## On “Recursive structures for Nahm sums’

The doctoral dissertation under review draws its motivation from a number of topics in mathematical physics. Its central objects are Nahm sums: a remarkable set of generalised $q$-hypergeometric series, which originally arose as CFT characters in work of Nahm and have appeared ubiquitously in several inter-related problems in mathematics and physics:

- in $SU(N)$ Chern–Simons topological gauge theory, they encode the generating function of vevs of symmetric traces of Wilson loops coloured in symmetric representations along a 1-dimensional connected defect;
- equivalently, their coefficients express the HOMFLY-PT polynomial of the corresponding knot in symmetric representations;
- furthermore, they correspond in certain cases to determinantal insertions in a random matrix model with a suitable $q$-deformed measure (such as the Stieltjes–Wiegert, or Rogers–Szabo matrix model);
- at large $N$, they have a topological string theory interpretation as an open partition function of a Lagrangian brane in the A-model (or a holomorphic one-dimensional brane in the B-model);
- and under the knots–quivers correspondence, they are suitable specialisations of a motivic Donaldson–Thomas partition function of a symmetric quiver.

All of these questions have received an enormous amount of attention in recent years from what’s increasingly referred to as the “Physical Mathematics” community. After a broad introduction on the appeareance of Nahm sums in the above mentioned contexts in Chapter 1, the dissertation turns to an in-depth study of some of their structural properties, and provides a slew of probing tests of some open speculations in the literature. I will describe these below, along with an appraisal of each of them.

Chapter 1 gives a rather systematic introduction to Nahm sums, motivating them as characters of Virasoro representations, and then proceeding to a description of their appearance in the Donaldson–Thomas theory of symmetric quivers after work of Reineke *et al.*. The dual picture of these motivic DT invariants as Ooguri–Vafa invariants, predicted by the knots-quivers correspondence, is then presented as well as with its large $N$ string interpretation in terms of open BPS invariants of certain special Lagrangians in the resolved conifold. The chapter is concluded by deducing the classical A-polynomial of knots from the semi-classical (saddle-point) limit of the Nahm series, and its quantum version from a canonical non-commutative deformation given by a $q$-difference operator annihlating the quiver partition function/Ooguri–Vafa wavefunction. The exposition, while a bit dry at parts, is exhaustive and reasonably self-contained.

Chapter 2 switches gear towards the treatment of original material. It deals with a highly suggestive conjecture of Gukov–Sułkowski from 2011: the conjecture states, in rough terms, that a topological B-model background is only non-pertubatively consistent if the periods of the holomorphic top form on the target are rational multiples of $\sqrt{-1}$. This is striking for a variety of reasons: in particular, perturbation theory is completely blind to this condition – any B-model vacuum, picked by an arbitrary choice of complex deformation of the target (leaving it smooth), is good enough from the vantage point of special Kähler geometry and its higher genus deformation by the perturbative expansion of the Kodaira–Spencer theory of gravity: it's therefore remarkable that such a condition can be already detected at string tree level (by computing periods). A natural question is then what spectral curves satisfy this condition – and in particular, whether that's true for the B-model geometries defined by classical quiver A-polynomials. Now, in this case – and in general for mirrors of local Calabi–Yau threefolds which admit a description in terms of family of plane curves – the condition has an algebraic avatar in the fact that the defining polynomial should be *tempered*: the face polynomial associated to the Newton polygon of the defining polynomial should be cyclotomic. This equivalence turns testing the quantisation condition of Gukov–Sulkowski – a difficult number-theoretic condition on periods of certain meromorphic differentials – into an algebraic (and in fact combinatorial) statement about Nahm series. The latter is much more manageable, but still requires a surprisingly vast toolkit from computational algebra. The author displays an impressive amount of technical prowess in tackling this problem using methods from the combinatorics of mixed resultants, leading him to a complete solution of the case of the case of diagonal quivers. I think this is a neat and interesting result, which has led to a solo publication from the author: the question being tackled fits in the grander scheme of fundamental problems any mathematical physicist ask themselves ('when does a classical system admit a consistent quantisation of which it is the limit?'), and while this is a somewhat basic toy model even within the restricted context of topological strings, and in particular of large $N$ duals to knots in Chern–Simons theory, (the vertices, which in the open string correspond to 'basic disks', do not interact with each other), the level of technical complexity required is quite formidable indeed. I view this as a welcome and interesting result.

Chapter 3 deals with a cognate problem, namely the relation between quantisation of quiver A-polynomials and the higher genus reconstruction offered by the topological recursion of Chekhov–Eynard–Orantin. Quantisation is well-known to be plagued by ambiguities: in particular it is a priori unclear that the quantisation of classical A-polynomials by the corresponding Nahm sum – essentially a process of $q$-deformation – is compatible with the perturbative reconstruction of B-model open wavefunctions using the topological recursion prescription, not even for the case in which the corresponding spectral curve has genus zero.

This chapter performs a series of tests in this context, in a 'l.h.s.=r.h.s.' kind of fashion: a good sample of quiver A-polynomials are picked for low number of vertices, the corresponding recursion relations are deduced for the WKB expansion of the Nahm sum, and these are then matched with the higher genus reconstruction of the open partition function from the CEO recursion up to some genus. Agreement is found in all cases but one (a diagonal quiver), where some extra tweaking is required. All calculations are carried out with plenty of detail, and are interesting – the relation between topological recursion for genus zero spectral curves and WKB has still a number of open questions, despite some systematic study by Bouchard–Eynard for curves in $\mathbb{C} \times \mathbb{C}$: the $\mathbb{C}^\star \times \mathbb{C}^\star$ setting is very interesting and largely untouched, save for the case of mirrors of toric Calabi–Yau threefolds (where the compatibility follows from combining the topological vertex and the proof of the remodeling conjecture), therefore the verifications carried out for A-polynomials are both nice and interesting. The Chapter concludes with a discussion of quantum Airy structure beyond the quadratic setting – from a logical point of view this part is only weakly connected to the rest of the chapter (mostly by sharing the key-phrase 'topological recursion') and contains a summary of work with the author with Borot, Bouchard, Chidambaram and Creutzig.

Chapter 4 finally deals with a study of the knots-quivers correspondence proper. It is known that the correspondence is very highly non-injective – multiple quivers were found to experimentally correspond to the same knot, and the chapter analyses the question of equivalence of quivers in this sense in more detail. A notion of local equivalence of quivers if formulated in terms of sequences of disjoint transpositions, and it is in turn applied to generate equivalent quivers to a given one. For low number of crossings (in particular for the trefoil and figure-8 knot), the presented classification of quivers up to local equivalence is complete. The results of this section are of a slightly technical nature, but of interest to experts in the topic.

My assessment is that the quantity and quality of material presented in this dissertation is broadly at the appropriate level for the award of a PhD. The form is on the whole appropriate, the problems tackle energetically a corner of some important questions in the field, and the technical level is generally strong. I was particularly pleased with the content of Chapter 2, which I understand is based wholly on solo work by the author. The three main conceptual units of the thesis have all been published into standard good quality peer-reviewed journals in theoretical physics (two on J. High Energy Phys., and one on Phys. Rev. D), and the tangential work on quantum Airy structures has appeared on Mem. Amer. Math. Soc. – a well-regarded journal in pure mathematics. I have minor corrections to suggest, which are listed at the end of this document.

Based on the above, my concluding recommendation is positive.

Sincerely,

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**List of recommended changes.**

(1) Equation (1.3): the denominator of the sum should presumably be the q-Pochhammer symbol $(q;q)_d$.
(2) Unnumbered equation below (1.25): the summation index should be $Q_1 \ni a : i \to j$ (there are indices $(i, j)$ in the summand, but no index $a$).
(3) Just above equation (1.34): "the space of connections" should be "a suitable space of connections modulo gauge transformations".
(4) Equation (1.36): in view of the text that follows, you probably want to normalise by the partition function at this point, otherwise it is not true that the vev is a rational function of $q^{1/2}$ and $a^{1/2}$ (in particular for $R = 0$ the trivial representation you want to get 1, not the partition function). Also on the r.h.s.: $W_K(A)$ should really be $W_R(K)$.
(5) Middle of page 24: "consistsing" should be "consisting".
(6) Last sentence before 1.3.1: it is stated in no uncertain terms that there are no polynomial invariants that can classify knots up to isotopy. I am not aware if it is known whether that's true or not. E.g. it's true that the HOMFLY doesn't detect mutants, but if I remember correctly the coloured HOMFLY does for sufficient number of boxes (at least for the the first pairs that have been checked at low numbers of crossings).
(7) First line at page 29, the "eigenvalue locus": please define/explain.
(8) Equation (1.61), "the leading term is": please justify.
(9) Page 33, last bullet point: "$(\mathbb{C}^*)^2 \times \mathbb{C}$" should be $(\mathbb{C}^*)^2 \times \mathbb{C}^2$.
(10) "a Hilbert space $\mathcal{H}$ of..." should be "a Hilbert space $\mathcal{H}$ carrying a representation of..."
(11) Page 37, "in other words, all their roots" should be "in particular, all their roots" (e.g. $x - e^{2\pi i/3}$ has only zeroes at roots of unity without being cyclotomic)
(12) Figure 3.2: this looks like a screen capture from Eynard's lecture notes – this is fine, however please cite source.