Report on the thesis of Shi Cheng

String theory is describing arbitrary gauge and gravity theories as low-energy excitations around its vacuum configurations, which can be represented either as Calabi-Yau compactifications or as brane networks (webs). The emerging theories appear related by numerous dualities, expressing the fact that the underlying theory is always the same, but realization and study of these generalized symmetries is still very complicated. In brane web picture the number of parameters is rather small, most of them are discrete and the dream is to encode them in terms of some quiver-like variables. Continuous are basically just two kinds of parameters, defining the sizes of rectangular cells in the net. They are naturally associated with q and t in Macdonald and DIM theories – in stringy context they are usually studied by the theory of refined topological strings. The origin of these parameters in Calabi-Yau description is more obscure, but they can be straightforwardly introduced in the description of peculiar "toric" manifolds. Altogether the subject is very interesting, but progress is rather slow, because there are not too many people, capable to simultaneously keep many algebraic ideas in mind – nothing to say about putting them in certain order and making a logically consistent presentation. Also there is a lot of specific terminology, which often hides rather simple senses under a cloud of mystery. This makes the topic and the task of the thesis important and challenging.

The thesis is devoted to refined topological string theory. Currently this is a technically involved domain, where one associates with the pictures of brane networks the convolutions of topological vertices, which are complicated sums over Young diagrams, and studies various symmetries and limits of these expressions. These are supposed to provide explicit realizations and examples of various dualities, expected between string vacua and other configurations. The thesis demonstrates that the author learned this sophisticated machinery and is capable of providing his own examples, interesting for the other experts.

The author begins from a review of Polyakov superstring, which is also known as topological string. The main emphasis is put on the target spaces which are toric Calabi-Yau threefolds, and can be easily related to brane networks. The main outcome is the construction of refined topological vertex for open topological string.

Another part of the story is a construction of partition functions for 3d N=2 gauge theories and relation of symmetries to configurations of branes.

Then topological theory is analyzed in the presence of Lagrangian brane on strip Calabi-Yau threefolds and $3d$ mirror symmetry is used to relate it to the previous consideration.

The most interesting part is a discussion of hidden quiver structure, its relation to mirror-equivalent $3d$ brane webs and its manifestation at the level of knot invariants in Chern-Simons description of emerging gauge theories.

The thesis reports particular new formulas which illustrate different aspects of above picture. No one of them is decisively important to clarify any of the general ideas in the background, but altogether they contribute to illuminating the entire pattern. Still, I think, a lot is left to be done for the next generation of researchers. Students can learn a lot from the comprehensive review, included into the thesis.

To make long story shorter, I picked just a sample result from every chapter – there are usually several more of the same quality and significance.

Chapter 2 discusses various expressions for topological partition function. The main one comes from the annulus diagram in Chern-Simons theory, then it can be refined and reexpressed

in terms of topological vertices. The main claim here is the integrality of certain Ooguri-Vafa coefficients, appearing in the pletistic logarithm of partition function and interpreted as multiplicities of BPS states in the intermediate channel. Concrete expression of this kind is eq.(2.122).

Chapter 3, among other things, connects closed and open Ooguri-Vafa multiplicities, eq.(3.57). Remarkably, some formulas for topological partition functions can be rewritten in terms of quivers (3.112), what expresses connection to Donaldson-Thomas invariants. It is not quite clear to me, if this is just an observation, first made in the theory of knot invariants, or the meaning of this transform is supposed to be fully transparent.

Chapter 4 provides more details about brane calculus and discusses symmetries under q-t^{-1} exchange. Technically they are related to the properties of Nekrasov factors, which enter expressions for partition functions. For certain configurations of branes expressions through quivers can be straightforwardly lifted to refined case. Of special interest are examples obtained by Hanany-Witten transitions, which are not directly described by toric diagrams.

Chapter 5 considers the quiver structures and the way mirror symmetries act on them. Also mentioned is the knot-quiver correspondence, i.e. the lifting of HOMFLY-PT polynomials of various knots to quiver-dependent series.

Finally in chapter 6 an attempt is made to find the correspondence between $3d$ branes and quivers.

To conclude, the thesis presents a lot of formulas for partition functions of various brane webs and analyzes their properties and inter-relations. It demonstrates an insight of the author in this technically and conceptually involved field, and proves that he is prepared to study and solve complicated problems in rather abstract areas of theoretical physics. The thesis is well written and reflects a good level of understanding. It matches all possible requirements and allows to highly evaluate the author and his qualification. The author, Shi Cheng, deserves the doctoral degree in theoretical physics.

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