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Analytical and Algebraic Structures of Special Functions in Exactly Solvable Systems

A tale of two functions

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Abstract

This thesis is concerned with two classes of functions arising in exactly solvable systems of physics. Special functions are often treated merely as instruments of solution, yet their deeper significance lies in their delicate identities, hidden symmetries, and intricate relations. To solve something exactly is not only to obtain an answer, but also to better understand the structure that lies beneath it. Guided by this perspective, we study the spectral theory of one-dimensional Schrödinger operators and zero-dimensional random matrix theories through the lens of special functions.

The first part is devoted to the spectral theory of one-dimensional Schrödinger operators with complex potentials that can be reduced to the hypergeometric equation. Exploiting the exact solvability of these operators, we construct their Green functions and analyze the corresponding operator domains, closed realizations, and boundary conditions. In particular, we classify nine families of Schrödinger operators arising from separation of variables on symmetric spaces. This part is based on the paper *Exactly solvable Schrödinger operators related to the hypergeometric equation*, written in collaboration with Jan Dereziński. We also include a new unpublished result concerning the exact WKB quantization of these operators, showing how exact WKB expansions lead to an exact quantization condition.

The second part of the thesis concerns symmetric polynomials and their role in the β-deformed eigenvalue model. We show that this model is not only exactly solvable, but that, when multipoint correlators are expanded in the appropriate basis given by Jack polynomials, they acquire a closed and factorized form. This remarkable phenomenon is known as superintegrability. We present a proof based on the Virasoro constraints of the model together with representation-theoretic methods. This part is based on the paper *Proving superintegrability in β-deformed eigenvalue models*, written in collaboration with Aditya Bawane and Piotr Sułkowski.

In addition, we include a new unpublished result establishing a more general framework for this phenomenon. The proof relies on the relation between multivariable hypergeometric functions and the Dunkl reproducing kernel. This broader perspective not only contains the superintegrability of β-deformed models as a special case, but also clarifies the underlying mechanism behind this factorization phenomenon through a simpler and more clean argument.

Streszczenie

Niniejsza praca dotyczy dwóch klas funkcji pojawiających się w dokładnie rozwiązywalnych układach fizycznych. Funkcje specjalne są często traktowane jedynie jako narzędzia rozwiązania, jednak ich głębsze znaczenie tkwi w subtelnych tożsamościach, ukrytych symetriach oraz złożonych relacjach między nimi. Dokładne rozwiązanie problemu oznacza nie tylko otrzymanie odpowiedzi, lecz także lepsze zrozumienie struktury, która się pod nim kryje. Kierując się tą perspektywą, badamy teorię spektralną jednowymiarowych operatorów Schrödingera oraz zerowymiarowe teorie macierzy losowych przez pryzmat funkcji specjalnych.

Pierwsza część poświęcona jest teorii spektralnej jednowymiarowych operatorów Schrödingera z zespolonymi potencjałami, które można sprowadzić do równania hipergeometrycznego. Wykorzystując dokładną rozwiązywalność tych operatorów, konstruujemy ich funkcje Greena oraz analizujemy odpowiadające im dziedziny operatorowe, domknięte realizacje oraz warunki brzegowe. W szczególności klasyfikujemy dziewięć rodzin operatorów Schrödingera powstających poprzez rozdzielenie zmiennych na przestrzeniach symetrycznych. Ta część oparta jest na pracy *Exactly solvable Schrödinger operators related to the hypergeometric equation*, napisanej we współpracy z Janem Derezińskim. Uwzględniamy również nowy, niepublikowany wynik dotyczący dokładnej kwantyzacji WKB tych operatorów, pokazujący, w jaki sposób dokładne rozwinięcia WKB prowadzą do ścisłego warunku kwantyzacji.

Druga część rozprawy dotyczy wielomianów symetrycznych oraz ich roli w β-deformowanym modelu wartości własnych. Pokazujemy, że model ten jest nie tylko dokładnie rozwiązywalny, lecz także że po rozwinięciu wielopunktowych korelatorów w odpowiedniej bazie danej przez wielomiany Jacka uzyskują one zamkniętą i sfaktoryzowaną postać. To wyjątkowe zjawisko znane jest jako supercałkowalność. Przedstawiamy dowód oparty na więzach Virasoro związanych z modelem oraz metodach teorii reprezentacji. Ta część oparta jest na pracy *Proving superintegrability in β-deformed eigenvalue models*, napisanej we współpracy z Adityą Bawane oraz Piotrem Sułkowskim.

Dodatkowo przedstawiamy nowy, niepublikowany wynik ustanawiający bardziej ogólne ramy dla tego zjawiska. Dowód opiera się na związku pomiędzy wielozmiennymi funkcjami hipergeometrycznymi a jądrem reprodukującym Dunkla. Szersza perspektywa nie tylko obejmuje supercałkowalność β-deformowanych modeli jako szczególny przypadek, lecz także wyjaśnia mechanizm stojący za tym zjawiskiem faktoryzacji w prostszym i bardziej przejrzystym ujęciu.

To my mom.

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Contents

Introduction

> I could have built the pyramids
> with the effort it takes me to
> cling to life and reason.
>
> F. Kafka

This thesis is devoted to the study of special functions and their applications to problems arising in physics. Special functions have always been closely tied to physics in one way or another. *What kind of function is special enough to be called a special function?* Richard Askey, in the foreword to the *Encyclopedia of Mathematics and Its Applications* [M2], gives a good answer: *Certain functions appear so often that it is convenient to give them names. These are collectively called special functions.*

Special functions often appear as a dry, almost taxonomic catalogue of identities and relations, yet it is precisely this richness of detail that reveals their beauty. Through calculation, they become bridges between distant parts of mathematics and physics, allowing hidden connections to emerge. This thesis embraces that very feature: to show that in calculation we do not merely compute, but connect, understand, and uncover the hidden unity of seemingly separate worlds.

The two works on which this thesis is based deal with exact solutions in two different settings: zero-dimensional random matrix models by Aditya Bawane, P. K., Piotr Sułkowski [BKS] and one-dimensional quantum-mechanical systems by Jan Dereziński, P. K. [DK]. Both classes of systems can be solved exactly, and in the case of random matrix theory, in a factorized form involving special functions. Special functions of one complex variable were a major topic of mathematics in the eighteenth and nineteenth centuries. During this period, many of the developments that one may call *formal* were achieved: the most important recurrence relations, differential equations, and limiting procedures were discovered. Let us discuss one of the most important special functions, namely the hypergeometric function, defined by

$$
{}_pF_q(a_1, \dots, a_p;\, b_1, \dots, b_q;\, z) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{0.0.1}
$$

where the Pochhammer symbol is given by $(a)_n := a(a+1)\cdots(a+n-1)$, with $(a)_0 = 1$. The series converges, for instance, when $p \le q$ and $|z| < 1$, and it can then be analytically continued to a larger domain. This function is one of the most thoroughly studied objects in mathematics. It was known, at least under the name *hypergeometric*, since the seventeenth century. However, it was Carl Friedrich Gauss who studied it systematically, in particular the specialization known as the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$. One of his most important contributions is the differential equation satisfied by this function [G],

$$
\big(z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab\big)\, {}_2F_1(a, b; c; z) = 0. \tag{0.0.2}
$$

Equation (0.0.2) had a profound impact on nineteenth-century mathematics. Since it is a second-order differential equation, one can derive three-term recurrence relations for the Gauss hypergeometric function. Even more significant is the study of the singularities of this equation, which will be discussed in detail in Chapter 2. One of the papers on which this thesis is based is devoted to the study of this equation in the modern language of functional analysis [DK]. This language was largely developed in the twentieth century in connection with the mathematical formulation of quantum mechanics. The most natural object in quantum mechanics is the Hilbert space, which is often infinite-dimensional. This brings entirely new concerns and possibilities. For instance, one can no longer make sense of unbounded differential operators solely through their algebraic form; one must carefully specify their domains.

In this sense, the first part of this thesis is devoted to the study of one-dimensional Schrödinger equations related to the Gauss hypergeometric equation through a careful examination of the corresponding operators. We realize them as closed operators in the sense of $L^2]a,b[$ and discuss the role of boundary conditions at the endpoints a and b. Surprisingly, we also found some new identities of an algebraic nature for hypergeometric functions [DK]. In the final section we added some new result related to exact WKB method in the context of hypergeoemtric Hamiltonians Sec. 1.7.

Thus, in the first part we use the rigorous methods of operator theory to understand the hypergeometric equation. The second part of the thesis is related to another important development in the theory of special functions: the search for multivariable analogues of hypergeometric functions. As emphasized in the historically important remarks of Askey [M2], *multivariable special functions are less well understood and less canonical.* Indeed, Eric Opdam, one of the pioneers of multivariable special functions, describes in his survey article [Op] several approaches to generalizing the notion of hypergeometric functions to the multivariable setting. This line of research began in the nineteenth century with the work of Appell, Lauricella, and Horn, but its more systematic development has taken place over the last fifty years.

Among these approaches, we mention two: the Gelfand–Kapranov–Zelevinsky (GKZ) theory, which is based on systems of partial differential equations [GKZ], and the theory of hypergeometric functions associated with root systems, defined via commuting differential-difference operators and developed by Macdonald [Mac3], Heckman-Opdam [HO].

In the first paper [BKS], and in the second part of this thesis Ch. 2, we focus on hypergeometric functions associated with root systems and use them to prove a conjecture arising in random matrix theory. We provide two proofs. The first is based on the symmetries of the integral representation of the β-ensemble. These symmetries generate an infinite family of constraints, known as Virasoro constraints or loop equations. We show, roughly speaking, that there exists a preferred basis that appears in the orbit of a differential operator constructed from these constraints. This preferred basis is given by the Jack polynomials, which form a distinguished basis of the ring of symmetric polynomials Sec. 2.1.

The second proof relies instead on the fact that Jack polynomials give rise to multivariable hypergeometric functions of Bessel type possessing a remarkable reproducing-kernel property Sec. 2.6. This allows us to obtain a new and rather direct proof, which does not rely on the more involved algebraic machinery of the first argument.

0.1 Part I: Exactly solvable Schrödinger operators in terms of the hypergeometric equation

The main objective is to study one-dimensional Schrödinger operators with complex potentials,

$$-\frac{d^2}{dx^2} + V(x). \tag{0.1.3}$$

The Schrödinger operator with a real potential is a classical topic covered in many monographs. However, as we will see shortly, the Schrödinger operators associated with the hypergeometric operator may have complex potentials. The theory of Schrödinger operators with complex potentials has been developed in [DeGe, DeRi].

Thus, we study the Schrödinger operators constructed from the hypergeometric operator

$$z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab. \tag{0.1.4}$$

These Schrödinger operators have been considered in physics since the 1940s because of their exact solvability. The Schrödinger operators associated with *Eqn.*(0.1.4) fall into two categories. We separately discuss the Schrödinger operators associated with the Gegenbauer operator which has extra algebraic relations due to their mirror symmetry. Therefore, we group the operators into three categories; with some abuse of terminology, we call them Hamiltonians:

1. Operators reducible to the Gegenbauer equation, called *Gegenbauer Hamiltonians.*
2. Operators reducible to the hypergeometric equation with behavior near the endpoint *e.g.* b of the form
$$V(x) = V_b + \left(m_b^2 - \frac{1}{4}\right)\frac{1}{(x-b)^2}, \tag{0.1.5}$$
called *hypergeometric Hamiltonians of the first kind.*
3. Operators reducible to the hypergeometric equation with behavior near the endpoint *e.g.* b of the form
$$V(x) = V_b + \left(m_b^2 - \frac{1}{4}\right)\frac{1}{(x-b)^2} - \frac{\beta_b}{|x-b|}, \tag{0.1.6}$$
called *hypergeometric Hamiltonians of the second kind.*

Furthermore, we divide each category into three families according to the choice of endpoints. Thus, in total, we obtain 3×3 classes of one-dimensional Schrödinger operators L arising from the Gauss hypergeometric operator. We prove that each family has a unique closed realization in the sense of $L^2]a,b[$. These operators depend holomorphically on one or more parameters, which allows us to extend them holomorphically to a larger domain [DK].

For each family, we determine the spectrum $\sigma(L_\bullet)$. We also obtain the exact form of the resolvent $(L_\bullet - z)^{-1}$; we often abuse notation and write this as $\frac{1}{L_\bullet - z}$. In each family, we determine the Green function, in the sense of an integral kernel, in terms of hypergeometric functions, and we prove its boundedness.

0.1.1 A review of the 3×3 families

The 9 families are parametrized by $\mathbb{C}$ or $\mathbb{C}\times\mathbb{C}$. In each case, we describe the domain of uniqueness of closed operators, denoted by $\mathcal{A}_{\rm un}$. We denote by $\mathcal{A}_{\rm hol}$ the domain of holomorphic extension of these closed operators, and by $\mathcal{A}_{\rm sa}$ the domain of self-adjointness.

1. Gegenbauer Hamiltonians
   (a) Spherical Gegenbauer Hamiltonian, $L^2]0,\pi[$:
$$\begin{gathered} L_\alpha^{\rm s} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\sin^2 r}, \\ \mathcal{A}_{\rm un} = \{\operatorname{Re}\alpha \geq 1\}, \quad \mathcal{A}_{\rm hol} = \{\operatorname{Re}\alpha > -1\}, \quad \mathcal{A}_{\rm sa} =]-1,+\infty[. \end{gathered} \tag{0.1.7}$$
   (b) Hyperbolic Gegenbauer Hamiltonian, $L^2(\mathbb{R}_+)$:
$$\begin{gathered} L_\alpha^{\rm h} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\sinh^2 r}, \\ \mathcal{A}_{\rm un} = \{\operatorname{Re}\alpha \geq 1\}, \quad \mathcal{A}_{\rm hol} = \{\operatorname{Re}\alpha > -1\}, \quad \mathcal{A}_{\rm sa} =]-1,+\infty[. \end{gathered} \tag{0.1.8}$$

(c) de Sitter Gegenbauer Hamiltonian, $L^2(\mathbb{R})$:

$$
\begin{aligned}
L_\alpha^{\mathrm{dS}} &:= -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r},\\
\mathcal{A}_{\mathrm{un}} &= \mathcal{A}_{\mathrm{hol}} = \mathbb{C}, \quad \mathcal{A}_{\mathrm{sa}} = \mathbb{R}.
\end{aligned}
\tag{0.1.9}
$$

2. Hypergeometric Hamiltonians of the first kind:

(a) Spherical hypergeometric Hamiltonian of the first kind, $L^2]0,\pi[$:

$$
\begin{aligned}
L_{\alpha,\beta}^{\mathrm{s}} &:= -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{4\sin^2\frac{r}{2}} + \left(\beta^2 - \frac{1}{4}\right)\frac{1}{4\cos^2\frac{r}{2}},\\
\mathcal{A}_{\mathrm{un}} &= \{\operatorname{Re}\alpha, \operatorname{Re}\beta \geq 1\}, \quad \mathcal{A}_{\mathrm{hol}} = \{\operatorname{Re}\alpha, \operatorname{Re}\beta > -1\}, \quad \mathcal{A}_{\mathrm{sa}} =]-1,+\infty[\times]-1,+\infty[.
\end{aligned}
\tag{0.1.10}
$$

(b) Hyperbolic hypergeometric Hamiltonian of the first kind, $L^2(]0,\infty[)$:

$$
\begin{aligned}
L_{\alpha,\beta}^{\mathrm{h}} &:= -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{4\sinh^2\frac{r}{2}} - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{4\cosh^2\frac{r}{2}},\\
\mathcal{A}_{\mathrm{un}} &= \{\operatorname{Re}\alpha \geq 1\}\times\mathbb{C}, \quad \mathcal{A}_{\mathrm{hol}} = \{\operatorname{Re}\alpha > -1\}\times\mathbb{C}, \quad \mathcal{A}_{\mathrm{sa}} =]-1,+\infty[\times\mathbb{R}.
\end{aligned}
\tag{0.1.11}
$$

(c) de Sitter hypergeometric Hamiltonian of the first kind, $L^2(\mathbb{R})$:

$$
\begin{aligned}
L_{\alpha,\beta}^{\mathrm{dS}} &:= -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}\left(\frac{1}{2} + \frac{\mathrm{i}\sinh r}{2}\right) - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}\left(\frac{1}{2} - \frac{\mathrm{i}\sinh r}{2}\right),\\
\mathcal{A}_{\mathrm{un}} &= \mathcal{A}_{\mathrm{hol}} = \mathbb{C}\times\mathbb{C}, \quad \mathcal{A}_{\mathrm{sa}} = \{\alpha = \overline{\beta}\}.
\end{aligned}
\tag{0.1.12}
$$

3. Hypergeometric Hamiltonians of the second kind:

(a) Spherical hypergeometric Hamiltonian of the second kind, $L^2]0,\pi[$:

$$
\begin{aligned}
K_{\tau,\mu}^{\mathrm{s}} &:= -\partial_u^2 + \tau\frac{\cos u}{\sin u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\sin^2 u},\\
\mathcal{A}_{\mathrm{un}} &= \mathbb{C}\times\{\operatorname{Re}\mu \geq 2\}, \quad \mathcal{A}_{\mathrm{hol}} = \mathbb{C}\times\{\operatorname{Re}\mu > -2\}\setminus\{(0,-1)\}, \quad \mathcal{A}_{\mathrm{sa}} = \mathbb{R}\times]-2,+\infty[.
\end{aligned}
\tag{0.1.13}
$$

(b) Hyperbolic hypergeometric Hamiltonian of the second kind, $L^2(]0,\infty[)$:

$$
\begin{aligned}
K_{\kappa,\mu}^{\mathrm{h}} &:= -\partial_u^2 + \kappa\frac{\cosh u}{\sinh u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\sinh^2 u},\\
\mathcal{A}_{\mathrm{un}} &= \mathbb{C}\times\{\operatorname{Re}\mu \geq 2\}, \quad \mathcal{A}_{\mathrm{hol}} = \mathbb{C}\times\{\operatorname{Re}\mu > -2\}\setminus\{(0,-1)\}, \quad \mathcal{A}_{\mathrm{sa}} = \mathbb{R}\times]-2,+\infty[.
\end{aligned}
\tag{0.1.14}
$$

(c) de Sitter hypergeometric Hamiltonian of the second kind, $L^2(\mathbb{R})$:

$$
\begin{aligned}
K_{\kappa,\mu}^{\mathrm{dS}} &:= -\partial_u^2 - \kappa\frac{\sinh w}{\cosh w} - \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\cosh^2 w},\\
\mathcal{A}_{\mathrm{un}} &= \mathcal{A}_{\mathrm{hol}} = \mathbb{C}\times\mathbb{C}, \quad \mathcal{A}_{\mathrm{sa}} = \mathbb{R}\times\mathbb{R}.
\end{aligned}
\tag{0.1.15}
$$

Gegenbauer Hamiltonians are special cases of hypergeometric Hamiltonians of both the first and second kinds. In fact, we have the following coincidences:

$$L^{\mathrm{s}}_{\alpha} = L^{\mathrm{s}}_{\alpha,\alpha} = K^{\mathrm{s}}_{0,2\alpha}, \tag{0.1.16}$$

$$L^{\mathrm{h}}_{\alpha} = L^{\mathrm{h}}_{\alpha,\alpha} = K^{\mathrm{h}}_{0,2\alpha}, \tag{0.1.17}$$

$$L^{\mathrm{dS}}_{\alpha} = L^{\mathrm{dS}}_{\alpha,\alpha} = K^{\mathrm{dS}}_{0,2\alpha}. \tag{0.1.18}$$

Let us also list the following identities that we proved [DK]:

$$K^{\mathrm{s}}_{\tau,-1} = K^{\mathrm{s}}_{\tau,1}, \quad \tau \neq 0, \tag{0.1.19}$$

$$K^{\mathrm{h}}_{\kappa,-1} = K^{\mathrm{h}}_{\kappa,1}, \quad \kappa \neq 0. \tag{0.1.20}$$

These identities imply that $(0,-1)$ is a singularity of the functions $(\tau,\mu) \mapsto K^{\mathrm{s}}_{\tau,\mu}$ and $(\kappa,\mu) \mapsto K^{\mathrm{h}}_{\kappa,\mu}$.

Going back to *Eqn.*(0.1.16) and *Eqn.*(0.1.17), note that

$$L^{\mathrm{s}}_{\alpha} = K^{\mathrm{s}}_{0,2\alpha}, \tag{0.1.21}$$

$$L^{\mathrm{h}}_{\alpha} = K^{\mathrm{h}}_{0,2\alpha}, \tag{0.1.22}$$

are identities of holomorphic functions only for $\alpha \neq -\frac{1}{2}$, because of the singularity mentioned above. Thus, the identities

$$L^{\mathrm{s}}_{-\frac{1}{2}} = K^{\mathrm{s}}_{0,-1}, \tag{0.1.23}$$

$$L^{\mathrm{h}}_{-\frac{1}{2}} = K^{\mathrm{h}}_{0,-1}, \tag{0.1.24}$$

should be understood as *definitions* of $K^{\mathrm{s}}_{0,-1}$ and $K^{\mathrm{h}}_{0,-1}$.

0.1.2 Transmutation identities

Despite the classical nature of the hypergeometric equation, we found a series of identities, which we call *transmutation identities*, referring to their somewhat miraculous nature [DK, DL]. In these identities, the roles of the spectral parameter and the coupling constant for a pair of operators are mutually interchanged Here is the list of the trasmutation identities proven in [DK].

Proposition 0.1.1. Gegenbauer spherical — Gegenbauer de Sitterian:

$$]0,\pi[\ni r \mapsto q \in \mathbb{R}, \quad \cot r = \sinh q; \tag{0.1.25}$$

$$\sin^{\frac{1}{2}} r \frac{1}{(L^{\mathrm{s}}_{\lambda} - \alpha^2)}(r,r') \sin^{\frac{1}{2}} r' = \frac{1}{(L^{\mathrm{dS}}_{\alpha} - \lambda^2)}(q,q'), \tag{0.1.26}$$

$$\frac{1}{(L^{\mathrm{s}}_{\lambda} - \alpha^2)}(r,r') = \cosh^{\frac{1}{2}} q \frac{1}{(L^{\mathrm{dS}}_{\alpha} - \lambda^2)}(q,q') \cosh^{\frac{1}{2}} q'. \tag{0.1.27}$$

Proposition 0.1.2. 1st kind spherical — 1st kind hyperbolic:

$$]0,\pi[\ni r \mapsto q \in \mathbb{R}_+, \quad \tan\frac{r}{2} = \sinh\frac{q}{2}; \tag{0.1.28}$$

$$\left(\cos\frac{r}{2}\right)^{-\frac{1}{2}} \frac{1}{\left(L^{\mathrm{s}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)}(r,r') \left(\cos\frac{r'}{2}\right)^{-\frac{1}{2}} = \frac{1}{\left(L^{\mathrm{h}}_{\alpha,\mu} + \frac{\beta^2}{4}\right)}(q,q'), \tag{0.1.29}$$

$$\frac{1}{\left(L^{\mathrm{s}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)}(r,r') = \left(\cosh\frac{q}{2}\right)^{-\frac{1}{2}} \frac{1}{\left(L^{\mathrm{h}}_{\alpha,\mu} + \frac{\beta^2}{4}\right)}(q,q') \left(\cosh\frac{q'}{2}\right)^{-\frac{1}{2}}. \tag{0.1.30}$$

Proposition 0.1.3. 2nd kind hyperbolic — 2nd kind de Sitterian:

$$\mathbb{R}_+ \ni u \mapsto w \in \mathbb{R}, \quad \mathrm{e}^{2u} = 1 + \mathrm{e}^{2w}; \tag{0.1.31}$$

$$(1-\mathrm{e}^{-2u})^{-\frac{1}{2}} \frac{1}{\left(K^{\mathrm{h}}_{\nu-\frac{\beta^2}{2},\mu} + \nu + \frac{\beta^2}{2}\right)}(u,u')(1-\mathrm{e}^{-2u'})^{-\frac{1}{2}} = \frac{1}{\left(K^{\mathrm{dS}}_{\nu+\frac{\mu^2}{2},\beta} + \nu - \frac{\mu^2}{2}\right)}(w,w'), \tag{0.1.32}$$

$$\frac{1}{\left(K^{\mathrm{h}}_{\nu-\frac{\beta^2}{2},\mu} + \nu + \frac{\beta^2}{2}\right)}(u,u') = (1+\mathrm{e}^{-2w})^{-\frac{1}{2}} \frac{1}{\left(K^{\mathrm{dS}}_{\nu+\frac{\mu^2}{2},\beta} + \nu - \frac{\mu^2}{2}\right)}(w,w')(1+\mathrm{e}^{-2w'})^{-\frac{1}{2}}. \tag{0.1.33}$$

Proposition 0.1.4. 1st kind spherical — 2nd kind de Sitterian:

$$]0,\pi[\ni r \mapsto u \in \mathbb{R}, \quad \cos r = \tanh u; \tag{0.1.34}$$

$$\sin^{\frac{1}{2}} r \, \frac{1}{\left(L^{\mathrm{s}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)} (r,r') \sin^{\frac{1}{2}} r' = \frac{1}{\left(K^{\mathrm{dS}}_{\kappa,\mu} + \delta\right)} (u,u'), \tag{0.1.35}$$

$$\frac{1}{\left(L^{\mathrm{s}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)} (r,r') = \cosh^{\frac{1}{2}} u \frac{1}{\left(K^{\mathrm{dS}}_{\kappa,\mu} + \delta\right)} (u,u') \cosh^{\frac{1}{2}} u', \tag{0.1.36}$$

$$\delta = \frac{\alpha^2+\beta^2}{2}, \quad \kappa = \frac{\alpha^2-\beta^2}{2}. \tag{0.1.37}$$

Proposition 0.1.5. 1st kind hyperbolic — 2nd kind hyperbolic:

$$\mathbb{R}_+ \ni r \mapsto u \in \mathbb{R}_+, \quad \cosh r = \coth u; \tag{0.1.38}$$

$$\sinh^{\frac{1}{2}} r \, \frac{1}{\left(L^{\mathrm{h}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)} (r,r') \sinh^{\frac{1}{2}} r' = \frac{1}{\left(K^{\mathrm{h}}_{\kappa,\mu} + \delta\right)} (u,u'), \tag{0.1.39}$$

$$\frac{1}{\left(L^{\mathrm{h}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)} (r,r') = \sinh^{\frac{1}{2}} u \frac{1}{\left(K^{\mathrm{h}}_{\kappa,\mu} + \delta\right)} (u,u') \sinh^{\frac{1}{2}} u', \tag{0.1.40}$$

$$\delta = \frac{\alpha^2+\beta^2}{2}, \quad \kappa = \frac{\alpha^2-\beta^2}{2}. \tag{0.1.41}$$

Proposition 0.1.6. 1st kind de Sitterian — 2nd kind spherical:

$$\mathbb{R} \ni r \mapsto u \in]0,\pi[, \quad \sinh r = -\cot u; \tag{0.1.42}$$

$$\cosh^{\frac{1}{2}} r \frac{1}{\left(L^{\mathrm{dS}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)} (r,r') \cosh^{\frac{1}{2}} r' = \frac{1}{\left(K^{\mathrm{s}}_{\tau,\mu} + \delta\right)} (u,u'), \tag{0.1.43}$$

$$\frac{1}{\left(L^{\mathrm{dS}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)} (r,r') = \sin^{\frac{1}{2}} u \frac{1}{\left(K^{\mathrm{s}}_{\tau,\mu} + \delta\right)} (u,u') \sin^{\frac{1}{2}} u', \tag{0.1.44}$$

$$\delta = \frac{\alpha^2+\beta^2}{2}, \quad \tau = \frac{\mathrm{i}(\alpha^2-\beta^2)}{2}. \tag{0.1.45}$$

0.1.3 Geometric interpretation and separation of variables

One of the main sources of special functions has always been separation of variables, in particular in the study of symmetric spaces. Probably the most important context in which hypergeometric Hamiltonians appear is geometry, more precisely the theory of symmetric spaces and Lie groups. Indeed, when we separate variables for invariant differential operators, for example (pseudo-)Laplacians on symmetric pseudo-Riemannian spaces, we often obtain certain forms of hypergeometric Hamiltonians.

This fact plays an important role in quantum field theory on curved spacetimes, where Green functions of the d'Alembertian on de Sitter and anti-de Sitter spaces appear naturally; see, for example, [DeGa].

This geometric interpretation is especially striking for Gegenbauer Hamiltonians. In the following list, we show how they arise after separation of variables for various d-dimensional (pseudo-)Laplacians and restriction to $(d-1)$-dimensional spherical harmonics of degree l. In all cases,

$$\alpha = \frac{d}{2} - 1 + l.$$

1. Δ_d^{s}, the Laplacian on the unit sphere $\mathbb{S}^d$, reduces to the spherical Gegenbauer Hamiltonian L_α^{s}:

$$(\sin r)^{\frac{d-1}{2}}\big(-\Delta_d^{\mathrm{s}}\big)(\sin r)^{-\frac{d-1}{2}} + \big(\tfrac{d-1}{2}\big)^2 = -\partial_r^2 + \frac{\big(\frac{d-2}{2}\big)^2 - \frac14 - \Delta_{d-1}^{\mathrm{s}}}{\sin^2 r}. \tag{0.1.46}$$

2. Δ_d^{h}, the Laplacian on hyperbolic space $\mathbb{H}^d$, reduces to the hyperbolic Gegenbauer Hamiltonian L_α^{h}:

$$(\sinh r)^{\frac{d-1}{2}}\big(-\Delta_d^{\mathrm{h}}\big)(\sinh r)^{-\frac{d-1}{2}} - \big(\tfrac{d-1}{2}\big)^2 = -\partial_r^2 + \frac{\big(\frac{d-2}{2}\big)^2 - \frac14 - \Delta_{d-1}^{\mathrm{s}}}{\sinh^2 r}. \tag{0.1.47}$$

3. $\Box_d^{\mathrm{dS}}$, the d'Alembertian on de Sitter space dS^d, reduces to the de Sitter Gegenbauer Hamiltonian L_α^{dS}:

$$(\cosh r)^{\frac{d-1}{2}}\Box_d^{\mathrm{dS}}(\cosh r)^{-\frac{d-1}{2}} - \big(\tfrac{d-1}{2}\big)^2 = -\partial_r^2 - \frac{\big(\frac{d-2}{2}\big)^2 - \frac14 - \Delta_{d-1}^{\mathrm{s}}}{\cosh^2 r}. \tag{0.1.48}$$

This geometric interpretation is the main motivation for our naming of these operators, rather than using their historical names [DK].

All three types of hypergeometric Hamiltonians of the first kind also have natural geometric interpretations. In the following list, we show how they arise from separation of variables for a (pseudo-)Laplacian on a (pseudo-)sphere in "double spherical coordinates." In all three examples, the coordinates in the ambient pseudo-Euclidean space are partitioned into two groups.

In the first two cases, these groups are of dimensions p and q, and spherical coordinates are considered within each group. After restriction to products of spherical harmonics of degrees l and j, a hypergeometric Hamiltonian arises, expressed in the relative variable. We have

$$\alpha = \frac{p}{2} - 1 + l, \qquad \beta = \frac{q}{2} - 1 + j.$$

The third case is somewhat different: here $p = q$ is the dimension of holomorphic and antiholomorphic complex spherical coordinates. The spherical harmonics are not the usual ones; rather, they are holomorphic and antiholomorphic harmonics on the complex $(p-1)$-dimensional sphere.

1. $\Delta^{\mathrm{s}}_{p+q-1}$, the Laplacian on the unit sphere $\mathbb{S}^{p+q-1}$, reduces to the spherical hypergeometric Hamiltonian of the first kind $L^{\mathrm{s}}_{\alpha,\beta}$:

$$\begin{aligned}&\left(\sin\frac{r}{2}\right)^{\frac{p-1}{2}}\left(\cos\frac{r}{2}\right)^{\frac{q-1}{2}}\left(-\Delta^{\mathrm{s}}_{p+q-1}\right)\left(\sin\frac{r}{2}\right)^{-\frac{p-1}{2}}\left(\cos\frac{r}{2}\right)^{-\frac{q-1}{2}}+\left(\tfrac{p+q-2}{2}\right)^2\\&=4\left(-\partial_r^2+\frac{\left(\frac{p-2}{2}\right)^2-\frac14-\Delta^{\mathrm{s}}_{p-1}}{4\sin^2\frac{r}{2}}+\frac{\left(\frac{q-2}{2}\right)^2-\frac14-\Delta^{\mathrm{s}}_{q-1}}{4\cos^2\frac{r}{2}}\right).\end{aligned}\tag{0.1.49}$$

2. $\Delta_{p-1,q}$, the pseudo-Laplacian on the hyperboloid $\mathbb{H}^{p-1,q}$, reduces to the hyperbolic hypergeometric Hamiltonian of the first kind $L^{\mathrm{h}}_{\alpha,\beta}$:

$$\begin{aligned}&\left(\cosh\frac{r}{2}\right)^{\frac{p-1}{2}}\left(\sinh\frac{r}{2}\right)^{\frac{q-1}{2}}\left(-\Delta_{p-1,q}\right)\left(\cosh\frac{r}{2}\right)^{-\frac{p-1}{2}}\left(\sinh\frac{r}{2}\right)^{-\frac{q-1}{2}}-\left(\tfrac{p+q-2}{2}\right)^2\\&=4\left(-\partial_r^2-\frac{\left(\frac{p-2}{2}\right)^2-\frac14-\Delta^{\mathrm{s}}_{p-1}}{4\cosh^2\frac{r}{2}}+\frac{\left(\frac{q-2}{2}\right)^2-\frac14-\Delta^{\mathrm{s}}_{q-1}}{4\sinh^2\frac{r}{2}}\right).\end{aligned}\tag{0.1.50}$$

3. $\Delta_{p-1,p}$, the pseudo-Laplacian on the hyperboloid $\mathbb{H}^{p-1,p}$, reduces to the de Sitter hypergeometric Hamiltonian of the first kind $L^{\mathrm{dS}}_{\alpha,\beta}$:

$$\begin{aligned}&(\cosh r)^{\frac{p-1}{2}}\left(\Delta_{p-1,p}\right)(\cosh r)^{-\frac{p-1}{2}}-(p-1)^2\\&=4\left(-\partial_r^2-\frac{\left(-\Delta^{\mathrm{s},\mathbb{C}}_{p-1}+\left(\frac{p-1}{2}\right)^2-\frac14\right)}{2(1+\mathrm{i}\sinh r)}-\frac{\left(-\overline{\Delta}^{\mathrm{s},\mathbb{C}}_{p-1}+\left(\frac{p-1}{2}\right)^2-\frac14\right)}{2(1-\mathrm{i}\sinh r)}\right).\end{aligned}\tag{0.1.51}$$

Now that we have clarified our terminology, we review the historical names for these operators.

Table 1: Names of hypergeometric Hamiltonians appearing in the literature.

Our terminology	Name in [DW]	Name in [CKS]	Alternative names
Spherical of 1st kind	Trigonometric Pöschl–Teller	Scarf I	Pöschl–Teller of the 1st kind, trigonometric Scarf
Hyperbolic of 1st kind	Hyperbolic Pöschl–Teller	Scarf II	Pöschl–Teller of the 2nd kind
de Sitterian of 1st kind	Scarf	Generalized Pöschl–Teller	Hyperbolic Scarf
Spherical of 2nd kind	Rosen–Morse	Rosen–Morse I	Trigonometric Rosen–Morse
Hyperbolic of 2nd kind	Eckart	Eckart	Generalized Morse, Hulthén
de Sitterian of 2nd kind	Manning–Rosen	Rosen–Morse II	Hyperbolic Rosen–Morse, Woods–Saxon

0.1.4 Exact WKB quantization

WKB (Wentzel–Kramers–Brillouin) method is often used in quantum mechanics as an approximation technique for determining the spectrum of an operator. The exact WKB method, developed by [Vo], is based on the ansatz for a solution of the Schr"odinger equation

$$\psi(x) = \exp\left(\frac{1}{\hbar}\sum_{n=0}^{\infty} \hbar^n S_n(x)\right). \tag{0.1.52}$$

Recall that keeping only the first term in the expansion, corresponding to order $\hbar^{-1}$, gives the usual WKB approximation. What we call exact WKB quantization is the condition

$$\oint \sum_{n=0}^{\infty} S_n'(z), dz = 2\pi \mathrm{i} k, \quad k \in \mathbb{N}_0, \tag{0.1.53}$$

where the contour is taken around the classical turning points. This condition should provide an approximation to the actual spectrum of the Schrödinger operator. However, even when all terms are included, the resulting series may fail to converge, and therefore the expression may be ill-defined.

In an additional section of this chapter Sec. 1.7, we explain and provide strong evidence that for hypergeometric Hamiltonians, exact WKB quantization leads to the correct discrete spectrum, at least in the self-adjoint case.

To clarify this point, we introduce what we call nodal quantization, which reads the zeros of eigenpolynomials:

$$\oint \frac{d}{dz} \log \psi(z), dz = 2\pi \mathrm{i} n, \tag{0.1.54}$$

where $\psi(z)$ is a solution of the eigenvalue problem. To avoid repetition, we prove three theorems for self-adjoint hypergeometric Hamiltonians of the first type by using the fact that the hypergeometric function becomes a polynomial, often a Jacobi polynomial, when one of the first two parameters is a negative integer. This turns out to be exactly the condition required for the existence of discrete spectrum in each case.

This provides strong evidence that hypergeometric Hamiltonians are exactly WKB-quantizable. To our knowledge, such a result had previously been confirmed only in the case of the de Sitterian Gegenbauer equation [BOW].

0.2 Part II: Random matrices and character expansion

The second part is devoted to a β-deformed eigenvalue model defined through the multivariable integral

$$Z_\beta(V; p_k) := \int_{\mathbb{R}^N} \left(\prod_{i=1}^N dz_i\right) [\Delta(z)]^{2\beta} \exp\left[-\sum_{i=1}^N V(z_i)\right] \exp\left[\beta \sum_{k=1}^{\infty} \frac{p_k}{k} \sum_{i=1}^N z_i^k\right]. \tag{0.2.55}$$

Here

$$\Delta(z) := \prod_{1 \le i < j \le N} (z_i - z_j), \tag{0.2.56}$$

is the Vandermonde determinant, $V(z) : \mathbb{R} \to \mathbb{R}$ is a potential, (p_k) are coupling constants, and $\beta \in \mathbb{R}$ is a parameter. For specific values of β, this integral is associated with classical matrix models:

- $\beta = 1$: unitary matrix model;
- $\beta = \frac{1}{2}$: orthogonal matrix model;

- $\beta = 2$: symplectic matrix model.

We proved a statement about the multipoint correlators with respect to Z_β namely

$$\left\langle \sum_{i=1}^{N} z_i^{\lambda_1} \cdots \sum_{i=1}^{N} z_i^{\lambda_l} \right\rangle_{Z_\beta}, \tag{0.2.57}$$

which are often computed using Wick's theorem. Note, however, that these correlators are invariant under permutations of the variables. This allows one to interpret the problem in terms of the ring of symmetric polynomials, consisting of multivariable polynomials in $\mathbb{C}[z_1, \dots, z_N]$ that are invariant under the action of the symmetric group.

There are many bases of this ring. In particular, there is the basis of power-sum polynomials,

$$p_\lambda(z) = p_{\lambda_1} \cdots p_{\lambda_l}, \qquad \text{where} \quad p_{\lambda_i} := \sum_{j=1}^{N} z_j^{\lambda_i}. \tag{0.2.58}$$

It is clear that the multipoint correlator can be written in terms of power-sum polynomials:

$$\left\langle \sum_{i=1}^{N} z_i^{\lambda_1} \cdots \sum_{i=1}^{N} z_i^{\lambda_l} \right\rangle_{Z_\beta} = \langle p_\lambda(z) \rangle_{Z_\beta}. \tag{0.2.59}$$

We may choose a different basis for the ring of symmetric polynomials. The natural question is: *Is there a preferred basis of the space of symmetric polynomials in which the multipoint correlators have a closed factorized form?* To answer this question, we must specify the model. We focus on the Gaussian potential with a linear term,

$$V(z) = \frac{a_2}{2} z^2 + a_1 z. \tag{0.2.60}$$

It was conjectured in [CLZ] that there exists a preferred basis in which the multipoint correlators are closed and factorized. This basis was suggested to be the Jack polynomials. Since Jack polynomials form a basis of the space of symmetric polynomials, we may write

$$p_\lambda = \sum_{\mu \vdash |\lambda|} c_{\mu,\lambda} P_\mu, \tag{0.2.61}$$

where μ and λ are partitions. In this basis, the multipoint correlators take the form

$$\big\langle P_\lambda(z_1, \dots, z_N) \big\rangle = \frac{P_\lambda\{N\} P_\lambda\{\beta^{-1}(a_2 \delta_{k,2} - a_1 \delta_{k,1})\}}{P_\lambda\{a_2 \beta^{-1} \delta_{k,1}\}}. \tag{0.2.62}$$

Here $P_\lambda\{N\}$ means that one expands the Jack polynomial in power-sum polynomials and then sets $p_k = N$; similarly for the other specializations.

We proved this statement using the symmetry of *Eqn.*(2.3.1) [BKS]. Indeed, a change of variables of the form $z \mapsto z + \epsilon z^{k+1}$ does not change the integral. This leads to differential operators $\mathcal{L}_k$, acting on the coupling constants p_k, such that

$$\mathcal{L}_k Z_\beta = 0. \tag{0.2.63}$$

These differential operators form an algebra known as the Virasoro algebra,

$$[\mathcal{L}_n, \mathcal{L}_m] = (n - m) \mathcal{L}_{n+m},$$

hence the name *Virasoro constraints*. Using these constraints, we can write *Eqn.*(2.3.1) as the action of a differential operator W_{-2} [CLZ, BKS]:

$$Z_\beta = \mathrm{e}^{\frac{W_{-2}}{2a_2}} \cdot 1. \tag{0.2.64}$$

Using representation-theoretic methods, we prove that Jack polynomials appear in the orbit of $\mathrm{e}^{\frac{W_{-2}}{2a_2}}$, thereby proving the conjecture [BKS].

Furthermore, we proved a character-preserving phenomenon for another class of potentials,

$$V(z) = -a_1 z + \nu \log z, \tag{0.2.65}$$

associated with the Wishart ensemble [BKS]. In this setting, we prove that the multipoint correlators are given by

$$\big\langle P_\lambda(z_1, \dots, z_N)\big\rangle = \frac{P_\lambda\{N\} P_\lambda\{N + \beta^{-1}\nu + \beta^{-1} - 1\}}{P_\lambda\{a_1 \beta^{-1}\delta_{k,1}\}}. \tag{0.2.66}$$

0.2.1 Kernel reproducing property and superintegrability

We also include in the thesis another unpublished result based on Okounkov's reproducing kernel argument [O1]. This part relies heavily on Dunkl theory [Du] and on the multivariable hypergeometric functions of Heckman–Opdam [HO]. For the root system of type A_{n-1}, the Dunkl operator, which forms a commuting family, is given by

$$T_i f(x) = \partial_{x_i} f(x) + \frac{1}{\alpha} \sum_{j \neq i} \frac{f(x) - f(s_{ij}x)}{x_i - x_j}, \tag{0.2.67}$$

where $s_{ij} \in \mathfrak{S}_N$. Heckman and Opdam defined their multivariable hypergeometric function as an eigenfunction of the spherical Dunkl Laplacian. Let $p(\mathbf{x}) = |\mathbf{x}|^2$. Then the Dunkl Laplacian acts on $F_{\frac{1}{\alpha}}(\mathbf{x}, \mathbf{y})$ as

$$p(T) F_k(x, y) = |\mathbf{y}|^2 F_k(x, y). \tag{0.2.68}$$

Olshanski and Okounkov proved [OO], using the fact that the Jack polynomials are eigenfunctions of the Calogero–Sutherland Hamiltonian [St]

$$\mathcal{H} := \frac{\alpha}{2} \sum_{i=1}^{N} (x_i \partial_{x_i})^2 + \frac{1}{2} \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} (x_i \partial_{x_i} - x_j \partial_{x_j}), \tag{0.2.69}$$

that Jack polynomials can be used to construct a Heckman-Opdam hypergeometric function of type A.

Using this function, we obtain a new result for a much more complicated multivariable expression:

$$\langle P_\mu(x;\alpha) P_\beta(x;\alpha)\rangle = \alpha^{-|\mu|-|\beta|} J_\mu(1;\alpha) J_\beta(1;\alpha) \left(P_\mu(y;\alpha), \left(P_\beta(z;\alpha), e^{|y|^2/2 + |z|^2/2} \sum_\lambda \frac{P_\lambda(y;\alpha) P_\lambda(z;\alpha)}{(n/\alpha)_\lambda \mathfrak{p}_\lambda} \right)_\alpha \right)_\alpha, \tag{0.2.70}$$

where $(\bullet, \bullet)_\alpha$ denotes the α-deformed Hall inner product. Although the answer appears complicated, we show that the previous superintegrability theorem is a special case of this result.

0.3 Content, notation, and organization

This thesis is based on two papers: [DK] in collaboration with Jan Dereziński, and [BKS] in collaboration with Aditya Bawane and Piotr Sułkowski. Therefore, the main content of this thesis overlaps substantially with those papers, but many details have been added in order to make the exposition as self-contained as possible. Unfortunately, there are limits both to the alphabet and to the author's creativity. For this reason, the two chapters should be read almost independently as far as notation is concerned. After all, we are working with only a finite collection of symbols.

The first chapter 1 is devoted to one-dimensional Schrödinger operators related to the hypergeometric equation. We begin by reviewing the differential equation in the complex domain and discussing in detail the algebraic properties of hypergeometric functions Sec. 1.1. We also explain carefully the procedure by which a Schrödinger equation can be constructed from the hypergeometric equation Sec. 1.2. The Gegenbauer equation is treated separately. We explain the inpiration behind the naming and classification using separation of variable in symmetric spaces Sec. 1.3. We then continue with a chapter devoted to operator theory, spectral theory, and boundary conditions for Schrödinger operators with complex potentials Sec. 1.5. This provides the necessary background for the final section, where the general theory is applied to hypergeometric Hamiltonians Sec. 1.6. We finish this chapter with some new result on exact WKB and explain shortly its connection to matrix model through topological recursion 1.7.

The second chapter 2 is related to the paper [BKS]. We begin with an introduction to the theory of symmetric polynomials, with emphasis on their linear-algebraic properties Sec. 2.1. We then present the first proof of superintegrability [BKS] Sec. 2.3. The chapter concludes with a section on Dunkl theory and the Dunkl reproducing kernel Sec. 2.6. In that section, we also present a new unpublished theorem that sheds light on why the superintegrability phenomenon occurs.

There are also several appendices Ap. 3. They include the relation between Gegenbauer and associated Legendre polynomials, specializations of hypergeometric functions with parameters $\pm\frac{1}{2}$, and realizations of root systems of types A, B, C, and D.

Chapter 1

Exactly solvable Schrödinger equation related to the hypergeometric equation

Starting with a geometric series,

$$S_n = \sum_{k=0}^{n} az^k, \tag{1.0.1}$$

for $z \in \mathbb{C}$, it is easy to see that the infinite series S_∞ converges if $|z| < 1$. The hypergeometric series was first introduced by John Wallis as an extension of the geometric series, which explains the name *hypergeometric*. The Gauss hypergeometric series is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.0.2}$$

where $a, b, c \in \mathbb{C}$. The series converges for $|z| < 1$ and can be analytically continued to the whole complex plane. The Pochhammer symbol is

$$(a)_n = \begin{cases} 1, & n = 0; \\ a(a+1)\ldots(a+n-1), & n > 0. \end{cases} \tag{1.0.3}$$

Later, we will drop the subscript and write the Gauss hypergeometric function as ${}_2F_1(a, b; c; z) = F(a, b; c; z)$.

The hypergeometric series contains all the elementary functions (polynomial, rational, exponential, and logarithmic functions) for particular values of the parameters. We discuss some of these identities in the appendix. Moreover, many special functions, such as Bessel and Legendre functions, can be seen, sometimes in a nontrivial way, as confluent cases of the hypergeometric function.

The name of Gauss is attached to the hypergeometric function because he was the one who studied this function using a second-order ordinary differential equation [G]. This is precisely the point of view that interests us. Thus, this section is a review and introduces differential equations in the complex domain, with a particular focus on the hypergeometric equation. This is a classical topic, and many books and articles have been written on it. One of the most notable references is the classical monograph of Whittaker and Watson [WW]. More modern books include [Ince]. In this section, we closely follow the notation of [D1], which was also used in [DK].

1.1 Second-order differential equations in the complex domain

We follow the notation and definitions of [DLI]. We study second-order differential equations of the form

$$\big(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta(z)\big)\, u(z) = 0, \tag{1.1.4}$$

where $\sigma(z)$, $\tau(z)$, and $\eta(z)$ are holomorphic functions of $z \in \mathbb{C}$. We say that a differential equation is in its *principal form* if it can be written as

$$\big(\partial_z^2 + p(z)\partial_z + q(z)\big)\, u(z) = 0. \tag{1.1.5}$$

In this work, we prefer to work with operators rather than equations. For instance, the corresponding *principal operator* is

$$\partial_z^2 + p(z)\partial_z + q(z). \tag{1.1.6}$$

This allows us to make more general statements. Passing from *Eqn.*(1.1.4) to *Eqn.*(1.1.5), we set

$$p(z) = \frac{\tau(z)}{\sigma(z)}, \qquad q(z) = \frac{\eta(z)}{\sigma(z)}. \tag{1.1.7}$$

Definition 1.1.1. The degree of a singularity of a function.
Let f be a function defined on an open subset of the Riemann sphere with values in $\mathbb{C} \cup \{\infty\}$. We say that $z_0 \in \mathbb{C}$ and ∞ are singularities of f if

$$f(z) = \sum_{k=-\infty}^{\infty} f_{z_0,k}(z - z_0)^k, \qquad |z - z_0| < r, \qquad z_0 \in \mathbb{C}, \tag{1.1.8}$$

$$f(z) = \sum_{k=-\infty}^{\infty} f_{\infty,k} z^{-k}, \qquad |z| > r. \tag{1.1.9}$$

The degree *of a singularity of f is defined by*

$$\deg(f, z_0) = -\min\{k \mid f_{z_0,k} \neq 0\}, \tag{1.1.10}$$

$$\deg(f, \infty) = \max\{k \mid f_{\infty,k} \neq 0\}. \tag{1.1.11}$$

If the degree of f at ∞ is positive, it coincides with the usual degree of a polynomial. If the degree of f at z_0 is positive, it coincides with the order of a pole of a meromorphic function.

We can now classify differential equations based on the degree of the functions appearing in the principal form *Eqn.*(1.1.5).

Definition 1.1.2. *We say that a differential equation of the form Eqn.*(1.1.5) *has* regular points *if, at every possible singularity on the Riemann sphere, we have*

$$\deg(p, z_0) \leq 0, \quad \deg(q, z_0) \leq 0, \qquad z_0 \in \mathbb{C}, \tag{1.1.12}$$

$$\deg\left(p - \frac{2}{z}, \infty\right) \leq -2, \quad \deg(q, \infty) \leq -4. \tag{1.1.13}$$

Note that the condition at ∞ can be obtained by the change of variable $z \to \frac{1}{w}$.

Definition 1.1.3. *We say that a differential equation of the form Eqn.*(1.1.5) *has* regular-singular points *if, at every possible singularity on the Riemann sphere, we have*

$$\deg(p, z_0) \leq 1, \quad \deg(q, z_0) \leq 2, \qquad z_0 \in \mathbb{C}, \tag{1.1.14}$$

$$\deg\left(p - \frac{2}{z}, \infty\right) \leq -1, \quad \deg(q, \infty) \leq -2. \tag{1.1.15}$$

Differential equations with regular-singular points are sometimes called *Fuchsian equations*. Any other singularities are usually called *irregular singularities*. We do not discuss them in this thesis. For references to their theory, see [SYL].

Definition 1.1.4. *We say that λ is an index of a singularity if it is a root of the following polynomial:*

$$P_{z_0}(\lambda) := \lambda(\lambda - 1) + p_{z_0,-1}\lambda + q_{z_0,-2}, \qquad z_0 \in \mathbb{C}, \tag{1.1.16}$$
$$P_\infty(\lambda) := \lambda(\lambda + 1) - p_{\infty,-1}\lambda + q_{\infty,-2}. \tag{1.1.17}$$

The index is sometimes also called the *exponent*, since it describes the exponent of the solution of *Eqn.*(1.1.5) near the singularity. More precisely, let

$$u(z) = (z - z_0)^\lambda \tilde{u}(z)$$

solve the second-order differential equation *Eqn.*(1.1.5), where $\tilde{u}(z)$ is holomorphic near z_0 and satisfies $\lim_{z\to z_0} \tilde{u}(z) = 1$. Then λ is determined by *Eqn.*(1.1.16).

Solutions of second-order differential equations with regular singular points can be obtained by the Frobenius method. The essence of this method is given by the following theorem.

Theorem 1.1.5. *Assume that $p(z_0) \neq 0$ and that p, q, r are holomorphic in an open, connected, simply connected set $\Omega \subset \mathbb{C}$ containing z_0. Let $\rho \in \mathbb{C}$ satisfy*

$$P_{z_0}(\rho) = 0, \quad P_{z_0}(\rho + n) \neq 0, \quad n = 1, 2, \ldots. \tag{1.1.18}$$

Then there exists a unique function $\tilde{u}(z)$, holomorphic in Ω, such that

$$u(z) := (z - z_0)^\rho \tilde{u}(z)$$

solves Eqn.(1.1.5) *and satisfies*

$$\tilde{u}(z_0) = 1.$$

Remark 1.1.6. *If the difference of the two indices is an integer, then the Frobenius method does not always provide two linearly independent solutions. In this case, one sometimes has to include logarithmic corrections.*

Riemann equation

A second-order differential equation on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with at most three singular points, all regular-singular, is classified by the positions of its singularities and the values of its indices. It is called the *Riemann equation*, or sometimes the *Papperitz equation*, and is described in the following classical theorem [WW, Ince]. For the Riemann operator, we use the notation introduced in [DW], which is inspired by the notation due to Papperitz used in [WW].

Theorem 1.1.7. (1) *Suppose that we are given a second-order differential equation on the Riemann sphere having at most three singular points z_1, z_2, z_3, all of them regular-singular, with the following indices:*

$$z_1 :\ \rho_1, \tilde{\rho}_1; \qquad z_2 :\ \rho_2, \tilde{\rho}_2; \qquad z_3 :\ \rho_3, \tilde{\rho}_3.$$

Then the following condition is satisfied:

$$\rho_1 + \tilde{\rho}_1 + \rho_2 + \tilde{\rho}_2 + \rho_3 + \tilde{\rho}_3 = 1. \tag{1.1.19}$$

Such an equation, normalized so that the coefficient of the second derivative is 1, is, for finite z_1, z_2, z_3, given by the operator

$$\mathcal{P}\begin{pmatrix} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix} := \partial_z^2 - \left(\frac{\rho_1+\tilde{\rho}_1-1}{z-z_1} + \frac{\rho_2+\tilde{\rho}_2-1}{z-z_2} + \frac{\rho_3+\tilde{\rho}_3-1}{z-z_3}\right)\partial_z$$

$$+\frac{\rho_1\tilde{\rho}_1(z_1-z_2)(z_1-z_3)}{(z-z_1)^2(z-z_2)(z-z_3)} + \frac{\rho_2\tilde{\rho}_2(z_2-z_3)(z_2-z_1)}{(z-z_2)^2(z-z_3)(z-z_1)} + \frac{\rho_3\tilde{\rho}_3(z_3-z_1)(z_3-z_2)}{(z-z_3)^2(z-z_1)(z-z_2)}.$$

For $z_3 = \infty$, the operator is

$$\begin{aligned} \mathcal{P}\begin{pmatrix} z_1 & z_2 & \infty & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix} &:= \partial_z^2 - \left(\frac{\rho_1+\tilde{\rho}_1-1}{z-z_1} + \frac{\rho_2+\tilde{\rho}_2-1}{z-z_2}\right)\partial_z \\ &+\frac{\rho_1\tilde{\rho}_1(z_1-z_2)}{(z-z_1)^2(z-z_2)} + \frac{\rho_2\tilde{\rho}_2(z_2-z_1)}{(z-z_2)^2(z-z_1)} + \frac{\rho_3\tilde{\rho}_3}{(z-z_1)(z-z_2)}. \end{aligned} \tag{1.1.20}$$

(2) *With the help of homographies (also called Möbius transformations), we can move the singularities around. More precisely, if*

$$w(z) = \frac{az+b}{cz+d},$$

where we may assume that $ad - bc = 1$, then

$$\mathcal{P}\begin{pmatrix} w(z_1) & w(z_2) & w(z_3) & \\ \rho_1 & \rho_2 & \rho_3 & w, \partial_w \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix} = (cz+d)^4 \mathcal{P}\begin{pmatrix} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix}.$$

(3) *By gauging with powers, we can shift the indices:*

$$\begin{aligned} &(z-z_1)^{-\lambda}(z-z_2)^{\lambda}\mathcal{P}\begin{pmatrix} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix}(z-z_1)^{\lambda}(z-z_2)^{-\lambda} \\ &= \mathcal{P}\begin{pmatrix} z_1 & z_2 & z_3 & \\ \rho_1-\lambda & \rho_2+\lambda & \rho_3 & z, \partial_z \\ \tilde{\rho}_1-\lambda & \tilde{\rho}_2+\lambda & \tilde{\rho}_3 & \end{pmatrix}, \end{aligned} \tag{1.1.21}$$

$$\begin{aligned} &(z-z_1)^{-\lambda}\mathcal{P}\begin{pmatrix} z_1 & z_2 & \infty & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \tilde{\rho}_3 & \end{pmatrix}(z-z_1)^{\lambda} \\ &= \mathcal{P}\begin{pmatrix} z_1 & z_2 & \infty & \\ \rho_1-\lambda & \rho_2 & \rho_3+\lambda & z, \partial_z \\ \tilde{\rho}_1-\lambda & \tilde{\rho}_2 & \tilde{\rho}_3+\lambda & \end{pmatrix}. \end{aligned} \tag{1.1.22}$$

The theorem can be generalized to the case of four regular-singular points, leading to the Heun equation [DLI]. This is a more exotic object, and we will not consider it in this work.

It is important to note that the second part of Theorem 1.1.7, namely 2, states that the Riemann equation is invariant under the action of the group $SL(2,\mathbb{C})$. We use this fact to reduce the equation to a simpler form.

From the Riemann equation to the hypergeometric equation

The Riemann equation can be simplified as follows. First, by Theorem 1.1.7(2), we may assume that the points z_1, z_2, z_3 are any three distinct points on the Riemann sphere. We choose them to be $0, 1, \infty$. Then, by Theorem 1.1.7(3), we may assume that ρ_1, ρ_2 are arbitrary numbers. We choose both of them to be 0. Since the sum of the remaining indices must be 1, we may assume that

$$0 \text{ has indices } 0,\ 1-c; \qquad 1 \text{ has indices } 0,\ c-a-b; \qquad \infty \text{ has indices } a,\ b.$$

The hypergeometric operator is

$$\begin{aligned} \mathcal{F}(a,b;c;z,\partial_z) \quad &:= \quad z(1-z)\mathcal{P}\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z, \partial_z \\ 1-c & c-a-b & b & \end{pmatrix} && (1.1.23) \\ &= \quad z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab. && (1.1.24) \end{aligned}$$

The hypergeometric function $F(a,b;c;z)$ is the unique solution of

$$\mathcal{F}(a,b;c;z,\partial_z)F(z) = 0, \qquad F(0) = 1. \tag{1.1.25}$$

Thus it is the solution of the hypergeometric equation obtained by the Frobenius method (Theorem 1.1.5) at the singularity $z = 0$ with index 0. It is often convenient to apply to $F(a,b;c;z)$ the so-called *Olver normalization*, which yields the function

$$\mathbf{F}(a,b;c;z) := \frac{F(a,b;c;z)}{\Gamma(c)} = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{\Gamma(c+j) j!} z^j. \tag{1.1.26}$$

Note that the symmetries of the hypergeometric equation are more transparent if we replace a, b, c with α, β, μ:

$$\begin{aligned} &\alpha = c-1, \qquad \beta = a+b-c, \qquad \mu = a-b, && (1.1.27) \\ &a = \frac{1+\alpha+\beta+\mu}{2}, \qquad b = \frac{1+\alpha+\beta-\mu}{2}, \qquad c = 1+\alpha. \end{aligned}$$

In fact, these new parameters coincide with the differences of the indices at the points $0, 1, \infty$:

$$\alpha = \rho_1 - \tilde{\rho}_1, \qquad \beta = \rho_2 - \tilde{\rho}_2, \qquad \mu = \rho_3 - \tilde{\rho}_3.$$

We call these the Lie-algebraic parameters [D1].

Thus the hypergeometric equation takes the form

$$\mathcal{F}\left(\tfrac{\alpha+\beta+\mu+1}{2}, \tfrac{\alpha+\beta-\mu+1}{2}; 1+\alpha; z, \partial_z\right) F(z) = 0. \tag{1.1.28}$$

The hypergeometric function is the main subject of our study. The following section is devoted to the beautiful identities in the theory of hypergeometric functions.

1.1.1 Identities for the hypergeometric function

In this section, we review the hypergeometric function, with emphasis on the connection formulas and Kummer's table. There are many reviews available, including [WW], [D1], and [D2]. A more encyclopedic reference is [NIST1], which follows a different notation. In this section, we closely follow the notation of [D1] and [D2]. For brevity, we introduce the notation

$$\mathbf{F}_{\alpha,\beta,\mu}(z) := \mathbf{F}\left(\frac{1+\alpha+\beta+\mu}{2}, \frac{1+\alpha+\beta-\mu}{2}; 1+\alpha, z\right). \tag{1.1.29}$$

Theorem 1.1.8. *Consider a parametrized curve* $[0,1] \ni t \mapsto \gamma(t)$ *such that*

$$t^{\frac{1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{1+\alpha-\beta+\mu}{2}}(t-z)^{-\frac{3+\alpha+\beta+\mu}{2}}|_{\gamma(0)}^{\gamma(1)} = 0. \tag{1.1.30}$$

Then the following integral is annihilated by the hypergeometric equation:

$$\mathcal{F}(\frac{1+\alpha+\beta-\mu}{2}, \frac{1+\alpha+\beta+\mu}{2}, 1+\alpha; ; z, \partial_z) \int_\gamma t^{\frac{-1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(t-z)^{-\frac{1+\alpha+\beta+\mu}{2}} = 0. \tag{1.1.31}$$

Remark 1.1.9. *There are different normalizations for the hypergeometric function. In particular, the integral representation is not normalized to* 1 *at* 0*. We denote this normalization by* $\mathbf{F}^I$*, where* I *stands for the integral representation.*

The normalization at 0 *can be found easily from Euler's integral formula:*

$$\mathbf{F}^I_{\alpha,\beta,\mu}(z) = \Gamma(\frac{1+\alpha-\beta+\mu}{2})\Gamma(\frac{1+\alpha-\beta+\mu}{2})\mathbf{F}_{\alpha,\beta,\mu}(z) = \frac{\Gamma(\frac{1+\alpha-\beta+\mu}{2})\Gamma(\frac{1+\alpha-\beta+\mu}{2})}{\Gamma(1+\alpha)}F_{\alpha,\beta,\mu}(z). \tag{1.1.32}$$

At 0*, we have* $F_{\alpha,\beta,\mu}(z) \sim 1$.

Remark 1.1.10. *It is clear from the definition that we have the following identity:*

$$\mathbf{F}_{\alpha,\beta,\mu}(z) = \mathbf{F}_{\alpha,\beta,-\mu}(z). \tag{1.1.33}$$

1.1.2 Various integral representations

To obtain the various solutions known as Kummer's table, we fix the endpoints of the contour and change the other two singularities of the integral. This process produces all solutions: there are 6 choices of contour and 4 permutations for each contour. Hence this gives $6 \times 4 = 24$ solutions.

Solutions at 0

Using Theorem 1.1.8, we set our first integral to be

$$\int_1^\infty t^{\frac{-1-\alpha+\beta+\mu}{2}}(t-1)^{\frac{-1+\alpha-\beta+\mu}{2}}(t-z)^{\frac{-1-\alpha-\beta-\mu}{2}}dt = \mathbf{F}^I_{\alpha,\beta,\mu}(z), \quad z \notin [1,+\infty[. \tag{1.1.34}$$

The integrability condition at the endpoints is

$$\mathrm{Re}(1+\alpha) > |\mathrm{Re}(\beta-\mu)|. \tag{1.1.35}$$

The identities are

$$\mathbf{F}^I_{\alpha,\beta,\mu}(z) = (1-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}^I_{\alpha,-\mu,-\beta}(\frac{z}{z-1}). \tag{1.1.36}$$

$$= (1-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}^I_{\alpha,\mu,\beta}(\frac{z}{z-1}) \tag{1.1.37}$$

$$= (1-z)^{-\beta}\mathbf{F}^I_{\alpha,-\beta,-\mu}(z). \tag{1.1.38}$$

Remark: Notice that the I-normalization constants are invariant under the exchange $\beta \longleftrightarrow \pm\mu$. Thus the identities are valid for this normalization as well.
The other solutions at 0, behaving as $\sim (-z)^{-\alpha}$, are given by the following integrals:

$$\int_z^0 (-t)^{\frac{-1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(t-z)^{\frac{-1-\alpha-\beta-\mu}{2}}dt = (-z)^{-\alpha}\mathbf{F}^I_{-\alpha,\beta,-\mu}(z), \quad z \notin [0,\infty[. \tag{1.1.39}$$

$$\int_0^z t^{\frac{-1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(z-t)^{\frac{-1-\alpha-\beta-\mu}{2}}dt = z^{-\alpha}\mathbf{F}^I_{-\alpha,\beta,-\mu}(z), \quad z \notin]-\infty,0]\cup[1,+\infty[. \quad (1.1.40)$$

and the integrability condition is

$$\mathrm{Re}(1-\alpha) > |\mathrm{Re}(\beta+\mu)|. \quad (1.1.41)$$

The identities are

$$(-z)^{\alpha}\mathbf{F}^I_{-\alpha,\beta,-\mu}(z) = (-z)^{\alpha}(1-z)^{\frac{-1+\alpha-\beta-\mu}{2}}\mathbf{F}^I_{-\alpha,\mu,-\beta}(\frac{z}{z-1}). \quad (1.1.42)$$

$$= (-z)^{\alpha}(1-z)^{\frac{-1+\alpha-\beta+\mu}{2}}\mathbf{F}^I_{-\alpha,-\mu,\beta}(\frac{z}{z-1}) \quad (1.1.43)$$

$$= (-z)^{\alpha}(1-z)^{-\beta}\mathbf{F}^I_{-\alpha,-\beta,\mu}(z). \quad (1.1.44)$$

Solutions at ∞

To derive the next one, we scale the previous integral *Eqn.*(1.1.48) by setting $t \to \frac{t}{z}$:

$$\int_z^{\infty} t^{\frac{-1-\alpha+\beta+\mu}{2}}(t-1)^{\frac{-1+\alpha-\beta+\mu}{2}}(t-z)^{\frac{-1-\alpha-\beta-\mu}{2}}dt = z^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}^I_{-\mu,\beta,-\alpha}(z^{-1}), \quad z \notin]-\infty,1]. \quad (1.1.45)$$

A different manipulation gives

$$\int_{\infty}^z (-t)^{\frac{-1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(z-t)^{\frac{-1-\alpha-\beta-\mu}{2}}dt = (-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}^I_{-\mu,\beta,-\alpha}(z^{-1}), \quad z \notin [1,\infty[. \quad (1.1.46)$$

The integrability condition at the endpoints is

$$\mathrm{Re}(1-\mu) > |\mathrm{Re}(\beta+\alpha)|. \quad (1.1.47)$$

The identities are

$$(-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}^I_{-\mu,\beta,-\alpha}(z^{-1}) = (1-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}^I_{-\mu,\alpha,-\beta}\left(\frac{1}{1-z}\right). \quad (1.1.48)$$

$$= (-z)^{-\alpha}(1-z)^{\frac{-1+\alpha-\beta-\mu}{2}}\mathbf{F}^I_{-\mu,-\alpha,\beta}\left(\frac{1}{1-z}\right) \quad (1.1.49)$$

$$= (-z)^{\frac{-1-\alpha-\beta+\mu}{2}}(1-z)^{-\beta}\mathbf{F}^I_{\alpha,-\beta,-\mu}\left(z^{-1}\right). \quad (1.1.50)$$

The next one is obtained by replacing $t \to \frac{1}{t}$ in *Eqn.*(1.1.48):

$$\int_0^1 t^{\frac{-1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(z-t)^{\frac{-1-\alpha-\beta-\mu}{2}}dt = z^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}^I_{\mu,\beta,\alpha}(z^{-1}), \quad z \notin]-\infty,1]. \quad (1.1.51)$$

In *Eqn.*(1.1.57), replace $t \to 1-\frac{1}{t}$. Then we get

$$\int_0^1 t^{\frac{-1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(t-z)^{\frac{-1-\alpha-\beta-\mu}{2}}dt = (-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}^I_{\mu,\beta,\alpha}(z^{-1}), \quad z \notin [0,\infty[. \quad (1.1.52)$$

The integrability condition at the endpoints is

$$\mathrm{Re}(1+\mu) > |\mathrm{Re}(\alpha-\beta)|. \quad (1.1.53)$$

The identities are

$$(-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}^I_{\mu,\beta,\alpha}(z^{-1}) = (1-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}^I_{\mu,-\alpha,-\beta}\left(\frac{1}{1-z}\right). \tag{1.1.54}$$

$$= (-z)^{-\alpha}(1-z)^{\frac{-1+\alpha-\beta+\mu}{2}}\mathbf{F}^I_{\mu,\alpha,\beta}\left(\frac{1}{1-z}\right) \tag{1.1.55}$$

$$= (-z)^{\frac{-1-\alpha-\beta-\mu}{2}}(1-z)^{-\beta}\mathbf{F}^I_{-\alpha,-\beta,\mu}\left(z^{-1}\right). \tag{1.1.56}$$

Solutions at 1

Sending $t \to 1-t$ in *Eqn.*(1.1.48), we get

$$\int_{-\infty}^{0}(-t)^{\frac{-1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(z-t)^{\frac{-1-\alpha-\beta-\mu}{2}}dt = \mathbf{F}^I_{\beta,\alpha,\mu}(1-z), \quad z \notin]-\infty,0]. \tag{1.1.57}$$

The integrability condition at the endpoints is

$$\mathrm{Re}(1+\beta) > |\mathrm{Re}(\alpha-\mu)|. \tag{1.1.58}$$

We have

$$\mathbf{F}^I_{\beta,\alpha,\mu}(1-z) = (-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}^I_{\beta,-\mu,-\alpha}(1-\frac{1}{z}) \tag{1.1.59}$$

$$= (-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}^I_{\beta,\mu,\alpha}(1-\frac{1}{z}) \tag{1.1.60}$$

$$= (-z)^{-\alpha}\mathbf{F}^I_{\beta,-\alpha,-\mu}(1-z). \tag{1.1.61}$$

We use another substitution in *Eqn.*(1.1.48), namely $t \to \frac{z-1}{t-1}$:

$$\int_{z}^{1} t^{\frac{-1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(t-z)^{\frac{-1-\alpha-\beta-\mu}{2}}dt = (1-z)^{-\beta}\mathbf{F}^I_{-\beta,\alpha,-\mu}(1-z), \quad z \notin]-\infty,0] \cup [1,+\infty[. \tag{1.1.62}$$

A different manipulation gives

$$\int_{1}^{z} t^{\frac{-1-\alpha+\beta+\mu}{2}}(t-1)^{\frac{-1+\alpha-\beta+\mu}{2}}(z-t)^{\frac{-1-\alpha-\beta-\mu}{2}}dt = (z-1)^{-\beta}\mathbf{F}^I_{-\beta,\alpha,\mu}(1-z), \quad z \notin]-\infty,1]. \tag{1.1.63}$$

The integrability condition for both integrals is

$$\mathrm{Re}(1-\beta) > |\mathrm{Re}(\alpha+\mu)|. \tag{1.1.64}$$

We have

$$(1-z)^{-\beta}\mathbf{F}^I_{-\beta,\alpha,-\mu}(1-z) = (-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}^I_{\beta,-\mu,-\alpha}(1-\frac{1}{z}) \tag{1.1.65}$$

$$= (-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}^I_{\beta,\mu,\alpha}(1-\frac{1}{z}) \tag{1.1.66}$$

$$= (-z)^{-\alpha}\mathbf{F}^I_{\beta,-\alpha,-\mu}(1-z). \tag{1.1.67}$$

1.1.3 Connection formulas for hypergeometric functions

To derive a connection formula, we use the integral representation. Consider the following contour. Let $Im(z) < 0$, and take the branches of the powers of t, $1-t$, and $t-z$ by continuation from the left in the clockwise direction. Then the following integral vanishes:

$$(\int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{\infty})(-t)^{\frac{-1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(z-t)^{-\frac{1+\alpha+\beta+\mu}{2}} = 0 \tag{1.1.68}$$

where

$$Re(\alpha-\beta\pm\mu) > -1, \quad \text{and} \quad Re(\alpha-\beta\pm\mu) > -1, \quad \text{and} \quad Re(\alpha+\beta\mp\mu) > -1. \tag{1.1.69}$$

The above conditions always admit a set of solutions. Using the integral representations, one obtains the following identity:

$$\mathbf{F}^{I}_{\beta,\alpha,\mu}(1-w) + \mathrm{e}^{\mathrm{i}\pi(\alpha+1)}(-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}^{I}_{-\mu,\beta,\alpha}(w^{-1}) + \mathrm{e}^{\mathrm{i}\pi\frac{3+\alpha+\beta-\mu}{2}}\mathbf{F}^{I}_{\alpha,\beta,\mu}(w) = 0. \tag{1.1.70}$$

We can employ the symmetry $\mu \to -\mu$ and change the normalization via

$$\mathbf{F}^{I}_{\alpha,\beta,\mu}(z) = \Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\mathbf{F}_{\alpha,\beta,\mu}(z), \tag{1.1.71}$$

and the symmetry $\mathbf{F}_{\alpha,\beta,\mu}(1-z) = \mathbf{F}_{\alpha,\beta,-\mu}(1-z)$ helps us obtain the first relation:

$$\mathbf{F}_{\alpha,\beta,\mu}(z) = \frac{-\pi(-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,\beta,\alpha}(z^{-1})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)} + \frac{\pi(-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\beta,\alpha}(z^{-1})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}. \tag{1.1.72}$$

To obtain the second connection formula, we consider the following contour. Let $Im(z) < 0$, and take the branches of the powers of t, $1-t$, and $t-z$ by continuation from the left in the clockwise direction. Then the following integral vanishes:

$$(\int_{-\infty}^{0} + \int_{0}^{z} + \int_{z}^{\infty})(-t)^{\frac{-1-\alpha+\beta+\mu}{2}}(1-t)^{\frac{-1+\alpha-\beta+\mu}{2}}(z-t)^{-\frac{1+\alpha+\beta+\mu}{2}}(z^{-1}) = 0, \tag{1.1.73}$$

where

$$\mathrm{Re}\,(\alpha-\beta+-\mu) < 1, \quad \mathrm{Re}\,(\alpha+\beta+\mu) < 1, \quad \mathrm{Re}\,(-\alpha-\beta+\mu) > 1. \tag{1.1.74}$$

Again, these conditions admit a set of solutions. We arrive at the following identity:

$$\mathbf{F}^{I}_{\beta,\alpha,\mu}(1-z) + \mathrm{e}^{-\mathrm{i}\pi\frac{1-\alpha-\beta-\mu}{2}}(-w)^{-\alpha}\mathbf{F}^{I}_{-\alpha,\beta,-\mu}(z) - (-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}^{I}_{-\mu,\beta,-\alpha}(z^{-1}) = 0. \tag{1.1.75}$$

Once again, we change $\mu \to -\mu$ and use the symmetry of the hypergeometric function to get

$$\mathbf{F}_{-\alpha,\beta,\mu}(z) = \frac{-\pi(-z)^{\frac{-1+\alpha-\beta-\mu}{2}}}{\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)}\mathbf{F}_{\mu,\beta,\alpha}(z^{-1}) + \frac{\pi(-z)^{\frac{-1+\alpha-\beta+\mu}{2}}}{\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)}\mathbf{F}_{-\mu,\beta,\alpha}(z^{-1}). \tag{1.1.76}$$

The following setup generates the other two connection formulas. First, in *Eqn.*(1.1.76) we fix α and subtract the formula obtained after replacing $\alpha \to -\alpha$. Then, in *Eqn.*(1.1.77), we set $\beta \to -\beta$ and again subtract the two resulting formulas, one with α and the other with $-\alpha$. The connection formulas are

$$\begin{aligned}&\mathbf{F}_{\beta,\alpha,\mu}(1-z) &(1.1.77)\\ &=\frac{\pi\ \mathbf{F}_{\alpha,\beta,\mu}(z)}{\sin(-\pi\alpha)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)}+\frac{\pi\ z^{-\alpha}\ \mathbf{F}_{-\alpha,\beta,-\mu}(z)}{\sin(\pi\alpha)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)},\\ &(1-z)^{-\beta}\mathbf{F}_{-\beta,\alpha,-\mu}(1-z) &(1.1.78)\\ &=\frac{\pi\ \mathbf{F}_{\alpha,\beta,\mu}(z)}{\sin(-\pi\alpha)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)}+\frac{\pi\ z^{-\alpha}\ \mathbf{F}_{-\alpha,\beta,-\mu}(z)}{\sin(\pi\alpha)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)},\end{aligned}$$

where $z\notin]-\infty,0]\cup[1,\infty[$ and $\mathrm{Im}(z)<0$. The inverse connection formulas are given below:

$$\begin{aligned}\mathbf{F}_{\alpha,\beta,\mu}(z)&=\frac{\pi\ \mathbf{F}_{\beta,\alpha,\mu}(1-z)}{\sin(-\pi\beta)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}+\frac{\pi\ (1-z)^{-\beta}\ \mathbf{F}_{-\beta,\alpha,-\mu}(1-z)}{\sin(\pi\beta)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)},\\ z^{-\alpha}\mathbf{F}_{-\alpha,\beta,\mu}(z)&=\frac{\pi\ \mathbf{F}_{\beta,\alpha,\mu}(z)}{\sin(-\pi\beta)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)}+\frac{\pi\ (1-z)^{-\beta}\ \mathbf{F}_{-\beta,\alpha,-\mu}(1-z)}{\sin(\pi\beta)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)}. \quad (1.1.79)\end{aligned}$$

And also

$$\begin{aligned}&(-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,\beta,\alpha}\left(z^{-1}\right) &(1.1.80)\\ &=\frac{-\pi\ \mathbf{F}_{\alpha,\beta,\mu}(z)}{\sin(\pi\alpha)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)}+\frac{\pi\ (-z)^{-\alpha}\mathbf{F}_{-\alpha,\beta,\mu}(z)}{\sin(\pi\alpha)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)};\\ &(-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\beta,\alpha}\left(z^{-1}\right) &(1.1.81)\\ &=\frac{-\pi\ \mathbf{F}_{\alpha,\beta,\mu}(z)}{\sin(\pi\alpha)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)}+\frac{\pi\ (-z)^{-\alpha}\ \mathbf{F}_{-\alpha,\beta,\mu}(z)}{\sin(\pi\alpha)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)}.\end{aligned}$$

1.1.4 Degenerate case

Let $\mu\in\mathbb{Z}$. Then we have the identity

$$\begin{aligned}\Big(\frac{\alpha+\beta-\mu+1}{2}\Big)_\mu\Big(\frac{\alpha-\beta-\mu+1}{2}\Big)_\mu F\Big(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha-\beta+\mu+1}{2};1+\mu;z\Big) \quad (1.1.82)\\ =z^{-\mu}F\Big(\frac{\alpha+\beta-\mu+1}{2},\frac{\alpha-\beta-\mu+1}{2};1-\mu;z\Big). \quad (1.1.83)\end{aligned}$$

1.1.5 Gegenbauer functions and their connection formula

The *Gegenbauer equation* is essentially a special case of the hypergeometric equation with the symmetry $w\to -w$ and the singular points placed at $-1,1,\infty$. It is given by the *Gegenbauer operator*

$$\mathcal{G}_{\alpha,\lambda}(w,\partial_w):=(1-w^2)\partial_w^2-2(1+\alpha)w\partial_w+\lambda^2-\big(\alpha+\tfrac{1}{2}\big)^2. \quad (1.1.84)$$

Here is its relation to the Riemann operator:

$$\mathcal{G}_{\alpha,\lambda}(w,\partial_w)=(1-w^2)\mathcal{P}\begin{pmatrix}-1 & 1 & \infty & \\ 0 & 0 & \alpha+\lambda+\frac{1}{2} & w,\partial_w\\ -\alpha & -\alpha & \alpha-\lambda+\frac{1}{2} & \end{pmatrix}. \quad (1.1.85)$$

The Gegenbauer equation annihilates the Gegenbauer functions [DGR]

$$\mathcal{G}_{\alpha,\lambda}(w,\partial_w)S_{\alpha,\lambda}(w)=0, \tag{1.1.86}$$

$$\mathcal{G}_{\alpha,\lambda}(w,\partial_w)Z_{\alpha,\lambda}(w)=0. \tag{1.1.87}$$

The Gegenbauer solution with value ~ 1 at 1 is defined by

$$S_{\alpha,\pm\lambda}(w)=F_{\alpha,\alpha,\lambda}\left(\frac{1-w}{2}\right). \tag{1.1.88}$$

It can be easily seen from *Eqn.*(1.1.36) that

$$S_{-\alpha,\pm\lambda}(w)=(\frac{1+w}{2})^{-\alpha}F_{-\alpha,\alpha,2\lambda}\left(\frac{1-w}{2}\right). \tag{1.1.89}$$

We introduce a well-defined function at ∞ by

$$Z_{\alpha,\lambda}(w)=(w\pm 1)^{-\frac{1}{2}-\alpha-\lambda}F_{2\lambda,\alpha,\alpha}\left(\frac{2}{1\pm w}\right). \tag{1.1.90}$$

It is useful to introduce Olver's normalization:

$$\mathbf{S}_{\alpha,\lambda}(w)=\frac{S_{\alpha,\lambda}(w)}{\Gamma(1+\alpha)},\qquad \mathbf{Z}_{\alpha,\lambda}(w)=\frac{Z_{\alpha,\lambda}(w)}{\Gamma(1+\lambda)}. \tag{1.1.91}$$

or, in terms of hypergeometric functions,

$$\mathbf{S}_{\alpha,\lambda}(w)=\frac{1}{\Gamma(\alpha+1)}F\left(\frac{1}{2}+\alpha+\lambda,\frac{1}{2}+\alpha-\lambda;\alpha+1;\frac{1-w}{2}\right), \tag{1.1.92}$$

$$\mathbf{Z}_{\alpha,\lambda}(w)=\frac{(w\pm 1)^{-\frac{1}{2}-\alpha-\lambda}}{\Gamma(\lambda+1)}F\Big(\frac{1}{2}+\alpha+\lambda,\frac{1}{2}+\lambda;2\lambda+1;\frac{2}{1\pm w}\Big). \tag{1.1.93}$$

We can read the connection formula from the connection formulas for hypergeometric functions *Eqn.*(1.1.76) and *Eqn.*(1.1.77), for $\mathrm{Im}(w)<0$ and $w\notin]-\infty,-1]\cup[1,\infty[$:

$$\mathbf{Z}_{\alpha,\lambda}(w)=-\frac{\sqrt{\pi}2^{-\alpha+\lambda-\frac{1}{2}}}{\sin\pi\alpha\;\Gamma\left(-\alpha+\lambda+\frac{1}{2}\right)}\mathbf{S}_{\alpha,\lambda}(w)+\frac{\sqrt{\pi}2^{\alpha+\lambda-\frac{1}{2}}(w^2-1)_\bullet^{-\alpha}}{\sin\pi\alpha\;\Gamma\left(\alpha+\lambda+\frac{1}{2}\right)}\mathbf{S}_{-\alpha,\lambda}(w), \tag{1.1.94}$$

$$\mathbf{S}_{\alpha,\lambda}(-w)=-\frac{\cos\pi\lambda}{\sin\pi\alpha}\mathbf{S}_{\alpha,\lambda}(w)+\frac{\pi\;(1-w^2)^{-\alpha}}{\sin\pi\alpha\;\Gamma\left(\frac{1}{2}+\alpha+\lambda\right)\Gamma\left(\frac{1}{2}+\alpha-\lambda\right)}\mathbf{S}_{-\alpha,\lambda}(w). \tag{1.1.95}$$

Note that the first connection formula is valid when $z\notin]-\infty,1]$, and the second connection formula is valid when $w\notin]-\infty,-1]\cup[1,\infty[$. Here we borrow the notation from [DGR]:

$$(w^2-1)_\bullet^\alpha := (w-1)^\alpha(w+1)^\alpha. \tag{1.1.96}$$

The functions $(w^2-1)^\alpha$ and $(w^2-1)_\bullet^\alpha$ coincide only if $\mathrm{Re}\,w>0$. In general, the function $(w^2-1)_\bullet^\alpha$ is holomorphic on $\mathbb{C}\backslash]-\infty,1]$, while $(w^2-1)^\alpha$ is holomorphic on $\mathbb{C}\setminus\{[-1,1]\cup \mathrm{i}\mathbb{R}\}$.

Whipple transformation

The Gegenbauer equation has an extra symmetry compared to the hypergeometric symmetry, called the Whipple transformation:

$$\mathbf{Z}_{\alpha,\lambda}(w)=(w^2-1)_\bullet^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}}\mathbf{S}_{\lambda,\alpha}\left(\frac{w}{(w^2-1)_\bullet^{\frac{1}{2}}}\right), \tag{1.1.97}$$

$$\mathbf{S}_{\alpha,\lambda}(w)=(w^2-1)_\bullet^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}}\mathbf{Z}_{\lambda,\alpha}\left(\frac{w}{(w^2-1)_\bullet^{\frac{1}{2}}}\right),\quad \mathrm{Re}(w)>0. \tag{1.1.98}$$

Eqn.(1.1.98) is obtained by inverting *Eqn.*(1.1.97) and using the fact that $w \mapsto \frac{w}{(w^2-1)^{\frac{1}{2}}_{\bullet}}$ is an involution if and only if $\operatorname{Re} w > 0$.

1.2 Natanzon's problem: construction of a one-dimensional Schrödinger operator

Definition 1.2.1. *Let x belong to a real curve, $x \in]a, b[$. We say that a Schrödinger operator is*

$$\mathbf{A}(x, \frac{d}{dx}) = \frac{d^2}{dx^2} + V(x), \tag{1.2.1}$$

where $V(x)$ may be a complex-valued function. We consider its domain to be $\mathrm{Dom}(\mathcal{A}) = L^2]a, b[$.

It is clear that the hypergeometric operator

$$\mathcal{F}(a, b; c; z) = z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab, \tag{1.2.2}$$

is not of the Schrödinger type. Therefore, we need to transform this operator into the form *Eqn.*(1.2.1). This is a mathematical problem known as Natanzon's problem, posed by Natanzon [Nat]. For the precise history of the topic and more details look at [DW].

1.2.1 Bose invariant and canonical form of an ODE

Any second-order differential operator of the form

$$\mathbf{A}(z, \partial_z) := \sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta(z), \tag{1.2.3}$$

can be transformed into the so-called canonical form

$$\sigma^{-1}(z)k(z)\mathbf{A}(z, \partial_z)k^{-1}(z) = \partial_z^2 + I(z), \tag{1.2.4}$$

where

$$k(z) := \exp \int^z \frac{\tau(w)}{2\sigma(w)}\, dw, \tag{1.2.5}$$

and

$$I(z) = \left(\frac{4\sigma\eta - 2\sigma\tau' + 2\sigma'\tau - \tau^2}{4\sigma^2} \right)(z), \tag{1.2.6}$$

is called the Bose invariant [Bos]. The Bose invariant is a fundamental object. In what follows, we introduce some of its properties through the lens of geometry.

Definition 1.2.2. *Let Σ be a Riemann surface, and let $z, w \in \Sigma$. A* projective connection, $\mathbf{P}$, is a quadratic form on a Riemann surface which, under the change of variable $z \mapsto w(z)$, transforms as

$$\mathbf{P}(w) = \mathbf{P}(z) \left(\frac{dz}{dw} \right)^2 + \frac{1}{2} \{z, w\}, \tag{1.2.7}$$

where $\{z, w\}$ is the Schwarzian derivative

$$\{z, w\} := \frac{z'''(w)}{z'(w)} - \frac{3}{2} \left(\frac{z''(w)}{z'(w)} \right)^2. \tag{1.2.8}$$

In the following lemma, we explain the relation between a projective connection and a second-order differential equation in canonical form.

Lemma 1.2.3. ***Liouville transformation:***
The canonical form of a differential equation

$$\left(\partial_z^2 + I(z)\right)\varphi(z) = 0, \tag{1.2.9}$$

retains its form under a change of variable if the Bose invariant $I(z)$ is a projective connection.

Proof. Set $w = w(z)$. We get

$$\left(\frac{1}{\left(w'(z)\right)^2}\partial_w^2 - \frac{w''(z)}{\left(w'(z)\right)^3}\partial_w + I\left(w(z)\right)\right)\varphi(w) = 0. \tag{1.2.10}$$

We can rearrange the expression as

$$\frac{\left(\partial_w^2 + J(w)\right)}{\left(w'(z)\right)^2}\varphi(w) = 0, \tag{1.2.11}$$

where the new Bose invariant $J(w)$ is

$$J(w) = \left(w'(z)\right)^2 I(w) + \frac{1}{2}\{w, z\}. \tag{1.2.12}$$

This completes the proof.

The above lemma, which is sometimes called the Liouville transformation, connects a second-order ODE with the geometry of a Riemann surface. In particular, any projective connection can be realized as a ratio of solutions of a second-order ODE.

Theorem 1.2.4. *Let $\varphi_1(z)$ and $\varphi_2(z)$ be two independent solutions of the second-order differential equation. Then*

$$\frac{1}{2}\left\{\frac{\varphi_1(z)}{\varphi_2(z)}, z\right\} = I(z). \tag{1.2.13}$$

Proposition 1.2.5. (1) *$\{f(z), z\} = 0$ if and only if $f(z) = \frac{az+b}{cz+d}$, with $ad - bc \neq 0$.*

(2) *Let $w \mapsto z(w)$ be the inverse of $z \mapsto w(z)$. Then*

$$\{w, z\} = -\frac{1}{(z')^2}\{z, w\}. \tag{1.2.14}$$

(3) *Let $v \mapsto w(v)$ and $w \mapsto z(w)$. Then*

$$\{z, v\} = \{z, w\}(w')^2 + \{w, v\}. \tag{1.2.15}$$

Proof. We prove only the implication from left to right in (1); the rest can be checked by direct computation. By Theorem 1.2.4, we have

$$\{f, z\} = 0 \implies P(z) = 0. \tag{1.2.16}$$

Hence

$$\partial_z^2 f(z) = 0, \tag{1.2.17}$$

which has two solutions, $f_1(z) = az+b$ and $f_2(z) = cz+d$, and they are linearly independent if $ad-bc \neq 0$. □

Indeed, this proposition justifies the name projective connection.

1.2.2 Natanzon's problem

We use the transformation to canonical form and then the Liouville transformation to construct all possible one-dimensional Schrödinger operators associated with a hypergeometric equation. This problem was posed and partially solved by Natanzon [Nat]. We state the problem first for a second-order differential equation in general, and then we discuss the hypergeometric case.

Let $I(a_1, \ldots, a_n; z)$ be the Bose invariant associated with a second-order differential equation with n parameters, $\mathcal{A}(a_1, \ldots, a_n; z, \partial_z)$. We ask for a decomposition of the Bose invariant of the form

$$I(b_1, \ldots, b_k; z) = \sum_{i=1}^{k} b_i I_i(z), \tag{1.2.18}$$

where $\{I_i\}$ are linearly independent and $k \leq n$.

Natanzon's problem: Find all potentials $V(x)$ in the Schrödinger equation

$$\left(-\partial_x^2 + V(x) - E\right) \varphi(E, x) = 0 \tag{1.2.19}$$

such that it can be transformed from $\mathcal{A}(a_1, \ldots, a_n; z, \partial_z)$, while allowing the following operations:

(1) Multiplication of the equation from both sides by sufficiently regular functions;

(2) decomposition of the eigenfunctions,

$$\tilde{\varphi}(x) = (g\varphi)(x); \tag{1.2.20}$$

(3) an arbitrary change of variable.

Algebraic formulation

We first write $\mathcal{A}(a_1, \ldots, a_n; z, \partial_z)$ in its canonical form:

$$-\partial_z^2 - I(b_1, \ldots, b_k; z). \tag{1.2.21}$$

Using the Liouville transformation, we have

$$I(b_1, \ldots, b_k; z) = (x'(z))^2 \left(E - V(x(z))\right) + \frac{1}{2}\{x, z\}. \tag{1.2.22}$$

We can linearly rearrange $(b_1, \ldots, b_k)$ as $(\check{b}_1, \ldots, \check{b}_{k-1}, E)$, where each $\check{b}_i$ is independent of E. Without loss of generality, due to the linear independence of the I_i, we can write

$$I(b_1, \ldots, b_k; z) = \check{b}_1 \check{I}_1 + \cdots + \check{b}_{k-1} \check{I}_{k-1} + E \check{I}_n. \tag{1.2.23}$$

Substituting this into *Eqn.*(1.2.22), we get

$$V(\check{b}_1, \ldots, \check{b}_{k-1}, x(z)) = -\frac{1}{(x'(z))^2} \left(\check{b}_1 \check{I}_1(z(x)) + \cdots + \check{b}_{k-1} \check{I}_{k-1}(z(x)) - \frac{1}{2}\{z(x), x\}\right), \tag{1.2.24}$$

$$(x'(z))^{-2} = \check{I}_n(z). \tag{1.2.25}$$

Using 3, we can invert these equations easily:

$$V(\check{b}_1, \ldots, \check{b}_{k-1}, z) = -(x'(z))^2 \left(\check{b}_1 \check{I}_1(z) + \cdots + \check{b}_{k-1} \check{I}_{k-1}(z)\right) - \frac{1}{2}\{x(z), z\}, \tag{1.2.26}$$

$$(z'(x))^2 = \check{I}_n(z(x)). \tag{1.2.27}$$

Remark 1.2.6. *There is an additional condition related to $\check{I}_n$. We assume that it has the form $\check{I}_n(z) = \prod_{i=1}^{N}(z - z_i)^{p_i}$, where $z_i \in \mathbb{C}$ are singularities of $\mathcal{A}(a_1, \ldots, a_n; z, \partial_z)$ and $p_i \in \mathbb{N}_0$.*

Natanzon's problem and Schrödinger operators related to the hypergeometric equation

Recall that the hypergeometric operator is

$$\mathcal{F}(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha+\beta-\mu+1}{2};1+\alpha;z,\partial_z)=z(1-z)\partial_z^2-(1+\beta)\partial_z-\frac{1}{4}(\alpha+\beta+\mu+1)(\alpha+\beta-\mu+1). \tag{1.2.28}$$

The Bose invariant associated with the hypergeometric operator is

$$I(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha+\beta-\mu+1}{2};1+\alpha;z)=\frac{(1-\mu^2)z^2+(\alpha^2-\beta^2+\mu^2-1)z+(1+\alpha)^2}{4z^2(1-z)^2}. \tag{1.2.29}$$

We can decompose the Bose invariant into at most three monomials:

$$I_1=\frac{1}{z^2(1-z)^2},\quad I_2=\frac{1}{z(1-z)^2},\quad I_3=\frac{1}{(1-z)^2}. \tag{1.2.30}$$

It is easy to see that $\check{I}_3$ should have one of the following forms:

$$\text{I.1}:\quad\frac{1}{z(1-z)},\qquad \text{I.2}:\quad\frac{1}{z(1-z)^2},\qquad (\text{I.3})\quad\frac{1}{(1-z)^2}; \tag{1.2.31}$$

$$\text{II.1}:\quad\frac{1}{z^2(1-z)^2},\qquad \text{II.2}:\quad\frac{1}{z^2(1-z)},\qquad (\text{II.3})\quad\frac{1}{z^2}. \tag{1.2.32}$$

As we will explain in the next section, the conditions (I) in *Eqn.*(1.2.31) are equivalent and lead to what we call hypergeometric Hamiltonians of the first kind, while the conditions (II) in *Eqn.*(1.2.32) lead to hypergeometric Hamiltonians of the second kind.

1.2.3 Schrödinger operators related to hypergeometric equations

The following lemma will be helpful.

Lemma 1.2.7. *The operator*

$$-t(z)\partial_z^2-\frac{1}{2}t'(z)\partial_z+v(z) \tag{1.2.33}$$

can be transformed into

$$-\partial_r^2+v\big(z(r)\big). \tag{1.2.34}$$

if there exists a change of variable from z to r such that

$$\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2=t(z). \tag{1.2.35}$$

Hypergeometric Hamiltonian of the first kind

We pick condition I.1 in *Eqn.*(1.2.31).

Let us transform the hypergeometric operator as follows:

$$\begin{aligned}
&-z^{\frac{\alpha}{2}+\frac14}(1-z)^{\frac{\beta}{2}+\frac14}\mathcal{F}\left(\tfrac{\alpha+\beta+\mu+1}{2},\tfrac{\alpha+\beta-\mu+1}{2};1+\alpha;z,\partial_z\right)z^{-\frac{\alpha}{2}-\frac14}(1-z)^{-\frac{\beta}{2}-\frac14}\\
=\quad&-z(1-z)\mathcal{P}\left(\begin{matrix} 0 & 1 & \infty & \\ \frac{\alpha}{2}+\frac14 & \frac{\beta}{2}+\frac14 & \frac{\mu}{2} & z,\partial_z \\ -\frac{\alpha}{2}+\frac14 & -\frac{\beta}{2}+\frac14 & -\frac{\mu}{2} & \end{matrix}\right)\\
&-z(1-z)\left(\partial_z^2+\left(\frac{1}{2z}-\frac{1}{2(1-z)}\right)\partial_z\right)\\
&+\left(\alpha^2-\frac14\right)\frac{1}{4z}+\left(\beta^2-\frac14\right)\frac{1}{4(1-z)}-\frac{\mu^2}{4}.
\end{aligned}\tag{1.2.36}$$

Thus, if we set

$$L_{\alpha,\beta}:=-z(1-z)\left(\partial_z^2+\left(\frac{1}{2z}-\frac{1}{2(1-z)}\right)\partial_z\right)+\left(\alpha^2-\frac14\right)\frac{1}{4z}+\left(\beta^2-\frac14\right)\frac{1}{4(1-z)},\tag{1.2.37}$$

and if $F(z)$ solves the hypergeometric equation *Eqn.*(1.1.28), then

$$\left(L_{\alpha,\beta}-\frac{\mu^2}{4}\right)z^{\frac{\alpha}{2}+\frac14}(1-z)^{\frac{\beta}{2}+\frac14}F(z)=0.\tag{1.2.38}$$

The hypergeometric equation has thus been interpreted as the eigenequation of the operator $L_{\alpha,\beta}$ with eigenvalue $\frac{\mu^2}{4}$ acting on functions on the complex plane. Let us reinterpret this operator as acting on functions on three intervals whose endpoints are singularities of the hypergeometric equation. We will see again that these choices lead to simple formulas for Green functions. In each of these cases, we perform the Liouville transformation, which yields a one-dimensional Hamiltonian. We consider three cases:

1. $z\in]0,1[$, which leads to an operator on $L^2]0,\pi[$, which we call the *spherical hypergeometric Hamiltonian of the 1st kind*;

2. $z\in]-\infty,0[$, which leads to an operator on $L^2(\mathbb{R}_+)$, which we call the *hyperbolic hypergeometric Hamiltonian of the 1st kind*;

3. $z\in\frac12+\mathrm{i}\mathbb{R}$, which leads to an operator on $L^2(\mathbb{R})$, which we call the *deSitterian hypergeometric Hamiltonian of the 1st kind*.

It is natural to introduce the parameters

$$\delta:=\frac12(\alpha^2+\beta^2),\tag{1.2.39}$$

$$\kappa:=\frac12(\alpha^2-\beta^2),\qquad \tau:=\frac{\mathrm{i}}{2}(\alpha^2-\beta^2)=\mathrm{i}\kappa.\tag{1.2.40}$$

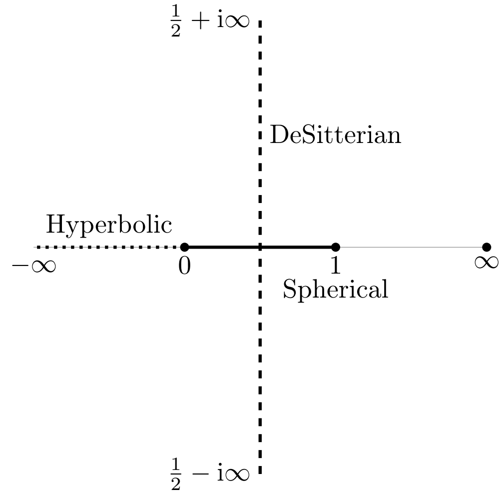


Figure 2: Hypergeometric Hamiltonians of the first kind on the z-plane.
The spherical Hamiltonian acts on the interval marked with a thick line, the hyperbolic Hamiltonian with a dotted line, and the deSitterian Hamiltonian with a dashed line.

We use Lemma 3 repeatedly to construct these Hamiltonians.

Spherical case

For $r\in]0,\pi[$, set in *Eqn.*(1.2.37)

$$z=\sin^2\frac{r}{2}=\frac{1-\cos r}{2},\quad \text{which solves}\ \ z'=z^{\frac12}(1-z)^{\frac12}. \tag{1.2.41}$$

This leads to the Schrödinger equation

$$\left(L^{\mathrm{s}}_{\alpha,\beta}-\frac{\mu^2}{4}\right)\phi(r)=0, \tag{1.2.42}$$

where

$$\begin{aligned} L^{\mathrm{s}}_{\alpha,\beta} &:= -\partial_r^2+\left(\alpha^2-\frac14\right)\frac{1}{4\sin^2\frac{r}{2}}+\left(\beta^2-\frac14\right)\frac{1}{4\cos^2\frac{r}{2}} \\ &= -\partial_r^2+\left(\delta-\frac14\right)\frac{1}{\sin^2 r}+\kappa\frac{\cos r}{\sin^2 r}. \end{aligned} \tag{1.2.43}$$

In [DW], $L^{\mathrm{s}}_{\alpha,\beta}$ is called the *trigonometric Pöschl-Teller Hamiltonian.*

The case $\alpha=\beta$ is especially important and coincides with the spherical Gegenbauer Hamiltonian:

$$L^{\mathrm{s}}_{\alpha}:=L^{\mathrm{s}}_{\alpha,\alpha}. \tag{1.2.44}$$

Hyperbolic case

For $r\in\mathbb{R}_+$, in *Eqn.*(1.2.37) we set

$$z=-\sinh^2\frac{r}{2}=\frac{1-\cosh r}{2},\quad \text{which solves}\ \ z'=-(-z)^{\frac12}(1-z)^{\frac12}. \tag{1.2.45}$$

This leads to the Schrödinger equation

$$\left(L^{\mathrm{h}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)\phi(r) = 0, \tag{1.2.46}$$

where

$$\begin{aligned} L^{\mathrm{h}}_{\alpha,\beta} &:= -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{4\sinh^2\frac{r}{2}} - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{4\cosh^2\frac{r}{2}} \\ &= -\partial_r^2 + \left(\delta - \frac{1}{4}\right)\frac{1}{\sinh^2 r} + \kappa\frac{\cosh r}{\sinh^2 r}. \end{aligned} \tag{1.2.47}$$

In [DW], $L^{\mathrm{h}}_{\alpha,\beta}$ is called the *hyperbolic Pöschl-Teller Hamiltonian.*

The case $\alpha = \beta$ is especially important and coincides with the hyperbolic Gegenbauer operator:

$$L^{\mathrm{h}}_{\alpha} := L^{\mathrm{h}}_{\alpha,\alpha}. \tag{1.2.48}$$

1.2.4 DeSitterian case

For $r \in \mathbb{R}$, in *Eqn.*(1.2.37) we set

$$z = \frac{1}{2} - \mathrm{i}\cosh\frac{r}{2}\sinh\frac{r}{2} = \frac{1 - \mathrm{i}\sinh r}{2}, \quad \text{which solves } \ z' = (-z)^{\frac{1}{2}}(1-z)^{\frac{1}{2}}. \tag{1.2.49}$$

This leads to the Schrödinger equation

$$\left(L^{\mathrm{dS}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)\phi(r) = 0, \tag{1.2.50}$$

where

$$L^{\mathrm{dS}}_{\alpha,\beta} := -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}\left(\frac{1}{2} + \frac{\mathrm{i}\sinh r}{2}\right) - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}\left(\frac{1}{2} - \frac{\mathrm{i}\sinh r}{2}\right) \tag{1.2.51}$$

$$= -\partial_r^2 - \left(\delta - \frac{1}{4}\right)\frac{1}{\cosh^2 r} - \tau\frac{\sinh r}{\cosh^2 r}. \tag{1.2.52}$$

This Hamiltonian was proposed and solved by F. Scarf [Sca], and in [DW] it is called the *Scarf Hamiltonian.*

The case $\alpha = \beta$ is especially important and coincides with the deSitterian Gegenbauer Hamiltonian:

$$L^{\mathrm{dS}}_{\alpha} := L^{\mathrm{dS}}_{\alpha,\alpha}. \tag{1.2.53}$$

Remark 1.2.8. *Conditions I.2 and I.3 give the same set of operators.*

1.2.5 Hypergeometric Hamiltonian of the second kind

We pick II.1 in *Eqn.*(1.2.32) and again use Lemma 3 repeatedly.

We transform the hypergeometric equation in a different way:

$$\begin{aligned} &-4z^{1+\frac{\alpha}{2}}(1-z)^{1+\frac{\beta}{2}}\mathcal{F}\left(\tfrac{\alpha+\beta+\mu+1}{2}, \tfrac{\alpha+\beta-\mu+1}{2}; 1+\alpha; z, \partial_z\right)z^{-\frac{\alpha}{2}}(1-z)^{-\frac{\beta}{2}} \\ = &-4z^2(z-1)^2\mathcal{P}\begin{pmatrix} 0 & 1 & \infty & \\ \frac{\alpha}{2} & \frac{\beta}{2} & \frac{\mu}{2}+\frac{1}{2} & z, \partial_z \\ -\frac{\alpha}{2} & -\frac{\beta}{2} & -\frac{\mu}{2}+\frac{1}{2} & \end{pmatrix} \\ = &-4z^2(1-z)^2\left(\partial_z^2 + \left(\frac{1}{z} - \frac{1}{1-z}\right)\partial_z\right) + \alpha^2(1-z) + \beta^2 z - (\mu^2-1)z(1-z). \end{aligned} \tag{1.2.54}$$

We rearrange the terms in *Eqn.*(1.2.54) containing α and β as follows:

$$\alpha^2(1-z)+\beta^2 z = \delta+\kappa(1-2z), \tag{1.2.55}$$

$$= \delta+\tau\mathrm{i}(2z-1), \tag{1.2.56}$$

where δ, κ, τ are defined in *Eqn.*(1.2.40). Thus if we set

$$K_{\kappa,\mu} := -4z^2(1-z)^2\left(\partial_z^2+\Big(\frac{1}{z}-\frac{1}{1-z}\Big)\partial_z\right)+\kappa(1-2z)-(\mu^2-1)z(1-z), \tag{1.2.57}$$

and if $F(z)$ solves the hypergeometric equation *Eqn.*(1.1.28), then

$$\Big(K_{\kappa,\mu}+\delta\Big)z^{\frac{\alpha}{2}}(1-z)^{\frac{\beta}{2}}F(z)=0. \tag{1.2.58}$$

We have therefore interpreted *Eqn.*(1.2.54) as the eigenequation of the operator $K_{\kappa,\mu}$ with eigenvalue $-\delta$. Again, we reinterpret this operator as acting on functions on an interval whose endpoints are singularities of the hypergeometric equation. In each case, we perform the Liouville transformation, which yields a one-dimensional Hamiltonian. We consider three cases:

1. $z \in \frac{1}{2}+\mathrm{i}\mathbb{R}$, which leads to an operator on $L^2]0,\pi[$, which we call the *spherical hypergeometric Hamiltonian of the 2nd kind*;
2. $z \in]-\infty, 0[$, which leads to an operator on $L^2]0,\infty[$, which we call the *hyperbolic hypergeometric Hamiltonian of the 2nd kind*;
3. $z \in]0,1[$, which leads to an operator on $L^2(\mathbb{R})$, which we call the *deSitterian hypergeometric Hamiltonian of the 2nd kind*.

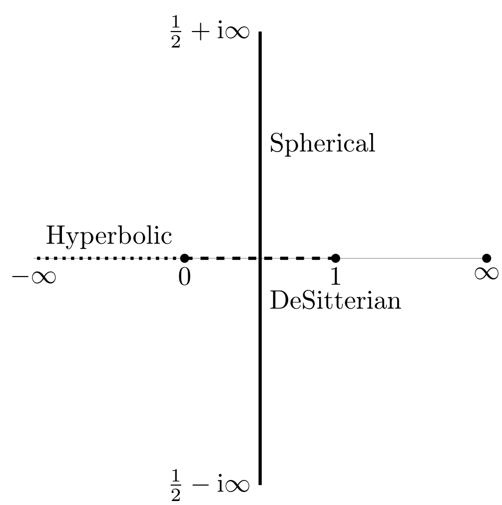


Figure 3: Hypergeometric Hamiltonians of the second kind on the z-plane.
The spherical Hamiltonian acts on the interval marked with a thick line, the hyperbolic one with a dotted line, and the DeSitterian one with a dashed line.

Spherical case

For $u \in]0,\pi[$, in *Eqn.*(1.2.54) and *Eqn.*(1.2.56) we set

$$z=\frac{1}{1-\mathrm{e}^{2\mathrm{i}u}}, \quad \text{which solves} \quad z' = 2\mathrm{i}z(1-z). \tag{1.2.59}$$

This leads to the Schrödinger equation

$$\left(K^{\mathrm{s}}_{\tau,\mu} - \delta\right) \phi(u) = 0, \tag{1.2.60}$$

where

$$K^{\mathrm{s}}_{\tau,\mu}(u) := -\partial_u^2 + \tau \frac{\cos u}{\sin u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right) \frac{1}{\sin^2 u}. \tag{1.2.61}$$

This Hamiltonian is known as the *Rosen-Morse Hamiltonian* (see [DW]).

In the case $\tau = 0$, we have the coincidence

$$L^{\mathrm{s}}_{\alpha} = K^{\mathrm{s}}_{0,2\alpha}. \tag{1.2.62}$$

Hyperbolic case

For $u \in \mathbb{R}_+$, in *Eqn.*(1.2.54) and *Eqn.*(1.2.55) we set

$$z = \frac{1}{1 - \mathrm{e}^{2u}}, \quad \text{which solves} \quad z' = 2z(z-1). \tag{1.2.63}$$

This leads to the Schrödinger equation

$$\left(K^{\mathrm{h}}_{\kappa,\mu} + \delta\right) \phi(u) = 0, \tag{1.2.64}$$

where

$$K^{\mathrm{h}}_{\kappa,\mu} := -\partial_u^2 + \kappa \frac{\cosh u}{\sinh u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right) \frac{1}{\sinh^2 u}. \tag{1.2.65}$$

In [DW], this is called the *Eckart Hamiltonian*.

In the case $\kappa = 0$, we have the coincidence

$$L^{\mathrm{h}}_{\alpha} = K^{\mathrm{h}}_{0,2\alpha}. \tag{1.2.66}$$

DeSitterian case

For $u \in \mathbb{R}$, in *Eqn.*(1.2.54) and *Eqn.*(1.2.55) we set

$$z = \frac{1}{1 + \mathrm{e}^{2u}}, \quad \text{which solves} \quad z' = 2z(z-1). \tag{1.2.67}$$

This leads to the Schrödinger equation

$$\left(K^{\mathrm{dS}}_{\kappa,\mu} + \delta\right) \phi(u) = 0, \tag{1.2.68}$$

where

$$K^{\mathrm{dS}}_{\kappa,\mu} := -\partial_u^2 - \kappa \frac{\sinh u}{\cosh u} - \left(\frac{\mu^2}{4} - \frac{1}{4}\right) \frac{1}{\cosh^2 u}. \tag{1.2.69}$$

In [DW], this is called the *Manning-Rosen Hamiltonian*.

In the case $\kappa = 0$, we have the coincidence

$$L^{\mathrm{dS}}_{\alpha} = K^{\mathrm{dS}}_{0,2\alpha}. \tag{1.2.70}$$

Remark 1.2.9. *Conditions II.2 and II.3 in Eqn.*(1.2.32) *give the same set of operators.*

1.2.6 Gegenbauer Hamiltonians

Although we explain the limiting process leading to the Gegenbauer Hamiltonian here, we also construct these Hamiltonians independently of the hypergeometric Hamiltonians.

Recall that the Gegenbauer equation is

$$\mathcal{G}_{\alpha,\lambda}(w,\partial_w) := (1-w^2)\partial_w^2 - 2(1+\alpha)w\partial_w + \lambda^2 - \big(\alpha+\tfrac{1}{2}\big)^2. \tag{1.2.1}$$

The Bose invariant for the Gegenbauer equation is

$$\mathcal{I}_{\alpha,\lambda}(w) = \frac{(1-4\lambda^2)z^2 + (3-4\alpha^2+4\lambda^2)}{(-1+w^2)^2}. \tag{1.2.2}$$

The Bose invariant decomposes into the following monomials:

$$\mathcal{I}_1(w) = \frac{1}{(-1+w^2)}, \quad \mathcal{I}_2 = \frac{1}{(-1+w^2)^2}. \tag{1.2.3}$$

These two lead to the following two possible conditions for $\check{\mathcal{I}}_2$:

$$\text{I.1}: \quad \frac{1}{(-1+w^2)}, \qquad \text{I.2}: \quad \frac{1}{(-1+w^2)^2}. \tag{1.2.4}$$

We pick the first condition, I.1, in *Eqn.*(1.2.4). Let us transform the Gegenbauer operator *Eqn.*(1.2.1) as follows:

$$\begin{aligned}
&-(1-w^2)^{\frac{\alpha}{2}+\frac{1}{4}}\mathcal{G}_{\alpha,\lambda}(w,\partial_w)(1-w^2)^{-\frac{\alpha}{2}-\frac{1}{4}} \\
=\;& -(1-w^2)\mathcal{P}\begin{pmatrix} -1 & 1 & \infty & \\ \frac{\alpha}{2}+\frac{1}{4} & \frac{\alpha}{2}+\frac{1}{4} & \lambda & w,\partial_w \\ -\frac{\alpha}{2}+\frac{1}{4} & -\frac{\alpha}{2}+\frac{1}{4} & -\lambda & \end{pmatrix} \\
=\;& -(1-w^2)\partial_w^2 + w\partial_w + \left(\alpha^2-\frac{1}{4}\right)\frac{1}{1-w^2} - \lambda^2.
\end{aligned} \tag{1.2.5}$$

Thus, if we set

$$L_\alpha := -(1-w^2)\partial_w^2 + w\partial_w + \left(\alpha^2-\frac{1}{4}\right)\frac{1}{1-w^2}, \tag{1.2.6}$$

and if $G(w)$ solves the Gegenbauer equation, then

$$\big(L_\alpha - \lambda^2\big)(1-w^2)^{\frac{\alpha}{2}+\frac{1}{4}}G(w) = 0. \tag{1.2.7}$$

We have therefore interpreted the Gegenbauer equation as the eigenequation of the operator L_α with eigenvalue λ^2, acting on functions on the complex plane. We would now like to reinterpret it as an operator acting on functions of a real variable $r \in]a,b[$ for some $-\infty \le a < b \le +\infty$. To this end, we need to embed the intervals $]a,b[$ into the complex plane.

We consider three intervals; in each case, their endpoints are among the singular points of the Gegenbauer equation, namely 1, -1, and ∞. For each of these intervals, we perform the Liouville transformation, which yields a one-dimensional Hamiltonian on $C_c^\infty]a,b[$. As we will see below, in each case we obtain a natural holomorphic family of closed realizations possessing elegant formulas for the Green functions.

Here are the three cases that we consider:

1. $w \in]-1,1[$. This leads to an operator on $L^2]0,\pi[$, which we call the *spherical Gegenbauer Hamiltonian.*

2. $w \in]1,\infty[$. This leads to an operator on $L^2(\mathbb{R}_+)$, which we call the *hyperbolic Gegenbauer Hamiltonian.*

3. $w \in \mathrm{i}\mathbb{R}$. This leads to an operator on $L^2(\mathbb{R})$, which we call the *deSitterian Gegenbauer Hamiltonian.*

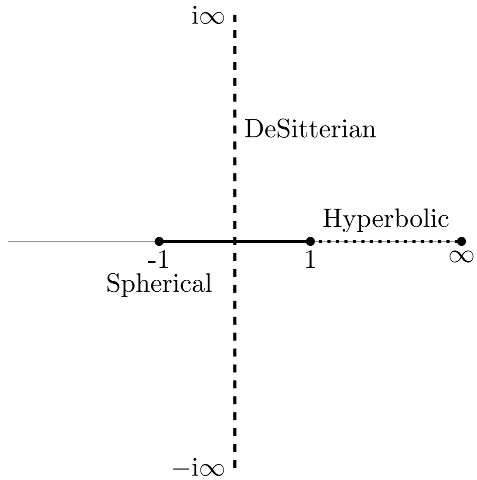


Figure 1: Gegenbauer equation on the w-plane.
The spherical Hamiltonian acts on the interval marked with a thick line, the hyperbolic Hamiltonian with a dotted line, and the deSitterian Hamiltonian with a dashed line.

Spherical case

For $r \in]0,\pi[$, in *Eqn.*(1.2.5) set

$$w = \cos r, \quad \text{which solves } \ w' = -(1-w^2)^{\frac{1}{2}}. \tag{1.2.8}$$

This leads to the Schrödinger equation

$$\left(L_\alpha^{\mathrm{s}} - \lambda^2\right)\phi(r) = 0, \tag{1.2.9}$$

where

$$L_\alpha^{\mathrm{s}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\sin^2 r}. \tag{1.2.10}$$

It has the mirror symmetry $r \to \pi - r$. It is obtained when we separate variables in the Laplacian on the sphere in any dimension; see, for example, Subsection 1.3.1. Hence our name "spherical".

Hyperbolic case

In *Eqn.*(1.2.5), for $r \in \mathbb{R}_+$, we set

$$w = \cosh r, \quad \text{which solves } \ w' = (w^2-1)^{\frac{1}{2}}. \tag{1.2.11}$$

This leads to the Schrödinger equation

$$\left(L_\alpha^{\mathrm{h}} + \lambda^2\right)\phi(r) = 0, \tag{1.2.12}$$

where

$$L_\alpha^{\mathrm{h}} = -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right) \frac{1}{\sinh^2 r}. \tag{1.2.13}$$

It is obtained when we separate variables in the Laplacian on the hyperbolic space of any dimension; see, for example, Subsection 1.3.2. Hence our name "hyperbolic".

DeSitterian case

For $r \in \mathbb{R}$, in *Eqn.*(1.2.5) we set

$$w = -\mathrm{i}\sinh r, \quad \text{which solves} \;\; w' = (w^2 - 1)^{\frac{1}{2}}. \tag{1.2.14}$$

This leads to the Schrödinger equation

$$\left(L_\alpha^{\mathrm{dS}} + \lambda^2\right) \phi(r) = 0, \tag{1.2.15}$$

where

$$L_\alpha^{\mathrm{dS}} := -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right) \frac{1}{\cosh^2 r}. \tag{1.2.16}$$

It has the mirror symmetry $r \to -r$. It is obtained when we separate variables in the d'Alembertian on deSitter space of any dimension; see Subsection 1.3.3. Hence our name "deSitterian".

Remark 1.2.10. *Condition I.2 in Eqn.*(1.2.4) *gives equivalent Hamiltonians.*

1.3 Geometric interpretation

One of the main motivations for studying special functions has been the separation of variables. To such an extent, in fact, that the first encyclopedic book on special functions was dedicated to this topic [M2]. On the other hand, we have suggested geometric names for our Hamiltonians, which require justification. Indeed, this justification comes from the geometric analysis of symmetric spaces. In this section, we interpret these Hamiltonians as arising from the separation of variables for Laplace equations on symmetric spaces. The interpretation of hypergeometric Hamiltonians of the second type as a separation of variables of some meaningful equation is still missing and, to our knowledge, remains an open question.

In our interpretations, we allow two kinds of manipulations motivated by operator theory:

1. Two operators, A and B, are equivalent if they are related by a similarity transformation U,

$$U^{-1}AU \sim B. \tag{1.3.1}$$

2. Two operators are equivalent up to a constant shift,

$$A + c\mathbb{I} \sim B, \tag{1.3.2}$$

where $\mathbb{I}$ is the identity operator and c is a constant.

We will obtain these Hamiltonians as the result of separation of variables for (pseudo-)Laplacians on various (pseudo-)spheres.

Recall that every (pseudo-)Riemannian manifold is equipped with a natural differential operator called the *(pseudo-)Laplacian.* Suppose we fix coordinates $x = x_1, \ldots, x_d$, and the (pseudo-)metric is given by a field of symmetric invertible matrices $[g_{ij}]$, so that the "line element" is

$$\mathrm{d}s^2 = \sum_{1\le i,j\le d} g_{ij}\mathrm{d}x_i\mathrm{d}x_j. \tag{1.3.3}$$

Then the pseudo-Laplacian is given by

$$\Delta = \frac{1}{\sqrt{|\det g|}} \partial_{x_i} g^{ij} \sqrt{|\det g|} \partial_{x_j}, \tag{1.3.4}$$

where $[g^{ij}]$ is the inverse of $[g_{ij}]$. In the case of Riemannian signature, Δ is called the *Laplacian* (or the *Laplace-Beltrami operator*). For Lorentzian signature, the usual name is the *d'Alembertian*.

(Pseudo-)spheres in a *(pseudo-)Euclidean space* $\mathbb{R}^{p,q}$ inherit a (pseudo-)Riemannian structure from the ambient space. If the ambient space is Euclidean, they are called *spheres*. Otherwise, they are various kinds of *hyperboloids*.

Below, we describe a few examples of separation of variables for a (pseudo-)Laplacian on a (pseudo-)sphere in appropriate coordinate systems. We will see that, after an appropriate gauge transformation, subtraction of a constant, and restriction to an invariant subspace, one obtains various hypergeometric Hamiltonians. These computations motivate the names "spherical", "hyperbolic", and "deSitterian" that we use in our paper for various types of hypergeometric Hamiltonians.

The best known among these Laplacians is Δ_d^{s}, the Laplacian on the d-dimensional sphere $\mathbb{S}^d$. As is well known, it has the spectrum

$$\sigma(\Delta_d^{\mathrm{s}}) = \{-l(l+d-1) \mid l \in \mathbb{N}_0\}. \tag{1.3.5}$$

(This is, incidentally, a consequence of the computations in Subsection 1.3.1 and the properties of the spherical Gegenbauer Hamiltonian.) Eigenfunctions of Δ_d^{s} with eigenvalue $-l(l+d-1)$ will be called *d-dimensional spherical harmonics of order l*.

1.3.1 Sphere

The unit d-dimensional sphere is defined by

$$\mathbb{S}^d := \{X \in \mathbb{R}^{d+1} \mid X_0^2 + X_1^2 + \cdots + X_d^2 = 1\}. \tag{1.3.6}$$

We denote elements of $\mathbb{S}^{d-1}$ by $\hat{X}$, and the corresponding line element by $\mathrm{d}\hat{X}^2$. We use the coordinates $(r, \hat{X})$ on $\mathbb{S}^d$ given by

$$X_0 = \cos r, \qquad X_i = \sin r\, \hat{X}_i, \qquad i = 1, \dots, d, \qquad \hat{X} \in \mathbb{S}^{d-1}. \tag{1.3.7}$$

One can also use slightly different coordinates $(w, \hat{X})$ with

$$\cos r = w, \qquad \sin r = \sqrt{1-w^2}. \tag{1.3.8}$$

In these coordinates, we first write the line element, and then the Laplacian:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \sin^2 r\, \mathrm{d}\hat{X}^2 = \frac{\mathrm{d}w^2}{1-w^2} + (1-w^2)\mathrm{d}\hat{X}^2, \tag{1.3.9}$$

$$\begin{aligned} \Delta_d^{\mathrm{s}} &= \partial_r^2 + (d-1)\cot r\, \partial_r + \frac{\Delta_{d-1}^{\mathrm{s}}}{\sin^2 r} \\ &= (1-w^2)\partial_w^2 - dw\, \partial_w + \frac{\Delta_{d-1}^{\mathrm{s}}}{1-w^2}. \end{aligned} \tag{1.3.10}$$

Finally, we perform an appropriate gauge transformation:

$$(\sin r)^{\frac{d-1}{2}} \big(-\Delta_d^{\mathrm{s}}\big)(\sin r)^{-\frac{d-1}{2}} + \big(\tfrac{d-1}{2}\big)^2 = -\partial_r^2 + \frac{\big(\frac{d-2}{2}\big)^2 - \frac14 - \Delta_{d-1}^{\mathrm{s}}}{\sin^2 r}. \tag{1.3.11}$$

Thus, the Laplacian on $\mathbb{S}^d$, restricted to $(d-1)$-dimensional spherical harmonics of order l, reduces by *Eqn.*(1.3.11) to the spherical Gegenbauer Hamiltonian L_α^{s} with

$$\alpha = \frac{d}{2} - 1 + l. \tag{1.3.12}$$

1.3.2 Hyperbolic space

The d-dimensional hyperbolic space is defined by

$$\mathbb{H}^d := \{X \in \mathbb{R}^{d+1} \mid -X_0^2 + X_1^2 + \cdots + X_d^2 = -1, \quad X_0 > 0\}. \tag{1.3.13}$$

We use the following coordinates on $\mathbb{H}^d$: $(r, \hat{X})$, where $\hat{X} \in \mathbb{S}^{d-1}$,

$$X_0 = \cosh r, \qquad X_i = \sinh r\, \hat{X}_i, \qquad i = 1, \dots, d. \tag{1.3.14}$$

One can also use slightly different coordinates $(w, \hat{X})$ with

$$\cosh r = w, \qquad \sinh r = \sqrt{w^2 - 1}. \tag{1.3.15}$$

In these coordinates, we first write the line element, and then the Laplacian:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2 r\, \mathrm{d}\hat{X}^2 = \frac{\mathrm{d}w^2}{w^2 - 1} + (w^2 - 1)\mathrm{d}\hat{X}^2, \tag{1.3.16}$$

$$\begin{aligned} \Delta_d^{\mathrm{h}} &= \partial_r^2 + (d-1)\coth r\, \partial_r + \frac{\Delta_{d-1}^{\mathrm{s}}}{\sinh^2 r} \\ &= (w^2 - 1)\partial_w^2 + dw\, \partial_w + \frac{\Delta_{d-1}^{\mathrm{s}}}{w^2 - 1}. \end{aligned} \tag{1.3.17}$$

Finally, we perform an appropriate gauge transformation:

$$(\sinh r)^{\frac{d-1}{2}} \big(-\Delta_d^{\mathrm{h}} \big) (\sinh r)^{-\frac{d-1}{2}} - \big(\tfrac{d-1}{2}\big)^2 = -\partial_r^2 + \frac{\big(\frac{d-2}{2}\big)^2 - \frac{1}{4} - \Delta_{d-1}^{\mathrm{s}}}{\sinh^2 r}. \tag{1.3.18}$$

Thus, the Laplacian on $\mathbb{H}^d$, restricted to $(d-1)$-dimensional spherical harmonics of order l, reduces to the hyperbolic Gegenbauer Hamiltonian L_α^{h} with

$$\alpha = \frac{d}{2} - 1 + l. \tag{1.3.19}$$

1.3.3 DeSitter space

The deSitter space is defined by

$$\mathrm{dS}^d := \{X \in \mathbb{R}^{d+1} \mid -X_0^2 + X_1^2 + \cdots + X_d^2 = 1\}. \tag{1.3.20}$$

We use the coordinates $(t, \hat{X})$ on dS^d given by

$$X_0 = \sinh t, \qquad X_i = \cosh t\, \hat{X}_i, \qquad i = 1, \dots, d, \qquad \hat{X} \in \mathbb{S}^{d-1}. \tag{1.3.21}$$

Alternative coordinates are

$$\sinh t = w, \qquad \cosh t = \sqrt{1 + w^2}. \tag{1.3.22}$$

In these coordinates, we first write the line element, and then the d'Alembertian:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \cosh^2 t\, \mathrm{d}\hat{X}^2 = -\frac{\mathrm{d}w^2}{w^2 + 1} + (w^2 + 1)\mathrm{d}\hat{X}^2, \tag{1.3.23}$$

$$\begin{aligned} \Box_d^{\mathrm{dS}} &= -\partial_t^2 + (d-1)\tanh t\, \partial_t + \frac{\Delta_{d-1}^{\mathrm{s}}}{\cosh^2 t} \\ &= -(w^2 + 1)\partial_w^2 - dw\, \partial_w + \frac{\Delta_{d-1}^{\mathrm{s}}}{w^2 + 1}. \end{aligned} \tag{1.3.24}$$

Finally, we perform an appropriate gauge transformation:

$$(\cosh t)^{\frac{d-1}{2}}\Box_d^{\mathrm{dS}}(\cosh t)^{-\frac{d-1}{2}} - \big(\tfrac{d-1}{2}\big)^2 = -\partial_t^2 - \frac{\big(\frac{d-2}{2}\big)^2 - \frac14 - \Delta_{d-1}^{\mathrm{s}}}{\cosh^2 t}. \tag{1.3.25}$$

Thus, the d'Alembertian on dS^d, restricted to $(d-1)$-dimensional spherical harmonics of order l, reduces to the deSitterian Gegenbauer Hamiltonian L_α^{dS} with

$$\alpha = \frac{d}{2} - 1 + l. \tag{1.3.26}$$

1.3.4 Sphere in double spherical coordinates

Consider the unit sphere of dimension $p+q-1$, with coordinates partitioned into two groups:

$$\mathbb{S}^{p+q-1} := \{(X,Y) \in \mathbb{R}^{p+q} \mid X_1^2 + \cdots + X_p^2 + Y_1^2 + \cdots + Y_q^2 = 1\}. \tag{1.3.27}$$

We also consider two spheres of dimensions $p-1$ and $q-1$:

$$\mathbb{S}^{p-1} = \{X \in \mathbb{R}^p \mid X_1^2 + \cdots + X_p^2 = 1\}, \tag{1.3.28}$$

$$\mathbb{S}^{q-1} = \{Y \in \mathbb{R}^q \mid Y_1^2 + \cdots + Y_q^2 = 1\}. \tag{1.3.29}$$

Then $\mathbb{S}^{p+q-1}$ is parametrized by $(\tau, \hat{X}, \hat{Y})$, with $0 \le \tau \le \frac{\pi}{2}$, $\hat{X} \in \mathbb{S}^{p-1}$, and $\hat{Y} \in \mathbb{S}^{q-1}$:

$$X_i = \sin\tau\, \hat{X}_i, \qquad i = 1, \dots, p; \qquad Y_j = \cos\tau\, \hat{Y}_j, \qquad j = 1, \dots, q. \tag{1.3.30}$$

Alternatively, one can use coordinates $(w, \hat{X}, \hat{Y})$ where

$$\sin^2\tau = w, \qquad \cos^2\tau = 1 - w. \tag{1.3.31}$$

We compute the line element and the Laplacian:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}w^2}{4w(1-w)} + w\,\mathrm{d}\hat{X}^2 + (1-w)\,\mathrm{d}\hat{Y}^2 \\ &= \mathrm{d}\tau^2 + \sin^2\tau\,\mathrm{d}\hat{X}^2 + \cos^2\tau\,\mathrm{d}\hat{Y}^2, \end{aligned} \tag{1.3.32}$$

$$\begin{aligned} \Delta_{p+q-1}^{\mathrm{s}} &= 4w(1-w)\partial_w^2 + 2\big(p(1-w) - qw\big)\partial_w + \frac{\Delta_{p-1}^{\mathrm{s}}}{w} + \frac{\Delta_{q-1}^{\mathrm{s}}}{1-w} \\ &= \partial_\tau^2 + \big((p-1)\cot\tau - (q-1)\tan\tau\big)\partial_\tau + \frac{\Delta_{p-1}^{\mathrm{s}}}{\sin^2\tau} + \frac{\Delta_{q-1}^{\mathrm{s}}}{\cos^2\tau}. \end{aligned} \tag{1.3.33}$$

We perform an appropriate gauge transformation:

$$\begin{aligned} &(\sin\tau)^{\frac{p-1}{2}}(\cos\tau)^{\frac{q-1}{2}}\big(-\Delta_{p+q-1}^{\mathrm{s}}\big)(\sin\tau)^{-\frac{p-1}{2}}(\cos\tau)^{-\frac{q-1}{2}} + \big(\tfrac{p+q-2}{2}\big)^2 \\ &= -\partial_\tau^2 + \frac{\big(\frac{p-2}{2}\big)^2 - \frac14 - \Delta_{p-1}^{\mathrm{s}}}{\sin^2\tau} + \frac{\big(\frac{q-2}{2}\big)^2 - \frac14 - \Delta_{q-1}^{\mathrm{s}}}{\cos^2\tau}. \end{aligned} \tag{1.3.34}$$

Finally, we make the substitution $\tau = \frac{r}{2}$:

$$Eqn.(1.3.34) = 4\left(-\partial_r^2 + \frac{\big(\frac{p-2}{2}\big)^2 - \frac14 - \Delta_{p-1}^{\mathrm{s}}}{4\sin^2\frac{r}{2}} + \frac{\big(\frac{q-2}{2}\big)^2 - \frac14 - \Delta_{q-1}^{\mathrm{s}}}{4\cos^2\frac{r}{2}}\right). \tag{1.3.35}$$

Thus, on products of spherical harmonics of orders j and l, we obtain the spherical hypergeometric Hamiltonian of the first kind $L_{\alpha,\beta}^{\mathrm{s}}$ with

$$\alpha = \frac{p}{2} - 1 + j, \qquad \beta = \frac{q}{2} - 1 + l. \tag{1.3.36}$$

1.3.5 Hyperboloid in double spherical coordinates

Consider the hyperboloid of signature $p-1, q$ embedded in the pseudo-Euclidean space of signature (p,q):

$$\mathbb{H}^{p-1,q} := \{(X,Y) \in \mathbb{R}^{p+q} \mid -X_1^2 - \cdots - X_p^2 + Y_1^2 + \cdots + Y_q^2 = -1\}. \tag{1.3.37}$$

Let $\mathbb{S}^{p-1}$ and $\mathbb{S}^{q-1}$ be as in *Eqn.*(1.3.29). Then $\mathbb{H}^{p-1,q}$ is parametrized by $(\tau, \hat{X}, \hat{Y})$, with $0 \leq \tau < \infty$, $\hat{X} \in \mathbb{S}^{p-1}$, and $\hat{Y} \in \mathbb{S}^{q-1}$:

$$X_i = \cosh\tau\,\hat{X}_i, \qquad i = 1,\dots,p; \qquad Y_j = \sinh\tau\,\hat{Y}_j, \qquad j = 1,\dots,q. \tag{1.3.38}$$

Alternatively, one can use coordinates $(w, \hat{X}, \hat{Y})$ where

$$\cosh^2\tau = w, \qquad \sinh^2\tau = w-1. \tag{1.3.39}$$

The line element and the pseudo-Laplacian in these coordinates are

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}w^2}{4w(w-1)} - w\,\mathrm{d}\hat{X}^2 + (w-1)\,\mathrm{d}\hat{Y}^2 \\ &= \mathrm{d}\tau^2 - \cosh^2\tau\,\mathrm{d}\hat{X}^2 + \sinh^2\tau\,\mathrm{d}\hat{Y}^2, \end{aligned} \tag{1.3.40}$$

$$\begin{aligned} \Delta_{p-1,q} &= 4w(w-1)\partial_w^2 + 2\big(p(w-1) + qw\big)\partial_w - \frac{\Delta^{\mathrm{s}}_{p-1}}{w} + \frac{\Delta^{\mathrm{s}}_{q-1}}{w-1} \\ &= \partial_\tau^2 + \big((p-1)\tanh\tau + (q-1)\coth\tau\big)\partial_\tau - \frac{\Delta^{\mathrm{s}}_{p-1}}{\cosh^2\tau} + \frac{\Delta^{\mathrm{s}}_{q-1}}{\sinh^2\tau}. \end{aligned} \tag{1.3.41}$$

We perform an appropriate gauge transformation:

$$\begin{aligned} &(\cosh\tau)^{\frac{p-1}{2}}(\sinh\tau)^{\frac{q-1}{2}}\big(-\Delta_{p-1,q}\big)(\cosh\tau)^{-\frac{p-1}{2}}(\sinh\tau)^{-\frac{q-1}{2}} - \big(\tfrac{p+q-2}{2}\big)^2 \\ &\quad = -\partial_\tau^2 - \frac{\big(\frac{p-2}{2}\big)^2 - \frac14 - \Delta^{\mathrm{s}}_{p-1}}{\cosh^2\tau} + \frac{\big(\frac{q-2}{2}\big)^2 - \frac14 - \Delta^{\mathrm{s}}_{q-1}}{\sinh^2\tau}. \end{aligned} \tag{1.3.42}$$

Finally, we substitute $\tau = \frac{r}{2}$:

$$\textit{Eqn.}(1.3.42) = 4\left(-\partial_r^2 - \frac{\big(\frac{p-2}{2}\big)^2 - \frac14 - \Delta^{\mathrm{s}}_{p-1}}{4\cosh^2\frac{r}{2}} + \frac{\big(\frac{q-2}{2}\big)^2 - \frac14 - \Delta^{\mathrm{s}}_{q-1}}{4\sinh^2\frac{r}{2}}\right). \tag{1.3.43}$$

Thus, on the product of spherical harmonics of orders j and l, we obtain the hyperbolic hypergeometric Hamiltonian of the first kind $L^{\mathrm{h}}_{\alpha,\beta}$ with

$$\alpha = \frac{q}{2} - 1 + l, \qquad \beta = \frac{p}{2} - 1 + j. \tag{1.3.44}$$

1.3.6 Complex manifolds

All the manifolds that we have used so far were real. In the next subsection, we will need a complex (analytic) manifold. Such manifolds have essentially the same formalism as real manifolds. Let us briefly sketch the main elements. For more details, see [LeB].

Suppose that a complex manifold is equipped with local complex coordinates $z = (z_1, \dots, z_d)$ and the holomorphic line element

$$\sum_{1\leq i,j\leq d} g_{ij}\mathrm{d}z_i\mathrm{d}z_j, \tag{1.3.45}$$

where g_{ij} is a complex symmetric invertible matrix. The corresponding complex Laplacian is defined by essentially the same formula as in the real case:

$$\Delta = \frac{1}{\sqrt{\det g}} \partial_{z_i} g^{ij} \sqrt{\det g} \partial_{z_j}. \tag{1.3.46}$$

(Note that there is no absolute value.)

Suppose that the manifold is equipped with a conjugation, which in these coordinates is given by $z_i \mapsto \overline{z}_i$. We then also have the anti-holomorphic line element

$$\sum_{1 \leq i,j \leq d} \overline{g}_{ij} \mathrm{d}\overline{z}_i \mathrm{d}\overline{z}_j, \tag{1.3.47}$$

and the corresponding conjugate Laplacian

$$\overline{\Delta} = \frac{1}{\sqrt{\det \overline{g}}} \partial_{\overline{z}_i} \overline{g}^{ij} \sqrt{\det \overline{g}} \partial_{\overline{z}_j}. \tag{1.3.48}$$

As our first example, consider the space $\mathbb{C}^{d+1}$ equipped with the line element

$$\mathrm{d}Z^2 = \mathrm{d}Z_0^2 + \mathrm{d}Z_1^2 + \cdots + \mathrm{d}Z_d^2. \tag{1.3.49}$$

The corresponding Laplacian is obviously

$$\Delta = \partial_{Z_0}^2 + \cdots + \partial_{Z_d}^2. \tag{1.3.50}$$

Note that our standard identification of $\mathbb{C}^{d+1}$ with $\mathbb{R}^{2(d+1)}$ will be

$$Z_i = \frac{1}{\sqrt{2}} (X_i + \mathbf{i} Y_i). \tag{1.3.51}$$

(and not $Z_i = X_i + \mathbf{i} Y_i$). Therefore,

$$\partial_{Z_i} = \frac{1}{\sqrt{2}} \big(\partial_{X_i} - \mathrm{i} \partial_{Y_i} \big), \tag{1.3.52}$$

(and not, as usual, $\partial_{Z_i} = \frac{1}{2} \big(\partial_{X_i} - \mathrm{i} \partial_{Y_i} \big)$). Clearly, with this definition, $\langle \mathrm{d}Z_i | \partial_{Z_j} \rangle = \delta_{ij}$.

Another example of a complex manifold is the unit sphere [HIU]

$$\mathbb{S}_{\mathbb{C}}^d := \{ Z \in \mathbb{C}^{d+1} \mid Z_0^2 + Z_1^2 + \cdots + Z_d^2 = 1 \}. \tag{1.3.53}$$

Introducing the complex polar coordinates

$$R := \sqrt{Z_0^2 + \cdots + Z_d^2}, \qquad \hat{Z}_i := \frac{Z_i}{R}, \tag{1.3.54}$$

we have the direct analogue of the formula from the real case:

$$\mathrm{d}Z^2 = \mathrm{d}R^2 + R^2 \mathrm{d}\hat{Z}^2, \tag{1.3.55}$$

where $\mathrm{d}\hat{Z}^2$ is the complex line element on $\mathbb{S}_{\mathbb{C}}^d$. The corresponding complex Laplacian is given by the same expressions as in the real case:

$$\begin{aligned} \Delta_{d,\mathbb{C}}^{\mathrm{s}} &= \sum_{0 \leq i < j \leq d} (Z_i \partial_{Z_j} - Z_j \partial_{Z_i})^2 \\ &= R^2 \Delta_{d+1,\mathbb{C}} - R^2 \partial_R^2 - d\, R\, \partial_R. \end{aligned} \tag{1.3.56}$$

1.3.7 Hyperboloid $\mathbb{H}^{p-1,p}$ in complex coordinates

Consider the hyperboloid $\mathbb{H}^{p-1,p}$ embedded in $\mathbb{R}^{2p}$, defined as in Subsection 1.3.5, with coordinates $X_i, Y_i \in \mathbb{R}$, $i = 1, \dots, p$. We identify $\mathbb{R}^{2p}$ with $\mathbb{C}^p$ as in *Eqn.*(1.3.51), so that we obtain two representations of $\mathbb{H}^{p-1,p}$, a real one and a complex one:

$$\begin{aligned}\mathbb{H}^{p-1,p} &= \{(X,Y) \in \mathbb{R}^{2p} \mid -Y_1^2 - \cdots - Y_p^2 + X_1^2 + \cdots + X_p^2 = -1\} \\ &= \{Z \in \mathbb{C}^p \mid Z_1^2 + \cdots + Z_p^2 + \overline{Z}_1^2 + \cdots + \overline{Z}_p^2 = 1\}.\end{aligned} \tag{1.3.57}$$

The real line element on $\mathbb{R}^{2p} = \mathbb{C}^p$ can be written as

$$\begin{aligned}-\mathrm{d}Y_1^2 - \cdots - \mathrm{d}Y_p^2 + \mathrm{d}X_1^2 + \cdots + \mathrm{d}X_p^2 &= \mathrm{d}Z_1^2 + \cdots + \mathrm{d}Z_p^2 + \mathrm{d}\overline{Z}_1^2 + \cdots + \mathrm{d}\overline{Z}_p^2 \\ &= \mathrm{d}R^2 + R^2\mathrm{d}\hat{Z}^2 + \mathrm{d}\overline{R}^2 + \overline{R}^2\mathrm{d}\overline{\hat{Z}}^2.\end{aligned} \tag{1.3.58}$$

Now on $\mathbb{H}^{p-1,p}$ we have $R^2 + \overline{R}^2 = 1$. Therefore

$$R^2 = \frac{1 + \mathrm{i}\sinh r}{2} \tag{1.3.59}$$

for a unique $r \in \mathbb{R}$. Thus we can parametrize $\mathbb{H}^{p-1,p}$ by $r \in \mathbb{R}$ and $\hat{Z} \in \mathbb{S}_{\mathbb{C}}^{p-1}$:

$$Z_i = \sqrt{\frac{1 + \mathrm{i}\sinh r}{2}}\hat{Z}_i, \tag{1.3.60}$$

where we take the principal branch of the square root. The real line element and the pseudo-Laplacian are

$$\mathrm{d}s^2 = -\frac{1}{4}\mathrm{d}r^2 + \frac{1 + \mathrm{i}\sinh r}{2}\mathrm{d}\hat{Z}^2 + \frac{1 - \mathrm{i}\sinh r}{2}\mathrm{d}\overline{\hat{Z}}^2, \tag{1.3.61}$$

$$\Delta_{p-1,p} = -4\partial_r^2 - 4(p-1)\tanh r\,\partial_r + \frac{2}{1 + \mathrm{i}\sinh r}\Delta_{p-1}^{\mathrm{s},\mathbb{C}} + \frac{2}{1 - \mathrm{i}\sinh r}\overline{\Delta}_{p-1}^{\mathrm{s},\mathbb{C}}. \tag{1.3.62}$$

We perform an appropriate gauge transformation:

$$\begin{aligned}&\big(\cosh r\big)^{\frac{p-1}{2}}\Delta_{p-1,p}\big(\cosh r\big)^{-\frac{p-1}{2}} - (p-1)^2 \\ &= 4\Bigg(-\partial_r^2 - \frac{-\Delta_{p-1}^{\mathrm{s},\mathbb{C}} + (\frac{p-1}{2})^2 - \frac14}{2(1 + \mathrm{i}\sinh r)} - \frac{-\overline{\Delta}_{p-1}^{\mathrm{s},\mathbb{C}} + (\frac{p-1}{2})^2 - \frac14}{2(1 - \mathrm{i}\sinh r)}\Bigg).\end{aligned} \tag{1.3.63}$$

Thus, on joint eigenvectors of $\Delta_{p-1}^{\mathrm{s},\mathbb{C}}$ and $\overline{\Delta}_{p-1}^{\mathrm{s},\mathbb{C}}$ of degrees l and j, respectively, we obtain the deSitterian hypergeometric Hamiltonian of the first kind $L_{\alpha,\beta}^{\mathrm{dS}}$ with

$$\alpha = \frac{p}{2} - 1 + l, \qquad \beta = \frac{p}{2} - 1 + j. \tag{1.3.64}$$

1.4 Historical remarks

One can roughly divide the mathematical literature related to the topic of this paper into two parts: algebraic and functional-analytic.

In algebraic papers, one considers differential expressions without a Hilbert space setting and without asking for self-adjointness or closedness. Functional-analytic papers treat differential operators as unbounded operators on a certain Hilbert space, usually self-adjoint and sometimes only closed.

Needless to say, the algebraic literature is vast. In fact, the hypergeometric equation is one of the most classical subjects in mathematics, with a history going back about three centuries. In this category, one should mention [CKS, DW], which contain an algebraic analysis of all 9 families of Schrödinger operators solvable in terms of the hypergeometric function.

One-dimensional Schrödinger operators are naturally a special case of Sturm–Liouville operators, whose history goes back to [Lio]. There exists a large contemporary literature on self-adjoint or closed realizations of Sturm–Liouville operators; see, for example, [GeZin, GTV, DuSch, EE]. In our paper we use mostly [DeGe], which is summarized in Section 2 of [DL].

Each Sturm–Liouville operator can appear in many equivalent forms, often with distinct names. By a unitary transformation, called the Liouville transformation, essentially each of them can be transformed into a Schrödinger operator, often called its *Liouville form* [Lio].

For instance, in the literature one often considers the Sturm–Liouville operator called the *Jacobi operator*,

$$-(1-x)^{-\alpha}(1+x)^{-\beta}\partial_x(1-x)^{\alpha+1}(1+x)^{\beta+1}\partial_x, \tag{1.4.1}$$

which acts on the Hilbert space $L^2\big(]-1,1[,(1-x)^{\alpha}(1+x)^{\beta}\big)$; see, for example, [GPLS, Koo]. Its eigenfunctions are the famous Jacobi polynomials. By a simple transformation, the Jacobi operator is unitarily equivalent to the trigonometric Pöschl–Teller Hamiltonian (which we call the spherical hypergeometric Hamiltonian of the first kind).

Another common differential operator is the Legendre operator

$$(1-x^2)\partial_x^2 - 2x\partial_x + \mu(\mu+1) - \frac{\alpha^2}{1-x^2}. \tag{1.4.2}$$

Acting on the Hilbert space $L^2\big(]-1,1[,\sqrt{1-x^2}\big)$, it is used in the study of spherical harmonics. It is also equivalent to the trigonometric Pöschl–Teller Hamiltonian with $\alpha = \beta$. Needless to say, spherical harmonics possess a very large literature.

The trigonometric Pöschl–Teller Hamiltonian itself is also often studied; see, for example, [FS].

The Scarf Hamiltonian with $\alpha = \beta$ (which we call the deSitterian Gegenbauer Hamiltonian) also has a large literature because of its special properties: for certain values of the parameters it is reflectionless.

An interesting review of various exactly solvable Schrödinger operators is contained in [Eve]. It does not, however, contain all 9 families that we consider.

To our knowledge, a complete analysis of all 9 families interpreted as closed operators, including formulas for the integral kernels of their resolvents and the computation of their spectra, seems to appear for the first time in the literature. The transmutation formulas described in our paper are probably new. They are analogous to the transmutation formulas for Hamiltonians related to the confluent equation [DL]. Another novelty of our paper is the identities *Eqn.*(0.1.19) and *Eqn.*(0.1.20). They are analogous to an identity for the Whittaker operators described in [DeRi].

Let us briefly outline the history of these Hamiltonians in physics. In the physics community, the study of Schrödinger operators exactly solvable in terms of hypergeometric functions started in the 1930s, when physicists studied the dynamics of biatomic and polyatomic molecules and sought exact solutions of the Schrödinger equation. The main reason these physicists considered such Hamiltonians was often the limitations of perturbation theory. Therefore, various exactly solvable Hamiltonians were proposed that fit the experimental data. This was the primary motivation behind the Hamiltonians proposed by Morse [Mor], Rosen–Morse [MoR], Eckart [Eck], and Manning–Rosen [MaR].

A second motivation for this line of research seems to have been the search for exact solutions of the Schrödinger equation. In the work of Eckart [Eck] and Rosen–Morse [MoR], it is clearly stated that these potentials are new exactly solvable potentials, although their main motivation was still the study of polyatomic molecules. It seems that for Pöschl and Teller [PT], the motivation leaned more toward the fact that their proposed potential was exactly solvable: they even referred to it as *exakt integrierbar*.

The case of the Scarf Hamiltonian *Eqn.*(0.1.12) is somewhat different. In the original paper, Scarf did not consider the Hamiltonian in *Eqn.*(1.6.5). We traced this name to [CKS], where the authors referred to it as the hyperbolic Scarf, or Scarf II. However, the origin of this terminology remains unclear to us.

The systematic study of the Schrödinger equation of hypergeometric type was initiated by Bose [Bos] and continued by Natanzon [Nat], Ginocchio [Gin], and Milson [Mil]. For a more systematic treatment and a detailed history of the topic, we refer to [DW]. In a separate line of research, the study of these potentials appeared in the factorization method of Infeld and Hull [HI], and later in the context of supersymmetric quantum mechanics and the so-called shape invariance [CKS, Cot].

Table 1 presents a comparison of the various names used in the literature and our suggested terminology.

1.5 Operator theory in the context of the spectral theory of the Schrödinger operator

In this section we provide the theoretical aspects of our study of the 1-dimensional Schrödinger operator with complex potential. This section helps us understand the functional analysis aspect of hypergeometric Hamiltonians. It is essential to notice that we have a peculiar case at hand. The theory is solvable. This is exciting since we can move away from extremely general results in the literature. At the same time, it is often more challenging, since the technical part is more demanding. Thus, this section is an attempt to make the abstract ideas more friendly. This subsection on operator theory is mainly based on the book by Kato [Kato]. The spectral theory of Schrödinger and Sturm–Liouville operators has been one of the most important chapters of mathematical physics, with many monographs and articles devoted to it [Eve]. However, the discussion of exactly solvable Schrödinger operators with complex potential somewhat escapes the general rules. The theory we use is developed in [DeRi] and [DeGe]; for a short survey see [DL].

Many of these materials might seem trivial to experienced readers, and therefore sound repetitive. However, due to the experience of the author, operator theory is not common knowledge among mathematicians. We hope this justifies the elementary definitions here. We closely follow [D3], and [Kato]. This is indeed a standard material covered in many textbooks such as [RS]

1.5.1 Short review on operator theory

Let $\mathcal{X}, \mathcal{Y}$ be complete normed vector spaces, *Banach spaces.* Then $\mathcal{X} \oplus \mathcal{Y}$ is also a Banach space equipped with the norm

$$\|(x, y)\|_1 := \|x\| + \|y\|, \tag{1.5.1}$$

for $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Theorem 1.5.1. *Let $A : \mathcal{X} \to \mathcal{Y}$ be an operator. Then the following conditions are equivalent:*

(1) $\mathrm{Gr\,A}$ *is closed in* $\mathcal{X} \oplus \mathcal{Y}$.

(2) *If* $x_n \to x$, $x_n \in \mathrm{D}om(A)$ *and* $Ax_n \to y$, *then* $y = Ax$.

(3) $\mathrm{D}om\, A$ *with the norm*

$$\|x\|_A = \|x\| + \|Ax\| \tag{1.5.2}$$

is a Banach space.

Proof. Note that $\mathrm{Gr\,A}$ is

$$\mathrm{Gr\,A} := \{(\mathrm{x}, \mathrm{Ax}) | \mathrm{x} \in \mathrm{Dom\,A}\}. \tag{1.5.3}$$

The graph being closed is exactly the second condition. Also, the third condition says precisely that $\mathrm{D}om(A)$ is complete with respect to the graph norm, which is equivalent to the graph being closed. □

Definition 1.5.2. *An operator $A : \mathcal{X} \to \mathcal{Y}$ is called a* closed operator *if* $\mathrm{Gr\,A}$ *is closed in* $\mathcal{X} \oplus \mathcal{Y}$.

Proposition 1.5.3. *If A is closed, then* $\mathrm{Ker}\, A$ *is also closed.*

Proof. Recall that

$$\mathrm{Ker}\, A := \{x \in \mathrm{D}om\, A | Ax = 0\}. \tag{1.5.4}$$

If $x_n \in \mathrm{Ker}\, A$ and $x_n \to x$, then $Ax_n = 0 \to 0$. Since A is closed, we obtain $x \in \mathrm{D}om\, A$ and $Ax = 0$. Hence $x \in \mathrm{Ker}\, A$, so $\mathrm{Ker}\, A$ is closed. □

Definition 1.5.4. *We say an operator $B : \mathcal{X} \to \mathcal{Y}$ is a* bounded operator *if*

$$\|Bx\| \leq \|x\|\|B\|, \tag{1.5.5}$$

where the norm of B, denoted by $\|B\|$, is

$$\|B\| := \sup_{x \in \mathrm{D}om\, B,\ \|x\|=1} \|Bx\|. \tag{1.5.6}$$

We denote the space of all bounded operators from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$, or sometimes just $\mathcal{B}$.

Theorem 1.5.5. Banach closed graph theorem. *A closed operator $A : \mathcal{X} \to \mathcal{Y}$ is a bounded operator if* $\mathrm{D}om\, A = \mathcal{X}$. *For a proof of this theorem see [Kato] page* 166.

Definition 1.5.6. *We say that an operator $A : \mathcal{X} \to \mathcal{Y}$ is invertible iff $A^{-1} \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$.*

Theorem 1.5.7. Closed range theorem. *Let $A : \mathcal{X} \to \mathcal{Y}$ be a closed operator, and suppose that for some $c > 0$,*

$$\|Ax\| \geq c\|x\|, \quad x \in \mathrm{D}om\, A. \tag{1.5.7}$$

Then $\mathrm{Ran}(A)$ *is closed. If* $\mathrm{Ran}(A) = \mathcal{Y}$, *then A is invertible and $\|A^{-1}\| \leq c^{-1}$.*

Proof. Let $y_n \in \mathrm{Ran}\, A$ and $y_n \to y$; we want to show that $y \in \mathrm{Ran}\, A$. Let $Ax_n = y_n$. We have

$$\|x_n - x_m\| \leq c^{-1}\|y_n - y_m\|. \tag{1.5.8}$$

Since y_n converges, x_n forms a Cauchy sequence. Hence there exists x such that $x_n \to x$. Since A is closed, we have $Ax = y$, which means that $\mathrm{Ran}\, A$ is closed. □

Proposition 1.5.8. *$A : \mathcal{X} \to \mathcal{Y}$ is a closed, bijective operator iff $A^{-1} \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$.*

Proof. By the closed range theorem, A being a closed and bijective operator implies that A^{-1} is bounded. Since A is closed and bijective, it means that $\mathrm{Gr\,A}$ is closed. Now $\mathrm{Gr\,A}^{-1}$ is just a swap of $\mathrm{Gr\,A}$, which means that A^{-1} is closed iff A is closed. □

Our main objective is to study the spectral theory of second-order differential operators. So we need to focus on spectral properties of operators.

1.5.2 Spectral theory

Definition 1.5.9. *Let A be an operator. The resolvent set of A is*

$$\rho(A) := \{z \in \mathbb{C} | A - z\mathbb{I}_{\mathrm{D}om\, A} \textit{ is invertible}\}, \tag{1.5.9}$$

where $\mathbb{I}_{\mathrm{D}om\, A}$ indicates the identity on the domain of A (we often drop the subscript).

Definition 1.5.10. *We define* the spectrum *of A as*

$$\sigma(A) := \mathbb{C} \setminus \rho(A). \tag{1.5.10}$$

We say that $x \in \mathcal{X}$ solves the eigenvalue problem if $Ax = \lambda x$, where $\lambda \in \mathbb{C}$. The set of eigenvalues is called the *point spectrum*, denoted by $\sigma_p(A)$.

We would like to have a cleaner definition for the set of eigenvalues in which we have isolated eigenvalues with finite multiplicities. By finite multiplicity we allow generalized eigenvalues with finite-dimensional generalized eigenspaces:

$$(A - \lambda\mathbb{I})^k x = 0, \tag{1.5.11}$$

such that

$$\dim \operatorname{Ker}(A - \lambda\mathbb{I})^k < \infty. \tag{1.5.12}$$

Lemma 1.5.11. *If $B^k x = 0$ for some positive integer k, then $k \leq \dim \operatorname{Ker} B$.*

Definition 1.5.12. *We define the* Riesz projection

$$P_\lambda = \frac{1}{2\pi \mathrm{i}} \oint_\gamma (z\mathbb{I} - A)^{-1}\, dz, \tag{1.5.13}$$

where γ is a simple curve encircling an open connected set $\Omega \subset \mathbb{C}$ such that $\sigma(A) \cap \overline{\Omega} = \{\lambda\}$, that is, the only intersection point of the spectrum with the closure of Ω is λ.

It is clear that the Riesz projection is a projection: $P_\lambda^2 = P_\lambda$.

Definition 1.5.13. *We say that a point λ is in the discrete spectrum of A, denoted by $\lambda \in \sigma_d(A)$, if*

(1) *λ is an isolated point in the point spectrum,*

(2) $\operatorname{rank} P_\lambda < \infty$, *where* $\operatorname{rank} P_\lambda = \dim \operatorname{Ran} P_\lambda$.

Lemma 1.5.14. *The equality* $\operatorname{rank} P_\lambda = \dim \operatorname{Ker}(A - \lambda\mathbb{I})^k$ *holds for $k < \infty$.*

Proof. Let $(A - \lambda\mathbb{I})^k x = 0$ for $x \in \operatorname{Ker}(A - \lambda\mathbb{I})^k$ and $k < \infty$. Then we can set $A = N + \lambda\mathbb{I}$ such that $N^k = 0$ on its domain.

$$(z\mathbb{I} - A)^{-1} P_\lambda = \sum_{j=0}^{k-1} \frac{N^j P_\lambda}{(z - \lambda)^{j+1}}. \tag{1.5.14}$$

It is clear that $\mathrm{Res}\, \mathrm{P}_\lambda \mathrm{x} = \mathrm{x}$ on $x \in \mathrm{D}om\, P_\lambda$. Also we have $N^k (z\mathbb{I} - A)^{-1} x = 0$, therefore

$$\mathrm{Ran} P_\lambda \subset \operatorname{Ker}(A - \lambda\mathbb{I})^k. \tag{1.5.15}$$

Note that this requires k to be finite. On the other hand, since $0 \in \mathrm{D}om P_\lambda$, we have $\operatorname{Ker}(A - \lambda\mathbb{I})^k \subset \mathrm{Ran} P_\lambda$. So we proved

$$\mathrm{Ran} P_\lambda = \operatorname{Ker}(A - \lambda\mathbb{I})^k. \tag{1.5.16}$$

In particular their dimensions match. □

We explained why the second condition 2 is equivalent to generalized eigenvalues with finite multiplicity. Moreover, it implies that the singularity of the resolvent is a pole and not an essential singularity at λ.

Here is a good point to introduce holomorphic families of operators. In this part we closely follow the book by Kato [Kato]. Here we often consider our operators to act on a Hilbert space, a Banach space equipped with a non-degenerate sesquilinear form, $(y, x) \in \mathbb{C}$, for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2^\star$.

Definition 1.5.15. *We say that an operator family $A(z) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a holomorphic family of bounded operators if for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$*

$$\Omega \ni z \mapsto (y, T(z)x) \in \mathbb{C}, \tag{1.5.17}$$

is a holomorphic function.

Example 1.5.16. *Let the resolvent operator*

$$R(z; A) = (A - z\mathbb{I})^{-1}, \tag{1.5.18}$$

be bounded. It is a holomorphic function in z, since $z \notin \sigma(A)$. Therefore, we say that $R(z; A)$ is a holomorphic family of bounded operators.

Remark 1.5.17. (1) *For us, the concept of holomorphicity is essentially similar whether we consider the strong or weak derivative of a function on an open set.*

(2) *We can also consider uniqueness of analytic continuation of a bounded operator similarly to the case of complex functions. Let $A_1(x_n) = A_2(x_n)$ for a sequence of points $x_n \in \Omega$ belonging to a connected open subset of $\Omega \subset \mathbb{C}$. Let $x_n \to x_0$ where $x_n \neq x_0 \in \Omega$, then $A_1(x) = A_2(x)$.*

(3) *Holomorphic families of operators and holomorphic extensions are often used interchangeably. This happens since one can consider a holomorphic family to be an extension of a real-holomorphic family of operators, where an operator can be expressed as a Taylor expansion.*

We have the following practical criterion for bounded holomorphic operators:

Theorem 1.5.18. *Suppose that $\Omega \ni z \mapsto H(z) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a function uniformly bounded on compact subsets of Ω. Suppose in addition that there exist dense subspaces $\mathcal{X}_1 \subset \mathcal{H}_1$, $\mathcal{X}_2 \subset \mathcal{H}_2$ such that for any $f \in \mathcal{X}_1$, $g \in \mathcal{X}_2$, the function*

$$\Omega \ni z \mapsto (f|H(z)g) \in \mathbb{C} \tag{1.5.19}$$

is holomorphic. Then $\Omega \ni z \mapsto H(z)$ is a holomorphic family of bounded operators.

The situation with unbounded operators is more subtle. There are various definitions of holomorphic families of unbounded operators, we show a set of unbounded operators from $\mathcal{H}_1 \to \mathcal{H}_2$ by $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ or sometimes $\mathcal{C}$. Here we use a natural concept of a holomorphic family of closed operators, due to Kato. It is, however, more subtle and less known. It is also known as type (B) [Kato].

Definition 1.5.19. *Consider a function*

$$\Omega \ni z \mapsto H(z) \in C(\mathcal{H}_1, \mathcal{H}_2). \tag{1.5.20}$$

We say that it is a holomorphic family of closed operators *if for each $z_0 \in \Omega$ there exists a neighborhood Ω_0 of z_0, a Hilbert space $\mathcal{K}$, and a holomorphic family of bounded operators $\Omega_0 \ni z \mapsto A(z) \in \mathcal{B}(\mathcal{K}, \mathcal{H}_1)$ that map bijectively $\mathcal{K}$ onto $\mathrm{Dom}(H(z))$ and*

$$\Omega_0 \ni z \mapsto H(z)A(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

We have the following practical criterion, which works for operators having non-empty resolvent sets:

Theorem 1.5.20. *Consider a function $\Omega \ni z \mapsto H(z) \in C(\mathcal{H})$. Suppose in addition that for any $z \in \Omega$ the resolvent set of $H(z)$ is non-empty. If for any $z_0 \in \Omega$ there exists $\lambda \in \mathbb{C}$ and a neighborhood Ω_0 of z_0 such that $\lambda \in \rho(H(z))$ for $z \in \Omega_0$ and $z \mapsto (H(z) - \lambda)^{-1} \in B(\mathcal{H})$ is holomorphic on Ω_0, then $z \mapsto H(z)$ is a holomorphic family of closed operators.*

Proof. We set $\mathcal{K} := \mathcal{H}$ and $A(z) := (H(z) - \lambda)^{-1}$, so that

$$H(z)A(z) = 1 + \lambda A(z). \tag{1.5.21}$$

□

We will apply this concept in the next section in our study of hypergeometric Hamiltonians.

Remark 1.5.21. *We can also consider uniqueness of analytic continuation of an unbounded operator similarly to the case of complex functions. Let $H_1(x_n) = H_2(x_n)$ for a sequence of points $x_n \in \Omega$ belonging to a connected open subset of $\Omega \subset \mathbb{C}$. Let $x_n \to x_0$ where $x_n \neq x_0 \in \Omega$, then $H_1(x) = H_2(x)$ if $H \in \mathcal{C}(\mathcal{H}, \mathcal{H})$ and the resolvent set of H is non-empty, $\rho(H) \neq \emptyset$.*

1.5.3 Boundary condition of the Schrödinger operator with complex potentials

Our goal is to realize certain 1-dimensional Schrödinger operators with complex potential as closed operators in the sense of $L^2]a, b[$. To do so, as explained in the previous section, we have to show that the spectrum is bounded. Our study relies heavily on the regular structure of these operators near the end-points. Notice that, since we are naturally interested in $L^2]a, b[$, the only singular points can appear at the end-points. Before going further into the abstract theory we pause momentarily to explain the idea in a simpler set-up. Consider that near the end-point a, which we set to $a = 0$ for simplicity, a singular potential behaves as a Bessel potential

$$V(x) \sim \left(m^2 - \frac{1}{4}\right) \frac{1}{x^2}. \tag{1.5.22}$$

The Schrödinger equation near the end-point can be solved by $|x|^{m+\frac{1}{2}}$ and $|x|^{\frac{1}{2}-m}$ for general m, and by $|x|$ and $\log x\,|x|$ for $m = 0$. For general m, these solutions belong to L^2 near the end-point 0 if $\operatorname{Re} m > -1$ and $\operatorname{Re} m < 1$ respectively. So for $\operatorname{Re} m > 1$ we have a unique operator via the solution $|x|^{\frac{1}{2}-m}$ near the end-point. This operator can be extended holomorphically to the larger domain $0 < \operatorname{Re} m \leq 1$ in the m-plane via a linear combination of $|x|^{m+\frac{1}{2}} + k|x|^{\frac{1}{2}-m}$ for $m \neq 0$ and via $|x|^{\frac{1}{2}} + \nu \log x|x|^{\frac{1}{2}}$ for $m = 0$. Thus the operator

$$H_m = -\frac{d^2}{dx^2} + \left(m^2 - \frac{1}{4}\right) \frac{1}{x^2}, \tag{1.5.23}$$

on the domain $L^2]0, b[$ is uniquely defined for $\operatorname{Re} m > 1$. We say that the index of the end-point is $\nu(0) = 0$. It can be extended to a 1-parameter holomorphic operator $H_{m,k}$ for $0 < \operatorname{Re} m \leq 1$. So the strip $0 < \operatorname{Re} m \leq 1$ is where one has a possibility to choose among the solutions, and we say that the index of the end-point is $\nu(0) = 2$. For some potentials the operator L initially defined on $C_c^\infty]a, b[$ possesses many closed realizations. These realizations $L_\bullet$ differ only by the behavior of elements of their domain near the endpoints—in other words, they differ by *boundary conditions.* The family $H_{m,k}$ is called Bessel type and has been studied in detail in [DeRi] and [DL].

Note that our study involves Schrödinger operators with complex potential. But this closely follows the case of real potential. For instance, for $m \in \mathbb{R}$, $m > 1$ is called limit circle and $0 < m \leq 1$ is called limit point [RS]. Since the hypergeometric Hamiltonians have complex potential, we have to implement the method of [DeGe], which we explain in a concise manner later.

We try to exploit the fact that our Hamiltonians are exactly solvable in terms of hypergeometric functions. We first summarize the behavior of hypergeometric Hamiltonians near the end-points.

1.5.4 Boundary conditions for hypergeometric Hamiltonians

The need for boundary conditions depends on the behavior of the potential V near the endpoints. Although we deal with closed operators, the theory is surprisingly similar to Weyl's theory [?].

Let us consider, for example, the endpoints a and b. There are three possibilities:

(1) There is a unique choice of square-integrable functions near a and a unique choice of square-integrable functions near b.

(2) There is a 1-parameter family of boundary conditions at one endpoint and only one choice at the other endpoint.

(3) There is a 1-parameter family of boundary conditions at a and a 1-parameter family of boundary conditions at b.

Analogous definitions are valid for the other endpoint a.

The 9 families considered in this paper illustrate Hamiltonians with various kinds of behaviors of the potential near endpoints. Let us list the behaviors encountered among these 9 families. We restrict ourselves to the right endpoint b; an analogous list applies to the left endpoint a.

1. **Short range potential.** $b = +\infty$ and $V(x)$ is integrable near $+\infty$ or constant plus integrable. Then $\nu_b(L) = 0$. Moreover, eigenfunctions in $\mathcal{D}(L_\bullet)$ with eigenvalue $-k^2 + V(+\infty)$, $\mathrm{Re}(k) > 0$, behave as e^{-kx}.

2. **Bessel type.** b is finite and
$$V(x) \sim \Big(m^2 - \frac{1}{4}\Big)\frac{1}{(x-b)^2}. \tag{1.5.24}$$
Then $\nu_b(L) = 2$ iff $|\mathrm{Re}\, m| < 1$, otherwise $\nu_b(L) = 0$. The behavior of elements of $\mathcal{D}(L_\bullet)$ near b is $|x-b|^{\frac{1}{2}+m}$ for $\mathrm{Re}\, m \geq 1$, a linear combination of $|x-b|^{\frac{1}{2}+m}$ and $|x-b|^{\frac{1}{2}-m}$ for $0 \leq \mathrm{Re}\, m < 1$, $m \neq 0$, and a linear combination of $|x-b|^{\frac{1}{2}}$ and $|x-b|^{\frac{1}{2}} \ln|x-b|$ for $m = 0$.

   The families of operators considered in this paper will always have only "homogeneous" or "basic" boundary conditions given by $|x-b|^{\frac{1}{2}+m}$, with $\mathrm{Re}\, m > -1$. Thus, in particular, for $|\mathrm{Re}\, m| < 1$, $m \neq 0$, each differential expression will have two closed realizations. We will not consider mixed boundary conditions, which are discussed e.g. in [GTV, DeGe, DeRi].

3. **Whittaker type.** b is finite and
$$V(x) \sim \left(m^2 - \frac{1}{4}\right)\frac{1}{(x-b)^2} - \frac{\beta}{|x-b|}. \tag{1.5.25}$$
The conditions are essentially the same as in the Bessel type, with one difference. For $\frac{1}{2} \leq \mathrm{Re}\, m < 1$, $(\beta, m) \neq \big(0, \frac{1}{2}\big)$, the behavior of functions in the domain of closed realizations of L are linear combinations of $|x-b|^{\frac{1}{2}+m}$ and
$$|x-b|^{\frac{1}{2}-m}\left(1 - \frac{\beta|x-b|}{1-2m}\right). \tag{1.5.26}$$
In the families of operators considered in this paper we will only consider "basic" boundary conditions given by
$$|x-b|^{\frac{1}{2}+m}\left(1 - \frac{\beta|x-b|}{1-2m}\right), \tag{1.5.27}$$
with $\mathrm{Re}\, m > -1$. See e.g. [DL, DeRi].

Table 1.1: Boundary behavior of the potentials.

Potential	Boundary type at a	Boundary type at b
Spherical of first kind	Bessel	Bessel
Hyperbolic of first kind	Bessel	Short range potential
DeSitterian of first kind	Short range potential	Short range potential
Spherical of second type	Whittaker	Whittaker
Hyperbolic of second type	Whittaker	Short range potential
DeSitterian of second type	Short range potential	Short range potential

1.5.5 1-dimensional Schrödinger operator: boundary condition

Here we review the theory of the 1-dimensional Schrödinger operator with complex potential developed in [DeGe]. In many ways this is a continuation of the well-studied method developed for Sturm–Liouville operators with boundary conditions [Eve]. The main difference is that the set-up we present is applicable to closed operators, whereas most of the literature is focused on self-adjoint operators [RS].

Let L be a 1-dimensional Schrödinger operator

$$L := -\frac{d^2}{dx^2} + V(x). \tag{1.5.28}$$

We study a closed realization of L, denoted by $L_\bullet$, on $L^2]a, b[$.

Definition 1.5.22. The maximal operator $L_{\max}$ *is*

$$\mathrm{D}om(L_{\max}) := \{f \in L^2]a, b[\cap \mathrm{AC}^1 | \mathrm{Lf} \in \mathrm{L}^2]\mathrm{a}, \mathrm{b}[\}, \tag{1.5.29}$$

$$L_{\max} := L|_{\mathrm{D}om(L_{\max})}. \tag{1.5.30}$$

where AC^1 *is the space of functions whose first derivative is absolutely continuous.*

We equip $\mathrm{D}om(L_{\max})$ with the graph norm

$$\|f\|_L^2 := \|f\|^2 + \|Lf\|^2. \tag{1.5.31}$$

Definition 1.5.23. The minimal operator $L_{\min}$ *is*

$$\mathrm{D}om(L_{\min}) := \{f \in \mathrm{AC}^1_\mathrm{c}]\mathrm{a}, \mathrm{b}[|\mathrm{Lf} \in \mathrm{L}^2]\mathrm{a}, \mathrm{b}[\}^{\mathrm{cl}}, \tag{1.5.32}$$

$$L_{\min} := L|_{\mathrm{D}om(L_{\min})}, \tag{1.5.33}$$

where AC^1_c *is the space of functions whose first derivative is compactly supported and absolutely continuous, and* cl *stands for the closure with respect to the operator norm.*

Note that the minimal operator contains elements which vanish near the end-points.

We denote by $\mathcal{N}(L - z)$ a null-space associated to the operator $L - z$:

$$\mathcal{N}(L - z) := \{f \in \mathrm{AC}^1 | (\mathrm{L} - \mathrm{z})\mathrm{f} = 0\}. \tag{1.5.34}$$

The null-space is 2-dimensional. We define the square-integrable subspace of the null-space around the endpoints by

$$\mathcal{U}_{a/b}(z) := \{\psi \in \mathcal{N}(L - z)_{z \to a/b} | \psi \in L^2]a, b[\}. \tag{1.5.35}$$

We set the index of the endpoint, $\nu_{a/b}$, to be

$$\begin{cases} \dim \mathcal{U}_a = 2 \iff \nu_a = 2, \\ \dim \mathcal{U}_a \leq 1 \iff \nu_a = 0. \end{cases} \tag{1.5.36}$$

A similar definition holds for ν_b. Note that if $\dim \mathcal{U}_\bullet(z) = 0$, the operator has an empty resolvent set. If an operator does not have a non-empty resolvent set, we say that the operator is not well-posed. Note that one cannot see the fact that some operators are ill-defined just from their algebraic form. This is indeed a feature coming from the rigorous construction of operators on Hilbert spaces.

Now, note that the space $\mathrm{D}om(L_{\max})$ can only be different from $\mathrm{D}om(L_{\min})$ if there are 2 independent solutions at the endpoints. Therefore it is easy to see that

$$\dim\, L_{\max}/L_{\min} = \nu_a + \nu_b. \tag{1.5.37}$$

We say that an operator is a closed operator contained in $L_{\max}$ if $\mathrm{D}om(L_\bullet)$ is closed in the operator norm, and we set

$$L_\bullet := L_{\max}|_{\mathrm{D}om(L_\bullet)}. \tag{1.5.38}$$

In addition we require the realization to satisfy

$$\mathrm{D}om(L_{\min}) \subset \mathrm{D}om(L_\bullet) \subset \mathrm{D}om(L_{\max}). \tag{1.5.39}$$

Moreover, as it was shown in [DeGe],

$$\dim\, L_{\max}/L_\bullet = \frac{1}{2}(\nu_a + \nu_b). \tag{1.5.40}$$

There are only three possibilities for a closed realization:

(1) $\frac{1}{2}(\nu_a + \nu_b) = 0$; in this case the closed realization is unique and $L_\bullet = L_{\max}$.

(2) $\frac{1}{2}(\nu_a + \nu_b) = 1$. In this case, if $\nu_b = 0$ we have to choose a boundary condition at a, and if $\nu_a = 0$ we have to choose a boundary condition at b.

(3) $\frac{1}{2}(\nu_a + \nu_b) = 2$. We have to choose a boundary condition at both end-points. In our work we only consider separate boundary conditions.

Similar to the story of the *limit circle* case of Weyl [We], when the closed realization $L_\bullet$ is not uniquely defined one has to pick a boundary condition. Note that the deficiency method [vN] is not applicable in the case of complex potential for obvious reasons.

Now we have to take into account the boundary condition.

Definition 1.5.24. *The* Wronskian *of two differentiable functions f, g is*

$$\mathcal{W}(f,g)(x) := (g'f - fg')(x). \tag{1.5.41}$$

Lemma 1.5.25. *$f, g \in \mathrm{AC}^1$ are linearly independent iff $\mathcal{W}(f,g) \neq 0$.*

Theorem 1.5.26. *For $f, g \in \mathrm{D}om(L_{\max})$ we have*

$$(Lfg - fLg)(x) = \mathcal{W}'(g,f)(x). \tag{1.5.42}$$

Moreover, if $f, g \in \mathcal{N}(L)$ then $\mathcal{W}'(f,g)(x) = 0$.

Definition 1.5.27. *A boundary functional is any functional on $\mathrm{D}om(L_{\max})$ such that it vanishes on $\mathrm{D}om(L_{\min})$. A boundary functional at a (b) is any functional on $\mathrm{D}om(L_{\max})$ such that for $f \in \mathrm{D}om(L_{\min})$, $\phi(f)(x) = 0$ near a (b).*

The set of all boundary functionals has dimension

$$\dim\left(\mathrm{D}om(L_{\max})/\mathrm{D}om(L_{\min})\right)' = \nu_a + \nu_b. \tag{1.5.43}$$

Definition 1.5.28. *For each $f \in \mathrm{D}om(L_{\max})$, we introduce a functional $F_x : \mathrm{D}om(L_{\max}) \to \mathbb{C}$ by*

$$F_x(g) := \mathcal{W}(f,g)(x). \tag{1.5.44}$$

Lemma 1.5.29. *The functional F_a is a well-defined functional, where a is an endpoint.*

Proof. For $f, g \in \mathrm{D}om(L_{\max})$ and $a < a_1 < b_1 < b$, using 1.5.26 we have

$$\int_{a_1}^{b_1} (Lfg - fLg)(x)\,dx = \mathcal{W}(g,f)(b_1) - \mathcal{W}(g,f)(a_1). \tag{1.5.45}$$

The l.h.s. converges when $a_1 \to a$ and $b_1 \to b$; therefore the Wronskian is well-defined at the endpoints. □

To deal with the problem of boundary conditions we have to address the cases 2 and 3 among the three possibilities. For the case 2:

If $\nu_a = 2$ and $\nu_b = 0$, we pick a functional F_a, such that $0 \neq f \in \mathrm{D}om(L_{\max})$ near a but $f \notin \mathrm{D}om(L_{\min})$. A good choice is an element of $\mathcal{U}_a$. In this case the domain corresponding to F_a is

$$\mathrm{D}om(L_\bullet) = \{g \in \mathrm{D}om(L_{\max}) | F_a(g) = 0\}. \tag{1.5.46}$$

A similar construction works for $\nu_b = 2$ and $\nu_a = 0$.

The third case 3 can be handled in a similar way. We fix boundary functionals F_a and F_b and set

$$\mathrm{D}om(L_\bullet) = \{g \in \mathrm{D}om(L_{\max}) | F_a(g) = F_b(g) = 0\}, \tag{1.5.47}$$

where F_a is a boundary condition at a and F_b is a boundary condition at b. This is called separated boundary conditions. In this work we do not consider mixed boundary conditions.

1.5.6 Green function of closed operators

When $\dim\mathcal{U}_a = \dim\mathcal{U}_b = 1$, we have the unique closed realization of L. In this case we do not need to fix a boundary condition. On top of that we can evaluate the resolvent and prove its boundedness. To do so we need to briefly review the theory of Green functions for the 1-dimensional Schrödinger operator. The Green function, for us, is an integral kernel of the right inverse of L.

Definition 1.5.30. *An operator $R_\bullet : L^1_c]a,b[\to \mathrm{AC}^1]\mathrm{a,b}[$ is called a (right) Green operator if*

$$LR_\bullet g = g. \tag{1.5.48}$$

One can define various species of Green functions. This is indeed closely related to the various Green functions for the Klein–Gordon equation. Here we only define the so-called two-sided Green function, and we call it the Green function for simplicity. For a discussion and definitions of other types and algebraic relations see [DeGe].

Definition 1.5.31. *The algebraic Green function associated to a 1-dimensional Schrödinger operator $L_\bullet$ is*

$$R_\bullet(z;x,y) := \frac{1}{\mathcal{W}(u_b,u_a)} \begin{cases} u_a(z;x)u_b(z;y), & a < x < y < b, \\ u_a(z;y)u_b(z;x), & a < y < x < b, \end{cases} \tag{1.5.49}$$

where $u_a \in \mathcal{U}_a(z)$ and $u_b \in \mathcal{U}_b(z)$, which immediately implies that $\mathcal{W}'(u_a,u_b)(x) = 0$; thus we denote the Wronskian by a constant.

It is an easy algebraic exercise to check that $LR_\bullet g = g$, for $f \in \mathrm{D}om(L_{\max})$. It is also common to call *Eqn.*(1.5.49) a candidate for the resolvent.

Theorem 1.5.32. *We say that an algebraic Green function is the resolvent of the unique closed operator $L_\bullet$ on $L^2]a, b[$ if one of the following equivalent conditions holds:*

(1) $\dim \mathcal{U}_a = \dim \mathcal{U}_b = 1$, *and $R_\bullet(z; x, y)$ is bounded.*

(2) $z \in \rho(L_\bullet)$.

Boundedness of a resolvent

To prove the boundedness of the resolvent we often use the following lemmas.

Lemma 1.5.33. *The Hilbert–Schmidt norm of an operator K is defined by*

$$\|K\|_2 = \sqrt{\mathrm{Tr} K^\star K} = \left(\int_a^b |K(x, y)|^2 dx\, dy \right)^{\frac{1}{2}}, \tag{1.5.50}$$

Then we have $\|K\| \leq \|K\|_2$.

Lemma 1.5.34. Schur's criterion *states that if*

$$\sup_{x \in]a,b[} \int_a^b K(x, y)\, dy = c_1, \qquad \sup_{y \in]a,b[} \int_a^b K(x, y)\, dx = c_2, \tag{1.5.51}$$

then

$$\|K\| \leq \sqrt{c_1 c_2}. \tag{1.5.52}$$

Lemma 1.5.35. Young's inequality. *If the integral kernel is translation invariant, $K(x, y) = K(x - y)$, then Schur's criterion simplifies to*

$$\|K\| \leq \int |K(x)| dx. \tag{1.5.53}$$

We can decompose the algebraic Green function as

$$R(x, y) = R_{++} + R_{+-} + R_{-+} + R_{--}, \tag{1.5.54}$$

where for $a < c < b$, we define

$$R_{\pm+}(x, y) = \theta\,(x \mp c)\,\theta(y - c)\,R(x, y), \tag{1.5.55}$$
$$R_{\pm-}(x, y) = \theta\,(x \mp c)\,\theta(y + c)\,R(x, y). \tag{1.5.56}$$

Lemma 1.5.36. *The operators R_{+-} and R_{-+} are bounded and the estimate is*

$$\|R_{\pm\mp}\| \leq C \int_a^c |\psi_a|^2 dx \int_c^b |\psi_b|^2 dx. \tag{1.5.57}$$

Corollary 1.5.37. *An integral kernel $R(x, y)$ is bounded if and only if R_{++} and R_{--} are bounded.*

Before concluding the discussion, we clarify the operator-theoretic construction of transmutation identities.

1.5.7 Transmutation identities as coordinate transformations of a Green function

Let $\psi : r \mapsto q$ be a 1-dimensional diffeomorphism. We implement a unitary implementation of ψ,

$$U_\psi : L^2(q) \to L^2(r), \tag{1.5.58}$$

This implies that

$$(U_\psi f)(r) = |J_\psi(r)|^{\frac{1}{2}} f(\psi(r)), \tag{1.5.59}$$

where the 1-dimensional Jacobian is

$$J_\psi(r) = \psi'(r). \tag{1.5.60}$$

We say that two operators

$$G_r : L^2(r) \to L^2(r), \qquad G_q : L^2(q) \to L^2(q), \tag{1.5.61}$$

are unitarily equivalent if

$$G_r = U_\psi^* G_q U_\psi. \tag{1.5.62}$$

Lemma 1.5.38. *Let U_ψ be the unitary implementation of the coordinate transformation $\psi : \mathbb{R} \to \mathbb{R}$. Then the integral kernel of $U_\psi^* G_q U_\psi$ is given by*

$$G_r(r, r') = |J_\psi(r)|^{\frac{1}{2}}\, G_q\big(\psi(r), \psi(r')\big)\, |J_\psi(r')|^{\frac{1}{2}}. \tag{1.5.63}$$

Proof. Let $f \in L^2(r)$. Then

$$(U_\psi f)(q) = |J_\psi(\psi^{-1}(q))|^{-\frac{1}{2}} f(\psi^{-1}(q)). \tag{1.5.64}$$

Hence

$$(G_q U_\psi f)(q) = \int G_q(q, q')\, |J_\psi(\psi^{-1}(q'))|^{-\frac{1}{2}} f(\psi^{-1}(q'))\, dq'. \tag{1.5.65}$$

The adjoint operator is given by

$$(U_\psi^* g)(q) = |J_\psi(q)|^{\frac{1}{2}} g(\psi(q)). \tag{1.5.66}$$

Now we change variables $q' = \psi(r')$, so that

$$dq' = |J_\psi(r')|\, dr'. \tag{1.5.67}$$

This proves that the integral kernel of $U_\psi^* G_q U_\psi$ is

$$G_r(r, r') = |J_\psi(r)|^{\frac{1}{2}}\, G_q\big(\psi(r), \psi(r')\big)\, |J_\psi(r')|^{\frac{1}{2}}. \tag{1.5.68}$$

□

In the literature, the name "transmutation" is sometimes used for identities of the form

$$A = VBV^{-1}, \tag{1.5.69}$$

where both A and B are differential operators; see, e.g., [Car, SS]. Note that our transmutation identities are not of this form, and in fact they seem to represent a novel contribution. Our transmutation identities connect the integral kernels of Green functions to one another. Indeed, they cannot be realized as similarity transformations of differential operators. This is the consequence of calculation.

1.6 Hypergeoemtric Hamiltonians as closed operators

In this section we study in details 3×3 families of Hamiltonian related to the Hypergeoemtric equation. In each case we solve the eigenvalue problem, then we write a theorem finding the Green function, resolvent, consequently obtaining the spectrum and the domain of uniqueness and holomorphicity.

1.6.1 Gegenbauer Hamiltonians

Let us recall the Gegenbauer Hamiltonians:

(1) Spherical Gegenbauer Hamiltonian

$$L_\alpha^{\mathrm{s}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\sin^2 r}. \tag{1.6.1}$$

Its corresponding Schrödinger operator acting on $L^2]0,\pi[$ is

$$\left(L_\alpha^{\mathrm{s}} - \lambda^2\right)\phi(r) = 0, \tag{1.6.2}$$

(2) Hyperbolic Gegenbauer Hamiltonian

$$L_\alpha^{\mathrm{h}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\sinh^2 r}. \tag{1.6.3}$$

Its corresponding Schrödinger operator acting on $L^2(\mathbb{R}_+)$ is

$$\left(L_\alpha^{\mathrm{h}} + \lambda^2\right)\phi(r) = 0, \tag{1.6.4}$$

(3) Desitterian Gegenbauer Hamiltonian

$$L_\alpha^{\mathrm{dS}} := -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}. \tag{1.6.5}$$

Its corresponding Schödinger operator acting on $L^2(\mathbb{R})$ is

$$\left(L_\alpha^{\mathrm{dS}} + \lambda^2\right)\phi(r) = 0. \tag{1.6.6}$$

In the following we solve the eigenvalue problem in each case and present the main theorems.

Spherical case

Let us define the function on $]0,\pi[$

$$\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r) \;:=\; \left(\frac{\sin r}{2}\right)^{\alpha+\frac{1}{2}}\mathbf{S}_{\alpha,\lambda}(\cos r). \tag{1.6.7}$$

It has the following asymptotic behaviour near 0:

$$\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r) \sim \frac{1}{\Gamma(1+\alpha)}\left(\frac{r}{2}\right)^{\frac{1}{2}+\alpha}. \tag{1.6.8}$$

The following four functions solve the eigenequation *Eqn.*(1.6.2):

$$\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r),\quad \mathcal{P}_{-\alpha,\lambda}^{\mathrm{s}}(r),\quad \mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(\pi-r),\quad \mathcal{P}_{-\alpha,\lambda}^{\mathrm{s}}(\pi-r). \tag{1.6.9}$$

The following symmetries hold

$$\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r) = \mathcal{P}_{\alpha,-\lambda}^{\mathrm{s}}(r), \qquad \mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(\pi-r) = \mathcal{P}_{\alpha,-\lambda}^{\mathrm{s}}(\pi-r). \tag{1.6.10}$$

The following theorem describes the basic family of closed realizations of L_α^{s} on $L^2]0,\pi[$.

Theorem 1.6.1. *For* $\operatorname{Re}\alpha \geq 1$ *there exists a unique closed operator* L_α^{s} *in the sense of* $L^2]0,\pi[$*, which on* $C_\mathrm{c}^\infty]0,\pi[$ *is given by Eqn.*(1.6.1)*. The family* $\alpha \mapsto L_\alpha^{\mathrm{s}}$ *is holomorphic and possesses a unique holomorphic extension to* $\operatorname{Re}\alpha > -1$*. It has only discrete spectrum:*

$$\sigma(L_\alpha^{\mathrm{s}}) = \sigma_{\mathrm{d}}(L_\alpha^{\mathrm{s}}) = \left\{ \left(k+\alpha\right)^2 \ : \ k \in \mathbb{N}_0 + \frac{1}{2} \right\}. \tag{1.6.11}$$

Outside of the spectrum its resolvent is

$$\begin{aligned} \frac{1}{(L_\alpha^{\mathrm{s}} - \lambda^2)}(x,y) =& \Gamma\left(\alpha - \lambda + \frac{1}{2}\right)\Gamma\left(\alpha + \lambda + \frac{1}{2}\right) \\ & \times \begin{cases} \mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(x)\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(\pi - y), & \text{if} \quad 0 < x < y < \pi; \\ \mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(y)\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(\pi - x), & \text{if} \quad 0 < y < x < \pi. \end{cases} \end{aligned} \tag{1.6.12}$$

Proof. Treating $\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r)$ and $\mathcal{P}_{-\alpha,\lambda}^{\mathrm{s}}(r)$ as a basis of solutions of *Eqn.*(1.6.2), we obtain a connection formula found using *Eqn.*(1.1.94):

$$\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(\pi - r) = -\frac{\cos\pi\lambda}{\sin\pi\alpha}\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r) + \frac{\pi}{\sin\pi\alpha}\frac{1}{\Gamma(\frac{1}{2}+\alpha+\lambda)\Gamma(\frac{1}{2}+\alpha-\lambda)}\mathcal{P}_{-\alpha,\lambda}^{\mathrm{s}}(r). \tag{1.6.13}$$

From

$$\mathcal{P}_{-\alpha,\lambda}^{\mathrm{s}}(r) \sim \frac{1}{\Gamma(1-\alpha)}\left(\frac{r}{2}\right)^{\frac{1}{2}-\alpha}, \tag{1.6.14}$$

and *Eqn.*(1.6.8) we obtain

$$\mathcal{W}(\mathcal{P}_{-\alpha,\lambda}^{\mathrm{s}}, \mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}) = \frac{\sin\pi\alpha}{\pi}. \tag{1.6.15}$$

This yields the Wronskians

$$\mathcal{W}\left(\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(\pi - r), \mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r)\right) = \frac{1}{\Gamma\left(\alpha - \lambda + \frac{1}{2}\right)\Gamma\left(\alpha + \lambda + \frac{1}{2}\right)} \tag{1.6.16}$$

$$\mathcal{W}\left(\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(\pi - r), \mathcal{P}_{-\alpha,\lambda}^{\mathrm{s}}(r)\right) = \frac{\cos(\pi\lambda)}{\pi}. \tag{1.6.17}$$

Thus *Eqn.*(1.6.12) is an integral kernel of the form *Eqn.*(1.5.49). Let us denote it by $R_\alpha^{\mathrm{s}}(-\lambda^2; x, y)$.

Let us fix temporarily λ. For $\operatorname{Re}\alpha > -1$ the integral kernel $R_\alpha^{\mathrm{s}}(-\lambda^2; x, y)$ is square integrable, and hence it defines a Hilbert-Schmidt operator, which we denote $R_\alpha^{\mathrm{s}}(-\lambda^2)$. By Sec. 1.5.6 there exists an operator L_α^{s}, which is a closed realization of the left hand side of *Eqn.*(1.6.1) such that $R_\alpha^{\mathrm{s}}(-\lambda^2) = (L_\alpha^{\mathrm{s}} + \lambda^2)^{-1}$.

The Hilbert-Schmidt norm of $R_\alpha^{\mathrm{s}}(-\lambda^2)$ is uniformly bounded for $\alpha \in K$ with $K \subset \{\operatorname{Re}\alpha > -1\}$ compact. Besides, $R_\alpha^{\mathrm{s}}(-\lambda^2; x, y)$ for any x, y depends analytically on α. Hence for any $f, g \in C_\mathrm{c}(]0,\pi[)$

$$\alpha \mapsto \int \overline{f(x)} R_\alpha^{\mathrm{s}}(-\lambda^2; x, y) g(y) \mathrm{d}x\mathrm{d}y \tag{1.6.18}$$

is analytic. Using Thm 1.5.18 we obtain that $\{\operatorname{Re}\alpha > 0\} \ni \alpha \mapsto R_\alpha^{\mathrm{s}}(-\lambda^2) \in B\big(L^2]0,\pi[\big)$ is an analytic family of bounded operators. Hence, by Thm 1.5.20, $\{\operatorname{Re}\alpha > 0\} \ni \alpha \mapsto L_\alpha^{\mathrm{s}} \in C\big(L^2]0,\pi[\big)$ is an analytic family of closed operators.

For $\operatorname{Re}\alpha \geq 1$, $r^{\frac{1}{2}-\alpha}$ is not L^2-integrable, and only $r^{\frac{1}{2}+\alpha}$ is. Hence for such α the right hand side of *Eqn.*(1.6.1) possesses a unique closed realization, which has to coincide with the operator L_α^{s} that we have just constructed. In particular, L_α^{s} does not depend on λ. By the uniqueness of analytic continuation, L_α^{s} does not depend on the choice of λ for all $\operatorname{Re}\alpha > -1$.

The singularities of the Gamma function yield the discrete spectrum of L_α^{s}. □

Hyperbolic case

Let us define functions on $\mathbb{R}_+$

$$\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r) := \Big(\frac{\sinh r}{2}\Big)^{\alpha+\frac12}\ \mathbf{S}_{\alpha,\lambda}(\cosh r), \tag{1.6.19}$$

$$\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r) := \frac{(\sinh r)^{\alpha+\frac12}}{2^{\lambda}}\ \mathbf{Z}_{\alpha,\lambda}(\cosh r). \tag{1.6.20}$$

They have the following asymptotic behavior:

$$\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r) \sim \frac{1}{\Gamma(1+\alpha)}\Big(\frac{r}{2}\Big)^{\frac12+\alpha}, \quad r \sim 0; \tag{1.6.21}$$

$$\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r) \sim \frac{1}{\Gamma(1+\lambda)}\mathrm{e}^{-\lambda r}, \quad r \to +\infty. \tag{1.6.22}$$

The following four functions solve the eigenequation (1.6.4):

$$\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r), \quad \mathcal{P}^{\mathrm{h}}_{-\alpha,\lambda}(r), \quad \mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r), \quad \mathcal{Q}^{\mathrm{h}}_{\alpha,-\lambda}(r). \tag{1.6.23}$$

The following symmetries are obvious:

$$\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r) = \mathcal{P}^{\mathrm{h}}_{\alpha,-\lambda}(r), \qquad \mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r) = \mathcal{Q}^{\mathrm{h}}_{-\alpha,\lambda}(r). \tag{1.6.24}$$

The following theorem describes the basic family of closed realizations of L^{h}_{α} on $L^2[0,\infty[$.

Theorem 1.6.2. *For* $\operatorname{Re}\alpha \geq 1$ *there exists a unique closed operator* L^{h}_{α} *in the sense of* $L^2(\mathbb{R}_+)$*, which on* $C_{\mathrm{c}}^{\infty}(\mathbb{R}_+)$ *is given by Eqn.*(1.6.3)*. The family* $\alpha \mapsto L^{\mathrm{h}}_{\alpha}$ *is holomorphic and possesses a unique holomorphic extension to* $\operatorname{Re}\alpha > -1$*. Here is its discrete spectrum and spectrum:*

$$\sigma_{\mathrm{d}}(L^{\mathrm{h}}_{\alpha}) = \Big\{-\Big(\frac12+\alpha\Big)^2\Big\}, \quad -1 < \operatorname{Re}\alpha < -\frac12; \tag{1.6.25}$$

$$\sigma_{\mathrm{d}}(L^{\mathrm{h}}_{\alpha}) = \emptyset, \quad -\frac12 \leq \operatorname{Re}\alpha; \tag{1.6.26}$$

$$\sigma(L^{\mathrm{h}}_{\alpha}) = [0,+\infty[\cup\sigma_{\mathrm{d}}(L^{\mathrm{h}}_{\alpha}). \tag{1.6.27}$$

Outside of the spectrum, for $\operatorname{Re}\lambda > 0$*, its resolvent is*

$$\begin{aligned}\frac{1}{(L^{\mathrm{h}}_{\alpha}+\lambda^2)}(x,y) =& \frac{\Gamma\big(\frac12+\alpha+\lambda\big)}{\sqrt{\pi}} \\ &\times\begin{cases}\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(x)\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(y) & \text{if } \ 0<x<y<\infty;\\ \mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(x)\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(y) & \text{if } \ 0<y<x<\infty.\end{cases}\end{aligned} \tag{1.6.28}$$

Proof. Consider $\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r)$ and $\mathcal{P}^{\mathrm{h}}_{-\alpha,\lambda}(r)$ as a basis of solutions of *Eqn.*(1.6.4). The connection formula is

$$\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r) = -\frac{\sqrt{\pi}}{\sin\pi\alpha\ \Gamma\big(\frac12-\alpha+\lambda\big)}\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r) + \frac{\sqrt{\pi}}{\sin\pi\alpha\ \Gamma\big(\frac12+\alpha+\lambda\big)}\mathcal{P}^{\mathrm{h}}_{-\alpha,\lambda}(r). \tag{1.6.29}$$

Similarly as in the spherical case, we obtain

$$\mathcal{W}(\mathcal{P}^{\mathrm{h}}_{-\alpha,\lambda}, \mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}) = \frac{\sin\alpha}{\pi}. \tag{1.6.30}$$

This yields the Wronskians

$$\mathcal{W}\left(\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r),\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r)\right)=\frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+\alpha+\lambda\right)}.$$

For $\operatorname{Re}\alpha>-1$ and $\operatorname{Re}\lambda>0$ the functions $\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r)$ resp. $\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r)$, are square integrable at the endpoints. Using the Schur test we check that under these conditions the integral kernel 1.6.28 defines a bounded operator, depending analytically on α.

For $\operatorname{Re}\alpha\geq 1$, $r^{\frac{1}{2}-\alpha}$ is not square integrable near 0. Therefore, for such α *Eqn.*(1.6.3) possesses a unique closed realization.

Looking for singularities of the Gamma function we find the discrete spectrum:

$$\sigma_{\mathrm{d}}(L^{\mathrm{h}}_{\alpha})=\left\{-\left(n+\frac{1}{2}+\alpha\right)^2\middle|n\in\mathbb{N}_0,\quad\operatorname{Re}\left(n+\frac{1}{2}+\alpha\right)<0\right\}.\tag{1.6.31}$$

It is easy to see that this coincides with *Eqn.*(1.6.26) □

The following theorem describes all closed realizations of L^{dS}_{α} on $L^2(\mathbb{R})$.

For $r\geq 0$ we introduce the following function which solves the eigenequation

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(r):=\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\frac{1}{2}+\alpha+\lambda)}\frac{(\cosh r)^{\alpha+\frac{1}{2}}}{2^{\lambda}}\ \mathbf{Z}_{\alpha,\lambda}(-\mathrm{i}\sinh r).\tag{1.6.32}$$

We extend it to $r\leq 0$ by analytic continuation. Here is its asymptotics:

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(r)\sim\frac{1}{\Gamma(\lambda+1)}\mathrm{e}^{-\lambda r},\quad r\to+\infty.\tag{1.6.33}$$

Thus the following functions solve the eigenequation (1.6.6):

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(r),\qquad\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-r),\qquad\mathcal{Q}^{\mathrm{dS}}_{\alpha,-\lambda}(r),\qquad\mathcal{Q}^{\mathrm{dS}}_{\alpha,-\lambda}(-r).\tag{1.6.34}$$

Let us describe closed realizations of L^{dS}_{α} on $L^2(\mathbb{R})$:

Theorem 1.6.3. *For any $\alpha\in\mathbb{C}$ there exists a unique closed operator L^{dS}_{α} in the sense of $L^2(\mathbb{R})$ that on $C_{\mathrm{c}}^{\infty}(\mathbb{R})$ is given by Eqn.*(1.6.5)*. The function $\mathbb{C}\ni\alpha\mapsto L^{\mathrm{dS}}_{\alpha}$ is holomorphic. It satisfies $L^{\mathrm{dS}}_{\alpha}=L^{\mathrm{dS}}_{-\alpha}$. Outside of the spectrum, for $\operatorname{Re}\lambda>0$, its resolvent is*

$$\begin{aligned}&\frac{1}{(L^{\mathrm{dS}}_{\alpha}+\lambda^2)}(x,y)\\=&\frac{\Gamma\left(\frac{1}{2}-\alpha+\lambda\right)\Gamma\left(\frac{1}{2}+\alpha+\lambda\right)}{2}\begin{cases}\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(x)\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-y) & -\infty<x<y<\infty;\\ \mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(y)\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-x) & -\infty<y<x<\infty.\end{cases}\end{aligned}\tag{1.6.35}$$

To describe the discrete spectrum and spectrum of L^{dS}_{α}. Then

$$\sigma_{\mathrm{d}}(L^{\mathrm{dS}}_{\alpha})=\left\{-(k-\alpha)^2\ \middle|\ k\in\mathbb{N}_0+\frac{1}{2},\quad k<|\operatorname{Re}\alpha|\right\}\tag{1.6.36}$$

$$\sigma(L^{\mathrm{dS}}_{\alpha})=[0,\infty[\cup\sigma_{\mathrm{d}}(L^{\mathrm{dS}}_{\alpha}).\tag{1.6.37}$$

Proof. We can use the proof of the theorem 1.6.7. Note that

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(r) = \frac{\Gamma\big(\frac{1}{2}+\lambda\big)}{\sqrt{\pi}}\mathcal{Q}^{\mathrm{dS}}_{\alpha,\alpha,2\lambda}(r) = \frac{\Gamma(1+2\lambda)}{2^{2\lambda}\Gamma(1+\lambda)}\mathcal{Q}^{\mathrm{dS}}_{\alpha,\alpha,2\lambda}(r). \tag{1.6.38}$$

Hence, the Wronskians are

$$\mathcal{W}\left(\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-r), \mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(r)\right) = \frac{2}{\Gamma\left(\frac{1}{2}+\alpha+\lambda\right)\Gamma\left(\frac{1}{2}-\alpha+\lambda\right)}, \tag{1.6.39}$$

$$\mathcal{W}\left(\mathcal{Q}^{\mathrm{dS}}_{\alpha,-\lambda}(-r), \mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(r)\right) = \frac{2\cos(\pi\alpha)}{\pi}. \tag{1.6.40}$$

Using the Schur test we check that the integral kernel *Eqn.*(1.6.35) defines a bounded operator.

The singularities of the Gamma function are at $\lambda = -\frac{1}{2}+\alpha-n$ and $\lambda = -\frac{1}{2}-\alpha-n$, $n \in \mathbb{N}_0$. This gives the following discrete spectrum:

$$\sigma_{\mathrm{d}}(L^{\mathrm{dS}}_{\alpha}) = \left\{-\left(\frac{1}{2}+\alpha+n\right)^2 \;\middle|\; n\in\mathbb{N}_0,\quad \mathrm{Re}\left(\frac{1}{2}+\alpha+n\right)<0\right\} \tag{1.6.41}$$

$$\cup\left\{-\left(\frac{1}{2}-\alpha+n\right)^2 \;\middle|\; n\in\mathbb{N}_0,\quad \mathrm{Re}\left(\frac{1}{2}-\alpha+n\right)<0\right\}. \tag{1.6.42}$$

For $\mathrm{Re}\alpha \geq 0$ the right hand side of *Eqn.*(1.6.41) is empty. This yields *Eqn.*(1.6.36). □

Proof of Prop. 0.1.1. The transformation $\cot r = \sinh q$ implies

$$\cos r = \tanh q, \quad \tan r = \sinh q, \tag{1.6.43}$$

$$\frac{\mathrm{d}q}{\mathrm{d}r} = -\frac{1}{\sin r} = \cosh q. \tag{1.6.44}$$

The Whipple transformation *Eqn.*(1.1.97), *Eqn.*(1.1.98) on the interval $w \in]-1,1[$ has two versions: above this interval and below, that is

$$\mathbf{Z}_{\alpha,\lambda}(w\pm\mathrm{i}0) = (\pm\mathrm{i}\sqrt{1-w^2})^{-\frac{1}{2}-\alpha-\lambda}\mathbf{S}_{\lambda,\alpha}\left(\frac{w}{\pm\mathrm{i}\sqrt{1-w^2}}\right). \tag{1.6.45}$$

This yields

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(q) = \left(\frac{2}{\sin r}\right)^{\frac{1}{2}}\mathcal{P}^{\mathrm{s}}_{\lambda,\alpha}(r), \tag{1.6.46}$$

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-q) = \left(\frac{2}{\sin r}\right)^{\frac{1}{2}}\mathcal{P}^{\mathrm{s}}_{\lambda,\alpha}(\pi-r). \tag{1.6.47}$$

This yields

$$(\sin r)^{\frac{1}{2}}\frac{1}{(L^{\mathrm{s}}_{\lambda}-\alpha^2)}(r,r')(\sin r')^{\frac{1}{2}} = \frac{1}{(L^{\mathrm{dS}}_{\alpha}-\lambda^2)}(q,q'), \tag{1.6.48}$$

which proves Prop. 0.1.1. □

Remark 1.6.4. *Using Eqn.*(1.6.44), *Eqn.*(1.6.46) *and Eqn.*(1.6.47) *we obtain*

$$\mathcal{W}\big(\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-q), \mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(q)\big) = 2\mathcal{W}\big(\mathcal{P}^{\mathrm{s}}_{\lambda,\alpha}(\pi-r), \mathcal{P}^{\mathrm{s}}_{\lambda,\alpha}(r)\big), \tag{1.6.49}$$

$$\mathcal{W}\big(\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-q), \mathcal{Q}^{\mathrm{dS}}_{\alpha,-\lambda}(q)\big) = 2\mathcal{W}\big(\mathcal{P}^{\mathrm{s}}_{\lambda,\alpha}(\pi-r), \mathcal{P}^{\mathrm{s}}_{-\lambda,\alpha}(r)\big). \tag{1.6.50}$$

Therefore, Eqn.(1.6.16) *and Eqn.*(1.6.17) *imply Eqn.*(1.6.39) *and Eqn.*(1.6.40). *This can be used in an alternative proof of Thm 1.6.3.*

1.6.2 Hypergeometric Hamiltonians of the first kind

In this section we solve the eigenvaalue problem and prove similar theorems to Gegenbauer case for hypergeoemntric Hamiltonian of the first kind. Let us review them here:

(1) Spherical hypergeoemtric Hamiltonian of the first kind is

$$\begin{aligned} L^{\mathrm{s}}_{\alpha,\beta} &:= -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{4\sin^2\frac{r}{2}} + \left(\beta^2 - \frac{1}{4}\right)\frac{1}{4\cos^2\frac{r}{2}} \\ &= -\partial_r^2 + \left(\delta - \frac{1}{4}\right)\frac{1}{\sin^2 r} + \kappa\frac{\cos r}{\sin^2 r}. \end{aligned} \tag{1.6.1}$$

Its the Schrödinger operator acting on $L^2]0,\pi[$

$$\left(L^{\mathrm{s}}_{\alpha,\beta} - \frac{\mu^2}{4}\right)\phi(r) = 0, \tag{1.6.2}$$

(2) Hyperbolic hypergeoemtric Hamiltonian of the first kind is

$$\begin{aligned} L^{\mathrm{h}}_{\alpha,\beta} &:= -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{4\sinh^2\frac{r}{2}} - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{4\cosh^2\frac{r}{2}} \\ &= -\partial_r^2 + \left(\delta - \frac{1}{4}\right)\frac{1}{\sinh^2 r} + \kappa\frac{\cosh r}{\sinh^2 r}. \end{aligned} \tag{1.6.3}$$

Its Schrödinger operator acting on $L^2(\mathbb{R}_+)$

$$\left(L^{\mathrm{h}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)\phi(r) = 0, \tag{1.6.4}$$

(3) Desiterian hypergeometric Hamiltonian of the first kind is

$$L^{\mathrm{dS}}_{\alpha,\beta} := -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}\left(\frac{1}{2} + \frac{\mathrm{i}\sinh r}{2}\right) - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}\left(\frac{1}{2} - \frac{\mathrm{i}\sinh r}{2}\right) \tag{1.6.5}$$

$$= -\partial_r^2 - \left(\delta - \frac{1}{4}\right)\frac{1}{\cosh^2 r} - \tau\frac{\sinh r}{\cosh^2 r}. \tag{1.6.6}$$

Its Schrödinger Operator acting on $L^2(\mathbb{R})$ is

$$\left(L^{\mathrm{dS}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)\phi(r) = 0, \tag{1.6.7}$$

If we set $\alpha = \beta$, we reduce these Hamiltonians to their Gegenbauer counterpart

$$L^{\mathrm{s}}_{\alpha} := L^{\mathrm{s}}_{\alpha,\alpha}, \quad L^{\mathrm{h}}_{\alpha} := L^{\mathrm{h}}_{\alpha,\alpha}, \quad L^{\mathrm{dS}}_{\alpha} := L^{\mathrm{dS}}_{\alpha,\alpha}. \tag{1.6.8}$$

Spherical case

For $r \in]0,\pi[$, define the function

$$\begin{aligned} P^{\mathrm{s}}_{\alpha,\beta,\mu}(r) &:= \left(\sin\frac{r}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{r}{2}\right)^{\beta+\frac{1}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha+\beta-\mu+1}{2}; 1+\alpha; \sin^2\left(\frac{r}{2}\right)\right) \\ &= \left(\sin\frac{r}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{r}{2}\right)^{-\alpha-\mu-\frac{1}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha-\beta+\mu+1}{2}; 1+\alpha; -\tan^2\left(\frac{r}{2}\right)\right). \end{aligned} \tag{1.6.9}$$

Note that

$$P^{\mathrm{s}}_{\alpha,\beta,\mu}(r) = P^{\mathrm{s}}_{\alpha,-\beta,\mu}(r) = P^{\mathrm{s}}_{\alpha,\beta,-\mu}(r). \tag{1.6.10}$$

Asymptotically our function behaves like

$$P^{\mathrm{s}}_{\alpha,\beta,\mu}(r) \sim \frac{1}{\Gamma(1+\alpha)}\left(\frac{r}{2}\right)^{\alpha+\frac{1}{2}}, \quad r \sim 0. \tag{1.6.11}$$

Now the following functions solve the eigenvalue problem *Eqn.*(1.6.2):

$$P^{\mathrm{s}}_{\alpha,\beta,\mu}(r), \quad P^{\mathrm{s}}_{-\alpha,\beta,\mu}(r), \quad P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-r), \quad P^{\mathrm{s}}_{-\beta,\alpha,\mu}(\pi-r). \tag{1.6.12}$$

The following theorem describes the basic holomorphic family of closed realizations of $L^{\mathrm{s}}_{\alpha,\beta}$ on $L^2]0,\pi[$.

Theorem 1.6.5. *For* $\operatorname{Re}\alpha, \operatorname{Re}\beta \geq 1$ *there exists a unique closed operator* $L^{\mathrm{s}}_{\alpha,\beta}$ *in the sense of* $L^2]0,\pi[$*, which on* $C_{\mathrm{c}}^{\infty}]0,\pi[$ *is given by Eqn.*(1.6.1)*. The family* $\alpha,\beta \mapsto L^{\mathrm{s}}_{\alpha,\beta}$ *is holomorphic and possesses a unique holomorphic extension to* $\operatorname{Re}\alpha, \operatorname{Re}\beta > -1$*. It has only discrete spectrum:*

$$\sigma(L^{\mathrm{s}}_{\alpha,\beta}) = \sigma_{\mathrm{d}}(L^{\mathrm{s}}_{\alpha,\beta}) = \left\{\left(k + \frac{\alpha+\beta}{2}\right)^2 \;:\; k \in \mathbb{N}_0 + \frac{1}{2}\right\}. \tag{1.6.13}$$

Outside of the spectrum its resolvent is

$$\begin{aligned}\frac{1}{\left(L^{\mathrm{s}}_{\alpha,\beta} - \frac{\mu^2}{4}\right)}(x,y) =& \Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\\ &\times \begin{cases} P^{\mathrm{s}}_{\alpha,\beta,\mu}(x)\ P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-y) & \text{if} \quad 0<x<y<\pi; \\ P^{\mathrm{s}}_{\alpha,\beta,\mu}(y)\ P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-x) & \text{if} \quad 0<y<x<\pi. \end{cases}\end{aligned} \tag{1.6.14}$$

Proof. Considering $P^{\mathrm{s}}_{\alpha,\beta,\mu}(r)$, and $P^{\mathrm{s}}_{-\alpha,\beta,\mu}(r)$ as a basis of solutions of *Eqn.*(1.6.2), we can rewrite the connection formula *Eqn.*(1.1.77) as

$$P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-r) = \frac{\pi\ P^{\mathrm{s}}_{\alpha,\beta,\mu}(r)}{\sin(-\pi\alpha)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)} + \frac{\pi\ P^{\mathrm{s}}_{-\alpha,\beta,\mu}(r)}{\sin(\pi\alpha)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)}. \tag{1.6.15}$$

Using *Eqn.*(1.6.11) and arguing as in the proof of *Eqn.*(1.6.15), we obtain

$$\mathcal{W}\left(P^{\mathrm{s}}_{-\alpha,\beta,\mu}(r), P^{\mathrm{s}}_{\alpha,\beta,\mu}(r)\right) = \frac{\sin\pi\alpha}{\pi}. \tag{1.6.16}$$

From the connection formula and *Eqn.*(1.6.16) we obtain the Wronskian

$$\mathcal{W}\left(P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-r), P^{\mathrm{s}}_{\alpha,\beta,\mu}(r)\right) = \frac{1}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)}. \tag{1.6.17}$$

The L^2 integrability conditions at the endpoints are $\operatorname{Re}\alpha > -1, \operatorname{Re}\beta > -1$, for $P^{\mathrm{s}}_{\alpha,\beta,\mu}(r)$ and $P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-r)$ respectively. With these conditions, we can write the candidate for the resolvent *Eqn.*(1.6.14). The L^2 norm of this integral kernel is finite. Hence it defines a bounded (even Hilbert-Schmidt) operator. For $\operatorname{Re}\alpha \geq 1, \operatorname{Re}\beta \geq 1$ it is a unique candidate for the resolvent. □

Hyperbolic case

For $r \in \mathbb{R}_+$, let us define

$$\begin{aligned}
P^{\mathrm{h}}_{\alpha,\beta,\mu}(r) &:= \Big(\sinh\frac{r}{2}\Big)^{\alpha+\frac12}\Big(\cosh\frac{r}{2}\Big)^{\beta+\frac12}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha+\beta-\mu+1}{2};1+\alpha;-\sinh^2\frac{r}{2}\right)\\
&= \Big(\sinh\frac{r}{2}\Big)^{\alpha+\frac12}\Big(\cosh\frac{r}{2}\Big)^{-\alpha-\mu-\frac12}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha-\beta+\mu+1}{2};1+\alpha;\tanh^2\frac{r}{2}\right),
\end{aligned} \tag{1.6.18}$$

$$\begin{aligned}
Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r) &:= \Big(\sinh\frac{r}{2}\Big)^{-\mu-\beta-\frac12}\Big(\cosh\frac{r}{2}\Big)^{\beta+\frac12}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{-\alpha+\beta+\mu+1}{2};1+\mu;-\sinh^{-2}\frac{r}{2}\right)\\
&= \Big(\sinh\frac{r}{2}\Big)^{\alpha+\frac12}\Big(\cosh\frac{r}{2}\Big)^{-\alpha-\mu-\frac12}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha-\beta+\mu+1}{2};1+\mu;\cosh^{-2}\frac{r}{2}\right).
\end{aligned} \tag{1.6.19}$$

Note that

$$P^{\mathrm{h}}_{\alpha,\beta,\mu}(r) = P^{\mathrm{h}}_{\alpha,-\beta,\mu}(r) = P^{\mathrm{h}}_{\alpha,\beta,-\mu}(r), \tag{1.6.20}$$

$$Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r) = Q^{\mathrm{h}}_{\alpha,-\beta,\mu}(r) = Q^{\mathrm{h}}_{-\alpha,\beta,\mu}(r), \tag{1.6.21}$$

Asymptotically,

$$P^{\mathrm{h}}_{\alpha,\beta,\mu}(r) \sim \frac{1}{\Gamma(1+\alpha)}\Big(\frac{r}{2}\Big)^{\alpha+\frac12}, \quad r\to 0; \tag{1.6.22}$$

$$Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r) \sim \frac{2^{\mu}}{\Gamma(1+\mu)}\mathrm{e}^{-\frac{\mu}{2}r}, \quad r\to+\infty. \tag{1.6.23}$$

Now the following functions solve the eigenvalue problem *Eqn.*(1.6.7):

$$P^{\mathrm{h}}_{\alpha,\beta,\mu}(r), \quad P^{\mathrm{h}}_{-\alpha,\beta,\mu}(r), \quad Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r), \quad Q^{\mathrm{h}}_{\alpha,\beta,-\mu}(r). \tag{1.6.24}$$

The following theorem describes the basic holomorphic family of closed realizations of $L^{\mathrm{h}}_{\alpha,\beta}$ on the Hilbert space $L^2(\mathbb{R}_+)$.

Theorem 1.6.6. *For $\operatorname{Re}\alpha \geq 1$ there exists a unique closed operator $L^{\mathrm{h}}_{\alpha,\beta}$ in the sense of $L^2(\mathbb{R}_+)$, which on $C_{\mathrm{c}}^\infty(\mathbb{R}_+)$ is given by Eqn.(1.6.3). The family $\alpha,\beta \mapsto L^{\mathrm{h}}_{\alpha,\beta}$ is holomorphic and possesses a unique holomorphic extension to $\operatorname{Re}\alpha > -1$. Its discrete spectrum and spectrum are*

$$\begin{aligned}
\sigma_{\mathrm{d}}(L^{\mathrm{h}}_{\alpha,\beta}) =& \left\{-\Big(\frac{\alpha+\beta}{2}+k\Big)^2 \;\middle|\; k\in\mathbb{N}_0+\frac12,\quad k<-\operatorname{Re}\frac{\alpha+\beta}{2}\right\},\\
&\cup\left\{-\Big(\frac{\alpha-\beta}{2}+k\Big)^2 \;\middle|\; k\in\mathbb{N}_0+\frac12,\quad k<-\operatorname{Re}\frac{\alpha-\beta}{2}\right\},
\end{aligned} \tag{1.6.25}$$

$$\sigma(L^{\mathrm{h}}_{\alpha,\beta}) = [0,\infty[\cup\sigma_{\mathrm{d}}(L^{\mathrm{h}}_{\alpha,\beta}). \tag{1.6.26}$$

Outside of the spectrum, for $\operatorname{Re}\mu > 0$, its resolvent is

$$\begin{aligned}
\frac{1}{(L^{\mathrm{h}}_{\alpha,\beta}+\frac{\mu^2}{4})}(x,y) =& \Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\\
&\times\begin{cases} P^{\mathrm{h}}_{\alpha,\beta,\mu}(x)\;Q^{\mathrm{h}}_{\alpha,\beta,\mu}(y) & \text{if}\quad 0<x<y<\infty;\\ P^{\mathrm{h}}_{\alpha,\beta,\mu}(y)\;Q^{\mathrm{h}}_{\alpha,\beta,\mu}(x) & \text{if}\quad 0<y<x<\infty.\end{cases}
\end{aligned} \tag{1.6.27}$$

Proof. Considering $P^{\rm h}_{-\alpha,\beta,\mu}(r)$, and $P^{\rm h}_{-\alpha,\beta,\mu}(r)$ as a basis of solutions of *Eqn.*(1.6.7), we can rewrite connection formula *Eqn.*(**??**) as

$$Q^{\rm h}_{\alpha,\beta,\mu}(r) = -\frac{\pi\ P^{\rm h}_{\alpha,\beta,\mu}(r)}{\sin\pi\alpha\ \Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} + \frac{\pi\ P^{h}_{-\alpha,\beta,\mu}(r)}{\sin(\alpha\pi)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}. \tag{1.6.28}$$

Using *Eqn.*(1.6.22) and again arguing as in the proof of *Eqn.*(1.6.15), we obtain we obtain

$$\mathcal{W}\left(P^{\rm h}_{-\alpha,\beta,\mu}(r), P^{\rm h}_{\alpha,\beta,\mu}(r)\right) = \frac{\sin\pi\alpha}{\pi}. \tag{1.6.29}$$

This yields the Wronskian

$$\mathcal{W}\left(Q^{\rm h}_{\alpha,\beta,\mu}(r), P^{\rm h}_{\alpha,\beta,\mu}(r)\right) = \frac{1}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}. \tag{1.6.30}$$

The L^2 integrability conditions at the endpoints are $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\mu > 0$ for $P^{\rm h}_{\alpha,\beta,\mu}(r)$ and $\mathcal{Q}^{\rm h}_{\alpha,\beta,\mu}(r)$, respectively. Using the Schur Test we check that integral kernel *Eqn.*(1.6.27) defines a bounded operator. We also see that for $\operatorname{Re}\alpha > -1, \operatorname{Re}\mu > 0$ it is a unique candidate for the resolvent. □

Proof of Prop. 0.1.2. The transformation

$$\tan\frac{r}{2} = \sinh\frac{q}{2}. \tag{1.6.31}$$

implies

$$\sin\frac{r}{2} = \tanh\frac{q}{2}, \quad \frac{{\rm d}q}{{\rm d}r} = \frac{1}{\cos\frac{r}{2}} = \cosh\frac{q}{2}; \tag{1.6.32}$$

$$P^{\rm s}_{\alpha,\beta,\mu}(r) = \frac{1}{\left(\cosh\frac{q}{2}\right)^{\frac12}} P^{\rm h}_{\alpha,\mu,\beta}(q), \tag{1.6.33}$$

$$P^{\rm s}_{\beta,\alpha,\mu}(\pi - r) = \frac{1}{\left(\cosh\frac{q}{2}\right)^{\frac12}} Q^{\rm h}_{\alpha,\mu,\beta}(q). \tag{1.6.34}$$

We obtain the transmutation relation

$$\frac{1}{\left(L^{\rm s}_{\alpha,\beta} + \frac{\mu^2}{4}\right)}(r,r') = \left(\cosh\frac{q}{2}\right)^{-\frac12} \frac{1}{\left(L^{\rm h}_{\alpha,\mu} + \frac{\beta^2}{4}\right)}(q,q') \left(\cosh\frac{q'}{2}\right)^{-\frac12}. \tag{1.6.35}$$

□

DeSitterian case

Define for $r \geq 0$

$$\begin{aligned} Q^{\rm dS}_{\alpha,\beta,\mu}(r) :=& \left(\frac{{\rm i} + \sinh r}{2}\right)^{-\frac{\beta}{2}-\frac{\mu}{2}-\frac14} \left(\frac{-{\rm i} + \sinh r}{2}\right)^{\frac{\beta}{2}+\frac14} \\ &\times \mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{-\alpha+\beta+\mu+1}{2}; 1+\mu; \frac{2}{1-{\rm i}\sinh r}\right) \end{aligned} \tag{1.6.36}$$

$$\begin{aligned} =& \left(\frac{{\rm i} + \sinh r}{2}\right)^{\frac{\alpha}{2}+\frac14} \left(\frac{-{\rm i} + \sinh r}{2}\right)^{-\frac{\alpha}{2}-\frac{\mu}{2}-\frac14} \\ &\times \mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha-\beta+\mu+1}{2}; 1+\mu; \frac{2}{1+{\rm i}\sinh r}\right). \end{aligned} \tag{1.6.37}$$

We extend $r \to Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)$ to $r<0$ by analyticity. It satisfies

$$Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \sim \frac{2^{\mu}\mathrm{e}^{-\frac{\mu}{2}r}}{\Gamma(1+\mu)}, \quad r\to+\infty. \tag{1.6.38}$$

Note that

$$Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) = Q^{\mathrm{dS}}_{\alpha,-\beta,\mu}(r) = Q^{\mathrm{dS}}_{-\alpha,\beta,\mu}(r). \tag{1.6.39}$$

Now the following functions solve the eigenvalue problem *Eqn.*(1.6.7):

$$Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r), \quad Q^{\mathrm{dS}}_{\alpha,\beta,-\mu}(r), \quad Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r), \quad Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r). \tag{1.6.40}$$

The following theorem describes all closed realizations of $L^{\mathrm{dS}}_{\alpha,\beta}$ on $L^2(\mathbb{R})$.

Theorem 1.6.7. *For any $\alpha,\beta\in\mathbb{C}$ there exists a unique closed operator L^{dS}_{α} in the sense of $L^2(\mathbb{R})$ that on $C_{\mathrm{c}}^{\infty}(\mathbb{R})$ is given by Eqn.(1.6.5). The function $\mathbb{C}\ni(\alpha,\beta)\mapsto L^{\mathrm{dS}}_{\alpha}$ is holomorphic. We have $L^{\mathrm{dS}}_{\alpha,\beta}=L^{\mathrm{dS}}_{-\alpha,\beta}=L^{\mathrm{dS}}_{\alpha,-\beta}$.*

Outside of the spectrum, for $\operatorname{Re}\mu>0$, its resolvent is

$$\begin{aligned}\frac{1}{(L^{\mathrm{dS}}_{\alpha,\beta}+\frac{\mu^2}{4})}(x,y) =& \frac{\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)}{2\pi}\\ &\times\begin{cases} Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(x)Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-y) & \text{if} \quad -\infty<x<y<\infty;\\ Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-y)Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(x) & \text{if} \quad -\infty<y<x<\infty.\end{cases}\end{aligned} \tag{1.6.41}$$

Then

$$\begin{aligned}\sigma_{\mathrm{d}}(L^{\mathrm{dS}}_{\alpha.\beta}) =& \left\{-\left(k-\frac{\alpha+\beta}{2}\right)^2 \;\middle|\; k\in\mathbb{N}_0+\frac12,\quad k<\left|\operatorname{Re}\frac{\alpha+\beta}{2}\right|\right\}\\ &\cup\left\{-\left(k-\frac{\alpha-\beta}{2}\right)^2 \;\middle|\; k\in\mathbb{N}_0+\frac12,\quad k<\left|\operatorname{Re}\frac{\alpha-\beta}{2}\right|\right\},\end{aligned} \tag{1.6.42}$$

$$\sigma(L^{\mathrm{dS}}_{\alpha,\beta}) = [0,\infty[\cup\sigma_{\mathrm{d}}(L^{\mathrm{dS}}_{\alpha}). \tag{1.6.43}$$

Proof. In the connection formula *Eqn.*(1.1.80) we insert $z=\frac{1-\mathrm{i}s}{2}$ and multiply it by $\left(\frac{1-\mathrm{i}s}{2}\right)^{\frac{\alpha}{2}+\frac14}\left(\frac{1+\mathrm{i}s}{2}\right)^{\frac{\beta}{2}+\frac14}$, obtaining

$$\begin{aligned}\left(\frac{1-\mathrm{i}s}{2}\right)^{\frac{\alpha}{2}+\frac14}\left(\frac{1+\mathrm{i}s}{2}\right)^{\frac{\beta}{2}+\frac14}\mathbf{F}_{\alpha,\beta,\mu}\left(\frac{1-\mathrm{i}s}{2}\right) =& \frac{\pi\left(\frac{1-\mathrm{i}s}{2}\right)^{\frac{\alpha}{2}+\frac14}\left(\frac{1+\mathrm{i}s}{2}\right)^{\frac{\beta}{2}+\frac14}\left(\frac{\mathrm{i}s-1}{2}\right)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,\beta,\alpha}(\frac{2}{1-\mathrm{i}s})}{\sin(-\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)}\\ &+\frac{\pi\left(\frac{1-\mathrm{i}s}{2}\right)^{\frac{\alpha}{2}+\frac14}\left(\frac{1+\mathrm{i}s}{2}\right)^{\frac{\beta}{2}+\frac14}\left(\frac{\mathrm{i}s-1}{2}\right)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\beta,\alpha}(\frac{2}{1-\mathrm{i}s})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}.\end{aligned} \tag{1.6.44}$$

We transform separately *Eqn.*(1.6.44) above and below the real line: obtaining resp.

$$\frac{\pi e^{i\frac{\pi}{2}(-\alpha-\frac{1}{2}-\frac{\mu}{2})}\left(\frac{s+i}{2}\right)^{-\frac{\beta}{2}-\frac{1}{4}-\frac{\mu}{2}}\left(\frac{s-i}{2}\right)^{\frac{\beta}{2}+\frac{1}{4}}\mathbf{F}_{\mu,\beta,\alpha}(\frac{2}{1-is})}{\sin(-\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)}$$
$$+\frac{\pi e^{i\frac{\pi}{2}(-\alpha-\frac{1}{2}+\frac{\mu}{2})}\left(\frac{s+i}{2}\right)^{-\frac{\beta}{2}-\frac{1}{4}+\frac{\mu}{2}}\left(\frac{s-i}{2}\right)^{\frac{\beta}{2}+\frac{1}{4}}\mathbf{F}_{-\mu,\beta,\alpha}(\frac{2}{1-is})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)},\quad s>0 \tag{1.6.45}$$

$$\frac{\pi e^{i\frac{\pi}{2}(\alpha+\frac{1}{2}+\frac{\mu}{2})}\left(\frac{-s-i}{2}\right)^{-\frac{\beta}{2}-\frac{1}{4}-\frac{\mu}{2}}\left(\frac{-s+i}{2}\right)^{\frac{\beta}{2}+\frac{1}{4}}\mathbf{F}_{\mu,\beta,\alpha}(\frac{2}{1-is})}{\sin(-\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)}$$
$$+\frac{\pi e^{i\frac{\pi}{2}(\alpha+\frac{1}{2}-\frac{\mu}{2})}\left(\frac{-s-i}{2}\right)^{-\frac{\beta}{2}-\frac{1}{4}+\frac{\mu}{2}}\left(\frac{-s+i}{2}\right)^{\frac{\beta}{2}+\frac{1}{4}}\mathbf{F}_{-\mu,\beta,\alpha}(\frac{2}{1-is})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}.\quad s<0 \tag{1.6.46}$$

Inserting $s=\sinh(-r)=-\sinh r$ into *Eqn.*(1.6.46), and using uniqueness of analytic continuation we get

$$\begin{aligned}p^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) :=& \left(\frac{1-i\sinh r}{2}\right)^{\frac{\alpha}{2}+\frac{1}{4}}\left(\frac{1+i\sinh r}{2}\right)^{\frac{\beta}{2}+\frac{1}{4}}\mathbf{F}_{\alpha,\beta,\mu}\left(\frac{1-i\sinh r}{2}\right)\\ =&\frac{\pi e^{\frac{i\pi}{2}\left(\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\sin(-\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)}Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r)\\ &+\frac{\pi e^{\frac{i\pi}{2}\left(\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r).\end{aligned} \tag{1.6.47}$$

By replacing α to $-\alpha$ we obtain the second identity

$$\begin{aligned}p^{\mathrm{dS}}_{-\alpha,\beta,\mu}(r) =&\frac{\pi e^{\frac{i\pi}{2}\left(-\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\sin(-\pi\mu)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)}Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r)\\ &+\frac{\pi e^{\frac{i\pi}{2}\left(-\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\sin(\pi\mu)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)}Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r).\end{aligned} \tag{1.6.48}$$

We can rewrite them via

$$\begin{aligned}\begin{bmatrix}p^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)\\ p^{\mathrm{dS}}_{-\alpha,\beta,\mu}(r)\end{bmatrix} =&\frac{\pi}{\sin(\pi\mu)}\begin{bmatrix}-\frac{e^{\frac{i\pi}{2}\left(\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)} & \frac{e^{\frac{i\pi}{2}\left(\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}\\ -\frac{e^{\frac{i\pi}{2}\left(-\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & \frac{e^{\frac{i\pi}{2}\left(-\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)}\end{bmatrix}\\ &\times\begin{bmatrix}Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r)\\ Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r)\end{bmatrix}.\end{aligned} \tag{1.6.49}$$

We evaluate

$$\det\begin{bmatrix}-\frac{e^{\frac{i\pi}{2}\left(\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)} & \frac{e^{\frac{i\pi}{2}\left(\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}\\ -\frac{e^{\frac{i\pi}{2}\left(-\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & \frac{e^{\frac{i\pi}{2}\left(-\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)}\end{bmatrix} = -i\frac{\sin(\pi\mu)\sin(\pi\alpha)}{\pi^2} \tag{1.6.50}$$

Therefore we have

$$\begin{bmatrix} Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) \\ Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r) \end{bmatrix} = \frac{-\mathrm{i}\pi}{\sin(\pi\alpha)} \begin{bmatrix} \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} & -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)} \\ \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)}. \end{bmatrix} \times \begin{bmatrix} p^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \\ p^{\mathrm{dS}}_{-\alpha,\beta,\mu}(r) \end{bmatrix} \tag{1.6.51}$$

In *Eqn.*(1.6.49), we exchange α and β, and replace r with $-r$, and we obtain

$$\begin{bmatrix} p^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) \\ p^{\mathrm{dS}}_{-\beta,\alpha,\mu}(-r) \end{bmatrix} = \frac{\pi}{\sin(\pi\mu)} \begin{bmatrix} -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\beta+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\beta-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} \\ -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\beta+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\beta-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} \end{bmatrix} \times \begin{bmatrix} Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \\ Q^{\mathrm{dS}}_{\alpha,\beta,-\mu}(r) \end{bmatrix}. \tag{1.6.52}$$

Using *Eqn.*(1.1.79), we know that

$$\begin{bmatrix} p^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \\ p^{\mathrm{dS}}_{-\alpha,\beta,\mu}(r) \end{bmatrix} = \frac{\pi}{\sin(\pi\beta)} \begin{bmatrix} -\frac{1}{\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)} & \frac{1}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)} \\ -\frac{1}{\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} & \frac{1}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} \end{bmatrix} \times \begin{bmatrix} p^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) \\ p^{\mathrm{dS}}_{-\beta,\alpha,\mu}(-r) \end{bmatrix}. \tag{1.6.53}$$

Thus we obtained the connection formula

$$\begin{aligned} \begin{bmatrix} Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) \\ Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r). \end{bmatrix} =& \frac{-\mathrm{i}\pi^3}{\sin(\pi\alpha)\sin(\pi\beta)\sin(\pi\mu)} \begin{bmatrix} \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} & -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)} \\ \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)} \end{bmatrix} \\ &\times \begin{bmatrix} -\frac{1}{\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)} & \frac{1}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)} \\ -\frac{1}{\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} & \frac{1}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} \end{bmatrix} \\ &\times \begin{bmatrix} -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\beta+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\beta-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} \\ -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\beta+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\beta-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} \end{bmatrix} \begin{bmatrix} Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \\ Q^{\mathrm{dS}}_{\alpha,\beta,-\mu}(r) \end{bmatrix}. \end{aligned} \tag{1.6.54}$$

The result of multiplication after simplification with the reflection formula for the gamma functions is

$$\begin{aligned} &\begin{bmatrix} -\frac{\mathrm{e}^{-\frac{\mathrm{i}\pi\mu}{2}}\cos(\pi\alpha)+\mathrm{e}^{\frac{\mathrm{i}\pi\mu}{2}}\cos(\pi\beta)}{\pi} & \frac{2\pi}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} \\ \frac{-2\pi}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} & \frac{\mathrm{e}^{-\frac{\mathrm{i}\pi\mu}{2}}\cos(\pi\beta)+\mathrm{e}^{\frac{\mathrm{i}\pi\mu}{2}}\cos(\pi\alpha)}{\pi} \end{bmatrix} \\ &\times \begin{bmatrix} Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \\ Q^{\mathrm{dS}}_{\alpha,\beta,-\mu}(r) \end{bmatrix} \frac{-\pi}{\sin(\pi\mu)} = \begin{bmatrix} Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) \\ Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r) \end{bmatrix}. \end{aligned} \tag{1.6.55}$$

Using

$$\mathcal{W}\left(Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r), Q^{\mathrm{dS}}_{\alpha,\beta,-\mu}(r)\right) = \frac{\sin\pi\mu}{\pi} \tag{1.6.56}$$

we obtain

$$\mathcal{W}\big(Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r), Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)\big) = \frac{2\pi}{\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)} \tag{1.6.57}$$

$$\mathcal{W}\big(Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r), Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)\big) = \frac{\mathrm{e}^{\mathrm{i}\pi\frac{\mu}{2}}\cos\pi\beta + \mathrm{e}^{-\mathrm{i}\pi\frac{\mu}{2}}\cos\pi\alpha}{\pi}. \tag{1.6.58}$$

The L^2 integrable condition at both endpoints is $\mathrm{Re}(\mu) > 0$ for both $Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)$, and $Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r)$. Therefore, for $\mathrm{Re}\,\mu > 0$, it is a unique candidate for the resolvent. Using the Schur Test we see that the integral kernel *Eqn.*(1.6.41) defines a bounded operator.

From the singularities of the Gamma function we obtain

$$\sigma_{\mathrm{d}}(L^{\mathrm{dS}}_{\alpha}) = \left\{ -\left(n + \frac{1+\alpha+\beta}{2}\right)^2 \;\middle|\; n \in \mathbb{N}_0, \quad \mathrm{Re}\left(n + \frac{1+\alpha+\beta}{2}\right) < 0 \right\} \tag{1.6.59}$$

$$\cup \left\{ -\left(n + \frac{1+\alpha-\beta}{2}\right)^2 \;\middle|\; n \in \mathbb{N}_0, \quad \mathrm{Re}\left(n + \frac{1+\alpha-\beta}{2}\right) < 0 \right\} \tag{1.6.60}$$

$$\cup \left\{ -\left(n + \frac{1-\alpha+\beta}{2}\right)^2 \;\middle|\; n \in \mathbb{N}_0, \quad \mathrm{Re}\left(n + \frac{1-\alpha+\beta}{2}\right) < 0 \right\} \tag{1.6.61}$$

$$\cup \left\{ -\left(n + \frac{1-\alpha-\beta}{2}\right)^2 \;\middle|\; n \in \mathbb{N}_0, \quad \mathrm{Re}\left(n + \frac{1-\alpha-\beta}{2}\right) < 0 \right\}. \tag{1.6.62}$$

If $\mathrm{Re}(\alpha+\beta) \geq 0$ and $\mathrm{Re}(\alpha-\beta) \geq 0$, then *Eqn.*(1.6.59) and *Eqn.*(1.6.60) are empty and we obtain *Eqn.*(1.7.23). □

1.6.3 Hypergeometric Hamiltonians of the second kind

In this section we solve the eigenvaalue problem and prove similar theorems to hyperegeometric Hamiltonian of the first kind for the hypergeoemntric Hamiltonian of the second kind. Let us review them here:

(1) Spherical hypergeoemtric Hamiltonian of the second kind is

$$K^{\mathrm{s}}_{\tau,\mu}(u) := -\partial_u^2 + \tau\frac{\cos u}{\sin u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\sin^2 u}. \tag{1.6.63}$$

Its Schrödinger Operator acting on $L^2]0,\pi[$

$$\left(K^{\mathrm{s}}_{\tau,\mu} - \delta\right)\phi(u) = 0. \tag{1.6.64}$$

(2) Hyperbolic hypergeoemtric Hamiltonian of the second kind is

$$K^{\mathrm{h}}_{\kappa,\mu} := -\partial_u^2 + \kappa\frac{\cosh u}{\sinh u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\sinh^2 u}. \tag{1.6.65}$$

Its Schrödinger operator acting on $L^2(\mathbb{R}_+)$ is

$$\left(K^{\mathrm{h}}_{\kappa,\mu} + \delta\right)\phi(u) = 0, \tag{1.6.66}$$

(3) Desiterian hypergeometric Hamiltonian of the first kind is

$$K^{\mathrm{dS}}_{\kappa,\mu} := -\partial_u^2 - \kappa\frac{\sinh u}{\cosh u} - \left(\frac{\mu^2}{4}-\frac{1}{4}\right)\frac{1}{\cosh^2 u}. \tag{1.6.67}$$

Its Schrödinger operator acting on $L^2(\mathbb{R})$ is

$$\left(K^{\mathrm{dS}}_{\kappa,\mu}+\delta\right)\phi(u)=0, \tag{1.6.68}$$

where

If we set $\kappa=\tau=0$ and $\mu=2\alpha$, we reduce these Hamiltonians to their Gegenbauer counterpart

$$L^{\mathrm{s}}_{\alpha} := K^{\mathrm{s}}_{0,2\alpha}, \quad L^{\mathrm{h}}_{\alpha} := K^{\mathrm{h}}_{0,2\alpha}, \quad L^{\mathrm{dS}}_{\alpha} := K^{\mathrm{dS}}_{0,2\alpha}. \tag{1.6.69}$$

Spherical case

We define for $u\in]0,\frac{\pi}{2}]$

$$\begin{aligned}
&\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u)\\
:=&\left(\frac{\mathrm{i}}{1-\mathrm{e}^{2\mathrm{i}u}}\right)^{\frac{-1-\beta-\mu}{2}}\left(\frac{-\mathrm{i}}{1-\mathrm{e}^{-2\mathrm{i}u}}\right)^{\frac{\beta}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{-\alpha+\beta+\mu+1}{2};\mu+1;1-\mathrm{e}^{2\mathrm{i}u}\right)\\
=&\left(\frac{\mathrm{i}}{1-\mathrm{e}^{2\mathrm{i}u}}\right)^{\frac{\alpha}{2}}\left(\frac{-\mathrm{i}}{1-\mathrm{e}^{-2\mathrm{i}u}}\right)^{\frac{-1-\alpha-\mu}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha-\beta+\mu+1}{2};\mu+1;1-\mathrm{e}^{-2\mathrm{i}u}\right).
\end{aligned} \tag{1.6.70}$$

For $u\in[\frac{\pi}{2},\pi[$ it is continued analytically. It has the asymptotics

$$\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u)\sim\frac{1}{\Gamma(\mu+1)}(2u)^{\frac{\mu}{2}+\frac{1}{2}}, \quad u\sim 0. \tag{1.6.71}$$

Note that

$$\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u)=\mathbb{Q}^{\mathrm{s}}_{-\alpha,\beta,\mu}(u)=\mathbb{Q}^{\mathrm{s}}_{\alpha,-\beta,\mu}(u), \tag{1.6.72}$$

$$\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-u)=\mathbb{Q}^{\mathrm{s}}_{\beta,-\alpha,\mu}(\pi-u)=\mathbb{Q}^{\mathrm{s}}_{-\beta,\alpha,\mu}(\pi-u), \tag{1.6.73}$$

$$\left(\frac{2}{\cosh r}\right)^{\frac{1}{2}}Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)=\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u), \tag{1.6.74}$$

$$\left(\frac{2}{\cosh r}\right)^{\frac{1}{2}}Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r)=\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-u), \quad \sinh r=-\cot u. \tag{1.6.75}$$

Now the following functions solve the eigenvalue problem *Eqn.*(1.6.64):

$$\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u), \quad \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,-\mu}(u), \quad \mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-u), \quad \mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,-\mu}(\pi-u). \tag{1.6.76}$$

The following theorem describes the basic closed realization of $K^{\mathrm{s}}_{\tau,\mu}$:

Theorem 1.6.8. *For $\operatorname{Re}\mu\geq 2$, $\tau\in\mathbb{C}$, there exists a unique closed operator $K^{\mathrm{s}}_{\tau,\mu}$ in the sense of $L^2]0,\pi[$, which on $C_{\mathrm{c}}^{\infty}]0,\pi[$ is given by Eqn.(1.6.63). The family $\tau,\mu\mapsto K^{\mathrm{s}}_{\tau,\mu}$ is holomorphic and possesses a unique holomorphic extension to $\operatorname{Re}\mu>-2$, except for a singularity at $(\tau,\mu)=(0,-1)$. It has only discrete spectrum:*

$$\sigma(K^{\mathrm{s}}_{\tau,\mu})=\left\{-\frac{\tau^2}{(2k+\mu)^2}+\left(k+\frac{\mu}{2}\right)^2 \quad\middle|\quad k\in\mathbb{N}_0+\frac{1}{2}\right\}. \tag{1.6.77}$$

Set

$$\alpha := \sqrt{\delta - \mathrm{i}\tau}, \quad \beta := \sqrt{\delta + \mathrm{i}\tau}. \tag{1.6.78}$$

(It does not matter which sign of the square root is taken). Outside of the spectrum the resolvent of $K^{\mathrm{s}}_{\tau,\mu}$ *is*

$$\begin{aligned} \frac{1}{\left(K^{\mathrm{s}}_{\tau,\mu} - \delta\right)}(x,y) = & \frac{\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)}{4\pi} \\ & \times \begin{cases} \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(x)\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi - y), & \textit{if} \quad 0 < x < y < \pi; \\ \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(y)\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi - x), & \textit{if} \quad 0 < y < x < \pi. \end{cases} \end{aligned} \tag{1.6.79}$$

Proof. The relation $\sinh r = -\cot u$ implies

$$\frac{2}{1 - \mathrm{i}\sinh r} = 1 - \mathrm{e}^{2\mathrm{i}u}, \quad \frac{2}{1 + \mathrm{i}\sinh r} = 1 - \mathrm{e}^{-2\mathrm{i}u}, \tag{1.6.80}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \sin u = \frac{1}{\cosh r}. \tag{1.6.81}$$

Therefore, using *Eqn.*(1.6.57) and *Eqn.*(1.6.58), we obtain

$$\begin{aligned} \mathcal{W}\left(\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi - u), \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u)\right) &= 2\mathcal{W}\left(Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r), Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)\right) \\ &= \frac{4\pi}{\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)}, \end{aligned} \tag{1.6.82}$$

$$\begin{aligned} \mathcal{W}\left(\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,-\mu}(\pi - u), \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u)\right) &= 2\mathcal{W}\left(Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r), Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)\right) \\ &= \frac{2\left(\mathrm{e}^{\mathrm{i}\pi\frac{\mu}{2}}\cos\pi\beta + \mathrm{e}^{-\mathrm{i}\pi\frac{\mu}{2}}\cos\pi\alpha\right)}{\pi}. \end{aligned} \tag{1.6.83}$$

The L^2 integrability condition for $\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u)$ at 0 and $\mathbb{Q}^{s}_{\beta,\alpha,\mu}(\pi - u)$ at π is $\mathrm{Re}\,\mu > -2$. The L^2 norm of this kernel 1.6.79 is finite. For $\mathrm{Re}(\mu) \geq 2$ it is a unique candidate for the resolvent.

As a byproduct we obtain a proof of Prop. 0.1.4 about the transmutation identity $L^{\mathrm{dS}} \to K^{\mathrm{s}}$.

The singularities of the Gamma functions in *Eqn.*(1.6.79) are at

$$1 + \epsilon_1\alpha + \epsilon_2\beta + \mu = -2n, \quad n \in \mathbb{N}_0, \tag{1.6.84}$$

where $\epsilon_1, \epsilon_2 \in \{1, -1\}$. This implies

$$\alpha^2 = (2n + 1 + \mu)^2 + 2\epsilon_2\beta(2n + 1 + \mu) + \beta^2. \tag{1.6.85}$$

Hence,

$$\beta = \frac{\epsilon_2\kappa}{2n + 1 + \mu} - \epsilon_2\Big(n + \frac{\mu}{2} + \frac{1}{2}\Big), \tag{1.6.86}$$

$$\alpha = -\frac{\epsilon_2\kappa}{2n + 1 + \mu} - \epsilon_2\Big(n + \frac{\mu}{2} + \frac{1}{2}\Big). \tag{1.6.87}$$

This shows

$$\delta = \frac{\kappa^2}{(2n + 1 + \mu)^2} + \Big(n + \frac{1}{2} + \frac{\mu}{2}\Big)^2. \tag{1.6.88}$$

Replacing κ^2 with $-\tau^2$ we obtain *Eqn.*(1.6.77). □

For $\mu \in \mathbb{Z}$ we have an additional identity for the $\mathbb{Q}^{\mathrm{s}}$ function, which follows directly from *Eqn.*(1.1.82):

$$\Big(\frac{\alpha+\beta-\mu+1}{2}\Big)_\mu \Big(\frac{\alpha-\beta-\mu+1}{2}\Big)_\mu \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u) = \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,-\mu}(u). \tag{1.6.89}$$

Using this with $\mu = 1$ we obtain the following unexpected identity:

Theorem 1.6.9. *For any $\tau \neq 0$, we have $K^{\mathrm{s}}_{\tau,1} = K^{\mathrm{s}}_{\tau,-1}$.*

Proof. Setting $\mu = 1$ in *Eqn.*(1.6.112) we obtain

$$\frac{\alpha^2-\beta^2}{4}\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,1}(u) = \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,-1}(u). \tag{1.6.90}$$

Setting $\mu = 1$ and $\mu = -1$ in the prefactor of the right hand side of *Eqn.*(1.6.79) we obtain

$$\frac{\Gamma\left(1-\frac{\alpha+\beta}{2}\right)\Gamma\left(1+\frac{\alpha-\beta}{2}\right)\Gamma\left(1-\frac{\alpha-\beta}{2}\right)\Gamma\left(1+\frac{\alpha+\beta}{2}\right)}{4\pi} = \frac{(\alpha^2-\beta^2)\pi}{16\sin\frac{\pi}{2}(\alpha+\beta)\sin\frac{\pi}{2}(\alpha-\beta)}, \tag{1.6.91}$$

$$\frac{\Gamma\left(-\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\alpha-\beta}{2}\right)\Gamma\left(-\frac{\alpha-\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta}{2}\right)}{4\pi} = \frac{\pi}{(\alpha^2-\beta^2)\sin\frac{\pi}{2}(\alpha+\beta)\sin\frac{\pi}{2}(\alpha-\beta)}. \tag{1.6.92}$$

We thus obtain

$$\frac{1}{(K^{\mathrm{s}}_{\tau,1}+\delta)} = \frac{1}{(K^{\mathrm{s}}_{\tau,-1}+\delta)}. \tag{1.6.93}$$

□

Note that *Eqn.*(1.6.79) is ill defined for $(\tau,\mu) = (0,-1)$. Moreover, we know that $\{2 < \operatorname{Re}\mu\} \ni \mu \mapsto L^{\mathrm{s}}_{\frac{\mu}{2}}$ is analytic, and for $\mu \neq -1$.

We have $K^{\mathrm{s}}_{0,\mu} = L^{\mathrm{s}}_{\frac{\mu}{2}}$. Therefore, it is natural to set

$$K^{\mathrm{s}}_{0,-1} := L^{\mathrm{s}}_{-\frac{1}{2}}. \tag{1.6.94}$$

We know that $L^{\mathrm{s}}_{-\frac{1}{2}} \neq L^{\mathrm{s}}_{\frac{1}{2}}$. Therefore, Thm 1.6.9 implies that the point $(\tau,\mu) = (0,-1)$ is a singularity of the function $(\tau,\mu) \mapsto K^{\mathrm{s}}_{\tau,\mu}$. See [DeRi], where a similar phenomenon is described for the Whittaker operator.

Hyperbolic case

We define

$$\begin{aligned}\mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(u) &:= \left(\mathrm{e}^{2u}-1\right)^{-\frac{\alpha}{2}}\left(1-\mathrm{e}^{-2u}\right)^{-\frac{\beta}{2}}\ \mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha+\beta-\mu+1}{2};\alpha+1;\frac{1}{1-\mathrm{e}^{2u}}\right)\\ &= \left(\mathrm{e}^{2u}-1\right)^{-\frac{\alpha}{2}}\left(1-\mathrm{e}^{-2u}\right)^{\frac{1+\alpha+\mu}{2}}\ \mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha-\beta+\mu+1}{2};\alpha+1;\mathrm{e}^{-2u}\right),\end{aligned} \tag{1.6.95}$$

$$\begin{aligned}\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(u) &:= \left(\mathrm{e}^{2u}-1\right)^{\frac{1+\beta+\mu}{2}}\left(1-\mathrm{e}^{-2u}\right)^{-\frac{\beta}{2}}\ \mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{-\alpha+\beta+\mu+1}{2};\mu+1;1-\mathrm{e}^{2u}\right)\\ &= \left(\mathrm{e}^{2u}-1\right)^{-\frac{\alpha}{2}}\left(1-\mathrm{e}^{-2u}\right)^{\frac{1+\alpha+\mu}{2}}\ \mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha-\beta+\mu+1}{2};\mu+1;1-\mathrm{e}^{-2u}\right).\end{aligned} \tag{1.6.96}$$

Note that

$$\mathbb{P}^{\mathrm{h}}_{\alpha,\beta\mu}(u) \sim \frac{1}{\Gamma(1+\alpha)}\mathrm{e}^{-\alpha u}, \quad u \sim +\infty; \tag{1.6.97}$$

$$\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta\mu}(u) \sim \frac{1}{\Gamma(1+\mu)}(2u)^{\frac{\mu}{2}+\frac{1}{2}}, \quad u \sim 0. \tag{1.6.98}$$

We have

$$\mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(u) = \mathbb{P}^{\mathrm{h}}_{\alpha,-\beta,\mu}(u) = \mathbb{P}^{\mathrm{h}}_{\alpha,\beta,-\mu}(u), \tag{1.6.99}$$

$$\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(u) = \mathbb{Q}^{\mathrm{h}}_{-\alpha,\beta,\mu}(u) = \mathbb{Q}^{\mathrm{h}}_{\alpha,-\beta,\mu}(u); \tag{1.6.100}$$

$$\left(\frac{2}{\sinh r}\right)^{\frac{1}{2}} P^{\mathrm{h}}_{\alpha,\beta,\mu}(r) = \mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(u), \tag{1.6.101}$$

$$\left(\frac{2}{\sinh r}\right)^{\frac{1}{2}} Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r) = \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(u), \qquad \cosh r = \coth u. \tag{1.6.102}$$

Now the following functions solve the eigenvalue problem *Eqn.*(1.6.66):

$$\mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(u), \quad \mathbb{P}^{\mathrm{h}}_{-\alpha,\beta,\mu}(u), \quad \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(u), \quad \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,-\mu}(u). \tag{1.6.103}$$

Let us describe the basic closed realization of $K^{\mathrm{h}}_{\kappa,\mu}$ on $L^2(\mathbb{R}_+)$.

Theorem 1.6.10. *For $\kappa \in \mathbb{C}, \operatorname{Re}\mu \geq 2$ there exists a unique closed operator $K^{\mathrm{h}}_{\kappa,\mu}$ in the sense of $L^2(\mathbb{R}_+)$, which on $C_{\mathrm{c}}^\infty(\mathbb{R}_+)$ is given by Eqn.*(1.6.65)*. The family $\kappa,\mu \mapsto K^{\mathrm{h}}_{\kappa,\mu}$ is holomorphic and possesses a unique holomorphic extension to $\operatorname{Re}\mu > -2$, $(\kappa,\mu) \neq (0,1)$. Its discrete spectrum and spectrum are*

$$\sigma_{\mathrm{d}}(K^{\mathrm{h}}_{\kappa,\mu}) = \left\{ -\frac{\kappa^2}{(2k+\mu)^2} - \left(k+\frac{\mu}{2}\right)^2 \;\middle|\; k \in \mathbb{N}_0 + \frac{1}{2}, \quad \operatorname{Re}\left(\frac{\kappa}{2k+\mu} + k + \frac{\mu}{2}\right) < 0 \right\}, \tag{1.6.104}$$

$$\sigma(K^{\mathrm{h}}_{\kappa,\mu}) = [0,+\infty[\cup \sigma_{\mathrm{d}}(K^{\mathrm{h}}_{\kappa,\mu}). \tag{1.6.105}$$

Set

$$\alpha := \sqrt{\delta+\kappa}, \quad \operatorname{Re}\alpha > 0, \quad \beta := \sqrt{\delta-\kappa} \tag{1.6.106}$$

(The choice of the square root for β does not matter). Outside of its spectrum the resolvent of $K^{\mathrm{h}}_{\kappa,\mu}$ is

$$\begin{aligned}\frac{1}{(K^{\mathrm{h}}_{\kappa,\mu}+\delta)}(x,y) =& \frac{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}{2} \\ &\times \begin{cases} \mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(x)\ \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(y) & \text{if} \quad 0<y<x<\infty, \\ \mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(y)\ \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(x) & \text{if} \quad 0<x<y<\infty. \end{cases}\end{aligned} \tag{1.6.107}$$

Proof. The relation $\cosh r = \coth u$ implies

$$\sinh^2\frac{r}{2} = \frac{1}{\mathrm{e}^{2u}-1}, \quad \cosh^2\frac{r}{2} = \frac{1}{\mathrm{e}^{-2u}-1}; \tag{1.6.108}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{\sinh r} = \sinh u, \tag{1.6.109}$$

Therefore,

$$\begin{aligned}\mathcal{W}\left(\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}, \mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}\right) &= 2\mathcal{W}\left(Q^{\mathrm{h}}_{\alpha,\beta,\mu}, P^{\mathrm{h}}_{\alpha,\beta,\mu}\right) \\ &= \frac{2}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}.\end{aligned} \tag{1.6.110}$$

The L^2 integrability conditions at endpoints are $\operatorname{Re}\alpha > 0$, and $\operatorname{Re}\mu > -2$ for $\mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(r)$ and $\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(r)$, respectively. For such parameters the integral kernel *Eqn.*(1.6.107) defines a bounded operator. For $\operatorname{Re}\alpha > 0$, and $\operatorname{Re}\mu \geq 2$, it is the unique candidate for the resolvent.

As a byproduct we obtain a proof of Prop. 0.1.5 about the transmutation identity $L^{\mathrm{h}} \to K^{\mathrm{h}}$.

The determination of the discrete spectrum follows by similar arguments as for Thm 1.6.9. $\square$

For $\mu \in \mathbb{Z}$ we have an additional identity for the $\mathbb{Q}^{\mathrm{h}}$ function, which follows directly from *Eqn.*(1.1.82):

$$\Big(\frac{\alpha+\beta-\mu+1}{2}\Big)_{\mu}\Big(\frac{\alpha-\beta-\mu+1}{2}\Big)_{\mu}\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(u) = \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,-\mu}(u). \tag{1.6.111}$$

Using this with $\mu = 1$ we obtain

Theorem 1.6.11. *For any $\kappa \neq 0$, we have $K^{\mathrm{h}}_{\kappa,1} = K^{\mathrm{h}}_{\kappa,-1}$.*

Proof. Setting $\mu = 1$ in *Eqn.*(1.6.112) we obtain

$$\frac{\alpha^2-\beta^2}{4}\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,1}(u) = \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,-1}(u). \tag{1.6.112}$$

Moreover,

$$\Gamma\left(1-\frac{\alpha+\beta}{2}\right)\Gamma\left(1+\frac{\alpha-\beta}{2}\right) = \frac{(\alpha^2-\beta^2)}{4}\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\alpha-\beta}{2}\right), \tag{1.6.113}$$

We thus obtain

$$\frac{1}{\big(K^{\mathrm{h}}_{\tau,1}+\delta\big)} = \frac{1}{\big(K^{\mathrm{h}}_{\tau,-1}+\delta\big)}. \tag{1.6.114}$$

$\square$

Similarly as in the spherical case, *Eqn.*(1.6.107) is ill defined for $(\kappa,\mu) = (0,-1)$. Moreover, we know that $\{2 < \operatorname{Re}\mu\} \ni \mu \mapsto L^{\mathrm{h}}_{\frac{\mu}{2}}$ is analytic, and for $\mu \neq -1$ we have $K^{\mathrm{h}}_{0,\mu} = L^{\mathrm{h}}_{\frac{\mu}{2}}$. Therefore, it is natural to set

$$K^{\mathrm{h}}_{0,-1} := L^{\mathrm{h}}_{-\frac{1}{2}}. \tag{1.6.115}$$

We know that $L^{\mathrm{h}}_{-\frac{1}{2}} \neq L^{\mathrm{h}}_{\frac{1}{2}}$. Therefore, Thm 1.6.11 implies that the point $(\kappa,\mu) = (0,-1)$ is a singularity of the function $(\kappa,\mu) \mapsto K^{\mathrm{h}}_{\kappa,\mu}$.

DeSitterian case

We define

$$\begin{aligned}\mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(u) &:= \left(1+\mathrm{e}^{2u}\right)^{-\frac{\alpha}{2}}\left(1+\mathrm{e}^{-2u}\right)^{-\frac{\beta}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha+\beta-\mu+1}{2}; \alpha+1; \frac{1}{1+\mathrm{e}^{2u}}\right)\\ &= \left(1+\mathrm{e}^{2u}\right)^{-\frac{\alpha}{2}}\left(1+\mathrm{e}^{-2u}\right)^{\frac{1+\alpha+\mu}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha-\beta+\mu+1}{2}; \alpha+1; -\mathrm{e}^{-2u}\right).\end{aligned} \tag{1.6.116}$$

We have the asymptotics

$$\mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(u) \sim \frac{1}{\Gamma(1+\alpha)}\mathrm{e}^{-\alpha u}, \quad u \sim +\infty. \tag{1.6.117}$$

Note that

$$\mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(u) = \mathbb{P}^{\mathrm{dS}}_{\alpha,-\beta,\mu}(u) = \mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,-\mu}(u), \tag{1.6.118}$$

$$\left(\frac{2}{\sin r}\right)^{\frac{1}{2}} P^{\mathrm{s}}_{\alpha,\beta,\mu}(r) = \mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(u), \quad \cos r = \tanh u. \tag{1.6.119}$$

Now the following functions solve the eigenvalue problem *Eqn.*(1.6.68):

$$\mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(u), \quad \mathbb{P}^{\mathrm{dS}}_{-\alpha,\beta,\mu}(u), \quad \mathbb{P}^{\mathrm{dS}}_{\beta,\alpha,\mu}(-u), \quad \mathbb{P}^{\mathrm{dS}}_{-\beta,\alpha,\mu}(-u). \tag{1.6.120}$$

We will find all closed realization of $K^{\mathrm{dS}}_{\kappa,\mu}$ in the sense of $L^2(\mathbb{R})$.

Theorem 1.6.12. *For any $\kappa, \mu \in \mathbb{C}$ there exists a unique closed operator $K^{\mathrm{dS}}_{\kappa,\mu}$ in the sense of $L^2(\mathbb{R})$ that on $C_{\mathrm{c}}^{\infty}(\mathbb{R})$ is given by Eqn.*(1.6.67)*. The function $\mathbb{C}^2 \ni (\kappa,\mu) \mapsto K^{\mathrm{dS}}_{\kappa,\mu}$ is holomorphic. The discrete spectrum and spectrum of $K^{\mathrm{dS}}_{\kappa,\mu}$ are*

$$\begin{aligned}
\sigma_{\mathrm{d}}(K^{\mathrm{dS}}_{\kappa,\mu}) = &\left\{ -\frac{\kappa^2}{(2k+\mu)^2} - \left(k + \frac{\mu}{2}\right)^2 \ \middle|\ k \in \mathbb{N}_0 + \frac{1}{2}, \quad k < -\left|\mathrm{Re}\,\frac{\kappa}{2k+\mu}\right| - \frac{\mu}{2} \right\} \\
&\cup \left\{ -\frac{\kappa^2}{(2k-\mu)^2} - \left(k - \frac{\mu}{2}\right)^2 \ \middle|\ k \in \mathbb{N}_0 + \frac{1}{2}, \quad k < -\left|\mathrm{Re}\,\frac{\kappa}{2k-\mu}\right| + \frac{\mu}{2} \right\},
\end{aligned} \tag{1.6.121}$$

$$\sigma(K^{\mathrm{dS}}_{\kappa,\mu}) = [\kappa, +\infty[\cup [-\kappa, +\infty[\cup \sigma_{\mathrm{d}}\left(K^{\mathrm{dS}}_{\kappa,\mu}\right). \tag{1.6.122}$$

Here, for $z \in \mathbb{C}$ we use the notation $[z, +\infty[:= \{z + t \quad | \quad t \in [0, +\infty[\}$.

Set

$$\alpha := \sqrt{\delta + \kappa}, \quad \mathrm{Re}\,\alpha > 0; \quad \beta := \sqrt{\delta - \kappa}, \quad \mathrm{Re}\,\beta > 0. \tag{1.6.123}$$

Outside of its spectrum, the resolvent of $K^{\mathrm{dS}}_{\kappa,\mu}$ is

$$\begin{aligned}
\frac{1}{\left(K^{\mathrm{dS}}_{\kappa,\mu} + \delta\right)}(x,y) = &\frac{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)}{2} \\
&\times \begin{cases} \mathbb{P}^{\mathrm{dS}}_{\beta,\alpha,\mu}(x)\ \mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(y), & \text{if} \quad -\infty < x < y < \infty; \\ \mathbb{P}^{\mathrm{dS}}_{\beta,\alpha,\mu}(y)\ \mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(x), & \text{if} \quad -\infty < y < x < \infty. \end{cases}
\end{aligned} \tag{1.6.124}$$

We have $K^{\mathrm{dS}}_{\kappa,\mu} = K^{\mathrm{dS}}_{\kappa,-\mu}$

Proof. The relation $\cos r = \tanh u$ implies

$$\sin^2 \frac{r}{2} = \frac{1}{1 + \mathrm{e}^{2u}}, \quad \cos^2 \frac{r}{2} = \frac{1}{1 + \mathrm{e}^{-2u}}; \tag{1.6.125}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{\sin r} = \cosh u. \tag{1.6.126}$$

This yields *Eqn.*(1.6.119). Using *Eqn.*(1.6.126) and *Eqn.*(1.6.119), and then *Eqn.*(1.6.17), we obtain

$$\begin{aligned}
\mathcal{W}\left(\mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(u), \mathbb{P}^{\mathrm{dS}}_{\beta,\alpha,\mu}(-u)\right) &= 2\mathcal{W}\left(P^{\mathrm{s}}_{\alpha,\beta,\mu}(r), P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi - r)\right) \\
&= \frac{2}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)}.
\end{aligned} \tag{1.6.127}$$

The L^2 integrability condition at $+\infty$ of $\mathbb{P}_{\alpha,\beta,\mu}(u)$ is $\mathrm{Re}(\alpha) > 0$, and at $-\infty$ of $\mathbb{P}_{\beta,\alpha,\mu}(-u)$ is $\mathrm{Re}(\beta) > 0$. For such parameters, the integral kernel *Eqn.*(1.6.124) defines a bounded operator and is a unique candidate for the resolvent.

As a byproduct we obtain a proof of Prop. 0.1.4 about the transmutation identity $L^{\mathrm{s}} \to L^{\mathrm{dS}}$.

The determination of the discrete spectrum is similar as in the hyperbolic case. □

Proof of Prop. 0.1.3. The change of variables

$$1 + \mathrm{e}^{2w} = \mathrm{e}^{2u} \tag{1.6.128}$$

implies

$$-\mathrm{e}^{2w} = 1 - \mathrm{e}^{2u}, \quad -\mathrm{e}^{-2w} = \frac{1}{1 - \mathrm{e}^{2u}}; \tag{1.6.129}$$

$$\frac{\mathrm{d}w}{\mathrm{d}u} = \frac{1}{1 - \mathrm{e}^{-2u}} = 1 + \mathrm{e}^{-2w}, \tag{1.6.130}$$

$$\mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(w) = \big(1 - \mathrm{e}^{-2u}\big)^{-\frac{1}{2}}\, \mathbb{P}^{\mathrm{h}}_{\alpha,\mu,\beta}(u), \tag{1.6.131}$$

$$\mathbb{P}^{\mathrm{dS}}_{\beta,\alpha,\mu}(-w) = \big(1 - \mathrm{e}^{-2u}\big)^{-\frac{1}{2}}\, \mathbb{Q}^{\mathrm{h}}_{\alpha,\mu,\beta}(u). \tag{1.6.132}$$

We redefine parameters

$$\delta' := \frac{1}{2}(\alpha^2 - \mu^2), \qquad \kappa' := \frac{1}{2}(\alpha^2 + \mu^2). \tag{1.6.133}$$

We obtain the transmutation identity

$$\big(1 - \mathrm{e}^{-2u}\big)^{-\frac{1}{2}} \frac{1}{\big(K^{\mathrm{h}}_{\kappa,\mu} + \delta\big)}(u, u')\big(1 - \mathrm{e}^{-2u'}\big)^{-\frac{1}{2}} = \frac{1}{\Big(K^{\mathrm{dS}}_{\kappa',\beta} + \delta'\Big)}(w, w'). \tag{1.6.134}$$

□

1.6.4 The Laplacian on an interval, halfline and line

The Laplacians on $]0, \pi[$, $\mathbb{R}_+$ and $\mathbb{R}$ with the Dirichlet or Neumann boundary conditions at endpoints belong to the most widely used operators. Their Green functions can be easily computed in terms of elementary functions, without using hypergeometric functions. In this section we will check that they are special cases of hypergeometric Hamiltonians. We will see that this coincidence is related to various identities for hypergeometric functions from Appendix A.1. This section provide a precise example for our discustion about the boundary condition in a clear manner.

Laplacian on an interval

Consider the Laplacian $-\partial_x^2$ on the interval $]0, \pi[$. The Dirichlet and Neumann boundary condition will be denoted D and N resp. Putting them at both 0 and π leads to 4 operators on $L^2]0, \pi[$. They are special cases of the spherical hypergeometric Hamiltonian of the first kind:

$$L_{\mathrm{DD}} := L^{\mathrm{s}}_{\frac{1}{2},\frac{1}{2}} = L^{\mathrm{s}}_{\frac{1}{2}}, \tag{1.6.135}$$

$$L_{\mathrm{ND}} := L^{\mathrm{s}}_{-\frac{1}{2},\frac{1}{2}}, \tag{1.6.136}$$

$$L_{\mathrm{DN}} := L^{\mathrm{s}}_{\frac{1}{2},-\frac{1}{2}}, \tag{1.6.137}$$

$$L_{\mathrm{NN}} := L^{\mathrm{s}}_{-\frac{1}{2},-\frac{1}{2}} = L^{\mathrm{s}}_{-\frac{1}{2}}. \tag{1.6.138}$$

Resolvents of these operators can be computed in terms of elementary functions. Indeed, the following functions solve

$$(-\partial_x^2 + k^2)\phi(x) = 0 \tag{1.6.139}$$

and satisfy the Dirichlet/Neumann boundary conditions at 0, resp. at π:

$$\text{Dirichlet:} \quad \sinh kx, \quad \sinh k(\pi - x); \tag{1.6.140}$$

$$\text{Neumann:} \quad \cosh kx, \quad \cosh k(\pi - x). \tag{1.6.141}$$

They have the Wronskians:

$$\mathcal{W}\big(\sinh k(\pi - x), \sinh kx\big) = k \sinh \pi k, \tag{1.6.142}$$

$$\mathcal{W}\big(\cosh k(\pi - x), \sinh kx\big) = k \cosh \pi k, \tag{1.6.143}$$

$$\mathcal{W}\big(\sinh k(\pi - x), \cosh kx\big) = k \cosh \pi k, \tag{1.6.144}$$

$$\mathcal{W}\big(\cosh k(\pi - x), \cosh kx\big) = k \sinh \pi k. \tag{1.6.145}$$

By the usual methods, we compute the spectra of operators *Eqn.*(1.6.135)–*Eqn.*(1.6.138), and for k^2 outside of the spectra their Green functions :

$$\sigma(L_{\mathrm{DD}}) = \{n^2 \mid n \in \mathbb{N}_0\},$$

$$\frac{1}{L_{\mathrm{DD}} + k^2}(x, y) = \frac{1}{k \sinh \pi k} \begin{cases} \sinh kx \, \sinh k(\pi - y), & \text{if} \quad 0 < x < y < \pi, \\ \sinh ky \, \sinh k(\pi - x), & \text{if} \quad 0 < y < x < \pi; \end{cases} \tag{1.6.146}$$

$$\sigma(L_{\mathrm{ND}}) = \{(n + \tfrac{1}{2})^2 \mid n \in \mathbb{N}_0\},$$

$$\frac{1}{L_{\mathrm{ND}} + k^2}(x, y) = \frac{1}{k \cosh \pi k} \begin{cases} \cosh kx \, \sinh k(\pi - y), & \text{if} \quad 0 < x < y < \pi, \\ \cosh ky \, \sinh k(\pi - x), & \text{if} \quad 0 < y < x < \pi; \end{cases} \tag{1.6.147}$$

$$\sigma(L_{\mathrm{DN}}) = \{(n + \tfrac{1}{2})^2 \mid n \in \mathbb{N}_0\},$$

$$\frac{1}{L_{\mathrm{DN}} + k^2}(x, y) = \frac{1}{k \cosh \pi k} \begin{cases} \sinh kx \, \cosh k(\pi - y), & \text{if} \quad 0 < x < y < \pi, \\ \sinh ky \, \cosh k(\pi - x), & \text{if} \quad 0 < y < x < \pi; \end{cases} \tag{1.6.148}$$

$$\sigma(L_{\mathrm{NN}}) = \{n^2 \mid n \in \mathbb{N}\},$$

$$\frac{1}{L_{\mathrm{NN}} + k^2}(x, y) = \frac{1}{k \sinh \pi k} \begin{cases} \cosh kx \, \cosh k(\pi - y), & \text{if} \quad 0 < x < y < \pi, \\ \cosh ky \, \cosh k(\pi - x), & \text{if} \quad 0 < y < x < \pi. \end{cases} \tag{1.6.149}$$

Let us check that *Eqn.*(1.6.146)–*Eqn.*(1.6.149) agree with the more general formula *Eqn.*(1.6.14) involving the hypergeometric function and the Gamma function. We identify $k = \frac{\mathrm{i}\mu}{2}$. By *Eqn.*(A.1.5) and *Eqn.*(A.1.6) we obtain

$$P^{\mathrm{s}}_{\frac{1}{2}, \pm\frac{1}{2}, \mu}(x) = \frac{\sinh kx}{k\sqrt{\pi}}, \tag{1.6.150}$$

$$P^{\mathrm{s}}_{-\frac{1}{2}, \pm\frac{1}{2}, \mu}(x) = \frac{\cosh kx}{\sqrt{\pi}}, \tag{1.6.151}$$

Finally, we use

$$\Gamma(1 + \mathrm{i}k)\, \Gamma(1 - \mathrm{i}k) = \frac{k\pi}{\sinh k\pi}, \tag{1.6.152}$$

$$\Gamma\left(\frac{1}{2} + \mathrm{i}k\right) \Gamma\left(\frac{1}{2} - \mathrm{i}k\right) = \frac{\pi}{\cosh \pi k}, \tag{1.6.153}$$

$$\Gamma(\mathrm{i}k)\, \Gamma(-\mathrm{i}k) = \frac{\pi}{k \sinh k\pi}. \tag{1.6.154}$$

Laplacian on the halfline

Consider the Laplacian $-\partial_x^2$ on the half-line $\mathbb{R}_+$. Setting the Dirichlet and Neumann boundary conditions at 0 we obtain 2 operators on $L^2(\mathbb{R}_+)$, which are special cases of the hyperbolic Gegenbauer Hamiltonian:

$$L_{\mathrm{D}} := L^{\mathrm{h}}_{\frac12,\frac12} = L^{\mathrm{h}}_{\frac12,-\frac12} = L^{\mathrm{h}}_{\frac12}, \tag{1.6.155}$$
$$L_{\mathrm{N}} := L^{\mathrm{h}}_{-\frac12,\frac12} = L^{\mathrm{h}}_{-\frac12,-\frac12} = L^{\mathrm{h}}_{-\frac12}. \tag{1.6.156}$$

Let us compute their resolvents. The following functions solve

$$(-\partial_x^2 + k^2)\phi(x) = 0 \tag{1.6.157}$$

and satisfy the Dirichlet/Neumann boundary conditions at 0 and decay at $+\infty$:

$$\text{Dirichlet:} \quad \sinh kx; \tag{1.6.158}$$
$$\text{Neumann:} \quad \cosh kx; \tag{1.6.159}$$
$$\text{decaying at } +\infty\text{:} \quad \mathrm{e}^{-kx}, \quad \operatorname{Re} k > 0. \tag{1.6.160}$$

They have the Wronskians:

$$\mathcal{W}\big(\mathrm{e}^{-kx}, \sinh kx\big) = k, \tag{1.6.161}$$
$$\mathcal{W}\big(\mathrm{e}^{-kx}, \cosh kx\big) = k. \tag{1.6.162}$$

Now for $\operatorname{Re} k > 0$,

$$\big(L_{\mathrm{D}} + k^2\big)^{-1}(x,y) = \frac{1}{k}\begin{cases} \sinh kx \ \mathrm{e}^{-ky}, & \text{if} \quad 0 < x < y, \\ \sinh ky \ \mathrm{e}^{-kx}, & \text{if} \quad 0 < y < x; \end{cases} \tag{1.6.163}$$

$$\big(L_{\mathrm{N}} + k^2\big)^{-1}(x,y) = \frac{1}{k}\begin{cases} \cosh kx \ \mathrm{e}^{-ky}, & \text{if} \quad 0 < x < y, \\ \cosh ky \ \mathrm{e}^{-kx}, & \text{if} \quad 0 < y < x. \end{cases} \tag{1.6.164}$$

To check that *Eqn.*(1.6.163) and *Eqn.*(1.6.164) agree with *Eqn.*(1.6.27), identify $k = \frac{\mu}{2}$. By *Eqn.*(A.1.7), *Eqn.*(A.1.8) and *Eqn.*(A.1.9), we have

$$P^{\mathrm{h}}_{\frac12,\pm\frac12,\pm\mu}(x) = \frac{\sinh kx}{k\sqrt{\pi}}, \tag{1.6.165}$$
$$P^{\mathrm{h}}_{-\frac12,\pm\frac12,\pm\mu}(x) = \frac{\cosh kx}{\sqrt{\pi}}, \tag{1.6.166}$$
$$Q^{\mathrm{h}}_{\pm\frac12,\pm\frac12,\mu}(x) = \frac{2^{2k}}{\Gamma(1+2k)}\mathrm{e}^{-kx}. \tag{1.6.167}$$

Finally, use

$$\Gamma(k)\Gamma\left(k + \frac12\right) = \frac{2^{-2k}\sqrt{\pi}}{k}\Gamma(2k+1), \tag{1.6.168}$$
$$\Gamma\left(k + \frac12\right)\Gamma\left(k+1\right) = 2^{-2k}\sqrt{\pi}\Gamma(2k+1). \tag{1.6.169}$$

Laplacian on the line

Consider the Laplacian $-\partial_x^2$ on the line $\mathbb{R}$, denoted L. It is a special case of the deSitterian hypergeometric Hamiltonian of the first kind:

$$L := L^{\mathrm{dS}}_{\pm\frac{1}{2},\pm\frac{1}{2}} = L^{\mathrm{dS}}_{\pm\frac{1}{2},\mp\frac{1}{2}} = L^{\mathrm{dS}}_{\pm\frac{1}{2}}. \tag{1.6.170}$$

It is well known how to compute its resolvent: The following functions solve

$$(-\partial_x^2 + k^2)\phi(x) = 0 \tag{1.6.171}$$

$$\text{decaying at } +\infty\text{:} \quad \mathrm{e}^{-kx}, \quad \mathrm{Re}(k) > 0, \tag{1.6.172}$$

$$\text{decaying at } -\infty\text{:} \quad \mathrm{e}^{kx}, \quad \mathrm{Re}(k) > 0. \tag{1.6.173}$$

They have the Wronskian:

$$\mathcal{W}\big(\mathrm{e}^{-kx}, \mathrm{e}^{kx}\big) = 2k. \tag{1.6.174}$$

Now for $\mathrm{Re}\, k > 0$,

$$\left(L + k^2\right)^{-1}(x,y) = \frac{1}{2k}\begin{cases} \mathrm{e}^{kx}\mathrm{e}^{-ky}, & \text{if} \quad x < y, \\ \mathrm{e}^{ky}\mathrm{e}^{-kx}, & \text{if} \quad y < x. \end{cases} \tag{1.6.175}$$

To see that *Eqn.*(1.6.175) agrees with *Eqn.*(1.6.41), we identify $k = \frac{\mu}{2}$, use

$$Q^{\mathrm{dS}}_{\pm\frac{1}{2},\pm\frac{1}{2},\mu}(x) = Q^{\mathrm{dS}}_{\pm\frac{1}{2},\mp\frac{1}{2},\mu}(x) = \frac{2^{2k}}{\Gamma(2k+1)}\mathrm{e}^{-kx}, \tag{1.6.176}$$

which follows from *Eqn.*(A.1.9), and

$$\Gamma(k)\,\Gamma\left(k + \frac{1}{2}\right)^2\Gamma\left(k+1\right) = \frac{2^{-4k}\pi}{k}\Gamma(2k+1)^2. \tag{1.6.177}$$

1.7 Exact WKB quantization of hypergeoemtric Hamiltonians

This section is an announcement of new results that have not yet been published. The quantization of the hypergeometric Hamiltonian leads to interesting results. By quantization, we mean a condition under which the discrete spectrum of the operator can be obtained. It was shown in various works, see [GJR] for a review, that WKB quantization of hypergeoemtric Hamiltonians is exact after imposing the Langer modification [L]. This was verified case by case for each Hamiltonian considered here by several research groups. Unfortunately, these works have not added much insight to the subject. We propose an alternative approach using the theory of orthogonal polynomial to show that a much stronger condition can be applied. If we consider the full WKB expansion, then the quantization is exact without requiring any modification. This was first observed in [BOW], but only in the case of what we call the Gegenbauer deSitterian Hamiltonian. The method presented here works only in the case of self-adjoint Schrödinger operators.

Our strategy is different: we use what we call nodal quantization. Consider a one-dimensional Schrödinger equation on $L^2]a,b[$:

$$\left(-\partial_x^2 - Q(x)\right)\psi(x) = 0, \quad \psi(a) = \psi(b) = 0, \tag{1.7.1}$$

$$Q(x) = V(x) - E. \tag{1.7.2}$$

The WKB ansatz (expansion) for a wave function is a formal expansion of the form

$$\psi(x) = \exp\left(\frac{1}{\hbar}\sum_{n=0}^{\infty} \hbar S_n(x)\right). \tag{1.7.3}$$

The terms with $n > 1$ are often referred to as Voros coefficients [Vo]. According to [BOW], the exact WKB quantization condition is

$$\frac{1}{2\mathrm{i}}\oint \sum_{n=0}^{\infty} S_n'(z)\,dz = k\pi, \qquad k \in \mathbb{N}_0, \tag{1.7.4}$$

where $\hbar$ is set equal to 1, sicne we are to see the result obtained in the previous sections. The contour is chosen to encircle the classical turning points on the real axis.

One can insert the WKB ansatz into the Schrödinger equation and solve it using a power series expansion. This is what the authors did for the deSitterian Gegenbauer Hamiltonian [BOW]. On the other hand, we can see immediately that the quantization condition is equivalent to the following expression:

$$k = \frac{1}{2\pi\mathrm{i}}\oint \frac{d}{dz}\log\psi(z)\,dz, \qquad k \in \mathbb{N}_0. \tag{1.7.5}$$

It is easy to see that this integral counts the number of zeros, N_ψ, minus the number of poles, P_ψ:

$$\frac{1}{2\pi\mathrm{i}}\oint \frac{d}{dz}\log\psi(z)\,dz = N_\psi - P_\psi. \tag{1.7.6}$$

1.7.1 Zeros of eigenpolynomials of Schrödinger operator

We prove two basic theorems about self-adjoint operators.

Theorem 1.7.1. *Let p_1, p_2 be elements of $L^2]a,b[$ associated with two distinct discrete eigenvalues of a self-adjoint operator L such that $Lp_1 = \lambda_1 p_1$, $Lp_2 = \lambda_2 p_2$, and $\lambda_1 \neq \lambda_2$. Then*

$$\langle p_1, p_2\rangle = \int_a^b \overline{p_1(x)}p_2(x)\,dx = 0. \tag{1.7.7}$$

Proof. Since L is self-adjoint,

$$\langle Lp_1, p_2\rangle - \langle p_1, Lp_2\rangle = 0 \implies (\lambda_1 - \lambda_2)\langle p_1, p_2\rangle = 0, \tag{1.7.8}$$

which proves the statement. □

Theorem 1.7.2. *Suppose $f_n \in L^2]a,b[$, with $a, b \neq \infty$, is a family of pairwise orthogonal eigenfunctions of a self-adjoint Schrödinger operator L, such that*

$$f_n(x) = w(x)p_n(x), \tag{1.7.9}$$

where $p_n(x)$ is a polynomial of degree n and $w(x) \neq 0$ on $]a,b[$. Then p_n has n simple zeros on $]a,b[$.

Proof. Since the functions f_n are pairwise orthogonal in $L^2]a,b[$, we have

$$\int_a^b \overline{p_n(x)}\,q(x)\,|w(x)|^2\,dx = 0 \tag{1.7.10}$$

for every polynomial $q(x)$ of degree strictly less than n.

Let $x_1, \ldots, x_m$ be the zeros of $p_n(x)$ in (a, b) at which $p_n(x)$ changes sign. Consider

$$q_m(x) = \prod_{j=1}^{m} (x - x_j). \tag{1.7.11}$$

Then the product $\overline{p_n(x)}\, q_m(x)$ does not change sign on $]a, b[$, and is not identically zero. Since $|w(x)|^2 > 0$ on $]a, b[$, it follows that

$$\int_a^b \overline{p_n(x)}\, q_m(x)\, |w(x)|^2\, dx \neq 0. \tag{1.7.12}$$

If $m < n$, then by orthogonality we would also have

$$\int_a^b \overline{p_n(x)}\, q_m(x)\, |w(x)|^2\, dx = 0, \tag{1.7.13}$$

which is a contradiction. Hence $m \geq n$.

On the other hand, since p_n has degree n, it can have at most n zeros. Therefore $m \leq n$, and so $m = n$.

Thus p_n has exactly n simple zeros in $]a, b[$. □

This is a specialized version of a known theorem in Sturm–Liouville theory, often referred to as an *oscillation theorem* [Z]. It is easy to see that the hypergeometric operator can be written in Sturm-Liouville form. Recall that

$$\mathcal{F}(-k, b; c; z, \partial_z) = z(1-z)\partial_z^2 + (c + (k - b + 1)z)\partial_z + kb. \tag{1.7.14}$$

The pure Sturm–Liouville form of the hypergeometric operator is

$$\partial_z \left(p(z) \partial_z \right) + \lambda w(z), \tag{1.7.15}$$

where

$$p(z) = z^c (1-z)^{-k+b-c+1}, \quad w(z) = z^{c-1}(1-z)^{-k+b-c}. \tag{1.7.16}$$

Theorem 1.7.2 is in fact the oscillation theorem associated with the Jacobi weight (1.7.16). In what follows, we use the fact that the hypergeometric function reduces to a polynomial if one of the first two parameters is a negative integer. More precisely, from the series expansion

$$\mathbf{F}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \Gamma(c)} \frac{z^n}{n!}, \tag{1.7.17}$$

the series truncates if $a = -k$ or $b = -k$. This gives the Jacobi polynomial. In the next section we prove the exact quantizability of hypergeometric Hamiltonians using nodal quantization.

1.7.2 Nodal quantization of hypergeometric Hamiltonians

Theorem 1.7.3. *Node counting produces the exact discrete spectrum of the self-adjoint spherical hypergeometric Hamiltonian of the first kind, given by*

$$\sigma_{\mathrm{d}} \left(L_{\alpha,\beta} \right) = \left\{ \left(k + \frac{\alpha + \beta}{2} \right)^2 : k \in \mathbb{N}_0 + \frac{1}{2}, \quad \alpha, \beta > 1 \right\}. \tag{1.7.18}$$

Proof. The integrability conditions at the endpoints 0 and π are $\alpha > 1$ and $\beta > 1$, respectively. Note that for $0 < r < \pi$, we have $0 < \sin^2\left(\frac{r}{2}\right) < 1$. Thus we choose the second expression as an eigenpolynomial in Eqn. (1.6.9) , where by Theorem 1.7.2, we have $k \in \mathbb{N}_0$ simple zeros if

$$\frac{\alpha+\beta+\mu+1}{2} = -k, \quad \text{or} \quad \frac{\alpha+\beta-\mu+1}{2} = -k. \tag{1.7.19}$$

This proves the statement. □

This exactly matches the result of [DK].

Theorem 1.7.4. *Node counting produces the exact discrete spectrum of the self-adjoint hyperbolic hypergeometric Hamiltonian of the first kind, given by*

$$\begin{aligned}\sigma_{\mathrm{d}}(L^{\mathrm{h}}_{\alpha,\beta}) = &\left\{-\Big(\frac{\alpha+\beta}{2}+k\Big)^2 \;\Big|\; k \in \mathbb{N}_0 + \frac{1}{2}, \quad k < -\frac{\alpha+\beta}{2}\right\}, &(1.7.20)\\ \cup &\left\{-\Big(\frac{\alpha-\beta}{2}+k\Big)^2 \;\Big|\; k \in \mathbb{N}_0 + \frac{1}{2}, \quad k < -\frac{\alpha-\beta}{2}\right\}, &(1.7.21)\end{aligned}$$

such that $\alpha > 1$.

Proof. The L^2 integrability conditions at the endpoints are $\alpha > 1$ and $\mu > 0$. Note that for $r \in]0,\infty[$, we have $0 < \tanh^2\frac{r}{2} < 1$ and $0 < \cosh^{-2}\frac{r}{2} < 1$. Hence the hypergeometric function reduces to a polynomial with $k \in \mathbb{N}_0$ on a finite interval by choosing the second expression in Eqn. (1.6.18) and Eqn. (1.6.19). Thus by Theorem 1.7.2, we have

$$\frac{\alpha+\beta+\mu+1}{2} = -k, \quad \text{or} \quad \frac{\alpha-\beta+\mu+1}{2} = -k, \tag{1.7.22}$$

subject to the condition $\mu > 0$. Solving these conditions leads to the correct spectrum. □

Theorem 1.7.5. *Node counting produces the exact discrete spectrum of the self-adjoint deSitterian hypergeometric Hamiltonian of the first kind, given by*

$$\begin{aligned}\sigma_{\mathrm{d}}(L^{\mathrm{dS}}_{\alpha.\beta}) = &\left\{-\left(k-\frac{\alpha+\beta}{2}\right)^2 \;\middle|\; k \in \mathbb{N}_0 + \frac{1}{2}, \quad k < \left|\operatorname{Re}\frac{\alpha+\beta}{2}\right|\right\}\\ \cup &\left\{-\left(k-\frac{\alpha-\beta}{2}\right)^2 \;\middle|\; k \in \mathbb{N}_0 + \frac{1}{2}, \quad k < \left|\operatorname{Re}\frac{\alpha-\beta}{2}\right|\right\}.\end{aligned} \tag{1.7.23}$$

Proof. The L^2 integrability condition at the endpoint is $\operatorname{Re}\mu > 0$. Since all the solutions are defined on an infinite interval, we cannot use Theorem 1.7.2. In this case we must provide an alternative argument. Let us consider one of the functions solving the eigenvalue problem and set $\sinh r = s$:

$$\begin{aligned}Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(s) =&\Big(\frac{\mathrm{i}+s}{2}\Big)^{-\frac{\beta}{2}-\frac{\mu}{2}-\frac{1}{4}}\Big(\frac{-\mathrm{i}+s}{2}\Big)^{\frac{\beta}{2}+\frac{1}{4}}\\ &\times \mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{-\alpha+\beta+\mu+1}{2}; 1+\mu; \frac{2}{1-\mathrm{i}s}\right) &(1.7.24)\\ =&\Big(\frac{\mathrm{i}+s}{2}\Big)^{\frac{\alpha}{2}+\frac{1}{4}}\Big(\frac{-\mathrm{i}+s}{2}\Big)^{-\frac{\alpha}{2}-\frac{\mu}{2}-\frac{1}{4}}\\ &\times \mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha-\beta+\mu+1}{2}; 1+\mu; \frac{2}{1+\mathrm{i}s}\right). &(1.7.25)\end{aligned}$$

Note that from the first line, $Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(s)$ is a polynomial of degree k in s if $\frac{\beta}{2}+\frac{1}{4}>k$ and $\frac{\alpha+\beta+\mu+1}{2}=-k$, or $\frac{-\alpha+\beta+\mu+1}{2}=-k$. We can easily see that by imposing $\mu\in\mathbb{R}$ and $\alpha=\overline{\beta}$, we have

$$\overline{Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(s)}=Q^{\mathrm{dS}}_{\overline{\beta},\overline{\alpha},\mu}(s)=Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(s), \tag{1.7.26}$$

therefore $Q^{\mathrm{dS}}_{\alpha,\beta,\mu}:\mathbb{R}\to\mathbb{R}$, and in particular its zeros are real. Since it is a polynomial of degree k, we obtain the following conditions:

$$\sigma_{\mathrm{d}}(L^{\mathrm{dS}}_{\alpha})=\left\{-\left(k+\frac{1+\alpha+\beta}{2}\right)^2\,\middle|\;k\in\mathbb{N}_0,\quad \mathrm{Re}\left(k+\frac{1+\alpha+\beta}{2}\right)<0\right\} \tag{1.7.27}$$

$$\cup\left\{-\left(k+\frac{1+\alpha-\beta}{2}\right)^2\,\middle|\;k\in\mathbb{N}_0,\quad \mathrm{Re}\left(k+\frac{1+\alpha-\beta}{2}\right)<0\right\} \tag{1.7.28}$$

$$\cup\left\{-\left(k+\frac{1-\alpha+\beta}{2}\right)^2\,\middle|\;k\in\mathbb{N}_0,\quad \mathrm{Re}\left(k+\frac{1-\alpha+\beta}{2}\right)<0\right\} \tag{1.7.29}$$

$$\cup\left\{-\left(k+\frac{1-\alpha-\beta}{2}\right)^2\,\middle|\;k\in\mathbb{N}_0,\quad \mathrm{Re}\left(k+\frac{1-\alpha-\beta}{2}\right)<0\right\}. \tag{1.7.30}$$

This produces the correct spectrum. □

One may still insist on using the concept of orthogonal polynomials to obtain a stronger statement, namely that the zeros are simple. This is indeed possible if we use the pseudo-Jacobi polynomials [JT].

The theorems for the other six families have similar proofs, so we state the following result without proof for each family.

Theorem 1.7.6. *Nodal quantization provides a complete and exact characterization of the discrete spectrum of a self-adjoint Schrödinger operator reducible to the hypergeometric equation.*

One may argue that we have not performed a WKB quantization, since we used the eigenfunctions to formulate this theorem. More importantly, the theorem is not an honest approximation method. This is indeed an important point, and we hope to address it soon. Preliminary calculations suggest the following conjecture.

Conjecture: *Hypergeometric Hamiltonians are exact WKB quantizable.*

Moreover, this statement can be made more precise. The Voros coefficients for the hypergeometric equation are constant. As mentioned before, this has only been shown by an ansatz argument in [BOW]. In particular, the Voros coefficient is equal to the Langer modification. This is indeed the reason why Langer modification works in this case. Instead of going further in this direction we will explain why this type of observation has been made separately for matrix models.

Here is a good moment to explain the context in which exact WKB theory has re-emerged in the modern literature. Since this thesis is not centered around these ideas, we only provide a brief overview of recent developments and explain why one-dimensional Schrödinger equations appear naturally in the study of matrix models.

Quantum curves, random matrices, and Schrödinger equations

Let

$$C=\left\{(x,y)\in\mathbb{C}^2\,|\,A(x,y)=0\right\}$$

be a plane curve. A *quantum curve* is a quantization of $A(x,y)$ such that

$$\hat{A}(\hat{x},\hat{y})\psi(p,\hbar)=0, \tag{1.7.31}$$

where $p \in C$, the operators satisfy

$$[\hat{x}, \hat{y}] = -\hbar,$$

and the wave function is constructed using the WKB ansatz

$$\psi(p, \hbar) = \exp\left(\frac{1}{\hbar}\sum_{n=0}^{\infty} \hbar^n S_n(p)\right). \tag{1.7.32}$$

The fundamental question is the following [N]: *Is there a quantization scheme, say $Q: A(x, y) \mapsto \hat{A}(\hat{x}, \hat{y})$, which produces a natural quantization independent of the WKB method?*

Note that the inverse map is, at least formally, easier to construct [N]:

$$\hat{A}(\hat{x}, \hat{y})\psi(p, \hbar) = (A(x, y) + \mathcal{O}(\hbar))\, \psi(p, \hbar). \tag{1.7.33}$$

One answer to this problem is provided by the method known as *topological recursion* [EO]. It suggests that the WKB expansion can be produced recursively from local geometric data. More precisely,

$$S_k(p) = \sum_{2g-1+n=k} \frac{1}{n!} \int^p \cdots \int^p w_{g,n}(p_1, \ldots, p_n), \tag{1.7.34}$$

where $w_{g,n}(p_1, \ldots, p_n)$ are multidifferentials for $g \geq 0$, $n > 0$, living on C, constructed through a universal recursion.

The origin of topological recursion lies in the attempt to solve random matrix models in the large N limit, where N denotes the size of the matrix. These models satisfy an infinite family of constraints known as Virasoro constraints. Indeed, in the next chapter we will use these constraints to solve the β-deformed eigenvalue model.

Chekhov, Eynard, and Orantin used these constraints to construct a recursive procedure on the spectral curve of the matrix model, allowing one to compute the full asymptotic expansion order by order. It was soon realized that this method is universal. For instance, it also recovers Mirzakhani's recursion for Weil–Petersson volumes [Mir]. Later, Gukov and Sułkowski [GS] showed that it can be applied in the framework of quantum curves to quantize the A-polynomials of knots.

The topological recursion is defined from a collection of data called a *spectral curve* (C, B, x, y), where C is a Riemann surface, B is a canonical bidifferential known as the Bergman kernel, and x, y are two meromorphic functions on C. For further details we refer to [BE].

Given this data, the topological recursion takes the form

$$w_{g,n}(p_1, p_{\mathbf{S}}) = \sum_a \mathrm{Res}_{p=a} K(p_1, p)\left(w_{g-1,n+1}(p, \hat{p}, p_{\mathbf{S}}) + \sum_{\substack{g_1+g_2=g \\ I \cup J = \mathbf{S}}}^{\circ} w_{g_1,|I|+1}(p, p_I)\, w_{g_2,|J|+1}(\hat{p}, p_J)\right), \tag{1.7.35}$$

where the sum is taken over the zeros a of dx, assumed to be simple. The symbol $\circ$ indicates that unstable terms involving $w_{0,1}$ are excluded. The initial data are

$$w_{0,1} = -y\, dx, \tag{1.7.36}$$

$$w_{0,2} = B, \qquad K(p_1, p) = \frac{\displaystyle\int_{\hat{p}}^{p} w_{0,2}(p_1, p')}{2\big(y(p) - y(\hat{p})\big)\, dx(p)}. \tag{1.7.37}$$

After the work of [GS], many examples appeared in which topological recursion was successfully applied to determine the corresponding quantum curve. Particularly relevant to our work are three papers where this method was applied to the Bessel equation [DN], the Weber equation [I], and the hypergeometric equation [I1].

Although the method is essentially formal, it has proven remarkably effective in these cases. The main limitation of both topological recursion and exact WKB theory comes from the technical difficulties associated with Riemann surfaces of higher genus, that is, $g > 0$.

Our method does not rely on the topology of the spectral curve. This feature provides a unique opportunity to study other Fuchsian differential equations, including equations of Heun type. With this brief overview of parallel developments, we conclude this chapter and move to the part devoted to random matrix models.

Chapter 2

Superintegrability of β-deformed matrix model

2.1 Symmetric polynomials

In this section, we briefly review the theory of symmetric polynomials. This topic can be approached in many different ways, since it is connected to many areas of mathematics. There is a saying, often attributed to I. G. Macdonald, that *the ultimate goal of algebra is to understand symmetric polynomials.* This section should be seen as an introduction to the main ideas rather than as a completely rigorous treatment. We often omit proofs for the sake of clarity.

As mentioned, symmetric polynomials appear in different fields, from representation theory, where they play the role of characters, to combinatorics, where one can compute them through beautiful problems such as lattice paths, to many-body physics and the determinantal form of fermionic wavefunctions. Therefore, the way one introduces them depends to some extent on the author's taste. For instance, one can start with determinantal formulas to define Schur polynomials and then prove the combinatorial formula in terms of partitions, or prove that they form an orthonormal basis for the Hall inner product. So this is ultimately a matter of choice. I often choose the linear-algebraic approach, since it produces more elegant definitions and clarifies the concepts underlying the theory. For proofs of almost all the facts and statements in this section, we refer to the books by Macdonald [Mac] and Stanley [St1]. There also exists a well-organized website called the *Catalog of Symmetric Polynomials*, by Per Alexandersson [A]. We add, at the end, the proof of Stanley [St] that Jack polynomials are eigenfunctions of the Calogero-Sutherland Hamiltonian.

2.1.1 Partitions

Definition 2.1.1. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a weakly decreasing sequence of positive integers,*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_\ell > 0, \tag{2.1.1}$$

such that

$$|\lambda| := \sum_{i=1}^{\ell} \lambda_i = n. \tag{2.1.2}$$

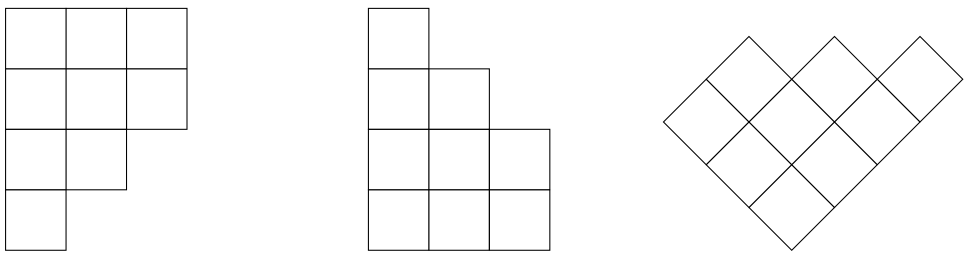
Then we say that λ is a partition of n, and we denote this by $\lambda \vdash n$.

The length of a partition is $\ell(\lambda) = \ell$. We sometimes use other notations for a partition, such as $\lambda = [1^{m_1}, \dots, i^{m_i} \dots]$, where m_i is the multiplicity of the number i in λ.

Definition 2.1.2. *The (English)* Young diagram *is a pictorial representation of a partition arranged from the top-left corner.*

Note that there are at least two other representations associated with a partition, known as the Russian and French Young diagrams. In this thesis, unfortunately, we use the English notation. As an example, we show the Young diagram associated with $\lambda = \{3, 3, 2, 1\}$ in all three common notations below.

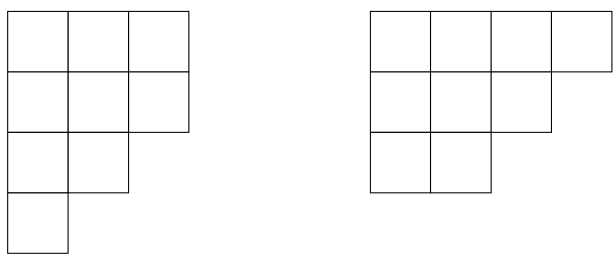
French



Russian

English

λ' is called the conjugate of λ and corresponds to the transpose of the English Young diagram. For instance, the conjugate of $\lambda = \{3, 3, 2, 1\}$ is $\lambda' = \{4, 3, 2\}$.



$\lambda = \{3, 3, 2, 1\}$ $\qquad\qquad$ $\lambda' = \{4, 3, 2\}$

We say $\mu \subseteq \lambda$ if $\lambda_i \geq \mu_i$ for all i.

Definition 2.1.3. *The dominance ordering of two partitions $\lambda \trianglerighteq \mu$ is defined by*

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

for all i.

The dominance order is a partial order. To illustrate this, consider $\lambda = \{4, 1, 1\}$ and $\mu = \{3, 3\}$.

Partitions are naturally connected to the symmetric group and its representations. For instance, there is an interesting number associated with the center of the symmetric group. The number z_λ is the size of the centralizer of a permutation of type λ in $\mathfrak{S}_N$:

$$z_\lambda := \left| \left\{ g \in \mathfrak{S}_N \,\middle|\, g h_\lambda g^{-1} = h_\lambda \right\} \right| . \tag{2.1.3}$$

This number can be evaluated easily by realizing that for each cycle of length i we can rotate it i times, which preserves the cycle, and there are $m_i!$ permutations of cycles of the same length. So

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!. \tag{2.1.4}$$

The number

$$c_{i,j} := j - i \tag{2.1.5}$$

is called the content of the box in the i-th row and j-th column.

Definition 2.1.4. *Standard Young tableaux are fillings of a Young diagram such that the rows and columns are strictly increasing.*

Definition 2.1.5. *Semistandard Young tableaux are fillings of a Young diagram such that they are weakly increasing along rows and strictly increasing along columns.*

The reading word of a (semi)standard Young tableau is the concatenation of its rows.

2.1.2 Ring of symmetric polynomials

Let $\mathbb{Q}[x_1, \ldots, x_N]$ be the ring of polynomials over the rational numbers in N variables $\{x_i\}_{i=1}^N$; we might also call $\{x_i\}_{i=1}^\infty$ an alphabet $\mathbf{x}$ (this often refers to infinitely many variables). For $\sigma \in \mathfrak{S}_N$, we define the action of the symmetric group on a polynomial $p \in \mathbb{Q}[x_1, \ldots, x_n]$ by

$$\sigma \cdot p(x_1, \ldots, x_n) := p\left(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\right). \tag{2.1.6}$$

A polynomial p is called symmetric if

$$\sigma \cdot p = p. \tag{2.1.7}$$

The ring of symmetric polynomials in n variables, denoted Λ_N, is graded with respect to the degree of a polynomial:

$$\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k, \tag{2.1.8}$$

where k is the degree. There is a clear forgetful morphism $f_{n,m} : \Lambda_N \to \Lambda_m$ for $n > m$. This morphism, together with the projection $\pi : \mathbf{x} \to \{x_i\}_{i=1}^n$, allows us to define an inverse limit $\varprojlim \Lambda_N^k = \Lambda^k$, and hence define the ring of symmetric functions

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k. \tag{2.1.9}$$

In the following we will spend a lot of time explaining distinguished members of the ring of symmetric polynomials. We often use formal mathematics and do not care about boundedness or convergence of these objects; in particular, the generating functions are treated formally. We start with the most obvious one. We abuse the notation $\mathbf{x}$ to denote a sequence of N variables.

Definition 2.1.6. Monomial symmetric polynomials $m_\lambda(\mathbf{x})$ *are polynomials indexed by a partition λ:*

$$m_\lambda(\mathbf{x}) = \sum_{\alpha \sim \lambda} \mathbf{x}^\alpha, \qquad \mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}, \tag{2.1.10}$$

where $\alpha \sim \lambda$ means all possible rearrangements of the parts of λ into a sequence α.

We can immediately see that monomial symmetric polynomials are stable:

$$m_\lambda(x_1, \ldots, x_{N-1}, 0) = m_\lambda(x_1, \ldots, x_{N-1}). \tag{2.1.11}$$

Theorem 2.1.7. *Monomial symmetric polynomials form a basis for the ring of symmetric polynomials.*

Proof. Pick any polynomial in the ring of symmetric polynomials $f \in \Lambda_N$:

$$f = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha \mathbf{x}^\alpha. \tag{2.1.12}$$

Since f is symmetric, for all $\sigma \in \mathfrak{S}_N$ we have $c_{\sigma\alpha} = c_\alpha$. Therefore, we can weakly order the sequence α. This divides the orbits into partitions, giving a partition λ with coefficient c_λ. This means that we can obtain all possible sequences of n integers by starting with all possible partitions and acting on them with the permutation group. Eventually we have

$$f = \sum_\lambda c_\lambda \sum_{\alpha \sim \lambda} \mathbf{x}^\alpha = \sum_\lambda c_\lambda m_\lambda. \tag{2.1.13}$$

So m_λ spans Λ_N^k. Indeed, they also form a basis, which means that if

$$\sum_\lambda a_\lambda m_\lambda = 0, \tag{2.1.14}$$

then $a_\lambda = 0$ for all λ. To see this, we look at the monomial $\mathbf{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_l^{\lambda_l}$ in m_λ; this cannot appear in m_μ for $\mu \neq \lambda$. Therefore $a_\lambda = 0$. □

We picked monomial symmetric polynomials as our building blocks. As we will show, there are other bases for the ring of symmetric polynomials. We proved in detail that monomial symmetric polynomials form a basis. We will not prove such a theorem for the other bases. One can find proofs in standard texts [Mac, St1]. As mentioned, we choose to introduce these polynomials through the linear-algebraic scheme. In this method the proofs often require showing triangularity of the coefficients and can be quite non-trivial. However, there is a clear conceptual reason behind this method: one can see the structure of the bases underlying the theory. To introduce other bases, we start with special cases of the monomial symmetric basis. Let $\lambda = (k)$ be a row of k boxes.

Definition 2.1.8. *The* power-sum polynomials *are defined by*

$$p_\lambda = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}; \quad \textit{where} \quad p_k := \sum_{i=1}^{n} x_i^k = m_{(k)}. \tag{2.1.15}$$

Unfortunately, both p_k and p_λ are called power-sum polynomials. But only p_λ forms a basis for the ring of symmetric polynomials.

An important object to construct, which is extremely helpful in calculations, is the generating function. The generating function for power-sum polynomials is

$$\mathcal{P}'(t) = \sum_{r \geq 1} p_r t^{r-1} = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t}, \tag{2.1.16}$$

where it is the derivative of another useful identity

$$\mathcal{P}(t) = \sum_{k \geq 1} \frac{p_k}{k} t^k = -\sum_{i=1}^{n} \log\left(1 - x_i t\right). \tag{2.1.17}$$

One can equivalently write this as

$$\mathrm{e}^{\sum_{k \geq 1} \frac{p_k}{k} t^k} = \prod_{i=1}^{n} \frac{1}{1 - x_i t}. \tag{2.1.18}$$

Now we set the partition in m_λ to be a column of k boxes only, $\lambda = \left[1^k\right]$.

Definition 2.1.9. Elementary symmetric polynomials *are defined by*

$$e_\lambda = \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}; \quad \textit{where} \quad e_k := m_{[1^k]}. \tag{2.1.19}$$

The e_λ form a basis for the space of symmetric polynomials. Note again that e_k alone does not form a basis. The generating function for elementary symmetric polynomials is

$$\mathcal{E}(t) := \sum_{k\geq 0} e_k t^k = \prod_{i=1}^{n} (1 + x_i t)\,, \tag{2.1.20}$$

with $e_0 = 1$.

Definition 2.1.10. Completely homogeneous symmetric polynomials *are defined by*

$$h_\lambda := \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i}; \quad \textit{where} \quad h_k := \sum_{\mu \vdash k} m_\mu. \tag{2.1.21}$$

h_λ forms a basis for the ring of symmetric polynomials. Indeed, the triangularity is quite interesting in this case. Let

$$h_\lambda = \sum_{\mu \vdash |\lambda|} K_{\mu\lambda} m_\mu. \tag{2.1.22}$$

$K_{\mu\lambda}$ is called the *Kostka coefficient*. One can prove that [Mac]

$$h_\lambda = m_\lambda + \sum_{\mu \lhd \lambda} K_{\mu\lambda} m_\mu. \tag{2.1.23}$$

This is indeed the triangularity relation. We list some properties of Kostka coefficients [Mac, St1]:

1. $K_{\mu\lambda}$ is the number of semistandard Young tableaux of shape μ with weight λ, which means that the number i appears λ_i times in the tableau.

2. $K_{\lambda\lambda} = 1$.

3. $K_{\mu\lambda} = 0$ unless $\lambda \unrhd \mu$.

The generating function for completely homogeneous symmetric polynomials is

$$\mathcal{H}(t) := \sum_{k\geq 0} h_k t^k = \prod_{i=1}^{n} \frac{1}{1 - x_i t} = \mathrm{e}^{\mathcal{P}(t)}. \tag{2.1.24}$$

Note that $\mathcal{E}(t)\mathcal{H}(-t) = 1$. We can use this identity to find an involution on the ring of symmetric functions. There exists a fundamental involution on the ring of symmetric functions $\omega : \Lambda \to \Lambda$ such that $\omega(e_r) = h_r$. This extends to the whole ring linearly and multiplicatively. One can immediately see that it is an involution:

$$\mathcal{E}(t)\mathcal{H}(-t) = 1, \tag{2.1.25}$$

$$\omega(\mathcal{E})(t)\omega(\mathcal{H})(-t) = 1, \tag{2.1.26}$$

$$\mathcal{H}(t)\omega(\mathcal{H}(-t)) = 1 \implies \omega(\mathcal{H}(t)) = \mathcal{E}(t) \implies \omega(h_k) = e_k. \tag{2.1.27}$$

In what follows, we add an inner-product structure to the ring of symmetric polynomials.

2.1.3 Hall inner product

Definition 2.1.11. *Let* $\langle \bullet, \bullet \rangle$ *be a bilinear form* $\Lambda \times \Lambda \to \mathbb{R}$ *defined by*

$$(p_\lambda, p_\mu) = z_\lambda \delta_{\lambda,\mu}. \tag{2.1.28}$$

We call this inner product the Hall inner product.

It is clear that the power-sum functions form an orthogonal basis with respect to the Hall inner product. Now we recall some definitions from linear algebra and apply them to this theory. Later, we will use these concepts for the deformation as well.

Definition 2.1.12. *We say that* $k(\mathbf{x}, \mathbf{y})$ *is a (Hall) reproducing kernel if for any member of the ring* $f \in \Lambda$ *we have*

$$\langle f(\mathbf{x}), k(\mathbf{x}, \mathbf{y}) \rangle = f(\mathbf{y}). \tag{2.1.29}$$

It is easy to see that the Hall reproducing kernel is given by

$$k(\mathbf{x}, \mathbf{y}) = \sum_\lambda \frac{p_\lambda(\mathbf{x}) p_\lambda(\mathbf{y})}{z_\lambda}. \tag{2.1.30}$$

We can rewrite the Hall reproducing kernel as the *Cauchy identity*. First note that

$$\mathrm{e}^{\sum_{k\geq 1} \frac{p_k(\mathbf{x}) p_k(\mathbf{y})}{k}} = \prod_{k \geq 1} \sum_{m_k \geq 0} \frac{1}{m_k!} \left(\frac{p_k(\mathbf{x}) p_k(\mathbf{y})}{k} \right)^{m_k}. \tag{2.1.31}$$

We use $p_\lambda = \prod_k p_k^{m_k}$ and $z_\lambda = \prod_{k \geq 1} m_k! k^{m_k}$ to get

$$k(\mathbf{x}, \mathbf{y}) = \mathrm{e}^{\sum_{k\geq 1} \frac{p_k(\mathbf{x}) p_k(\mathbf{y})}{k}} = \prod_{i,j} \frac{1}{1 - x_i y_j}, \tag{2.1.32}$$

which is the *Cauchy kernel*. Note that $\mathcal{P}(t)$ is the Cauchy kernel when $\mathbf{y} = (t, 0, \ldots, 0)$.

Theorem 2.1.13. *Let* $f_\lambda, g_\lambda \in \Lambda$ *be dual bases with respect to the Hall inner product. Then the reproducing kernel is given by*

$$k(\mathbf{x}, \mathbf{y}) = \sum_\lambda f_\lambda(\mathbf{x}) g_\lambda(\mathbf{y}). \tag{2.1.33}$$

By rearranging the Cauchy kernel we can easily obtain the following form of the kernel:

$$k(\mathbf{x}, \mathbf{y}) = \sum_\lambda h_\lambda(\mathbf{x}) m_\lambda(\mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j}. \tag{2.1.34}$$

This is the closest expression we can find for the generating function of the monomial symmetric basis; since they are labelled by partitions, one cannot find a simpler object. Also, we can clearly see that the monomial symmetric polynomials m_λ and the completely homogeneous symmetric polynomials h_λ are dual bases.

Definition 2.1.14. Schur polynomials *form an orthonormal (self-dual) basis for the ring of symmetric polynomials, which means that they satisfy*

1. *orthonormality:*

$$(S_\lambda, S_\mu) = \delta_{\lambda,\mu}. \tag{2.1.35}$$

2. *triangularity:*

$$S_\lambda = m_\lambda + \sum_{\mu \lhd \lambda} K_{\lambda\mu} m_\mu. \tag{2.1.36}$$

The self-duality manifests itself in the kernel identity

$$k(\mathbf{x}, \mathbf{y}) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda S_\lambda(\mathbf{x}) S_\lambda(\mathbf{y}). \tag{2.1.37}$$

This identity can be used to evaluate the Schur polynomial. I would like to clarify that this is not the usual definition of the Schur polynomial. People often prefer to define it via a determinant, since in that approach it is clearer how to evaluate the Schur polynomial. I hope my approach justifies itself by the clarity it brings to the subject. We list some of the crucial properties of Schur polynomials. It is important to mention that Schur polynomials, as one of the most fundamental objects in mathematics, have enough features to fill a whole book dedicated to them. However, one has to be selective.

1. In the triangularity of Schur polynomials, $K_{\lambda\mu}$ is a Kostka coefficient.
2. S_λ is homogeneous of degree $|\lambda|$, since the Hall inner product respects the grading.
3. A Schur polynomial is the character of a finite-dimensional irreducible polynomial representation of $GL(N)$ and of its compact real form $U(N)$.
4. The following identity captures what is known as Schur–Weyl duality:

$$p_\mu = \sum_{\lambda \vdash |\mu|} \chi_\mu^\lambda S_\lambda, \tag{2.1.38}$$

where χ_μ^λ is the irreducible character of $\mathfrak{S}_{|\lambda|}$ indexed by λ, evaluated on the conjugacy class of type μ. One can easily invert this identity using the Hall inner product:

$$S_\lambda = \sum_{\mu \vdash |\lambda|} \frac{\chi_\mu^\lambda}{z_\mu} p_\mu. \tag{2.1.39}$$

2.2 Jack polynomials and α-deformation

We deform the Hall inner product with a real parameter $\alpha \in \mathbb{R}$ as

$$(p_\lambda, p_\mu)_\alpha = \alpha^{\ell(\lambda)} z_\lambda \delta_{\lambda,\mu}. \tag{2.2.1}$$

Definition 2.2.1. *Jack-P polynomials are an orthogonal basis for the α-deformed Hall inner product with a particular normalization. This is equivalent to:*

1. *orthogonality:*

$$(P_\lambda, P_\mu)_\alpha = 0, \text{ for } \lambda \neq \mu. \tag{2.2.2}$$

2. *triangularity:*

$$P_\lambda = \sum_{\mu \unlhd \lambda} C_{\lambda\mu} m_\mu. \tag{2.2.3}$$

3. *P-normalization:*

$$[m_\lambda] \, P_\lambda = 1. \tag{2.2.4}$$

There is another common Jack function, especially used in combinatorics, known as the Jack-J polynomials. The Jack-J polynomials have the same orthogonality and triangularity conditions, but the normalization is changed to the J-normalization:

$$\left[m_{[1^{|\lambda|}]}\right] J_\lambda = |\lambda|!. \tag{2.2.5}$$

It is noteworthy that there is an alternative way of looking at Jack polynomials using the combinatorics of partitions [Mac]. Jack polynomials do not play the same fundamental role in representation theory as Schur polynomials. The main reason is that Lie groups do not possess such a deformation. Thus, many representation-theoretic tools are not available for Jack polynomials. However, they appear in mathematical physics as eigenfunctions of certain many-body Hamiltonians.

2.2.1 Calogero-Sutherland Hamiltonian

The trigonometric Calogero-Sutherland Hamiltonian is a differential operator associated with n variables, defined by [Cal, Sut]

$$\tilde{H} = -\frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2}{\partial \theta_i^2} + \frac{1-\alpha}{\alpha^2}\sum_{i<j} \frac{1}{\sin^2 \frac{\theta_i - \theta_j}{2}}. \tag{2.2.6}$$

This Hamiltonian describes the dynamics of n particles on the unit circle; for further discussion see [P]. We are interested here in some formal calculations involving this operator. We set $z_i := \mathrm{e}^{\mathrm{i}\theta_i}$ and define the following factor:

$$\tilde{\Delta} = \prod_{i<j} \sin\left(\frac{\theta_i - \theta_j}{2}\right), \tag{2.2.7}$$

which is proportional to a Vandermonde:

$$\tilde{\Delta} = (\prod_{i=1}^{n} z_i)^{-\frac{n-1}{2}} \Delta(z); \quad \Delta(z) = \prod_{i<j}(z_i - z_j). \tag{2.2.8}$$

After gauging the Calogero–Sutherland Hamiltonian we get

$$\tilde{\Delta}^{\alpha} \tilde{H} \tilde{\Delta}^{-\alpha} =: \mathcal{H} = \frac{\alpha}{2}\sum_{i=1}^{n} (z_i \partial_{z_i})^2 + \frac{1}{2}\sum_{i<j} \frac{z_i + z_j}{z_i - z_j}(z_i\partial_{z_i} - z_j\partial_{z_j}). \tag{2.2.9}$$

We often refer to *Eqn.*(2.2.9) as the Calogero–Sutherland Hamiltonian, while the physical Hamiltonian for the n-body system is clearly *Eqn.*(2.2.6).

Our goal is to show that Jack-P polynomials are the eigenpolynomials of this operator. This is a result due to Stanley [St]. We will later see the connection to the Dunkl Laplacian and the Heckman–Opdam hypergeometric function. For now, we want to solve the eigenvalue problem for H_α.

But before showing that, we recall some small theorems from spectral theory.

Definition 2.2.2. *Let $\mathcal{W}$ be a vector space equipped with a bilinear form $(\bullet, \bullet) : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$. We say that an operator $A : \mathcal{W} \to \mathcal{W}$ is a symmetric operator if*

$$(Aw_1, w_2) = (w_1, Aw_2), \quad \forall w_1, w_2 \in \mathcal{W}. \tag{2.2.10}$$

It is important to clarify that sometimes people call a symmetric operator a Hermitian operator. We reserve the term Hermitian operator for a sesquilinear form. For finite-dimensional vector spaces, a symmetric operator is self-transposed. We avoid using this terminology in order to decrease confusion. For the clear explanation of the bilinear and sesquilinear form on Hilbert spaces look at [DeGe].

Theorem 2.2.3. *If A is symmetric, then eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the bilinear form $(\bullet, \bullet)$. Conversely, if A admits an orthogonal basis of eigenfunctions, then A is symmetric.*

Proof. *Let $Af_i = a_i f_i$ and $a_i \neq a_j$ for $i \neq j$. If A is symmetric, then*

$$(Af_i, f_j) - (f_i, Af_j) = 0 \implies E(f_i, f_j)(a_i - a_j) = 0, \tag{2.2.11}$$

which means that f_i and f_j are orthogonal. Conversely, if A admits an orthogonal basis of eigenfunctions, then by bilinearity we see that A is a symmetric operator. □

To see that the eigenfunctions of the Calogero–Sutherland Hamiltonian are Jack-P polynomials, we first show that Jack polynomials are eigenfunctions. First notice that H_α preserves the grading on the ring of symmetric polynomials. Then we will show that

$$\mathcal{H} m_\lambda = \epsilon_\lambda m_\lambda + \sum_{\mu \lhd \lambda} c_{\lambda\mu} m_\mu. \tag{2.2.12}$$

To see this we first define another operator $D_i = z_i \partial_{z_i}$. It is easy to see that

$$\mathcal{D}_i \mathbf{x}^\lambda = \lambda_i \mathbf{x}^\lambda, \quad \sum_{i=1}^{n} \mathcal{D}_i \mathbf{x}^\lambda = |\lambda| \mathbf{x}^\lambda. \tag{2.2.13}$$

So we get the first term in H_α:

$$\sum_{i=1}^{n} \mathcal{D}_i^2 m_\lambda = \sum_{i=1}^{n} \lambda_i^2 m_\lambda. \tag{2.2.14}$$

The action of the interaction term on the diagonal term can be seen easily:

$$\sum_{i<j} \frac{z_i + z_j}{z_i - z_j} (\mathcal{D}_i - \mathcal{D}_j) \mathbf{x}^\lambda = \sum_{i<j} \left(1 + \frac{2z_j}{z_i - z_j}\right) (\lambda_i - \lambda_j) \mathbf{x}^\lambda \tag{2.2.15}$$

$$= \sum_{i=1}^{n} (n + 1 - 2i) \lambda_i \mathbf{x}^\lambda + \text{off-diagonal terms}. \tag{2.2.16}$$

Thus the eigenvalue is

$$\epsilon_\lambda = \sum_{i=1}^{n} \left(\frac{\alpha}{2} \lambda_i^2 + \frac{n + 1 - 2i}{2} \lambda_i \right). \tag{2.2.17}$$

To see the off-diagonal term we have to consider two terms in the action. We pick $z_i^{\mu_1} z_j^{\mu_2} + z_i^{\mu_2} z_j^{\mu_1}$, for $\mu_1 > \mu_2$:

$$\frac{z_j}{z_i - z_j} (\mathcal{D}_i - \mathcal{D}_j)(z_i^{\mu_1} z_j^{\mu_2} + z_i^{\mu_2} z_j^{\mu_1}) = (\mu_1 - \mu_2) \frac{z_i^{\mu_1} z_j^{\mu_2 + 1} - z_i^{\mu_2} z_j^{\mu_1 + 1}}{z_i - z_j} \tag{2.2.18}$$

$$= (\mu_1 - \mu_2) \sum_{r=0}^{\mu_1 - \mu_2 - 1} z_i^{\mu_1 - 1 - r} z_j^{\mu_2 + 1 + r}. \tag{2.2.19}$$

This is clearly decreasing with respect to the dominance ordering. Thus, H_α respects both the dominance ordering and the degree. So we see that the eigenfunctions of the Calogero–Sutherland Hamiltonian have the triangular relation of the Jack polynomials with the normalization of Jack-P.

We still have to show that the Calogero–Sutherland Hamiltonian is a symmetric operator with respect to the α-deformed Hall inner product. To do so we first need the following lemma.

Lemma 2.2.4. *Let $f, g \in \Lambda_N$. The adjoint of the power-sum polynomial is determined by*

$$(f, p_n g)_\alpha = \left(p_n^\perp f, g\right)_\alpha, \quad \text{where} \quad p_n^\perp = \alpha n \partial_{p_n}. \tag{2.2.20}$$

We want to use the above lemma, so we write the Calogero–Sutherland Hamiltonian in terms of power-sum polynomials in the alphabet $\mathbf{z}$. To do so, we use $\partial_{z_i} = \sum_{k\geq 1} z_i^{k-1} k \partial_{p_k}$ and obtain

$$\mathcal{H}_\alpha = \frac{\alpha-1}{2}\sum_{k\geq 1} k(k-1)p_k\partial_{p_k} + \frac{\alpha}{2}\sum_{k,m\geq 1} kmp_{k+m}\partial_{p_k}\partial_{p_m} + \frac{1}{2}\sum_{k,m\geq 1}(k+m)p_kp_m\partial_{p_{k+m}}. \tag{2.2.21}$$

The first term is self-transposed, and after taking the transposed the second term becomes the third term. Therefore the Calogero–Sutherland Hamiltonian is a symmetric operator with distinct eigenvalues, and thus the eigenpolynomials are the Jack-P polynomials. We use this fact later to prove the superintegrability expression in the matrix model.

Remark 2.2.5. *We end our discussion of symmetric polynomials with a remark on β-, α-, and b-deformations. There are three related deformations in mathematics associated with the Hall inner product, mathematical physics, and unoriented surfaces. In random matrix theory people use $\alpha = \frac{1}{\beta}$ [EKR]. In combinatorics, people often use $\alpha = 1 + b$, hence the term b-deformation [GJ].*

2.3 A proof of superintegrability for the gaussian model

The partition functions of interest, involve a potential $V(z)$ and are given by

$$Z_\beta(V;p_k) \overset{\text{def}}{=} \int \left(\prod_{i=1}^N dz_i\right)[\Delta(z)]^{2\beta}\exp\left[-\sum_{i=1}^N V(z_i)\right]\exp\left[\beta\sum_{k=1}^\infty \frac{p_k}{k}\sum_{i=1}^N z_i^k\right] \tag{2.3.1}$$

where $\Delta(z) = \prod_{1\leq i<j\leq N}(z_i - z_j)$ is the Vandermonde factor. As is common practice, p_k plays a double role: it denotes the coupling constants in the action, and also the power-sum polynomials in terms of which various other symmetric functions are expanded. The integral converges for p_k sufficiently small [Gu]. In this section we consider the purely Gaussian potential

$$V(z) = \frac{a_2}{2}z^2. \tag{2.3.2}$$

It follows from the invariance of the partition function under the transformation $z_i \to z_i + \epsilon z_i^{n+1}$, $n \geq -1$, for all $i \in \{1, \dots, N\}$, that the partition function satisfies the Virasoro constraints

$$\mathcal{L}_n Z_\beta = 0, \quad n \geq -1, \tag{2.3.3}$$

where the Virasoro operators $\mathcal{L}_n$, $n \geq -1$, satisfy $[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m}$. The choice of the space of functions $n \geq 1$ is not arbitrary. The space of functions for the coordinate transformation is uniquely fixed by the condition that the variation of the partition function should be a polynomial function. This is a highly nontrivial condition. Indeed, for values $n < -1$ we obtain a rational function of z_i, which cannot be expressed as a derivative with respect to the coupling constants p_k. But why should we care? The Virasoro algebra is the unique central extension of the Witt algebra [Scho]

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{c}{12}n(n-1)(n+1)\delta_{n,-m}\mathbf{1}, \tag{2.3.4}$$

where c is the central extension and $\mathbf{1}$ is the identity element of the algebra. Note that for $n \geq -1$, one cannot see the central charge c. One can extend the Lie algebra generators $\mathcal{L}_m$ coming from *Eqn.*(2.3.1) to all integers using the free bosonic vertex algebra, and find that the central charge is [AOS]

$$c = 1 - 6\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)^2. \tag{2.3.5}$$

Thus, as we can see, the name Virasoro is justified, since the algebra indeed possesses a nontrivial central charge $c \neq 0$.

An explicit representation of the Virasoro generators in the β-deformed eigenvalue model is given by

$$\mathcal{L}_n = -\frac{a_2(n+2)}{\beta}\frac{\partial}{\partial p_{n+2}} + L_n \tag{2.3.6}$$

where the L_n's are the potential-independent part of the Virasoro operators:

$$\begin{aligned} L_{-1} &= \beta p_1 N + \sum_{k=1}^{\infty} k p_{k+1}\frac{\partial}{\partial p_k}, \\ L_0 &= \beta N^2 + (1-\beta)N + \sum_{k=1}^{\infty} k p_k \frac{\partial}{\partial p_k}, \\ L_{n>0} &= \sum_{k=1}^{\infty}(n+k)p_k\frac{\partial}{\partial p_{n+k}} + 2Nn\frac{\partial}{\partial p_n} \\ &\quad + \frac{1}{\beta}\sum_{k=1}^{n-1} k(n-k)\frac{\partial^2}{\partial p_k \partial p_{n-k}} + \frac{1-\beta}{\beta}n(n+1)\frac{\partial}{\partial p_n}. \end{aligned} \tag{2.3.7}$$

It follows *a fortiori* that

$$-a_2^{-1}\beta\sum_{n=1}^{\infty} p_n \mathcal{L}_{n-2} Z_\beta \stackrel{\text{def}}{=} \left(\mathcal{D} - a_2^{-1}W_{-2}\right) Z_\beta = 0, \tag{2.3.8}$$

where we have defined $\mathcal{D} \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} np_n \frac{\partial}{\partial p_n}$ and

$$\begin{aligned} W_{-2} \stackrel{\text{def}}{=} &\beta^2 N p_1^2 + \beta^2 N^2 p_2 + (1-\beta)\beta N p_2 \\ &+ \sum_{k,l=1}^{\infty}\left[\beta(k+l-2)p_k p_l \frac{\partial}{\partial p_{k+l-2}} + klp_{k+l+2}\frac{\partial^2}{\partial p_k \partial p_l}\right] \\ &+ \sum_{k=1}^{\infty}\left[2\beta N k p_{k+2}\frac{\partial}{\partial p_k} + (1-\beta)k(k+1)p_{k+2}\frac{\partial}{\partial p_k}\right]. \end{aligned} \tag{2.3.9}$$

It was proven in [BCD] that *Eqn.*(2.3.8) is equivalent to *Eqn.*(2.3.3) using an induction argument. This is indeed related to the Airy structure and topological recursion; for more details see [KS].

It can be verified that $[\mathcal{D}, W_{-2}] = 2W_{-2}$. The partition function *Eqn.*(2.3.1) can then be written as

$$Z_\beta = e^{W_{-2}/2a_2} \cdot 1. \tag{2.3.10}$$

Our goal is to evaluate the right-hand side above. It turns out that the Jack polynomials constitute a natural basis of special functions in which to describe the action of W_{-2}.

When a symmetric function, P_λ for instance, is written in terms of the basic variables, we denote its arguments by parentheses: $P_\lambda(z_1, \ldots, z_N)$. More often, we will need to write it in terms of the power-sum polynomials $p_k = \sum_{i=1}^{N} z_i^k$, in which case we denote it by $P_\lambda\{p_k\}$. In particular, $P_\lambda\{f(k)\}$ is shorthand for $P_\lambda\{p_k = f(k)\}$.

As we discussed in the previous section, Jack polynomials are known to be eigenpolynomials of the following operators [CJ]:

$$\begin{aligned}\mathcal{D} &= \sum_{k=1}^{\infty} k p_k \frac{\partial}{\partial p_k},\\ \mathcal{H} &= \frac{1}{2}\sum_{k,l=1}^{\infty}\left[\beta(k+l)p_k p_l \frac{\partial}{\partial p_{k+l}} + kl p_{k+l}\frac{\partial^2}{\partial p_k \partial p_l}\right] + \frac{1-\beta}{2}\sum_{k=1}^{\infty} k(k-1)p_k \frac{\partial}{\partial p_k},\end{aligned} \tag{2.3.11}$$

so that

$$\begin{aligned}\mathcal{D}P_\lambda &= |\lambda| P_\lambda,\\ \mathcal{H}P_\lambda &= c_\lambda^{(\beta)} P_\lambda,\end{aligned} \tag{2.3.12}$$

and

$$c_\lambda^{(\beta)} = \sum_{(i,j)\in\lambda} (j-1-\beta(i-1)) = n(\lambda') - \beta n(\lambda). \tag{2.3.13}$$

Also, note that the operator $\mathcal{H}$ defined above differs from a similar operator defined in [WZC] and [CJ] by a shift of $\mathcal{D}$. This makes the eigenvalues of the resulting operator more natural, while also simplifying the expression *Eqn.*(2.3.14) for W_{-2} given below.

A crucial step in the proof is determining the action of W_{-2} on Jack polynomials. In [WZC], W_{-2} was presented in terms of $\mathcal{H}$ when $\beta = 1$, which lets us almost immediately read off the action of W_{-2} on Schur polynomials. Following a similar strategy for an indeterminate β, one finds

$$\begin{aligned}W_{-2} =& \frac{1}{4}\left[\mathcal{H}, [\mathcal{H}, p_2]\right] + N\beta\left[\mathcal{H}, p_2\right]\\ &+ N^2\beta^2 p_2 - \frac{1}{4}(1-\beta+\beta^2)p_2 - \frac{\beta(1-\beta)}{4}p_1^2.\end{aligned} \tag{2.3.14}$$

We will also need the following Pieri rule for Jack polynomials (see Theorem 6.3 of [KS], and [Naq] for additional explanations and examples):

$$P_{(1^r)}P_\mu = \sum_\lambda c^\lambda_{\mu,(1^r)} P_\lambda \tag{2.3.15}$$

where the coefficients $c^\lambda_{\mu,(1^r)}$ are known and to which we will return later. For now, we simply note that the sum on the right runs over partitions λ such that $\lambda - \mu$ is a vertical r-strip (i.e. a skew diagram consisting of r boxes, no two of which are in the same row). We also note that $P_{(1^r)} = s_{(1^r)} = e_r$, where s and e are the Schur and elementary symmetric polynomials, respectively.

There is a similar Pieri rule for Jack polynomials indexed by row partitions (see Proposition 5.3 and Theorem 6.1 in [St]), given in terms of the J_λ:

$$J_r J_\mu = \sum_\lambda \frac{g^\lambda_{\mu,r}}{j_\lambda} J_\lambda \tag{2.3.16}$$

where now the sum runs over partitions λ such that $\lambda - \mu$ is a horizontal r-strip (i.e. a skew diagram consisting of r boxes, no two of which are in the same column). The constants $g^\lambda_{\mu,r}$ and j_λ are known, but we will not need to use them.

In particular, for $r = 1$, the diagrams λ on the right are those obtained by adding one box to μ. Multiplying two factors of $P_1 = p_1$ with P_μ therefore gives a sum over all partitions where two boxes are added to μ without any constraint. For $r = 2$, multiplying with $P_{(1,1)} = (p_1^2 - p_2)/2$ gives a sum over partitions where the two additional boxes cannot lie in the same row. Consequently, expanding $p_2 P_\mu$ over

Jack polynomials gives a sum over partitions where two boxes are added to μ without any constraint. One may therefore write

$$p_2 P_\mu = \sum_{\lambda=\mu+\square+\square} C_{\mu\lambda} P_\lambda. \tag{2.3.17}$$

One may also write

$$p_1^2 P_\mu = \sum_{\lambda=\mu+\square+\square} A_{\mu\lambda} C_{\mu\lambda} P_\lambda. \tag{2.3.18}$$

Using this notation and *Eqn.*(2.3.14), the action of W_{-2} on a Jack polynomial is

$$\begin{aligned} W_{-2} P_\mu = \sum_{\lambda=\mu+\square_1+\square_2} & \Big[(j_1 - 1 + \beta(N - i_1 + 1))(j_2 - 1 + \beta(N - i_2 + 1)) \\ & + \frac{1}{4}(j_2 - j_1 + \beta(i_1 - i_2))^2 - \frac{1}{4}(1 - \beta + \beta^2) \\ & - \frac{\beta(1-\beta)}{4} A_{\mu\lambda} \Big] C_{\mu\lambda} P_\lambda, \end{aligned} \tag{2.3.19}$$

where (i_1, j_1) and (i_2, j_2) are the coordinates of the boxes $\square_1$ and $\square_2$, respectively. We observe that the first line on the right-hand side corresponds precisely to the expression in *Eqn.*(35) of **[?]** (noting that they count rows and columns starting from 0, while we begin counting from 1). We now show, by computing $A_{\mu\lambda}$, that the terms in the second and third lines cancel.

Observe that

$$A_{\mu\lambda} = \frac{\langle P_\lambda, p_1^2 P_\mu \rangle}{\langle P_\lambda, p_2 P_\mu \rangle}, \tag{2.3.20}$$

where $\langle \cdot, \cdot \rangle$ denotes the Jack scalar product. The constraint in the summation in *Eqn.*(2.3.15) tells us that $\langle P_{\mu+\square\square}, P_{(1,1)} P_\mu \rangle = 0$, implying that $A_{\mu,\mu+\square\square} = 1$. Likewise, *Eqn.*(2.3.16) tells us that $\langle J_{\mu+\begin{smallmatrix}\square\\\square\end{smallmatrix}}, J_2 J_\mu \rangle = 0$. Using $J_2 = p_1^2 + \beta^{-1} p_2$ we see that

$$-\beta = \frac{\langle J_{\mu+\begin{smallmatrix}\square\\\square\end{smallmatrix}}, p_1^2 J_\mu \rangle}{\langle J_{\mu+\begin{smallmatrix}\square\\\square\end{smallmatrix}}, p_2 J_\mu \rangle} = \frac{\langle P_{\mu+\begin{smallmatrix}\square\\\square\end{smallmatrix}}, p_1^2 P_\mu \rangle}{\langle P_{\mu+\begin{smallmatrix}\square\\\square\end{smallmatrix}}, p_2 P_\mu \rangle} = A_{\mu,\mu+\begin{smallmatrix}\square\\\square\end{smallmatrix}}. \tag{2.3.21}$$

More generally, one can write

$$A_{\mu\lambda} = \frac{\langle P_\lambda, p_1^2 P_\mu \rangle}{\langle P_\lambda, p_2 P_\mu \rangle} = \frac{1}{1 - 2\frac{\langle P_\lambda, P_{(1,1)} P_\mu \rangle}{\langle P_\lambda, P_1^2 P_\mu \rangle}}. \tag{2.3.22}$$

We now consider the case where the two boxes are staggered, i.e. added to neither the same row nor the same column. The above expression can be related to the coefficients appearing in the Pieri rule *Eqn.*(2.3.15) as

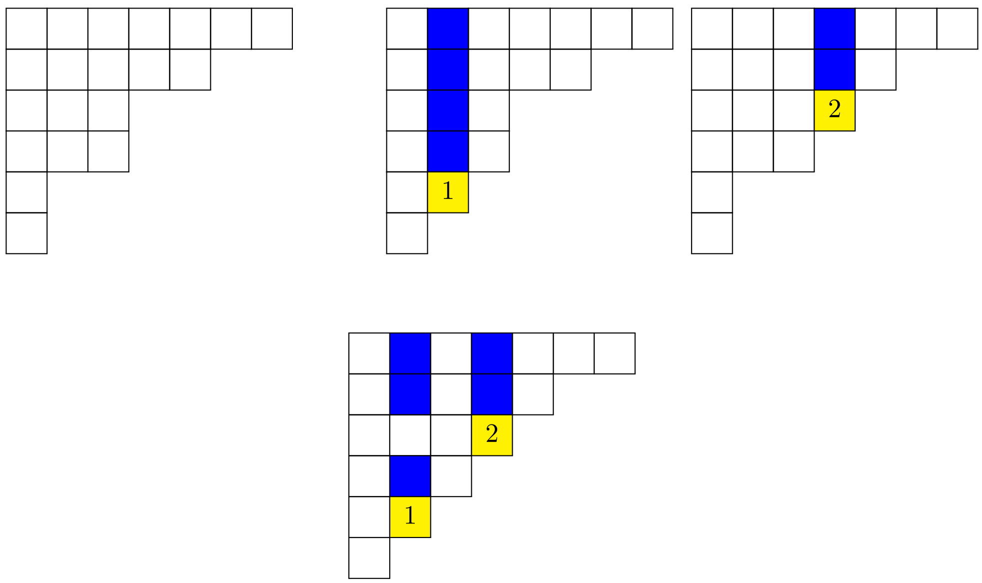
$$\frac{\langle P_\lambda, P_1^2 P_\mu \rangle}{\langle P_\lambda, P_{(1,1)} P_\mu \rangle} = \frac{\sum_{\sigma=\mu+\square} c^\lambda_{\sigma,1} c^\sigma_{\mu,1}}{c^\lambda_{\mu,(1,1)}} = \frac{c^{\mu+\square_1+\square_2}_{\mu+\square_2,1} c^{\mu+\square_2}_{\mu,1} + c^{\mu+\square_1+\square_2}_{\mu+\square_1,1} c^{\mu+\square_1}_{\mu,1}}{c^{\mu+\square_1+\square_2}_{\mu,(1,1)}}. \tag{2.3.23}$$

The sum over σ in the middle expression consists of two terms, corresponding to the order in which the two boxes are added to reach the same final shape $\lambda = \mu + \square_1 + \square_2$. Without loss of generality, we may label the box on the left as box 1 ($\square_1 = (i_1, j_1)$) and the one on the right as box 2 ($\square_2 = (i_2, j_2)$), so that $i_1 > i_2$ and $j_1 < j_2$.

We now need the explicit form of the coefficients:

$$c^{\lambda}_{\mu,(1^r)} = \prod_{s \in X(\lambda/\mu)} \frac{h^{\lambda}_{*}(s) h^{*}_{\mu}(s)}{h^{\mu}_{*}(s) h^{*}_{\lambda}(s)} \tag{2.3.24}$$

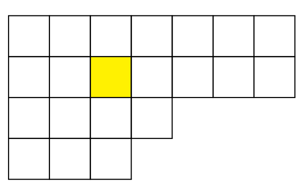
where $X(\lambda/\mu)$ is the set of boxes $(i,j) \in \mu$ such that $\mu_i = \lambda_i$ and $\mu_j < \lambda_j$. For example, in the figure below, $\mu = (7,5,3,3,1,1)$, and the yellow squares are the ones that have been added to obtain various λ's, while the blue squares constitute the set $X(\lambda/\mu)$. Note how, on adding two boxes together, as in the last figure, we get one blue box fewer than the union of the set of blue boxes when only one box is added at a time.



Furthermore, in *Eqn.*(2.3.24), $h^{*}_{\lambda}(s)$ and $h^{\lambda}_{*}(s)$ are respectively the upper and lower hook lengths of the box s:

$$\begin{aligned} h^{*}_{\lambda}(s) &= \beta^{-1}(a_\lambda(s)+1) + l_\lambda(s), \\ h^{\lambda}_{*}(s) &= \beta^{-1} a_\lambda(s) + l_\lambda(s) + 1, \end{aligned} \tag{2.3.25}$$

where $a_\lambda(i,j) = \lambda_i - j$ is the arm length and $l_\lambda(i,j) = \lambda'_j - i$ is the leg length of $(i,j) \in \lambda$. (A useful mnemonic is that in the definition of the lower hook length of a given box, the box itself is treated as part of the leg, going downwards.) For example, the yellow square in the diagram below has coordinates $(i,j) = (2,3)$, with $a_\lambda = 4$, $l_\lambda = 2$, $h^{*}_{\lambda} = 5\beta^{-1} + 2$, and $h^{\lambda}_{*} = 4\beta^{-1} + 3$.



Consider now the first term of *Eqn.*(2.3.23), where the box on the right ($\square_2$) is added first:

$$t_1(\square_1, \square_2) \stackrel{\text{def}}{=} \frac{c^{\mu+\square_1+\square_2}_{\mu+\square_2,1}\, c^{\mu+\square_2}_{\mu,1}}{c^{\mu+\square_1+\square_2}_{\mu,(1,1)}}. \tag{2.3.26}$$

Due to the product form of *Eqn.*(2.3.24), one can expect a great deal of cancellation in the above expression. However, exactly one factor survives: the contribution to $c^{\mu+\square_1+\square_2}_{\mu+\square_2,1}$ due to the box (i_2, j_1), which does not appear in the denominator. Therefore

$$
\begin{aligned}
t_1(\square_1,\square_2) &= \frac{h_*^{\mu+\square_1+\square_2}(i_2,j_1)h^*_{\mu+\square_2}(i_2,j_1)}{h_*^{\mu+\square_2}(i_2,j_1)h^*_{\mu+\square_1+\square_2}(i_2,j_1)} \\
&= \frac{\left[\beta^{-1}(j_2-j_1)+i_1-i_2+1\right]\left[\beta^{-1}(j_2-j_1+1)+i_1-i_2-1\right]}{\left[\beta^{-1}(j_2-j_1+1)+i_1-i_2\right]\left[\beta^{-1}(j_2-j_1)+i_1-i_2\right]}.
\end{aligned}
\tag{2.3.27}
$$

(It helps to note that $i_1 = \mu'_{j_1}+1$, $j_1 = \mu_{i_1}+1$, $i_2 = \mu'_{j_2}+1$, and $j_2 = \mu_{i_2}+1$.) In the second term of *Eqn.*(2.3.23), where the left box ($\square_1$) is added first,

$$
t_2(\square_1,\square_2) \stackrel{\text{def}}{=} \frac{c^{\mu+\square_1+\square_2}_{\mu+\square_1,1}c^{\mu+\square_1}_{\mu,1}}{c^{\mu+\square_1+\square_2}_{\mu,(1,1)}},
\tag{2.3.28}
$$

one has similar cancellations, leaving a contribution to $c^{\mu+\square_1}_{\mu,1}$ due to the box (i_2, j_1):

$$
\begin{aligned}
t_2(\square_1,\square_2) &= \frac{h_*^{\mu+\square_1}(i_2,j_1)h^*_{\mu}(i_2,j_1)}{h_*^{\mu}(i_2,j_1)h^*_{\mu+\square_1}(i_2,j_1)} \\
&= \frac{\left[\beta^{-1}(j_2-j_1-1)+i_1-i_2+1\right]\left[\beta^{-1}(j_2-j_1)+i_1-i_2-1\right]}{\left[\beta^{-1}(j_2-j_1)+i_1-i_2\right]\left[\beta^{-1}(j_2-j_1-1)+i_1-i_2\right]}.
\end{aligned}
\tag{2.3.29}
$$

We digress momentarily to point out that $t_1(\square_1,\square_2) = t_2(\square_2,\square_1)$. This is not *a priori* obvious because the labels 1 and 2 were assigned based on the relative positions of the boxes, and indeed the intermediate expressions in terms of the hook lengths h^* and h_* are not manifestly equal under this exchange.

We now have the expression for $A_{\mu\lambda}$:

$$
A_{\mu\lambda} = \frac{\beta-1+(j_2-j_1+\beta(i_1-i_2+1))(j_2-j_1+\beta(i_1-i_2-1))}{\beta(1-\beta)}.
\tag{2.3.30}
$$

While we derived this under the assumption that $\lambda-\mu$ was neither a row nor a column, we see that these two cases are also captured by this expression, as can be seen by setting $i_1 = i_2$, $j_2 - j_1 = 1$ for a row, or $j_1 = j_2$, $i_1 - i_2 = 1$ for a column. This is therefore the general expression.

It is now straightforward to see that

$$
\beta(1-\beta)A_{\mu\lambda} = (j_2-j_1+\beta(i_1-i_2))^2-(1-\beta+\beta^2),
\tag{2.3.31}
$$

thus proving

$$
W_{-2}P_\mu = \sum_{\lambda=\mu+\square_1+\square_2} (j_1-1+\beta(N-i_1+1))(j_2-1+\beta(N-i_2+1))C_{\mu\lambda}P_\lambda.
\tag{2.3.32}
$$

To calculate *Eqn.*(2.3.10), we need to act with W_{-2} multiple times on $P_\phi = 1$, which can now be done using *Eqn.*(2.3.32):

$$
\begin{aligned}
(W_{-2})^n\cdot 1 = \sum_{\lambda=\lambda^{(2n)}} \Bigg[&\prod_{(i,j)\in\lambda^{(2n)}} (j-1+\beta(N-i+1)) \\
&\times \sum_{\lambda^{(2)}\ldots\lambda^{(2n-2)}} C_{\phi\lambda^{(2)}}C_{\lambda^{(2)}\lambda^{(4)}}\ldots C_{\lambda^{(2n-2)}\lambda^{(2n)}}P_\lambda\Bigg]
\end{aligned}
\tag{2.3.33}
$$

where $\lambda^{(2m)}$ is obtained by adding 2 boxes successively m times starting from the initial empty partition ϕ, so that $|\lambda^{(2m)}| = 2m$. The rest of the proof consists of writing the product over the cells in the first line, and the sum of products of the C's in the second line, in terms of Jack polynomials $P_\lambda\{p_k\}$ evaluated at special values of the power-sum polynomials p_k.

We begin with the C's. The Cauchy identity for Jack polynomials reads

$$\exp\left[\beta \sum_{k=1}^{\infty} \frac{p_k \overline{p}_k}{k}\right] = \sum_{\lambda} \frac{P_\lambda\{p_k\} P_\lambda\{\overline{p}_k\}}{\langle P_\lambda, P_\lambda \rangle}. \tag{2.3.34}$$

Setting $\overline{p}_k = a_2 \beta^{-1} \delta_{k,2}$, we see that

$$\exp\left[\frac{a_2 p_2}{2}\right] = \sum_{\lambda} \frac{P_\lambda \cdot P_\lambda\{a_2 \beta^{-1} \delta_{k,2}\}}{\langle P_\lambda, P_\lambda \rangle}. \tag{2.3.35}$$

On the other hand, directly from the definition of $C_{\mu\lambda}$ in *Eqn.*(2.3.17), we see that

$$\exp\left[\frac{a_2 p_2}{2}\right] = \sum_{n \geq 0} \frac{a_2^n P_{\lambda^{(2n)}}}{2^n n!} \sum_{\lambda^{(2)} \ldots \lambda^{(2n-2)}} C_{\phi \lambda^{(2)}} C_{\lambda^{(2)} \lambda^{(4)}} \ldots C_{\lambda^{(2n-2)} \lambda^{(2n)}}. \tag{2.3.36}$$

The above expressions together give

$$\sum_{\lambda^{(2)} \ldots \lambda^{(2n-2)}; \lambda^{(2n)} = \lambda} C_{\phi \lambda^{(2)}} C_{\lambda^{(2)} \lambda^{(4)}} \ldots C_{\lambda^{(2n-2)} \lambda^{(2n)}} = \frac{2^{|\lambda|/2} (|\lambda|/2)!}{a_2^{|\lambda|/2}} \frac{P_\lambda\{a_2 \beta^{-1} \delta_{k,2}\}}{\langle P_\lambda, P_\lambda \rangle}. \tag{2.3.37}$$

For the first line of *Eqn.*(2.3.33), we turn to [Mac]. By *Eqn.*(10.20) in this reference,

$$P_\lambda\{N\} = \prod_{(i,j) \in \lambda} \frac{\beta^{-1}(j-1) + N - i + 1}{h_*^\lambda(i,j)}, \tag{2.3.38}$$

while eqns. (10.22) and (10.29) tell us that

$$P_\lambda\{\delta_{k,1}\} = \prod_{(i,j) \in \lambda} \frac{1}{h_*^\lambda(i,j)}. \tag{2.3.39}$$

Putting these together, we can write

$$\prod_{(i,j) \in \lambda} (j - 1 + \beta(N - i + 1)) = \frac{P_\lambda\{N\}}{P_\lambda\{\beta^{-1} \delta_{k,1}\}}. \tag{2.3.40}$$

We now have all the ingredients to write *Eqn.*(2.3.10) entirely in terms of Jack polynomials:

$$Z_\beta(p_k) = e^{W_{-2}/2a_2} \cdot 1 = \sum_{\lambda} \frac{P_\lambda\{p_k\}}{\langle P_\lambda, P_\lambda \rangle} \frac{P_\lambda\{N\} P_\lambda\{a_2 \beta^{-1} \delta_{k,2}\}}{P_\lambda\{a_2 \beta^{-1} \delta_{k,1}\}}. \tag{2.3.41}$$

(Note that $P_\lambda\{x\delta_{k,2}\} = x^{|\lambda|/2} P_\lambda\{\delta_{k,2}\}$ and $P_\lambda\{x\delta_{k,1}\} = x^{|\lambda|} P_\lambda\{\delta_{k,1}\}$.) On the other hand, again by the Cauchy identity *Eqn.*(2.3.34), one can expand the partition function *Eqn.*(2.3.1) as

$$Z_\beta(V; p_k) = \sum_{\lambda} \frac{P_\lambda\{p_k\}}{\langle P_\lambda, P_\lambda \rangle} \langle P_\lambda(z_1, \ldots, z_N) \rangle, \tag{2.3.42}$$

regardless of the potential in question. The orthogonality of the Jack polynomials gives us the desired result

$$\langle P_\lambda(z_1, \ldots, z_N) \rangle = \frac{P_\lambda\{N\} P_\lambda\{a_2 \beta^{-1} \delta_{k,2}\}}{P_\lambda\{a_2 \beta^{-1} \delta_{k,1}\}}. \tag{2.3.43}$$

2.4 Gaussian model with a linear term

We now consider a slightly more general potential

$$V(z) = \frac{a_2}{2} z^2 + a_1 z \tag{2.4.1}$$

and denote the corresponding partition function *Eqn.*(2.3.1) by $Z_\beta(a_2, a_1; p_k)$. The Virasoro operators annihilating the partition function are modified to

$$\mathcal{L}_n = -\frac{a_2(n+2)}{\beta}\frac{\partial}{\partial p_{n+2}} - \frac{a_1(n+1)}{\beta}\frac{\partial}{\partial p_{n+1}} - a_1 N \delta_{n,-1} + L_n, \quad n \geq -1, \tag{2.4.2}$$

where the L_n's are given in *Eqn.*(2.3.7). The Virasoro constraints now imply the single equation

$$\left(\mathcal{D} - \frac{W_{-2}}{a_2} + \frac{a_1 L_{-1}}{a_2}\right) Z_\beta(a_2, a_1; p_k) = 0, \tag{2.4.3}$$

where W_{-2} is the same as in *Eqn.*(2.3.9), and $[\mathcal{D}, L_{-1}] = L_{-1}$. A crucial observation is that $[W_{-2}, L_{-1}] = 0$. The partition function can now be written as

$$Z_\beta(a_2, a_1; p_k) = \exp\left[\frac{W_{-2}}{2a_2} - \frac{a_1 L_{-1}}{a_2}\right] \cdot 1 = e^{W_{-2}/2a_2} \cdot e^{-a_1 L_{-1}/a_2} \cdot 1. \tag{2.4.4}$$

We now follow a similar strategy as in the previous section: we work out the action of L_{-1} on Jack polynomials. This, as we have seen, is facilitated by writing it in terms of the operator $\mathcal{H}$, defined in *Eqn.*(2.3.11), which has the Jack polynomials as its eigenfunctions:

$$L_{-1} = [\mathcal{H}, p_1] + \beta N p_1. \tag{2.4.5}$$

Analogously to *Eqn.*(2.3.17), let us define constants $B_{\mu\lambda}$ as coefficients in the expansion

$$p_1 P_\mu = \sum_{\lambda=\mu+\square} B_{\mu\lambda} P_\lambda. \tag{2.4.6}$$

These constants are, of course, known from *Eqn.*(2.3.24), but we will not need their explicit expression, just as in the case of $C_{\mu\lambda}$ in the previous section. Using *Eqn.*(2.4.5) and *Eqn.*(2.4.6) we get

$$L_{-1} P_\mu = \sum_{\lambda=\mu+\square} (j - 1 + \beta(N - i + 1)) B_{\mu\lambda} P_\lambda, \tag{2.4.7}$$

where (i, j) are the coordinates of the additional box $\square$.

On expanding the exponentials on the right-most side of *Eqn.*(2.4.4), one has to deal with terms of the type

$$\begin{aligned}(W_{-2})^n (L_{-1})^m \cdot 1 = &\sum_{\lambda=\lambda^{(2n+m)}} \Big[\prod_{(i,j)\in\lambda^{(2n+m)}} (j - 1 + \beta(N - i + 1)) \\ &\times \sum_{\lambda^{(1)}\ldots\lambda^{(m)}} B_{\phi\lambda^{(1)}} B_{\lambda^{(1)}\lambda^{(2)}} \ldots B_{\lambda^{(m-1)}\lambda^{(m)}} \\ &\times \sum_{\lambda^{(m+2)}\ldots\lambda^{(m+2n-2)}} C_{\lambda^{(m)}\lambda^{(m+2)}} \ldots C_{\lambda^{(m+2n-2)}\lambda^{(m+2n)}} P_{\lambda^{(m+2n)}} \Big].\end{aligned} \tag{2.4.8}$$

Setting $\overline{p}_k = a_2\beta^{-1}\delta_{k,2} - a_1\beta^{-1}\delta_{k,1}$ in the Cauchy identity *Eqn.*(2.3.34), we see that

$$\exp\left[\frac{a_2 p_2}{2} - a_1 p_1\right] = \sum_\lambda \frac{P_\lambda \cdot P_\lambda\{\beta^{-1}(a_2\delta_{k,2} - a_1\delta_{k,1})\}}{\langle P_\lambda, P_\lambda\rangle}. \tag{2.4.9}$$

On the other hand, by definition of $B_{\mu\lambda}$ and $C_{\mu\lambda}$,

$$\begin{aligned}\exp\left[\frac{a_2 p_2}{2} - a_1 p_1\right] &= \\ &= \sum_{n,m\geq 0}\Bigg[P_{\lambda^{(m+2n)}}\frac{(a_2/2)^n(-a_1)^m}{n!m!}\sum_{\lambda^{(1)}\ldots\lambda^{(m)}} B_{\phi\lambda^{(1)}}B_{\lambda^{(1)}\lambda^{(2)}}\ldots B_{\lambda^{(m-1)}\lambda^{(m)}} \\ &\quad\times\sum_{\lambda^{(m+2)}\ldots\lambda^{(m+2n-2)}} C_{\lambda^{(m)}\lambda^{(m+2)}}\ldots C_{\lambda^{(m+2n-2)}\lambda^{(m+2n)}}\Bigg].\end{aligned} \tag{2.4.10}$$

The above two equations together imply, for $\lambda = \lambda^{(m+2n)}$,

$$\begin{aligned}\frac{P_\lambda\{\beta^{-1}(a_2\delta_{k,2} - a_1\delta_{k,1})\}}{\langle P_\lambda, P_\lambda\rangle} &= \\ &= \frac{(a_2/2)^n(-a_1)^m}{n!m!}\sum_{\lambda^{(1)}\ldots\lambda^{(m)}} B_{\phi\lambda^{(1)}}B_{\lambda^{(1)}\lambda^{(2)}}\ldots B_{\lambda^{(m-1)}\lambda^{(m)}} \\ &\quad\times\sum_{\lambda^{(m+2)}\ldots\lambda^{(m+2n-2)}} C_{\lambda^{(m)}\lambda^{(m+2)}}\ldots C_{\lambda^{(m+2n-2)}\lambda^{(m+2n)}}\end{aligned} \tag{2.4.11}$$

Summing *Eqn.*(2.4.8) over n and m with appropriate coefficients, while also recalling *Eqn.*(2.3.40), we can finally write

$$\begin{aligned}Z_\beta(a_2, a_1; p_k) &= e^{W_{-2}/2a_2}\cdot e^{-a_1 L_{-1}/a_2}\cdot 1 \\ &= \sum_\lambda \frac{P_\lambda\{p_k\}}{\langle P_\lambda, P_\lambda\rangle}\frac{P_\lambda\{N\}P_\lambda\{\beta^{-1}(a_2\delta_{k,2} - a_1\delta_{k,1})\}}{P_\lambda\{a_2\beta^{-1}\delta_{k,1}\}}.\end{aligned} \tag{2.4.12}$$

Using *Eqn.*(2.3.42), we get the desired result for the correlators

$$\big\langle P_\lambda(z_1,\ldots,z_N)\big\rangle = \frac{P_\lambda\{N\}P_\lambda\{\beta^{-1}(a_2\delta_{k,2} - a_1\delta_{k,1})\}}{P_\lambda\{a_2\beta^{-1}\delta_{k,1}\}}. \tag{2.4.13}$$

This relation was conjectured and verified for $|\lambda| \leq 9$ in [CLZ], and here we have provided its complete proof.

2.5 Linear and logarithmic potential

Finally, following an analogous strategy, we consider the eigenvalue model *Eqn.*(2.3.1) with a potential containing linear and logarithmic terms. The superintegrability property of a linear theory $V(z) = z$ of $N \times N$ complex matrices was proven in [WZC]. We generalize that proof and include a determinant factor in the action, which can also be interpreted as an additional logarithmic term in the potential:

$$Z_\beta \overset{\text{def}}{=} \int_0^\infty \left(\prod_{i=1}^N dz_i\right)[\Delta(z)]^{2\beta}\left(\prod_{i=1}^N z_i^\nu\right) e^{-a_1\sum_{i=1}^N z_i}\exp\left[\beta\sum_{k=1}^\infty \frac{p_k}{k}\sum_{i=1}^N z_i^k\right]. \tag{2.5.1}$$

This integral can be viewed as the β-deformation of the (anti-)Wishart model over the space of complex matrices of size $N_1 \times N_2$, with $N = \min(N_1, N_2)$ and $\nu = |N_1 - N_2|$; see, for example, Chapter 13 of [LN].

The Virasoro operators that annihilate the above partition function are given by

$$\mathcal{L}_n = -\frac{a_1(n+1)}{\beta}\frac{\partial}{\partial p_{n+1}} - a_1 N\delta_{n,-1} + \nu\left(\frac{n}{\beta}\frac{\partial}{\partial p_n} + N\delta_{n,0}\right) + L_n, \quad n \geq -1. \tag{2.5.2}$$

The single constraint in this case is $(\mathcal{D} - a_1^{-1}W_{-1})Z_\beta = 0$, so that $Z_\beta = e^{W_{-1}/a_1}\cdot 1$, where

$$\begin{aligned} W_{-1} \overset{\text{def}}{=} & \beta N(\nu+\beta N+1-\beta)p_1 \\ & + \sum_{k,l=1}^{\infty}\left[\beta(k+l-1)p_k p_l\frac{\partial}{\partial p_{k+l-1}} + klp_{k+l+1}\frac{\partial^2}{\partial p_k\partial p_l}\right] \\ & + \sum_{k=1}^{\infty}(k-1)(\nu+2\beta N+k-k\beta)p_k\frac{\partial}{\partial p_{k-1}}, \end{aligned} \tag{2.5.3}$$

which can be written as

$$\begin{aligned} W_{-1} = & [\mathcal{H},[\mathcal{H},p_1]] + (1-\beta+\nu+\beta N)\,[\mathcal{H},p_1] \\ & + \beta N(\nu+\beta N+1-\beta)p_1. \end{aligned} \tag{2.5.4}$$

This lets us write

$$\begin{aligned} W_{-1}P_\mu = \sum_{\lambda=\mu+\square} & \Big[(j-1+\beta(N-i+1)) \\ & \times\left(j-1+\beta((N+\beta^{-1}\nu+\beta^{-1}-1)-i+1)\right)B_{\mu\lambda}P_\lambda\Big], \end{aligned} \tag{2.5.5}$$

where (i,j) are the coordinates of $\square$. Once again, expanding $e^{W_{-1}/a_1}\cdot 1$, we find terms of the type

$$\begin{aligned} W_{-1}^n\cdot 1 = \sum_{\lambda=\lambda^{(n)}} & \Big[\prod_{(i,j)\in\lambda^{(n)}}(j-1+\beta(N-i+1)) \\ & \times\prod_{(i,j)\in\lambda^{(n)}}\left(j-1+\beta((N+\beta^{-1}\nu+\beta^{-1}-1)-i+1)\right) \\ & \times\sum_{\lambda^{(1)}\ldots\lambda^{(n-1)}}B_{\phi\lambda^{(1)}}B_{\lambda^{(1)}\lambda^{(2)}}\ldots B_{\lambda^{(n-1)}\lambda^{(n)}}P_{\lambda^{(n)}}\Big]. \end{aligned} \tag{2.5.6}$$

Expanding e^{p_1} in two ways, as demonstrated in the previous two sections, gives

$$\sum_{\lambda^{(1)}\ldots\lambda^{(n-1)};\lambda^{(n)}=\lambda} B_{\phi\lambda^{(1)}}B_{\lambda^{(1)}\lambda^{(2)}}\ldots B_{\lambda^{(n-1)}\lambda^{(n)}} = \frac{P_\lambda\{\beta^{-1}\delta_{k,1}\}}{\langle P_\lambda,P_\lambda\rangle}, \tag{2.5.7}$$

while the products in the first and second lines of *Eqn.*(2.5.6) can be evaluated using *Eqn.*(2.3.40). Altogether, we find

$$Z_\beta = e^{W_{-1}/a_1}\cdot 1 = \sum_\lambda \frac{P_\lambda\{p_k\}}{\langle P_\lambda,P_\lambda\rangle}\frac{P_\lambda\{N\}P_\lambda\{N+\beta^{-1}\nu+\beta^{-1}-1\}}{P_\lambda\{a_1\beta^{-1}\delta_{k,1}\}}, \tag{2.5.8}$$

so that, in this theory, the correlators are

$$\big\langle P_\lambda(z_1,\ldots,z_N)\big\rangle = \frac{P_\lambda\{N\}P_\lambda\{N+\beta^{-1}\nu+\beta^{-1}-1\}}{P_\lambda\{a_1\beta^{-1}\delta_{k,1}\}}. \tag{2.5.9}$$

This relation was conjectured and verified for $|\lambda|\leq 9$ in [CLZ], and here we have provided its complete proof. In particular, when $\beta=1$, for a complex linear model $\int d^2Ze^{-\mathrm{Tr}ZZ^\dagger}$, where Z is of size $N_1\times N_2$, one finds

$$\big\langle s_\lambda(ZZ^\dagger)\big\rangle = \frac{s_\lambda\{N_1\}s_\lambda\{N_2\}}{s_\lambda\{\delta_{k,1}\}}. \tag{2.5.10}$$

2.6 Dunkl theory and character expansion in the β-deformed matrix model

In this section we provide another proof of the character expansion in the β-deformed matrix model. The proof for the Gaussian potential and a single Jack polynomial was originally given by Okounkov **??**. We extend his result to include a linear term and also the insertion of two Jack polynomials, as suggested by him.

In the previous chapter, we proved the superintegrability of the β-ensemble eigenvalue model using representation-theoretic techniques. One may ask for a more analytic proof. This chapter deals with that question. We provide the necessary background to understand Okounkov's lemma [O1], and we also present some unpublished results.

At the same time, this chapter connects the two papers because of its differential-operator nature. Indeed, there are various ways of generalizing hypergeometric functions in a multivariable setting. For instance, the Macdonald hypergeometric function is defined using Jack polynomials [Mac3].

Let $\alpha > 0$ and let λ be a partition. Then the α-Macdonald hypergeometric function [Mac3] is

$$ {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{x}) := \sum_{\lambda} \frac{(a_1)_\lambda \dots (a_p)_\lambda}{(b_1)_\lambda \dots (b_q)_\lambda} \frac{P_\lambda(x;\alpha)}{j_\lambda(\alpha)}, \tag{2.6.1} $$

where the generalized Pochhammer symbol is

$$ (a)_\lambda^{(\alpha)} = \prod_{(i,j)\in\lambda} \left(a + j - 1 - \frac{i-1}{\alpha} \right). \tag{2.6.2} $$

Instead, we work with another definition of multivariable hypergeometric function, namely the Heckman–Opdam hypergeometric function.

Definition 2.6.1. *[HO] Let p be a Weyl-invariant polynomial associated with a root system. Then $F_k(\mathbf{x}, \mathbf{y})$ is called a multivariable hypergeometric function associated with a root system if*

$$ p(T_{\mathbf{x}})F_k(\mathbf{x}, \mathbf{y}) = p(\mathbf{y})F_k(\mathbf{x}, \mathbf{y}), \tag{2.6.3} $$

where $T_{\mathbf{x}}$ denotes the Dunkl operator.

This chapter deals with this definition and with applications of this object to the root system of type A, related to the β-deformed eigenvalue model. We also explain that this object resembles the Bessel function; therefore, it is sometimes called a generalized Bessel function.

This is a short review of Dunkl differential-difference operators, mainly based on the reviews by Rösler [Ro] and Opdam [Op1]. We give a brief detour through the theory up to the point where we can state a new result.

2.6.1 Root system

We briefly review the root system that appears in the definition of the Dunkl operator. We always work with the root system of type A, but Dunkl theory can be formulated for any root system. Although the Bessel function of rank higher than 2 for other root systems is not explicitly known, this is the main reason we cannot develop the character expansion further. In the appendix we describe realizations of the root systems of type A, B, C, and D.

Let $E = \mathbb{R}^N$ be a Euclidean space equipped with the standard inner product

$$ (x, y)_E = \sum_{i=1}^{N} x_i y_i. $$

The reflection with respect to the hyperplane orthogonal to a vector $a \in E$ is defined by

$$\sigma_a(x) := x - 2\frac{(a,x)_E}{(a,a)_E}a. \tag{2.6.4}$$

Any reflection is an element of $O(N,\mathbb{R})$.

Definition 2.6.2. *Let $R \subset \mathbb{R}^N \setminus \{0\}$ be a finite set of vectors, called roots. We say that R is a root system if it satisfies the following conditions:*

(1) *It spans $E = \mathbb{R}^N$.*

(2) *Reducedness: for every $a \in R$, one has $R \cap \mathbb{R}a = \{a, -a\}$.*

(3) *Closedness: for every $a \in R$, one has $\sigma_a(R) = R$.*

(4) *Crystallographic condition: for every $a, b \in R$,*

$$2\frac{(a,b)_E}{(a,a)_E} \in \mathbb{Z}.$$

Condition 4 is not necessary in general, since a root system may be non-crystallographic.

The subgroup $G \subset O(N,\mathbb{R})$ generated by the reflections σ_a, $a \in R$, is called the reflection group, and is sometimes also called the *Weyl group* or *Coxeter group.*

2.6.2 Dunkl theory

Definition 2.6.3. *Let W be the Weyl group associated with a root system R. A map $k : R \to \mathbb{C}$ is called a* multiplicity function *if it is invariant under the action of W on R.*

For the root system of type A_{N-1} the Weyl group is $\mathfrak{S}_N$, and there is only one orbit. Hence every multiplicity function is constant. For the precise reaqlization look at appendix B.

Definition 2.6.4. *[Du] Let k be a multiplicity function and let $\zeta \in \mathbb{R}^N$. Then the* Dunkl operator *$T_\zeta = T_\zeta(k)$ is defined on $C^1(\mathbb{R}^N)$ by*

$$T_\zeta f(x) := \partial_\zeta f(x) + \sum_{a\in R_+} k(a)\,(a,\zeta)_E \frac{f(x) - f(\sigma_a x)}{(a,x)_E}, \tag{2.6.5}$$

where $(\cdot,\cdot)_E$ is the standard Euclidean inner product.

We define the action of a group element $h \in G$ on functions by

$$(h \cdot f)(x) := f(h^{-1}x). \tag{2.6.6}$$

Then we have the following lemma.

Lemma 2.6.5. (1) *T_ζ is equivariant:*

$$g \circ T_\zeta \circ g^{-1} = T_{g\zeta}, \qquad g \in G.$$

(2) *Let $f, g \in C^1(\mathbb{R}^N)$, and assume that at least one of them is G-invariant. Then*

$$T_\zeta(fg) = T_\zeta(f)\,g + f\,T_\zeta(g). \tag{2.6.7}$$

Theorem 2.6.6 (Dunkl's theorem). *For fixed k, the Dunkl operators commute [Du]:*

$$[T_{\zeta_i}, T_{\zeta_j}] = 0. \tag{2.6.8}$$

The previous theorem has an important consequence. There exists an algebra homomorphism

$$\begin{aligned} &\Phi_k : \Lambda \to \mathrm{End}_{\mathbb{C}}(\Lambda), \\ &x_i \mapsto T_i, \quad 1 \mapsto \mathrm{id}. \end{aligned} \tag{2.6.9}$$

For $p \in \Lambda$, we write

$$p(T) := \Phi_k(p). \tag{2.6.10}$$

This is analogous to the basic property of partial derivatives, namely $[\partial_{x_i}, \partial_{x_j}] = 0$.

Definition 2.6.7. *Let $p(x) = |x|^2$. We define the* Dunkl Laplacian *by*

$$\Delta_k := p(T). \tag{2.6.11}$$

Lemma 2.6.8. Δ_k *is G-invariant:*

$$g \circ \Delta_k = \Delta_k \circ g, \qquad g \in G.$$

Proof. *This follows from the G-equivariance of T_ζ.* □

Recall that the Fischer product is the bilinear form on Λ defined by

$$[p, q]_0 := \big(p(\partial)\, q\big)(0), \qquad p, q \in \Lambda. \tag{2.6.12}$$

The integral representation of the Fischer product was discovered by Macdonald [Mac1]:

$$[p, q]_0 = (2\pi)^{-N/2} \int_{\mathbb{R}^N} \left(e^{-\Delta/2} p(\mathbf{x})\right) \left(e^{-\Delta/2} q(\mathbf{x})\right) e^{-|\mathbf{x}|^2/2}\, dx, \tag{2.6.13}$$

where

$$e^{-\Delta/2} p = \sum_{n \geq 0} \frac{(-1)^n}{2^n n!} \Delta^n p. \tag{2.6.14}$$

This bilinear form generalizes naturally to the Dunkl setting.

Definition 2.6.9. *The* generalized Fischer product *is defined by*

$$[p, q]_k := \big(p(T) q\big)(0), \qquad p, q \in \Pi. \tag{2.6.15}$$

Lemma 2.6.10. (1) *If $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_m$ with $n \neq m$, then $[p, q]_k = 0$.*

(2) $[x_i p, q]_k = [p, T_i q]_k$.

(3) $[gp, gq]_k = [p, q]_k$ *for $g \in G$.*

Proof.

(1) The Dunkl operator respects the grading. In particular, if $f \in \mathcal{P}_n$, then $T_i f \in \mathcal{P}_{n-1}$.

(2) This follows directly from the definition.

(3) This follows from the equivariance of T.

□

For any root system we define the weight

$$w_k(x) := \prod_{\alpha \in R_+} |(\alpha, x)|^{2k(\alpha)}. \tag{2.6.16}$$

The Macdonald–Mehta–Selberg integral is

$$c_k := \int_{\mathbb{R}^N} e^{-|x|^2/2} w_k(x)\, dx. \tag{2.6.17}$$

Macdonald showed that the coefficient c_k can be obtained using the Selberg integral.

This is a good point to mention one of the most interesting episodes in the interaction between physics and mathematics. In 1944, Selberg computed the multivariable version of the beta integral [Se]:

$$\int_{[0,1]^N} \prod_{i=1}^{N} \left(dx_i\, x_i^{\mu-1}(1-x_i)^{\nu-1} \right) \prod_{1\le i<j\le N} |x_i - x_j|^{2\gamma} = \prod_{j=0}^{N-1} \frac{\Gamma(\mu+j\gamma)\Gamma(\nu+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\mu+\nu+(N+j-1)\gamma)\Gamma(1+\gamma)}. \tag{2.6.18}$$

The integrability conditions are

$$\operatorname{Re}\mu > 0, \qquad \operatorname{Re}\nu > 0, \qquad \operatorname{Re}\gamma > -\min\left\{ \frac{1}{N}, \frac{\operatorname{Re}\mu}{N-1}, \frac{\operatorname{Re}\nu}{N-1} \right\}, \tag{2.6.19}$$

up to the precise normalization convention. Since Selberg believed that this integral was already known, he published his work in Norwegian [Se]. This was one of the main reasons it remained relatively unknown until the 1970s, when Selberg revealed his integral to Dyson. The next major step was taken by Macdonald, who immediately recognized the root-system structure behind these integrals and formulated a series of conjectures that led to many important developments.

Theorem 2.6.11. *For all $p, q \in \Lambda$, the generalized Fischer product admits the integral representation*

$$[q,p]_k = c_k^{-1} \int_{\mathbb{R}^N} e^{-\Delta_k/2} p(x)\, e^{-\Delta_k/2} q(x)\, e^{-|x|^2/2} w_k(x)\, dx. \tag{2.6.20}$$

For $k \ge 0$, $[\bullet,\bullet]_k$ is a symmetric, non-degenerate bilinear form on Λ. This is due to Dunkl [Du], de Jeu [dJ], and Opdam [Op1].

2.6.3 Dunkl's intertwining operator

It was shown that the commutative algebra of Dunkl operators intertwines with the algebra of ordinary partial differential operators via a unique linear isomorphism. This can be summarized as follows.

Definition 2.6.12. *Let*

$$K^{\mathrm{reg}} := \left\{ k \in K : \bigcap_{\zeta\in\mathbb{R}^N} \mathrm{Ker} T_\zeta(k) = \mathbb{C}\,\mathrm{id} \right\}. \tag{2.6.21}$$

Then the following statements are equivalent:

(1) $k \in K^{\mathrm{reg}}$.

(2) *There exists a unique linear isomorphism, called the* intertwining operator V_k *on* Π*, such that*

$$V_k(\mathcal{P}_n) = \mathcal{P}_n, \qquad V_k|_{\mathcal{P}_0} = \mathrm{id}, \qquad T_\zeta V_k = V_k \partial_\zeta \quad \textit{for all } \zeta \in \mathbb{R}^N. \tag{2.6.22}$$

We immediately see that V_k is G-invariant:

$$g^{-1} \circ V_k \circ g = V_k. \tag{2.6.23}$$

2.6.4 Dunkl kernel

Definition 2.6.13. *For $y \in \mathbb{C}^N$, define*

$$E_k(x,y) := V_k\big(e^{(\bullet,y)_E}\big)(x). \tag{2.6.24}$$

We may equivalently define the kernel as follows.

Proposition 2.6.14. *Let $k \geq 0$ and $y \in \mathbb{C}^N$. Then $f = E_k(\bullet, y)$ is the unique real-analytic solution of*

$$T_\zeta f = (\zeta, y)_E f \tag{2.6.25}$$

satisfying $f(0) = 1$.

Some properties of Dunkl's kernel are listed below:

(1) $E_k(x,y) = E_k(y,x)$.

(2) $E_k(\lambda x, y) = E_k(x, \lambda y)$ for $\lambda \in \mathbb{C}$.

(3) $E_k(gx, gy) = E_k(x,y)$ for $g \in G$.

(4) $\overline{E_k(x,y)} = E_k(\overline{x}, \overline{y})$.

The kernel is known explicitly only in a few cases. An important example is the rank-one root system.

2.6.5 Spherical Bessel function inside the Dunkl kernel of rank 1

Using the definition

$$E_k(x,y) := V_k\big(e^{(\bullet,y)}\big)(x), \tag{2.6.26}$$

we obtain

$$E_k(x,y) = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k)} \int_{-1}^{1} e^{txy}(1-t)^{k-1}(1+t)^k \, dt = e^{xy} \, {}_1F_1(k, 2k+1, -2xy). \tag{2.6.27}$$

It is known that this hypergeometric function can be expressed in terms of Bessel functions. Let $j_k(z)$ denote the spherical Bessel function:

$$j_k(z) := \Gamma(k+1)\frac{J_k(z)}{z^k}. \tag{2.6.28}$$

Then

$$E_k(x,y) = j_{k-\frac{1}{2}}(ixy) + \frac{xy}{2k+1} j_{k+\frac{1}{2}}(ixy). \tag{2.6.29}$$

Hence

$$\frac{1}{2}\big(E_k(x,y) + E_k(-x,y)\big) = j_{k-\frac{1}{2}}(ixy). \tag{2.6.30}$$

Evaluating the kernel is a non-trivial task and its mostly an open question for higher ranks. But what we will need a weaker version of kernel, often referred to as the spherical part of the kernel.

2.6.6 Generalized k-Bessel function

Motivated by the rank-one calculation, Heckman and Opdam [HO] defined the generalized hypergeometric (k-Bessel) function.

Definition 2.6.15. *The k-Bessel function is*

$$F_k(x,y) := \frac{1}{|G|} \sum_{g \in G} E_k(gx, y). \tag{2.6.31}$$

Let $p(x) = |x|^2$. Then

$$p(T)E_k(x, y) = (y, y)_E \, E_k(x, y), \tag{2.6.32}$$

and therefore

$$p(T)F_k(x, y) = (y, y)_E \, F_k(x, y). \tag{2.6.33}$$

Set $k = \frac{1}{\alpha} \in \mathbb{C}$. For the root system of type A_{N-1}, this is indeed the only multiplicity function. Then the Dunkl operator takes the form

$$T_i f(x) = \partial_{x_i} f(x) + \frac{1}{\alpha} \sum_{j \neq i} \frac{f(x) - f(s_{ij}x)}{x_i - x_j}. \tag{2.6.34}$$

Applying this to a symmetric function $p(x) \in C_c^\infty(\mathbb{R}^N)$, we get

$$\sum_{i=1}^{N} T_i^2 p(x) = \sum_{i=1}^{N} \partial_{x_i}^2 p(x) + \frac{2}{\alpha} \sum_{i \neq j} \frac{1}{x_i - x_j} \partial_{x_i} p(x). \tag{2.6.35}$$

2.6.7 Jack polynomials and the Bessel function

Here we review the derivation of Olshanski-Okounkov Bessel function [OO]. Recall that the Jack polynomial is the eigenfunction of

$$\mathcal{H} := \frac{\alpha}{2} \sum_{i=1}^{N} (x_i \partial_{x_i})^2 + \frac{1}{2} \sum_{i<j} \frac{x_i + x_j}{x_i - x_j} (x_i \partial_{x_i} - x_j \partial_{x_j}), \tag{2.6.36}$$

with eigenvalues

$$\mathcal{H} P_\lambda(x; \alpha) = \sum_{i=1}^{N} \left(\frac{\alpha}{2} \lambda_i^2 + \frac{N + 1 - 2i}{2} \lambda_i \right) P_\lambda(x; \alpha). \tag{2.6.37}$$

We now change variables by setting $x \mapsto 1 - \frac{x}{M}$ and then taking the limit $M \to \infty$ while scaling the partition label as $\lambda \mapsto [M\lambda]$, where $[M\lambda]$ denotes the componentwise integer part. Then, at leading order, we obtain

$$\lim_{M \to \infty} \mathcal{H} = \left(\frac{\alpha}{2} \sum_{i=1}^{N} \partial_{x_i}^2 + \sum_{i \neq j} \frac{1}{x_i - x_j} \partial_{x_i} \right) M^2 + \mathcal{O}(M). \tag{2.6.38}$$

Interpreting λ_i as variables in a new alphabet, namely $\lambda_i = y_i$, we connect the two theories. To determine the kernel completely, we also need the initial condition

$$F_k(x, 0) = 1. \tag{2.6.39}$$

Thus we obtain the Bessel function of type A_{N-1} for the multiplicity function $k = \frac{1}{\alpha}$ [OO]:

$$F_{1/\alpha}(x, y) = \lim_{M \to \infty} \frac{J_{[My]}(1 - \frac{x}{M}; \alpha)}{J_{[My]}(1; \alpha)}. \tag{2.6.40}$$

Remark 2.6.16. *It does not matter which normalization of Jack polynomials one chooses; the Bessel function is independent of the normalization.*

Remark 2.6.17. *There is analogous expression in the theory of hypergoemetric function of* 1*-variable. Gabor Szego found following identity [Sz]*

$$\lim_{n \to \infty} n^{-\alpha} \mathcal{P}_n^{(\alpha,\beta)}(\cos \frac{z}{n}) = (\frac{z}{2})^{-\alpha} j_\alpha(z), \tag{2.6.41}$$

here $\mathcal{P}^{(\alpha,\beta)}(z)$ is Jacobi polynomial and $j_\alpha(z)$ is the Bessel function of second kind.

To our knowledge, the Bessel function associated with other root systems has been defined in a similar fashion by Heckman and Opdam [HO]. However, no explicit form in terms of known symmetric polynomials is available in general.

2.6.8 Matrix model and the Bessel function

Here we show that the correct operator to insert into the matrix model is the Bessel function rather than the generating function of Jack polynomials. For this, we recall the following theorem of Dunkl.

Theorem 2.6.18. *Let $k \geq 0$. Then*

(1)
$$\int_{\mathbb{R}^N} e^{-\Delta/2} p(x)\, E_k(x,y)\, e^{-|x|^2/2} w_k(x)\, dx = c_k e^{(y,y)_E/2} p(y).$$

(2)
$$\int_{\mathbb{R}^N} E_k(x,y) E_k(x,z)\, e^{-|x|^2/2} w_k(x)\, dx = c_k e^{(y,y)_E/2+(z,z)_E/2} E_k(y,z).$$

Proof. *See [Ro].*

Using the first integral, Okounkov proved Jackson's conjecture [O1], known in physics as superintegrability. He also suggested the existence of a similar formula for the average of two Jack polynomials [O1]. We now provide such a result.

Theorem 2.6.19.
$$F_{1/\alpha}(x,y) = \sum_{\lambda} \frac{P_\lambda(x;\alpha) P_\lambda(y;\alpha)}{(N/\alpha)_\lambda\, p_\lambda}, \tag{2.6.42}$$
where
$$(u)_\lambda = \prod_{(i,j)\in\lambda} \left(u + j - 1 - \frac{i-1}{\alpha}\right),$$
and $p_\lambda = (P_\lambda, P_\lambda)$. This is due to [OO].

Theorem 2.6.20. *The average of two Jack polynomials in the β-deformed eigenvalue model is*

$$\langle P_\mu(x;\alpha) P_\beta(x;\alpha)\rangle = \alpha^{-|\mu|-|\beta|} J_\mu(1;\alpha) J_\beta(1;\alpha) \left(P_\mu(y;\alpha), \left(P_\beta(z;\alpha), e^{|y|^2/2+|z|^2/2} \sum_\lambda \frac{P_\lambda(y;\alpha) P_\lambda(z;\alpha)}{(N/\alpha)_\lambda\, \mathfrak{p}_\lambda} \right)_\alpha \right)_\alpha . \tag{2.6.43}$$

Proof.
$$\frac{1}{c_{1/\alpha}} \int_{\mathbb{R}^N} E_{1/\alpha}(x,y) E_{1/\alpha}(x,z) e^{-|x|^2/2} w_k(x)\, dx = e^{|y|^2/2+|z|^2/2} E_{1/\alpha}(y,z). \tag{2.6.44}$$

Symmetrizing with respect to the action of the symmetric group on the alphabets y and z, we obtain

$$\frac{1}{c_{1/\alpha}} \int_{\mathbb{R}^N} F_{1/\alpha}(x,y) F_{1/\alpha}(x,z) e^{-|x|^2/2} w_k(x)\, dx = e^{|y|^2/2+|z|^2/2} F_{1/\alpha}(y,z). \tag{2.6.45}$$

Next we use the identity *Eqn.*(10.25) [Mac]

$$J_\lambda(\underline{1};\alpha) = \alpha^{|\lambda|} (N/\alpha)_\lambda. \tag{2.6.46}$$

We abbreviate the average by

$$\langle \bullet \rangle = \frac{1}{c_{1/\alpha}} \int_{\mathbb{R}^N} (\bullet) e^{-|x|^2/2} w_k(x)\, dx. \tag{2.6.47}$$

Using Okounkov's expansion, we get

$$\sum_{\mu,\beta} \frac{P_\mu(y;\alpha) P_\beta(z;\alpha)}{p_\mu p_\beta (N/\alpha)_\mu (N/\alpha)_\beta} \langle P_\mu(x;\alpha) P_\beta(x;\alpha) \rangle = e^{|y|^2/2+|z|^2/2} \sum_\lambda \frac{P_\lambda(y;\alpha) P_\lambda(z;\alpha)}{(N/\alpha)_\lambda\, \mathfrak{p}_\lambda}. \tag{2.6.48}$$

Finally, taking the Hall inner product in the variables y and z, we obtain *Eqn.*(2.6.43). □

Corollary 2.6.21. *If $|\mu| - |\beta| = 2k+1$ for some $k \in \mathbb{N}$, then the integral vanishes.*

2.6.9 Adjoint action under the deformed Hall inner product and superintegrable form

As defined before, the α-deformed Hall inner product is

$$(p_\mu, p_\nu)_\alpha = \alpha^{\ell(\mu)} \delta_{\mu,\nu} z_\mu. \tag{2.6.49}$$

Recal that the adjoint of the power-sum polynomial p_n with respect to the Hall inner product is (lemma 2.2.4)

$$p_n^\perp = \frac{n}{\alpha} \partial_{p_n}. \tag{2.6.50}$$

Using this result, we can write the formula in a compact form:

$$\begin{aligned}\langle P_\mu(x;\alpha) P_\beta(x;\alpha) \rangle &= \alpha^{-|\mu|-|\beta|} J_\mu(1;\alpha) J_\beta(1;\alpha) \\ &\times \sum_\lambda \left[P_\mu(p(y)^\perp;\alpha) P_\beta(p(z)^\perp;\alpha) e^{|y|^2/2+|z|^2/2} \frac{P_\lambda(p(y);\alpha) P_\lambda(p(z);\alpha)}{(N/\alpha)_\lambda\, \mathfrak{p}_\lambda} \right](0,0).\end{aligned} \tag{2.6.51}$$

Here $f(p(y)) = f(p_1, p_2, \dots)$ means the symmetric function written in the power-sum variables, and the adjoint is obtained by replacing each p_n by its adjoint:

$$f(p(y)^\perp) = f\left(\alpha \partial_{p_1}, 2\alpha \partial_{p_2}, 3\partial_{p_3}, \dots\right). \tag{2.6.52}$$

The notation $[\bullet](0,0)$ means that after evaluating the differential operators one sets $\underline{p}(y) = \underline{p}(z) = 0$. We also fix the number of variables to be N.

Remark 2.6.22. *The sum over partitions λ is restricted to those satisfying $|\lambda| \le \min(|\mu|, |\beta|)$.*

2.6.10 Examples

In the next sections, we give some important examples and then discuss the evaluation of Jack polynomials in terms of the power sums.

Okounkov's theorem

In *Eqn.*(2.6.43), set $\beta = \emptyset$. Then

$$\langle P_\mu(x;\alpha)\rangle = \alpha^{-|\mu|} J_\mu(1;\alpha)\Big(P_\mu(y;\alpha), e^{|y|^2/2}\Big)_\alpha. \tag{2.6.53}$$

The Hall inner product can be evaluated directly:

$$\big(P_\mu(y;\alpha), e^{|y|^2/2}\big) = \alpha^{|\mu|/2}\,[|y|^{|\mu|}]\,P_\mu(y;\alpha) = P_\mu\{\delta_{k,2}\}, \tag{2.6.54}$$

where $|\mu| = 2d$.

Finally,

$$\langle J_\mu(x;\alpha)\rangle = \alpha^{-|\mu|/2} J_\mu(1;\alpha)\,[|y|^{|\mu|}]\,P_\mu(y;\alpha). \tag{2.6.55}$$

This was obtained by Okounkov.

Another example

In *Eqn.*(2.6.43), set $\beta = \{1\}$. Since $P_{\{1\}}(x;\alpha) = p_1$, we get

$$\langle P_\mu(x;\alpha)P_{\{1\}}(x;\alpha)\rangle = \alpha^{-|\mu|-1}J_\mu(1;\alpha)J_{\{1\}}(1;\alpha)\Bigg(P_\mu(y;\alpha), \bigg(P_{\{1\}}(z;\alpha), e^{|y|^2/2+|z|^2/2}\sum_\lambda \frac{P_\lambda(y;\alpha)P_\lambda(z;\alpha)}{(N/\alpha)_\lambda\,\mathfrak{p}_\lambda}\bigg)_\alpha\Bigg)_\alpha. \tag{2.6.56}$$

This simplifies to

$$\langle P_\mu(x;\alpha)P_{\{1\}}(x;\alpha)\rangle = \alpha^{-|\mu|} J_\mu(1;\alpha)\big(P_\mu(y;\alpha), e^{|y|^2/2}P_{\{1\}}(y;\alpha)\big). \tag{2.6.57}$$

Since

$$\big(p_{\{2^k,1\}}, p_{\{2^k,1\}}\big) = k!\,2^k\,\alpha^{k+1}, \tag{2.6.58}$$

we conclude that

$$\langle P_\mu(x;\alpha)P_{\{1\}}(x;\alpha)\rangle = \alpha^{\frac{-|\mu|+1}{2}} J_\mu(1;\alpha)\,\big[p_{2^{\frac{|\mu|-1}{2}}}\,p_1\big]\,J_\mu(y;\alpha), \tag{2.6.59}$$

provided $|\mu|$ is odd.

Gaussian plus linear potential

As shown in [BKS], the character-preservation formula also works for a quadratic plus linear potential. Here we explain how this fits into our theorem *Eqn.*(2.6.43).

Lemma 2.6.23. *The generating function for Jack polynomials is given by the Cauchy identity:*

$$\exp\left[\frac{1}{\alpha}\sum_{k\geq 1}\frac{p_k\overline{p}_k}{k}\right] = \sum_\lambda \frac{P_\lambda\{p_k\}P_\lambda\{\overline{p}_k\}}{(P_\lambda, P_\lambda)}. \tag{2.6.60}$$

Using this identity, we obtain

$$e^{-p_1(x)} = \sum_\lambda \frac{P_\lambda(x;\alpha)P_\lambda(-\alpha\delta_{k,1})}{(P_\lambda, P_\lambda)}. \tag{2.6.61}$$

Therefore

$$\sum_{\beta}\left\langle P_\mu(x;\alpha)\,\frac{P_\beta(-\alpha\delta_{k,1})P_\beta(x;\alpha)}{(P_\beta,P_\beta)}\right\rangle = \sum_{\beta}\alpha^{-|\mu|-|\beta|}J_\mu(1;\alpha)J_\beta(1;\alpha)$$
$$\times\left(P_\mu(y;\alpha),\left(\frac{P_\beta(-\alpha\delta_{k,1})P_\beta(z;\alpha)}{(P_\beta,P_\beta)},e^{|y|^2/2+|z|^2/2}\sum_{\lambda}\frac{P_\lambda(y;\alpha)P_\lambda(z;\alpha)}{p_\lambda(N/\alpha)_\lambda}\right)\right). \tag{2.6.62}$$

We use the evaluations

1.
$$J_\beta(1;\alpha)=\frac{P_\beta(1;\alpha)}{P_\beta(\delta_{k,1};\alpha)}.$$

2.
$$\sum_{\beta}\frac{P_\beta(1;\alpha)P_\beta(z;\alpha)}{(P_\beta,P_\beta)}=\exp\left[\frac{1}{\alpha}\sum_{k\geq 1}\frac{Np_k}{k}\right].$$

Substituting these identities into *Eqn.*(2.6.62), we obtain

$$\langle P_\mu(x;\alpha)e^{-p_1(x)}\rangle=\alpha^{-|\mu|}J_\mu(1;\alpha)\Bigg(P_\mu(y;\alpha),\left(e^{-\sum_{k\geq 1}\frac{N}{k}p_k},e^{|y|^2/2+|z|^2/2}\sum_{\lambda}\frac{P_\lambda(y;\alpha)P_\lambda(z;\alpha)\alpha^{|\lambda|}P_\lambda(\delta_{k,1})}{P_\lambda(1)\,(P_\lambda,P_\lambda)}\right)\Bigg). \tag{2.6.63}$$

Using the adjoint action with respect to the deformed Hall inner product,

$$(p_n^\dagger p_\lambda,p_\mu)=(p_\lambda,p_np_\mu), \tag{2.6.64}$$

together with $p_n^\dagger=n\partial_{p_n}$, we can construct the shift operator

$$e^{-\sum_{k\geq 1}N\partial_{p_k}}$$

and simplify the expression to

$$\langle P_\mu(x;\alpha)e^{-p_1(x)}\rangle=\alpha^{-|\mu|}J_\mu(1;\alpha)\Bigg(P_\mu(y;\alpha),\left(1,e^{|y|^2/2+|z|^2/2-N/2}\sum_{\lambda}\frac{P_\lambda(z-1;\alpha)}{P_\lambda(1;\alpha)}\,\frac{P_\lambda(y;\alpha)\alpha^{|\lambda|}P_\lambda(\delta_{k,1})}{(P_\lambda,P_\lambda)}\right)\Bigg). \tag{2.6.65}$$

Finally, after simplifying the Hall inner product using the generalized binomial theorem, we obtain

$$\langle P_\mu(x;\alpha)e^{-p_1(x)}\rangle=\alpha^{-|\mu|}J_\mu(1;\alpha)e^{-N/2}\big[p_2^{|\mu|/2}-p_1^{|\mu|}\big]P_\mu. \tag{2.6.66}$$

This was proved in [BKS]. In many physics applications one considers normalized correlation functions, so the factor $e^{-N/2}$ can be absorbed into the Macdonald-Mehta-Selberg normalization.

Chapter 3

Conclusions

This thesis develops a systematic and unified study of exactly solvable systems along two complementary directions, based on the works [BKS] and [DK]. While a central theme in modern mathematical physics is to explain the origin of solvability, typically through hidden symmetries, we adopt a deliberately different standpoint. We take solvability as a given and extract from it its full structural and analytic consequences. This perspective proves to be both robust and generative.

In Chapter 1, we revisited classical exactly solvable quantum mechanical potentials within the framework of modern operator theory. Starting from explicit eigenfunctions, we carried out a complete spectral analysis of the associated operators. In particular, we characterized the resolvent (Green function), established criteria for boundedness, and identified the precise role of boundary conditions in defining the operators uniquely. A key feature of our approach is that it naturally extends beyond the self-adjoint setting, thereby providing a unified treatment of one-dimensional Schrödinger operators of hypergeometric type.

A central contribution of this chapter is the formulation and proof of a family of 3×3 theorems for hypergeometric Hamiltonians. While fragments of these results exist in the literature in restricted settings, our work provides the first complete and systematic treatment that incorporates non-self-adjoint operators. This substantially enlarges the scope of the classical theory and places it within a coherent operator-theoretic framework.

A particularly striking outcome of our analysis is the emergence of *transmutation identities*. These identities arise directly from symmetries of hypergeometric functions and establish nontrivial correspondences between spectral parameters and coupling constants. They do not appear in the existing literature and represent a genuinely new structural feature of exactly solvable systems. Their existence points to an underlying algebraic mechanism that is not captured before.

We further reconsidered the classical Laplacian on an interval using the modern language of boundary value problems [DeRi]. Although the associated Green functions are classical, our explicit reduction of Gegenbauer-type operators to boundary conditions requires a nontrivial and technically precise analysis (see Appendix A.1). This demonstrates that classical objects, when treated systematically, continue to yield new insights.

The classification of Hamiltonians obtained in this thesis is grounded in a geometric interpretation via separation of variables on symmetric spaces. This perspective not only explains the appearance of special functions but also organizes the theory conceptually. While certain aspects of the geometric interpretation of Gegenbauer Hamiltonians are known, our treatment consolidates and sharpens these results. More importantly, for hypergeometric Hamiltonians of the first kind, we obtain probably known constructions. However, the realization of deSitterian Hamiltonians in this framework is most-likely new and lies beyond the scope of real symmetric spaces.

The second part of the thesis (Chapter 2) is devoted to symmetric polynomials in random matrix theory. Building on the classical Itzykson–Zuber integral [IZ], we proved in [BKS] a β-deformation of this formula, originally conjectured in [CLZ] in the context of supersymmetric localization in $\mathcal{N} = 2$ gauge theories. Our proof relies entirely on the intrinsic symmetry of the β-deformed matrix model and provides a direct and conceptually transparent derivation. We emphasize that the purely Gaussian case had been previously obtained by Okounkov [O1], while we were not aware of this, our result is more general than Okounkov's lemma.

Motivated by a question of Okounkov on averages of 2 Jack polynomials, we introduced in Section 2.6 a new analytic framework compare to our paper [BKS]. This leads to Theorem 2.6.20, which unifies and extends known results on β-deformed matrix models. Conceptually, this result illustrates a central principle of this thesis: problems that appear algebraically complex often admit a significantly simpler formulation when recast in an analytic framework, in this case based on multivariable hypergeometric functions.

Although the two main parts of the thesis develop along different lines, they are connected at a deeper structural level. Recent developments show that one-dimensional Schrödinger operators play a central role in topological recursion and exact WKB analysis associated with matrix models [EO, DN, No, GS, I, I1]. While topological recursion has been successfully applied to the Gauss hypergeometric equation [I1], the current theory remains incomplete.

Our discussion on exact-WKB quantization of hypergoemetric Hamiltonian in Sec. 1.7, is not only a new result in this field but it is also quite surprising. The result we have already shown using a simple analysis of zeros of hypergeometric function implies that the Voros coefficient associated with the Gauss hypergeometric equation is trivial. A natural next step is the extension to differential equations with four regular-singular points, namely the Heun equation. Despite substantial progress in the hypergeometric case, existing techniques encounter serious obstructions in this setting.[1]

The spectral-theoretic approach developed in this thesis provides a concrete and promising alternative. In particular, it offers a viable route toward the quantization of the Heun curve, a problem that remains open and structurally significant. Establishing this connection would create a direct bridge between classical operator theory and modern developments such as quantum curves and topological recursion.

The results of this thesis demonstrate that classical mathematical physics, when developed systematically and combined with modern perspectives, leads not only to new results but also to new structural insights. This synthesis is the central theme of the present work and forms the basis for future developments.

[1] Private communication with Kohei Iwaki.

Appendix A

Hypergeometric function: specialization and different notation

A.1 Half integer case

Gegenbauer functions with $\alpha = \pm\frac{1}{2}$ have simple expressions in terms of elementary functions. To see this we change variables in Gegenbauer operators. For $w \in]-1, 1[$, we substitute $w = \cos\phi$:

$$\mathcal{G}_{-\frac{1}{2},\lambda}(w, \partial_w) = \partial_\phi^2 + \lambda^2; \tag{A.1.1}$$

$$\mathcal{G}_{\frac{1}{2},\lambda}(w, \partial_w) = \frac{1}{\sin\phi}\big(\partial_\phi^2 + \lambda^2\big)\sin\phi. \tag{A.1.2}$$

For $w \in]1, +\infty[$, we substitute $w = \cosh\theta$:

$$\mathcal{G}_{-\frac{1}{2},\lambda}(w, \partial_w) = -\partial_\theta^2 + \lambda^2; \tag{A.1.3}$$

$$\mathcal{G}_{\frac{1}{2},\lambda}(w, \partial_w) = \frac{1}{\sinh\theta}\big(-\partial_\theta^2 + \lambda^2\big)\sinh\theta. \tag{A.1.4}$$

Eqn.(A.1.1)—*Eqn.*(A.1.4) easily imply the first column of the following identities for Gegenbauer functions. (The second column simply follows from the definitions of Gegenbauer functions).

$$\cos\lambda\phi = S_{-\frac{1}{2},\lambda}(\cos\phi) = F\Big(\lambda, -\lambda; \frac{1}{2}; \sin^2\frac{\phi}{2}\Big), \tag{A.1.5}$$

$$\frac{\sin\lambda\phi}{\lambda\sin\phi} = S_{\frac{1}{2},\lambda}(\cos\phi) = F\Big(1+\lambda, 1-\lambda; \frac{3}{2}; \sin^2\frac{\phi}{2}\Big), \tag{A.1.6}$$

$$\cosh\lambda\theta = S_{-\frac{1}{2},\lambda}(\cosh\theta) = F\Big(\lambda, -\lambda; \frac{1}{2}; \sinh^2\frac{\theta}{2}\Big), \tag{A.1.7}$$

$$\frac{\sinh\lambda\theta}{\lambda\sinh\theta} = S_{\frac{1}{2},\lambda}(\cosh\theta) = F\Big(1+\lambda, 1-\lambda; \frac{3}{2}; \sinh^2\frac{\theta}{2}\Big), \tag{A.1.8}$$

$$2^\lambda e^{-\lambda\theta} = Z_{-\frac{1}{2},\lambda}(\cosh\theta) = \Big(2\sinh^2\frac{\theta}{2}\Big)^{-\lambda} F\Big(\lambda, \lambda+\frac{1}{2}; 1+2\lambda; -\frac{1}{\sinh^2\frac{\theta}{2}}\Big), \tag{A.1.9}$$

$$\frac{2^\lambda e^{-\lambda\theta}}{\sinh\theta} = Z_{\frac{1}{2},\lambda}(\cosh\theta) = \Big(2\sinh^2\frac{\theta}{2}\Big)^{-\lambda-1} F\Big(\lambda+\frac{1}{2}, \lambda+1; 1+2\lambda; -\frac{1}{\sinh^2\frac{\theta}{2}}\Big). \tag{A.1.10}$$

The above formulas can be found e.g. in equations (4.25)–(4.28) of [DGR] (in a slightly different form). They are straightforward generalizations of well-known formulas for Chebyshev polynomials. In fact, for $n \in \mathbb{N}_0$ the usual Chebyshev polynomials are special cases of Gegenbauer functions with $\alpha = \pm\frac{1}{2}$:

$$T_n(w) = S_{-\frac{1}{2},n}(w), \quad U_n(w) = (n+1)S_{\frac{1}{2},n+1}(w). \tag{A.1.11}$$

A.2 Associated Legendre functions vs. Gegenbauer functions

In the literature, many authors use a Legendre function instead of Gegenbauer functions. Here, we briefly discuss the relations between these functions and Gegenbauer functions. For Legendre function we use [NIST2] as our reference and [D1] as our reference for Gegenbauer function.

The Legendre differential operator is

$$\mathcal{L}_\mu^\alpha := (1+z^2)\partial_z^2 - 2z\partial_z + \mu(\mu+1) - \frac{\alpha^2}{1-z^2}. \tag{A.2.1}$$

It is equivalent to the Gegenbauer operator, in fact

$$\begin{aligned}&(1-w^2)^{\mp\frac{\alpha}{2}}\mathcal{L}_\mu^\alpha(1-w^2)^{\pm\frac{\alpha}{2}}\\ =&(1-w^2)\partial_w^2 - 2(\pm\alpha+1)w\partial_w + (\mu\mp\alpha)(\mu\pm\alpha+1) = \mathcal{G}_{\pm\alpha,\mu+\frac{1}{2}}.\end{aligned} \tag{A.2.2}$$

Certain distinguished funnctions annihilated by this operator are called associated Legendre functions. There various choices for these functions, which we quote following [NIST2]:
The associated Legendre function of the first kind is

$$\begin{aligned}\mathbf{P}_\mu^\alpha(z) &= \left(\frac{z+1}{z-1}\right)^{\frac{\alpha}{2}}\mathbf{F}\left(\mu+1, -\mu; 1-\alpha; \frac{1-z}{2}\right)\\ &= \frac{2^\alpha}{(z^2-1)_\bullet^\alpha}\mathbf{S}_{-\alpha,\mu+\frac{1}{2}}(z)\end{aligned} \tag{A.2.3}$$

The Ferrers function of the first kind is

$$\begin{aligned}\mathcal{P}_\mu^\alpha(z) &= \left(\frac{z+1}{1-z}\right)^{\frac{\alpha}{2}}\mathbf{F}\left(\mu+1, -\mu; 1-\alpha; \frac{1-z}{2}\right)\\ &= \frac{2^\alpha}{(1-z^2)_\bullet^\alpha}\mathbf{S}_{-\alpha,\mu+\frac{1}{2}}(z)\end{aligned} \tag{A.2.4}$$

And *the associated Legendre function of the second kind* is

$$\begin{aligned}\mathbf{Q}_\mu^\alpha(z) &= \mathrm{e}^{\mathrm{i}\pi\alpha}\frac{\pi^{\frac{1}{2}}\Gamma(\alpha+\mu+1)\left(z^2-1\right)^{\frac{\alpha}{2}}}{2^{\mu+1}z^{\alpha+\mu+1}}\mathbf{F}\left(\frac{\alpha+\mu}{2}+1, \frac{\alpha+\mu}{2}+\frac{1}{2}; \frac{3}{2}+\mu; z^{-2}\right)\\ &= \mathrm{e}^{\mathrm{i}\pi\alpha}\frac{\pi^{\frac{1}{2}}\Gamma(\alpha+\mu+1)\left(z^2-1\right)^{\frac{\alpha}{2}}}{2^{\mu+1}}\mathbf{Z}_{\alpha,\mu+\frac{1}{2}}(z).\end{aligned} \tag{A.2.5}$$

Note that Legendre functions are represented with upper and lower indices. One should not confuse them with other, quite analogous functions defined in the text, whose parameters are lower indices.

The Legendre functions are closely related to the functions $\mathcal{P}^{\mathrm{s}}$, $\mathcal{P}^{\mathrm{h}}$, and $\mathcal{Q}^{\mathrm{h}}$ that we introduced in the section on Gegenbauer Hamiltonians, which you can see on the right of the following comparison:

$$\mathcal{P}_{\mu}^{\alpha}(\cos r) = \left(\frac{2}{\sin r}\right)^{\frac{1}{2}} \mathcal{P}^{\mathrm{s}}_{-\alpha,\mu+\frac{1}{2}}(r), \tag{A.2.6}$$

$$\mathbf{P}_{\mu}^{\alpha}(\cosh r) = \left(\frac{2}{\sinh r}\right)^{\frac{1}{2}} \mathcal{P}^{\mathrm{h}}_{-\alpha,\mu+\frac{1}{2}}(r), \tag{A.2.7}$$

$$\mathbf{Q}_{\mu}^{\alpha}(\cosh r) = \frac{1}{(2\sinh r)^{\frac{1}{2}}} \mathrm{e}^{\mathrm{i}\pi\alpha} \sqrt{\pi}\Gamma(\alpha+\mu+1) \mathcal{Q}^{\mathrm{h}}_{-\alpha,\mu+\frac{1}{2}}(r). \tag{A.2.8}$$

Appendix B

Realizations of A, B, C, D root systems

In this appendix we describe explicit realizations of the root systems of type A, B, C, and D. In all cases, $\epsilon_1, \dots, \epsilon_N$ denote the canonical basis of $\mathbb{R}^N$.

Root system A_{N-1}

The root system of type A_{N-1} is

$$R^{A_{N-1}} = \{\epsilon_i - \epsilon_j \mid i \neq j\},$$

and the set of positive roots is

$$R_+^{A_{N-1}} = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq N\}.$$

It is a simply laced root system. The corresponding weight function is

$$w_k^{A_{N-1}}(x) := \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2k}, \tag{B.0.1}$$

where k is a constant multiplicity.

Root system B_N

The root system of type B_N is

$$R^{B_N} = \{\pm\epsilon_i \pm \epsilon_j \mid i \neq j\} \cup \{\pm\epsilon_i \mid 1 \leq i \leq N\},$$

and the set of positive roots is

$$R_+^{B_N} = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq N\} \cup \{\epsilon_i \mid 1 \leq i \leq N\}.$$

The squared lengths of the roots are 1 and 2. The corresponding weight function is

$$w_k^{B_N}(x) := \prod_{1 \leq i < j \leq N} |x_i^2 - x_j^2|^{2k_1} \prod_{i=1}^{N} |x_i|^{2k_2}, \tag{B.0.2}$$

where k_1 and k_2 are constants.

Root system C_N

The root system of type C_N is

$$R^{C_N} = \{\pm\epsilon_i \pm \epsilon_j \mid i \neq j\} \cup \{\pm 2\epsilon_i \mid 1 \leq i \leq N\},$$

and the set of positive roots is

$$R_+^{C_N} = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq N\} \cup \{2\epsilon_i \mid 1 \leq i \leq N\}.$$

The squared lengths of the roots are 2 and 4. The corresponding weight function is

$$w_k^{C_N}(x) := \prod_{1\leq i<j\leq N} |x_i^2 - x_j^2|^{2k_1} \prod_{i=1}^{N} |2x_i|^{2k_2}, \tag{B.0.3}$$

where k_1 and k_2 are constants.

Root system D_N

The root system of type D_N is

$$R^{D_N} = \{\pm\epsilon_i \pm \epsilon_j \mid i \neq j\},$$

and the set of positive roots is

$$R_+^{D_N} = \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq N\}.$$

It is also a simply laced root system. The corresponding weight function is

$$w_k^{D_N}(x) := \prod_{1\leq i<j\leq N} |x_i^2 - x_j^2|^{2k}, \tag{B.0.4}$$

where k is a constant multiplicity.

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