mu \in \mathfrak{g}^*$ is a regular value of $J^{\widetilde{\Phi}}_\eta$. If $\ker \mu + \mathfrak{g}_\mu \neq \mathfrak{g}$, then $\mathfrak{g}_\mu \subset \ker \mu$ and $\dim G_\mu = \dim K_\mu$. If G_μ acts in a free and proper manner on $J^{\widetilde{\Phi}-1}_\theta(\mu)$, one has that $\dim J^{\widetilde{\Phi}-1}_\theta(\mu)/G_\mu = \dim J^{\widetilde{\Phi}-1}_\theta(\mu) - \dim G_\mu = 2\ell$ for some $\ell \in \mathbb{N}$ due to the symplectic reduction theorem 2.1.8. For $\mu \neq 0$ and assuming the transversality of $J^{\widetilde{\Phi}}_\theta$ relative to $\mathbb{R}^\times \mu$ and $\dim J^{\widetilde{\Phi}-1}_\theta(\mathbb{R}^\times \mu) = \dim J^{\widetilde{\Phi}-1}_\theta(\mu) + 1$, one gets

$$\dim J^{\widetilde{\Phi}-1}_\theta(\mathbb{R}^\times \mu)/K_\mu = \dim J^{\widetilde{\Phi}-1}_\theta(\mu) + 1 - \dim G_\mu = 2\ell + 1\,,$$

This is a contradiction, as $J^{\widetilde{\Phi}-1}_\theta(\mathbb{R}^\times \mu)/K_\mu$ must be even-dimensional for $(\tau(J^{\widetilde{\Phi}-1}_\theta(\mathbb{R}^\times \mu))/K_\mu, \eta_\mu)$ to be a contact manifold.

2.6.3 Comparisons to previous contact reductions

This section demonstrates through simple examples that the one-contact Marsden–Meyer–Weinstein reduction Theorem 2.5.29 method applies, where the contact Marsden–Meyer–Weinstein reduction approaches from [4, 150] do not. The method is also more general than the one in [70], since some of the technical assumptions required there fail in the examples below, as explained in the previous Subsection 2.6.2. Furthermore, the examples clearly illustrate that the Lie group used in the reduction in [70] must be modified.

Example 2.6.5. Consider the contact manifold $(\mathbb{R}^7, \eta)$ with

$$\eta = \mathrm{d}t - x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2 - x_4\mathrm{d}x_3 + x_6\mathrm{d}x_5\,.$$

Define a contact Lie group action of $G = \mathrm{GL}_2 \simeq \mathrm{SL}_2 \times \mathbb{R}$ on $\mathbb{R}^7$ of the form

$$\Phi\colon G \times \mathbb{R}^7 \ni ((g,\lambda); t, x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (t, (x_1, x_2)g^T, x_3, x_4, x_5 + \lambda, x_6) \in \mathbb{R}^7,$$

where g^T is transpose of the matrix $g \in \mathrm{SL}_2$. A direct calculation shows that $\Phi^*_{(g,\lambda)}\eta = \eta$ for every $(g,\lambda) \in \mathrm{SL}_2 \times \mathbb{R}$. A basis of the fundamental vector fields related to Φ reads

$$\vartheta_{1M} = x_2\frac{\partial}{\partial x_1}\,, \qquad \vartheta_{2M} = x_1\frac{\partial}{\partial x_2}\,, \qquad \vartheta_{3M} = x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}\,, \qquad \vartheta_{4M} = \frac{\partial}{\partial x_5}\,.$$

Let $\mathfrak{gl}_2^* = \langle \widetilde{\mu}^1, \widetilde{\mu}^2, \widetilde{\mu}^3, \widetilde{\mu}^4 \rangle$, where $\{\widetilde{\mu}^1, \widetilde{\mu}^2, \widetilde{\mu}^3, \widetilde{\mu}^4\}$ is the dual basis to the basis $\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ of $\mathfrak{gl}_2 \simeq \mathfrak{sl}_2 \oplus \mathbb{R}$ with non-vanishing commutation relations given by

$$[\vartheta_1, \vartheta_2] = \vartheta_3, \qquad [\vartheta_1, \vartheta_3] = -2\vartheta_1, \qquad [\vartheta_2, \vartheta_3] = 2\vartheta_2.$$

Then, the contact momentum map $J^\Phi_\eta\colon \mathbb{R}^7 \to \mathfrak{gl}_2^*$ in the basis $\{\widetilde{\mu}^1, \widetilde{\mu}^2, \widetilde{\mu}^3, \widetilde{\mu}^4\}$ reads

$$J^\Phi_\eta(x) = (\iota_{\vartheta_{1M}}\eta(x), \iota_{\vartheta_{2M}}\eta(x), \iota_{\vartheta_{3M}}\eta(x)) = (-x_2^2, x_1^2, -2x_1x_2, x_6) \in \mathfrak{gl}_2^*\,.$$

Fix $\mu = \widetilde{\mu}^2 \in \mathfrak{gl}_2^*$. Then,

$$J^{\Phi-1}_\eta(\mathbb{R}^\times \widetilde{\mu}^2) = \{x = (t, x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^7 \mid x_2 = x_6 = 0,\, x_1 \neq 0\}$$

and

$$\mathrm{T}_x J^{\Phi-1}_\eta(\mathbb{R}^\times \widetilde{\mu}^2) = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5} \right\rangle\,.$$

Moreover, $\ker \widetilde{\mu}^2 = \langle \vartheta_1, \vartheta_3, \vartheta_4 \rangle$, $\mathfrak{g}_{\widetilde{\mu}^2} = \langle \vartheta_1, \vartheta_4 \rangle$, and $\mathfrak{g}_{[\widetilde{\mu}^2]} = \langle \vartheta_1, \vartheta_3, \vartheta_4 \rangle$ since $\mathrm{ad}^*_{\vartheta^4}\vartheta = 0$ for every $\vartheta \in \mathfrak{gl}_2^*$ and

$$\begin{aligned}
&\mathrm{ad}^*_{\vartheta_1}\widetilde{\mu}^1 = -2\widetilde{\mu}_3\,, && \mathrm{ad}^*_{\vartheta_2}\widetilde{\mu}^1 = 0\,, && \mathrm{ad}^*_{\vartheta_3}\widetilde{\mu}^1 = 2\widetilde{\mu}^1\,,\\
&\mathrm{ad}^*_{\vartheta_1}\widetilde{\mu}^2 = 0\,, && \mathrm{ad}^*_{\vartheta_2}\widetilde{\mu}^2 = 2\widetilde{\mu}^3\,, && \mathrm{ad}^*_{\vartheta_3}\widetilde{\mu}^2 = -2\widetilde{\mu}^2\,,\\
&\mathrm{ad}^*_{\vartheta_1}\widetilde{\mu}^3 = \widetilde{\mu}_2\,, && \mathrm{ad}^*_{\vartheta_2}\widetilde{\mu}^3 = -\widetilde{\mu}^1\,, && \mathrm{ad}^*_{\vartheta_3}\widetilde{\mu}^3 = 0\,.
\end{aligned}$$

Consequently, $\mathfrak{k}_{\widetilde{\mu}^2} = \ker \widetilde{\mu}^2 \cap \mathfrak{g}_{\widetilde{\mu}^2} = \langle \vartheta_1, \vartheta_4 \rangle$ and $\mathfrak{k}_{[\widetilde{\mu}^2]} = \langle \vartheta_1, \vartheta_3, \vartheta_4 \rangle$ yield the following

$$\mathrm{T}_x(K_{\widetilde{\mu}^2} x) = \langle \vartheta_{1M}(x), \vartheta_{4M}(x) \rangle \quad \text{and} \quad \mathrm{T}_x(K_{[\widetilde{\mu}^2]} x) = \langle \vartheta_{1M}(x), \vartheta_{3M}(x), \vartheta_{4M}(x) \rangle \ .$$

The condition required by Willett's contact reduction [150], namely $\ker \widetilde{\mu}^2 + \mathfrak{g}_{\widetilde{\mu}^2} = \mathfrak{gl}_2$ is not satisfied, making his results inapplicable. Attempting the contact reduction introduced in [70], one gets that the assumptions of Theorem 2.6.3 are not satisfied, since ϑ_{1M} restricted to $J_\eta^{\Phi-1}(\mathbb{R}^\times \widetilde{\mu}^2)$ vanishes and the associated action of $K_{\widetilde{\mu}^2}$ is not free on it. Nevertheless, the remaining assumptions are indeed satisfied. In particular, $J_\eta^{\Phi-1}(\mathbb{R}^\times \widetilde{\mu}^2)$ is transversal to the contact distribution and $\widetilde{\mu}^2 \in \mathfrak{gl}_2^*$ is a weak regular value of the exact symplectic momentum map obtained by lifting J_η^Φ to $\mathbb{R}^\times \times \mathbb{R}^7$.

Applying Theorem 2.6.4 shows that the quotient with respect to $K_{[\widetilde{\mu}^2]}$ yields

$$\mathrm{T}_{\pi_{[\widetilde{\mu}^2]}(x)}[J_\eta^{\Phi-1}(\mathbb{R}^\times \widetilde{\mu}^2)/(K_{[\widetilde{\mu}^2]} x)] \simeq \mathrm{T}_x J_\eta^{\Phi-1}(\mathbb{R}^\times \widetilde{\mu}^2)/\mathrm{T}_x(K_{[\widetilde{\mu}^2]} x) = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\rangle,$$

and $(J_\eta^{\Phi-1}(\mathbb{R}^\times \widetilde{\mu}^2)/K_{[\widetilde{\mu}^2]}, \eta_{[\widetilde{\mu}^2]})$ becomes a three-dimensional contact manifold with

$$\eta_{[\widetilde{\mu}^2]} = \mathrm{d}t - x_4 \mathrm{d}x_3 \ .$$

△

This example can be slightly modified to illustrate that the reduction group in [70] is incorrect.

Example 2.6.6. Consider the restriction, Φ', of Φ to the action of the Lie subgroup

$$H_2 \times \mathbb{R} = \left\{ \left(\begin{pmatrix} \lambda_1 & 0 \\ \lambda_2 & 1/\lambda_1 \end{pmatrix}, \lambda \right) \mid \lambda_1 \in \mathbb{R}_+ \,, \ \lambda_2 \in \mathbb{R} \,, \ \lambda \in \mathbb{R} \right\}$$

on $\mathbb{R}^7$. Then, Φ' is Hamiltonian with respect to the contact form given above, and the new contact momentum map $J_\eta^{\Phi'} : M \to (\mathfrak{h}_2 \oplus \mathbb{R})^*$ is

$$J_\eta^{\Phi'}(x) = (\iota_{\vartheta_{2M}} \eta(x), \iota_{\vartheta_{3M}} \eta(x), \iota_{\vartheta_{4M}} \eta(x)) = (x_1^2, -2x_1x_2, x_6) \in (\mathfrak{h}_2 \oplus \mathbb{R})^* \ .$$

Consider $(\mathfrak{h}_2 \oplus \mathbb{R})^* = \langle \widehat{\mu}^2, \widehat{\mu}^3, \widehat{\mu}^4 \rangle$, where $\{\widehat{\mu}^2, \widehat{\mu}^3, \widehat{\mu}^4\}$ is the dual basis to the basis $\{\vartheta_2, \vartheta_3, \vartheta_4\}$ of $\mathfrak{h}_2 \oplus \mathbb{R}$ with a non-vanishing commutation relation

$$[\vartheta_2, \vartheta_3] = 2\vartheta_2 \ .$$

Fix $\mu = \widehat{\mu}^2 \in (\mathfrak{h}_2 \oplus \mathbb{R})^*$. Then,

$$J_\eta^{\Phi'-1}(\mathbb{R}^\times \widehat{\mu}^2) = \{ x = (t, x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^7 \mid x_2 = x_6 = 0, x_1 \neq 0 \}$$

and

$$\mathrm{T}_x J_\eta^{\Phi'-1}(\mathbb{R}^\times \widehat{\mu}^2) = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5} \right\rangle \ .$$

Moreover, $\ker \widehat{\mu}^2 = \langle \vartheta_3, \vartheta_4 \rangle$, $\mathfrak{g}_{\widehat{\mu}^2} = \langle \vartheta_4 \rangle$, and $\mathfrak{g}_{[\widehat{\mu}^2]} = \langle \vartheta_3, \vartheta_4 \rangle$ since

$$\mathrm{ad}^*_{\vartheta_2} \widehat{\mu}^2 = 2\widehat{\mu}^3 \,, \qquad \mathrm{ad}^*_{\vartheta_3} \widehat{\mu}^2 = -2\widehat{\mu}^2 \ .$$

Thus, $\mathfrak{k}_{\widehat{\mu}^2} = \ker \widehat{\mu}^2 \cap \mathfrak{g}_{\widehat{\mu}^2} = \langle \vartheta_4 \rangle$, $\mathfrak{k}_{[\widehat{\mu}^2]} = \langle \vartheta_3, \vartheta_4 \rangle$ yields

$$\mathrm{T}_x(K_{[\widehat{\mu}^2]} x) = \langle \vartheta_{3M}(x), \vartheta_{4M}(x) \rangle \ .$$

The condition $\ker \widehat{\mu}^2 + \mathfrak{g}_{\widehat{\mu}^2} = \mathfrak{h}_2 \oplus \mathbb{R}$ does not hold, making Willett's results inapplicable. Then, the reduction introduced in [70] gives $J_\eta^{\Phi'-1}(\mathbb{R}^\times \widehat{\mu}^2)/K_{\widehat{\mu}^2}$ which is even dimensional. Assumptions in Theorem 2.6.3 are indeed satisfied. In particular, $J_\eta^{\Phi'-1}(\mathbb{R}^\times \widehat{\mu}^2)$ is transversal to the contact distribution and $\widehat{\mu}^2 \in (\mathfrak{h}_2 \oplus \mathbb{R})^*$ is a weak regular value of an exact symplectic momentum map. △

The final example revisits the previous one, but is reformulated using the framework and notation introduced in [70]. To address non-co-oriented contact manifolds, the authors in [70] focus on one-homogeneous symplectic $\mathbb{R}^\times$-principal bundles over a contact manifold M. Consequently, the example below is the symplectic extension of a particular case of the earlier example, illustrating that the theory in [70] requires modifications to remain valid.

Example 2.6.7. Consider the exact symplectic manifold $(P = \mathbb{R}^\times \times \mathbb{R}^7, s\,\tau^*\eta)$, where $\tau\colon P \to \mathbb{R}^7$ is the canonical projection, s is the fibre variable on $\mathbb{R}^\times$, and the contact form is defined as

$$\eta = \mathrm{d}t - x_2\mathrm{d}x_1 + x_1\mathrm{d}x_2 - x_4\mathrm{d}x_3 + x_6\mathrm{d}x_5$$

for the linear coordinates $t, x_1, \ldots, x_6$ on $\mathbb{R}^7$. Then, $\theta = s\tau^*\eta$ and denote by H_2 the Lie group consisting of 2×2 lower triangular unimodular matrices with real entries and a positive diagonal. The positive diagonal ensures that H_2 is connected, thus preventing subsequent technical complications. The Lie group action $\Phi'\colon (H_2 \times \mathbb{R}) \times \mathbb{R}^7 \to \mathbb{R}^7$ is given by

$$\Phi'\left(\left(h = \begin{pmatrix} \lambda_3 & 0 \\ \lambda_2 & 1/\lambda_3 \end{pmatrix}, \lambda_4\right), (t, x_1, \ldots, x_6)\right) = (t, \lambda_3 x_1, \lambda_2 x_1 + x_2/\lambda_3, x_3, x_4, x_5 + \lambda_4, x_6)\,,$$

where $\lambda_3 > 0$ and $\lambda_2, \lambda_4 \in \mathbb{R}$. This action leaves η invariant and admits a lift to a Hamiltonian Lie group action $\widetilde{\Phi}'\colon (H_2 \times \mathbb{R}) \times P \to P$ of the form

$$\widetilde{\Phi}'\colon (H_2 \times \mathbb{R}) \times P \to P, \qquad \widetilde{\Phi}'((h, \lambda_4), (s, t, x_1, \ldots, x_6)) = (s, \Phi'((h, \lambda_4), (t, x_1, \ldots, x_6)))\,,$$

where $s, t, x_1, \ldots, x_6$ naturally form a coordinate system on P. Indeed, the lifted action $\widetilde{\Phi}'$ is Hamiltonian with respect to the symplectic form $\mathrm{d}\theta$ on P and $\tau \circ \widetilde{\Phi}'_{(h,\lambda_4)} = \Phi'_{(h,\lambda_4)} \circ \tau$ for every $(h, \lambda_4) \in H_2 \times \mathbb{R}$. A basis of fundamental vector fields of $\widetilde{\Phi}'$ reads

$$\nu_{2P} = x_1\frac{\partial}{\partial x_2}\,, \qquad \nu_{3P} = x_2\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_1}\,, \qquad \nu_{4P} = \frac{\partial}{\partial x_5}\,.$$

Each vector field ν_{iP} is the unique Hamiltonian vector field on P projecting onto a contact Hamiltonian vector field $\vartheta_{iM} = \tau_*\nu_{iP}$ on $\mathbb{R}^7$ for $i = 2, 3, 4$. Moreover, each ν_{iP} admits one-homogeneous Hamiltonian function $-\iota_{\nu_{iP}}\theta$ with $i = 2, 3, 4$.

The exact symplectic momentum map $J_\theta^{\widetilde{\Phi}'}\colon P \to (\mathfrak{h}_2 \oplus \mathbb{R})^*$ associated with $\widetilde{\Phi}'$, where $\mathfrak{h}_2$ is the Lie algebra of H_2, is given by

$$J_\theta^{\widetilde{\Phi}'}(p) = (\iota_{\nu_{2P}}\theta(p), \iota_{\nu_{3P}}\theta(p), \iota_{\nu_{4P}}\theta(p)) = (sx_1^2, 2sx_1x_2, sx_6) \in (\mathfrak{h}_2 \oplus \mathbb{R})^*\,, \qquad \forall p \in P\,,$$

in a basis $\{e^1, e^2, e^3\}$ of $\mathfrak{h}_2^* \oplus \mathbb{R}^*$ dual to a basis $\{e_1, e_2, e_3\}$ adapted to the decomposition $\mathfrak{h}_2 \oplus \mathbb{R}$ and closing opposite commutation relations to $\nu_{2P}, \nu_{3P}, \nu_{4P}$, respectively[2]. Note that the exact symplectic momentum map $J_\theta^{\widetilde{\Phi}'}$ is Ad^*-equivariant. For fixed element $\mu = e^1 \in \mathfrak{h}_2^* \oplus \mathbb{R}^*$, it follows

$$J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1) = \{p = (s, t, x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^\times \times \mathbb{R}^7 \mid x_2 = x_6 = 0, s \neq 0, x_1 \neq 0\}$$

and

$$\mathrm{T}_p J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1) = \left\langle \nabla = s\frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5} \right\rangle_p\,,$$

for any $p \in J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1)$. Additionally,

$$J_\theta^{\widetilde{\Phi}'-1}(\lambda e^1) = \{p \in P \mid sx_1^2 = \lambda,\ x_2 = 0,\ x_6 = 0\}, \qquad \lambda \in \mathbb{R}^\times$$

and

$$\mathrm{T}_p J_\theta^{\widetilde{\Phi}'-1}(\lambda e^1) = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, 2\nabla - x_1\frac{\partial}{\partial x_1} \right\rangle_p\,, \qquad \lambda \in \mathbb{R}^\times\,,$$

---

[2] Recall that the definition of fundamental vector fields induces a Lie algebra anti-homomorphism $\xi \in \mathfrak{h}_2 \oplus \mathbb{R} \simeq \mathfrak{g} \mapsto \xi_P \in \mathfrak{X}(P)$.

for any $p \in J_\theta^{\widetilde{\Phi}'-1}(\lambda e^1)$. Then, $\nu_{4P}(p) \in \mathrm{T}_p J_\theta^{\widetilde{\Phi}'-1}(\lambda e^1)$ for every $p \in J_\theta^{\widetilde{\Phi}'-1}(\lambda e^1)$ and

$$\mathrm{T}_p J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1) = \langle \nabla_p \rangle \oplus \mathrm{T}_p J_\theta^{\widetilde{\Phi}'-1}(\lambda e^1)\,,$$

for any $p \in J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1)$ and $J_\theta^{\widetilde{\Phi}'}(p) = \lambda e^1$ for some $\lambda \in \mathbb{R}^\times$. Recall that

$$\omega = \mathrm{d}s \wedge (\mathrm{d}t + x_1 \mathrm{d}x_2 - x_4 \mathrm{d}x_3) + s(2\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4 - \mathrm{d}x_5 \wedge \mathrm{d}x_6)$$

and then

$$\left(\mathrm{T}_p J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1)\right)^{\perp_\omega} = \langle \nu_{3P}(p), \nu_{4P}(p) \rangle\,, \qquad \forall p \in J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1)\,.$$

Since $[e_1, e_2] = -2e_1$ and $[e_3, e_i] = 0$, it follows that $\mathrm{ad}^*_{e_3} e^1 = 0$ and $\mathrm{ad}^*_{e_2} e^1 = 2e^1$. Consequently, the connected Lie subgroups of $H_2 \times \mathbb{R}$ whose Lie algebras are $\ker e^1 \cap \mathfrak{g}_{[e^1]}$ and $\ker e^1 \cap \mathfrak{g}_{e^1}$ are

$$K_{[e^1]} = \left\{ \left(\begin{pmatrix} \lambda_3 & 0 \\ 0 & 1/\lambda_3 \end{pmatrix}, \lambda_4 \right) \mid \lambda_3 > 0, \lambda_4 \in \mathbb{R} \right\}, \quad K_{e^1} = \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \lambda_4 \right) \mid \lambda_4 \in \mathbb{R} \right\}.$$

Hence,

$$\left(\mathrm{T}_p J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1)\right)^{\perp_\omega} \cap \mathrm{T}_p J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1) = \langle \nu_{3P}(p), \nu_{4P}(p) \rangle = \mathrm{T}_p(K_{[e^1]}p) \neq \mathrm{T}_p(K_{e^1}p), \tag{2.6.3}$$

for any $p \in J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1)$. Therefore $(J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1)/K_{[e^1]}, \theta_{[e^1]})$ is an exact symplectic manifold that is a symplectic cover of the co-oriented contact manifold $(\tau(J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1)/K_{[e^1]}, \eta_{[e^1]})$, where

$$\theta_{[e^1]} = s(\mathrm{d}t - x_4 \mathrm{d}x_3)\,, \qquad \eta_{[e^1]} = \mathrm{d}t - x_4 \mathrm{d}x_3\,.$$

One can verify that the assumptions of Theorem 2.6.3 are satisfied in this example, namely the tangent space to $\tau(J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1))$ is transversal to the contact distribution, $e^1 \in \mathfrak{h}_2^* \oplus \mathbb{R}^*$ is a weak regular value of an exact symplectic momentum map $J_\theta^{\widetilde{\Phi}'}: P \to \mathfrak{h}_2^* \oplus \mathbb{R}^*$, and the restriction of the Lie group action of K_{e^1} on $J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1)$ is free and proper. However, by (2.6.3) the quotient manifold obtained in [70] is not a symplectic manifold, so the formula in [70] for the symplectic orthogonal to $\mathrm{T}J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times \mu)$ is incorrect. Furthermore, $\tau(J_\theta^{\widetilde{\Phi}'-1}(\mathbb{R}^\times e^1))/K_{e^1}$ is even-dimensional and can not be a contact manifold. Instead of taking the quotient by the Lie subgroup K_μ, the correct approach requires using $K_{[\mu]}$ under previously considered regularity conditions on the Lie group actions and the momentum maps. △

Chapter 3

Energy-momentum methods

The classical energy-momentum method is a technique for analysing Hamiltonian systems on symplectic manifolds, particularly in the neighbourhood near solutions whose evolution is induced by the Lie symmetries of the Hamiltonian system (see [18] for a historical introduction and [113] for one of its foundational works). More specifically, it studies whether solutions approach or diverge from the solutions associated with the Lie symmetries of the Hamiltonian system. The classical energy-momentum method is based on the symplectic Marsden–Meyer–Weinstein reduction theory and stability analysis techniques.

The main ideas behind the energy-momentum method can be traced back to Routh, Poincaré, Lyapunov, Arnold, Lewis, and Smale, among others (see [18, Section 3.14]). Then, the classical energy-momentum method, devised and developed mainly by J.C. Simo and J.E. Marsden [113], was successfully applied to a wide range of problems by numerous researchers [1, 84, 110, 112, 114, 127, 137, 152]. Over the years, the energy-momentum method has been extended to deal with more general differential equations, e.g. discrete systems [112, 138].

Recall that the symplectic Marsden–Meyer–Weinstein reduction theorem uses a symplectic Lie group action $\Phi\colon G\times P \to P$ and a symplectic form $\omega \in \Omega^2(P)$ to define a *symplectic momentum map* $\mathbf{J}^\Phi\colon P \to \mathfrak{g}^*$ [109], see Chapter 2. The momentum map allows for reducing the Hamiltonian system h on P to a Hamiltonian system k_μ on the manifold $P_\mu := \mathbf{J}^{\Phi-1}(\mu)/G_\mu$ of smaller dimension for a weak regular value $\mu \in \mathfrak{g}^*$ of $\mathbf{J}^\Phi$ and an appropriate Lie subgroup $G_\mu \subset G$ acting freely and properly on $\mathbf{J}^{\Phi-1}(\mu)$. Then, there exists a canonical symplectic form, $\omega_\mu \in \Omega^2(P_\mu)$ induced by ω, while k_μ is univocally defined by the relation $k_\mu \circ \pi_\mu := h$ on $\mathbf{J}^{\Phi-1}(\mu)$, where $\pi_\mu\colon \mathbf{J}^{\Phi-1}(\mu) \to P_\mu$ is the canonical projection. The reduced Hamiltonian system admits the *equilibrium points*, i.e. stable points relative to the evolution given by the Hamilton equations for k_μ, that are the projection of not necessarily equilibrium points of the initial Hamiltonian system generated by h. Such points in P are referred to as *relative equilibrium points* [113]. The geometric situation is illustrated schematically in Figure 3.1.

This final chapter presents applications of the previously established Marsden–Meyer–Weinstein reduction theorems to the study of stability through energy-momentum methods. These methods are formulated in symplectic, cosymplectic, and k-polysymplectic frameworks, with emphasis on time-dependent and non-autonomous systems. The analysis focuses on the characterisation and analysis of the so-called relative equilibrium points via Lyapunov stability theory in reduced spaces introduced in Section 1.1.

Several physical systems are studied, including quantum models, the circular restricted three-body problem, and systems of coupled vibrating strings. These examples illustrate the effectiveness of the geometric approach in simplifying stability analysis and determining relative equilibrium points. The results confirm the relevance of the extended Marsden–Meyer–Weinstein reduction theory in addressing problems arising in mathematical physics.

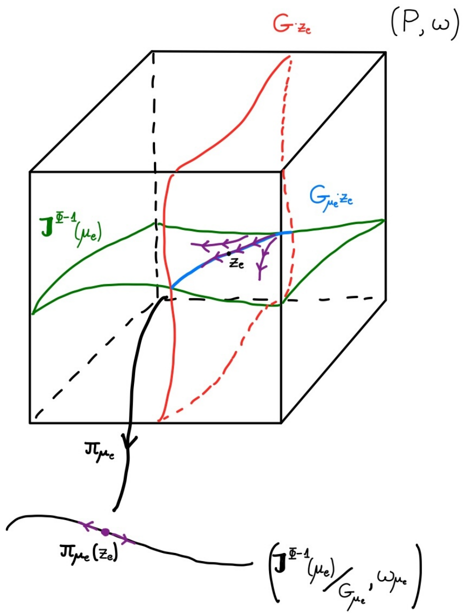

Figure 3.1: The visualisation of the main idea of the energy-momentum method. An example of a relative equilibrium point that, after the reduction via π_μ gives an unstable equilibrium point.

3.1 Symplectic time–dependent energy-momentum method

This section presents some generalisations of results concerning autonomous Hamiltonian systems to the t-dependent realm. Then, it defines and analyses relative equilibrium points for t-dependent Hamiltonian systems. Moreover, it establishes the relation between the manifold of relative equilibrium points and foliated Lie systems. In particular, it is proven that Hamiltonian vector fields X_{h_t} are tangent to the manifold of relative equilibrium points and give rise to a foliated Lie system [30]. Finally, the stability of trajectories around an equilibrium point on the reduced space is analysed. It also gives the necessary conditions on the function H_{z_e} to obtain stability or asymptotic stability. Next, it links the properties of stable points in P_{μ_e} and their associated relative equilibrium points in $\mathbf{J}^{\Phi-1}(\mu_e)$ and P. Finally, an example of an almost-rigid body is examined.

3.1.1 Symplectic geometry in a time-dependent setting

This subsection extends the notions introduced in Section 1.2 to a non-autonomous setting. The Hamiltonian function h, previously defined $h\colon P \to \mathbb{R}$, is now considered to be time-dependent, namely

$$h\colon (t,p) \in \mathbb{R} \times P \mapsto h(t,p) =: h_t(p) \in \mathbb{R}\,.$$

Definition 3.1.1. A *non-autonomous* Ad^**-equivariant* G*-invariant symplectic Hamiltonian system* is a 5-tuple $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$, where $\Phi\colon G \times P \to P$ is a symplectic Lie group action with an Ad^*-equivariant symplectic momentum map $\mathbf{J}^\Phi\colon P \to \mathfrak{g}^*$, and h is a time-dependent Hamiltonian function on $\mathbb{R} \times P$ satisfying $h(t, \Phi(g,p)) = h(t,p)$ for every $g \in G$, $t \in \mathbb{R}$, and $p \in P$.

In this section, to simplify the notation, a 5-tuple $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$ is hereafter assumed to be a non-autonomous Ad^*-equivariant symplectic Hamiltonian system. The time-dependent Hamiltonian function $h\colon \mathbb{R} \times P \to \mathbb{R}$ gives rise to a *t-dependent vector field* on P of the form $X_h\colon \mathbb{R} \times P \to \mathrm{T}P$ such that each vector field

$$X_{h_t}\colon p\in P\mapsto X_h(t,p)\in \mathrm{T}P\,,$$

with $t\in\mathbb{R}$, is the Hamiltonian vector field associated with the Hamiltonian function $h_t\colon p\in P\mapsto h(t,p)\in\mathbb{R}$. Then, a *particular solution, $p(t)$, to $(P,\omega,h,\Phi,\mathbf{J}^\Phi)$* is then defined as a solution of the non-autonomous system of differential equations

$$\frac{\mathrm{d}p}{\mathrm{d}t}=X_{h_t}(p)=X_h(t,p)\,,\qquad \forall t\in\mathbb{R}\,.$$

Proposition 3.1.2 analyses the evolution of $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^*$ under the dynamics given by X_{h_t}. In particular, it shows that even in the time-dependent setting $\mathbf{J}^\Phi\colon P\to\mathfrak{g}^*$ is conserved along the flow of X_{h_t}.

Proposition 3.1.2. *Let $(P,\omega,h,\Phi,\mathbf{J}^\Phi)$ be a non-autonomous G-invariant symplectic Hamiltonian system. Then, $\mathbf{J}^\Phi$ is invariant relative to the evolution of h, i.e. if $F\colon\mathbb{R}\times P\to P$ is the flow of the t-dependent vector field $X_h\colon(t,p)\in\mathbb{R}\times P\mapsto X_h(t,p)\in\mathrm{T}P$, then*

$$\mathbf{J}^\Phi(F(t,p))=\mathbf{J}^\Phi(p),\qquad\forall p\in P,\quad\forall t\in\mathbb{R}\,.$$

Proof. Define $F_t\colon p\in P\mapsto F(t,p)\in P$ for every $t\in\mathbb{R}$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}J^\Phi_\xi(F_t)=(X_{h_t}J^\Phi_\xi)\circ F_t=\{J^\Phi_\xi,h_t\}\circ F_t=(-X_{J^\Phi_\xi}h_t)\circ F_t=-(\xi_P h_t)\circ F_t=0,\forall\xi\in\mathfrak{g},\forall t\in\mathbb{R}\,.$$

The final equality follows from the fact that each h_t, for $t\in\mathbb{R}$, is assumed to be G-invariant. Consequently, $\mathbf{J}^\Phi$ is invariant under the evolution in time of the time-dependent symplectic Hamiltonian system determined by h. □

The G-invariance property of h under the symplectic Lie group action $\Phi\colon G\times P\to P$ also yields, by Theorem 2.1.8 and Proposition 2.1.11, that F induces canonically a Hamiltonian flow on the reduced phase space $P_\mu=\mathbf{J}^{\Phi-1}(\mu)/G_\mu$ associated with a Hamiltonian function $k_\mu\colon\mathbb{R}\times P_\mu\to\mathbb{R}$ defined in a unique way via the equation $k_\mu(t,\pi_\mu(p))=h(t,p)$ for every $p\in\mathbf{J}^{\Phi-1}(\mu)$ and $t\in\mathbb{R}$.

3.1.2 Relative equilibrium points

This subsection extends Poincaré's terminology of a *relative equilibrium point* (see [2, p 306]) for a time-independent Hamiltonian function to the realm of time-dependent Hamiltonian systems on symplectic manifolds [54].

Definition 3.1.3. A point $z_e\in P$ is a *relative equilibrium point* of a symplectic time-dependent Hamiltonian system $(P,\omega,h,\Phi,\mathbf{J}^\Phi)$ if there exists a curve $\xi(t)$ in $\mathfrak{g}$ such that

$$(X_{h_t})_{z_e}=(\xi(t)_P)_{z_e},\qquad\forall t\in\mathbb{R}.$$

Definition 3.1.3 recovers the classical notion of relative equilibrium in the autonomous case. The following proposition justifies the terminology in the time-dependent setting.

Proposition 3.1.4. *Let $z_e\in P$ be a relative equilibrium point of $(P,\omega,h,\Phi,\mathbf{J}^\Phi)$ and let $\mu_e:=\mathbf{J}^\Phi(z_e)$. Then, any solution $p(t)$, to $(P,\omega,h,\Phi,\mathbf{J}^\Phi)$ passing through a relative equilibrium point $z_e\in P$, namely $p(t_0)=z_e$ for some $t_0\in\mathbb{R}$, projects onto the point $\pi_{\mu_e}(z_e)$, that is*

$$\pi_{\mu_e}(p(t))=\pi_{\mu_e}(z_e)\qquad\forall t\in\mathbb{R}\,.$$

Proof. According to Proposition 3.1.2, each solution $p(t)$ to the Hamilton equations of h is fully contained within $\mathbf{J}^{\Phi-1}(\mu_e)$ and projects, under π_{μ_e}, to a curve in the reduced manifold $P_{\mu_e}=\mathbf{J}^{\Phi-1}(\mu_e)/G_{\mu_e}$, where, recall, G_{μ_e} denotes the isotropy subgroup of μ_e under the coadjoint action. Such a curve is a solution to the Hamiltonian system $(P_{\mu_e}, \omega_{\mu_e}, k_{\mu_e})$, where $k_{\mu_e}: \mathbb{R}\times P_{\mu_e} \to \mathbb{R}$ is the unique time-dependent function satisfying

$$k_{\mu_e}(t, \pi_{\mu_e}(p)) = h(t,p)\,, \qquad \forall p \in \mathbf{J}^{\Phi -1}(\mu_e)\,, \quad \forall t \in \mathbb{R}\,.$$

Since z_e is a relative equilibrium point, one has

$$0 = \mathrm{T}\mathbf{J}^{\Phi}(X_{h_t})_{z_e} = \mathrm{T}\mathbf{J}^{\Phi}(\xi(t)_P)_{z_e} = (\xi(t))_{\mathfrak{g}^*}(\mu_e), \qquad \forall t \in \mathbb{R},$$

for some curve $\xi(t)$ in $\mathfrak{g}$. Consequently, $\xi(t) \in \mathfrak{g}_{\mu_e}$ for every $t \in \mathbb{R}$.

Note that $\pi_{\mu_e}(p(t))$ is the integral curve of the t-dependent vector field Y_{μ_e} on P_{μ_e} given by the t-parametric family of vector fields on P_{μ_e} of the form $(Y_{\mu_e})_t := \pi_{\mu_e *}(X_{h_t})$ for every $t \in \mathbb{R}$. Since $X_{h_t} = \xi(t)_P$, for a certain curve $\xi(t)$ contained in $\mathfrak{g}_{\mu_e}$, then $((Y_{\mu_e})_t)_{\pi_{\mu_e}(z_e)} = \mathrm{T}_{z_e}\pi_{\mu_e}(\xi(t)_P)_{z_e} = 0$ for every $t \in \mathbb{R}$. As a consequence, $\pi_{\mu_e}(z_e)$ is an equilibrium point of Y_{μ_e} and the integral curve of the t-dependent vector field Y_{μ_e} passing through $\pi_{\mu_e}(z_e)$ is just $\pi_{\mu_e}(z_e)$. Hence, $\pi_{\mu_e}(p(t)) = \pi_{\mu_e}(z_e)$ for every $t \in \mathbb{R}$ and $p(t) \in \pi_{\mu_e}^{-1}(z_e)$ for every $t \in \mathbb{R}$. Then, the projection of every solution passing through z_e is the equilibrium point $\pi_{\mu_e}(z_e)$ of the reduced Hamiltonian system related to Y_{μ_e} on P_{μ_e}. ☐

Proposition 3.1.4 implies that every solution passing through a relative equilibrium z_e with $\mathbf{J}^{\Phi}(z_e) = \mu_e$ is of the form $p(t) = g(t)z_e$ for some curve $g(t)$ in G_{μ_e}. The converse also holds, as the following proposition shows.

Proposition 3.1.5. *If every solution $p(t)$ to $(P, \omega, h, \Phi, \mathbf{J}^{\Phi})$ passing through a point $z_e \in P$, with $\mu_e := \mathbf{J}^{\Phi}(z_e)$, projects onto $\pi_{\mu_e}(z_e)$, then z_e is a relative equilibrium point.*

Proof. Let $p(t)$ be the solution to $(P, \omega, h, \Phi, \mathbf{J}^{\Phi})$ passing through z_e at $t = t_0$. By assumption, $\pi_{\mu_e}(p(t))$ projects onto $\pi_{\mu_e}(z_e)$. Consequently, there exists a curve $g(t)$ in G_{μ_e} such that $p(t) = \Phi(g(t), p(t_0))$ and $g(t_0) = e$. Therefore,

$$(X_{h_{t_0}})_{z_e} = \frac{\mathrm{d}p}{\mathrm{d}t}(t_0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0}(\Phi(g(t), z_e)) = \mathrm{T}_e\Phi_{z_e}\left(\frac{\mathrm{d}g}{\mathrm{d}t}(t_0)\right) = (\nu(t_0))_P(z_e),$$

for a certain $\nu(t_0) \in \mathfrak{g}_{\mu_e}$. As this holds for all $t_0 \in \mathbb{R}$, the point z_e is thus a relative equilibrium point. ☐

Note that if a solution $p(t)$ to $(P, \omega, h, \Phi, \mathbf{J}^{\Phi})$ satisfies $p(t) = g(t)p$ for some curve $g(t)$ in G and $p \in P$, then Proposition 3.1.2 ensures that $\mathbf{J}^{\Phi}(p(t)) = \mathbf{J}^{\Phi}(p)$. Hence, by Lemma 2.1.7, $g(t)$ must belong to the isotropy subgroup G_{μ_e} of $\mu_e = \mathbf{J}^{\Phi}(p)$.

The equivalence established above is summarised in the following corollary.

Corollary 3.1.6. *The following two conditions are equivalent:*

- *The point $z_e \in P$ is a relative equilibrium point of $(P, \omega, h, \Phi, \mathbf{J}^{\Phi})$,*
- *Every particular solution to $(P, \omega, h, \Phi, \mathbf{J}^{\Phi})$ passing through $z_e \in P$ is of the form $p(t) = g(t)z_e$ for a curve $g(t)$ in G_{μ_e}.*

In time-dependent Hamiltonian systems, the Hamiltonian function is not generally conserved along solutions. In fact,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial t} + \{h, h\} = \frac{\partial h}{\partial t}\,.$$

However, Corollary 3.1.6 implies that for solutions of the form $p(t) = g(t)z_e$ through a relative equilibrium point z_e, one has $h(t, p(t)) = h(t, z_e)$, although h does not need to be conserved. This complicates the stability analysis of solutions to the reduced Hamiltonian system $(P_{\mu_e}, \omega_{\mu_e}, k_{\mu_e})$, since k_{μ_e} is not, in general, autonomous, and classical stability methods must be replaced by more general techniques (cf. [113]).

The following result provides a more practical criterion for identifying relative equilibria.

Theorem 3.1.7. (Time-Dependent Relative Equilibrium Theorem) *A point $z_e \in P$ is a relative equilibrium point of $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$ if and only if there exists a curve $\xi(t)$ in $\mathfrak{g}$ such that z_e is a critical point of $h_{\xi,t}\colon P \to \mathbb{R}$ given by*

$$h_{\xi,t} := h_t - [J^\Phi_{\xi(t)} - \langle \mu_e, \xi(t)\rangle] = h_t - \langle \mathbf{J}^\Phi - \mu_e,\ \xi(t)\rangle\,,$$

for every $t \in \mathbb{R}$ and $\mu_e := \mathbf{J}^\Phi(z_e)$.

Proof. Assume first that $z_e \in P$ is a relative equilibrium point. The definition of the momentum map and Corollary 3.1.6 yield

$$(X_{h_t})_{z_e} - (X_{J^\Phi_{\xi(t)}})_{z_e} = 0\,, \qquad \forall t \in \mathbb{R}\,.$$

Since (P, ω) is symplectic, the latter is equivalent to z_e being a critical point of $h_t - J^\Phi_{\xi(t)}$ for every $t \in \mathbb{R}$, which implies that z_e is a critical point of $h_{\xi,t}$ for every $t \in \mathbb{R}$, i.e. $(\mathrm{d}h_{\xi,t})_{z_e} = 0$.

Conversely, assume that $z_e \in P$ is a critical point of $h_{\xi,t}$. Then z_e is a stationary point of the dynamical system $X_{h_t - J_{\xi(t)}}$ for every $t \in \mathbb{R}$. Hence, the evolution of every particular solution of X_h passing through z_e at time t_0 is of the form $g(t)z_e$ for a certain curve in G with $g(t_0) = e$ and, in view of Corollary 3.1.6, one has that z_e is a relative equilibrium point. □

In view of Theorem 3.1.7, to find relative equilibrium points, one can consider the family of functions $h^e_t\colon (p, \xi) \in P \times \mathfrak{g} \mapsto h_t - \langle \mathbf{J}^\Phi - \mu_e,\ \xi\rangle \in \mathbb{R}$, for every $t \in \mathbb{R}$, and look for $z_e \in P$ such that $(z_e, \xi(t))$ is a critical point of h^e_t for each $t \in \mathbb{R}$ and a certain curve $\xi(t)$ in $\mathfrak{g}$. Evidently, $\xi(t)$ plays the role of a Lagrange multiplier depending on t. Note that the term $\langle \mathbf{J}^\Phi - \mu_e, \xi\rangle$ in h^e_t ensures that the described relative equilibrium points belong to $\mathbf{J}^{\Phi-1}(\mu_e)$.

3.1.3 Foliated Lie systems and relative equilibrium submanifolds

This subsection presents that the set of relative equilibrium points for a G-invariant time-dependent Hamiltonian system $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$ is given by a union of immersed submanifolds. Furthermore, assuming a certain condition on the Lie algebra of fundamental vector fields of the action of G on P, it is proven that the restriction of the original t-dependent Hamiltonian system to such immersed submanifolds can be described via a foliated Lie system [30].

Proposition 3.1.8. *Let z_e be a relative equilibrium point of a G-invariant time-dependent Hamiltonian system $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$. Then, $\mathcal{O}_{z_e} := Gz_e$ is an immersed submanifold of P consisting of relative equilibrium points.*

Proof. Since z_e is a relative equilibrium point, every solution passing through z_e is of the form $z(t) = g(t)z_e$ for a certain curve $g(t)$ in G. Since $h(t, \Phi_g(p)) = h(t, p)$ for every $t \in \mathbb{R}$ and $p \in P$, and also $\Phi_g^*\omega = \omega$ for every $g \in G$, one gets

$$\begin{aligned}\iota_{X_{h_t}}\omega = \mathrm{d}h_t \Rightarrow (\iota_Y \iota_{\Phi_{g*}X_{h_t}}\omega)(gp) &= [(\Phi_g^*\omega)(X_{h_t}, \Phi_{g^{-1}*}Y)](p)\\ &= \omega(X_{h_t}, \Phi_{g^{-1}*}Y)(p) = \langle \mathrm{d}h_t, \Phi_{g^{-1}*}Y\rangle(p) = \langle \mathrm{d}\Phi^*_{g^{-1}}h_t, Y\rangle(gp) = \langle \mathrm{d}h_t, Y\rangle(gp),\end{aligned}$$

for every $Y \in \mathfrak{X}(P)$, $g \in G$, $p \in P$ and $t \in \mathbb{R}$. Therefore, $\Phi_{g*}X_{h_t} = X_{h_t}$ for every $t \in \mathbb{R}$. Hence, every solution $z'(t)$ passing through gz_e is such that $z(t) := g^{-1}z'(t)$ is a solution to X_{h_t} passing through z_e. Thus, $z'(t) = gz(t) = gg(t)g^{-1}gz_e$. In other words, gz_e is a relative equilibrium point for $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$. Since Gz_e is the orbit of a Lie group action, it is an immersed submanifold of P (see [2, 95, 128]). Then, the proposition follows. □

Definition 3.1.9. A *foliated Lie system* [30] on a manifold P is a first-order system of differential equations of the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = X(t, p), \qquad \forall t \in \mathbb{R}, \qquad \forall p \in P\,,$$

such that

$$X(t,p)=\sum_{\alpha=1}^{r} f_\alpha(t,p)X_\alpha(p), \qquad \forall t\in\mathbb{R}, \quad \forall p\in P\,, \tag{3.1.1}$$

where $X_1,\ldots,X_r$ span an r-dimensional Lie algebra of vector fields, that is

$$[X_\alpha,X_\beta]=\sum_{\gamma=1}^{r} c_{\alpha\beta}^{\gamma}X_\gamma, \qquad \alpha,\beta=1,\ldots,r\,,$$

for certain structure constants $c_{\alpha\beta}^{\gamma}$. The functions $f_{\alpha,t}\colon p\in P\mapsto f_\alpha(t,p)\in\mathbb{R}$, for every $t\in\mathbb{R}$ and $\alpha=1,\ldots,r$, are first integrals of $X_1,\ldots,X_r$. The Lie algebra $\langle X_1,\ldots,X_r\rangle$ is called a *Vessiot–Guldberg Lie algebra of the foliated Lie system* X [52].

Foliated Lie systems naturally arise in the analysis of relative equilibrium points for G-invariant time-dependent Hamiltonian systems when the dynamics is restricted to the union of group orbits through such points, as the following theorem shows.

Theorem 3.1.10. *Let z_e be a relative equilibrium point of $(P,\omega,h,\Phi,\mathbf{J}^\Phi)$ and denote $\mu_e:=\mathbf{J}^\Phi(z_e)$. Suppose that G_{μ_e} is abelian. Then, X_{h_t} is tangent to $\mathcal{O}_{z_e}:=Gz_e$ for every fixed $t\in\mathbb{R}$, Furthermore, X_h gives rise, by restriction, to a t-dependent vector field $X_h|_{\mathcal{O}_{z_e}}$ on $\mathcal{O}_{z_e}$ and $X_h|_{\mathcal{O}_{z_e}}$ becomes a foliated Lie system with an abelian Vessiot–Guldberg Lie algebra of dimension equal to $\dim\mathfrak{g}_{\mu_e}$.*

Proof. Assume that $z_e'\in\mathcal{O}_{z_e}$ and denote $\mu_e':=\mathbf{J}^\Phi(z_e')$. The aim is to prove that $X_h|_{\mathcal{O}_{z_e}}$ exists and can be written as (3.1.1) for certain functions g_α, with $\alpha=1,\ldots,r$, that depend only on time on the submanifolds of the form $G_{\mu_e'}z_e'$, with $G_{\mu_e'}$ denoting the isotropy subgroup of the coadjoint action of G at μ_e', and certain vector fields tangent to $\mathcal{O}_{z_e}$ closing on a finite-dimensional Lie algebra of vector fields. This establishes that $X_h|_{\mathcal{O}_{z_e}}$ defines a foliated Lie system.

By Proposition 3.1.8, a point z_e' is a relative equilibrium point. Then, every integral curve of X_h passing through z_e' takes the form $z(t)=g(t)z_e'$ for a certain curve $g(t)$ in G. Hence, X_h is tangent to $\mathcal{O}_{z_e}$ and thus restricts to it. Proposition 3.1.2 yields that $\mathbf{J}^\Phi$ is constant along integral curves of X_h. Consequently, the integral curves of X_h passing through z_e' are contained in $\mathbf{J}^{\Phi-1}(\mu_e')$. Assuming that $z(t_0)=z_e'$, one has $z(t)=g(t)z_e'$ for a curve $g(t)$ in G with $g(t_0)=e$. Then,

$$0=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0}\mathbf{J}^\Phi(z(t))=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0}\mathbf{J}^\Phi(g(t)z_e')=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0}\mathrm{Ad}^*_{g(t)^{-1}}(\mathbf{J}^\Phi(z_e'))=[\xi(t_0)]_{\mathfrak{g}^*}(\mu_e')\,,$$

where $\mathrm{d}g(t)/\mathrm{d}t|_{t=t_0}=\xi(t_0)$. Therefore, $\xi(t_0)\in\mathfrak{g}_{\mu_e'}$.

Let $\{\xi_1^{\mu_e},\ldots,\xi_r^{\mu_e}\}$ be a basis for $\mathfrak{g}_{\mu_e}$. Since G_{μ_e} is abelian, it follows that $\mathfrak{g}_{\mu_e}$ is abelian as well. Define the vector fields on $\mathcal{O}_{z_e}$ of the form

$$Y_\alpha(gz_e):=[\mathrm{T}_{z_e}\Phi_g(\xi_\alpha^{\mu_e})_P](z_e)\,,$$

for $\alpha=1,\ldots,r$. Each Y_α is well defined because G_{z_e} is a subgroup of G_{μ_e}, which acts freely on $\mathbf{J}^{\Phi-1}(\mu_e)$, so $gz_e=g'z_e$ implies $g^{-1}g'\in G_{z_e}=\{e\}$ and $g'=g$. Since the action of G_{μ_e} is assumed to be free on $\mathbf{J}^{\Phi-1}(\mu_e)$, the tangent vectors $Y_1(z_e),\ldots,Y_r(z_e)$ are linearly independent. Furthermore, $Y_\alpha(gz_e)=\mathrm{T}_{z_e}\Phi_g[Y_\alpha(z_e)]$ for every $g\in G$, implies that $Y_1\wedge\ldots\wedge Y_r\neq 0$ on $\mathcal{O}_{z_e}$. As $\mathfrak{g}_{\mu_e}$ is abelian, for every $g_{\mu_e}\in G_{\mu_e}$, one gets

$$Y_\alpha(gg_{\mu_e}z_e)=\mathrm{T}_{g_{\mu_e}z_e}\Phi_g\circ\mathrm{T}_{z_e}\Phi_{g_{\mu_e}}[(\xi_\alpha^{\mu_e})_P(z_e)]=\mathrm{T}_{g_{\mu_e}z_e}\Phi_g(\xi_\alpha^{\mu_e})_P(g_{\mu_e}z_e)=(\mathrm{Ad}_g(\xi_\alpha^{\mu_e}))_P(gg_{\mu_e}z_e)\,, \tag{3.1.2}$$

for $\alpha=1,\ldots,r$. Moreover, $\mathrm{Ad}_g(\xi_\alpha^{\mu_e})$, with $\alpha=1,\ldots,r$, is a basis of the Lie algebra $\mathfrak{g}_{\mathbf{J}^\Phi(gg_{\mu_e}z_e)}$. Thus,

$$X_h(t,z)=\sum_{\alpha=1}^{r} f_\alpha(t,z)Y_\alpha(z)\,,$$

for every $z \in G_{\mu'_e} z'_e$ for a unique set of functions $f_1(t,z), \ldots, f_r(t,z)$. If $z'_e := g z_e$ and, since G_{μ_e} is abelian, it follows that $G_{\mu'_e} = g G_{\mu_e} g^{-1}$ is abelian too. Then,

$$\mathrm{T}_{z'_e} \Phi_{g_{\mu'_e}} (\mathrm{Ad}_g(\xi^{\mu_e}_\alpha))_P(z'_e) = (\mathrm{Ad}_{g_{\mu'_e} g}(\xi^{\mu_e}_\alpha))_P(g_{\mu'_e} z'_e) = (\mathrm{Ad}_{g g'_{\mu_e}}(\xi^{\mu_e}_\alpha))_P(g_{\mu'_e} z'_e) = (\mathrm{Ad}_g(\xi^{\mu_e}_\alpha))_P(g_{\mu'_e} z'_e)\,.$$

The last equality stems from $g^{-1} G_{\mu'_e} g = G_{\mu_e}$ and $g^{-1} g_{\mu'_e} g = g'_{\mu_e}$ for some $g'_{\mu_e} \in G_{\mu_e}$. Equation (3.1.2) yields

$$\begin{aligned} X_h(t, g_{\mu'_e} z'_e) = \mathrm{T}_{z'_e} \Phi_{g_{\mu'_e}} X_h(t, z'_e) &= \sum_{\alpha=1}^r f_\alpha(t, z'_e) \mathrm{T}_{z'_e} \Phi_{g_{\mu'_e}} (\mathrm{Ad}_g(\xi^{\mu_e}_\alpha))_P(z'_e) \\ &= \sum_{\alpha=1}^r f_\alpha(t, z'_e) (\mathrm{Ad}_g(\xi^{\mu_e}_\alpha))_P(g_{\mu'_e} z'_e) = \sum_{\alpha=1}^r f_\alpha(t, z'_e) Y_\alpha(g_{\mu'_e} z'_e)\,, \end{aligned}$$

for every $g_{\mu'_e} \in G_{\mu'_e}$. Then,

$$\begin{aligned} Y_\alpha(g_{\mu'_e} z'_e) = Y(g_{\mu'_e} g z_e) = Y(g g^{-1} g_{\mu'_e} g z_e) = Y(g g'_{\mu_e} z_e) &= (\mathrm{Ad}_g(\xi^{\mu_e}_\alpha))_P(g g'_{\mu_e} z_e) \\ &= (\mathrm{Ad}_g(\xi^{\mu_e}_\alpha))_P(g_{\mu'_e} g z_e) = (\mathrm{Ad}_g(\xi^{\mu_e}_\alpha))_P(g_{\mu'_e} z'_e)\,. \end{aligned}$$

From (3.1.3) and the fact that $X_h(t, g_{\mu'_e} z') = \sum_{\alpha=1}^r f_\alpha(t, g_{\mu'_e} z') Y_\alpha(g_{\mu'_e} z')$, one has that $f_\alpha(t, z'_e) = f_\alpha(t, g_{\mu'_e} z'_e)$ for every $g_{\mu'_e} \in G_{\mu'_e}$ and $\alpha = 1, \ldots, r$. Consequently,

$$X_h(t,z) = \sum_{\alpha=1}^r f_\alpha(t,z) Y_\alpha(z), \qquad \forall z \in \mathcal{O}_{z_e}, \quad \forall t \in \mathbb{R}\,,$$

for some functions $f_1, \ldots, f_r$ on $\mathbb{R} \times \mathcal{O}_{z_e}$ that depend only on t when restricted to $G_{\mu'_e} z'_e$. The vector fields $Y_1, \ldots, Y_r$ are tangent to the submanifolds $G_{\mu'_e} z'_e$ and span an abelian Lie algebra. Since the functions $f_1, \ldots, f_r$ are just t-dependent on the submanifolds $G_{\mu'_e} z'_e$, they become first integrals of the vector fields in $\langle Y_1, \ldots, Y_r \rangle$. Therefore, $X_h|_{\mathcal{O}_z}$ becomes a foliated Lie system with an abelian Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{g}_{\mu_e}$. □

3.1.4 Stability on the reduced symplectic manifold

Theorem 3.1.7 characterises the relative equilibrium points of G-invariant time-dependent Hamiltonian systems as the extrema of the Hamiltonian function constrained to level sets of the symplectic momentum map. Accordingly, the function

$$h_{\xi,t} := h_t - \langle \mathbf{J}^\Phi - \mu_e, \xi(t) \rangle$$

has to be optimised and $\xi(t) \in \mathfrak{g}$ is a t-dependent Lagrange multiplier.

The study of the stability of equilibrium points in $\mathbf{J}^{\Phi -1}(\mu_e)/G_{\mu_e}$ for non-autonomous Hamiltonian systems requires the use of a t-dependent Lyapunov stability analysis. This is more complicated than studying the stability of autonomous Hamiltonian systems, which frequently relies on searching for a minimum of the Hamiltonian [113], although such a condition is not always necessary [2, p 207]. To address the non-autonomous Hamiltonians, Theorem 1.1.6 is applied along with a more general approach, which easily retrieves the standard results used in the energy-momentum method for autonomous Hamiltonian systems.

Let z_e be a relative equilibrium point of $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$. Define the function $h_{z_e} : \mathbb{R} \times P \to \mathbb{R}$ as

$$h_{z_e}(t,z) := h(t,z) - h(t,z_e), \qquad \forall (t,z) \in \mathbb{R} \times P\,.$$

Then, $h_{z_e}(t, z_e) = 0$ for every $t \in \mathbb{R}$. Let $z(t)$ denote a solution to the Hamiltonian system passing through some $z \in P$ at time t_0. Then,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} h_{z_e}(t, z(t)) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} h(t, z(t)) - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} h(t, z_e)\,.$$

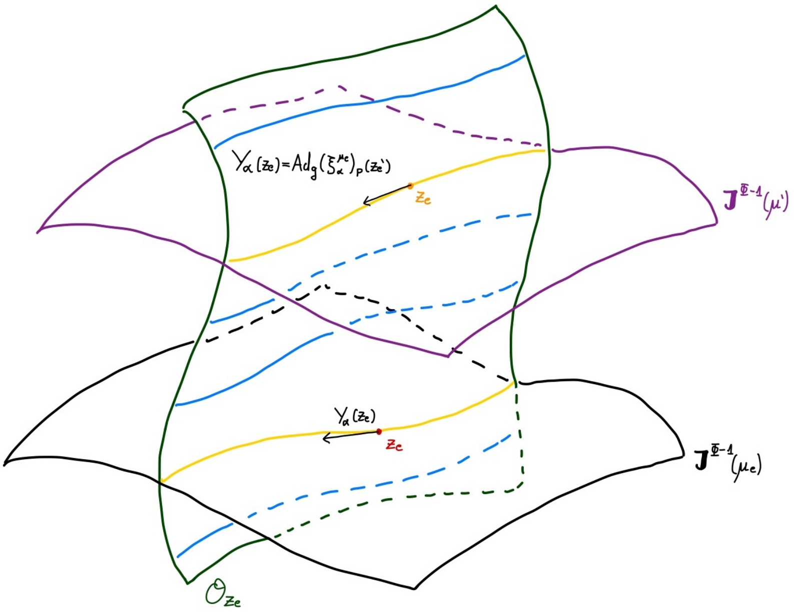

Figure 3.2: A visualisation of a foliated Lie system $X_h|_{\mathcal{O}_{z_e}}$.

Recall that the time derivative of a Hamiltonian function h along the solutions of its Hamilton equations is given by

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\partial h}{\partial t} + \{h_t, h_t\} = \frac{\partial h}{\partial t}.$$

Consequently,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} h_{z_e}(t, z(t)) := \frac{\partial h}{\partial t}(t_0, z) - \frac{\partial h}{\partial t}(t_0, z_e) = \frac{\partial h_{z_e}}{\partial t}(t_0, z)\,.$$

Note that $h_{z_e}(t, gz) = h_{z_e}(t, z)$ for every $g \in G$ and every $(t, z) \in \mathbb{R} \times P$, which shows that $h_{z_e}(t, z)$ is G-invariant. Then, one can define a function $H_{z_e} : \mathbb{R} \times P_{\mu_e} \to \mathbb{R}$ of the form

$$H_{z_e}(t, [z]) := h_{z_e}(t, z), \qquad \forall z \in \mathbf{J}^{\Phi-1}(\mu_e), \quad \forall t \in \mathbb{R}\,,$$

where $[z]$ stands for the equivalence class of $z \in \mathbf{J}^{\Phi-1}(\mu_e)$ in $\mathbf{J}^{\Phi-1}(\mu_e)/G_{\mu_e}$. Since $H_{z_e}(t, [z]) - k_{\mu_e}(t, [z])$ depends only on time, the point $[z_e]$ is an equilibrium point of H_{z_e}. Furthermore,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} H_{z_e}(t, [z(t)]) = \frac{\partial h_{z_e}}{\partial t}(t_0, z), \qquad \forall t_0 \in \mathbb{R}, \qquad \forall [z] \in \mathbf{J}^{\Phi-1}(\mu_e)/G_{\mu_e}\,.$$

The function H_{z_e} is used to study the stability of the point $[z_e]$ in the reduced space P_{μ_e}. In particular, the conditions on h to ensure when H_{z_e} gives rise to different types of stable equilibrium points at $[z_e]$ are provided. With this aim, consider a coordinate system $\{x_1, \ldots, x_n\}$ on an open neighbourhood U of $[z_e] \in P_{\mu_e}$ such that $x_i([z_e]) = 0$ for $i = 1, \ldots, n$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_1, \ldots, \alpha_n \in \mathbb{N} \cup \{0\}$, be a multi-index with $n := \dim \mathbf{J}^{\Phi-1}(\mu_e)/G_{\mu_e}$ and let $|\alpha| := \sum_{i=1}^n \alpha_i$ and $D^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

Lemma 3.1.11. *Let $M(t)$ denote the t-dependent parametric family of $n \times n$ matrices defined by*

$$[M(t)]_i^j := \frac{1}{2}\frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]), \qquad \forall t \in \mathbb{R}, \qquad i, j = 1, \ldots, n,$$

and let $\operatorname{spec}(M(t))$ denote the spectrum of $M(t)$ at $t \in \mathbb{R}$. Suppose that there exists a constant λ such that

$$0 < \lambda < \inf_{t \in I_{t_0}} \min \operatorname{spec}(M(t))\,,$$

for some $t^0 \in \mathbb{R}$. *Moreover, assume that there exists a constant* $c \in \mathbb{R}$ *satisfying*

$$c \geq \frac{1}{6} \sup_{t \in I_{t^0}} \max_{|\alpha|=3} \max_{[y] \in \mathcal{B}} |D^\alpha H_{z_e}(t, [y])|$$

for a certain compact neighbourhood $\mathcal{B}$ *of* $[z_e]$. *Then, there exists an open neighbourhood* $\mathcal{U}$ *of* $[z_e]$ *where the function* $H_{z_e} : \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ *is lpdf from* t^0. *If, in addition, there exists a constant* $\Lambda \in \mathbb{R}$ *such that*

$$\sup_{t \in I_{t^0}} \max \operatorname{spec}(M(t)) < \Lambda \,,$$

then $H_{z_e} : \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ *is a decrescent function from* t^0.

Proof. Since z_e is a relative equilibrium point of $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$, it follows that $H_{z_e}(t, \cdot)$ admits a critical point at $[z_e]$ for every $t \in \mathbb{R}$. By the Taylor expansion of $H_{z_e}(t, \cdot)$ around $[z_e]$ and the fact that z_e is a relative equilibrium point of each $h_{z_e}(t, \cdot)$ for $t \in \mathbb{R}$, one gets

$$H_{z_e}(t, [z]) = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]) x_i x_j + R_t([z]), \qquad [z] \in U, \quad t \in \mathbb{R},$$

where $R_t([z])$ denotes the third-order remainder function of the Taylor expansion for $H_{z_e}(t, [z])$ at a fixed $t \in \mathbb{R}$ around $[z_e]$. The coefficients of the quadratic part coincide with the entries of the matrix $M(t)$ in coordinates $\{x_1, \ldots, x_n\}$. Since $M(t)$ is symmetric, it can be diagonalised via an orthogonal transformation O_t for each $t \in \mathbb{R}$. Let $\lambda_1(t), \ldots, \lambda_n(t)$ be the (possibly repeated) n eigenvalues of $M(t)$ and let $\mathbf{w} = (w_1, \ldots, w_n)^T$ be the coordinate vector corresponding to $\mathbf{z} = (x_1, \ldots, x_n)^T$ in the diagonalising basis induced by O_t, namely $\mathbf{w} = O_t \mathbf{z}$. Although the explicit construction of O_t is not required, it can be obtained by forming orthogonal bases of eigenvectors of $M(t)$ at each time t. Then, $\mathbf{z}^T M(t) \mathbf{z} = \mathbf{w}^T D(t) \mathbf{w}$, where $D(t) = \operatorname{diag}(\lambda_1(t), \ldots, \lambda_n(t))$. Thus, $\mathbf{w}^T D(t) \mathbf{w} = \sum_{i=1}^n \lambda_i(t) w_i^2$ and

$$\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]) x_i x_j = \mathbf{z}^T M(t) \mathbf{z} = \mathbf{w}^T D(t) \mathbf{w} \geq \lambda(t) \|\mathbf{w}\|^2,$$

where $\lambda(t) := \min_{i=1,\ldots,n} \lambda_i(t)$ for each $t \in \mathbb{R}$. Using the assumption on the existence of $\lambda > 0$ and the fact that O_t is orthogonal, it follows that

$$\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]) x_i x_j \geq \lambda(t) \|\mathbf{z}\|^2 \geq \lambda \|\mathbf{z}\|^2$$

on the neighbourhood U.

The third-order Taylor remainder $R_t([z])$ around $[z_e]$ can be expressed as

$$R_t([z]) = \sum_{|\beta|=3} B_\beta(t, [z]) \mathbf{z}^\beta, \qquad \mathbf{z}^\beta := x_1^{\beta_1} \cdots x_n^{\beta_n},$$

at points $[z]$ of the open coordinate subset U, $t \in \mathbb{R}$, and for certain functions $B_\beta : \mathbb{R} \times U \to \mathbb{R}$.

Note that $R_t([z])$ is not a third-degree polynomial in general, due to the fact that the functions $B_\beta(t, [z])$ depend on the coordinates of $[z]$. Although $R_t([z])$ can be bounded by a third-order polynomial in the coordinates of $[z]$ for each fixed time t, O_t, the open subsets U_t, and, in addition, the coefficients of the polynomials used to bound $R_t([z])$ depend on t. This may potentially lead to problems since, for example, to bound $R_t([z])$ for every $t \in \mathbb{R}$, one has to restrict to $\bigcap_{t \in I_{t^0}} U_t$, which may give rise to a single point. An alternative approach is presented, more appropriate but complicated, to bound all $R_t([z])$ for $t \in I_{t^0}$.

The coefficients B_β satisfy

$$|B_\beta(t, [z])| \leq \frac{1}{3!} \max_{|\alpha|=3} \max_{[y] \in \mathcal{C}} |D^\alpha H_{z_e}(t, [y])|, \qquad \forall [z] \in \mathcal{C} \,,$$

on any compact neighbourhood $\mathcal{C}$ of $[z_e]$ for each $t \in \mathbb{R}$. By the assumption, there exists a constant $c > 0$ such that

$$c \geq \frac{1}{3!} \max_{|\alpha|=3} \max_{y \in \mathcal{B}} |D^\alpha H_{z_e}(t, [y])|, \qquad \forall t \in I_{t^0} ,$$

for some compact neighbourhood $\mathcal{B}$ of $[z_e]$. The aim now is to prove that

$$\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]) x_i x_j + R_t([z]) - \frac{1}{2} \lambda \|\mathbf{z}\|^2$$

is bigger or equal to zero for every $t \in I_{t^0}$ and every $[z] \in \mathcal{U}$ for a certain open neighbourhood $\mathcal{U}$ of $[z_e]$. Recall that there exists $\lambda < \inf_{t \in I_{t^0}} \lambda(t)$ and note that $\lambda_i(t) - \lambda \geq \lambda(t) - \lambda$ and $\lambda(t) - \lambda$ is larger than a certain properly chosen $\lambda' > 0$ for every $t \in I_{t^0}$. Then,

$$\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]) x_i x_j - \lambda \|\mathbf{z}\|^2 = \mathbf{w}^T \mathrm{diag}(\lambda_1(t) - \lambda, \ldots, \lambda_n(t) - \lambda) \mathbf{w} \geq \lambda' \|\mathbf{w}\|^2 = \lambda' \|\mathbf{z}\|^2 .$$

Then, the first bracket in the following expression

$$\left(\frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]) x_i x_j - \lambda \|\mathbf{z}\|^2 - \lambda' \|\mathbf{z}\|^2 \right) + \left(\lambda' \|\mathbf{z}\|^2 + R_t([z]) \right) ,$$

is larger than or equal to zero on U. To prove the same for the second bracket on a neighbourhood of $[z_e]$, note that

$$|R_t([z])| \leq \sum_{|\beta|=3} |B_\beta(t, [z])| |x_1|^{\beta_1} \cdots |x_n|^{\beta_n} \leq c \sum_{|\beta|=3} |x_1|^{\beta_1} \cdots |x_n|^{\beta_n}, \qquad \forall t \in I_{t^0} ,$$

on $\mathcal{B}$. The function

$$\lambda' \|\mathbf{z}\|^2 - c \sum_{|\beta|=3} \lambda_\beta \mathbf{z}^\beta ,$$

where the $\{\lambda_\beta\}$ is any set of constants such that $\lambda_\beta \in \{\pm 1\}$ for every multi-index β with $|\beta| = 3$, admits a minimum at $[z_e]$ as follows from standard differential calculus arguments. Consequently, the above function is bigger than or equal to zero on a neighbourhood $U_{\{\lambda_\beta\}}$ of zero. Considering the intersection of all the possible open subsets $U_{\{\lambda_\beta\}}$ for every set of constants λ_β, one obtains an open neighbourhood $\mathcal{U}$ of $[z_e]$. Assume that $[z] \in \mathcal{U}$ satisfies

$$0 > \lambda' \|\mathbf{z}\|^2 - c \sum_{|\beta|=3} |x_1|^{\beta_1} \cdots |x_n|^{\beta_n} .$$

Then,

$$0 > \lambda' \|\mathbf{z}\|^2 - c \sum_{|\beta|=3} \mathrm{sgn} \left(\prod_{i=1}^{n} x_i^{\beta_i} \right) \mathbf{z}^\beta ,$$

where sgn(a) is the sign of the constant a. Then, $[z]$ cannot belong to $\mathcal{U}$. In other words,

$$\lambda' \|\mathbf{z}\|^2 - c \sum_{|\beta|=3} |x_1|^{\beta_1} \cdots |x_n|^{\beta_n} \geq 0 , \tag{3.1.3}$$

on $\mathcal{U}$. Since $|R_t([z])| \leq c \sum_{|\beta|=3} |x_1|^{\beta_1} \cdots |x_n|^{\beta_n}$ on $\mathcal{U}$ and $t \in I_{t^0}$, then

$$\lambda' \|\mathbf{z}\|^2 + R_t([z]) \geq 0$$

for every $[z] \in \mathcal{U}$ and $t \in I_{t^0}$. Finally, one gets that

$$H_{z_e}(t, [z]) \geq \lambda \|\mathbf{z}\|^2, \qquad \forall [z] \in \mathcal{U}, \qquad \forall t \in I_{t^0} .$$

Hence, the restriction of $H_{z_e} : \mathbb{R} \times P_{\mu_e} \to \mathbb{R}$ to $I_{t^0} \times \mathcal{U}$ is a lpdf function.

Now, the orthogonal change of variables O_t allows for writing

$$\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]) x_i x_j = \mathbf{z}^T M(t)\mathbf{z} = \mathbf{w}^T D(t)\mathbf{w} \leq \Lambda(t)\|\mathbf{w}\|^2 = \Lambda(t)\|\mathbf{z}\|^2 ,$$

for $\Lambda(t) := \max_{i=1,\ldots,n} \lambda_i(t)$ and every $t \in \mathbb{R}$. By assumption, $\Lambda > \Lambda(t)$ for every $t \in I_{t^0}$. Hence,

$$\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]) x_i x_j \leq \Lambda\|\mathbf{z}\|^2, \qquad \forall t \in I_{t^0} .$$

Recall the expression (3.1.3) for every $t \in I_{t^0}$ and $[z] \in \mathcal{U}$. Then,

$$H_{z_e}(t, [z]) \leq \Lambda\|\mathbf{z}\|^2 + \lambda'\|\mathbf{z}\|^2 ,$$

and H_{z_e} is decrescent on $I_{t^0} \times \mathcal{U}$. □

The eigenvalues of the matrix $M(t)$ depend on the choice of coordinates in a neighbourhood of $[z_e]$. By choosing a suitable coordinate system, the matrix $M(t)$ may be simplified at certain values of $t \in \mathbb{R}$, for example, by expressing $M(t)$ in a canonical form. However, it is generally not possible to simplify $M(t)$ simultaneously at all times $t \in I_{t^0}$. Although a time-dependent change of coordinates may allow one to achieve such a simplification for all t, constructing such a transformation is typically difficult and may not be compatible with the symplectic framework, which is formulated in terms of time-independent changes of variables. Therefore, attention is restricted to determining conditions relative to a fixed coordinate system.

The above lemma implies the following.

Theorem 3.1.12. *Suppose that there exist $\lambda, c > 0$ and an open neighbourhood U of $[z_e] \in P_{\mu_e}$ so that the following hold*

$$\lambda < \min(\operatorname{spec}(M(t))), \qquad c \geq \frac{1}{3!}\max_{|\alpha|=3}\sup_{[x]\in U} |D^\alpha H_{z_e}(t, [x])|, \qquad \left.\frac{\partial H_{z_e}}{\partial t}\right|_U \leq 0,$$

for every $t \in I_{t^0}$. Then, $[z_e]$ is a stable equilibrium point of the Hamiltonian system k_{μ_e} on $\mathbf{J}^{\Phi-1}(\mu_e)/G_{\mu_e}$ from t^0. Furthermore, if there exists Λ such that $\max(\operatorname{spec}(M(t))) < \Lambda$ for every $t \in I_{t^0}$, then $[z_e]$ is uniformly stable from t^0.

Proof. Lemma 3.1.11 ensures that $H_{z_e}(t, [z])$ is a locally positive definite $\mathscr{C}^1$-function. Since $\partial H_{z_e}/\partial t \leq 0$, Theorem 1.1.6, point 1., implies that $[z_e]$ is stable from t^0. If additionally Λ exists, then again Theorem 1.1.6, point 2., yields that $[z_e]$ is uniformly stable from t^0. □

For geometric purposes, the following corollary is stated as a consequence of Theorem 3.1.12, assuming a stronger condition on the derivatives of H_{z_e}. In particular, Corollary 3.1.13 establishes coordinate-independent criteria ensuring the stability of $[z_e]$.

Corollary 3.1.13. *If there exist $\lambda, c > 0$ and an open neighbourhood U of $[z_e] \in P_{\mu_e}$ satisfying*

$$\lambda < \min\left(\operatorname{spec}\left(M(t)\right)\right), \qquad c \geq \frac{1}{3!}\max_{1\leq|\alpha|\leq 3}\sup_{[x]\in U} \left|D^\alpha H_{z_e}(t, [x])\right|, \qquad \left.\frac{\partial H_{z_e}}{\partial t}\right|_U \leq 0 ,$$

for every $t \in I_{t^0}$, then $[z_e]$ is a uniformly stable equilibrium point of the Hamiltonian system k_{μ_e} on $\mathbf{J}^{\Phi-1}(\mu_e)/G_{\mu_e}$ from t^0.

The existence of c in Corollary 3.1.13 implies that $\max(\operatorname{spec}(M(t)))$ for every $t \in I_{t^0}$ is bounded from above, namely

$$v^T M(t) v \leq \sum_{i,j=1}^{n} |v_i v_j M_i^j(t)| \leq \sum_{i,j=1}^{n} |v_i||v_j||M_j^i(t)| \leq 6c\sum_{i,j=1}^{n} \|v\|^2 = 6cn^2\|v\|^2, \qquad \forall v \in \mathbb{R}^n .$$

Hence, $v^T M(t) v < \Lambda v^T v$ for $v \in \mathbb{R}^n \backslash \{0\}$ and $\Lambda > 6cn^2$.

Note that previous results use a distance defined on an open neighbourhood of $[z_e]$, induced by a standard norm on $\mathbb{R}^n$. Since the topology induced by this norm on the open neighbourhood of $[z_e]$ coincides with the one induced by any other Riemannian metric on the neighbourhood, the presented results concerning the stability of $[z_e]$ are independent of the used metric.

The conditions formulated in Corollary 3.1.13 admit a geometric interpretation. Specifically, if these conditions are satisfied within a given coordinate chart on a neighbourhood of $[z_e]$, then they remain valid under any other such coordinate chart, possibly with modified values of the constants λ and c. Furthermore, the same observations partially extend to the setting of Theorem 3.1.12. In particular, the requirement involving the time derivative of H_{z_e} is intrinsically defined and thus independent of the choice of coordinates on P_{μ_e}. In contrast, the remaining conditions in Corollary 3.1.13 require a more detailed analysis.

Lemma 3.1.14. *If the t-dependent matrix $M(t)$, which is defined in a local coordinate system $\{x_1, \ldots, x_n\}$ on an open neighbourhood of an equilibrium point $[z_e] \in P_{\mu_e}$, satisfies that $0 < \lambda < \inf_{t \in I_{t^0}} \min \operatorname{spec} M(t)$ for some λ (resp. $\sup_{t \in I_{t^0}} \max \operatorname{spec} M(t) < \Lambda$ for some Λ), then $M_{\mathcal{B}'}(t)$, defined as $M(t)$ but in another coordinate system $\mathcal{B}' := \{\tilde{x}_1, \ldots, \tilde{x}_n\}$ on another neighbourhood in P_{μ_e} of $[z_e]$, satisfies that $0 < \lambda' < \inf_{t \in I_{t^0}} \min \operatorname{spec} M_{\mathcal{B}'}(t)$ for some λ' (resp. $\sup_{t \in I_{t^0}} \max \operatorname{spec} M_{\mathcal{B}'}(t) < \Lambda'$ for some Λ').*

Proof. Since every symmetric matrix can be orthogonally diagonalised via a t-dependent orthogonal matrix O_t, the condition for $M(t)$ amounts to the fact that

$$v^T M(t) v = v^T O_t^T D(t) O_t v > \lambda v^T O_t^T O_t v = \lambda v^T v, \qquad \forall v \in \mathbb{R}^n \backslash \{0\}, \qquad \forall t \in I_{t^0},$$

where $D(t)$ is a diagonal matrix consisting the eigenvalues of $M(t)$.

Since $[z_e]$ is an equilibrium point of H_{z_e}, there exists an invertible $n \times n$ time-independent matrix A such that

$$M_{\mathcal{B}'}(t) = A^T M(t) A, \qquad \forall t \in \mathbb{R}.$$

Hence,

$$v^T M_{\mathcal{B}'}(t) v = (Av)^T M(t) Av > \lambda (Av)^T Av, \qquad \forall v \in \mathbb{R}^n \backslash \{0\}, \qquad \forall t \in I_{t^0}.$$

Since A is invertible, the positive function $f\colon v \in \mathbb{S}^{n-1} \mapsto (Av)^T (Av) \in \mathbb{R}$ on the ball $\mathbb{S}^{n-1} = \{v \in \mathbb{R}^n \mid \|v\| := \sqrt{v^T v} = 1\}$, which is compact, admits a maximum and a minimum $M_S, m_S > 0$, respectively. Then, $(Av)^T (Av) \geq m_S v^T v$ for every $v \in \mathbb{R}^n$. Thus,

$$v^T M_{\mathcal{B}'}(t) v > \lambda m_S v^T v, \qquad \forall v \in \mathbb{R}^n \backslash \{0\}, \qquad \forall t \in I_{t^0}.$$

Similarly, since $(Av)^T (Av) \leq M_S v^T v$ for every $v \in \mathbb{R}^n$, then, the existence of Λ implies

$$v^T M_{\mathcal{B}'}(t) v < \Lambda M_S v^T v, \qquad \forall v \in \mathbb{R}^n \backslash \{0\}, \qquad \forall t \in I_{t^0}.$$

Therefore, choosing $\lambda' = \lambda m_S$ and $\Lambda' = \Lambda M_S$, the lemma follows. ☐

Note that the condition for c in Corollary 3.1.13 is independent of the particular choice of a coordinate system. The same condition holds after possibly modifying the constant to some $c' > 0$ and restricting attention to a smaller open neighbourhood of $[z_e]$ where both the original and the new coordinate systems are defined.

3.1.5 Stability, reduced symplectic manifold, and relative equilibrium points

The energy-momentum method aims to determine conditions on the Hamiltonian function h in a neighbourhood of a relative equilibrium point $z_e \in P$ that ensure a particular type of stability for the corresponding equilibrium point in the reduced phase space P_{μ_e}, associated with the reduced Hamiltonian system k_{μ_e}. In particular, the conditions are provided on the family of functions

$$h^t_{\mu_e} \colon z \in \mathbf{J}^{\Phi-1}(\mu_e) \mapsto h(t, z) \in \mathbb{R},$$

and $\partial h^t_{\mu_e}/\partial t$ with $t \in I_{t^0}$, such that the hypothesis of Theorem 3.1.12 and/or Corollary 3.1.13 are satisfied. Rather than examining the family of matrices $M(t)$, the analysis is based on conditions formulated in terms of the functions $h_{\xi,t}$ for $t \in I_{t^0}$, which is more practical since these functions are defined on the original manifold P rather than on a quotient space.

The ideas used in the proofs of Proposition 3.1.15 and Corollary 3.1.16 below are a generalisation of the t-independent formulation of the classical energy-momentum method developed in [113]. First, define

$$(\delta^2 f)(X,Y) := \iota_Y \mathrm{d}(\iota_X \mathrm{d}f)\,, \tag{3.1.4}$$

for every $X, Y \in \mathfrak{X}(P)$ and $f \in \mathscr{C}^\infty(P)$. If $\mathrm{d}f_p = 0$ for a certain $p \in P$, then $[\delta^2 f(X,Y)](p)$ depends only on the values of X, Y at p and gives rise to a well-defined bilinear map on T_pP of the form

$$(\delta^2 f)_p(v,w) := (\iota_Y \mathrm{d}(\iota_X \mathrm{d}f))(p), \qquad \forall v, w \in \mathrm{T}_pP,$$

for $X, Y \in \mathfrak{X}(P)$ such that $X(p) = v$ and $Y(p) = w$. Moreover, $(\delta^2 f)_p$ becomes a symmetric bilinear form.

Proposition 3.1.15. *Let $z_e \in P$ be a relative equilibrium point for $(P, \omega, h, \Phi, \mathbf{J}^\Phi)$. Then,*

$$(\delta^2 h_{\xi,t})_{z_e}((\eta_P)_{z_e}, v_{z_e}) = 0, \quad \forall \eta \in \mathfrak{g}, \quad \forall v_{z_e} \in \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\mu_e), \quad \forall t \in \mathbb{R}\,.$$

Proof. The G-invariance of $h\colon \mathbb{R} \times P \to \mathbb{R}$, together with the Ad^*-equivariance of the symplectic momentum map $\mathbf{J}^\Phi\colon P \to \mathfrak{g}^*$, imply

$$h_{\xi,t}(gp) = h(t,gp) - \langle \mathbf{J}^\Phi(gp), \xi(t)\rangle + \langle \mu_e, \xi(t)\rangle = h(t,p) - \langle \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}^\Phi(p)), \xi(t)\rangle + \langle \mu_e, \xi(t)\rangle$$

and

$$h_{\xi,t}(gp) = h(t,p) - \langle \mathbf{J}^\Phi(p), \mathrm{Ad}_{g^{-1}}(\xi(t))\rangle + \langle \mu_e, \xi(t)\rangle,$$

for any $g \in G$ and $p \in P$. Substituting $g := \exp(s\eta)$, for some $\eta \in \mathfrak{g}$, and differentiating with respect to the parameter s, one obtains

$$(\iota_{\eta_P}\mathrm{d}h_{\xi,t})(p) = -\left\langle \mathbf{J}^\Phi(p), \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathrm{Ad}_{\exp(-s\eta)}(\xi(t)) \right\rangle = \langle \mathbf{J}^\Phi(p), [\eta, \xi(t)]\rangle.$$

Taking variations relative to $p \in P$, evaluating at z_e, and using that $(\mathrm{d}h_{\xi,t})_{z_e} = 0$ since $z_e \in P$ is a critical point of $h_{\xi,t}$, it follows that

$$(\delta^2 h_{\xi,t})_{z_e}((\eta_P)_{z_e}, v_{z_e}) = \langle \mathrm{T}_{z_e}\mathbf{J}^\Phi(v_{z_e}), [\eta, \xi(t)]\rangle\,.$$

This vanishes if $\mathrm{T}_{z_e}\mathbf{J}^\Phi(v_{z_e}) = 0$, i.e. when $v_{z_e} \in \ker \mathrm{T}_{z_e}\mathbf{J}^\Phi = \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\mu_e)$. □

Proposition 3.1.15 and Lemma 2.1.7 yield the following.

Corollary 3.1.16. *The mapping $(\delta^2 h_{\xi,t})_{z_e}$ vanishes identically on $\mathrm{T}_{z_e}(G_{\mu_e} z_e)$ for every $t \in \mathbb{R}$.*

Proof. By Lemma 2.1.7 one has that $\mathrm{T}_{z_e}(G_{\mu_e} z_e) = \mathrm{T}_{z_e}(G z_e) \cap \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\mu_e)$. Since $\mathrm{T}_{z_e}(G_{\mu_e} z_e) \subset \mathrm{T}_{z_e}(G z_e)$, the result follows from (3.1.15) by taking $v_{z_e} := (\xi_P)_{z_e}$ for some $\xi \in \mathfrak{g}_{\mu_e}$. □

Consider the particular case of the presented theory, where there is no time dependence. Then $h_{\xi,t}$ becomes just h_ξ. By Corollary 3.1.16, the *formal stability* of a symplectic relative equilibrium point requires *positive definiteness* of the second variation $(\delta^2 h_\xi)_{z_e}$ on $\mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\mu_e)$ modulo the so-called *gauge directions*

$$\mathrm{T}_{z_e}(G_{\mu_e} z_e) = \{(\eta_P)_{z_e} \mid \eta \in \mathfrak{g}_{\mu_e}\}\,.$$

In summary, the formal stability of a symplectic relative equilibrium point in an autonomous setting is equivalent to

$$(\delta^2 h_\xi)_{z_e}(v,v) > 0, \qquad \forall v \in \mathcal{S}\backslash\{0\}\,,$$

for some subspace $\mathcal{S} \subset \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\mu_e)$ supplementary to $\mathrm{T}_{z_e}(G_{\mu_e}z_e)$.

The fact that the definiteness of the second variation is to be examined restricted to the quotient space $\mathcal{S}$ is an essential aspect of the energy-momentum method. This is justified by the standard test for constrained optimisation problems together with Corollary 3.1.16. The type of stability one gets in P_{μ_e} is time-independent Lyapunov stability, while in P it is orbital stability of the symplectic relative equilibrium orbit $\Phi(\exp(t\xi), z_e)$, see [113] for more details. The step-by-step verification of a relative equilibrium point $z_e \in P$ together with its formal stability, proceeds as follows:

1. Momentum map - Compute a symplectic momentum map $\mathbf{J}^\Phi : P \to \mathfrak{g}^*$ associated with a symplectic Lie group action $\Phi \colon G \times P \to P$.

2. First variation - Define $h_\xi := h - [J^\Phi_\xi - \langle \mu_e,\ \xi\rangle]$ and determine $z_e \in P$ and $\xi \in \mathfrak{g}$ such that
$$\langle (\mathrm{d}h_\xi)_{z_e}, v_{z_e}\rangle = 0\,, \qquad \mathbf{J}^\Phi(z_e) - \mu_e = 0\,, \qquad \forall v_{z_e} \in \mathrm{T}_{z_e}P\,.$$

3. Admissible variations for second variation test - Choose a linear subspace $\mathcal{S} \subset \mathrm{T}_{z_e}P$ such that
$$\mathrm{T}_{z_e}\mathbf{J}^\Phi(v_{z_e}) = 0,\ \forall v_{z_e} \in \mathcal{S}, \quad \mathcal{S} \oplus \mathrm{T}_{z_e}(G_{\mu_e}z_e) = \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\mu_e)\,.$$

4. Check $\delta^2 h_\xi$ for definiteness on $\mathcal{S}$, namely
$$(\delta^2 h_\xi)_{z_e}(v,v) > 0\,,$$
for all $v \in \mathcal{S}\setminus\{0\}$. The positivity of the second variation on $\mathcal{S}$ implies the formal stability of the relative equilibrium point $z_e \in P$.

The above procedure summarises the classical energy–momentum method originally established in [113]. In particular, the time-independent case of the presented time-dependent energy-momentum method recovers these classical results as a direct consequence.

Now, continuing with the non-autonomous symplectic Hamiltonian systems, recall that it is assumed that G_{μ_e} acts freely and properly on $\mathbf{J}^{\Phi-1}(\mu_e)$. Consider a set of coordinates $\{z_1,\ldots,z_q\}$ on an open subset $\mathcal{A} \subset \mathbf{J}^{\Phi-1}(\mu_e)$ containing z_e. Let $\{\pi^*_{\mu_e}x_1,\ldots,\pi^*_{\mu_e}x_n\}$ be the coordinates on $\mathcal{A}$ given by the pullback to $\mathcal{A}$ of certain coordinates $\{x_1\ldots,x_n\}$ on $\mathcal{O} := \pi_{\mu_e}(\mathcal{A})$ [1] and let $\{y_1,\ldots,y_s\}$ be additional coordinates giving rise to a coordinate system $\{z_1,\ldots,z_q\}$ on $\mathcal{A}$. Due to the G_{μ_e}-invariance of $h_{\mu_e} := h\circ i_{\mu_e}\colon \mathbf{J}^{\Phi-1}(\mu_e) \to \mathbb{R}$, where $i_{\mu_e}\colon \mathbf{J}^{\Phi-1}(\mu_e) \to \mathbf{J}^{\Phi-1}(\mu_e)/G_{\mu_e}$ is the natural embedding, one has that there exists a constant $c > 0$ such that
$$c \geq \frac{1}{3!}\max_{3=|\vartheta|}\sup_{z\in\mathcal{A}}|D^\vartheta h_{\mu_e}(t,y)|, \qquad \forall t \in I_{t^0},$$
where ϑ is a multi-index $\vartheta := (\vartheta_1,\ldots,\vartheta_q)$, if and only if
$$c \geq \frac{1}{3!}\max_{3=|\alpha|}\sup_{x\in\mathcal{O}}|D^\alpha H_{z_e}(t,x)|, \qquad \forall t \in I_{t^0}\,, \tag{3.1.5}$$
for $\mathcal{O}$, which is an open neighbourhood of $[z_e]$ since π_{μ_e} is an open map. Indeed, since h_{μ_e} is constant on the submanifolds where $x_1,\ldots,x_n$ take constant values, it follows that $h_{\mu_e}(t,x_1,\ldots,x_n,y_1,\ldots,y_s) - h(t,z_e) = H_{z_e}(t,x_1,\ldots,x_n)$. Consequently, Equation (3.1.5) holds.

Consider again the coordinate system $\{z_1,\ldots,z_q\}$ on $\mathbf{J}^{\Phi-1}(\mu_e)$. Define $[\widehat{M}(t)]$ as the t-dependent $q\times q$ matrix of the form
$$[\widehat{M}(t)]^j_i := \frac{\partial^2 h_{\mu_e}}{\partial z_i \partial z_j}(t,z_e), \qquad i,j = 1,\ldots,q\,.$$

---

[1] To simplify the notation, $\{x_1,\ldots,x_n\}$ denotes a set of coordinates on a certain neighbourhood of $[z_e]$ and their pullbacks to $\mathbf{J}^{\Phi-1}(\mu_e)$ via π_{μ_e}.

By Lemma 3.1.14, the existence of constants $\lambda > 0$ and $\Lambda > 0$ is equivalent, geometrically, to the condition that the t-dependent bilinear symmetric form $K(t)\colon \mathrm{T}_{[z_e]}P_{\mu_e} \times \mathrm{T}_{[z_e]}P_{\mu_e} \to \mathbb{R}$ defined as

$$K(t) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e])\mathrm{d}x_i|_{[z_e]} \otimes \mathrm{d}x_j|_{[z_e]}\,,$$

satisfies that

$$K(t)(w, w) > \lambda(w|w)_{\mathcal{B}}, \qquad \forall w \in T_{[z_e]}P_{\mu_e}\backslash\{0\}, \quad \forall t \in I_{t^0}, \tag{3.1.6}$$

where $(\cdot|\cdot)_{\mathcal{B}}$ is the Euclidean inner product on $\mathrm{T}_{[z_e]}P_{\mu_e}$, for which the basis $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$ is orthonormal. Indeed, if v is the column vector describing the coordinates of $w \in \mathrm{T}_{[z_e]}P_{\mu_e}$ in the chosen orthonormal basis, then condition (3.1.6) can be rewritten as

$$K(t)(w, w) = v^T M(t) v > \lambda v^T v = \lambda(w|w)_{\mathcal{B}}, \qquad \forall w \in \mathrm{T}_{[z_e]}P_{\mu_e}\backslash\{0\}, \quad \forall t \in I_{t^0}.$$

Note that this condition is independent of the choice of inner product. Namely, for any other inner product $(\cdot|\cdot)_{\mathcal{B}'}$ on $\mathrm{T}_{[z_e]}P_{\mu_e}$, there exists $m_i, m_s > 0$ such that $m_s(w|w)_{\mathcal{B}'} \geq (w|w)_{\mathcal{B}} \geq m_i(w|w)_{\mathcal{B}'}$ for all $w \in \mathrm{T}_{[z_e]}P_{\mu_e}$. Consequently, if condition (3.1.6) is satisfied for some inner product in $\mathrm{T}_{[z_e]}P_{\mu_e}$, then it is also satisfied for any other inner product in $\mathrm{T}_{[z_e]}P_{\mu_e}$ with another positive λ. An analogous argument applies to the relation $\Lambda(w|w)_{\mathcal{B}} > K(t)(w, w)$ for some $\Lambda > 0$, for all $t \in I_{t^0}$ and every $w \in \mathrm{T}_{[z_e]}P_{\mu_e}\backslash\{0\}$.

The introduction of the inner product $(\cdot|\cdot)_{\mathcal{B}}$ is motivated both theoretically and practically. From a computational perspective, in order to verify whether the eigenvalues of the t-dependent matrix $M(t)$ can be bounded from below simultaneously for every time $t \in I_{t^0}$, it is convenient to use the eigenvalues of the matrix representation of $K(t)$ and $(\cdot|\cdot)_{\mathcal{B}}$, which are geometric objects. Since the choice of inner product on the finite-dimensional vector space $\mathrm{T}_{[z_e]}P_{\mu_e}$ is arbitrary, choosing $(\cdot|\cdot)_{\mathcal{B}}$ simplifies the verification of the condition.

Condition (3.1.6) can be verified through an object defined directly on the level set $\mathbf{J}^{\Phi-1}(\mu_e)$. Since h_{μ_e} admits a critical point at each relative equilibrium point $z_e \in \mathbf{J}^{\Phi-1}(\mu_e)$, one can define a t-dependent symmetric bilinear form $\widehat{M}(t)\colon \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\mu_e) \times \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\mu_e) \to \mathbb{R}$, as follows

$$\widehat{M}(t) := \frac{1}{2}\sum_{i,j=1}^{q} \frac{\partial^2 h_{\mu_e}}{\partial z_i \partial z_j}(t, z_e)\mathrm{d}z_i|_{z_e} \otimes \mathrm{d}z_j|_{z_e}, \qquad \forall t \in I_{t^0},$$

where $\mathcal{B} = \{z_1, \ldots, z_q\}$ is any coordinate system in an open neighbourhood of $z_e \in \mathbf{J}^{\Phi-1}(\mu_e)$.

Consider the coordinate system $\{x_1, \ldots, x_n, y_1, \ldots, y_s\}$ on the open neighbourhood z_e in $\mathbf{J}^{\Phi-1}(\mu_e)$ defined above. In this coordinate system, one has

$$\frac{\partial^2 h_{\mu_e}}{\partial x_k \partial y_j}(t, z_e) = \frac{\partial^2 h_{\mu_e}}{\partial y_i \partial y_j}(t, z_e) = 0, \qquad i, j = 1, \ldots, s, \qquad k = 1, \ldots, n, \qquad \forall t \in \mathbb{R}.$$

In the chosen coordinate system, one has $\pi^*_{\mu_e} K(t) = \widehat{M}(t)$ and

$$\mathrm{T}_{z_e}(G_{\mu_e} z_e) \subset \ker \widehat{M}(t) \qquad \forall t \in \mathbb{R}\,.$$

This inclusion holds for any other coordinate system as well. Consequently, the bilinear form $K(t)$ can be considered as the induced bilinear form by $\widehat{M}(t)$ on the quotient space

$$S_{z_e} := \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\mu_e)/\mathrm{T}_{z_e}(G_{\mu_e} z_e) \simeq \mathrm{T}_{[z_e]}P_{\mu_e}\,.$$

Therefore, the conditions imposed on $M(t)$ can be equivalently verified through the bilinear form $\widehat{M}(t)$ defined on $\mathbf{J}^{\Phi-1}(\mu_e)$. Furthermore, note that if the dimension of $\ker \widehat{M}(t)$ is greater than $\dim \mathrm{T}_{z_e}(G_{\mu_e} \cdot z_e)$, then the hypothesis of Lemma 3.1.11 do not hold.

Corollary 3.1.13 together with the previous remarks yield the following theorem.

Theorem 3.1.17. *Assume that there exist* $\lambda, c > 0$ *and an open coordinate neighbourhood* $\mathcal{A} \subset \mathbf{J}^{\Phi-1}(\mu_e)$ *of* z_e *so that*

$$\lambda < \min(\operatorname{spec}([\widehat{M}(t)]|_{S_{z_e}}), \quad c \geq \frac{1}{3!} \max_{1\leq|\vartheta|\leq 3} \sup_{y\in\mathcal{A}} |D^{\vartheta} h_{\mu_e}(t,y)|, \quad \left.\frac{\partial h_{\mu_e}}{\partial t}\right|_{\mathcal{A}} \leq 0, \tag{3.1.7}$$

for every $t \in I_{t^0}$. *Then,* $[z_e] \in P_{\mu_e}$ *is a uniformly stable equilibrium point of the Hamiltonian system* k_{μ_e} *on* $\mathbf{J}^{\Phi-1}(\mu_e)/G_{\mu_e}$ *from* t^0.

Recall that in the case of an autonomous Hamiltonian, the third condition in (3.1.7) is immediately satisfied. Moreover, still in the case of autonomous systems, if h is sufficiently smooth, there always exists a constant c and a suitable open neighbourhood $\mathcal{A}$ of z_e such that the second condition in (3.1.7) holds. Finally, the condition on λ reduces to the standard requirement on the positiveness of the eigenvalues of the matrix $\widehat{M}$, which is time-independent by assumption, up to the subspaces on which it vanishes identically due to Corollary 3.1.16 (cf. [113]).

In the non-autonomous case, the second condition in (3.1.7) can also be readily verified for sufficiently smooth functions h whose spatial partial derivatives remain uniformly bounded in time; thus, this requirement is generally easy to satisfy.

To relate the properties of $h_{\xi,t}$ with H_{μ_e} for the study of relative equilibrium points and their associated equilibria in P_{μ_e}, observe that $h_{\xi,t}$ has a critical point at each relative equilibrium point $z_e \in P$ for every $t \in \mathbb{R}$. Hence, one can define the t-dependent bilinear symmetric form on $\mathrm{T}_{z_e}P$ given by

$$T_{z_e}(t) := \frac{1}{2}\sum_{i,j=1}^{\chi} \frac{\partial^2 h_{\xi,t}}{\partial u_i \partial u_j}(t, z_e)\mathrm{d}u_i|_{z_e} \otimes \mathrm{d}u_j|_{z_e}, \qquad \forall t \in \mathbb{R},$$

where $u_1, \ldots, u_\chi$, with $\chi = \dim P$, is a coordinate system on an open neighbourhood of z_e in P. The objective is to determine the relation between $T_{z_e}(t)$ and the matrix $\widehat{M}(t)$, so that $\widehat{M}(t)$ can be studied via $T_{z_e}(t)$. Importantly, $T_{z_e}(t)$ is a geometric object that is straightforward to construct, being defined directly on $\mathrm{T}_{z_e}P$ and depending only on h and $\mathbf{J}^{\Phi}$.

Suppose that $\mathbf{J}^{\Phi}(z_e) = \mu_e$ is a regular value of a symplectic momentum map $\mathbf{J}^{\Phi}\colon P \to \mathfrak{g}^*$. Then its coordinate functions $\mu_1, \ldots, \mu_r$ form $\dim \mathfrak{g}$ functionally independent functions on P. Consider a coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_s)$ on a neighbourhood of z_e in $\mathbf{J}^{\Phi-1}(\mu_e)$, as previously defined. These functions can be smoothly extended to an open neighbourhood of z_e in P. By regularity of $\mathbf{J}^{\Phi}$, the differentials $\mathrm{d}\mu_1, \ldots, \mathrm{d}\mu_r$ are linearly independent at z_e, so that $(x_1, \ldots, x_n, y_1, \ldots, y_s, \mu_1, \ldots, \mu_r)$ forms a local coordinate system on P around z_e.

Taking this into account, one obtains that

$$\left.\frac{\partial h_t}{\partial y_i}\right|_{\mathbf{J}^{\Phi-1}(\mu_e)} = 0, \qquad \frac{\partial\langle \mathbf{J}^{\Phi} - \mu_e,\ \xi(t)\rangle}{\partial y_i} = 0, \qquad \forall t \in \mathbb{R}, \quad i = 1, \ldots, s\,.$$

It is relevant to recall that the derivatives $\partial h_t/\partial y_i$, with $i = 1, \ldots, s$, do not need to vanish away from $\mathbf{J}^{\Phi-1}(\mu_e)$, since $y_1, \ldots, y_s$ were defined as a smooth extension beyond $\mathbf{J}^{\Phi-1}(\mu_e)$ without imposing any specific properties away of $\mathbf{J}^{\Phi-1}(\mu_e)$. Furthermore,

$$\left.\left(\frac{\partial}{\partial y_j}\frac{\partial h_t}{\partial y_i}\right)\right|_{\mathbf{J}^{\Phi-1}(\mu_e)} = 0, \qquad \left.\left(\frac{\partial}{\partial x_k}\frac{\partial h_t}{\partial y_i}\right)\right|_{\mathbf{J}^{\Phi-1}(\mu_e)} = 0\,,$$

$$\frac{\partial}{\partial y_j}\frac{\partial\langle \mathbf{J}^{\Phi} - \mu_e,\ \xi(t)\rangle}{\partial y_i} = 0, \qquad \frac{\partial}{\partial x_k}\frac{\partial\langle \mathbf{J}^{\Phi} - \mu_e,\ \xi(t)\rangle}{\partial y_i} = 0\,,$$

for all $t \in \mathbb{R}$, $i, j = 1, \ldots, s$, and $k = 1, \ldots, n$. Note that the first and second identities above hold because, at points of $\mathbf{J}^{\Phi-1}(\mu_e)$, the derivative on the left depends only on the values of $\partial h_t/\partial y_i$ restricted to $\mathbf{J}^{\Phi-1}(\mu_e)$.

Consequently, in this chosen coordinate system, the Hessian of $h_{\xi,t}$ at z_e, denoted $\mathbf{H}h_{\xi,t}$, coincides with $\widehat{M}(t)$ when restricted to $\mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\mu_e)$. This is the crucial point: the function $h_{\xi,t}$ can be used to study the matrices $\widehat{M}(t)$ and $M(t)$. Note that, in general, h does not have a critical point at z_e, so its Hessian at z_e does not directly define a bilinear symmetric form; however, in the chosen coordinate system, it reproduces the matrix of $T_{z_e}(t)$.

The reasoning presented above serves as a conceptual template for the subsequent sections, which adapt these ideas to other geometric structures such as cosymplectic and k-polysymplectic. The construction becomes significantly more technical, but the core idea remains the same.

3.1.6 Example: The almost-rigid body

In this subsection, the symplectic time-dependent energy-momentum method is illustrated via a generalisation of the classical freely spinning rigid body studied in [113]. The goal is to determine relative equilibrium points and analyse the second-order variation of the extended Hamiltonian $h_{\xi,t}$, generalising the autonomous results in [113]. The main results are expressed in (3.1.12) and (3.1.13).

Let $t^0 = 0$ and let SO_3 denote the Lie group of orthogonal unimodular linear automorphisms of $\mathbb{R}^3$, with the Lie algebra $\mathfrak{so}_3$ identified with $\mathbb{R}^3$ via the standard isomorphism

$$\phi\colon \mathbb{R}^3 \to \mathfrak{so}_3,\ \omega \mapsto \widehat{\omega} := \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix},$$

where $\omega := (\omega^1, \omega^2, \omega^3)^T$.

With the vector product "$\times$" in $\mathbb{R}^3$, one has $\widehat{\omega}\mathbf{r} = \omega \times \mathbf{r}$, $[\widehat{\omega}, \widehat{\Theta}] = \widehat{\omega \times \Theta}$, and $\Lambda\widehat{\Theta}\Lambda^T = \widehat{\Lambda\Theta}$ for every $\Lambda \in \mathrm{SO}_3$, and every $\Theta, \omega \in \mathbb{R}^3$. Thus, ϕ is a Lie algebra isomorphism between $\mathbb{R}^3$, which is a Lie algebra relative to the vector product and $\mathfrak{so}_3$ with the commutator of matrices.

The adjoint action $\mathrm{Ad}\colon \mathrm{SO}_3 \times \mathfrak{so}_3 \to \mathfrak{so}_3$, defined geometrically in (1.2.2), reduces to the expression $\mathrm{Ad}_\Lambda\widehat{\Theta} = \Lambda\widehat{\Theta}\Lambda^T$, as $\Lambda^{-1} = \Lambda^T$, for all $\Lambda \in \mathrm{SO}_3$ and $\Theta \in \mathbb{R}^3$. Moreover,

$$\widehat{\Lambda(\mathbf{r}{\times}\mathbf{s})} = \Lambda\widehat{\mathbf{r}{\times}\mathbf{s}}\Lambda^T = \Lambda[\widehat{\mathbf{r}}, \widehat{\mathbf{s}}]\Lambda^T = [\Lambda\widehat{\mathbf{r}}\Lambda^T, \Lambda\widehat{\mathbf{s}}\Lambda^T] = [\widehat{\Lambda\mathbf{r}}, \widehat{\Lambda\mathbf{s}}] = \widehat{\Lambda\mathbf{r}{\times}\Lambda\mathbf{s}}, \qquad \forall \mathbf{r}, \mathbf{s} \in \mathbb{R}^3.$$

One can identify $\mathrm{T}_\Lambda\mathrm{SO}_3$ with $\mathfrak{so}_3$ via two isomorphisms. Recall that $L_\Lambda\colon \Theta \in \mathrm{SO}_3 \mapsto \Lambda\Theta \in \mathrm{SO}_3$ and $R_\Lambda\colon \Theta \in \mathrm{SO}_3 \mapsto \Theta\Lambda \in \mathrm{SO}_3$ are diffeomorphisms for every $\Lambda \in \mathrm{SO}_3$. Then, $\mathrm{T}_{\mathrm{Id}_3}L_\Lambda\colon \mathrm{T}_{\mathrm{Id}_3}\mathrm{SO}_3 \simeq \mathfrak{so}_3 \mapsto \mathrm{T}_\Lambda\mathrm{SO}_3$ and $\mathrm{T}_{\mathrm{Id}_3}R_\Lambda\colon \mathrm{T}_{\mathrm{Id}_3}\mathrm{SO}_3 \simeq \mathfrak{so}_3 \mapsto \mathrm{T}_\Lambda\mathrm{SO}_3$ are isomorphisms, where Id_3 is 3×3 identity matrix.

Then, for every $\Theta \in \mathbb{R}^3$, the *left-invariant extension* of $\widehat{\Theta}$ is defined by $(\mathrm{T}_{\mathrm{Id}_3}L_\Lambda)\widehat{\Theta} =: (\Lambda, \Lambda\widehat{\Theta})$, for every $\Theta \in \mathbb{R}^3$. Meanwhile, the *right-invariant extension* of $\widehat{\theta}$ is defined as $(\mathrm{T}_{\mathrm{Id}_3}R_\Lambda)\widehat{\theta} := (\Lambda, \widehat{\theta}\Lambda)$, for every $\theta \in \mathbb{R}^3$. When the base point Λ is clear from the context, the notation $\Lambda\widehat{\Theta}$ and $\widehat{\theta}\Lambda$ is used instead of $(\Lambda, \Lambda\widehat{\Theta})$ and $(\Lambda, \widehat{\theta}\Lambda)$, respectively. Since $\mathfrak{so}_3$ is a simple Lie algebra, its Killing metric, κ, is non-degenerate and induces an isomorphism

$$\widehat{\Theta} \in \mathfrak{so}_3 \mapsto \kappa(\widehat{\Theta}, \cdot) \in \mathfrak{so}_3^*\,.$$

The Killing form, up to a non-zero constant factor, is given by

$$\kappa(\widehat{\Theta}, \widehat{\omega}) = \frac{1}{2}\mathrm{tr}(\widehat{\Theta}^T\widehat{\omega})\,, \qquad \forall \Theta, \omega \in \mathbb{R}^3\,.$$

Moreover, one has

$$\Pi \cdot \Upsilon = \kappa(\widehat{\Pi}, \widehat{\Upsilon})\,, \qquad \Pi, \Upsilon \in \mathbb{R}^3\,.$$

where "$\cdot$" denotes the standard Euclidean scalar product in $\mathbb{R}^3$. Thus,

$$\langle \Lambda\widehat{\Pi}, \Lambda\widehat{\Theta} \rangle := \frac{1}{2}\mathrm{tr}((\Lambda\widehat{\Pi})^T\Lambda\widehat{\Theta}) = \frac{1}{2}\mathrm{tr}(\widehat{\Pi}^T\widehat{\Theta}) = \Pi \cdot \Theta, \quad \forall \Theta, \Pi \in \mathbb{R}^3\,,$$

and analogously

$$\langle \widehat{\Pi}\Lambda, \widehat{\Theta}\Lambda \rangle := \frac{1}{2}\mathrm{tr}((\widehat{\Pi}\Lambda)^T\widehat{\Theta}\Lambda) = \frac{1}{2}\mathrm{tr}(\widehat{\Pi}^T\widehat{\Theta}) = \Pi \cdot \Theta, \quad \forall \Theta, \Pi \in \mathbb{R}^3\,.$$

To simplify the notation, $\widehat{\Pi} \in \mathfrak{so}_3^*$ denotes $\kappa(\widehat{\Pi}, \cdot) \in \mathfrak{so}_3^*$ and elements of $\mathrm{T}^*_\Lambda \mathrm{SO}_3$ are written either as $(\Lambda, \widehat{\pi}\Lambda)$ or $(\Lambda, \Lambda\widehat{\Pi})$. If $(\Lambda, \widehat{\pi}\Lambda) = (\Lambda, \Lambda\widehat{\Pi})$, then $\widehat{\pi} = \Lambda\widehat{\Pi}\Lambda^T$, which matches the coadjoint action. Indeed,

$$\begin{aligned}\langle \mathrm{Ad}^*_{\Lambda^T}\widehat{\Pi}, \cdot\rangle = \frac{1}{2}\mathrm{tr}(\widehat{\Pi}^T \mathrm{Ad}_{\Lambda^T}(\cdot)) = \frac{1}{2}\mathrm{tr}(\widehat{\Pi}^T \Lambda^T(\cdot)\Lambda) \\ = \frac{1}{2}\mathrm{tr}(\Lambda\widehat{\Pi}^T\Lambda^T(\cdot)) = \frac{1}{2}\mathrm{tr}((\Lambda\widehat{\Pi}\Lambda^T)^T(\cdot)) = \langle \widehat{\pi}, \cdot\rangle \,.\end{aligned}$$

The mechanical framework is defined as follows. The configuration manifold is the Lie group SO_3, and the phase space is its cotangent bundle $\mathrm{T}^*\mathrm{SO}_3$, endowed with the canonical symplectic structure. Remarkably, this framework recovers, as a particular autonomous case, the classical dynamics of a rigid body in the absence of external forces.

Consider a time-dependent *Hamiltonian* $h\colon \mathbb{R} \times \mathrm{T}^*\mathrm{SO}_3 \to \mathbb{R}$ of the form

$$h(t, \Lambda, \widehat{\pi}) := \frac{1}{2}\pi \cdot \mathbb{I}_t^{-1}\pi \,, \qquad \mathbb{I}_t := \Lambda \mathbb{J}_t \Lambda^T \,, \tag{3.1.8}$$

where $\mathbb{I}_t$ is the *time-dependent inertia tensor* (in spatial coordinates) and $\mathbb{J}_t$ is the *inertia dyadic* given by

$$\mathbb{J}_t = \int_{\mathbb{R}^3} \varrho_\nu(t, X)[\|X\|^2\mathbb{1} - X \otimes X]\mathrm{d}^3X \,.$$

Here, $\varrho_\nu\colon \mathbb{R} \times \mathcal{B} \to \mathbb{R}$ denotes the time-dependent reference density. The inertia dyadic $\mathbb{J}_t$ is thus a matrix depending only on time, which at each $t \in \mathbb{R}$ yields the natural inertia tensor corresponding to the mass distribution. Indeed, it gives rise to a natural generalisation of its time-independent analogue, as introduced in [113]. The formalism for almost rigid bodies developed here is independent of the explicit form of $\mathbb{J}_t$.

The Hamiltonian function h given by (3.1.8) is interpreted as a function

$$h\colon \mathbb{R} \times \mathrm{SO}_3 \times \mathfrak{so}_3^* \to \mathbb{R} \,,$$

with $\mathfrak{so}_3^* \simeq \mathbb{R}^{3*}$ since it is more convenient for calculations. The Hamiltonian has the interpretation of the kinetic energy of the mechanical system, which is later on referred to as a *quasi-rigid body* (cf. [113]).

To study the invariance properties of the Hamiltonian function, recall that $\widehat{\pi} = \Lambda\widehat{\Pi}\Lambda^T$, then

$$\begin{aligned}h(t, \Lambda, \widehat{\pi}) = \frac{1}{4}\mathrm{tr}(\widehat{\pi}^T\Lambda\mathbb{J}_t^{-1}\Lambda^T\widehat{\pi}) = \frac{1}{4}\mathrm{tr}((\Lambda^T\widehat{\pi})^T\mathbb{J}_t^{-1}\Lambda^T\widehat{\pi}) = \\ \frac{1}{4}\mathrm{tr}((\widehat{\Pi}\Lambda^T)^T\mathbb{J}_t^{-1}\widehat{\Pi}\Lambda^T) = \frac{1}{4}\mathrm{tr}(\widehat{\Pi}^T\mathbb{J}_t^{-1}\widehat{\Pi}) = \frac{1}{2}\Pi \cdot \mathbb{J}_t^{-1}\Pi,\end{aligned}$$

which shows that h is *left invariant* under the action of SO_3. Consequently, the *left reduction by* SO_3 induces a function on the quotient $\mathbb{R} \times \mathrm{T}^*\mathrm{SO}_3/\mathrm{SO}_3 \simeq \mathbb{R} \times \mathfrak{so}_3^*$.

As a result, each h_t is a quadratic function of the momenta $\widehat{\pi}$. Choosing an appropriate coordinate system adapted to the $\mathbf{J}^{\Phi-1}(\mu)/\mathrm{SO}_3$ and an appropriate t-dependent dependence, the second condition in (3.1.7) is satisfied.

Consider the action of $G = \mathrm{SO}_3$ on $Q = \mathrm{SO}_3$ by left translations given by

$$\Psi\colon (A, \Lambda) \in G \times Q \longmapsto A\Lambda \in Q \,.$$

The induced *cotangent lift* of Ψ, denoted $\widehat{\Psi}$, also acts by left translations. Explicitly,

$$\widehat{\Psi}(\Lambda', (\Lambda, \widehat{\pi}\Lambda)) = (\Lambda'\Lambda, \widehat{\Lambda'\pi}\Lambda'\Lambda) \,, \qquad \forall \Lambda', \Lambda \in \mathrm{SO}_3 \,, \qquad \forall \pi \in (\mathbb{R}^3)^*$$

The momentum map associated with this action is defined as a map

$$\mathbf{J}^{\widehat{\Psi}}\colon \mathrm{SO}_3 \times \mathfrak{so}_3^* \to \mathfrak{so}_3^* \,,$$

where the identification of $\mathrm{T}^*\mathrm{SO}_3$ with $\mathrm{SO}_3 \times \mathfrak{so}_3^*$ was used via the right-translations R_Λ, with $\Lambda \in \mathrm{SO}_3$. Since

$$(\widehat{\xi}_{\mathfrak{so}_3})_\Lambda = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\widehat{\xi})\Lambda = \widehat{\xi}\Lambda\,,$$

for every $\xi \in \mathfrak{so}_3$, Proposition 2.1.3 yields

$$J^{\widehat{\Psi}}_{\xi}(\widehat{\pi}\Lambda) = \frac{1}{2}\mathrm{tr}[(\Lambda\widehat{\pi})^T\widehat{\xi}_{\mathfrak{so}_3}] = \frac{1}{2}\mathrm{tr}[\Lambda^T\widehat{\pi}^T\widehat{\xi}\Lambda] = \frac{1}{2}\mathrm{tr}[\widehat{\pi}^T\widehat{\xi}] = \pi \cdot \xi.$$

Thus,

$$\mathbf{J}^{\widehat{\Psi}}(\Lambda, \widehat{\pi}) = \widehat{\pi}\,, \qquad \text{and} \qquad J^{\widehat{\Psi}}_{\xi}(\widehat{\pi}\Lambda) = \pi \cdot \xi\,.$$

It follows that every $\widehat{\pi} \in \mathfrak{so}_3^*$ is a regular value of a symplectic momentum map $\mathbf{J}^{\widehat{\Psi}}$. Moreover, G_π consists of those elements of SO_3 that leave π invariant. Hence, $G_\pi \simeq \mathrm{SO}_2$ for $\pi \neq 0$ and $G_0 = \mathrm{SO}_3$. Furthermore,

$$\mathbf{J}^{\widehat{\Psi}-1}(\widehat{\pi}) = \mathrm{SO}_3 \times \{\widehat{\pi}\}\,, \qquad \forall \widehat{\pi} \in \mathfrak{so}_3^*\,.$$

Since each G_π is always compact, it acts properly on $\mathbf{J}^{\widehat{\Psi}-1}(\widehat{\pi})$. Moreover, the action of G_π on $\mathbf{J}^{\widehat{\Psi}-1}(\widehat{\pi})$ is always free. Therefore, the quotient $\mathbf{J}^{\widehat{\Psi}-1}(\widehat{\pi})/G_\pi$ is always a well-defined two-dimensional manifold, a sphere, for $\widehat{\pi} \neq 0$ and a zero-dimensional manifold for $\widehat{\pi} = 0$.

Consider the modified Hamiltonian function of the form

$$h_{\xi,t} = h_t - [J^{\widehat{\Psi}}_{\xi} - \pi_e \cdot \xi_t] = \frac{1}{2}\pi \cdot \mathbb{I}_t^{-1}\pi - \xi_t \cdot (\pi - \pi_e),$$

and study its critical points. To derive the first variation, it is appropriate to regard $h_{\xi,t}$ as a function of $(\Lambda, \pi) \in \mathrm{SO}_3 \times \mathfrak{so}_3^*$. Assume that $(\Lambda_e, \widehat{\pi}_e\Lambda_e) \in \mathrm{T}^*\mathrm{SO}_3$ is a relative equilibrium point. Then, for arbitrary $\delta\theta \in \mathbb{R}^3$, define a curve in SO_3 given by

$$\epsilon \mapsto \Lambda_\epsilon := \exp[\epsilon\widehat{\delta\theta}]\Lambda_e\,.$$

Similarly, for $\widehat{\delta\pi} \in \mathfrak{so}_3^*$ define a curve in $\mathfrak{so}_3^*$ as

$$\epsilon \mapsto \widehat{\pi}_\epsilon := \widehat{\pi}_e + \epsilon\widehat{\delta\pi} \in \mathfrak{so}_3^*\,.$$

These constructions induce a curve

$$\epsilon \mapsto (\Lambda_\epsilon, \widehat{\pi}_\epsilon\Lambda_\epsilon) \in \mathrm{T}^*\mathrm{SO}_3\,.$$

Consider $\delta h_{\xi,t} := \mathrm{d}h_{\xi,t}(\widehat{\delta\theta}, \widehat{\delta\pi})$. By applying the chain rule and introducing $\mathbb{I}_{t,\epsilon} := \Lambda_\epsilon\mathbb{J}_t\Lambda_\epsilon^T$, one gets

$$0 = \delta h_{\xi,t}\big|_e = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\left(\frac{1}{2}\pi_\epsilon \cdot \mathbb{I}_{t,\epsilon}^{-1}\pi_\epsilon - \xi_t \cdot (\pi_\epsilon - \pi_e)\right), \tag{3.1.9}$$

where $\mathbb{I}_{t,\epsilon}^{-1} := \Lambda_\epsilon\mathbb{J}_t^{-1}\Lambda_\epsilon^T$. Interpreting $h_{\xi,t}$ as a function on $\mathrm{T}^*\mathrm{SO}_3 \times \mathfrak{so}_3$, at the equilibrium point, the condition arising from the variation with respect to the Lagrange multiplier takes the form

$$(\pi - \pi_e) \cdot \eta = 0\,, \qquad \forall \eta \in \mathbb{R}^3\,.$$

Moreover,

$$\begin{aligned} \frac{1}{2}\pi_e \cdot \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\mathbb{I}_{t,\epsilon}^{-1}\pi_e &= \frac{1}{2}\pi_e \cdot [\widehat{\delta\theta}\mathbb{I}_{t,e}^{-1} - \mathbb{I}_{t,e}^{-1}\widehat{\delta\theta}]\pi_e \\ &= \frac{1}{2}[\pi_e \cdot (\delta\theta \times \mathbb{I}_{t,e}^{-1}\pi_e) - \mathbb{I}_{t,e}^{-1}\pi_e \cdot (\delta\theta \times \pi_e)] = \delta\theta \cdot (\mathbb{I}_{t,e}^{-1}\pi_e \times \pi_e)\,, \end{aligned} \tag{3.1.10}$$

where elementary vector product identities are used. By (3.1.10), expression (3.1.9) yields the following

$$\delta h_{\xi,t}\big|_e = \delta\pi \cdot [\mathbb{I}_{t,e}^{-1}\pi_e - \xi_t] + \delta\theta \cdot [\mathbb{I}_{t,e}^{-1}\pi_e \times \pi_e] = 0\,. \tag{3.1.11}$$

At critical points, expression (3.1.11) must vanish for every $\delta\pi$ and $\delta\theta$. Consequently, the following conditions are obtained

$$\mathbb{I}_{t,e}^{-1}\pi_e = \xi_t\,, \qquad \mathbb{I}_{t,e}^{-1}\pi_e \times \pi_e = 0\,.$$

Substituting the first condition into the second yields $\xi_t \times \pi_e = 0$, which implies that ξ_t and π_e are proportional. Thus, $\xi_t = \sigma_t \pi_t$ for a certain t-dependent function σ_t. Hence,

$$\xi_t \times \pi_e = 0\,, \qquad \mathbb{I}_{t,e}^{-1}\xi_t = \lambda_t \xi_t\,, \tag{3.1.12}$$

where $\lambda_t > 0$ due to the positive definiteness of $\mathbb{I}_{t,e}$. These conditions imply that π_e lies along a principal axis, that is, in the subspace spanned by an eigenvector of $\mathbb{I}_t$. Consequently, the rotation of the almost rigid body is around this axis.

To determine the stability, one has to study the second variation of $h_{\xi,t}$. By (3.1.11), at equilibrium point, one gets

$$(\delta^2 h_{\xi,t})\big|_e := \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \left[\delta\pi \cdot (\mathbb{I}_{t,\epsilon}^{-1}\pi_\epsilon - \xi_t) + \delta\theta \cdot (\mathbb{I}_{t,\epsilon}^{-1}\pi_\epsilon \times \pi_\epsilon)\right].$$

Note that the matrix of second-order derivatives is determined by its evaluation on pairs of equal tangent vectors. Proceeding analogously to the derivation of (3.1.11) and using (3.1.12), it follows that

$$(\delta^2 h_{\xi,t})\big|_e((\delta\pi,\delta\theta),(\delta\pi,\delta\theta)) = \; [\delta\pi^T \delta\theta^T] \begin{bmatrix} \mathbb{I}_{t,e}^{-1} & (\mathbb{I}_{t,e}^{-1} - \lambda_t \mathbb{1})\widehat{\pi}_e \\ -\widehat{\pi}_e(\mathbb{I}_{t,e}^{-1} - \lambda_t \mathbb{1}) & -\widehat{\pi}_e(\mathbb{I}_{t,e}^{-1} - \lambda_t \mathbb{1})\widehat{\pi}_e \end{bmatrix} \begin{bmatrix} \delta\pi \\ \delta\theta \end{bmatrix}. \tag{3.1.13}$$

Consider $(\delta\pi,\delta\theta) \in \mathbb{R}^{3*} \times \mathbb{R}^3$. Since $\mathbf{J}^{\widehat{\Psi}}(\widehat{\pi}\Lambda) = \widehat{\pi}$, it follows that $\mu_e = \widehat{\pi}_e$, and therefore $\mathrm{T}_{z_e}(G_{\mu_e} z_e)$ coincides with the infinitesimal generators of rotations about the axis determined by π_e. Consequently, distinct forms of $\mathbb{I}_{t,e}$ may be chosen such that the application of the above results guarantees stability of the reduced system at the projection of a relative equilibrium.

As a basic case, if $\mathbb{I}_{t,e}$ is independent of t, the stability criterion reduces to the classical analysis carried out in [113]. In this situation, the conditions involving third-order spatial derivatives of h_{μ_e}, as well as their partial time derivatives, are satisfied identically. More involved examples concern diagonal matrices $\mathbb{I}_{t,e}$ with positive nonincreasing eigenvalues which are properly bounded from below and, in some cases, also from above.

3.2 Cosymplectic energy-momentum method

This section develops the energy-momentum method in the setting of cosymplectic geometry, providing an alternative framework for the analysis of time-dependent Hamiltonian systems. The cosymplectic formulation significantly extends the scope of the classical energy-momentum method by allowing for a broader class of symmetries than those in the time-dependent symplectic approach. The applicability of this extension is demonstrated through certain important physical examples.

3.2.1 Cosymplectic relative equilibrium points

Assume that $\mu \in \mathfrak{g}^*$ is a weak regular value of a cosymplectic momentum map $\mathbf{J}^\Phi$. Furthermore, assume that the isotropy subgroup G_μ^Δ of μ relative to the cosymplectic affine Lie group action introduced in Proposition 2.2.8, acts on $\mathbf{J}^{\Phi-1}(\mu)$ through Φ in a quotientable manner, that is, the quotient $P_\mu^\Delta = \mathbf{J}^{\Phi-1}(\mu)/G_\mu^\Delta$ is a manifold and the canonical projection $\pi_\mu \colon \mathbf{J}^{\Phi-1}(\mu) \to P_\mu^\Delta$ is a submersion. Recall that the sufficient condition for $\mathbf{J}^{\Phi-1}(\mu)/G_\mu^\Delta$ to be a manifold is that G_μ^Δ acts freely and properly on $\mathbf{J}^{\Phi-1}(\mu)$, although weaker assumptions are admissible (cf. [2]). In the subsequent discussion, attention is restricted to cosymplectic manifolds of the form $(T \times P, \omega_P, \tau_T)$. For notational simplicity, the subscripts on the differential forms ω_P and τ_T are omitted.

Poincaré's notion of a relative equilibrium point for time-independent Hamiltonian systems (cf. [2, p 306]) is now extended to cosymplectic Hamiltonian systems. Multiple extensions of this concept are possible; an alternative formulation is introduced in Section 3.2.7.

Definition 3.2.1. A point $z_e \in P$ is a *cosymplectic relative equilibrium point* of $((T\times P)^{\omega}_{\tau}, h, \mathbf{J}^{\Phi})$ if there exists a curve $\xi(t) \in \mathfrak{g}$ so that

$$(X_h)_{(t,z_e)} = (\xi(t)_M)_{(t,z_e)}, \qquad \forall t \in T\,.$$

If $T = \mathbb{R}$, Definition 3.2.1 can be reformulated in terms of an integral curve of an evolution vector field. In fact, a point $z_e \in P$ is a cosymplectic relative equilibrium point of $((\mathbb{R}\times P)^{\omega}_{\tau}, h, \mathbf{J}^{\Phi})$ if, for each $t_0 \in \mathbb{R}$, there exists some curve $\xi_{t_0}(s)$ in $\mathfrak{g}$ such that

$$s \in \mathbb{R} \mapsto \Phi(\exp(\xi_{t_0}(s)), (t_0 + s, z_e)) \in \mathbb{R}\times P, \tag{3.2.1}$$

is the integral curve of E_h with initial condition (t_0, z_e). Equivalently, the trajectory $\Phi(\exp(\xi_{t_0}(t), z_e))$ is a solution of the Hamilton equations related to h, with initial condition z_e at $t = t_0$. This characterisation shows that the evolution for the Hamilton equations of h is given by the symmetries of the problem encoded in Φ.

An analogous statement holds for general T, although a local version of (3.2.1) must be used, since T may not admit a global coordinate system, for example, when $T = \mathbb{S}^1$. In what follows, unless otherwise stated, it is assumed that $t_0 = 0$, and the notation $\xi_{t_0=0}(t)$ will be abbreviated as $\xi(t)$.

Cosymplectic relative equilibrium points $z_e \in P$ giving rise to equilibria (t, z_e) of the Hamilton equations of $((T\times P)^{\omega}_{\tau}, h, \mathbf{J}^{\Phi})$ for all $t \in T$ are particular cases of relative equilibria. Indeed, in this case $(X_h)_{(t,z_e)} = 0$ for every $t \in T$, so (t, z_e) remains fixed under the dynamics generated by X_h, and consequently $\xi(t)$ in (3.2.1) can be taken to be identically zero.

Proposition 3.2.2. *Every integral curve, $m(t) = (t, z(t))$ of E_h with respect to $((T\times P)^{\omega}_{\tau}, h, \mathbf{J}^{\Phi})$ such that $z(t_0) = z_e$ for a cosymplectic relative equilibrium point $z_e \in P$ with $\mu_e = \mathbf{J}^{\Phi}(t_0, z_e)$ and $t_0 \in T$, projects onto the single point $(\pi_{P^{\Delta}_{\mu_e}} \circ \pi_{\mu_e})(t_0, z_e)$, i.e.*

$$(\pi_{P^{\Delta}_{\mu_e}} \circ \pi_{\mu_e})(m(t)) = (\pi_{P^{\Delta}_{\mu_e}} \circ \pi_{\mu_e})(t_0, z_e)\,,$$

for every $t \in T$, where $\pi_{\mu_e}\colon \mathbf{J}^{\Phi-1}(\mu_e) \to \mathbf{J}^{\Phi-1}(\mu_e)/G^{\Delta}_{\mu_e}$ and $\pi_{P^{\Delta}_{\mu_e}}\colon T\times P^{\Delta}_{\mu_e} \to P^{\Delta}_{\mu_e}$ are the canonical projections, see Corollary 2.2.14.

Proof. Proposition 2.2.10 ensures that every integral curve $m(t)$ of E_h is entirely contained within the the level set $\mathbf{J}^{\Phi-1}(\mu_e)$. By Proposition 2.2.15, such curve projects, via π_{μ_e}, onto a curve in $M^{\Delta}_{\mu_e} := \mathbf{J}^{\Phi-1}(\mu_e)/G^{\Delta}_{\mu_e} \simeq T\times P^{\Delta}_{\mu_e}$, where $G^{\Delta}_{\mu_e}$ denotes the isotropy subgroup of $\mu_e \in \mathfrak{g}^*$ relative to the cosymplectic affine Lie group action Δ.

Since $z_e \in P$ is a cosymplectic relative equilibrium point and $\mathbf{J}^{\Phi}$ is Δ-equivariant, it follows that

$$0 = \mathrm{T}_{(t,z_e)}\mathbf{J}^{\Phi}(E_h)_{(t,z_e)} = \mathrm{T}_{(t,z_e)}\mathbf{J}^{\Phi}(R + \xi(t)_M)_{(t,z_e)} = (\xi(t)^{\Delta}_{\mathfrak{g}^*})_{\mu_e}, \qquad \forall t \in T,$$

for some curve $\xi(t)$ in $\mathfrak{g}$. Consequently, the curve $\xi(t)$ is contained in $\mathfrak{g}^{\Delta}_{\mu_e}$.

Note that the curve $\pi_{\mu_e}(m(t))$ is the integral curve of the reduced vector field on $T\times P^{\Delta}_{\mu_e}$ of the form

$$R_{\mu_e} + Y_{\mu_e} := \pi_{\mu_e *}(E_h)\,,$$

where R_{μ_e} is the Reeb vector field associated with the reduced cosymplectic manifold $(M^{\Delta}_{\mu_e}, \omega_{\mu_e}, \tau_{\mu_e})$. Since $(X_h)_{(t,z_e)} = (\xi(t)_M)_{(t,z_e)}$, for a certain curve $\xi(t)$ in $\mathfrak{g}^{\Delta}_{\mu_e}$ and $\pi_{\mu_e *}R|_{\mathbf{J}^{\Phi-1}(\mu_e)} = R_{\mu_e}$, then

$$(Y_{\mu_e})_{\pi_{\mu_e}(m(t))} = (\mathrm{T}_{(t,z_e)}\pi_{\mu_e})(\xi(t)_M)_{(t,z_e)} = 0\,.$$

Hence, $\pi_{\mu_e}(m(t))$ consists of equilibrium points of Y_{μ_e}. The integral curve of the vector field Y_{μ_e} passing through $(\pi_{P^{\Delta}_{\mu_e}} \circ \pi_{\mu_e})(t_0, z_e)$ is just that point. Hence,

$$(\pi_{P^{\Delta}_{\mu_e}} \circ \pi_{\mu_e})(m(t)) = (\pi_{P^{\Delta}_{\mu_e}} \circ \pi_{\mu_e})(t_0, z_e)\,,$$

for every $t \in T$. Then, the projection of every solution passing through z_e is just the equilibrium point $(\pi_{P^{\Delta}_{\mu_e}} \circ \pi_{\mu_e})(t_0, z_e)$ of the Hamilton vector field Y_{μ_e} on $P^{\Delta}_{\mu_e}$. Equivalently, this point is an equilibrium point of the reduced Hamilton equations in $M^{\Delta}_{\mu_e}$. □

From Proposition 3.2.2 follows that $z_e \in \pi_P(\mathbf{J}^{\Phi-1}(\mu_e))$ is a cosymplectic relative equilibrium point of $((T\times P)^\omega_\tau, h, \mathbf{J}^\Phi)$ if and only if every solution to the Hamilton equations associated with h passing through z_e projects onto an equilibrium point in $P^\Delta_{\mu_e}$ of a reduced Hamiltonian system.

Cosymplectic relative equilibrium points of $((T\times P)^\omega_\tau, h, \mathbf{J}^\Phi)$ can be characterised via Lagrange multipliers as critical points of h restricted to $\mathbf{J}^{\Phi-1}(\mu_e)$, as done in Theorem 3.1.7 and initially for the classical energy-momentum method in [2, p 307] and [140].

Theorem 3.2.3. *A point $z_e \in P$ is a cosymplectic relative equilibrium point of $((T\times P)^\omega_\tau, h, \mathbf{J}^\Phi)$ if and only if there exists a curve $\xi(t)$ in $\mathfrak{g}$ such that, for every $t\in T$, the point z_e is a critical point of the restriction to $\{t\}\times P$ of the function $h_{\xi(t)}\colon T\times P \to \mathbb{R}$ of the form*

$$h_{\xi(t)}(t',z) := h(t',z) - \langle \mathbf{J}^\Phi(t',z) - \mathbf{J}^\Phi(t',z_e), \xi(t)\rangle\,.$$

Proof. Suppose $z_e \in P$ is a cosymplectic relative equilibrium point. Then, there exists a curve $\xi(t)$ in $\mathfrak{g}$ such that $(\xi(t)_M)_{(t,z_e)} = (X_h)_{(t,z_e)}$ for every $t\in T$. Due to the definition of the cosymplectic momentum map $\mathbf{J}^\Phi$, it turns out that

$$(\xi(t)_M\,)_{(t,z_e)} = (X_{J^\Phi_{\xi(t)}}\,)_{(t,z_e)}\,, \qquad \text{and} \qquad (X_{h-J^\Phi_{\xi(t)}})_{(t,z_e)} = 0\,,$$

for every $t\in T$. Since $J^\Phi_{\xi(t)}(t',z_e)$ is independent on t', it follows that

$$0 = [\flat(X_{h-J^\Phi_{\xi(t)}})]_{(t,z_e)} = (\mathrm{d}h_{\xi(t)})_{(t,z_e)} - (Rh_{\xi(t)})_{(t,z_e)}\tau_{(t,z_e)}, \qquad \forall t\in T.$$

Hence, $(\mathrm{d}h_{\xi(t)})_{(t,z_e)} \upharpoonright_{\ker \tau_{(t,z_e)}} = 0$, and therefore (t,z_e) is a critical point of $h_{\xi(t)} \upharpoonright_{\{t\}\times P}$ for every $t\in T$.

Conversely, assume that $(t,z_e)\in T\times P$ is a critical point of $h_{\xi(t)} \upharpoonright_{\{t\}\times P}$ for every $t\in T$. Then,

$$(\mathrm{d}h_{\xi(t)})_{(t,z_e)} \upharpoonright_{\ker \tau_{(t,z_e)}} = \mathrm{d}(h - J^\Phi_{\xi(t)})_{(t,z_e)} \upharpoonright_{\ker \tau_{(t,z_e)}} = (\iota_{X_{h-J^\Phi_{\xi(t)}}}\omega)_{(t,z_e)} \upharpoonright_{\ker \tau_{(t,z_e)}} = 0, \qquad \forall t\in T\,.$$

Since $X_{h-J^\Phi_{\xi(t)}}(t,z_e)$ takes values in $\ker\tau$, it follows that $(X_{h-J^\Phi_{\xi(t)}})_{(t,z_e)} = 0$ for every $t\in T$. Consequently, $(X_h)_{(t,z_e)} = (X_{J^\Phi_{\xi(t)}})_{(t,z_e)} = (\xi(t)_M)_{(t,z_e)}$ for every $t\in T$, proving that z_e is a cosymplectic relative equilibrium point. □

The above theorem can be equivalently rewritten as follows.

Corollary 3.2.4. *A point $z_e\in P$ is a cosymplectic relative equilibrium point of $((T\times P)^\omega_\tau, h, \mathbf{J}^\Phi)$ if and only if there exists a curve $\xi(t)$ in $\mathfrak{g}$ such that $(z_e,\xi(t))\in P\times\mathfrak{g}$, for every $t\in T$, are critical points of the functions $\widehat{h}_t\colon P\times\mathfrak{g}\to\mathbb{R}$ of the form*

$$\widehat{h}_t(z,\nu) := h(t,z) - \langle \mathbf{J}^\Phi(t,z) - \mathbf{J}^\Phi(t,z_e), \nu\rangle.$$

Note that $\xi(t)$ plays the role of a t-dependent Lagrange multiplier in Corollary 3.2.4.

Let z_e be a cosymplectic relative equilibrium point of $((T\times P)^\omega_\tau, h, \mathbf{J}^\Phi)$. The second variation of $h_{\xi(t_e)}$ at (t_e,z_e), for any $t_e\in T$ is defined as the mapping

$$(\delta^2 h_{\xi(t_e)})_{(t_e,z_e)}\colon\ \ker\tau_{(t_e,z_e)} \times \ker\tau_{(t_e,z_e)} \to \mathbb{R}\,,$$

of the form

$$(\delta^2 h_{\xi(t)})_{(t_e,z_e)}(v_1,v_2) := \iota_Y(\mathrm{d}(\iota_X \mathrm{d}h_{\xi(t_e)}))_{(t_e,z_e)}\,, \tag{3.2.2}$$

for some vector fields X,Y on M defined on a neighbourhood of (t_e,z_e) taking values in $\ker\tau$ and such that $v_1 = X_{(t_e,z_e)}$, $v_2 = Y_{(t_e,z_e)}$. Note that, for each pair v_1,v_2, it is always possible to find some X,Y satisfying the given conditions.

In cosymplectic Darboux coordinates $\{t,x_1,\dots,x_{2n}\}$ in an open neighbourhood U of (t_e,z_e), one can write

$$X = \sum_{i=1}^{2n} f_i\frac{\partial}{\partial x_i}\,, \qquad Y = \sum_{i=1}^{2n} g_i\frac{\partial}{\partial x_i}\,,$$

where $\iota_{\frac{\partial}{\partial x_i}}\tau = 0$ for $i = 1, \ldots, 2n$, and $f_1, \ldots, f_{2n}, g_1, \ldots, g_{2n} \in \mathscr{C}^\infty(U)$ may depend on t.

Notice that the definition of the second variation is analogous to that used in the time-dependent energy-momentum method, see Equation (3.1.4). The key distinction here is that the variable $t \in \mathbb{R}$ is treated not as a parameter but as a coordinate of the associated cosymplectic manifold.

Proposition 3.2.5. *Let $z_e \in P$ be a cosymplectic relative equilibrium point of $((T \times P)^\omega_\tau, h, \mathbf{J}^\Phi)$. If $\{t, x_1, \ldots, x_{2n}\}$ are cosymplectic Darboux coordinates on a neighbourhood of $(t_e, z_e) \in T \times P$, then*

$$(\delta^2 h_{\xi(t_e)})_{(t_e,z_e)}(w, v) = \sum_{i,j=1}^{2n} \frac{\partial^2 h_{\xi(t_e)}}{\partial x_i \partial x_j}(t_e, z_e) w_i v_j, \quad \forall v, w \in \ker \tau_{(t_e,z_e)}, \tag{3.2.3}$$

where $w = \sum_{i=1}^{2n} w_i \partial/\partial x_i$ and $v = \sum_{i=1}^{2n} v_i \partial/\partial x_i$.

Proof. From (3.2.2) and the fact that the vector fields X, Y associated with the tangent vectors w, v take values in $\ker \tau$, one gets

$$\begin{aligned}(\delta^2 h_{\xi(t_e)})_{(t_e,z_e)}(w, v) = \iota_Y(\mathrm{d}\iota_X \mathrm{d}h_{\xi(t_e)})_{(t_e,z_e)} &= \sum_{i,j=1}^{2n} \frac{\partial^2 h_{\xi(t_e)}}{\partial x_i \partial x_j}(t_e, z_e) w_i v_j \\ &+ \sum_{i,j=1}^{2n} \frac{\partial h_{\xi(t_e)}}{\partial x_i}(t_e, z_e) \frac{\partial X_i}{\partial x_j}(t_e, z_e) v_j = \sum_{i,j=1}^{2n} \frac{\partial^2 h_{\xi(t_e)}}{\partial x_i \partial x_j}(t_e, z_e) w_i v_j,\end{aligned}$$

where $X = \sum_{i=1}^{2n} X^i \partial/\partial x^i$, $X(t_e, z_e) = w$. The second equality uses the fact that z_e is a cosymplectic relative equilibrium point. □

It follows from (3.2.3) that, for each $t \in T$, the maps $(\delta^2 h_{\xi(t_e)})_{(t,z_e)}$ are symmetric.

Proposition 3.2.6. *Let $z_e \in P$ be a cosymplectic relative equilibrium point for $((T \times P)^\omega_\tau, h, \mathbf{J}^\Phi)$. Then, for every $t \in T$, one has*

$$(\delta^2 h_{\xi(t)})_{(t,z_e)}((\zeta_M)_{(t,z_e)}, v_{(t,z_e)}) = 0, \quad \forall \zeta \in \mathfrak{g}, \quad \forall v_{(t,z_e)} \in \mathrm{T}_{(t,z_e)}\mathbf{J}^{\Phi-1}(\mu_e) \cap \ker \tau_{(t,z_e)} .$$

Proof. The G-invariance of a Hamiltonian function $h\colon T \times P \to \mathbb{R}$ together with the equivariance condition for $\mathbf{J}^\Phi$ relative to the cosymplectic affine Lie group action, Δ, imply that for every $g \in G$ and all $(t', z) \in T \times P$, with $\mu_e = \mathbf{J}^\Phi(t', z_e)$, one has

$$\begin{aligned}h_{\xi(t)}(\Phi_g(t', z)) &= h(\Phi_g(t', z)) - \left\langle \Delta_g \mathbf{J}^\Phi(t', z), \xi(t) \right\rangle + \langle \mu_e, \xi(t) \rangle \\ &= h(t', z) - \left\langle \mathbf{J}^\Phi(t', z), \Delta_g^T \xi(t) \right\rangle + \langle \mu_e, \xi(t) \rangle ,\end{aligned}$$

where $\Delta_g^T\colon \mathfrak{g} \to \mathfrak{g}$ is the transpose of Δ_g for every $g \in G$.

Fix any $t \in T$ and let $g = \exp(s\zeta)$, with $\zeta \in \mathfrak{g}$. Differentiating with respect to s at $s = 0$ yields

$$(\iota_{\zeta_M} \mathrm{d}h_{\xi(t)})(t', z) = -\left\langle \mathbf{J}^\Phi(t', z), \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Delta^T_{\exp(s\zeta)}\xi(t) \right\rangle = \left\langle \mathbf{J}^\Phi(t', z), (\zeta_{\mathfrak{g}}^\Delta)_{\xi(t)} \right\rangle ,$$

where $(\zeta_{\mathfrak{g}}^\Delta)_{\xi(t)}$ is the fundamental vector field associated with a Lie group action $\Delta^T\colon G \times \mathfrak{g} \to \mathfrak{g}$ related to $\zeta \in \mathfrak{g}$ at $\xi(t) \in \mathfrak{g}$, for a fixed $t \in T$, namely

$$\zeta_{\mathfrak{g}}^\Delta(v) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Delta^T_{\exp(-s\zeta)} v, \qquad \forall v \in \mathfrak{g} .$$

Since $(\zeta_M)_{(t,z_e)}$ and $v_{(t,z_e)}$ take values in $\ker \tau_{(t,z_e)}$, taking variations with respect to $z \in P$ and evaluating at (t, z_e) gives

$$(\delta^2 h_{\xi(t)})_{(t,z_e)}((\zeta_M)_{(t,z_e)}, v_{(t,z_e)}) = \left\langle \mathrm{T}_{(t,z_e)}\mathbf{J}^\Phi(v_{(t,z_e)}), (\zeta_{\mathfrak{g}}^\Delta)_{\xi(t)} \right\rangle .$$

This vanishes whenever $\mathrm{T}_{(t,z_e)}\mathbf{J}^\Phi(v_{(t,z_e)}) = 0$, that is, if $v_{(t,z_e)} \in \ker \mathrm{T}_{(t,z_e)}\mathbf{J}^\Phi = \mathrm{T}_{(t,z_e)}\mathbf{J}^{\Phi-1}(\mu_e)$, which proves the claim. □

The next corollary is an immediate consequence of Proposition 3.2.6.

Corollary 3.2.7. *Let z_e be a cosymplectic relative equilibrium point of $((T\times P)^{\omega}_{\tau}, h, \mathbf{J}^{\Phi})$. Then, for every $t\in T$, the subspace $\mathrm{T}_{(t,z_e)}(G^{\Delta}_{\mu_e}(t,z_e))$ is contained in the kernel of the restriction of $(\delta^2 h_{\xi(t)})_{(t,z_e)}$ to $\mathrm{T}_{(t,z_e)}\mathbf{J}^{\Phi-1}(\mu_e)\cap\ker\tau_{(t,z_e)}$.*

3.2.2 Stability on the reduced cosymplectic manifold

Subsection 3.2.1 introduced the fundamental results of a cosymplectic energy-momentum method, which provides a systematic approach to finding cosymplectic relative equilibrium points of $((T\times P)^{\omega}_{\tau}, h, \mathbf{J}^{\Phi})$.

In this section, the stability of these points on the reduced space is analysed by applying and interpreting the results similarly as in Section 3.1.4 within the cosymplectic framework. From now on, it is assumed that $T=\mathbb{R}$ in order to employ Definition 1.1.1, which serves as the foundation for establishing conditions that ensure various types of stability on manifolds.

Recall that the cosymplectic Marsden–Meyer–Weinstein reduction for $((\mathbb{R}\times P)^{\omega}_{\tau}, h, \mathbf{J}^{\Phi})$ consists of reducing the cosymplectic manifold $(\mathbb{R}\times P,\omega,\tau)$ to a cosymplectic manifold $(\mathbb{R}\times P^{\Delta}_{\mu},\omega_\mu,\tau_\mu)$, where ω_μ and τ_μ are defined by

$$i_\mu^*\omega=\pi_\mu^*\omega_\mu\,,\qquad i_\mu^*\tau=\pi_\mu^*\tau_\mu\,,$$

where $i_\mu\colon \mathbf{J}^{\Phi-1}(\mu)\hookrightarrow\mathbb{R}\times P$ is the natural immersion, $\pi_\mu\colon\mathbf{J}^{\Phi-1}(\mu)\to M^{\Delta}_{\mu}=\mathbf{J}^{\Phi-1}(\mu)/G^{\Delta}_{\mu}$ and $\pi_T\colon T\times P\to P$ are the canonical projections. Recall that $M^{\Delta}_{\mu}\simeq\mathbb{R}\times P^{\Delta}_{\mu}$ for a certain manifold P^{Δ}_{μ} introduced in Corollary 2.2.14. The following analysis relies on the ideas developed in Subsection 3.1.4.

Consider the function $h_{z_e}\colon\mathbb{R}\times P\to\mathbb{R}$ given by

$$h_{z_e}(t,z):=h(t,z)-h(t,z_e)\,.$$

Then, $h_{z_e}(t,z_e)=0$ for every $t\in\mathbb{R}$. This is done to study $h_{z_e}(t,z)$ with lpdf functions and other functions of the sort, analogously as in Subsection 3.1.4. If $(t,z(t))$ is the particular solution to the G-invariant cosymplectic Hamiltonian system $((\mathbb{R}\times P)^{\omega}_{\tau}, h, \mathbf{J}^{\Phi})$ with the initial condition $(0,z_e)$, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}h_{z_e}(t,z(t))=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}h(t,z(t))-\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}h(t,z_e)\,.$$

Since the integral curves of E_h for $(\mathbb{R}\times P,\omega,\tau)$ are given by (1.3.6a), the derivative with respect to t of a function h_{z_e} along the solutions of the Hamilton equations for h reads

$$\frac{\mathrm{d}h_{z_e}}{\mathrm{d}t}=E_h h_{z_e}=Rh_{z_e}+\{h_{z_e},h\}_{\omega,\tau}=Rh_{z_e}=\frac{\partial h_{z_e}}{\partial t}\,.$$

Note that this relation is independent of the particular choice of the variable t in cosymplectic Darboux coordinates on $\mathbb{R}\times P$. Furthermore, one has that h_{z_e} is G-invariant, namely $h_{z_e}\circ\Phi_g=h_{z_e}$ for every $g\in G$. Since, by assumption, $\pi_T\circ\Phi_g=\pi_T$, there exists a reduced function $H_{z_e}\colon\mathbb{R}\times P^{\Delta}_{\mu_e}\to\mathbb{R}$ of the form

$$H_{z_e}(t,[z]):=h_{z_e}(t,z),\qquad\forall(t,z)\in\mathbf{J}^{\Phi-1}(\mu_e),$$

where $(t,[z])$ stands for the equivalence class of $(t,z)\in\mathbf{J}^{\Phi-1}(\mu_e)$ in $\mathbf{J}^{\Phi-1}(\mu_e)/G^{\Delta}_{\mu_e}$. Similarly as in Subsection 3.1.4, the function $H_{z_e}(t,[z])-k_{\mu_e}(t,[z])$ depends only on t, because $k_{\mu_e}(t,[z])$ satisfies $\pi^*_{\mu_e}k_{\mu_e}=i^*_{\mu_e}h$. Since $\mathbb{R}\times P^{\Delta}_{\mu_e}$ is a cosymplectic manifold, the relation $\pi_{\mu_e*}(R+X_h)=R_{\mu_e}+X_{k_{\mu_e}}$ holds. Consequently, $[z_e]$ is an equilibrium point of $X_{k_{\mu_e}}$ and thus $H_{z_e}|_{\{t\}\times P^{\Delta}_{\mu_e}}$ has an equilibrium point at $[z_e]$. Additionally,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}H_{z_e}(t,[z(t)])=(Rh_{z_e})(0,z(0)),\qquad\forall(0,[z(0)])\in\mathbf{J}^{\Phi-1}(\mu_e)/G^{\Delta}_{\mu_e}\simeq\mathbb{R}\times P^{\Delta}_{\mu_e},$$

where $z(t)$ is any solution to the initial Hamiltonian equations of h within $\mathbf{J}^{\Phi-1}(\mu_e)$ with initial condition $z(0)$.

Now, the function H_{z_e} is used to study the stability of $[z_e]$ in $P^{\Delta}_{\mu_e}$. In particular, conditions on h are derived to guarantee that H_{z_e} induces different types of stable equilibrium points at $[z_e]$. Therefore, following Subsection 3.1.4, consider a coordinate system $\{x_1, \dots, x_n\}$ on an open neighbourhood $\mathcal{U}$ of $[z_e] \in P^{\Delta}_{\mu_e}$ such that $x_i([z_e]) = 0$ for $i = 1, \dots, n$. Let $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_1, \dots, \alpha_n \in \mathbb{N} \cup \{0\}$, be a multi-index with $n = \dim \mathbf{J}^{\Phi-1}(\mu_e)/G^{\Delta}_{\mu_e} - 1$. To understand this, recall that $M^{\Delta}_{\mu_e} \simeq \mathbb{R} \times P^{\Delta}_{\mu_e}$. However, in the stability analysis, the t-dependence is treated as a parameter, not a variable. Let $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and $D^{\alpha} = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$ for every α. The proof of the following lemma follows the same way as Lemma 3.1.11.

Lemma 3.2.8. *Consider the t-dependent parametric family of $n \times n$ matrices $M(t)$ with entries*

$$[M(t)]^j_i = \frac{1}{2} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]), \qquad \forall t \in \mathbb{R}, \qquad i, j = 1, \dots, n,$$

and let $\operatorname{spec}(M(t))$ be the spectrum of the matrix $M(t)$ at $t \in \mathbb{R}$. Assume that there exists a $\lambda \in \mathbb{R}$ such that $0 < \lambda < \inf_{t \in I_{t^0}} \min \operatorname{spec}(M(t))$ for some $t^0 \in \mathbb{R}$. Suppose also that there exists a real constant c such that

$$c \geq \frac{1}{6} \sup_{t \in I_{t^0}} \max_{|\alpha|=3} \max_{[y] \in \mathcal{B}} |D^{\alpha} H_{z_e}(t, [y])|$$

for a certain compact neighbourhood $\mathcal{B}$ of $[z_e]$. Then, there exists an open neighbourhood $\mathcal{U}$ of $[z_e]$, where the function $H_{z_e} \colon \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ is lpdf from t^0. If there exists additionally a constant Λ such that

$$\sup_{t \in I_{t^0}} \max \operatorname{spec}(M(t)) < \Lambda,$$

then, $H_{z_e} \colon \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ is a decrescent function from t^0.

Recall that the eigenvalues of $M(t)$ depend on the chosen coordinate system around $[z_e]$. However, as shown later on, the stability analysis does not depend on the choice of the coordinate system.

Similarly to Section 3.1.4, an appropriate coordinate system may simplify $M(t)$ at certain values of t, e.g. by writing $M(t)$ in a canonical form. Nevertheless, the simplification of $M(t)$ at every time $t \in I_{t^0}$ for a certain coordinate system in $P^{\Delta}_{\mu_e}$ around $[z_e]$ is impossible, in general. Consequently, the analysis must be restricted to a specific coordinate system.

Lemma 3.2.8 yields the following theorem.

Theorem 3.2.9. *Suppose that there exist $\lambda, c > 0$ and an open neighbourhood U of $[z_e]$ so that*

$$\lambda < \min(\operatorname{spec}(M(t))), \qquad c \geq \frac{1}{3!} \max_{|\alpha|=3} \sup_{[x] \in U} |D^{\alpha} H_{z_e}(t, [x])|, \qquad \left. \frac{\partial H_{z_e}}{\partial t} \right|_U \leq 0,$$

for every $t \in I_{t^0}$. Then $[z_e]$ is a stable point of the Hamiltonian vector field related to k_{μ_e} on $\mathbf{J}^{\Phi-1}(\mu_e)/G^{\Delta}_{\mu_e}$ from t^0. If there exists Λ such that $\max(\operatorname{spec}(M(t))) < \Lambda$ for every $t \in I_{t^0}$, then $[z_e]$ is uniformly stable from t^0.

Under stronger hypotheses on the higher-order derivatives of H_{z_e} than those appearing in Theorem 3.2.9, one obtains Corollary 3.2.10, whose conditions can be shown to hold independently of the chosen coordinate system, analogously to Lemma 3.1.14, which gives them an intrinsic geometric character.

Corollary 3.2.10. *If there exist $\lambda, c > 0$ and an open neighbourhood U of $[z_e]$ such that*

$$\lambda < \min\left(\operatorname{spec}\left(M(t)\right)\right), \qquad c \geq \frac{1}{3!} \max_{1 \leq |\alpha| \leq 3} \sup_{[x] \in U} \left|D^{\alpha} H_{z_e}(t, [x])\right|, \qquad \left. \frac{\partial H_{z_e}}{\partial t} \right|_U \leq 0, \tag{3.2.4}$$

for every $t \in I_{t^0}$, then $[z_e]$ is a uniformly stable point of the Hamiltonian system k_{μ_e} on $\mathbf{J}^{\Phi-1}(\mu_e)/G^{\Delta}_{\mu_e}$ from t^0.

The existence of a constant c as required in Corollary 3.2.10, together with the first and second inequalities in (3.2.4), implies that $\max(\operatorname{spec}(M(t))) \leq 6cn^2$ for every $t \in I_{t^0}$. Indeed,

$$v^T M(t)v \leq \sum_{i,j=1}^{n} |v_i||v_j||M^i_j(t)| \leq 6c \sum_{i,j=1}^{n} \|v\|^2 = 6cn^2\|v\|^2, \quad \forall v \in \mathbb{R}^n \,.$$

Consequently, $v^T M(t)v < \Lambda v^T v$ for every non-zero $v \in \mathbb{R}^n$ and $\Lambda > 6cn^2$.

The results established above employ a distance on an open coordinate neighbourhood of $[z_e]$ that was induced by a standard norm on $\mathbb{R}^n$. Since the topology induced by this norm coincides with the one induced by any Riemannian metric on the neighbourhood of $[z_e]$, the stability properties obtained are independent of the chosen Riemannian metric.

The following lemma, whose proof is the same as Lemma 3.1.14, confirms that Corollary 3.2.10 is of the geometric nature, i.e. the conditions remain valid regardless of the choice of coordinates. The values of the constants λ and c may vary with the coordinates, but their existence is preserved, which is the essential property in the stability analysis.

Lemma 3.2.11. *If $M(t)$, which is defined in a local coordinate system $\{x_1, \ldots, x_n\}$ on an open neighbourhood of an equilibrium point $[z_e] \in P^{\Delta}_{\mu_e}$, is such that $0 < \lambda < \inf_{t^0 \leq t} \min \operatorname{spec} M(t)$ for some λ (resp. $\sup_{t^0 \leq t} \max \operatorname{spec} M(t) < \Lambda$ for some Λ), then $M_{\mathcal{B}'}(t)$, which is determined like $M(t)$ but in another coordinate system $\mathcal{B}' = \{\tilde{x}_1, \ldots, \tilde{x}_n\}$ around $[z_e] \in P^{\Delta}_{\mu_e}$, holds that $0 < \lambda' < \inf_{t^0 \leq t} \min \operatorname{spec} M_{\mathcal{B}'}(t)$ for some λ' (resp. $\sup_{t^0 \leq t} \max \operatorname{spec} M_{\mathcal{B}'}(t) < \Lambda'$ for some Λ').*

3.2.3 Stability, reduced cosymplectic manifold, and cosymplectic relative equilibrium points

The cosymplectic energy-momentum method determines properties of a Hamiltonian function h on a neighbourhood of a cosymplectic relative equilibrium point $m_e = (t, z_e) \in \mathbb{R} \times P$ that guarantee a certain type of stability around an associated equilibrium point of the Hamilton equations related to k_{μ_e} in $\mathbb{R} \times P^{\Delta}_{\mu_e}$. Similarly as in Subsection 3.1.5, conditions on $h_{\mu_e} \colon (t, x) \in \mathbf{J}^{\Phi-1}(\mu_e) \mapsto h(t, x) \in \mathbb{R}$, and $\partial h_{\mu_e}/\partial t$ with $t \in \mathbb{R}$ are derived to ensure that the hypotheses of Theorem 3.2.9 and/or Corollary 3.2.10 are satisfied. Instead of investigating $M(t)$, one is focused on the conditions on the functions $h_{\xi(t)} \upharpoonright_{\{t\} \times P}$ for $t \in \mathbb{R}$, which is more practical, since these functions are defined directly on P, rather than on the reduced manifold.

Consider a coordinate system $\{t, z_1, \ldots, z_q\}$ on an open subset $\mathbb{R} \times \mathcal{A}_{\mu_e} \subset \mathbf{J}^{\Phi-1}(\mu_e)$ containing $m_e = (t_e, z_e)$ for some $t_e \in \mathbb{R}$. Let $\{t, \pi^*_{\mu_e} x_1, \ldots, \pi^*_{\mu_e} x_n\}$ be the coordinates on $\mathbb{R} \times \mathcal{A}_{\mu_e}$ obtained by pull-back to $\mathbb{R} \times \mathcal{A}_{\mu_e}$ some coordinates $\{t, x_1 \ldots, x_n\}$ on $\mathbb{R} \times \mathcal{O} = \pi_{\mu_e}(\mathbb{R} \times \mathcal{A}_{\mu_e})$ [2], since the cosymplectic Marsden–Meyer–Weinstein reduction does not "reduce" the space component $\mathbb{R}$ (see Corollary 2.2.14), and let $\{y_1, \ldots, y_s\}$ be additional coordinates giving rise to a coordinate system $\{t, z_1, \ldots, z_q\}$ on $\mathbb{R} \times \mathcal{A}_{\mu_e}$. Due to the $G^{\Delta}_{\mu_e}$-invariance of $h_{\mu_e} = h \circ i_{\mu_e} \colon \mathbf{J}^{\Phi-1}(\mu_e) \to \mathbb{R}$ there exists c such that

$$c \geq \frac{1}{3!} \max_{3 \geq |\vartheta| \geq 1} \sup_{z \in \mathcal{A}_{\mu_e}} |D^{\vartheta} h_{\mu_e}(t, y)| \,, \qquad \forall t \in I_{t^0} \,,$$

where ϑ is a multi-index $\vartheta = (\vartheta_1, \ldots, \vartheta_q)$ if and only if

$$c \geq \frac{1}{3!} \max_{3 \geq |\alpha| \geq 1} \sup_{[x] \in \mathcal{O}} |D^{\alpha} H_{z_e}(t, x)|, \qquad \forall t \in I_{t^0}, \tag{3.2.5}$$

where $\mathbb{R} \times \mathcal{O}$ is an open neighbourhood of $[m_e] = (t_e, [z_e])$ because π_{μ_e} is an open mapping. Indeed, since h_{μ_e} is constant on the submanifolds where $t, x_1, \ldots, x_n$ take constant values, $h_{\mu_e}(t, x_1, \ldots, x_n, y_1, \ldots, y_s) - h(t, z_e) = H_{z_e}(t, x_1, \ldots, x_n)$ and (3.2.5) follows.

---

[2] To simplify the notation, $\{x_1, \ldots, x_n\}$ stands for a set of coordinates on a neighbourhood of $[z_e]$ and their pull-backs to $\mathbf{J}^{\Phi-1}(\mu_e)$ via π_{μ_e} simultaneously, the same as in Subsection 3.1.5.

Consider again the local coordinate system $\{t, z_1, \ldots, z_q\}$ on $\mathbf{J}^{\Phi-1}(\mu_e)$. Let $[\widehat{M}(t)]$ stands for the t-dependent $q \times q$ matrix given by the t-dependent coefficients of the form

$$[\widehat{M}(t)]_i^j := \frac{\partial^2 h_{\mu_e}}{\partial z_i \partial z_j}(t, z_e), \qquad i, j = 1, \ldots, q.$$

The coordinate system is constructed in accordance with the natural local decomposition $\mathbf{J}^{\Phi-1}(\mu_e)$ of the form $\mathbb{R} \times \mathcal{A}_{\mu_e}$. Moreover, for $(t, z) \in \mathbf{J}^{\Phi-1}(\mu)$ the equality $h_{\mu_e}(t, z) = h_{\xi(t)}(t, z)$ holds.

According to Lemma 3.2.11, the existence of constants λ and Λ is equivalent to the fact that the t-dependent symmetric bilinear form $K(t)\colon \mathrm{T}_{[z_e]} P^{\Delta}_{\mu_e} \times \mathrm{T}_{[z_e]} P^{\Delta}_{\mu_e} \to \mathbb{R}$ of the form

$$K(t) := \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 H_{z_e}}{\partial x_i \partial x_j}(t, [z_e]) \mathrm{d}x_i|_{[z_e]} \otimes \mathrm{d}x_j|_{[z_e]}$$

satisfies the following inequality

$$K(t)(w, w) > \lambda (w|w)_{\mathcal{B}}\,, \qquad \forall w \in \mathrm{T}_{[z_e]} P^{\Delta}_{\mu_e} \backslash \{0\}, \qquad \forall t \in I_{t^0}\,, \tag{3.2.6}$$

where $(\cdot|\cdot)_{\mathcal{B}}$ is the Euclidean product in $\mathrm{T}_{[z_e]} P^{\Delta}_{\mu_e}$ for which $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$ is an orthonormal basis. Indeed, if v stands for the column vector of the coordinates of $w \in \mathrm{T}_{[z_e]} P^{\Delta}_{\mu_e}$ in $\{\partial_{x_1}, \ldots, \partial_{x_n}\}$, then

$$K(t)(w, w) = v^T M(t) v > \lambda v^T v = \lambda (w|w)_{\mathcal{B}}\,, \qquad \forall w \in \mathrm{T}_{[z_e]} P^{\Delta}_{\mu_e} \backslash \{0\}\,, \quad \forall t \in I_{t^0}\,.$$

Moreover, for any another inner product $(\cdot|\cdot)_{\mathcal{B}'}$ on $\mathrm{T}_{[z_e]} P^{\Delta}_{\mu_e}$ there exist constants $m_l, m_s > 0$ such that [3]

$$m_s (w|w)_{\mathcal{B}'} > (w|w)_{\mathcal{B}} > m_l (w|w)_{\mathcal{B}'}\,, \qquad \forall w \in \mathrm{T}_{[z_e]} P^{\Delta}_{\mu_e} \setminus \{0\}\,.$$

Consequently, if (3.2.6) holds for a given inner product on $\mathrm{T}_{[z_e]} P^{\Delta}_{\mu_e}$, then it also holds for any other choice of inner product, possibly after a change of the value of λ. The same reasoning applies to the upper bound

$$\Lambda (w|w)_{\mathcal{B}} > K(t)(w, w)\,,$$

for a $\Lambda > 0$, for all $t \in I_{t^0}$, and every $w \in \mathrm{T}_{[z_e]} P^{\Delta}_{\mu_e} \backslash \{0\}$.

It is worth noting that the inner product $(\cdot|\cdot)_{\mathcal{B}}$ is introduced to effectively characterise whether the t-dependent matrix $M(t)$ has eigenvalues that can be bounded from below simultaneously for every time $t \in I_{t^0}$.

A geometric approach to the verification of condition (3.2.6) can be formulated as follows. Since each $h_{\mu_e}|_{\{t\} \times \mathcal{A}_{\mu_e}}$, with $t \in \mathbb{R}$, admits a critical point at the cosymplectic relative equilibrium $z_e \in \mathcal{A}_{\mu_e}$, there exists a t-dependent symmetric bilinear form

$$\widehat{M}(t)\colon \mathrm{T}_{z_e} \mathcal{A}_{\mu_e} \times \mathrm{T}_{z_e} \mathcal{A}_{\mu_e} \to \mathbb{R}\,,$$

of the form

$$\widehat{M}(t) := \frac{1}{2} \sum_{i,j=1}^{q} \frac{\partial^2 h_{\mu_e}}{\partial z_i \partial z_j}(t, z_e) \mathrm{d}z_i|_{z_e} \otimes \mathrm{d}z_j|_{z_e}\,, \qquad \forall t \in I_{t^0}\,,$$

where $\mathcal{B} = \{t, z_1, \ldots, z_q\}$ is any coordinate system in an open neighbourhood of $(t_e, z_e) \in \mathbf{J}^{\Phi-1}(\mu_e)$ adapted to $\mathbb{R} \times \mathcal{A}_{\mu_e}$.

Let $\{t, x_1, \ldots, x_n, y_1, \ldots, y_s\}$ be the coordinate system on the open neighbourhood of (t_e, z_e) in $\mathbf{J}^{\Phi-1}(\mu_e)$ introduced above. Then,

$$\frac{\partial^2 h_{\mu_e}}{\partial x_k \partial y_j}(t, z_e) = \frac{\partial^2 h_{\mu_e}}{\partial y_i \partial y_j}(t, z_e) = 0\,, \qquad i, j = 1, \ldots, s, \qquad k = 1, \ldots, n\,, \qquad \forall t \in \mathbb{R}\,.$$

---

[3] Recall that in finite-dimensional spaces all metrics induced by norms are strong equivalent [2, 95].

Furthermore, $\pi_{\mu_e}^* K(t) = \widehat{M}(t)$ and $\mathrm{T}_{z_e}(G_{\mu_e}^{\Delta} z_e) \subset \ker \widehat{M}(t)$ for every $t \in \mathbb{R}$. Since these objects are geometric, the bilinear form $K(t)$ can be considered as the induced bilinear form by $\widehat{M}(t)$ on $S_{z_e} \simeq \mathrm{T}_{z_e}\mathcal{A}_{\mu_e}/\mathrm{T}_{z_e}(G_{\mu_e}^{\Delta} z_e) \simeq \mathrm{T}_{[z_e]}P_{\mu_e}^{\Delta}$. Thus, the conditions for $M(t)$ can be equivalently verified via $\widehat{M}(t)$.

Corollary 3.2.10 together with the previous remarks lead to the following theorem that serves as an analogue of Theorem 3.1.17 from Subsection 3.1.5.

Theorem 3.2.12. *Suppose that there exist $\lambda, c > 0$ and an open coordinate neighbourhood $\mathcal{A}_{\mu_e}$ of z_e so that $\mathbb{R} \times \mathcal{A}_{\mu_e} \subset \mathbf{J}^{\Phi-1}(\mu_e)$ and*

$$\lambda < \min(\operatorname{spec}([\widehat{M}(t)]|_{S_{z_e}}), \qquad c \geq \frac{1}{3!} \max_{1 \leq |\vartheta| \leq 3} \sup_{y \in \mathcal{A}_{\mu_e}} |D^{\vartheta} h_{\mu_e}(t,y)|, \qquad \left.\frac{\partial h_{\mu_e}}{\partial t}\right|_{\mathcal{A}_{\mu_e}} \leq 0,$$

for every $t \in I_{t^0}$ and a subspace $S_{z_e} \subset \mathrm{T}_{z_e}\mathcal{A}_{\mu_e}$ suplementary to $\mathrm{T}_{z_e}(G_{\mu_e}^{\Delta} z_e)$, then $[z_e]$ is a uniformly stable point of the Hamiltonian system k_{μ_e} on $\mathbf{J}^{\Phi-1}(\mu_e)/G_{\mu_e}^{\Delta}$ from t^0.

Finally, the properties of $h_{\xi(t)}|_{\{t\}\times P}$ can be related to H_{μ_e} in order to analyse cosymplectic relative equilibrium points in P together with their associated equilibrium points in $P_{\mu_e}^{\Delta}$. Since $h_{\xi(t)} \upharpoonright_{\{t\}\times P}$ admits a critical point at each cosymplectic relative equilibrium point $z_e \in P$ for every $t \in \mathbb{R}$, one may introduce the t-dependent bilinear symmetric form on $\mathrm{T}_{z_e}P$ defined by

$$T_{z_e}(t) := \frac{1}{2} \sum_{i,j=1}^{\chi} \frac{\partial^2 h_{\xi(t)}}{\partial u_i \partial u_j}(t, z_e) \mathrm{d}u_i|_{z_e} \otimes \mathrm{d}u_j|_{z_e}\,, \qquad \forall t \in \mathbb{R}\,,$$

where $\{t, u_1, \ldots, u_\chi\}$, with $\chi = \dim P$, is a coordinate system on an open neighbourhood of $m_e = (t, z_e)$ in $\mathbb{R} \times P$. The relation of $T_{z_e}(t)$ and $\widehat{M}(t)$ is crucial, since the latter can be studied through the former. Furthermore, $T_{z_e}(t)$ is a geometric object naturally constructed on $\mathrm{T}_{z_e}P$ essentially depending only on h and $\mathbf{J}^{\Phi}$.

Following the reasoning from Section 3.1.5, for a regular value $\mathbf{J}^{\Phi}(z_e) = \mu_e \in \mathfrak{g}^*$, one gets the following. Since μ_e is a regular value, the coordinates of $\mathbf{J}^{\Phi}$ around $\mathbf{J}^{\Phi-1}(\mu_e)$, e.g. $\mu_1, \ldots, \mu_r$, give rise to $\dim \mathfrak{g}$ functionally independent functions on P. Consider now the coordinate system on a neighbourhood $\mathcal{A}_{\mu_e}$ of z_e so that $\mathbb{R} \times \mathcal{A}_{\mu_e} \subset \mathbf{J}^{\Phi-1}(\mu_e)$ given by $\{t, x_1, \ldots, x_n, y_1, \ldots, y_s\}$. Extend these coordinates smoothly to an open neighbourhood in M containing $\mathbb{R} \times \{z_e\}$. Since $\mathbf{J}^{\Phi}$ is regular at each (t, z_e) for $t \in \mathbb{R}$, the functions $\mu_1, \ldots, \mu_r$, which are constant on the level sets of $\mathbf{J}^{\Phi}$, satisfy

$$\mathrm{d}\mu_1 \wedge \cdots \wedge \mathrm{d}\mu_r \neq 0\,,$$

on each (t, z_e) for every $t \in \mathbb{R}$. This yields a coordinate system $\{t, x_1, \ldots, x_n, y_1, \ldots, y_s, \mu_1, \ldots, \mu_r\}$ on an open neighbourhood in $\mathbb{R} \times P$ containing $\mathbb{R} \times \{z_e\}$. Consequently,

$$\left.\frac{\partial h}{\partial y_i}\right|_{\mathbf{J}^{\Phi-1}(\mu_e)} = 0\,, \qquad \frac{\partial \langle \mathbf{J}^{\Phi} - \mu_e,\ \xi(t)\rangle}{\partial y_i} = 0\,, \qquad \forall t \in \mathbb{R}\,, \qquad i = 1, \ldots, s\,.$$

It is worth noting that these identities are not required to hold away from $\mathbf{J}^{\Phi-1}(\mu_e)$ since $y_1, \ldots, y_s$ were extended smoothly from $\mathbf{J}^{\Phi-1}(\mu_e)$. Furthermore,

$$\left.\left(\frac{\partial}{\partial y_j}\frac{\partial h}{\partial y_i}\right)\right|_{\mathbf{J}^{\Phi-1}(\mu_e)} = 0, \qquad \left.\left(\frac{\partial}{\partial x_k}\frac{\partial h}{\partial y_i}\right)\right|_{\mathbf{J}^{\Phi-1}(\mu_e)} = 0\,,$$

and

$$\frac{\partial}{\partial y_j}\frac{\partial \langle \mathbf{J}^{\Phi} - \mu_e, \xi(t)\rangle}{\partial y_i} = 0\,, \qquad \frac{\partial}{\partial x_k}\frac{\partial \langle \mathbf{J}^{\Phi} - \mu_e\,, \xi(t)\rangle}{\partial y_i} = 0\,,$$

for all $t \in \mathbb{R}$ with $i, j = 1, \ldots, s$ and $k = 1, \ldots, n$.

The first two relations follow because the derivatives on the left depend only on $\partial h/\partial y_i$ within $\mathbf{J}^{\Phi-1}(\mu_e)$. In the chosen coordinate system, the Hessian of $h_{\xi(t)}$ restricted to $t \times P$ on $\mathrm{T}_{(t,z_e)}\mathbf{J}^{\Phi-1}(\mu_e) \cap \ker \tau_{(t,z_e)}$ coincides with $\widehat{M}(t)$. Hence, the functions $h_{\xi(t)}$ can be used to study both $\widehat{M}(t)$ and $M(t)$.

3.2.4 Example: Two-state quantum system

Consider a quantum mechanical system defined by a time-dependent Schrödinger equation on a finite-dimensional Hilbert space, to study its cosymplectic relative equilibrium points with respect to the group of symmetries of the time-dependent Schrödinger equation given by multiplication by non-zero complex numbers.

In this section, attention is restricted to a two-level quantum system subjected to a time-dependent Hermitian Hamiltonian operator $\widehat{H}(t)$, which may arise, for instance, from a spin-magnetic interaction with an additional drift term. In particular, the techniques introduced previously in this section are applied to this system.

The states of the two-level system are represented by elements of the Hilbert space $\mathbb{C}^2$, where only non-zero vectors are physically relevant. Since $\mathbb{C}^n$ admits a real differential structure globally homeomorphic to $\mathbb{R}^{2n}$, the Hilbert space describing the two-level system is two-dimensional as a complex manifold or, equivalently a four-dimensional as a real manifold. The evolution of the system is determined by the action of the Lie group U_2 of unitary automorphisms on $\mathbb{C}^2$. More precisely, the solution of the time-dependent Schrödinger equation generated by $\widehat{H}(t)$, with an initial state $\Psi_0 \in \mathbb{C}^2$ at $t = 0$, is given by $\Psi(t) = U_t\Psi_0$ for a curve $\mathbb{R} \ni t \mapsto U_t \in \mathrm{U}_2$. Recall that the time-dependent Schrödinger equation associated with $\widehat{H}(t)$ reads

$$i\frac{\mathrm{d}\Psi(t)}{\mathrm{d}t} = \widehat{H}(t)\Psi(t), \tag{3.2.7}$$

where $\widehat{H}(t)$, for every $t \in \mathbb{R}$, is assumed to be a Hermitian Hamiltonian operator on $\mathbb{C}^2$.

Consider the real vector space of Hermitian operators on $\mathbb{C}^2$, denoted by $\mathfrak{u}_2^*$. A convenient basis of this space is given by $\{\widehat{S}_j := \frac{1}{2}\sigma_j\}_{j=1,2,3}$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices, and the 2×2 identity matrix $\widehat{I}$. To simplify computations, introduce a Lie bracket on $\mathfrak{u}_2^*$ defined by

$$[\![A, B]\!] := -i[A, B]\,, \qquad \forall A, B \in \mathfrak{u}_2^*\,,$$

where $[\cdot, \cdot]$ is the operator commutator for endomorphisms on $\mathbb{C}^2$. In this basis, the commutation relations are

$$[\![\widehat{I}, \widehat{S}_j]\!] = 0, \quad [\![\widehat{S}_j, \widehat{S}_k]\!] = \sum_{l=1}^{3} \epsilon_{jkl}\widehat{S}_l, \quad k, j = 1, 2, 3,$$

where ϵ_{jkl}, with $j, k, l = 1, 2, 3$, are the *Levi-Civita symbols*.

In the presence of an external magnetic field $\vec{B}(t) := B(t)(B_1, B_2, B_3)$, where $B(t)$ is an arbitrary t-dependent function, applied to a spin 1/2 particle and under the additional drift term of the form $B(t)B_0\widehat{I}$, the time-dependent Hamiltonian operator reads

$$\widehat{H}(t) = B(t)B_0\widehat{I} + \vec{B}(t) \cdot \vec{S}\,,$$

where $\vec{S} := (\widehat{S}_1, \widehat{S}_2, \widehat{S}_3)$. By construction, $\widehat{H}(t)$ is Hermitian for every $t \in \mathbb{R}$. Recall that each operator $\widehat{H}(t)$ is Hermitian and therefore admits only real eigenvalues.

Since $\mathbb{C}^2$ is diffeomorphic to $\mathbb{R}^4$ as a manifold, a point $(z_1, z_2) \in \mathbb{C}^2$ can be represented by $\Psi := (q_1, p_1, q_2, p_2) \in \mathbb{R}^4$, where $q_i = \mathfrak{Re}(z_i)$ and $p_i = \mathfrak{Im}(z_i)$ for $i = 1, 2$. Thus, the time-dependent Schrödinger equation (3.2.7) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{bmatrix} = \frac{1}{2}B(t)\begin{bmatrix} 0 & 2B_0 + B_3 & -B_2 & B_1 \\ -2B_0 - B_3 & 0 & -B_1 & -B_2 \\ B_2 & B_1 & 0 & 2B_0 - B_3 \\ -B_1 & B_2 & -2B_0 + B_3 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{bmatrix}. \tag{3.2.8}$$

The manifold $\mathbb{R} \times \mathbb{C}^2 \simeq \mathbb{R}^5$ is a natural cosymplectic manifold $(\mathbb{R} \times \mathbb{R}^2, \omega_S := \mathrm{d}q_1 \wedge \mathrm{d}p_1 + \mathrm{d}q_2 \wedge \mathrm{d}p_2, \tau_S := \mathrm{d}t)$, where t is the natural coordinate on $\mathbb{R}$ understood as a coordinate on $\mathbb{R} \times \mathbb{R}^4$. The solutions of system

(3.2.8) can be geometrically described as the integral curves, parametrised by t, of the evolution vector field on $\mathbb{R}\times\mathbb{R}^4$ given by

$$R+B(t)\left(B_0X_0+B_1X_1+B_2X_2+B_3X_3\right),$$

where $R=\partial/\partial t$ denotes the Reeb vector field associated with $(\mathbb{R}\times\mathbb{R}^4,\omega_S,\tau_S)$ and $X_0,\ldots,X_3$ are the vector fields on $\mathbb{R}\times\mathbb{R}^4$ of the form

$$X_0:=p_1\frac{\partial}{\partial q_1}-q_1\frac{\partial}{\partial p_1}+p_2\frac{\partial}{\partial q_2}-q_2\frac{\partial}{\partial p_2},\quad X_1:=\frac{1}{2}\left(p_2\frac{\partial}{\partial q_1}-q_2\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial q_2}-q_1\frac{\partial}{\partial p_2}\right),$$
$$X_2:=\frac{1}{2}\left(-q_2\frac{\partial}{\partial q_1}-p_2\frac{\partial}{\partial p_1}+q_1\frac{\partial}{\partial q_2}+p_1\frac{\partial}{\partial p_2}\right),\quad X_3:=\frac{1}{2}\left(p_1\frac{\partial}{\partial q_1}-q_1\frac{\partial}{\partial p_1}-p_2\frac{\partial}{\partial q_2}+q_2\frac{\partial}{\partial p_2}\right).$$

Their commutation relations are given by

$$[X_0,X_j]=0\,,\quad [X_1,X_2]=-X_3\,,\quad [X_2,X_3]=-X_1\,,\quad [X_3,X_1]=-X_2\,,\qquad j=1,2,3\,.$$

On the cosymplectic manifold $(\mathbb{R}\times\mathbb{R}^4,\omega_S=\mathrm{d}q_1\wedge\mathrm{d}p_1+\mathrm{d}q_2\wedge\mathrm{d}p_2,\tau_S=\mathrm{d}t)$ the vector fields $X_0,\ldots,X_3$ are Hamiltonian with related Hamiltonian functions $h_0,\ldots,h_3$ given by

$$h_0(\Psi)=\frac{1}{2}\langle\Psi,\widehat{I}\Psi\rangle=\frac{1}{2}\left(q_1^2+q_2^2+p_1^2+p_2^2\right),\qquad h_1(\Psi)=\frac{1}{2}\langle\Psi,\widehat{S}_1\Psi\rangle=\frac{1}{2}(p_1p_2+q_1q_2)\,,$$
$$h_2(\Psi)=\frac{1}{2}\langle\Psi,\widehat{S}_2\Psi\rangle=\frac{1}{2}(q_1p_2-q_2p_1)\,,\qquad h_3(\Psi)=\frac{1}{2}\langle\Psi,\widehat{S}_3\Psi\rangle=\frac{1}{4}\left(p_1^2+q_1^2-p_2^2-q_2^2\right).$$

The functions h_1,h_2,h_3 are functionally independent and $h_0^2=4(h_1^2+h_2^2+h_3^2)$.

Accordingly, the time-dependent Schrödinger equation, in coordinates given by (3.2.8), can be associated with an evolution vector field, E_h, on $\mathbb{R}\times\mathbb{R}^4$ induced by the Hamiltonian function $h\in\mathscr{C}^\infty(\mathbb{R}\times\mathbb{R}^4)$ given by

$$h(t,\Psi):=B(t)\sum_{\alpha=0}^{3}B_\alpha h_\alpha(\Psi),\qquad t\in\mathbb{R},\qquad \Psi\in\mathbb{R}^4\,. \tag{3.2.9}$$

In other words, the solutions $(q_1(t),p_1(t),q_2(t),p_2(t))$ to (3.2.8) correspond to the integral curves

$$t\longmapsto(t,q_1(t),p_1(t),q_2(t),p_2(t))\,,$$

of the evolution vector field E_h. Recall that equivalently $\Psi=(z_1,z_2)$, with $z_1,z_2\in\mathbb{C}$. Then define a Lie group action

$$\Phi\colon \mathrm{U}_1\times\mathbb{R}\times\mathbb{C}^2\ni(e^{i\theta};t,z_1,z_2)\mapsto(t,e^{-i\theta}z_1,e^{-i\theta}z_2)\in\mathbb{R}\times\mathbb{C}^2\,.$$

This Lie group action gives rise to a Lie group of symmetries of (3.2.7). Its fundamental vector field is spanned by X_0 (considered as a vector field on $\mathbb{C}^2$).

Equivalently, by $\mathbb{R}\times\mathbb{C}^2\simeq\mathbb{R}\times\mathbb{R}^4$, one gets the Lie group action with a fundamental vector field X_0 of the form

$$\Phi\colon \mathrm{SO}_2\times\mathbb{R}\times\mathbb{R}^4\ni(\theta;t,q_1,p_1,q_2,p_2)\mapsto(t,(R_\theta\otimes R_\theta)(q_1,p_1,q_2,p_2))\in\mathbb{R}\times\mathbb{R}^4\,,$$

where SO_2 is the special orthogonal 2×2 matrix group and R_θ satisfies

$$R_\theta=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\in\mathrm{SO}_2,\qquad R_\theta\begin{pmatrix}q_j\\ p_j\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}q_j\\ p_j\end{pmatrix},\qquad j=1,2\,.$$

Note that Φ leaves invariant the Hamiltonian function h given by (3.2.9). Furthermore, Φ is a cosymplectic Lie group action, that is, $\Phi_g^*\omega_S=\omega_S$ and $\Phi_g^*\tau_S=\tau_S$ for every $g\in\mathrm{SO}_2$. Additionally, it admits an associated momentum map $\mathbf{J}^\Phi\colon\mathbb{R}\times\mathbb{R}^4\to\mathfrak{so}_2^*$ given by

$$\mathbf{J}^\Phi(t,q_1,p_1,q_2,p_2):=h_0(q_1,p_1,q_2,p_2)\,,$$

where $\mathfrak{so}_2^* \simeq \mathbb{R}^*$. Note that $0 \neq \mu \in \mathfrak{so}_2^*$ is a regular value of $\mathbf{J}^\Phi$ and $\mu = 0$ is not even a weak regular value of $\mathbf{J}^\Phi$ because $\mathrm{T}_{(t,0,0,0,0)}\mathbf{J}^\Phi = 0$ but $\mathbf{J}^{\Phi-1}(0) = \{(t,0,0,0,0) \mid t \in \mathbb{R}\}$. Thus,

$$\mathrm{T}_{(t,0,0,0,0)}\mathbf{J}^{\Phi-1}(0) \neq \ker \mathrm{T}_{(t,0,0,0,0)}\mathbf{J}^\Phi \,.$$

Therefore, for $\mu \neq 0$, the level set $\mathbf{J}^{\Phi-1}(\mu)$ is a submanifold of M, given by

$$\mathbf{J}^{\Phi-1}(\mu) = \{(t, q_1, p_1, q_2, p_2) \mid q_1^2 + p_1^2 + q_2^2 + p_2^2 = 2\mu, \quad t \in \mathbb{R}\} = \mathbb{R} \times A_\mu,$$

where

$$A_\mu = \{(q_1, p_1, q_2, p_2) \in \mathbb{R}^4 \mid q_1^2 + p_1^2 + q_2^2 + p_2^2 = 2\mu\},$$

is a three-dimensional sphere in $\mathbb{R}^4 \simeq \mathbb{C}^2$ centred at the origin and of radius $\sqrt{2\mu}$. Hence $A_\mu \simeq \mathbb{S}^3$. Since $\mathfrak{so}_2^*$ is isomorphic to $\mathbb{R}^*$ and SO_2 is abelian, the coadjoint action of SO_2 on $\mathfrak{so}_2^*$ is trivial (every element of SO_2 acts as the identity in $\mathfrak{so}_2^*$). Because h_0 is invariant under the action of SO_2, the momentum map $\mathbf{J}^\Phi$ is Ad^*-equivariant. Moreover, the isotropy group of every $\mu \in \mathbb{R}^*\backslash\{0\}$ is SO_2, i.e. $G_\mu = \mathrm{SO}_2$ for every $\mu \neq 0$. Since SO_2 is diffeomorphic to the one-dimensional sphere in $\mathbb{R}^2$, that is, the circle with radius one and centre at 0, $\mathbb{S}^1$, in $\mathbb{R}^2$, it follows

$$(\mathbb{R} \times A_\mu)/G_\mu \simeq \mathbb{R} \times (\mathbb{S}^3/\mathbb{S}^1).$$

It is known that $\mathbb{S}^1$ acting on $\mathbb{S}^3$ gives rise to a space of orbits diffeomorphic to $\mathbb{S}^2$. Therefore,

$$\mathbf{J}^{\Phi-1}(\mu)/G_\mu \simeq \mathbb{R} \times \mathbb{S}^2 \,.$$

In particular, the manifold $\mathbf{J}^{\Phi-1}(\mu)$ admits coordinates $\{t, \varphi, \theta_1, \theta_2\}$ such that the points in $\mathbf{J}^{\Phi-1}(\mu)$ can be parametrised by

$$\begin{aligned} q_1 =& \sqrt{2\mu}\sin\varphi\cos\theta_1\,, \quad p_1 = \sqrt{2\mu}\sin\varphi\sin\theta_1\,, \\ q_2 =& \sqrt{2\mu}\cos\varphi\cos\theta_2\,, \quad p_2 = \sqrt{2\mu}\cos\varphi\sin\theta_2\,, \end{aligned}$$

with $t \in \mathbb{R}$, $\varphi \in]0, \pi/2[$, $\theta_1 \in [0, 2\pi[$ and $\theta_2 \in [0, 2\pi[$. In these coordinates, one has

$$i_\mu^*\omega = \mu\sin(2\varphi)\mathrm{d}\varphi \wedge \mathrm{d}(\theta_1 - \theta_2)\,,$$

where $i_\mu \colon \mathbf{J}^{\Phi-1}(\mu) \to P$ is the natural embedding. The form $i_\mu^*\omega$ becomes degenerate at $\varphi \in \{0, \pi/2\}$. This degeneracy arises from the fact that the chosen coordinate system is not properly defined at these values. Then, the Lie group action of SO_2 on $\mathbf{J}^{\Phi-1}(\mu)$ is of the form

$$e^{i\theta}(t, \varphi, \theta_1, \theta_2) = (t, \varphi, \theta_1 - \theta, \theta_2 - \theta)\,,$$

and

$$\pi_\mu \colon (t, \varphi, \theta_1, \theta_2) \in \mathbf{J}^{\Phi-1}(\mu) \mapsto (t, \varphi, \theta_1 - \theta_2) \in (\mathbb{R} \times A_\mu)/G_\mu\,.$$

Hence, $\{t, \varphi, \theta := \theta_1 - \theta_2\}$ are local coordinates on $\mathbf{J}^{\Phi-1}(\mu)/G_\mu$, and the reduced cosymplectic structure is given by

$$\tau_\mu := \mathrm{d}t, \qquad \omega_\mu := \mu\sin(2\varphi)\mathrm{d}\varphi \wedge \mathrm{d}\theta\,.$$

Indeed, $i_\mu^*\omega = \pi_\mu^*\omega_\mu$.

The function $h_{\xi(t)}$ has the following form

$$h_{\xi(t)} = B(t)\left[\sum_{\alpha=0}^{3} B_\alpha h_\alpha\right] - (h_0 - \mu)\xi(t),$$

for certain $B_0, B_1, B_2, B_3 \in \mathbb{R}$. The critical points of $h_{\xi(t)}$ for a fixed t correspond to the solutions to the system of equations

$$\begin{aligned}
q_1(2B_0 + B_3 - 2\xi(t)/B(t)) + B_1 q_2 + B_2 p_2 = 0\,,\\
q_2(2B_0 - B_3 - 2\xi(t)/B(t)) + B_1 q_1 - B_2 p_1 = 0\,,\\
p_1(2B_0 + B_3 - 2\xi(t)/B(t)) + B_1 p_2 - B_2 q_2 = 0\,,\\
B_1 p_1 + 2B_0 p_2 + B_2 q_1 - p_2(B_3 + 2\xi(t)/B(t)) = 0\,,
\end{aligned} \tag{3.2.10}$$

which is equivalent to

$$B(t)\left(B_0\widehat{I} + \sum_{\alpha=1}^{3} B_\alpha \widehat{S}_\alpha\right)\begin{bmatrix} q_1 + ip_1 \\ q_2 + ip_2 \end{bmatrix} = \xi(t)\begin{bmatrix} q_1 + ip_1 \\ q_2 + ip_2 \end{bmatrix}. \tag{3.2.11}$$

According to Theorem 3.2.3, Equation (3.2.11) characterises the cosymplectic relative equilibrium points. Indeed, these are the eigenvectors of the operators

$$\widehat{H}(t) = B(t)\left(B_0\widehat{I} + B_1\widehat{S}_1 + B_2\widehat{S}_2 + B_3\widehat{S}_3\right),$$

for every $t \in \mathbb{R}$. This observation is consistent with the fact that the cosymplectic relative equilibrium points with respect to Φ are precisely given by the eigenvectors corresponding to real eigenvalues of $\widehat{H}(t)$ that remain constant for every $t \in \mathbb{R}$.

The reduced Hamiltonian system admits reduced Hamiltonian functions of the form

$$k_0 = \mu\,, \quad k_1 = \frac{1}{2}\mu\sin(2\varphi)\cos\theta\,, \quad k_2 = -\frac{1}{2}\mu\sin(2\varphi)\sin\theta\,, \quad k_3 = -\frac{1}{2}\mu\cos(2\varphi)\,.$$

Then, the reduced Hamiltonian function on $\mathbf{J}^{\Phi-1}(\mu)/G_\mu$ is given by

$$k_\mu(t, [\Psi]) := B(t)\sum_{\alpha=0}^{3} B_\alpha k_\alpha([\Psi])\,, \qquad [\Psi] \in \mathbb{S}^2\,.$$

Consider now the case where $B_0 = B_1 = B_2 = 0$ and $B_3 = 1$. In view of (3.2.10) and (3.2.11), the cosymplectic relative equilibrium points are given by points in $\mathbb{C}^2$ of the form

$$\langle(1,0)\rangle_{\mathbb{C}} \cup \langle(0,1)\rangle_{\mathbb{C}}\,.$$

The stability of these equilibrium points in the projected space is determined, within this framework, by the Hessian of k_3. However, the standard criteria [2, 113] do not provide a conclusive result in this case, since the Hessian of k_3 is degenerate as it depends only on the variable φ. From a geometric perspective, it can be shown that the evolution on $\mathbb{S}^2$ preserves a Riemannian metric [28]. This invariance ensures that the dynamics on the reduced system conserve the distance between any trajectory and the equilibrium points, thereby ensuring the stability of the reduced cosymplectic relative equilibrium points.

3.2.5 Example: Cosymplectic relative equilibrium points of n-state quantum system

Consider a more general quantum mechanical system than in the previous subsection, namely a system given by the time-dependent Schrödinger equation on a finite-dimensional Hilbert space $\mathbb{C}^n$, associated with a time-dependent Hermitian Hamiltonian operator $\widehat{H}(t)$ of the form

$$i\frac{\mathrm{d}\psi}{\mathrm{d}t} = \widehat{H}(t)\psi, \qquad \forall\psi \in \mathbb{C}^n, \qquad \forall t \in \mathbb{R}\,. \tag{3.2.12}$$

The following analysis focuses on finding the cosymplectic relative equilibrium points of this system.

The states of an n-level quantum system are represented by the elements of the Hilbert space $\mathbb{C}^n$, and any orthonormal basis in $\mathbb{C}^n$ determines a real global chart on $\mathbb{C}^n$. Indeed, let $\{e_j\}_{1,\ldots,n}$ be an orthonormal basis of $\mathbb{C}^n$ relative to its canonical inner product $\langle\cdot,\cdot\rangle\colon \mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$. Then, the functions $q_j, p_j\colon \mathbb{C}^n\to\mathbb{R}$, with $j=1,\ldots,n$, defined by

$$\langle e_j,\psi\rangle =: q_j(\psi)+ip_j(\psi)\,,\qquad j=1,\ldots,n\,,\quad \forall\psi\in\mathbb{C}^n\,,$$

define a real global chart on $\mathbb{C}^n$. Recall that for every $t\in\mathbb{R}$, the operator $\widehat{H}(t)$ is Hermitian with respect to the inner product above. Since $\mathbb{C}^n\simeq\mathbb{R}^{2n}$, it follows that at each $\tilde{\psi}\in\mathbb{C}^n$, there exists a canonical $\mathbb{R}$-linear isomorphism $\psi\in\mathbb{R}^{2n}\simeq\mathbb{C}^n\mapsto\psi_{\tilde{\psi}}\in \mathrm{T}_{\tilde{\psi}}\mathbb{R}^{2n}\simeq\mathrm{T}_{\tilde{\psi}}\mathbb{C}^n$, where

$$\psi_{\tilde{\psi}}f := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f(\tilde{\psi}+t\psi),\qquad \forall f\in\mathscr{C}^\infty(\mathbb{C}^n)\,.$$

Therefore, an antisymmetric and non-degenerate two-form ω can be introduced on $\mathbb{C}^n$ and is defined as

$$\omega_\psi(\psi_{1\tilde{\psi}},\psi_{2\tilde{\psi}}) := \mathfrak{Im}\langle\psi_1,\psi_2\rangle\,,\qquad \forall\psi,\psi_1,\psi_2\in\mathbb{C}^n\,.$$

In the coordinate $\{q_j,p_j\}_{j=1,\ldots,n}$, one has

$$\omega=\sum_{j=1}^n \mathrm{d}q_j\wedge\mathrm{d}p_j\,.$$

Since ω is closed, it follows that ω is a symplectic form on $\mathbb{C}^n$ with symplectic Darboux coordinates $\{q_1,\ldots,q_n,p_1,\ldots,p_n\}$.

Let $\mathfrak{u}_n^*$ denote the real vector space of Hermitian operators on $\mathbb{C}^n$. Then, each observable on $\mathbb{C}^n$, namely $\widehat{A}\in\mathfrak{u}_n^*$ induces a real smooth function on $\mathbb{C}^n$ of the form

$$f_{\widehat{A}}(\psi):=\frac{1}{2}\langle\psi,\widehat{A}\psi\rangle,\qquad\forall\psi\in\mathbb{C}^n\,,$$

giving rise to the Hamiltonian vector field

$$X_{\widehat{A}}:=\{\cdot,f_{\widehat{A}}\}\,,$$

where the Poisson bracket $\{f,g\}$ of two smooth real-valued functions $f,g\in\mathscr{C}^\infty(\mathbb{C}^n)$ is given by $\{f,g\}:=\omega(X_f,X_g)$.

The integral curves of the time-dependent Hamiltonian vector field $X_{\widehat{H}(t)}$, associated with $f_{\widehat{H}(t)}$, coincide with the solutions of the time-dependent Schrödinger equation (3.2.12) (see [28] and references therein for details).

Proceeding to the cosymplectic setting, consider the manifold $\mathbb{C}^n$ embedded in $(t,z_1,\ldots,z_n)\in\mathbb{R}\times\mathbb{C}^n\simeq\mathbb{R}^{2n+1}\ni(t,q_1,p_1,\ldots,q_n,p_n)$. Then, $(\mathbb{R}\times\mathbb{R}^{2n},\mathrm{pr}^*_{\mathbb{R}^{2n}}\omega,\mathrm{d}t)$ is a cosymplectic manifold, where $\mathrm{pr}_{\mathbb{R}^{2n}}\colon\mathbb{R}\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is the canonical projection onto the second factor and t is the pull-back to $\mathbb{R}\times\mathbb{R}^{2n}$ of the natural variable in $\mathbb{R}$. The solutions of (3.2.12) are the curves $z(t)$ such that $(t,z(t))$ is an integral curve of the evolution vector field

$$E_{f_{\widehat{H}(t)}}=R+X_{f_{\widehat{H}(t)}}\,,$$

where $R=\frac{\partial}{\partial t}$ is the Reeb vector field associated with $(\mathbb{R}\times\mathbb{R}^{2n},\mathrm{pr}^*_{\mathbb{R}^{2n}}\omega,\mathrm{d}t)$.

The Lie group action of the form

$$\begin{aligned}\Phi\colon \mathrm{SO}_2\times\mathbb{R}\times\mathbb{R}^{2n}&\longrightarrow\mathbb{R}\times\mathbb{R}^{2n}\,,\\ (R_\theta,t,q_1,p_1,\ldots,q_n,p_n)&\longmapsto(t,(R_\theta\otimes\cdots\otimes R_\theta)(q_1,p_1,\ldots,q_n,p_n))\,,\end{aligned}$$

gives rise to a Lie group of symmetries of $E_{\widehat{H}(t)}$. Moreover, the action of each element of SO_2 leaves invariant the canonical inner product on $\mathbb{R}^{2n}$. Therefore, Φ leaves ω and $\tau=\mathrm{d}t$ invariant, and Φ becomes a cosymplectic Lie group action. In addition, Φ leaves invariant the Hamiltonian function $f_{\widehat{H}(t)}$.

A cosymplectic momentum map $\mathbf{J}^{\Phi}$ associated with Φ is given by

$$\mathbf{J}^{\Phi} : \mathbb{R} \times \mathbb{R}^{2n} \ni (t, q_1, p_1, \ldots, q_n, p_n) \mapsto \frac{1}{2} \sum_{i=1}^{n} (q_i^2 + p_i^2) \in \mathfrak{so}_2^*,$$

where $\mathfrak{so}_2^* \simeq \mathbb{R}^*$. Similarly, $\mu \in \mathfrak{so}_2^*$ is a regular value of $\mathbf{J}^{\Phi}$ provided that $\mu \neq 0$. Otherwise, $\mathrm{T}\mathbf{J}^{\Phi-1}(0) \neq \ker \mathrm{T}\mathbf{J}^{\Phi}|_{\mathbf{J}^{\Phi-1}(0)}$ and consequently $\mu = 0$ is not a weak regular value of $\mathbf{J}^{\Phi}$. Thus, for $\mu \neq 0$, one gets

$$\mathbf{J}^{\Phi-1}(\mu) = \left\{ (t, q_1, p_1, \ldots, q_n, p_n) \mid \sum_{i=1}^{n} (q_i^2 + p_i^2) = 2\mu, \quad t \in \mathbb{R} \right\} = \mathbb{R} \times \mathbb{S}^{2n-1} .$$

Since the coadjoint action of SO_2 on $\mathfrak{so}_2^*$ is trivial, a cosymplectic momentum map $\mathbf{J}^{\Phi}$ is Ad^*-equivariant. Therefore, the isotropy group of every non-zero $\mu \in \mathfrak{so}_2^*$ is $G_\mu = \mathrm{SO}_2$. Theorem 2.2.13 and Corollary 2.2.14 yield that the reduced space

$$(\mathbb{R} \times \mathbb{S}^{2n-1})/\mathrm{SO}_2 \simeq \mathbb{R} \times \left(\mathbb{S}^{2n-1}/\mathbb{S}^1 \right) ,$$

is a cosymplectic manifold. It is well known that $\mathbb{S}^1$ acting on $\mathbb{S}^{2n-1}$ gives rise to a space of orbits diffeomorphic to the projective space $\mathbb{PC}^n \simeq \mathbb{C}^{n\times}/\mathbb{C}^{\times}$, where $\mathbb{C}^{\times} := \mathbb{C} \backslash \{0\}$ and $\mathbb{C}^{n\times} := \mathbb{C}^n \backslash \{0\}$ [2, 95]. Therefore, by $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, one obtains

$$M_\mu^\Delta = \mathbf{J}^{\Phi-1}(\mu)/G_\mu \simeq \mathbb{R} \times \mathbb{PC}^n ,$$

for every non-zero $\mu \in \mathfrak{so}_2^*$. From Proposition 3.2.2, it follows that the cosymplectic relative equilibrium points are of the form

$$(t, \psi(t)) = \Phi_{g(t)}(t, \psi_e), \qquad g(t) \in G_{\mu_e} = \mathrm{U}_1 ,$$

where $g(t) \in \mathrm{SO}_2 \simeq U_1$ is the evolution operator of the Schrödinger equation (3.2.12) and Ψ_e is an eigenvector for each $\widehat{H}(t)$. Consequently, analogously to the previous example, the cosymplectic relative equilibrium points correspond to the constant eigenvalues of a Hamiltonian operator $\widehat{H}(t)$ for every $t \in \mathbb{R}$.

The reduction of (3.2.12) to the projective space $\mathbb{PC}^n$ is stable at its cosymplectic equilibrium points as a consequence of the same arguments discussed in the previous example, together with the results presented in [28].

3.2.6 Cosymplectic-to-symplectic reduction and gradient relative equilibrium points

This section introduces a novel cosymplectic-to-symplectic reduction together with an associated class of relative equilibrium points, referred to as gradient relative equilibrium points. The proposed reduction differs from the standard cosymplectic Marsden–Meyer–Weinstein procedure in that it does not rely on Lie symmetries taking values in the kernel of the one-form τ of a cosymplectic manifold, thereby extending the applicability of the method to a broader range of physical systems. Furthermore, the present construction generalises the cosymplectic-to-symplectic reduction developed by Albert in [4, p 640], which is recovered here as a particular case. Finally, the reduction is a modification of a Poisson reduction that cannot be entirely described within the framework of standard Poisson theory, for several reasons that are discussed in detail below.

The cosymplectic-to-symplectic reduction introduced by Albert [4, p 640] is recalled now.

Theorem 3.2.13. *Let (M, ω, τ) be a cosymplectic manifold and let Y be a vector field on M satisfying*

$$\iota_Y \tau = 1, \qquad \iota_Y \omega = -\mathrm{d}f , \tag{3.2.13}$$

for some $f \in \mathscr{C}^\infty(M)$. Suppose that the space M/Y of orbits of Y in M is a manifold and $\pi_Y : M \to M/Y$ is a submersion. Then there exists a symplectic form ω_Y on M/Y and a unique function $f_Y \in \mathscr{C}^\infty(M/Y)$ such that the Reeb vector field R projects onto the Hamiltonian vector field X_{f_Y} on M/Y relative to ω_Y and $\pi_Y^* f_Y = f$.

It is worth noting that the conditions (3.2.13) imply that $Y = R - X_f$, i.e. Y is an evolution vector field and $Rf = 0$. Consequently, this reduction is rather restrictive, although it allows for a reduction relative to a vector field that does not take values in $\ker \tau$.

Proposition 3.2.14. *Every cosymplectic manifold* (M, ω, τ) *induces a unique Poisson bivector* $\Lambda_{\omega,\tau}$ *on* M *that is tangent to the leaves of* $\ker \tau$ *and coincides with the Poisson bivector associated with* ω *on each such a leaf.*

Proof. By definition, a cosymplectic manifold (M, ω, τ) is such that the restriction of ω to each leaf of the integral distribution $\ker \tau$ is symplectic. Therefore, on each leaf, one can define the Poisson bivector associated with the restriction of ω, which naturally gives rise to a Poisson bivector on M. Indeed, this coincides with the Poisson bivector corresponding to the Poisson bracket (1.3.8) canonically defined on cosymplectic manifolds. □

In cosymplectic Darboux coordinates $\{t, x^i, p_i\}$ for (M, ω, τ), one obtains

$$\Lambda_{\omega,\tau} = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_i} .$$

In these coordinates, one sees that the Reeb vector field R and any Hamiltonian vector field X_h are Lie symmetries of $\Lambda_{\omega,\tau}$. Nevertheless, R and the gradient vector fields ∇h, for $h \in \mathscr{C}^\infty(M)$ with $Rh \neq 0$, are not Hamiltonian vector fields relative to the Poisson bivector $\Lambda_{\omega,\tau}$. Indeed, they do not lie in the image of the induced vector bundle morphism

$$\Lambda^\sharp_{\omega,\tau} \colon \vartheta_p \in \mathrm{T}^*M \mapsto (\Lambda_{\omega,\tau})_p(\vartheta_p, \cdot) \in \mathrm{T}M .$$

This implies that the standard cosymplectic Marsden–Meyer–Weinstein reduction cannot be directly applied to reduce the dynamics of such vector fields.

Prior to proving the main result of this subsection, the following lemma is established.

Lemma 3.2.15. *Let* $\Lambda_{\omega,\tau}$ *be the Poisson bivector on* M *associated with* (M, ω, τ). *Then,*

$$\mathscr{L}_{\nabla\Upsilon} \Lambda_{\omega,\tau} = 0 ,$$

for $\Upsilon \in \mathscr{C}^\infty(M)$ *such that* $\iota_{\mathrm{d}(R\Upsilon)} \Lambda_{\omega,\tau} = 0$.

Proof. Recall that $\nabla\Upsilon = (R\Upsilon)R + X_\Upsilon$. Since $\iota_{X_\Upsilon}\tau = 0$, one has that X_Υ is tangent to the leafs of the integrable distribution $\ker \tau$. Moreover, its restriction to any such leaf of the distribution $\ker \tau$ is a Hamiltonian vector field with respect to the restriction of $\Lambda_{\omega,\tau}$ to this leaf, which is symplectic. Indeed, for a vector field X taking values in $\ker \tau$, one has $\iota_X \iota_{X_\Upsilon} \omega = X\Upsilon$ and then, on each integral leaf of $\ker \tau$, it follows that $\iota_{X_\Upsilon}\omega = \mathrm{d}\Upsilon_\tau$, where Υ_τ denotes the restriction of Υ to the particular integral leaf of $\ker \tau$. Therefore, $\mathscr{L}_{X_\Upsilon} \Lambda_{\omega,\tau} = 0$. The assumption $\iota_{\mathrm{d}(R\Upsilon)} \Lambda_{\omega,\tau} = 0$ yields that

$$\mathscr{L}_{(R\Upsilon)R} \Lambda_{\omega,\tau} = (R\Upsilon)\mathscr{L}_R \Lambda_{\omega,\tau} + (\iota_{\mathrm{d}(R\Upsilon)} \Lambda_{\omega,\tau}) \wedge R = 0 ,$$

as claimed. □

Lemma 3.2.15 can also be proven using a coordinate-dependent approach via cosymplectic Darboux coordinates. However, the proof presented above is intrinsic and illustrates the geometric properties of cosymplectic manifolds.

Proposition 3.2.16. (The cosymplectic-to-symplectic reduction theorem) *Let* (M, ω, τ) *be a cosymplectic manifold with a Reeb vector field* R. *Let* $\Upsilon \in \mathscr{C}^\infty(M)$ *satisfy* $\iota_{\mathrm{d}(R\Upsilon)} \Lambda_{\omega,\tau} = 0$ *and* $R\Upsilon \neq 0$ *at any point of* M. *Assume that* $M/\nabla\Upsilon$ *is a manifold and* $\pi_\Upsilon \colon M \to M/\nabla\Upsilon$ *is a submersion. Then,* $\Lambda_{\omega,\tau}$ *projects onto a bivector field* Λ_Υ *on* $M/\nabla\Upsilon$ *giving rise to a symplectic manifold.*

Moreover, if $h \in \mathscr{C}^\infty(M)$ *is such that* $[\nabla\Upsilon, E_h] = 0$*, then* E_h *projects via* π_Υ *onto a vector field* Y_k *on* $M/\nabla\Upsilon$*, which is a Hamiltonian vector field relative to the symplectic form induced by* Λ_Υ *on* $M/\nabla\Upsilon$*. In this case,* $R\Upsilon$ *is a constant and* Y_k *admits a uniquely defined Hamiltonian function* $k \in \mathscr{C}^\infty(M/\nabla\Upsilon)$ *given by*

$$\pi_\Upsilon^* k = h - \Upsilon/c - \int^t [(\nabla\Upsilon)(h - \Upsilon/c)]\mathrm{d}t\,.$$

Proof. Recall that $\nabla\Upsilon = (R\Upsilon)R + X_\Upsilon$ whenever $R\Upsilon \neq 0$. By Lemma 3.2.15, one has $\mathscr{L}_{\nabla\Upsilon}\Lambda_{\omega,\tau} = 0$, hence $\Lambda_{\omega,\tau}$ projects onto $M/\nabla\Upsilon$ via the projection $\pi\colon M \to M/\nabla\Upsilon$ to a Poisson bivector Λ_Υ.

To prove that Λ_Υ gives rise to a symplectic form, suppose, by contradiction, that Λ_Υ is degenerate at some point $y \in M/\nabla\Upsilon$. Then there exists a nonzero covector $\vartheta_y \in \mathrm{T}^*_y(M/\nabla\Upsilon)$ such that $(\Lambda_\Upsilon)_y(\vartheta_y, \cdot) = 0$. Hence, for every $x \in \pi^{-1}(y)$ and every $\vartheta'_y \in \mathrm{T}^*_y(M/\nabla\Upsilon)$, one has

$$(\Lambda_{\omega,\tau})_x(\vartheta_y \circ \pi_{*x}, \vartheta'_y \circ \pi_{*x}) = 0\,.$$

Thus, $\vartheta_y \circ \pi_{x*}$ is orthogonal relative to $\Lambda_{\omega,\tau}$ at x to the annihilator of $\langle\nabla\Upsilon\rangle_x$. Let $\langle\nabla\Upsilon\rangle_x^{\Lambda_{\omega,\tau}}$ denotes the orthogonal to annihilator of $\langle\nabla\Upsilon\rangle_x$ relative to $\Lambda_{\omega,\tau}$. Since Λ_Υ is a bivector field on an even-dimensional manifold, its kernel must be even-dimensional. Therefore, in addition to $\vartheta_y \in \ker\Lambda_\Upsilon$, there exists a linearly independent covector $\vartheta'_y \in \mathrm{T}^*_y(M/\nabla\Upsilon)$ belonging to $\ker\Lambda_\Upsilon$. Their pullbacks via π^*_x give two linearly independent elements in $\langle\nabla\Upsilon\rangle_x^{\Lambda_{\omega,\tau}}$.

Furthermore, $\mathrm{d}t_x$ belongs to $\langle\nabla\Upsilon\rangle_x^{\Lambda_{\omega,\tau}}$ at x, because it belongs to $\ker(\Lambda_{\omega,\tau})_x$. However, $\mathrm{d}t_x$ cannot be the pull-back via π_{x*} of any element of $\mathrm{T}^*_y(M/\nabla\Upsilon)$, since $\iota_{\nabla\Upsilon}\mathrm{d}t = R\Upsilon \neq 0$. Therefore, $\langle\nabla\Upsilon\rangle_x^{\Lambda_{\omega,\tau}}$ has dimension at least three. This contradicts the fact that $\Lambda_{\omega,\tau}$ has rank $2n$ and that the orthogonal complement relative to $\Lambda_{\omega,\tau}$ of a codimension k subspace has dimension at most $k+1$. Hence, Λ_Υ is nondegenerate and defines a symplectic form on $M/\nabla\Upsilon$.

Consider now a vector field E_h. Since $[E_h, \nabla\Upsilon] = 0$ by assumption, then E_h projects onto a vector field Z on $M/\nabla\Upsilon$. Hence, $\mathscr{L}_Z\Lambda_\Upsilon = 0$ and Z is locally Hamiltonian with respect to the symplectic form associated with Λ_Υ. Note that $\iota_{\mathrm{d}(R\Upsilon)}\Lambda_{\omega,\tau} = 0$ implies that $R\Upsilon$ is, in cosymplectic Darboux coordinates, a function depending only on time. Then, $X_h(R\Upsilon) = 0$ and

$$0 = [E_h, \nabla\Upsilon] = [R + X_h, (R\Upsilon)R + X_\Upsilon] = (R^2\Upsilon)R + [R, X_\Upsilon] + R\Upsilon[X_h, R] + [X_h, X_\Upsilon]\,.$$

Hence, $R^2\Upsilon = 0$ and since $R\Upsilon$ depends only on t in cosymplectic Darboux coordinates, it follows that $R\Upsilon$ is a nonzero constant c. Thus,

$$0 = [E_h, \nabla\Upsilon] = X_{R\Upsilon} - cX_{Rh} - X_{\{h,\Upsilon\}} = X_c - cX_{Rh} - X_{\{h,\Upsilon\}} = -cX_{Rh} - X_{\{h,\Upsilon\}} = X_{-cRh-\{h,\Upsilon\}}. \tag{3.2.14}$$

Consequently, $-\{h,\Upsilon\} - cRh$ depends only on time in cosymplectic Darboux coordinates.

Since $\nabla\Upsilon$ projects onto zero vector field on $M/\nabla\Upsilon$, the projection of E_h onto $M/\nabla\Upsilon$ coincides with the projection of $E_h - \nabla\Upsilon/c = X_{h-\Upsilon/c}$. The fact that $X_{h-\Upsilon/c}$ is a Hamiltonian vector field relative to $\Lambda_{\omega,\tau}$ and (3.2.14) give

$$\nabla\Upsilon(h - \Upsilon/c) = \{h, \Upsilon\} + cRh - c = g(t)\,,$$

for a certain function $g(t)$ in cosymplectic Darboux coordinates. Hence,

$$\nabla\Upsilon\left(h - \Upsilon/c - \frac{1}{c}\int^t g(t')\mathrm{d}t'\right) = 0$$

and

$$h - \Upsilon/c - \int^t [\nabla\Upsilon(h - \Upsilon/c)](t')dt'/c$$

is the pull-back of a function on $M/\nabla\Upsilon$, namely $\pi^*_\Upsilon k = h - \Upsilon/c - \int^t[(\nabla\Upsilon)(h - \Upsilon/c)]\mathrm{d}t$ for some $k \in \mathscr{C}^\infty(M/\nabla\Upsilon)$. Moreover, $\Lambda_{\omega,\eta}(\mathrm{d}(h - \Upsilon/c - \int^t g(t')\mathrm{d}t'/c), \cdot)$ is projectable onto $M/\nabla\Upsilon$ giving rise to a vector field

$$\Lambda_\Upsilon(\mathrm{d}k, \cdot) = Y_k = Z\,.$$

This completes the proof. □

Note that the presented reduction allows for studying general evolution vector fields, thus providing a more general framework than Albert's cosymplectic-to-symplectic reduction, which applies only to the case where the Hamiltonian is the first integral of the Reeb vector field [4]. Moreover, the cosymplectic-to-symplectic reduction developed here makes it possible to reduce an evolution vector field relative to another vector field, yielding a more general reduction than the one presented by Albert in [4]. In particular, Albert's reduction arises as a particular case of this reduction for $E_h = R$, $Y = R - X_f = \nabla(t-f)$, and $\Upsilon = t-f$, where f is a function such that $Rf = 0$ and t denotes a cosymplectic Darboux time coordinate. Since R is a Hamiltonian vector field with zero Hamiltonian function $h = 0$ and $c = R\Upsilon = 1$, it follows that

$$\nabla(t-f)(0-(t-f)) = -1 \;\Rightarrow\; \pi_Y^* k = -t + f + \int^t \mathrm{d}t' = f \,.$$

Consequently, this reduction ensures that R projects onto a vector field on M/Y with a Hamiltonian function k such that $f = \pi_\Upsilon^* k$.

Then, the following definition of a *gradient relative equilibrium point* can be introduced.

Definition 3.2.17. Let (M, ω, τ) be a cosymplectic manifold, let $h \in \mathscr{C}^\infty(M)$ be a Hamiltonian function, and let $\Phi\colon G \times M \to M$ be a cosymplectic Lie group action on M whose fundamental vector fields are of the form $\xi_M = \nabla\Upsilon$ for some $\Upsilon \in \mathscr{C}^\infty(M)$ and $\xi \in \mathfrak{g}$ such that $\iota_{\mathrm{d}(R\Upsilon)}\Lambda_{\omega,\tau} = 0$. Then, a *gradient relative equilibrium point* of h is a point $z_e \in M$ such that

$$\nabla h_{z_e} = (\xi_M)_{z_e} \,,$$

for a certain fundamental vector field $\nabla\Upsilon$ associated with Φ.

Since the evolution vector field E_h, by definition, satisfies $\iota_{E_h}\tau \neq 0$, it follows that, at gradient relative equilibrium points and in an open neighbourhood of such points, the fundamental vector field $\nabla\Upsilon$ satisfies the condition required for the cosymplectic-to-symplectic reduction described above. After the reduction, standard symplectic methods can be applied to analyse the stability of the corresponding projected system.

It is worth observing that the initial step of the cosymplectic-to-symplectic reduction, namely the projection of the Poisson bivector $\Lambda_{\omega,\tau}$, can be described as a particular case of the Poisson reduction. However, the general Poisson reduction framework does not take into account the specific features of the bivector $\Lambda_{\omega,\tau}$ arising from a cosymplectic structure, nor does it cover the reduction scheme based on gradient vector fields that are not Hamiltonian with respect to $\Lambda_{\omega,\tau}$.

3.2.7 Example: Reduced circular three-body problem

This subsection shows that the cosymplectic Marsden–Meyer–Weinstein reduction presented in Subsection 2.2.3 is not always sufficient for the analysis of certain physically relevant systems, and that the cosymplectic-to-symplectic reduction developed in the previous subsection becomes necessary.

Consider a planar three-body system consisting of three masses μ, $1-\mu$, and m, which are moving on a plane and interact via Newtonian gravitation. For convenience, the gravitational constant is taken to be equal to one. Furthermore, assume that μ is much larger than $1-\mu$. Without loss of generality, one may set $m = 1$, since the extension to arbitrary m follows straightforwardly. In addition, suppose that the mass $1-\mu$ moves around μ in a stable circular orbit with constant angular frequency ϖ, and that the presence of m does not influence the motion of μ and $1-\mu$. Physically, this happens when m is negligible in comparison with μ and $1-\mu$. Collisions are excluded from the discussion. The resulting model corresponds to the circular restricted three-body problem [4, 61], a standard framework in celestial mechanics (see, for instance, [2, p 663] or [60] and references therein).

Under these assumptions, the centre of mass of the system lies on the line connecting the bodies of mass μ and $1-\mu$, at distances $r_1 = 1-\mu$ and $r_2 = \mu$, respectively. This approximation accurately describes, for example, the Sun-Earth-satellite configuration, where the Earth moves around the Sun on a circular orbit of radius one, i.e. $r_1 + r_2 = 1$, with frequency ϖ, and the satellite is affected by the gravitational attraction of the Sun and the Earth while having no significant influence on the motion of the Sun and the Earth. Several other astronomical systems can also be modelled in this manner [60].

Mathematically, this model is described by a time-dependent Hamiltonian function h defined on the phase space of a particle moving in the plane. Formally, the phase space is $\mathbb{R}\times \mathrm{T}^*\mathbb{R}^2$, where the factor $\mathbb{R}$ represents time, and the Hamiltonian is given by a function $h\colon \mathbb{R}\times\mathrm{T}^*\mathbb{R}^2\to\mathbb{R}$. In coordinates adapted to polar variables on $\mathbb{R}^2$, namely r,φ, together with the corresponding canonical momenta p_r, p_φ and the time variable t, the Hamiltonian takes the form

$$h(t,r,\varphi,p_r,p_\varphi) = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2r^2} - \frac{\mu}{[r^2+r_1^2+2rr_1\cos(\varphi-\varpi t)]^{1/2}} - \frac{1-\mu}{[r^2+r_2^2-2rr_2\cos(\varphi-\varpi t)]^{1/2}}\,.$$

Technical issues related to the lack of differentiability of h at collision points will not be considered, since they are irrelevant for the subsequent discussion.

The dynamics is described on the cosymplectic manifold $(\mathbb{R}\times\mathrm{T}^*\mathbb{R}^2, \omega_{TB}, \tau_{TB} = \mathrm{d}t)$, where ω_{TB} is the pull-back to $\mathbb{R}\times\mathrm{T}^*\mathbb{R}^2$ of the canonical symplectic form on $\mathrm{T}^*\mathbb{R}^2$, namely $\omega_{TB} = \mathrm{d}r\wedge\mathrm{d}p_r + \mathrm{d}\varphi\wedge\mathrm{d}p_\varphi$ in the chosen coordinates (see [4] for a different approach using old techniques in cosymplectic geometry).

The evolution vector field associated with h is of the form $R_{TB}+X_h$, namely

$$\frac{\partial}{\partial t} - \left(\frac{\mu(r+r_1\cos(\varphi-\varpi t))}{(r^2+2rr_1\cos(\varphi-\varpi t)+r_1^2)^{3/2}} + \frac{(1-\mu)(r-r_2\cos(\varphi-\varpi t))}{(r^2-2rr_2\cos(\varphi-\varpi t)+r_2^2)^{3/2}} - \frac{p_\varphi^2}{r^3}\right)\frac{\partial}{\partial p_r} + p_r\frac{\partial}{\partial r}$$
$$+\frac{p_\varphi}{r^2}\frac{\partial}{\partial\varphi} + \left(\frac{\mu r r_1\sin(\varphi-\varpi t)}{(r^2+2rr_1\cos(\varphi-\varpi t)+r_1^2)^{3/2}} - \frac{(1-\mu)rr_2\sin(\varphi-\varpi t)}{(r^2-2rr_2\cos(\varphi-\varpi t)+r_2^2)^{3/2}}\right)\frac{\partial}{\partial p_\varphi}.$$

The Hamilton equations corresponding to h read

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}t} &= p_r, \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{p_\varphi}{r^2},\\ \frac{\mathrm{d}p_r}{\mathrm{d}t} &= \frac{p_\varphi^2}{r^3} - \frac{\mu(r+r_1\cos(\varphi-\varpi t))}{(r^2+2rr_1\cos(\varphi-\varpi t)+r_1^2)^{3/2}} - \frac{(1-\mu)(r-r_2\cos(\varphi-\varpi t))}{(r^2-2rr_2\cos(\varphi-\varpi t)+r_2^2)^{3/2}},\\ \frac{\mathrm{d}p_\varphi}{\mathrm{d}t} &= \frac{\mu rr_1\sin(\varphi-\varpi t)}{(r^2+2rr_1\cos(\varphi-\varpi t)+r_1^2)^{3/2}} - \frac{(1-\mu)rr_2\sin(\varphi-\varpi t)}{(r^2-2rr_2\cos(\varphi-\varpi t)+r_2^2)^{3/2}}.\end{aligned} \tag{3.2.15}$$

Consider the vector field on $\mathbb{R}\times\mathbb{R}^2$ of the form

$$Y = \frac{\partial}{\partial t} + \varpi\frac{\partial}{\partial\varphi}\,.$$

Denote by $\widehat{Y}$ the fundamental vector field on $\mathbb{R}\times\mathrm{T}^*\mathbb{R}^2$ corresponding to the lifted action of the Lie group $\mathbb{R}$ on $\mathbb{R}\times\mathbb{R}^2$ induced by the flow of Y, associated with the same element of the Lie algebra of $\mathbb{R}$ (see Section 2.2.1).

The vector field $\widehat{Y}$ is a cosymplectic vector field, namely $\mathscr{L}_{\widehat{Y}}\omega_{TB} = 0$ and $\mathscr{L}_{\widehat{Y}}\tau_{TB} = 0$. In fact, $\widehat{Y}$ is the gradient vector field associated with the function $\Upsilon = t + p_\varphi\varpi$, for which $R_{TB}\Upsilon$ is a constant. Note that $\widehat{Y}$ is not a Hamiltonian vector field relative to the cosymplectic manifold $(\mathbb{R}\times\mathrm{T}^*\mathbb{R}^2,\omega_{TB},\tau_{TB}=\mathrm{d}t)$, since $\iota_{\widehat{Y}}\tau_{TB}\neq 0$. Moreover, $\widehat{Y}$ is a Lie symmetry of the Hamiltonian function h, which follows from the fact that $\widehat{Y}$ takes the same form as Y but in the coordinates $\{t,r,\varphi,p_r,p_\varphi\}$. At this point, it becomes clear that Theorem 2.2.13 is not applicable.

It is relevant to find the gradient relative equilibrium points of h. Physically, it corresponds to the situation where the mass m moves around the centre of mass at a fixed distance r and frequency ϖ. The standard notion of a cosymplectic relative equilibrium point for $(\mathbb{R}\times\mathrm{T}^*\mathbb{R}^2,\omega_{TB},\tau_{TB}=\mathrm{d}t)$ does not apply in this case, because $\widehat{Y}$ is not a Hamiltonian vector field. Similarly, the techniques developed in Section 3.1 are not applicable, as $\widehat{Y}$ is not tangent to $\mathrm{T}^*\mathbb{R}^2$. For this reason, Theorem 3.2.16 is employed.

By Definition 3.2.17, a point $z_e \in \mathbb{R} \times \mathrm{T}^*\mathbb{R}^2$ is a gradient relative equilibrium point if $R_{TB} + X_h$ and $\widehat{Y}$ are proportional to each other at z_e. If this occurs at a point $(t, r, \varphi, p_r, p_\varphi)$, then the last expression in the Hamilton equations (3.2.15) is equal to zero and $\varphi = \varpi t + k\pi$, with $k \in \mathbb{Z}$, or $\varphi - \varpi t$ is such that the distance between the mass from m to μ and from m to $1-\mu$ are the same. In the latter case, one can prove that $\varphi - \varpi t = \Delta$ and $r\cos\Delta = \mu - 1/2$. Note that it is sufficient to restrict to the cases where $k \in \{0, 1\}$. The remaining equations for the gradient relative equilibrium points read as follows

$$\frac{p_\varphi^2}{r^3} - \frac{\mu(r + r_1\cos(\varphi - \varpi t))}{(r^2 + 2rr_1\cos(\varphi - \varpi t) + r_1^2)^{3/2}} - \frac{(1-\mu)(r - r_2\cos(\varphi - \varpi t))}{(r^2 - 2rr_2\cos(\varphi - \varpi t) + r_2^2)^{3/2}} = 0,$$

$$p_r = 0, \qquad \frac{p_\varphi}{r^2} = \varpi.$$

Since the masses $1-\mu$ and μ spin around their centre of mass, located at $r = 0$, with constant angular velocity ϖ due to their gravitational attraction, one gets

$$\frac{\mu}{(r_1 + r_2)^2} = \varpi^2 r_2 \;\Rightarrow\; \varpi = \pm 1.$$

Note that the force is determined by the relative distance between the masses, while the centripetal force is considered relative to the inertial reference system at the centre of mass of the system of μ and $1-\mu$.

Consider then the three cases for the relations between φ and t for gradient relative equilibrium points, namely

$$\varphi = \varpi t\,, \qquad \varphi = \varpi t + \pi\,, \qquad \varphi = \varpi t + \Delta\,.$$

In the first case, one gets

$$r = \frac{\mu}{(r + 1 - \mu)^2} \mp \frac{1-\mu}{(\mu - r)^2}\,, \tag{3.2.16}$$

which correspond precisely to the equations for the centripetal force of a circular motion induced by the gravitational force of the masses μ and $1-\mu$ when the three objects move in circles with a frequency ϖ, while remaining collinear along a line rotating with this frequency about the origin.

Then, Equation (3.2.16) leads to two quintic equations

$$\begin{aligned} P_\pm(r,\mu) := r^5 &+ (2-4\mu)r^4 + \left(6\mu^2 - 6\mu + 1\right) r^3 + \left(-4\mu^3 + 6\mu^2 - (3\pm 1)\mu \pm 1\right) r^2 \\ &+ \left(\mu^4 - 2\mu^3 + (3\pm 2)\mu^2 \mp (4\mu - 2)\right) r - \mu^3 \pm (1-\mu)^3 = 0\,, \end{aligned} \tag{3.2.17}$$

for the gradient relative equilibrium position of r, which has always a root in $]0, \infty[$ since the polynomial has negative value at $r = 0$; the value of μ is approximately equal to 1 with $\mu < 1$; and the value of the polynomial (3.2.17) tends to infinity when r does so. Each of the above two equations in (3.2.16) has just one real solution.

The analysis of the gradient relative equilibrium points can be carried out approximately as follows. The quintic polynomial has a triple root $r = 1$ for $\mu = 1$. Write $r = 1 + \sum_{n\in\mathbb{N}} \delta^{n/3} x_n$ for certain constants $\{x_n\}_{n\in\mathbb{N}}$ and a parameter $\delta \geq 0$. Then, consider the quintic polynomial as

$$P_\pm(r,\mu) = \sum_{n=0}^{\infty} P_{\pm n}(r)\delta^{1+n/3}\,,$$

for $\delta = 1-\mu$, and look for solutions of $P_\pm(r(\delta), \delta) = 0$ for every δ in some $[0, \delta_{\max}[$. For $\delta = 0$, one gets that $P_\pm(r, 1)$ has a triple root $r = 1$ and

$$0 = P_\pm(r(\delta), \delta) = (\pm 1 + 3x^3)\delta + (\pm 2x + 3x^4 + 9x^2 y)\delta^{4/3} + \ldots,$$

The convergence of solutions of $P_\pm(r(\delta), \delta) = 0$ can be obtained by the implicit function theorem and writing $P_\pm(r(\delta), \delta)$ in an appropriate manner. Then, the equilibrium points of order $\delta^{1/3}$ are given by

$$r = 1 \mp \sqrt[3]{\frac{1-\mu}{3}}\,.$$

This expression agrees with the well-known approximation of the Hill radii of the Hill spheres. Since $r > 0$, both solutions are admissible. Therefore, the model reproduces two gradient relative equilibrium points for $k = 0$, which, by convention, are denoted by L_2 and L_1, corresponding to the positive and negative signs in (3.2.16), respectively.

Meanwhile, for $k = 1$, the equations determining the gradient relative equilibrium points take the form

$$r = \pm\frac{\mu}{(r-1+\mu)^2} + \frac{1-\mu}{(\mu+r)^2}. \tag{3.2.18}$$

The case of (3.2.18) with the minus sign in ± amounts to one of the equations in (3.2.16) with $-r$. Since (3.2.16) admits only one real positive solution for each choice of signs, it follows that (3.2.18) has no positive solution with the minus sign in ±. Consequently, only the following equation is of physical interest

$$r = \frac{\mu}{(r-1+\mu)^2} + \frac{1-\mu}{(\mu+r)^2}\,.$$

It can be written as a polynomial in terms of r and $\delta = 1-\mu$ of the form

$$0 = (1-r)S(r) + \delta Q(r,\delta)\,, \qquad S(r) = -(1+r)^2(1+r+r^2)\,,$$

$$Q(r,\delta) = 3 + 4r - 2r^2 - 6r^3 - 4r^4 + (-3+r+6r^2+6r^3)\delta + (-2r-4r^2)\delta^2 + r\delta^3\,.$$

For $\mu = 1$, this gives a solution $r = 1$. Assume now that $r = 1 + \lambda\delta$. Neglecting second and higher-order terms in δ, one obtains the approximate equation

$$0 = \lambda\delta S(1) + \delta Q(1,0) = \lambda 12\delta - \delta 5\,,$$

which yields

$$r = 1 + (1-\mu)\frac{5}{12}\,.$$

This expression coincides with the approximated value of the Lagrange point L_3. Two additional Lagrange points appear when considering the case $\varphi - \varpi t = \Delta$. It can be directly verified that these correspond to the well-known Lagrange points L_4 and L_5.

The main conclusion is that, at every Lagrange point, the following condition holds

$$R_{TB} + X_h = \nabla\Upsilon\,,$$

where $\Upsilon = t + \varpi p_\varphi$, for every $t \in \mathbb{R}$. Recall that $\nabla\Upsilon$ is a symmetry of both τ_{TB} and ω_{TB}. However, it does not take values in $\ker \tau_{TB}$, which makes the standard cosymplectic reduction inapplicable. Consequently, Theorem 3.2.16 must be used.

The projection from $\mathbb{R} \times \mathrm{T}^*\mathbb{R}^2$ onto the quotient space $(\mathbb{R} \times \mathrm{T}^*\mathbb{R}^2)/\nabla\Upsilon \simeq \mathbb{R}^2 \times \mathbb{R}^2$ corresponding to the orbit space of the integral curves of $\nabla\Upsilon$, is given by

$$\pi\colon (t, r, \varphi, p_r, p_\varphi) \in \mathbb{R} \times \mathrm{T}^*\mathbb{R}^2 \mapsto (r, \varphi - t\varpi, p_r, p_\varphi) \in \mathbb{R}^2 \times \mathbb{R}^2\,,$$

where $\{r, \varphi', p_r, p_\varphi\}$ is the chosen global coordinate system on $\mathbb{R}^2 \times \mathbb{R}^2$.

The manifold $\mathbb{R} \times \mathrm{T}^*\mathbb{R}^2$ admits a Poisson bivector on $\mathbb{R} \times \mathrm{T}^*\mathbb{R}^2$ induced by its cosymplectic structure. In the above coordinates, it reads

$$\Lambda_{TB} = \frac{\partial}{\partial\varphi} \wedge \frac{\partial}{\partial p_\varphi} + \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial p_r}.$$

By Theorem 3.2.16, the bivector field Λ_{TB} projects onto the quotient space of the orbits of the gradient vector field $\nabla\Upsilon$ under the projection π, and it reads

$$\pi_*\Lambda_{TB} = \frac{\partial}{\partial\varphi'} \wedge \frac{\partial}{\partial p_\varphi} + \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial p_r}\,, \qquad \pi_* R_{TB} = -\varpi\frac{\partial}{\partial\varphi'}\,, \qquad \pi_*\nabla\Upsilon = 0$$

and

$$\begin{aligned}\pi_*(R_{TB}+X_h) = p_r\frac{\partial}{\partial r} + \left(-\varpi + \frac{p_\varphi}{r^2}\right)\frac{\partial}{\partial \varphi'} \\ - \left(\frac{\mu(r+r_1\cos\varphi')}{(r^2+2rr_1\cos\varphi'+r_1^2)^{3/2}} + \frac{(1-\mu)(r-r_2\cos\varphi')}{(r^2-2rr_2\cos\varphi'+r_2^2)^{3/2}} - \frac{p_\varphi^2}{r^3}\right)\frac{\partial}{\partial p_r} \\ - r_1 r r_2 \sin\varphi'\left(\frac{1}{(r^2+2rr_1\cos\varphi'+r_1^2)^{3/2}} + \frac{1}{(r^2-2rr_2\cos\varphi'+r_2^2)^{3/2}}\right)\frac{\partial}{\partial p_\varphi}\,.\end{aligned}$$

The vector field $\pi_*(R_{TB}+X_h)$ vanishes only at the image under π of the gradient relative equilibrium points. Furthermore, the quotient manifold is a symplectic manifold, and $\pi_*(R_{TB}+X_h)$ is a Hamiltonian vector field with a Hamiltonian function

$$k(r,\varphi',p_r,p_\varphi) = -\varpi p_\varphi + \frac{p_r^2}{2} + \frac{p_\varphi^2}{2r^2} - \frac{\mu}{[r^2+r_1^2+2rr_1\cos\varphi']^{1/2}} - \frac{1-\mu}{[r^2+r_2^2-2rr_2\cos\varphi']^{1/2}}\,.$$

Physically, this system corresponds to an autonomous Hamiltonian system obtained by fixing a coordinate frame rotating around the centre of mass with angular frequency ϖ. From the mathematical perspective, the function k is precisely the Hamiltonian function from Theorem 3.2.16.

This example demonstrates that the cosymplectic framework provides a broader reduction scheme, opening possibilities that extend beyond the scope of the classical theory of the energy-momentum method. In particular, the cosymplectic-to-symplectic reduction of the Poisson bivector field Λ_{TB} can be interpreted as a Poisson reduction with respect to the distribution generated by $\nabla\Upsilon$ (cf. [111]). However, the initial dynamical system is determined by the vector field $R_{TB}+X_h$, which is not Hamiltonian with respect to the Poisson bivector Λ_{TB}. Consequently, its reduction cannot be described by the standard cosymplectic Marsden–Meyer–Weinstein reduction scheme.

3.3 k-Polysymplectic energy momentum-method

This section introduces the concept of the k-polysymplectic relative equilibrium points associated with an $\boldsymbol{\omega}$-Hamiltonian vector field X, see Definition 1.4.10. The notion is devised to investigate the relative stability of $\boldsymbol{\omega}$-Hamiltonian vector fields, thereby extending the classical notion of relative equilibrium points from the symplectic setting to the k-polysymplectic framework.

3.3.1 k-Polysymplectic relative equilibrium points

The following definition introduces the notion of a k-polysymplectic relative equilibrium point. It is worth noting that the idea remains the same as in the classical symplectic setting. That is, a relative equilibrium point of a dynamical system determined by a vector field is a point whose trajectory is entirely described by the action of a Lie group of symmetries of that vector field and the geometric structure.

Definition 3.3.1. Let $(P,\boldsymbol{\omega},\boldsymbol{h},\mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system. A point $z_e\in P$ is a *k-polysymplectic relative equilibrium point* of the $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ if there exists $\xi\in\mathfrak{g}$ so that

$$(X_{\boldsymbol{h}})(z_e) = (\xi_P)(z_e)\,.$$

For $k=1$, this definition recovers the classical notion of a relative equilibrium point for symplectic Hamiltonian systems from Section 3.1, when there is no time-dependence. Moreover, Lemma 2.3.12 together with the fact that $X_{\boldsymbol{h}}$ is tangent to the level sets of $\mathbf{J}^\Phi$ implies that the element $\xi\in\mathfrak{g}$ appearing in Definition 3.3.1 necessarily belongs to the Lie subalgebra $\mathfrak{g}^{\Delta}_{\boldsymbol{\mu}_e}$, where $\boldsymbol{\mu}_e=\mathbf{J}^\Phi(z_e)$.

Note that a k-polysymplectic relative equilibrium point $z_e\in P$ projects onto $\pi_{\boldsymbol{\mu}_e}(z_e)$, with $\boldsymbol{\mu}_e = \mathbf{J}^\Phi(z_e)$. This projected point becomes an equilibrium point of the vector field $X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}}$, obtained by projection of $X_{\boldsymbol{h}}$ onto the reduced space $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$ via k-polysymplectic Marsden–Meyer–Weinstein reduction Theorem 2.3.14. As in the time-dependent symplectic setting, this explains the term *relative* used in the definition of *relative equilibrium points.*

The following theorem characterises k-polysymplectic relative equilibrium points of an $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ in terms of the critical points of a modified $\mathbb{R}^k$-valued function $\boldsymbol{h}_\xi$ on P. Analogously to the previous sections, this is an application of the Lagrange multiplier theorem, where the role of the multiplier is played by $\xi \in \mathfrak{g}$.

Theorem 3.3.2. *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system. Then, $z_e \in P$ is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if there exists $\xi \in \mathfrak{g}$ such that z_e is a critical point of the following $\mathbb{R}^k$-valued function*

$$\boldsymbol{h}_\xi := \boldsymbol{h} - \langle \mathbf{J}^\Phi - \boldsymbol{\mu}_e, \xi \rangle \,, \tag{3.3.1}$$

where $\boldsymbol{\mu}_e := \mathbf{J}^\Phi(z_e) \in \mathfrak{g}^{*k}$.

Proof. Let z_e be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$, i.e. $X_{\boldsymbol{h}}(z_e) = \xi_P(z_e)$ for some $\xi \in \mathfrak{g}$. Then,

$$\mathrm{d}\boldsymbol{h}_\xi(z_e) = \mathrm{d}(\boldsymbol{h} - \langle \mathbf{J}^\Phi, \xi \rangle)(z_e) = (\iota_{X_{\boldsymbol{h}} - \xi_P} \boldsymbol{\omega})(z_e) = 0 \,.$$

Hence, $z_e \in P$ is a critical point of the $\mathbb{R}^k$-valued function $\boldsymbol{h}_\xi$.

Conversely, assume that z_e is a critical point of some $\boldsymbol{h}_\xi$ with $\xi \in \mathfrak{g}$. Then,

$$0 = \mathrm{d}\boldsymbol{h}_\xi(z_e) = (\iota_{X_{\boldsymbol{h}} - \xi_P} \boldsymbol{\omega})(z_e) = 0 \,,$$

and $(X_{\boldsymbol{h}} - \xi_P)(z_e) \in \ker \boldsymbol{\omega}_{z_e}$. Since $\ker \boldsymbol{\omega} = 0$, it follows that $X_{\boldsymbol{h}}(z_e) = \xi_P(z_e)$. Hence, z_e is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$. □

The following example illustrates the k-polysymplectic energy-momentum method.

Example 3.3.3. Consider a two-polysymplectic manifold $(\mathbb{R}^6, \boldsymbol{\omega})$ with the two-polysymplectic form

$$\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 = (\mathrm{d}x_1 \wedge \mathrm{d}x_3 + \mathrm{d}x_2 \wedge \mathrm{d}x_4) \otimes e_1 + (\mathrm{d}x_1 \wedge \mathrm{d}x_5 + \mathrm{d}x_2 \wedge \mathrm{d}x_6) \otimes e_2 \,.$$

Note that

$$\ker \omega^1 = \left\langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\rangle \,, \qquad \ker \omega^2 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\rangle \,,$$

and $\ker \omega^1 \cap \ker \omega^2 = 0$. Define the Lie group action $\Phi \colon \mathbb{R} \times \mathbb{R}^6 \to \mathbb{R}^6$ by

$$\Phi \colon (\lambda; x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R} \times \mathbb{R}^6 \longmapsto (x_1 + \lambda, x_2 + \lambda, x_3 + \lambda, x_4 + \lambda, x_5 + \lambda, x_6 + \lambda) \in \mathbb{R}^6 \,.$$

The fundamental vector field associated with the Lie group action Φ is spanned by

$$\xi_P = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \,.$$

This Lie group action is two-polysymplectic since $\mathscr{L}_{\xi_P} \boldsymbol{\omega} = 0$. Then, Φ gives rise to a two-polysymplectic momentum map $\mathbf{J}^\Phi$ for $\boldsymbol{\mu} = (\mu^1, \mu^2)$ given by

$$\mathbf{J}^\Phi \colon (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \longmapsto (x_3 + x_4 - x_1 - x_2, x_5 + x_6 - x_1 - x_2) = \boldsymbol{\mu} \in \mathbb{R}^{*2} \,.$$

Therefore, the level set of the two-polysymplectic momentum map $\mathbf{J}^\Phi$ is of the form

$$\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \left\{ (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \mid x_3 + x_4 - x_1 - x_2 = \mu^1, \; x_5 + x_6 - x_1 - x_2 = \mu^2 \right\} \,. \tag{3.3.2}$$

Note that every $\boldsymbol{\mu} \in \mathbb{R}^{*2}$ is a regular two-value of a two-polysymplectic momentum map $\mathbf{J}^\Phi$ and $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \simeq \mathbb{R}^4$. In addition, $\mathbf{J}^\Phi$ is an Ad^{*2}-equivariant. Then,

$$\begin{aligned} \mathrm{T}_p(G_{\boldsymbol{\mu}}p) = \mathrm{T}_p(G_{\mu^1}p) = \mathrm{T}_p(G_{\mu^2}p) &= \left\langle \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \right\rangle , \\ \ker \mathrm{T}_p\mathbf{J}^\Phi_1 &= \left\langle \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\rangle , \\ \ker \mathrm{T}_p\mathbf{J}^\Phi_2 &= \left\langle \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_6} \right\rangle , \\ \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) &= \left\langle \sum_{i=1}^{6} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right\rangle , \end{aligned}$$

and one can verify that conditions (2.3.3) and (2.3.4) are satisfied.

Since the Lie group $\mathbb{R}$ act by translations on $\mathbb{R}^6$ via Φ, Theorem 2.3.14 implies that the reduced manifold $(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}} \simeq \mathbb{R}^3, \boldsymbol{\omega}_{\boldsymbol{\mu}})$ is a two-polysymplectic manifold with coordinates $(y_1, y_2, y_3) \in \mathbb{R}^3$, satisfying that

$$\begin{array}{lll} y_1 = x_1 - x_2 \,, & y_2 = x_3 - x_1 \,, & y_3 = x_5 - x_1 \,, \\ y_4 = x_1 + x_2 - x_3 - x_4 \,, & y_5 = x_1 + x_2 - x_5 - x_6 \,, & y_6 = x_1 \,, \end{array}$$

with

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \omega^1_{\boldsymbol{\mu}} \otimes e_1 + \omega^2_{\boldsymbol{\mu}} \otimes e_2 = \mathrm{d}y_1 \wedge \mathrm{d}y_2 \otimes e_1 + \mathrm{d}y_1 \wedge \mathrm{d}y_3 \otimes e_2 \,.$$

Next, consider an $\boldsymbol{\omega}$-Hamiltonian vector field, $X_{\boldsymbol{h}}$, on $P = \mathbb{R}^6$ whose $\boldsymbol{\omega}$-Hamiltonian function is $\mathbb{R}$-invariant. Then, $X_{\boldsymbol{h}}$ is tangent to each level set $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, and can be expressed as

$$X_{\boldsymbol{h}} = F_1 \sum_{i=1}^{6} \frac{\partial}{\partial x_i} + F_2 \left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \right) + F_3 \left(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5} \right) + F_4 \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) ,$$

for certain uniquely defined G-invariant functions $F_1, \ldots, F_4 \in \mathscr{C}^\infty(P)$. By definition, a point $z_e \in P$ is a two-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if $X_{\boldsymbol{h}}(z_e) = \xi_P(z_e)$, which holds, if and only if, $F_1(z_e) = 1$ and $F_2(z_e) = F_3(z_e) = F_4(z_e) = 0$. The next step is to verify this using Theorem 3.3.2.

First, one has that

$$\begin{aligned} \mathrm{d}h^1 &= \iota_{X_{\boldsymbol{h}}}\omega^1 = -\left(F_1 + F_2 + F_3\right)\mathrm{d}x_1 - \left(F_1 - F_2\right)\mathrm{d}x_2 + \left(F_1 + F_4\right)\mathrm{d}x_3 + \left(F_1 + F_3 - F_4\right)\mathrm{d}x_4 \,, \\ \mathrm{d}h^2 &= \iota_{X_{\boldsymbol{h}}}\omega^2 = -\left(F_1 + F_3\right)\mathrm{d}x_1 - F_1\mathrm{d}x_2 + \left(F_1 + F_4\right)\mathrm{d}x_5 + \left(F_1 + F_3 - F_4\right)\mathrm{d}x_6 \,. \end{aligned}$$

Then, Theorem 3.3.2 yields that $z_e \in P$ is a two-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if $\mathrm{d}h^1_\xi(z_e) = 0$ and $\mathrm{d}h^2_\xi(z_e) = 0$ for some $\xi \in \mathbb{R}$. Indeed, using (3.3.2), one has

$$\begin{aligned} \mathrm{d}h^1_\xi = \mathrm{d}h^1 - \mathrm{d}J^1_\xi = -\left(F_1 + F_2 + F_3 - \xi\right)\mathrm{d}x_1 - \left(F_1 - F_2 - \xi\right)\mathrm{d}x_2 \\ + \left(F_1 + F_4 - \xi\right)\mathrm{d}x_3 + \left(F_1 + F_3 - F_4 - \xi\right)\mathrm{d}x_4 \,, \end{aligned} \tag{3.3.3}$$

$$\begin{aligned} \mathrm{d}h^2_\xi = \mathrm{d}h^2 - \mathrm{d}J^2_\xi = -\left(F_1 + F_3 - \xi\right)\mathrm{d}x_1 - \left(F_1 - \xi\right)\mathrm{d}x_2 \\ + \left(F_1 + F_4 - \xi\right)\mathrm{d}x_5 + \left(F_1 + F_3 - F_4 - \xi\right)\mathrm{d}x_6 \,, \end{aligned} \tag{3.3.4}$$

for $\xi \in \mathbb{R}$. Since at z_e both (3.3.3) and (3.3.4) must vanish, one gets that this occurs if and only if $F_1(z_e) = \xi$ and $F_2(z_e) = F_3(z_e) = F_4(z_e) = 0$. Therefore, $z_e \in P$ is a two-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ under the above-mentioned conditions.

To verify that $\pi_{\boldsymbol{\mu}_e}(z_e)$ is a critical point of the $f^\alpha_{\boldsymbol{\mu}_e} \in \mathscr{C}^\infty(\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)/G_{\boldsymbol{\mu}_e})$, note that the reduced vector field $X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}}$ is of the form

$$X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}} = (2\widetilde{F}_4 - \widetilde{F}_3)\frac{\partial}{\partial y_1} + (\widetilde{F}_2 + \widetilde{F}_3 - \widetilde{F}_4)\frac{\partial}{\partial y_2} - \widetilde{F}_4\frac{\partial}{\partial y_3}\,,$$

where $F_i = \pi^*_{\boldsymbol{\mu}_e}\widetilde{F}_i$ for $i = 2, 3, 4$. Note that the projection exists because F_2, F_3, F_4 are assumed to be G-invariant functions on P. Then,

$$\begin{aligned} \mathrm{d}f^1_{\boldsymbol{\mu}_e}(\pi_{\boldsymbol{\mu}_e}(z_e)) &= \left(\iota_{X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}}}\omega^1_{\boldsymbol{\mu}_e}\right)_{\pi_{\boldsymbol{\mu}_e}(z_e)} = \\ &= \left(2\widetilde{F}_4(\pi_{\boldsymbol{\mu}_e}(z_e)) - \widetilde{F}_3(\pi_{\boldsymbol{\mu}_e}(z_e))\right)\mathrm{d}y_2 + \left(\widetilde{F}_4(\pi_{\boldsymbol{\mu}_e}(z_e)) - \widetilde{F}_2(\pi_{\boldsymbol{\mu}_e}(z_e)) - \widetilde{F}_3(\pi_{\boldsymbol{\mu}_e}(z_e))\right)\mathrm{d}y_1 = 0\,, \end{aligned}$$

and

$$\mathrm{d}f^2_{\boldsymbol{\mu}_e}(\pi_{\boldsymbol{\mu}_e}(z_e)) = \left(\iota_{X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}}}\omega^2_{\boldsymbol{\mu}_e}\right)_{\pi_{\boldsymbol{\mu}_e}(z_e)} = \left(2\widetilde{F}_4(\pi_{\boldsymbol{\mu}_e}(z_e)) - \widetilde{F}_3(\pi_{\boldsymbol{\mu}_e}(z_e))\right)\mathrm{d}y_3 + \widetilde{F}_4(\pi_{\boldsymbol{\mu}_e}(z_e))\mathrm{d}y_1 = 0\,.$$

Indeed, $\pi_{\boldsymbol{\mu}_e}(z_e)$ is a critical point of $\boldsymbol{f}_{\boldsymbol{\mu}_e}$, hence $z_e \in P$ is a two-polysymplectic relative equilibrium point of $X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}}$.

3.3.2 Stability in the k-polysymplectic energy momentum-method

This subsection develops the stability analysis associated with the k-polysymplectic energy-momentum method relative to a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. Recall that Theorem 3.3.2 characterises k-polysymplectic relative equilibrium points as the critical points of the $\mathbb{R}^k$-valued function (3.3.1). However, as in the precious setting, when studying the stability of k-polysymplectic relative equilibrium points, the presence of symmetry requires to investigate how the second variation of $\boldsymbol{h}_\xi$ along directions tangent to the isotropy group $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$ influence the positive definiteness of $\boldsymbol{h}_\xi$. Furthermore, the results presented in this subsection apply exclusively to situations where k-polysymplectic reduction is possible and conditions (2.3.3) and (2.3.3) are satisfied.

Define the second variation of $\boldsymbol{h}_\xi$ at a k-polysymplectic relative equilibrium point $z_e \in \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ as the mapping $(\delta^2\boldsymbol{h}_\xi)_{z_e} \colon \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e) \times \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e) \to \mathbb{R}$, with $\boldsymbol{\mu}_e = \mathbf{J}^\Phi(z_e)$, of the form

$$\left(\delta^2\boldsymbol{h}_\xi\right)_{z_e}(v_1, v_2) = \sum_{\alpha=1}^k \iota_Y\left(\mathrm{d}\left(\iota_X\mathrm{d}h^\alpha_\xi\right)\right)_{z_e} \otimes e_\alpha\,, \tag{3.3.5}$$

for some vector fields X, Y on P defined on a neighbourhood of $z_e \in P$ and such that $v_1 = X_{z_e}$, $v_2 = Y_{z_e}$. Note that (3.3.5) serve as a generalisation of (3.1.4) to k-polysymplectic setting. The following proposition shows that, since z_e is a k-polysymplectic relative equilibrium point, the above definition does not depend on the value of the particular chosen vector fields X and Y out of z_e and $(\delta^2\boldsymbol{h}_\xi)_{z_e}$ is well-defined.

Proposition 3.3.4. *Let $z_e \in P$ be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ on a k-polysymplectic manifold $(P, \boldsymbol{\omega})$. If $\{x_1, \ldots, x_n\}$ are local coordinates on a neighbourhood of $z_e \in P$, then*

$$(\delta^2h^\alpha_\xi)_{z_e}(w, v) = \sum_{i,j=1}^n \frac{\partial^2 h^\alpha_\xi}{\partial x_i\partial x_j}(z_e)w_iv_j\,, \qquad \forall w, v \in \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)\,, \qquad \alpha = 1, \ldots, k\,,$$

where $w = \sum_{i=1}^n w_i\partial/\partial x_i$ and $v = \sum_{i=1}^n v_i\partial/\partial x_i$.

Proof. From (3.3.5) for $\alpha = 1, \ldots, k$, one has

$$\begin{aligned} (\delta^2h^\alpha_\xi)_{z_e}(w, v) &= \iota_Y(\mathrm{d}\iota_X\mathrm{d}h^\alpha_\xi)_{z_e} \\ &= \sum_{i,j=1}^n \frac{\partial^2 h^\alpha_\xi}{\partial x_i\partial x_j}(z_e)w_iv_j + \sum_{i,j=1}^n \frac{\partial h^\alpha_\xi}{\partial x_i}(z_e)\frac{\partial X_i}{\partial x_j}(z_e)v_j \\ &= \sum_{i,j=1}^n \frac{\partial^2 h^\alpha_\xi}{\partial x_i\partial x_j}(z_e)w_iv_j\,, \end{aligned}$$

where $X = \sum_{i=1}^{n} X_i \partial/\partial x_i$ with $X(z_e) = w$, and it has been used that z_e is a k-polysymplectic relative equilibrium point and, therefore, $\boldsymbol{h}_\xi$ admits a critical point at z_e, namely $(Zh_\xi^\alpha)(z_e) = 0$ for every vector field Z on P and $\alpha = 1, \ldots, k$. □

Note that each map $(\delta^2 h_\xi^\alpha)_{z_e}$ is symmetric bilinear form for $\alpha = 1, \ldots, k$. Therefore, $(\delta^2 \boldsymbol{h}_\xi)_{z_e}$ is symmetric.

Proposition 3.3.5. *Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system and let $z_e \in P$ be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$. Then,*

$$(\delta^2 \boldsymbol{h}_\xi)_{z_e}((\zeta_P)_{z_e}, v_{z_e}) = 0\,, \qquad \forall \zeta \in \mathfrak{g}^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}\,, \qquad \forall v_{z_e} \in \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)\,,$$

with $\boldsymbol{\mu}_e = \mathbf{J}^\Phi(z_e)$. Moreover,

$$(\delta^2 h_\xi^\alpha)_{z_e}(Y_{z_e}, \cdot) = 0\,, \qquad \forall\, Y_{z_e} \in \ker \omega^\alpha_{z_e} \cap \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)\,, \qquad \alpha = 1, \ldots, k\,. \tag{3.3.6}$$

Proof. First, since $\boldsymbol{h} \in \mathscr{C}^\infty(P, \mathbb{R}^k)$ is G-invariant and k-polysymplectic momentum map $\mathbf{J}^\Phi$ is equivariant with respect to the k-polysymplectic affine Lie group action $\boldsymbol{\Delta}\colon G \times \mathfrak{g}^{*k} \to \mathfrak{g}^{*k}$, then for every $g \in G$ and $p \in P$, one has

$$\begin{aligned}\boldsymbol{h}_\xi(\Phi_g(p)) &= \boldsymbol{h}(\Phi_g(p)) - \langle \mathbf{J}^\Phi(\Phi_g(p)), \xi\rangle + \langle \boldsymbol{\mu}_e, \xi\rangle \\ &= \boldsymbol{h}(p) - \langle \boldsymbol{\Delta}_g \mathbf{J}^\Phi(p), \xi\rangle + \langle \boldsymbol{\mu}_e, \xi\rangle = \boldsymbol{h}(p) - \sum_{\alpha=1}^{k} \langle \mathbf{J}^\Phi_\alpha(p), \Delta^T_{g\alpha}\xi\rangle \otimes e_\alpha + \langle \boldsymbol{\mu}_e, \xi\rangle,\end{aligned}$$

where $\boldsymbol{\Delta}^T_g\colon \mathfrak{g}^k \to \mathfrak{g}^k$ is the transpose of $\boldsymbol{\Delta}_g$ for $g \in G$ and $\Delta_{g1}, \ldots, \Delta_{gk}$ are its components. Substituting $g = \exp(t\zeta)$, with $\zeta \in \mathfrak{g}$, and differentiating with respect to t, one gets

$$(\iota_{\zeta_P} \mathrm{d}\boldsymbol{h}_\xi)_{z_e} = -\sum_{\alpha=1}^{k} \left\langle \mathbf{J}^\Phi_\alpha(p), \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Delta^T_{\exp(t\zeta)\alpha}\xi \right\rangle \otimes e_\alpha = -\sum_{\alpha=1}^{k} \left\langle \mathbf{J}^\Phi_\alpha(p), (\zeta_{\mathfrak{g}}^{\Delta_\alpha})_\xi \right\rangle \otimes e_\alpha\,, \tag{3.3.7}$$

where $(\zeta_{\mathfrak{g}}^{\Delta_\alpha})_\xi$ is the fundamental vector field of $\Delta^T_\alpha\colon G \times \mathfrak{g} \to \mathfrak{g}$ at $\xi \in \mathfrak{g}$ for $\alpha = 1, \ldots, k$. Taking the second variation of (3.3.7) relative to $p \in P$, evaluating at $z_e \in P$, and contracting with v_{z_e} gives

$$\left(\delta^2 \boldsymbol{h}_\xi\right)_{z_e}((\zeta_P)_{z_e}, v_{z_e}) = -\sum_{\alpha=1}^{k} \left\langle \mathrm{T}_{z_e}\mathbf{J}^\Phi_\alpha\left(v_{z_e}\right), (\zeta_{\mathfrak{g}}^{\Delta_\alpha})_\xi \right\rangle \otimes e_\alpha.$$

Since $v_{z_e} \in \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e) \subset \ker \mathrm{T}_{z_e}\mathbf{J}^\Phi_\alpha$ the second variation $\left(\delta^2 \boldsymbol{h}_\xi\right)_{z_e}((\zeta_P)_{z_e}, v_{z_e})$ vanishes.

The identity (3.3.6) follows from (3.3.5) and the fact that, for every vector field Y on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ taking values in $\ker \omega^\alpha \cap \mathrm{T}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$, one obtains

$$\iota_Y \mathrm{d}h^\alpha = \omega^\alpha(X_{\boldsymbol{h}}, Y) = 0\,, \qquad \iota_Y \mathrm{d}\langle \mathbf{J}^\Phi_\alpha, \xi\rangle = \omega^\alpha(\xi_P, Y) = 0\,,$$

for $\alpha = 1, \ldots, k$ and every $\xi \in \mathfrak{g}$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$. □

Proposition 3.3.5 and Proposition 2.3.12 establish that $\left(\delta^2 \boldsymbol{h}_\xi\right)_{z_e}$ is degenerate along the directions tangent to $\mathrm{T}_{z_e}\left(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e\right)$, while each $(\delta^2 h_\xi^\alpha)_{z_e}$ is degenerate in the directions of $\ker \omega^\alpha_{z_e} \cap \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$. Moreover, since $\ker(\delta^2 \boldsymbol{h}_\xi)_{z_e}$ contains $\ker \mathrm{T}_{z_e}\pi_{\boldsymbol{\mu}_e}$, it is possible to define a bilinear two-form on $\mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)}P^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$, with $P^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)/G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$, by reducing the bilinear two-form $(\delta^2 \boldsymbol{h}_\xi)_{z_e}$ to that space. By using an adapted coordinate system, one can prove that the reduction of $(\delta^2 \boldsymbol{h}_\xi)_{z_e}$ to $\mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)}P^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$ yields the the Hessian of $\boldsymbol{f}_{\boldsymbol{\mu}_e}$ on $P_{\boldsymbol{\mu}_e}$.

It is worth noting that the reduction $\boldsymbol{f}_{\mu_e}$ to $P^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}}$ of $\boldsymbol{h}_\xi$ on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ is independent on ξ, since the value of $\boldsymbol{h}_\xi$ on points of $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ does not actually depend on ξ, being just the restriction of $\boldsymbol{h}$ to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$. Furthermore, only the directions transverse to the orbit of $G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$ play a role in determining, via the variation of $\boldsymbol{h}_\xi$, the stability properties of $\boldsymbol{f}_{\boldsymbol{\mu}_e}$ at an equilibrium point.

There exist several approaches for establishing the stability of a k-polysymplectic reduced Hamiltonian system. This motivates the following definition of formal stability. In the case of a symplectic manifold, this definition recovers the standard criterion for stability in reduced symplectic systems as presented in Section 3.1 when there is no time-dependence. In other words, it retrieves the standard results from [113].

Definition 3.3.6. Let $(P, \boldsymbol{\omega}, \boldsymbol{h}, \mathbf{J}^\Phi)$ be a G-invariant $\boldsymbol{\omega}$-Hamiltonian system and let $z_e \in P$ be a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$. Then, z_e is a *formally stable k-polysymplectic relative equilibrium point* if, for a family of supplementary spaces $\mathcal{S}^\alpha$ satisfying

$$\mathcal{S}^\alpha \oplus \left(\mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e) + \ker \omega^\alpha_{z_e} \cap \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)\right) = \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)\,,$$

and

$$\mathcal{S}^1 + \cdots + \mathcal{S}^k + \mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e) = \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)\,,$$

it follows that

$$\left(\delta^2 h^\alpha_\xi\right)_{z_e}(v_{z_e}, v_{z_e}) > 0\,, \qquad \forall v_{z_e} \in \mathcal{S}^\alpha \backslash \{0\}\,, \qquad \alpha = 1, \ldots, k\,. \tag{3.3.8}$$

It is important to note that if a family of subspaces $W_1, \ldots, W_k$ of a vector space E satisfies $\cap_{\alpha=1}^k W_\alpha = 0$, it does not necessarily follow that for any choice of supplementary spaces V_α such that $V_\alpha \oplus W_\alpha = E$ implies $V_1 + \cdots + V_k = E$. This observation underlies the necessity of the condition

$$\mathcal{S}^1 + \cdots + \mathcal{S}^k + \mathrm{T}_{z_e}(G^{\Delta}_{\boldsymbol{\mu}_e} z_e)\,.$$

Indeed, in order to guarantee stability on the reduced manifold, it is necessary to ensure that the projection of $\mathcal{S}^1 + \cdots + \mathcal{S}^k$ onto the tangent space at the equilibrium point in the reduced manifold spans the entire tangent space at that point.

If a system satisfies the condition of formal stability from Definition 3.3.6, then $\sum_{\alpha=1}^k f^\alpha_{\boldsymbol{\mu}_e}$ admits a strict minimum at $\pi_{\boldsymbol{\mu}_e}(z_e)$. Moreover, the function is invariant relative to the evolution of the reduced $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian system. Consequently, the reduced system is stable at that point. The converse, however, does not hold, as in the symplectic case.

The proof of the above-mentioned fact relies on using a coordinate system on $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ adapted to its fibration over $P^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$, together with the fact that the obtained results involve geometric objects that are independent of the coordinate system. In the adapted coordinate system, the Hessian of $\boldsymbol{f}_{\boldsymbol{\mu}_e}$ on the reduced space $P^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}$ at $\pi_{\boldsymbol{\mu}_e}(z_e)$ is recovered from the Hessian of $\boldsymbol{h}_\xi$ on directions of $\mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ that are transversal to $\ker \mathrm{T}_{z_e}\pi_{\boldsymbol{\mu}_e}$. The Hessian of the reduced function $\boldsymbol{f}_{\boldsymbol{\mu}_e}$ can be decomposed into k components. The vector subspaces $\mathcal{S}^1, \ldots, \mathcal{S}^k$ project onto a family of subspaces whose sum spans $\mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)} P_{\boldsymbol{\mu}_e}$. Condition (3.3.8) then implies that

$$\begin{aligned}
&\frac{\partial^2 f^\alpha_{\boldsymbol{\mu}_e}}{\partial z_i \partial z_j}(\pi_{\boldsymbol{\mu}_e}(z_e)) v^i v^j > 0\,, \qquad \forall v \in \mathrm{Im}\mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)}\pi_{\boldsymbol{\mu}_e}(\mathcal{S}^\alpha)\backslash\{0\}\,,\\
&\frac{\partial^2 f^\alpha_{\boldsymbol{\mu}_e}}{\partial z_i \partial z_j}(\pi_{\boldsymbol{\mu}_e}(z_e)) v^i v^j \geq 0\,, \qquad \forall v \in \mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)} P^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}\,,
\end{aligned}$$

for $\alpha = 1, \ldots, k$. Then,

$$\sum_{\alpha=1}^k \frac{\partial^2 f^\alpha_{\boldsymbol{\mu}_e}}{\partial z_i \partial z_j}(\pi_{\boldsymbol{\mu}_e}(z_e)) v^i v^j > 0\,, \qquad \forall v \in \mathrm{T}_{\pi_{\boldsymbol{\mu}_e}(z_e)} P^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e}\backslash\{0\}\,.$$

Consequently, the second-order Taylor part of $\sum_{\alpha=1}^k f^\alpha_{\boldsymbol{\mu}_e}$ is definite-positive, which ensures the existence of a strict minimum. Since the components $f^\alpha_{\boldsymbol{\mu}_e}$ are first integrals $X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}}$, the flow of $X_{\boldsymbol{f}_{\boldsymbol{\mu}_e}}$, for an initial condition close enough to $\pi_{\boldsymbol{\mu}_e}(z_e)$ can be restricted to an open neighbourhood of $\pi_{\boldsymbol{\mu}_e}(z_e)$.

It is worth noting that the term *formally stable k-polysymplectic relative equilibrium points* also refers to points for which each (3.3.8) is negative-definite, since analogous results can be established in this case.

It is possible to obtain many other stability criteria. However, a comprehensive analysis of methods for establishing stability of the reduced k-polysymplectic $\boldsymbol{\omega}$-Hamiltonian systems is not discussed here and is left for future research.

3.4 Applications and examples

This section demonstrates how the theoretical framework developed in the previous subsections concerning the k-polysymplectic energy-momentum method can be applied to examples of both physical and mathematical relevance.

3.4.1 Example: Complex Schwarz equations

The first example illustrates how locally automorphic Lie systems [72] can be interpreted as $\boldsymbol{\omega}$-Hamiltonian systems relative to a k-polysymplectic structure.

Consider the t-dependent complex differential equation given by

$$\frac{\mathrm{d}z}{\mathrm{d}t}=v\,,\qquad \frac{\mathrm{d}v}{\mathrm{d}t}=a\,,\qquad \frac{\mathrm{d}a}{\mathrm{d}t}=\frac{3}{2}\frac{a^2}{v}+2b(t)v\,,\qquad z,v,a\in\mathbb{C}\,, \tag{3.4.1}$$

for a certain complex t-dependent function $b(t)$ defined on $\mathcal{O}=\{(z,v,a)\in\mathrm{T}^2\mathbb{C}\mid v\neq 0\}$.

The system (3.4.1) can be understood as the complex analogue of the Lie system on $\mathcal{O}_{\mathbb{R}}=\{(z,v,a)\in\mathrm{T}^2\mathbb{R}\mid v\neq 0\}$ studied in [53]. More precisely, (3.4.1) is a first-order representation for the third-order complex differential equation of the form

$$\frac{\mathrm{d}^3z}{\mathrm{d}t^3}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^{-1}-\frac{3}{2}\left(\frac{\mathrm{d}^2z}{\mathrm{d}t^2}\right)^2\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^{-2}=2b(t)\,.$$

The left-hand side of the above expression coincides, for $z\in\mathbb{R}$, with the real version of the *Schwarzian derivative* (also known as the *Schwarz equation*) of a function $z(t)$, commonly denoted by $\{z(t),t\}_{sc}$, which appears in many research problems [77, 83, 96].

The methods developed in this work, together with (3.4.1), provide a potential framework for extending to the complex setting the results obtained for the real third-order Kummer–Schwarz equation and Schwarzian derivatives via Lie systems (see [20, 52] and references therein). It is worth noting that the Schwarzian derivative plays a significant role in the study of linearisation of time-dependent systems, projective systems, the theory of special functions, and related areas (cf. [77, 83, 96]).

In real coordinates

$$v_1=\mathfrak{Re}(z)\,,\quad v_2=\mathfrak{Im}(z)\,,\quad v_3=\mathfrak{Re}(v)\,,\quad v_4=\mathfrak{Im}(v)\,,\quad v_5=\mathfrak{Re}(a)\,,\quad v_6=\mathfrak{Im}(a)\,,$$

the system (3.4.1) is associated with the t-dependent vector field

$$X=X_1+2b_R(t)X_2+2b_I(t)X_3\,,$$

where $b_R(t)=\mathfrak{Re}(b(t))$, $b_I(t)=\mathfrak{Im}(b(t))$, and

$$X_1=v_3\frac{\partial}{\partial v_1}+v_4\frac{\partial}{\partial v_2}+v_5\frac{\partial}{\partial v_3}+v_6\frac{\partial}{\partial v_4}+\frac{3}{2}\frac{2v_4v_5v_6+(v_5^2-v_6^2)v_3}{v_3^2+v_4^2}\frac{\partial}{\partial v_5}+\frac{3}{2}\frac{2v_3v_5v_6-v_4(v_5^2-v_6^2)}{v_3^2+v_4^2}\frac{\partial}{\partial v_6}\,,$$
$$X_2=v_3\frac{\partial}{\partial v_5}+v_4\frac{\partial}{\partial v_6}\,,\qquad X_3=-v_4\frac{\partial}{\partial v_5}+v_3\frac{\partial}{\partial v_6}\,,$$
$$X_4=-v_3\frac{\partial}{\partial v_3}-v_4\frac{\partial}{\partial v_4}-2v_5\frac{\partial}{\partial v_5}-2v_6\frac{\partial}{\partial v_6}\,,\qquad X_5=v_4\frac{\partial}{\partial v_3}-v_3\frac{\partial}{\partial v_4}+2v_6\frac{\partial}{\partial v_5}-2v_5\frac{\partial}{\partial v_6}\,,$$
$$X_6=-v_4\frac{\partial}{\partial v_1}+v_3\frac{\partial}{\partial v_2}-v_6\frac{\partial}{\partial v_3}+v_5\frac{\partial}{\partial v_4}-\frac{3}{2}\frac{2v_3v_5v_6-v_4(v_5^2-v_6^2)}{(v_3^2+v_4^2)}\frac{\partial}{\partial v_5}+\frac{3}{2}\frac{2v_4v_5v_6+v_3(v_5^2-v_6^2)}{(v_3^2+v_4^2)}\frac{\partial}{\partial v_6}\,.$$

These vector fields satisfy the following commutation relations

$$\begin{array}{lllll}
[X_1, X_2] = X_4\,, & [X_1, X_3] = X_5\,, & [X_1, X_4] = X_1\,, & [X_1, X_5] = X_6\,, & [X_1, X_6] = 0\,, \\
 & [X_2, X_3] = 0\,, & [X_2, X_4] = -X_2\,, & [X_2, X_5] = -X_3\,, & [X_2, X_6] = -X_5\,, \\
 & & [X_3, X_4] = -X_3\,, & [X_3, X_5] = X_2\,, & [X_3, X_6] = X_4\,, \\
 & & & [X_4, X_5] = 0\,, & [X_4, X_6] = -X_6\,, \\
 & & & & [X_5, X_6] = X_1\,,
\end{array}$$

Hence, the vector fields $X_1, \ldots, X_k$ give rise to a Lie algebra of vector fields V_{sc} isomorphic, as a real vector space, to $\mathbb{C} \otimes \mathfrak{sl}_2$. Indeed,

$$\langle X_1, X_2, X_4 \rangle \simeq \mathfrak{sl}(2, \mathbb{R}) \simeq \langle X_3, X_4, X_6 \rangle\,.$$

Additionally, $\mathbb{C} \otimes \mathfrak{sl}_2$ admits the decomposition of the form $\langle X_1, X_4, X_2 \rangle \oplus \langle X_6, X_5, X_3 \rangle$. Then, V_{sc} is graded as $V_{sc} = E_{-1} \oplus E_0 \oplus E_1$, where $E_{-1} = \langle X_6, X_1 \rangle$, $E_0 = \langle X_4, X_5 \rangle$, and $E_1 = \langle X_3, X_2 \rangle$, with $[E_i, E_j] = E_{i+j}$, where the sum is in the additive group $\{-1, 0, 1\}$.

Furthermore, a direct computation shows that $X_1 \wedge \cdots \wedge X_6 \neq 0$ almost everywhere. This linear independence, together with the fact that $X_1, \ldots, X_6$ span a Lie algebra of vector fields generating $\mathrm{T}\mathcal{O}$, explains why system (3.4.1) is related to a locally automorphic Lie system (cf. [72]).

Meanwhile, the Lie algebra of Lie symmetries of the system (3.4.1) associated with the Lie algebra V_{sv} is generated by

$$\begin{aligned}
2Y_1 = {} & (v_1^2 - v_2^2)\frac{\partial}{\partial v_1} + 2v_1v_2\frac{\partial}{\partial v_2} + 2(v_1v_3 - v_2v_4)\frac{\partial}{\partial v_3} + 2(v_3v_2 + v_1v_4)\frac{\partial}{\partial v_4} \\
& +2(v_3^2 + v_1v_5 - v_4^2 - v_2v_6)\frac{\partial}{\partial v_5} + 2(v_5v_2 + 2v_3v_4 + v_2v_6)\frac{\partial}{\partial v_6}\,,
\end{aligned}$$

$$Y_2 = \frac{\partial}{\partial v_1}\,, \qquad Y_3 = \frac{\partial}{\partial v_2}\,,$$

$$Y_4 = -v_1\frac{\partial}{\partial v_1} - v_2\frac{\partial}{\partial v_2} - v_3\frac{\partial}{\partial v_3} - v_4\frac{\partial}{\partial v_4} - v_5\frac{\partial}{\partial v_5} - v_6\frac{\partial}{\partial v_6}\,,$$

$$Y_5 = v_2\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial v_2} + v_4\frac{\partial}{\partial v_3} - v_3\frac{\partial}{\partial v_4} + v_6\frac{\partial}{\partial v_5} - v_5\frac{\partial}{\partial v_6}\,.$$

$$\begin{aligned}
2Y_6 = {} & -2v_1v_2\frac{\partial}{\partial v_1} + (v_1^2 - v_2^2)\frac{\partial}{\partial v_2} - 2(v_2v_3 + v_1v_4)\frac{\partial}{\partial v_3} + 2(v_1v_3 - v_2v_4)\frac{\partial}{\partial v_4} \\
& -2(2v_3v_4 + v_2v_5 + v_1v_6)\frac{\partial}{\partial v_5} + 2(v_3^2 - v_4^2 + v_1v_5 - v_2v_6)\frac{\partial}{\partial v_6}\,.
\end{aligned}$$

In other words, $[X_i, Y_j] = 0$ for every $i, j = 1, \ldots, 6$. The commutation relations among the vector fields $Y_1, \ldots, Y_6$ are

$$\begin{array}{lllll}
[Y_1, Y_2] = Y_4\,, & [Y_1, Y_3] = Y_5\,, & [Y_1, Y_4] = Y_1\,, & [Y_1, Y_5] = Y_6\,, & [Y_1, Y_6] = 0\,, \\
 & [Y_2, Y_3] = 0\,, & [Y_2, Y_4] = -Y_2\,, & [Y_2, Y_5] = -Y_3\,, & [Y_2, Y_6] = -Y_5\,, \\
 & & [Y_3, Y_4] = -Y_3\,, & [Y_3, Y_5] = Y_2\,, & [Y_3, Y_6] = Y_4\,, \\
 & & & [Y_4, Y_5] = 0\,, & [Y_4, Y_6] = -Y_6\,, \\
 & & & & [Y_5, Y_6] = Y_1\,.
\end{array}$$

The vector fields $Y_1, \ldots, Y_6$ admit identical structure constants as $X_1, \ldots, X_6$. It is possible to choose one-forms $\eta^1, \ldots, \eta^6$ dual to $Y_1, \ldots, Y_6$. Their existence is ensured by the condition $Y_1 \wedge \cdots \wedge Y_6 \neq 0$ and the fact that $Y_1, \ldots, Y_6$ span $\mathrm{T}\mathcal{O}$. These dual one-forms remain invariant under the Lie derivatives with respect to the vector fields $X_1 \ldots, X_6$, namely $\mathscr{L}_{X_i}\eta^j = 0$ for $i, j = 1, \ldots, 6$.

Moreover, the differential forms $\mathrm{d}\eta^1, \ldots, \mathrm{d}\eta^6$, as well as their linear combinations, are closed differential forms that are invariant relative to the Lie derivatives along $X_1, \ldots, X_6$. These properties ensure that the vector fields $X_1, \ldots, X_6$ are Hamiltonian vector fields relative to the presymplectic forms $\mathrm{d}\eta^1, \ldots, \mathrm{d}\eta^6$. The appropriate linear combinations of these forms yield a family of presymplectic forms with the zero intersection of their kernels. Consequently, the vector fields $X_1, \ldots, X_6$ become $\boldsymbol{\omega}$-Hamiltonian vector fields relative to this k-polysymplectic structure.

In particular, if

$$\begin{aligned}
\mathrm{d}\eta^1 &= -\eta^5 \wedge \eta^6 - \eta^1 \wedge \eta^4\,, & \mathrm{d}\eta^2 &= -\eta^3 \wedge \eta^5 - \eta^4 \wedge \eta^2\,,\\
\mathrm{d}\eta^3 &= -\eta^4 \wedge \eta^3 - \eta^5 \wedge \eta^2\,, & \mathrm{d}\eta^4 &= -\eta^1 \wedge \eta^2 - \eta^3 \wedge \eta^6\,,\\
\mathrm{d}\eta^5 &= -\eta^1 \wedge \eta^3 - \eta^6 \wedge \eta^2\,, & \mathrm{d}\eta^6 &= -\eta^1 \wedge \eta^5 - \eta^6 \wedge \eta^4\,,
\end{aligned}$$

then every vector field in $\langle X_1, \ldots, X_6 \rangle$ becomes an $\boldsymbol{\omega}$-Hamiltonian vector field relative to the two-polysymplectic form $\mathrm{d}\eta^1 \otimes e_1 + \mathrm{d}\eta^2 \otimes e_2$. The same statement holds for the two-polysymplectic form $\mathrm{d}\eta^5 \otimes e_1 + \mathrm{d}\eta^6 \otimes e_2$, and many other forms. This naturally extends to three-polysymplectic forms, such as $\mathrm{d}\eta^1 \otimes e_1 + \mathrm{d}\eta^2 \otimes e_2 + \mathrm{d}\eta^3 \otimes e_3$, provided that the kernels of their presymplectic components have zero intersection.

Consider the three-polysymplectic form defined by

$$\boldsymbol{\omega} = \omega^1 \otimes e_1 + \omega^2 \otimes e_2 + \omega^3 \otimes e_3 = \mathrm{d}\eta^1 \otimes e_1 + \mathrm{d}\eta^2 \otimes e_2 + \mathrm{d}\eta^4 \otimes e_3\,.$$

A three-polysymplectic Marsden–Meyer–Weinstein reduction, Theorem 2.3.14 and Theorem 2.3.16, can be performed by taking, for instance, the $\boldsymbol{\omega}$-Hamiltonian vector field X_1 and the Lie symmetry X_6, which satisfies $[X_1, X_6] = 0$. Then, a three-polysymplectic momentum map $\mathbf{J}^\Phi \colon \mathcal{O} \to \mathbb{R}^{*3}$ takes the form

$$\iota_{X_6}\mathrm{d}\mathbf{J}^\Phi = \iota_{X_6}\omega^1 \otimes e_1 + \iota_{X_6}\omega^2 \otimes e_2 + \iota_{X_6}\omega^3 \otimes e_3 = \mathrm{d}J_1^\Phi \otimes e_1 + \mathrm{d}J_2^\Phi \otimes e_2 + \mathrm{d}J_3^\Phi \otimes e_3\,.$$

A direct calculation shows that $\mathrm{d}J_1^\Phi \wedge \mathrm{d}J_2^\Phi \wedge \mathrm{d}J_3^\Phi \neq 0$ almost everywhere based on the fact that $\partial(J_1^\Phi, J_2^\Phi, J_3^\Phi)/\partial(v_1, v_2, v_3) \neq 0$ almost everywhere. Consequently, $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is a three-dimensional submanifold for some weak regular three-value $\boldsymbol{\mu} \in \mathbb{R}^{*3}$. Moreover, since $\iota_{X_6}\mathrm{d}\mathbf{J}^\Phi = 0$, the reduced manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/X_6$ is two-dimensional.

The vector field X_1 is tangent to the level set $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ since

$$\iota_{X_1}\iota_{X_6}\mathrm{d}\eta^\alpha = X_1 J_\alpha^\Phi = 0\,, \qquad \alpha = 1, 2, 3\,.$$

Therefore, by Theorem 2.3.16 the vector field X_1 projects onto the reduced manifold $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/X_6$.

After performing the necessary calculations, condition (2.3.3) is verified. To check condition (2.3.4), which reads

$$\mathrm{T}_p(G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}} p) = \bigcap_{\alpha=1}^{k} \left(\ker \omega_p^\alpha + \mathrm{T}_p(G_{\mu^\alpha}^{\Delta^\alpha} p)\right) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})\,,$$

observe that

$$\mathrm{T}_p(G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}} p) = \langle X_6 \rangle \subset \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) \subset \mathrm{T}_p P\,.$$

Moreover,

$$\ker \omega^1 = \langle Y_2, Y_3 \rangle\,, \quad \ker \omega^2 = \langle Y_1, Y_6 \rangle\,, \quad \ker \omega^3 = \langle Y_4, Y_5 \rangle\,.$$

Then it suffices to show that no element of $\ker \omega^\alpha$ belongs to $\mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. This can be done by computing three determinants, each of which is non-zero at some generic point, namely

$$\det\begin{pmatrix} Y_2 J_2 & Y_2 J_3 \\ Y_3 J_2 & Y_3 J_3 \end{pmatrix} \neq 0\,, \qquad \det\begin{pmatrix} Y_1 J_1 & Y_1 J_3 \\ Y_6 J_1 & Y_6 J_3 \end{pmatrix} \neq 0\,, \qquad \det\begin{pmatrix} Y_4 J_1 & Y_4 J_2 \\ Y_5 J_1 & Y_5 J_2 \end{pmatrix} \neq 0\,.$$

Consequently, condition (2.3.4) is satisfied, that is

$$\big(\langle Y_2, Y_3 \rangle + \langle X_6 \rangle\big) \cap \big(\langle Y_1, Y_6 \rangle + \langle X_6 \rangle\big) \cap \big(\langle Y_4, Y_5 \rangle + \langle X_6 \rangle\big) \cap \mathrm{T}_p\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}) = \langle X_6 \rangle\,.$$

Hence, Theorem 2.3.14 can be applied to perform the three-polysymplectic Marsden–Meyer–Weinstein reduction.

3.4.2 k-Polysymplectic manifold given by the product of k symplectic manifolds

This subsection presents an illustrative example of the k-polysymplectic Marsden–Meyer–Weinstein reduction of a product of k symplectic manifolds (see Subsection 2.3.7). It shows different types of systems of differential equations that can be understood as Hamiltonian systems relative to a k-polysymplectic manifold and describes their reductions. In particular, the so-called diagonal prolongations of Lie–Hamilton systems, which appear also in the multidimensional generalisations of certain integral systems, such as the Winternitz–Smorodinsky oscillator on $\mathrm{T}^*\mathbb{R}$ (see [52]), can be viewed as Hamiltonian systems relative to a k-polysymplectic manifold, including higher-dimensional Winternitz–Smorodinsky oscillators.

Recall the formalism introduced in Subsection 2.3.7. Let $P = P_1 \times \cdots \times P_k$ for some k symplectic manifolds $(P_\alpha, \omega^\alpha)$, where $\alpha = 1, \ldots, k$. This gives rise to a k-polysymplectic manifold $(P, \mathrm{pr}_\alpha^* \omega^\alpha \otimes e_\alpha)$. Assume that each Lie group action $\Phi^\alpha \colon G_\alpha \times P_\alpha \to P_\alpha$ admits a symplectic momentum map $\mathbf{J}^{\Phi^\alpha} : P_\alpha \to \mathfrak{g}_\alpha^*$ for $\alpha = 1, \ldots, k$. Define the Lie group action of $G = G_1 \times \cdots \times G_k$ on P as (2.3.8). If $\mathfrak{g} = \bigoplus_{\alpha=1}^k \mathfrak{g}_\alpha$, then there exists a k-polysymplectic momentum map associated with Φ given by

$$\mathbf{J} \colon (x_1, \ldots, x_k) \in P \longmapsto (0, \ldots, \mathbf{J}^\alpha, \ldots, 0) \otimes e_\alpha = \begin{pmatrix} \mathbf{J}^1 & 0 & \cdots & 0 \\ 0 & \mathbf{J}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{J}^k \end{pmatrix} \in \mathfrak{g}^{*k} ,$$

where there is a summation over α and $\mathbf{J}^\alpha(x_1, \ldots, x_k) = \mathbf{J}^{\Phi^\alpha}(x_\alpha)$ for $\alpha = 1, \ldots, k$. The above matrix array should be understood as a convenient representation of the image of $\mathbf{J}$. Note that $\boldsymbol{\mu} = (0, \ldots, \mu^\alpha, \ldots, 0) \otimes e_\alpha \in \mathfrak{g}^{*k}$ is a weak regular k-value of $\mathbf{J}$ if and only if each $\mu^\alpha \in \mathfrak{g}_\alpha^*$ is a weak regular value of its corresponding $\mathbf{J}^{\Phi^\alpha}$. Assume that some $G_{\boldsymbol{\mu}}^{\Delta}$ acts in a quotientable manner on the associated level $\mathbf{J}^{-1}(\boldsymbol{\mu})$. This happens if and only if every $G_{\mu^\alpha}^{\Delta^\alpha}$ acts on a quotientable manner on each $\mathbf{J}^{\Phi^\alpha -1}(\mu^\alpha)$ for $\alpha = 1, \ldots, k$.

Subsection 2.3.7 established that the conditions (2.3.3) and (2.3.4) hold. Thus, by Theorem 2.3.14, these equations guarantee that, the reduced manifold $\mathbf{J}^{-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}}$ admits a unique k-polysymplectic structure. Explicitly, one gets

$$\left(\mathbf{J}^{-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}^{\boldsymbol{\Delta}} \simeq \mathbf{J}^{\Phi^1 -1}(\mu^1)/G_{1\boldsymbol{\mu}}^{\Delta^1} \times \cdots \times \mathbf{J}^{\Phi^k -1}(\mu^k)/G_{k\boldsymbol{\mu}}^{\Delta^k} \ , \ \boldsymbol{\omega}_{\boldsymbol{\mu}} = \sum_{\alpha=1}^k \omega^{\mu^\alpha} \otimes e_\alpha \right)$$

where each ω_{μ^α} denotes the reduced presymplectic form induced on $\mathbf{J}^{\Phi^\alpha -1}(\mu^\alpha)/G_{\alpha\mu^\alpha}^{\Delta^\alpha}$, for $\alpha = 1, \ldots, k$.

Consider a vector field X on P that is both $\boldsymbol{\omega}$-Hamiltonian and G-invariant. For instance, a vector field X can be written as

$$X = \sum_{\alpha=1}^k X_\alpha \,,$$

where each X_α is a vector field on P_α that is tangent to $\mathbf{J}^{\Phi^\alpha -1}(\mu^\alpha)$ for $\alpha = 1, \ldots, k$. Recall that $\iota_{X_\alpha} \omega^\beta = \delta_\alpha^\beta \mathrm{d}h^\alpha$ for $\alpha, \beta = 1, \ldots, k$. This frequently happens in diagonal prolongations of Lie–Hamilton systems, where a vector field $X^{[m]}$ defined on a manifold of the form N^m is considered as a copy of a Hamiltonian system on each N relative to a symplectic manifold on that N (cf. [52]). Then,

$$\mathrm{d}\boldsymbol{h} = \sum_{\alpha=1}^k \mathrm{d}h^\alpha \otimes e_\alpha = \sum_{\alpha=1}^k \iota_X \omega^\alpha \otimes e_\alpha \,.$$

Next, recall that $\boldsymbol{h}_\xi = \boldsymbol{h} - \langle \mathbf{J} - \boldsymbol{\mu}_e, \xi \rangle$ for $\xi \in \mathfrak{g}$. By Theorem 3.3.2, a point $z_e = (z_{1e}, \ldots, z_{ke}) \in P$ is a k-polysymplectic relative equilibrium point if and only if each $z_{\alpha e}$ is a symplectic relative equilibrium point of a Hamiltonian vector field X_α on the symplectic manifold $(P_\alpha, \omega^\alpha)$ relative to some $\xi_\alpha \in \mathfrak{g}_\alpha$, see Definition 3.1.3 where there is no time-dependence.

Therefore, a k-polysymplectic relative equilibrium point $z_e \in P$ is formally stable if there exists a series of supplementary spaces $\mathcal{S}^\alpha$ to $\mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e) \oplus \left(\ker \omega^\alpha_{z_e} \cap \mathrm{T}_{z_e}\mathbf{J}^{-1}(\boldsymbol{\mu}_e)\right)$ in $\mathrm{T}_{z_e}\mathbf{J}^{-1}(\boldsymbol{\mu}_e)$, with $\alpha = 1, \ldots, k$, such that

$$\left(\delta^2 h^\alpha_\xi\right)_{z_e}(v_{z_e}, v_{z_e}) > 0\,, \qquad \forall v_{z_e} \in \mathcal{S}^\alpha \setminus \{0\}\,, \qquad \alpha = 1, \ldots, k\,,$$

and $\mathcal{S}^1 + \cdots + \mathcal{S}^k + \mathrm{T}_{z_e}(G^{\boldsymbol{\Delta}}_{\boldsymbol{\mu}_e} z_e) = \mathrm{T}_{z_e}\mathbf{J}^{-1}(\boldsymbol{\mu}_e)$.

3.4.3 Example: Product of oscillators

A practical application of the above formalism can be illustrated by considering the product of k isotropic three-dimensional harmonic oscillators. The equations read

$$\frac{\mathrm{d}^2 x^i_\alpha}{\mathrm{d}t^2} = -b^2_\alpha x^i_\alpha\,, \qquad \alpha = 1, \ldots, k\,, \qquad i = 1, 2, 3\,,$$

where each $b_\alpha > 0$ is a constant. The above system of second-order differential equations can be written as a first-order system of differential equations

$$\begin{cases} \dfrac{\mathrm{d}x^i_\alpha}{\mathrm{d}t} = p^i_\alpha\,, \\ \dfrac{\mathrm{d}p^i_\alpha}{\mathrm{d}t} = -b^2_\alpha x^i_\alpha\,, \end{cases} \qquad \alpha = 1, \ldots, k\,, \qquad i = 1, 2, 3\,, \tag{3.4.2}$$

defined on the product manifold $P = (\mathrm{T}^*\mathbb{R}^3)^k$. The α-th factor $\mathrm{T}^*\mathbb{R}^3$ in P is a symplectic manifold equipped with the canonical symplectic form

$$\omega^\alpha = \sum_{i=1}^{3} \mathrm{d}x^i_\alpha \wedge \mathrm{d}p^i_\alpha\,,$$

where the index α is not summed over. Then, P is a k-polysymplectic manifold when endowed with the $\mathbb{R}^k$-valued form

$$\boldsymbol{\omega} = \sum_{\alpha=1}^{k} \omega^\alpha \otimes e_\alpha\,,$$

where $\omega^1, \ldots, \omega^k$ are considered as pulled back to P in the natural way. Moreover, system (3.4.2) corresponds to the integral curves of the vector field of the form

$$X_{\boldsymbol{h}} = \sum_{\alpha=1}^{k}\sum_{i=1}^{3}\left(p^i_\alpha \frac{\partial}{\partial x^i_\alpha} - b^2_\alpha x^i_\alpha \frac{\partial}{\partial p^i_\alpha}\right)\,,$$

which is $\boldsymbol{\omega}$-Hamiltonian vector field admitting an $\boldsymbol{\omega}$-Hamiltonian function

$$\boldsymbol{h} = \frac{1}{2}\sum_{\alpha=1}^{k}\left(p^2_\alpha + b^2_\alpha x^2_\alpha\right) \otimes e_\alpha\,, \qquad p^2_\alpha = \sum_{i=1}^{3}(p^i_\alpha)^2, \qquad x^2_\alpha = \sum_{i=1}^{3}(x^i_\alpha)^2\,.$$

Consider now the Lie group action $\Phi^\alpha\colon \mathrm{SO}_3 \times (\mathrm{T}^*\mathbb{R}^3)_\alpha \to (\mathrm{T}^*\mathbb{R}^3)_\alpha$, where each Φ^α is the lift of the natural Lie group action $\Psi\colon \mathrm{SO}_3 \times \mathbb{R}^3 \to \mathbb{R}^3$ induced by rotations on $\mathbb{R}^3$ to the α-th copy of $\mathrm{T}^*\mathbb{R}^3$ in P. Then, the resulting Lie group action Φ on $(\mathrm{T}^*\mathbb{R}^3)^k$, given by (2.3.8), reads

$$\Phi\colon \mathrm{SO}_3^k \times (\mathrm{T}^*\mathbb{R}^3)^k \longrightarrow (\mathrm{T}^*\mathbb{R}^3)^k\,.$$

The Lie algebra of fundamental vector fields of Φ is spanned by the basis of vector fields on P of the form

$$\xi^1_{\alpha P} = \left(x^1_\alpha \frac{\partial}{\partial x^2_\alpha} - x^2_\alpha \frac{\partial}{\partial x^1_\alpha} + p^1_\alpha \frac{\partial}{\partial p^2_\alpha} - p^2_\alpha \frac{\partial}{\partial p^1_\alpha}\right), \qquad \xi^2_{\alpha P} = \left(x^2_\alpha \frac{\partial}{\partial x^3_\alpha} - x^3_\alpha \frac{\partial}{\partial x^2_\alpha} + p^2_\alpha \frac{\partial}{\partial p^3_\alpha} - p^3_\alpha \frac{\partial}{\partial p^2_\alpha}\right),$$
$$\xi^3_{\alpha P} = \left(x^3_\alpha \frac{\partial}{\partial x^1_\alpha} - x^1_\alpha \frac{\partial}{\partial x^3_\alpha} + p^3_\alpha \frac{\partial}{\partial p^1_\alpha} - p^1_\alpha \frac{\partial}{\partial p^3_\alpha}\right),$$

with $\alpha = 1, \ldots, k$. These vector fields are Lie symmetries of $\boldsymbol{\omega}$ and $\boldsymbol{h}$. Furthermore, the corresponding k-polysymplectic momentum map $\mathbf{J}\colon (\mathrm{T}^*\mathbb{R}^3)^k \to (\mathfrak{so}_3^k)^{*k}$ associated with Φ is given by

$$\mathbf{J}(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k) = \sum_{\alpha=1}^{k} (0,0,0; \ldots; J^1_\alpha, J^2_\alpha, J^3_\alpha; \ldots; 0,0,0) \otimes e_\alpha$$

where $\boldsymbol{q}_\alpha = (x^1_\alpha, x^2_\alpha, x^3_\alpha, p^1_\alpha, p^2_\alpha, p^3_\alpha) \in \mathrm{T}^*\mathbb{R}^3$ for $\alpha = 1, \ldots, k$, while

$$(J^1_\alpha, J^2_\alpha, J^3_\alpha) = (x^1_\alpha p^2_\alpha - x^2_\alpha p^1_\alpha, x^2_\alpha p^3_\alpha - x^3_\alpha p^2_\alpha, x^3_\alpha p^1_\alpha - x^1_\alpha p^3_\alpha)\,,$$

and $\alpha = 1, \ldots, k$. The function $x^1_\alpha p^2_\alpha - x^2_\alpha p^1_\alpha$ is the angular momentum, $p_{\alpha\varphi}$, of the α-th particle in the corresponding spherical coordinates $\{r_\alpha, \theta_\alpha, \varphi_\alpha\}$. Meanwhile, $L^2_\alpha = (J^1_\alpha)^2 + (J^2_\alpha)^2 + (J^3_\alpha)^2$ is the square of the total angular momentum of the α-th particle. Both quantities are conserved by the flow of $X_{\boldsymbol{h}}$.

Note that the momentum map $\mathbf{J}$ is Ad^{*k}-equivariant. Recall that $\mathbf{J} = (0, \ldots, \mathbf{J}^\alpha, \ldots, 0) \otimes e_\alpha$. Then, $\boldsymbol{\mu} = (0,0,0; \ldots; J^1_\alpha, J^2_\alpha, J^3_\alpha; \ldots; 0,0,0) \otimes e_\alpha$ is a weak regular k-value of $\mathbf{J}$ if and only if each triple $\mu^\alpha = (J^1_\alpha, J^2_\alpha, J^3_\alpha) \in \mathfrak{so}_3^*$ is a weak regular value of $\mathbf{J}^{\Phi^\alpha}$, where $\alpha = 1, \ldots, k$. Fix a weak regular k-value $\boldsymbol{\mu}$. Then,

$$\mathrm{T}_{\boldsymbol{q}}\mathbf{J}^{-1}(\boldsymbol{\mu}) = \bigoplus_{\alpha=1}^{k} \mathrm{T}_{\boldsymbol{q}_\alpha}\mathbf{J}^{\Phi^\alpha\,-1}_\alpha(\mu^\alpha)\,, \qquad \forall \mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_k) \in P\,.$$

Furthermore,

$$\xi^i_{\alpha P} J^j_\beta = -\delta_{\alpha\beta}\epsilon_{ijk} J^k_\beta\,, \qquad i,j = 1,2,3, \qquad \alpha,\beta = 1, \ldots, k\,.$$

The isotropy subgroup of Φ at $\boldsymbol{\mu}$ is given by the Cartesian product of the isotropy subgroups corresponding to each μ^α relative to Φ^α, for $\alpha = 1, \ldots, k$. To determine G_{μ^α}, one requires that $\sum_{i=1}^3 \lambda_i(\xi^i_{\alpha P})$ belongs to $\mathrm{T}_{\boldsymbol{q}_\alpha}(\mathbf{J}^{\Phi^\alpha\,-1}(\mu_\alpha))$, namely $\sum_{i=1}^3 \lambda_i(\xi^i_P)_\alpha J^j_\alpha = 0$ for $j = 1,2,3$ (with no summation over α), which occurs if and only if

$$\begin{pmatrix} 0 & -J^3_\alpha & J^2_\alpha \\ J^3_\alpha & 0 & -J^1_\alpha \\ -J^2_\alpha & J^1_\alpha & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The matrix of coefficients has rank two for $L^2_\alpha \neq 0$. In this case, the k-polysymplectic momentum map $\mathbf{J}^{\Phi^\alpha}$ has a regular value at μ_α for every $\alpha = 1, \ldots, k$. Therefore, assume that $L^2_\alpha \neq 0$. Then, each isotropy subgroup G_{μ^α} has dimension one. Consequently, the reduced manifold $\mathbf{J}^{\Phi-1}\boldsymbol{\mu}/G_{\boldsymbol{\mu}}$ is of dimension $6k - 3k - k = 2k$. Since the conditions for the k-polysymplectic Marsden–Meyer–Weinstein reduction follow from the ones for the symplectic reduction on each component, as commented in Subsection 2.3.7, the k-polysymplectic Marsden–Meyer–Weinstein reduction theorem can be applied.

Note that in spherical coordinates on each factor $\mathrm{T}^*\mathbb{R}^3$ of the product manifold $(\mathrm{T}^*\mathbb{R}^3)^k$, the Hamiltonian function is given by

$$\boldsymbol{h} = \frac{1}{2}\sum_{\alpha=1}^{k}(p^2_{\alpha r} + p^2_{\alpha\varphi}/(r^2_\alpha \sin^2\theta_\alpha) + p^2_{\alpha\theta}/r^2_\alpha + b^2_\alpha r^2_\alpha) \otimes e_\alpha\,,$$

and the symplectic forms read

$$\omega^\alpha = \mathrm{d}r_\alpha \wedge \mathrm{d}p_{\alpha r} + \mathrm{d}\theta_\alpha \wedge \mathrm{d}p_{\alpha\theta} + \mathrm{d}\varphi_\alpha \wedge \mathrm{d}p_{\alpha\varphi}\,,$$

for each $\alpha = 1, \ldots, k$. Then, the differential equations for the integral curves of $X_{\boldsymbol{h}}$ read

$$\frac{\mathrm{d}p_{\alpha r}}{\mathrm{d}t} = \frac{p^2_{\alpha\varphi}}{r^3_\alpha \sin^2\theta_\alpha} + \frac{p^2_{\alpha\theta}}{r^3_\alpha} - b^2_\alpha r_\alpha\,, \qquad \frac{\mathrm{d}p_{\alpha\varphi}}{\mathrm{d}t} = 0\,, \qquad \frac{\mathrm{d}p_{\alpha\theta}}{\mathrm{d}t} = \frac{p^2_{\alpha\varphi}\cos\theta_\alpha}{r^2_\alpha \sin^3\theta_\alpha}\,,$$
$$\frac{\mathrm{d}r_\alpha}{\mathrm{d}t} = p_{\alpha r}\,, \qquad \frac{\mathrm{d}\theta_\alpha}{\mathrm{d}t} = \frac{p_{\alpha\theta}}{r^2_\alpha}\,, \qquad \frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}t} = \frac{p_{\alpha\varphi}}{r^2_\alpha \sin^2\theta_\alpha}\,.$$

A k-polysymplectic relative equilibrium point is a point $z_e \in P$ such that the vector field $X_{\boldsymbol{h}}$ at z_e is proportional to one of the fundamental vector fields associated with the Lie group action Φ. In particular, consider a point $z_e = (r_\alpha, \theta_\alpha{=}\frac{\pi}{2}, \varphi_\alpha, p_{\alpha r}{=}0, p_{\alpha\theta} = 0, p_{\alpha\varphi})$ and $L_\alpha = b_\alpha r_\alpha^2 = p_{\alpha\varphi}$ for every $\alpha = 1, \ldots, k$. At such points, the $\boldsymbol{\omega}$-Hamiltonian vector field $X_{\boldsymbol{h}}$ takes the form

$$X_{\boldsymbol{h}} = \sum_{\alpha=1}^{k} \frac{p_{\alpha\varphi}}{r_\alpha^2} \frac{\partial}{\partial \varphi_\alpha}\,.$$

This implies that $z_e \in P$ is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$.

This result can also be obtained via Theorem 3.3.2, which ensures that $z_e \in P$ is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$ if and only if there exists $\xi \in \mathfrak{so}_3^k$ such that $\boldsymbol{h}_\xi = \boldsymbol{h} - \langle \mathbf{J} - \boldsymbol{\mu}_e, \xi \rangle$ admits a critical point at z_e. Indeed, for

$$\xi = (p_{1\varphi}/r_1^2, 0, 0; \ldots; p_{k\varphi}/r_k^2, 0, 0) \in \mathfrak{so}_3^k\,,$$

the $\mathbb{R}^k$-valued function

$$\boldsymbol{h}_\xi = \boldsymbol{h} - \langle \mathbf{J} - \boldsymbol{\mu}_e, \xi \rangle = \sum_{\alpha=1}^{k} (h^\alpha - \langle (0, \ldots, \mathbf{J}^{\Phi_\alpha} - (L_\alpha, 0, 0), \ldots, 0), \xi \rangle) \otimes e_\alpha\,,$$

has a critical point at z_e. Therefore, z_e is a k-polysymplectic relative equilibrium point of $X_{\boldsymbol{h}}$.

By Theorem 2.3.14, the reduced manifold is $(\mathrm{T}^*\mathbb{R})^k$ with coordinates $\{r_\alpha, p_{\alpha r}\}$ for $\alpha = 1, \ldots, k$. The reduced k-polysymplectic form is given by

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \sum_{\alpha=1}^{k} \mathrm{d}r_\alpha \wedge \mathrm{d}p_{\alpha r} \otimes e_\alpha\,,$$

and the reduced $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian $\mathbb{R}^k$-valued function reads

$$\boldsymbol{f}_{\boldsymbol{\mu}} = \frac{1}{2} \sum_{\alpha=1}^{k} \left(p_{\alpha r}^2 + \frac{L_\alpha^2}{r_\alpha^2} + b_\alpha^2 r_\alpha^2 \right) \otimes e_\alpha\,.$$

Furthermore, the reduced equations of motion take the following form

$$\frac{\mathrm{d}p_{\alpha r}}{\mathrm{d}t} = -b_\alpha^2 r_\alpha + \frac{L_\alpha^2}{r_\alpha^3}\,, \qquad \frac{\mathrm{d}r_\alpha}{\mathrm{d}t} = p_{\alpha r}\,, \qquad \alpha = 1, \ldots, k\,.$$

Consequently, the equilibrium points of $X_{\boldsymbol{f}_{\boldsymbol{\mu}}}$ are determined by the conditions

$$p_{\alpha r} = 0, \qquad -b_\alpha^2 r_\alpha + \frac{L_\alpha^2}{r_\alpha^3} = 0\,,$$

for $\alpha = 1, \ldots, k$. It follows that the equilibrium configurations of the reduced k-polysymplectic $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian system are precisely those points of $(\mathrm{T}^*\mathbb{R})^k$ that correspond to circular motions of the oscillators with fixed radii r_α and angular momenta L_α. Note that an equilibrium point of the reduced system is the projection of a k-polysymplectic relative equilibrium point $z_e \in P$.

The Hessian of the reduced functions $f_{\boldsymbol{\mu}}^\alpha$ is positive-definite on a supplementary subspace to $\ker \omega_{\mu^\alpha}$ at the equilibrium point. Indeed,

$$\mathrm{Hess}(f_{\boldsymbol{\mu}}^\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & 4b_\alpha^2 \end{pmatrix}\,.$$

Moreover, the function $\sum_{\alpha=1}^{k} f_{\boldsymbol{\mu}}^\alpha$ admits a positive-definite Hessian, and the equilibrium point becomes a strict minimum. This implies that the reduced k-polysymplectic relative equilibrium point is stable in the Lyapunov sense. In other words, the orbits through a k-polysymplectic relative equilibrium point are contained within the preimage of an open neighbourhood of the projection of the k-polysymplectic relative equilibrium point.

3.4.4 Example: k-Polysymplectic affine Lie systems

Consider now the application of the developed techniques to a family of affine inhomogeneous systems of first-order differential equations. It is well known that every such system gives rise to a Lie system [29]. In what follows, such systems are referred to as *affine Lie systems*. Many affine Lie systems arise in control theory as well as in various other areas of applied mathematics [31]. The affine Lie systems whose associated Vessiot–Guldberg Lie algebra is spanned by $\boldsymbol{\omega}$-Hamiltonian vector fields with respect to the given k-polysymplectic form $\boldsymbol{\omega}$ are called *k-polysymplectic affine Lie systems.*

Consider the following system of first-order differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}b_1(t)\\b_2(t)\\b_3(t)\\b_4(t)\\b_5(t)\end{pmatrix}+b_6(t)\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&-1&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}, \tag{3.4.3}$$

where $b_1(t),\dots,b_6(t)$ are arbitrary t-dependent functions. This system corresponds to differential equations whose solutions are the integral curves of the t-dependent vector field

$$X=\sum_{\alpha=1}^{6}b_\alpha(t)X_\alpha\,,$$

where

$$X_1=\frac{\partial}{\partial x_1}\,,\quad X_2=\frac{\partial}{\partial x_2}\,,\quad X_3=\frac{\partial}{\partial x_3}\,,\quad X_4=\frac{\partial}{\partial x_4}\,,\quad X_5=\frac{\partial}{\partial x_5}\,,\quad X_6=x_5\frac{\partial}{\partial x_3}-x_3\frac{\partial}{\partial x_5}\,.$$

These vector fields span a six-dimensional Lie algebra V of vector fields. The non-vanishing commutation relations are

$$[X_3,X_6]=-X_5\,,\qquad [X_5,X_6]=X_3\,.$$

Consider the particular case in which the functions $b_1(t),\dots,b_6(t)$ are constants, denoted by $c_1,\dots,c_6\in\mathbb{R}$, respectively. Since $X_1\wedge\dots\wedge X_6=0$, the methods presented in Subsection 3.4.1 for describing k-polysymplectic forms compatible with Lie systems can not be directly applied to system (3.4.3). Nevertheless, a two-polysymplectic form can be defined on $\mathbb{R}^5$ as

$$\boldsymbol{\omega}=(\mathrm{d}x_3\wedge\mathrm{d}x_5+\mathrm{d}x_4\wedge\mathrm{d}x_1)\otimes e_1+(\mathrm{d}x_3\wedge\mathrm{d}x_5+\mathrm{d}x_4\wedge\mathrm{d}x_2)\otimes e_2\,,$$

which turns all the vector fields $X_1,\dots,X_6$ into $\boldsymbol{\omega}$-Hamiltonian vector fields. Indeed, the corresponding $\boldsymbol{\omega}$-Hamiltonian functions for $X_1,\dots,X_6$ read

$$\begin{aligned}&\boldsymbol{h}_1=-x_4\otimes e_1\,, &&\boldsymbol{h}_2=-x_4\otimes e_2\,, &&\boldsymbol{h}_3=x_5\otimes e_1+x_5\otimes e_2\,,\\ &\boldsymbol{h}_4=x_1\otimes e_1+x_2\otimes e_2\,, &&\boldsymbol{h}_5=-x_3\otimes e_1-x_3\otimes e_2\,, &&\boldsymbol{h}_6=\frac{1}{2}(x_3^2+x_5^2)\otimes e_1+\frac{1}{2}(x_3^2+x_5^2)\otimes e_2\,.\end{aligned}$$

The flow of the vector field X_4 gives rise to a two-polysymplectic Lie group action $\Phi\colon\mathbb{R}\times\mathbb{R}^5\to\mathbb{R}^5$. Moreover, X_4, which spans the space of fundamental vector fields of Φ, is a Lie symmetry of the system (3.4.3). Then, the two-polysymplectic momentum map associated with Φ is given by

$$\mathbf{J}^\Phi\colon(x_1,x_2,x_3,x_4,x_5)\in\mathbb{R}^5\mapsto(x_1,x_2)=\boldsymbol{\mu}\in\mathbb{R}^{*2}\,.$$

Note that $\boldsymbol{\mu}\in\mathbb{R}^{*2}$ is a regular two-value of $\mathbf{J}^\Phi$, and $\mathbf{J}^\Phi$ is Ad^{*2}-equivariant two-polysymplectic momentum map. Furthermore, one can check that the example satisfies the conditions (2.3.3) and (2.3.4). Consequently, Theorem 2.3.14 can be applied.

The vector field X_4 is tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and $\mathrm{T}_x(G_{\boldsymbol{\mu}}x) = \langle\frac{\partial}{\partial x_4}\rangle$ for $x \in \mathbb{R}^5$. Therefore, $P_{\boldsymbol{\mu}} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathbb{R}$ is a two-dimensional manifold with natural coordinates $\{x_3, x_5\}$. Thus, the reduced two-polysymplectic form reads

$$\boldsymbol{\omega}_{\boldsymbol{\mu}} = \omega^1_{\boldsymbol{\mu}} \otimes e_1 + \omega^2_{\boldsymbol{\mu}} \otimes e_2 = \mathrm{d}x_3 \wedge \mathrm{d}x_5 \otimes e_1 + \mathrm{d}x_3 \wedge \mathrm{d}x_5 \otimes e_2 \,.$$

To apply Theorem 2.3.16, the affine Lie system must be tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$. This condition can be ensured by requiring that the associated $\boldsymbol{\omega}$-Hamiltonian function is invariant relative to X_4. It is satisfied by imposing $c_1 = c_2 = 0$. The resulting vector field reads

$$X_{\boldsymbol{\omega}} = c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6 \,.$$

This vector field projects onto $P_{\boldsymbol{\mu}} = \mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathbb{R}$, yielding an $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian vector field of the form

$$X_{\boldsymbol{\mu}} = c_6\left(x_5\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_5}\right) + c_3\frac{\partial}{\partial x_3} + c_5\frac{\partial}{\partial x_5} \,.$$

The $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian function associated with $X_{\boldsymbol{\mu}}$ reads

$$\boldsymbol{f}_{\boldsymbol{\mu}} = \left(c_3x_5 - c_5x_3 + c_6\left(\frac{x_3^2}{2} + \frac{x_5^2}{2}\right)\right) \otimes e_1 + \left(c_3x_5 - c_5x_3 + c_6\left(\frac{x_3^2}{2} + \frac{x_5^2}{2}\right)\right) \otimes e_2 \,.$$

Next, the methods developed in Section 3.3 are applied to determine the two-polysymplectic relative equilibrium points of the $\boldsymbol{\omega}$-Hamiltonian vector field

$$Y = X_4 + X_6$$

and to analyse their stability properties.

By Theorem 3.3.2, a two-polysymplectic relative equilibrium point $z_e \in P$ is a point for which there exists $\xi \in \mathfrak{g} \simeq \mathbb{R}$ such that z_e is a critical point of each component of the $\mathbb{R}^2$-valued function of the form

$$\boldsymbol{h}_\xi = \left(x_1 - \xi(x_1 - \mu^1) + \frac{1}{2}(x_3^2 + x_5^2)\right) \otimes e_1 + \left(x_2 - \xi(x_2 - \mu^2) + \frac{1}{2}(x_3^2 + x_5^2)\right) \otimes e_1 \,.$$

This is satisfied for $\xi = 1$, and two-polysymplectic relative equilibrium points are of the form $z_e = (x_1, x_2, x_3 = 0, x_4, x_5 = 0) \in \mathbb{R}^5$, where x_1, x_2, x_4 are arbitrary.

To analyse the stability of the projection of a k-polysymplectic relative equilibrium point $z_e \in \mathbb{R}^5$, note that the supplementary spaces to $\mathrm{T}_{z_e}(G_{\boldsymbol{\mu}_e}z_e) + \ker\omega^1_{z_e}$ and $\mathrm{T}_{z_e}(G_{\boldsymbol{\mu}_e}z_e) + \ker\omega^2_{z_e}$ in $\mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ are given by

$$\mathcal{S}^1 = \left\langle\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5}\right\rangle, \qquad \mathcal{S}^2 = \left\langle\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5}\right\rangle,$$

respectively. Then, $\mathcal{S}^1 + \mathcal{S}^2 + \mathrm{T}_{z_e}(G_{\boldsymbol{\mu}_e}z_e) = \mathrm{T}_{z_e}\mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e)$ and the Hessian of $(\delta^2 h^1_\xi)_{z_e}$ at z_e in the subspace $\mathcal{S}^1$ and the Hessian of $(\delta^2 h^2_\xi)_{z_e}$ in the subspace $\mathcal{S}^2$ are definite-positive. Therefore, a two-polysymplectic relative equilibrium point $z_e \in \mathbb{R}^5$ is relatively stable. That is, its projection to the reduced manifold $P_{\boldsymbol{\mu}_e}$ is stable. More concretely, the reduced system admits an $\boldsymbol{\omega}_{\boldsymbol{\mu}_e}$-Hamiltonian function whose components $f^1_{\boldsymbol{\mu}_e}, f^2_{\boldsymbol{\mu}_e}$ have positive-definite Hessians at the equilibrium points $\pi_{\boldsymbol{\mu}_e}(z_e) = (x_3 = 0, x_5 = 0)$ in the directions of $\ker(\omega^1_{\boldsymbol{\mu}_e})_{z_e}$ and $\ker(\omega^2_{\boldsymbol{\mu}_e})_{z_e}$, respectively.

Indeed, the reduced $\boldsymbol{\omega}_{\boldsymbol{\mu}_e}$-Hamiltonian function reads

$$\boldsymbol{f}_{\boldsymbol{\mu}_e} = \frac{1}{2}(x_3^2 + x_5^2) \otimes e_1 + \frac{1}{2}(x_3^2 + x_5^2) \otimes e_2$$

and the function

$$f^1_{\boldsymbol{\mu}_e} + f^2_{\boldsymbol{\mu}_e} = x_3^2 + x_5^2 \,,$$

is invariant under the dynamics of $Y_{\boldsymbol{\mu}_e}$ and admits a strict minimum at $\pi_{\boldsymbol{\mu}_e}(z_e) = (x_3 = 0, x_5 = 0)$. Hence, the reduced two-polysymplectic Hamiltonian system is stable at $\pi_{\boldsymbol{\mu}_e}(z_e)$.

3.4.5 Example: Quantum quadratic Hamiltonian operators

Next, an example is analysed based on the Wei–Norman equations for the automorphic Lie system associated with quantum mechanical models described by quadratic Hamiltonian operators. These models include, as particular cases, quantum harmonic oscillators with or without dissipation [29, 149]. In this framework, the differential system under consideration is the one determining the integral curves of the time-dependent vector field

$$X = \sum_{\alpha=1}^{6} b_\alpha(t) X_\alpha^R, \tag{3.4.4}$$

for certain t-dependent functions $b_1(t), \ldots, b_6(t)$ and the vector fields

$$\begin{aligned}
X_1^R &= \frac{\partial}{\partial v_1} + v_5 \frac{\partial}{\partial v_4} - \frac{1}{2} v_5^2 \frac{\partial}{\partial v_6}\,, & X_2^R &= v_1 \frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2} + \frac{1}{2} v_4 \frac{\partial}{\partial v_4} - \frac{1}{2} v_5 \frac{\partial}{\partial v_5}\,,\\
X_3^R &= v_1^2 \frac{\partial}{\partial v_1} + 2v_1 \frac{\partial}{\partial v_2} + e^{v_2} \frac{\partial}{\partial v_3} - v_4 \frac{\partial}{\partial v_5} + \frac{1}{2} v_4^2 \frac{\partial}{\partial v_6}\,, & X_4^R &= \frac{\partial}{\partial v_4}\,,\\
X_5^R &= \frac{\partial}{\partial v_5} - v_4 \frac{\partial}{\partial v_6}\,, & X_6^R &= \frac{\partial}{\partial v_6}\,.
\end{aligned}$$

The commutation relations between the above vector fields are

$$\begin{array}{lllll}
[X_1^R, X_2^R] = X_1^R\,, & & & & \\
[X_1^R, X_3^R] = 2\,X_2^R\,, & [X_2^R, X_3^R] = X_3^R\,, & & & \\
[X_1^R, X_4^R] = 0\,, & [X_2^R, X_4^R] = -\dfrac{1}{2}\,X_4^R\,, & [X_3^R, X_4^R] = X_5^R\,, & & \\
[X_1^R, X_5^R] = -X_4^R\,, & [X_2^R, X_5^R] = \dfrac{1}{2}\,X_5^R\,, & [X_3^R, X_5^R] = 0\,, & [X_4^R, X_5^R] = -X_6^R\,, & \\
[X_1^R, X_6^R] = 0\,, & [X_2^R, X_6^R] = 0\,, & [X_3^R, X_6^R] = 0\,, & [X_4^R, X_6^R] = 0\,, & [X_5^R, X_6^R] = 0\,.
\end{array}$$

The Lie algebra of Lie symmetries of $\langle X_1^R, \ldots, X_6^R \rangle$ is spanned by

$$\begin{gathered}
X_1^L = e^{v_2} \frac{\partial}{\partial v_1} + 2v_3 \frac{\partial}{\partial v_2} + v_3^2 \frac{\partial}{\partial v_3}\,, \qquad X_2^L = \frac{\partial}{\partial v_2} + v_3 \frac{\partial}{\partial v_3}\,, \qquad X_3^L = \frac{\partial}{\partial v_3}\,,\\
X_4^L = e^{-v_2/2}(e^{v_2} - v_1 v_3) \frac{\partial}{\partial v_4} - e^{-v_2/2} v_3 \frac{\partial}{\partial v_5} - e^{-v_2/2}(e^{v_2} - v_1 v_3) v_5 \frac{\partial}{\partial v_6}\,,\\
X_5^L = v_1 e^{-v_2/2} \frac{\partial}{\partial v_4} + e^{-v_2/2} \frac{\partial}{\partial v_5} - v_1 v_5 e^{-v_2/2} \frac{\partial}{\partial v_6}\,, \quad X_6^L = \frac{\partial}{\partial v_6}\,.
\end{gathered}$$

In particular, consider the system (3.4.4) of the form

$$X_5^R = \frac{\partial}{\partial v_5} - v_4 \frac{\partial}{\partial v_6}\,.$$

A Lie symmetry of this system is given by

$$Y = \frac{\partial}{\partial v_5}\,.$$

A two-polysymplectic form on $\mathbb{R}^6$ can be defined as

$$\begin{aligned}
\boldsymbol{\omega} &= \omega^1 \otimes e_1 + \omega^2 \otimes e_2\\
&= (\mathrm{d}v_1 \wedge \mathrm{d}v_3 + \mathrm{d}v_2 \wedge \mathrm{d}v_4 + \mathrm{d}v_5 \wedge \mathrm{d}v_1 + \mathrm{d}v_4 \wedge \mathrm{d}v_6) \otimes e_1 + (\mathrm{d}v_4 \wedge \mathrm{d}v_6 - \mathrm{d}v_3 \wedge \mathrm{d}v_5) \otimes e_2\,.
\end{aligned}$$

Then,

$$\ker \omega^1 = \left\langle \frac{\partial}{\partial v_3} + \frac{\partial}{\partial v_5}, \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_6} \right\rangle, \qquad \ker \omega^2 = \left\langle \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\rangle,$$

and $\ker \omega^1 \cap \ker \omega^2 = 0$. Consequently, $(\mathbb{R}^6, \boldsymbol{\omega})$ becomes a two-polysymplectic manifold.

The vector field Y is a Lie symmetry of the two-polysymplectic form, since $\mathscr{L}_Y\boldsymbol{\omega}=0$. Then,

$$\iota_{Y_3}\omega^1=\mathrm{d}v_1\,,\qquad \iota_{Y_3}\omega^2=\mathrm{d}v_3\,,$$

and a two-polysymplectic momentum map $\mathbf{J}^\Phi$ associated with the Lie group action given by the flow of Y reads

$$\mathbf{J}^\Phi\colon x\in\mathbb{R}^6\longmapsto\boldsymbol{\mu}=(v_1,v_3)\in\mathfrak{g}^{*2}\simeq\mathbb{R}^{*2}\,.$$

Note that every $\boldsymbol{\mu}=(\mu^1,\mu^2)\in\mathbb{R}^{*2}$ is a weak regular two-value of $\mathbf{J}^\Phi$, which is Ad^{*2}-equivariant. The isotropy group related to each $\boldsymbol{\mu}\in\mathbb{R}^{*2}$ is given by $G_{\boldsymbol{\mu}}=\mathbb{R}$. Hence, $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is a submanifold of $\mathbb{R}^6$, as well as $\mathbf{J}_1^{\Phi-1}(\mu^1)$ and $\mathbf{J}_2^{\Phi-1}(\mu^2)$. Since Y_5 is tangent to $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$, then $P_{\boldsymbol{\mu}}=\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/G_{\boldsymbol{\mu}}$ admits local coordinates given by $\{v_2,v_4,v_6\}$.

The vector field X_5^R is $\boldsymbol{\omega}$-Hamiltonian with

$$\iota_{X_5^R}\boldsymbol{\omega}=\iota_{X_5^R}\omega^1\otimes e_1+\iota_{X_5^R}\omega^2\otimes e_2=\mathrm{d}\left(v_1+\frac{v_4^2}{2}\right)\otimes e_1+\mathrm{d}\left(v_3+\frac{v_4^2}{2}\right)\otimes e_2=\mathrm{d}\boldsymbol{h}_5^R\,.$$

Then, the reduced two-forms read

$$\omega_{\boldsymbol{\mu}}^1=\mathrm{d}v_2\wedge\mathrm{d}v_4+\mathrm{d}v_4\wedge\mathrm{d}v_6\,,\qquad \omega_{\boldsymbol{\mu}}^2=\mathrm{d}v_4\wedge\mathrm{d}v_6\,.$$

Furthermore, one has

$$\ker\omega_{\boldsymbol{\mu}}^1=\left\langle\frac{\partial}{\partial v_2}+\frac{\partial}{\partial v_6}\right\rangle\,,\qquad \ker\omega_{\boldsymbol{\mu}}^2=\left\langle\frac{\partial}{\partial v_2}\right\rangle\,,$$

and therefore $\omega_{\boldsymbol{\mu}}^1$ and $\omega_{\boldsymbol{\mu}}^2$ give rise to a two-polysymplectic form on $P_{\boldsymbol{\mu}}$. Moreover, the $\boldsymbol{\omega}$-Hamiltonian function of X_5^R is invariant relative to Y. Then, Theorem 2.3.16 ensures that the projection of X_5^R onto $P_{\boldsymbol{\mu}}$ exists and is given by

$$X_{\boldsymbol{\mu}}=-v_4\frac{\partial}{\partial v_6}\,,$$

which is the $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian vector field of the $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian $\mathbb{R}^2$-function of the form

$$\boldsymbol{f}_{\boldsymbol{\mu}}=\left(\mu^1+\frac{v_4^2}{2}\right)\otimes e_1+\left(\mu^2+\frac{v_4^2}{2}\right)\otimes e_2\,.$$

It admits a critical point at every point of the form $(v_4=0,v_6)$, where v_6 is arbitrary. Such points are not stable equilibrium points. In particular, this $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian function does not satisfy that $f_{\boldsymbol{\mu}}^1+f_{\boldsymbol{\mu}}^2$ has a strict minimum at the equilibrium point: it has only a minimum. The points in $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ projecting onto the above-mentioned equilibrium points are two-polysymplectic relative equilibrium points.

3.4.6 Example: Equilibrium points and vector fields with polynomial coefficients

To illustrate certain aspects of the k-polysymplectic energy-momentum method, consider vector fields with polynomial coefficients. Moreover, the following example highlights some features of weak regular k-values of k-polysymplectic momentum maps and the character of their associated k-polysymplectic Marsden–Meyer–Weinstein reductions.

Consider the coordinates $\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\}$ on $\mathbb{R}^8$ and define the vector field X on $\mathbb{R}^8$ by

$$X=x_6^a\frac{\partial}{\partial x_2}+x_4^b\frac{\partial}{\partial x_3}-x_3^c\frac{\partial}{\partial x_4}+x_8^d\frac{\partial}{\partial x_7}-x_7^e\frac{\partial}{\partial x_8}\,,$$

where $a,b,c,d,e\in\mathbb{N}$. A two-polysymplectic form $\boldsymbol{\omega}$ on $\mathbb{R}^8$ is of the form

$$\boldsymbol{\omega}=\omega^1\otimes e_1+\omega^2\otimes e_2=(\mathrm{d}x_3\wedge\mathrm{d}x_4+\mathrm{d}x_1\wedge\mathrm{d}x_5)\otimes e_1+(\mathrm{d}x_2\wedge\mathrm{d}x_6+\mathrm{d}x_7\wedge\mathrm{d}x_8)\otimes e_2\,.$$

Indeed,

$$\ker\omega^1_x=\left\langle\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_6},\frac{\partial}{\partial x_7},\frac{\partial}{\partial x_8}\right\rangle,\qquad \ker\omega^2_x=\left\langle\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4},\frac{\partial}{\partial x_5}\right\rangle,\qquad \ker\omega^1_x\cap\ker\omega^2_x=0$$

for any $x\in\mathbb{R}^8$, and thus $\boldsymbol{\omega}$ becomes a two-polysymplectic form on $\mathbb{R}^8$.

The vector field X admits the Lie symmetries of the form

$$Y_1=\frac{\partial}{\partial x_2}\,,\; Y_2=\frac{\partial}{\partial x_1}\,,\; Y_3=\frac{\partial}{\partial x_5}\,,$$

which span a three-dimensional abelian Lie algebra of vector fields. These Lie symmetries are the infinitesimal generators of the translations along the x_2, x_1, and x_5, respectively. Moreover, they also leave the two-polysymplectic structure invariant, since $\mathscr{L}_{Y_i}\omega^\alpha=0$ for $i=1,2,3$ and $\alpha=1,2$. Therefore, they give rise to a two-polysymplectic Lie group action $\Phi\colon\mathbb{R}^3\times\mathbb{R}^8\to\mathbb{R}^8$.

Since

$$\begin{aligned}&\iota_{Y_1}\omega^1=0\,, &&\iota_{Y_2}\omega^1=\mathrm{d}x_5\,, &&\iota_{Y_3}\omega^1=-\mathrm{d}x_1\,,\\ &\iota_{Y_1}\omega^2=\mathrm{d}x_6\,, &&\iota_{Y_2}\omega^2=0\,, &&\iota_{Y_3}\omega^2=0\,,\end{aligned}$$

a two-polysymplectic momentum map $\mathbf{J}^\Phi$ associated with Φ is of the form

$$\mathbf{J}^\Phi\colon x\in\mathbb{R}^8\longmapsto\mathbf{J}^\Phi(x)=(0,x_5,-x_1;x_6,0,0)\in(\mathbb{R}^{3*})^2\simeq(\mathbb{R}^3)^2\,.$$

Then, for each $x\in\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ and $\boldsymbol{\mu}\in(\mathbb{R}^3)^2$, it follows that

$$\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})=\left\langle\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4},\frac{\partial}{\partial x_7},\frac{\partial}{\partial x_8}\right\rangle.$$

The two-polysymplectic momentum map $\mathbf{J}^\Phi$ is Ad^{*2}-equivariant. Moreover, every $\boldsymbol{\mu}\in(\mathbb{R}^3)^2$ is a weak regular two-value of $\mathbf{J}^\Phi$. Indeed, each $\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$ is a five-dimensional submanifold of $\mathbb{R}^8$, and its tangent space at each point coincides with the kernel of $\mathbf{J}^\Phi$ at that point. On the other hand, $\mathbf{J}^\Phi$ has no regular two-values.

The vector fields Y_2 and Y_3, at x, do not take values in $\mathrm{T}_x(G_{\boldsymbol{\mu}}x)$, whereas Y_1 does. The assumptions of Theorem 2.3.14 are satisfied, and the quotient space $\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_x\,(G_{\boldsymbol{\mu}}x)$ is a two-dimensional subspace

$$\mathrm{T}_x\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})/\mathrm{T}_x\,(G_{\boldsymbol{\mu}}x)=\left\langle\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4},\frac{\partial}{\partial x_7},\frac{\partial}{\partial x_8}\right\rangle,\qquad\forall x\in\mathbf{J}^{\Phi-1}(\boldsymbol{\mu})$$

and

$$\boldsymbol{\omega}_{\boldsymbol{\mu}}=\omega^1_{\boldsymbol{\mu}}\otimes e_1+\omega^2_{\boldsymbol{\mu}}\otimes e_2=(\mathrm{d}x_3\wedge\mathrm{d}x_4)\otimes e_1+(\mathrm{d}x_7\wedge\mathrm{d}x_8)\otimes e_2\,.$$

The vector field X is $\boldsymbol{\omega}$-Hamiltonian relative to

$$\begin{aligned}\mathrm{d}\boldsymbol{h}&=\iota_X\boldsymbol{\omega}=\iota_X\omega^1\otimes e_1+\iota_X\omega^2\otimes e_2\\&=\mathrm{d}\left(\frac{1}{1+b}x_4^{b+1}+\frac{1}{c+1}x_3^{c+1}\right)\otimes e_1+\mathrm{d}\left(\frac{1}{1+a}x_6^{a+1}+\frac{1}{d+1}x_8^{d+1}+\frac{1}{1+e}x_7^{e+1}\right)\otimes e_2\,.\end{aligned}$$

The $\mathbb{R}^2$-valued function $\boldsymbol{h}$ is invariant relative to the Lie symmetries Y_1, Y_2, and Y_3. By Theorem 2.3.16, the vector field X projects onto the quotient manifold and its projection $X_{\boldsymbol{\mu}}$ reads

$$X_{\boldsymbol{\mu}}=x_4^b\frac{\partial}{\partial x_3}-x_3^c\frac{\partial}{\partial x_4}+x_8^d\frac{\partial}{\partial x_7}-x_7^e\frac{\partial}{\partial x_8}\,,$$

which is an $\boldsymbol{\omega}_{\boldsymbol{\mu}}$-Hamiltonian vector field since

$$\mathrm{d}\boldsymbol{f}_{\boldsymbol{\mu}}=\iota_{X_\mu}\boldsymbol{\omega}_{\boldsymbol{\mu}}=\mathrm{d}\left(\frac{1}{1+b}x_4^{b+1}+\frac{1}{c+1}x_3^{c+1}\right)\otimes e_1+\mathrm{d}\left(\frac{1}{d+1}x_8^{d+1}+\frac{1}{1+e}x_7^{e+1}\right)\otimes e_2\,.$$

Then, according to Theorem 3.3.2, a point z_e is a two-polysymplectic relative equilibrium point if it is a critical point of $\boldsymbol{h}_\xi$ for some $\xi = (\xi_1, \xi_2, \xi_3) \in \mathfrak{g} \simeq \mathbb{R}^3$. Then,

$$\begin{aligned} \mathrm{d}\boldsymbol{h}_\xi &= \mathrm{d}f_\xi^1 \otimes e_1 + \mathrm{d}f_\xi^2 \otimes e_2 \\ &= \left(x_4^b \mathrm{d}x_4 + x_3^c \mathrm{d}x_3 - \xi_2 \mathrm{d}x_5 + \xi_3 \mathrm{d}x_1\right) \otimes e_1 + \left((x_6^a - \xi_1)\mathrm{d}x_6 + x_8^d \mathrm{d}x_8 + x_7^e \mathrm{d}x_7\right) \otimes e_2 \,. \end{aligned}$$

Thus, $\xi_2 = \xi_3 = 0$ and the two-polysymplectic relative equilibrium points of X are of the form $z_e = (x_1, x_2, 0, 0, x_5, x_6, 0, 0)$ for $x_6^a = \xi_1$ where x_1, x_2, x_5, x_6 are arbitrary. Indeed, $(X_{\boldsymbol{\mu}_e})_{[z_e]} = 0$ for $\boldsymbol{\mu}_e = \mathbf{J}^\Phi(z_e)$.

To study the stability of these two-polysymplectic relative equilibrium points, one needs to analyse the second derivatives of $\boldsymbol{h}_\xi$ at z_e. Then,

$$\begin{aligned} (\delta^2 \boldsymbol{h}_\xi)_{z_e} &= (\delta^2 h_\xi^1)_{z_e} \otimes e_1 + (\delta^2 h_\xi^2)_{z_e} \otimes e_2 \\ &= \left(c x_3^{c-1} \mathrm{d}x_3 \otimes \mathrm{d}x_3 + b x_4^{b-1} \mathrm{d}x_4 \otimes \mathrm{d}x_4\right) \otimes e_1 + \left(e x_7^{e-1} \mathrm{d}x_7 \otimes \mathrm{d}x_7 + d x_8^{d-1} \mathrm{d}x_8 \otimes \mathrm{d}x_8\right) \otimes e_2 \,. \end{aligned}$$

Taking into account that the supplementary spaces to

$$\mathrm{T}_{z_e}(G_{\boldsymbol{\mu}_e} z_e) + \ker \omega_{z_e}^1 \cap \mathrm{T}_{z_e} \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e) \,, \qquad \mathrm{T}_{z_e}(G_{\boldsymbol{\mu}_e} z_e) + \ker \omega_{z_e}^2 \cap \mathrm{T}_{z_e} \mathbf{J}^{\Phi-1}(\boldsymbol{\mu}_e) \,,$$

can be chosen as

$$\mathcal{S}_{z_e}^1 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\rangle , \qquad \mathcal{S}_{z_e}^2 = \left\langle \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle ,$$

respectively. Definition 3.3.6 yields that the two-polysymplectic relative equilibrium points z_e are stable if

$$(\delta^2 h_\xi^1)_{z_e}(v_{z_e}, v_{z_e}) > 0 \,, \qquad \forall v_{z_e} \in \mathcal{S}_{z_e}^1 \setminus \{0\} \,,$$

and

$$(\delta^2 h_\xi^2)_{z_e}(v_{z_e}, v_{z_e}) > 0 \,, \qquad \forall v_{z_e} \in \mathcal{S}_{z_e}^2 \setminus \{0\} \,.$$

These inequalities hold if and only if $b, c, d, e = 1$. Consequently, points z_e are formally stable two-polysymplectic relative equilibrium points of X for $b, c, d, e = 1$.

Conclusions

This dissertation presents a systematic development of reduction theory, energy-momentum methods, and stability analysis for Hamiltonian systems with symmetries, with a particular emphasis on non-autonomous settings and the geometric frameworks of k-polysymplectic, k-polycosymplectic, and k-contact structures.

This PhD thesis generalises Marsden–Meyer–Weinstein reductions to new, more general, geometric realms. A significant generalisation of the classical Marsden–Meyer–Weinstein reduction framework has been achieved by developing its generalisation within the k-polycosymplectic and k-contact settings. It also removes the technical conditions that momentum maps have to be Ad^*-equivariant by the introduction of affine Lie group actions. In addition, this PhD thesis discusses some inaccuracies found in the literature concerning k-polycosymplectic and k-contact MMW reductions. In particular, it is shown that both technical assumptions in the k-polycosymplectic MMW reduction theorem are sufficient and are independent of each other. In the case of a k-contact MMW reduction theorem, it is explained in detail how the quotient Lie subgroup is obtained and what the reduced manifold should be to get the reduction in the most general case, correcting some results in the literature. All constructions are illustrated through detailed examples and applications to physical models, such as coupled vibrating strings and vibrating membranes, thereby demonstrating the practical relevance and applicability of the theoretical developments.

The dissertation presents a new energy-momentum framework within cosymplectic geometry, which generalises the classical symplectic approach by adjusting Hamiltonian, gradient, and evolution symmetries. This setting permits the definition and analysis of new types of relative equilibria, notably gradient relative equilibria, and is applied to physically relevant problems, such as the restricted circular three-body problem and time-dependent Schrödinger equation. A novel cosymplectic-to-symplectic reduction scheme is also formulated, further enriching the methodological toolkit for the study of time-dependent Hamiltonian systems.

The final contribution of this dissertation is the construction of a k-polysymplectic energy-momentum method, complemented by new techniques for the stability analysis of Hamiltonian systems defined on k-polysymplectic manifolds. This framework is applied to a range of systems of both physical and mathematical interest, including integrable Hamiltonian systems, quantum oscillators with dissipation, polynomial dynamical systems, and equations related to the Schwarzian derivative.

Altogether, this PhD thesis contributes significantly to the geometric theory of reduction and stability of Hamiltonian systems, offering generalisations that bridge mathematical theory with physical applications. The tools and insights developed herein are expected to inform and inspire further research in geometric mechanics, dynamical systems, and mathematical models of complex physical phenomena.

This PhD thesis opens new ways of research.

Presently, there are still several works in progress that are related to the topic of this PhD thesis.

1. A. Lopez-Gordon, J. de Lucas, and B.M. Zawora, "*Stability of contact Hamiltonian systems*", 2026.

2. B.M. Zawora, "*Modified Witt-Artin decomposition of exact symplectic manifolds*", 2026.

The first article concerns the generalisation of the energy-momentum method for contact Hamiltonian systems. It is a very difficult topic, and the main problem lies in the fact that stable contact Hamiltonian systems cannot be extended in a natural manner to stable symplectic Hamiltonian systems in the usual way. Moreover, it is challenging to analyse the relative equilibrium points of contact Hamiltonian systems, since contact Hamiltonian systems describe dissipative Hamiltonian systems, where the energy of the systems is not conserved. The dissipated quantities are used to investigate the stability of contact Hamiltonian systems.

The second work is about the Witt–Artin decomposition theorem. Originally established by E. Witt in 1937, provides a particular decomposition of a vector space. One of its most significant applications in symplectic geometry is the decomposition of the tangent space at some point of a symplectic manifold endowed with a proper Lie group action that preserves the symplectic form. This decomposition is an essential step in proving the Symplectic Slice Theorem, a fundamental result in the singular reduction theory. The modified version of the Witt–Artin decomposition theorem is adapted to exact symplectic manifolds with a proper Lie group action that leaves the primitive one-form invariant. Since contact manifolds can be considered a special class of exact symplectic manifolds, one gets an analogous decomposition theorem for contact manifolds with a proper Lie group action preserving the contact structure. This generalisation may lead to the possibility of generalising the Symplectic Slice Theorem and the Marle–Guillemin–Sternberg Normal Form Theorem to the contact setting, providing a framework for singular contact reduction in full generality.

Moreover, there are other questions to be analysed:

1. The stability of PDEs is an open topic that is just very briefly analysed in [84]. The method could be potentially used in new examples of Hamiltonian systems, such as the Chaplygin gas model and Born-Infeld model [71]. It is concerned with the Casimir-energy method, which could be extended to more general geometric settings, e.g. to Dirac manifolds, but this topic has been left for further work. Nevertheless, a first work in this direction has recently been accepted for publication [71].

2. Another promising direction concerning the reduction by symmetries of Hamiltonian systems involves the reduction of space-time variables for k-contact structures. Such a generalisation is expected to be feasible for k-contact manifolds, as the reduction of space-time variables has already been achieved in the context of the k-polycosymplectic Marsden–Meyer–Weinstein reduction, namely the reduction from k-polycosymplectic to ℓ-polysymplectic for $\ell < k$ (see [50]). It is well known that the Reeb vector fields constitute symmetries of the k-contact structure; they take values in the kernel of $\mathrm{d}\boldsymbol{\eta}$, and the contraction of each Reeb vector field with $\boldsymbol{\eta}$ equals one. The underlying idea is analogous to that employed in the reduction from k-polycosymplectic to ℓ-polycosymplectic structures. The results are completely novel and seem to be achievable, leading to new and interesting applications.

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