Review of the doctoral dissertation of Mr. Dmitry Noshchenko, M.Sc., entitled "Recursive Structures for Nahm Sums"

General comments

The doctoral dissertation of Mr. Dmitry Noshchenko is an interesting and, I would say outstanding, work that presents original results relevant for various subjects of theoretical physics including conformal field theories, topological string theory, Chern-Simons field theory and matrix models. A central aspect of the work is the so-called *topological recursion*, a technique that first emerged in matrix models in order to solve recursively the loop equations satisfied by the $n-$point correlation functions. Later, it has been applied to enumerative geometry, string theory and other problems of mathematical physics in which the relevant quantities to be computed, e. g. generating functions or partition functions, is a functional ψ involving meromorphic functions on a plane curve C. The present doctoral work focuses on the relation between the quantized version of these curves and topological recursion. Quantum curves play the same role of the Schrödinger operator in quantum mechanics. The computation of ψ is similar to that of a wave function in quantum mechanics and can be performed using various methods like the WKB method. Schrödinger-like operators are linear, while topological recursion is nonlinear. This is one of the advantages offered by the introduction of quantum curves with respect to topological recursion.

The dissertation treats some aspects of recursive structures with particular applications to the knot-quiver correspondence. In this correspondence the generating series of a powerful class of knot invariants called colored HOMFLY-PT polynomials is identified with the generating function of the motivic Donaldson-Thomas invariants of symmetric quivers. The latter generating function is in the form of a Nahm sum, a type of series that describes the characters of certain conformal field theories and is a basic ingredient in the construction of their partition function. Using the saddle-point approximation, it turns out that the leading contribution to the Nahm sum for quivers is determined by a set of equations (the Nahm equations). The solutions of such equations define plane curves as the zero locus of polynomials called the *A-polynomials*. The associated quantum curve is obtained after a nontrivial procedure of quantization.

The work presented in the reviewed PhD thesis has connections to many models for theoretical and mathematical physics. Moreover, the completion of the work for obtaining the PhD degree has required the application of several concepts and methods from mathematics. As a consequence, there is a wealth of material in the thesis and sometimes it is not easy to distinguish the results of the PhD work from those coming from the research conducted by the Supervisor and his network of collaborators. Nonetheless, I believe that it is possible to claim that the author of the PhD thesis has scored several innovative and interesting contributions in the subjects of topological recursion, quantum curves and their applications. First of all, it has convincingly been shown using the WKB method that the quantum curve approach based on the A-polynomials successfully reproduces the expansion of the Nahm sum for quivers obtained via topological recursion. To confirm this conjecture, at least in the case of a restricted class of A-polynomials, an extensive number of different quivers models has been explored. This investigation is relevant for knot theory because some of the analysed quiver models correspond to the so-called extremal colored invariants of knots. Besides, a systematic analysis and classification of general quantum A-polynomials, not strictly related to knot theory or topological string theory, has been performed. Another important contribution is the proof that quiver A-polynomials are tempered, a property that ensures the *quantizability* of these polynomials. The proof has been made using methods from K-theory and includes very general quivers, in which the associated quiver A-polynomials are not restricted to curves of genus zero. This important achievement has been published in an article of which Mr. Noshchenko is the sole author. A third noteworthy contribution is the generalization of the topological recursion based on quadratic plane curves C to higher order curves with non-simple ramification points. More precisely, topological recursion may be cast in the form of an infinite family of differential operators, called quantum Airy structures. The solutions of the related differential equations generate series of enumerative invariants. An example is the Witten-Kontsevich generating series. As a result of the PhD work, together with co-workers, the quadratic quantum Airy structures, which are associated to conformal field theories and the Virasoro algebra, have been extended to the higher order quantum Airy structures. The latter are relevant for generalizations of the Virasoro algebra called $\mathcal{W}-$algebras. Some interesting applications to enumerative geometry have been discussed in the thesis. Finally, in the last chapter the uniqueness of the correspondence between quivers and knots has been investigated. A set of sufficient conditions that identify equivalent quivers corresponding to the same knot has been provided. The transformations that relate equivalent quivers are called local symmetries. These local symmetries have been investigated and classified for three infinite families of knots and the particular knots $6_2, 6_3$ and 7_3.

The structure of the doctoral dissertation is not standard. At least from my experience, usually the first chapters are devoted to the presentation of the basic concepts, theories and methods that have been used in the performed research. These introductory chapters are also useful to illustrate the state-of-art in the field together with the main assumptions made. The original material coming from the performed research is the subject of the later chapters. This is not the case of the reviewed dissertation. The propaedeutic part is divided within all chapters and, apart from Chapter 1, is mixed with more technical discussions. For instance, Chapter 2 contains a brief introduction on quantization which is followed by a rather technical and extensive discussion on the K-theory criterion for quantization. Next, in Chapter 3 the basics of topological recursion are explained, but after that there are again rather technical sections dedicated to the reconstruction of the WKB expansion from the topological recursion and to Airy quantum structures. In Chapter 4 the phenomenon of quiver degeneracy is outlined very briefly and the remaining part of the chapter is dominated by formulas and tables. In my opinion this particular organization of the presented material is not helpful for the reader to get an overview over the performed research. Despite this, it is possible to state that, overall, the exposed material is sufficiently clear. Moreover, the bibliography is excellent and exhaustive

Detailed remarks

Below a list of more detailed remarks about the submitted dissertation is reported. These remarks are just minor shortcomings and do not lower the value of the work:

1. In the Introduction the publications on which the doctoral dissertation is based have been listed and there is also a brief but clear summary of the chapters of the thesis. However, in the whole thesis there is not a sufficient description of the significance of the obtained results in theoretical and mathematical physics.

2. The explanations and notations about quantum curves and topological recursion are sometimes confusing. For instance, in Eq. (3.62) it is not

explained what are $\hat{l}$ and $\hat{m}$. Moreover, the general formula of the topological recursion that allows to compute the correlation differentials $\omega_{g,n}$ has been given in Eq. (3.52). Later it is claimed that the $\omega'_{g,n}$s may be integrated and re-assembled into the topological recursion wave function defined by equations (3.79)-(3.81). The procedure to arrive to these equations does not seem to be entirely trivial. A reference on such procedure should have been added. For instance, in Eq. (3.81) a meromorphic differential is integrated along a path defined on a plane curve. I believe that some comments about possible ambiguities in computing the integrals in (3.81) would have been appropriate. Finally, how Eq. (3.81) is related to the specific case of the Gromov-Witten invariants described by Eqs. (3.54)-(3.55)? In the expression of the free energy of Eq. (3.54) the $\omega'_{g,n}$s are not integrated as in the general formula (3.81).

3. Page 15: "The Hilbert space $\mathcal{H}$ of a CFT consists of the states....". This sentence seems not to be complete.

4. On page 18: it should have been explained for clarity that ta means $t(a)$. Moreover, the reference to figure 1.2 is not correct. It should be figure 1.3.

Concluding remarks

Concluding, I consider the work of Mr. Dmitry Noshchenko to be a valuable contribution and to meet the criteria prescribed by the law for a doctoral dissertation. Therefore, I request that this dissertation will be admitted to the public defense. In addition, in view of the outstanding results obtained, I would like to kindly ask that the doctoral dissertation will be considered for distinction.

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