Abstract

This thesis is concerned with two classes of functions arising in exactly solvable systems of physics. Special functions are often treated merely as instruments of solution, yet their deeper significance lies in their delicate identities, hidden symmetries, and intricate relations. To solve something exactly is not only to obtain an answer, but also to better understand the structure that lies beneath it. Guided by this perspective, we study the spectral theory of one-dimensional Schrödinger operators and zero-dimensional random matrix theories through the lens of special functions.

The first part is devoted to the spectral theory of one-dimensional Schrödinger operators with complex potentials that can be reduced to the hypergeometric equation. Exploiting the exact solvability of these operators, we construct their Green functions and analyze the corresponding operator domains, closed realizations, and boundary conditions. In particular, we classify nine families of Schrödinger operators arising from separation of variables on symmetric spaces. This part is based on the paper *Exactly solvable Schrödinger operators related to the hypergeometric equation*, written in collaboration with Jan Dereziński. We also include a new unpublished result concerning the exact WKB quantization of these operators, showing how exact WKB expansions lead to an exact quantization condition.

The second part of the thesis concerns symmetric polynomials and their role in the β-deformed eigenvalue model. We show that this model is not only exactly solvable, but that, when multipoint correlators are expanded in the appropriate basis given by Jack polynomials, they acquire a closed and factorized form. This remarkable phenomenon is known as superintegrability. We present a proof based on the Virasoro constraints of the model together with representation-theoretic methods. This part is based on the paper *Proving superintegrability in β-deformed eigenvalue models*, written in collaboration with Aditya Bawane and Piotr Sułkowski.

In addition, we include a new unpublished result establishing a more general framework for this phenomenon. The proof relies on the relation between multivariable hypergeometric functions and the Dunkl reproducing kernel. This broader perspective not only contains the superintegrability of β-deformed models as a special case, but also clarifies the underlying mechanism behind this factorization phenomenon through a simpler and more clean argument.