Geometric and topological aspects of non-trivial solutions of Maxwell and Einstein equations

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Abstract

Both theories, Maxwell source-free electromagnetism and vacuum Einstein theory of gravity allow for solutions with a rich topological structure. In the dissertation, the author generalizes original description of electromagnetism with the help of conformal Yano–Killing tensors to the case of the Plebański–Demiański black hole. The proposed formalism is used to describe electromagnetism and weak gravitational field with a non-trivial topological structure in Minkowski space-time. These solutions, called Hopfions, are very simply and effectively described in the proposed formalism. Topological structure of electromagnetic fields is characterized by the concept of helicity. In the dissertation, the author proposes an analogical quantity for linearized gravitational field. Quasi-local (super-)energy densities are compared for gravitational Hopfion. The generating function for Hopfions displays a close relationships with the so-called Magic Field. Magic Field is obtained from the electromagnetic field of Kerr–Newman black hole in the $m \to 0$ limit, see section 4.1.1. The author has analyzed the relationships between Hopfions and Magic Field.

Streszczenie

Obie teorie, bezźródłowa teoria elektromagnetyzmu Maxwella i próżniowa teoria grawitacji Einsteina pozwalają na istnienie rozwiązań o bogatej strukturze topologicznej. W pracy uogólniono autorski opis elektromagnetyzmu z wykorzystaniem konforemnych tensorów Yano-Killinga do czasoprzestrzeni uogólnionej czarnej dziury Plebańskiego – Demiańskiego. Wykorzystano zaproponowany formalizm do opisu elektromagnetyzmu i słabego pola grawitacyjnego o nietrywialnej strukturze topologicznej w czasoprzestrzeni Minkowskiego. Rozwiązania te, nazywane Hopfionami, bardzo prosto i efektywnie są opisywane w zaproponowanym formalizmie. Strukturę topologiczną rozwiązań elektromagnetycznych charakteryzuję wielkość nazywana skrętnością (z ang. *helicity*). W pracy autor zaproponował analogiczną wielkość dla zlinearyzowanej grawitacji. Funkcja generująca dla Hopfionów wykazuje związki z tzw. elektromagnetycznym Polem Magicznym. Pole Magiczne otrzymuje się z pola elektromagnetycznego czarnej dziury Kerra–Newmana w granicy masy dążącej do zera, zobacz rozdział 4.1.1. Autor przeanalizował związki pomiędzy Hopfionami i Polem Magicznym.

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Contents

Chapter 1

Introduction

Electric and magnetic fields were described by Faraday, Gauss, Maxwell and others in the XIX century. It was a beginning of consistent approach to electromagnetism, nowadays known as Maxwell theory. Topology has been, and still is, fundamental in the development of electromagnetism. Oriented field lines and field fluxes are, up to today, basic concepts to understand electromagnetic system. In 1833 Gauss [24] considered two linked circuits and established the relation between the magnetic field induced by the currents along the circuits and a topological invariant known as the linking number. An extension of this concept led to the definition of magnetic helicity. Moreover, Maxwell equations admit curious solutions with rich topological structure. For example, Hopfion (see section 3.1) is one of the physical realization of Hopf fibration (see section 3.1.1) which can be obtained experimentally [30]. Chapter 3 of the thesis is a detailed study of such concepts.

In XX century, theoretical astrophysics has been dominated by Theory of Relativity. During that time, Maxwell theory has been generalized for the case of curved spacetime. One of the crucial issues has been, and still is, a precise description of propagation of electromagnetic wave around a black hole. In particular, the electrodynamics around Kerr background has been examined in 1970s using Newman–Penrose formalism [13, 14]. Chapter 2 is devoted for that aspects and their generalization.

1.1 Content

The thesis consists of introduction and three main chapters. Each of main chapters has a structure of separate project which is weakly related to the others. To clarify the exposition, a full explanation of a complex, scalar framework for electromagnetism/linearized gravity has been placed in the appendix. The appendices contains also some technical results and proofs. The work is organized as follows:

Chapter 1 contains overview of the thesis, used notation and conventions. Index of symbols is given in the end of the thesis, see appendix E.

In chapter 2, we propose a geometric construction of Klein–Gordon like equation for reduced electromagnetic data on Plebański–Demiański generalized black hole. Our construction is based on conformal Yano–Killing (CYK) two-form. CYK tensors were often investigated as a tool to study symmetries and construct conserved quantities (see [38]

and references therein). In electrodynamics there are two kinds of conserved quantities which are defined with the help of CYK tensors. The first one corresponds to electric or magnetic charge and is linear with respect to the Maxwell field. The second kind is quadratic and expresses energy, linear momentum or angular momentum of the electromagnetic tensor.
The chapter 2 begins with a brief introduction to CYK two-forms, section 2.1. In particular, we present properties of CYK two-forms which are useful in electrodynamics, section 2.2. The next section 2.3.1 is focused on properties of Plebański–Demiański spacetime. We analyze non-standard properties of CYK tensor in Plebański–Demiański background (paragraph 2.3.3) which enable one to reduce Maxwell's equation to a single generalized Klein–Gordon equation for a complex scalar.

In chapter 3, we analyze topological aspects of electromagnetism and linearized Einstein theory. The chapter is started with a short presentation of Hopf fibration (section 3.1.1). The Hopf fibration is one of the simplest non-trivial fibration of three-dimensional sphere. We will study electromagnetic and gravitational solutions based on the Hopf projection which is a surjective map sending circles on S^3 to points on S^2. These circles weave nested toroidal surfaces and each is linked with every other circle exactly once, creating the characteristic Hopf fibration.
Hopfion or Hopf soliton (paragraph 3.1.2) is a 'solitonary' solution of spin-N field which has rich topological structure related to Hopf fibration. The characteristic structure of hopfion can be easily seen on the integration curves of the vector field (see figures 3.2 and 3.3). The structure of closed, linked field lines of hopfions propagates without intersections along the light cone. Relation with Hopf index is discussed (section 3.1.3).
In 1977 Trautman [66] proposed the first electromagnetic solutions which were derived from the Hopf fibration. Rañada developed them to propagating solutions in refs [51, 52]. Last years, these little known solutions become more interesting because hopfions have successful applications in many areas of physics including electromagnetism [53, 30], magnetohydrodynamics [40], hadronic physics [57] and Bose–Einstein condensate [41]. The definition of hopfion was extended in [65]. It includes a class of spin N-fields and uses this to classify the electromagnetic and gravitational hopfions by algebraic type.
Next, we generalize electromagnetic Hopfions (section 3.3) with the help of original framework which is proposed by Prof. Jezierski [32]. The advantages of used description is given in introduction to section 3.2. In particular, properties of topological charge – helicity (paragraph 3.3.3) has been investigated in the framework. Most interesting results in this chapter has been obtained for gravitational Hopfions (section 3.4). Quasi-local (super-)energy densities are compared for gravitational Hopfion (paragraph 3.4.2). The analog of helicity for weak gravitational field is proposed in section 3.4.5.

In chapter 4, we try to obtain a new electromagnetic solution by generalization of the idea of imaginary shift in time for fundamental solution of wave equation (section 4.2). It turns out that Magic Field electromagnetic solution, described in section 4.1.2 (see potential (4.16)), can be obtained, analogically to Hopfions, by an imaginary shift in spatial direction $z \to z - \imath a$, applied to Columb field potential. The imaginary shift has been successfully used many times in different contexts. One of the most significant result obtained in this way is the Kerr–Newman metric. Kerr–Newman black hole is generated from Reissner–Nordström metric with the help of Newman–Janis algorithm [21].

The chapter 4 begins with a brief introduction to Magic Field (section 4.1.1). Magic Field is obtained as a limit of Kerr–Newman potential when mass tends to zero. Next, description of electromagnetism in Newman–Penrose formalism is given (paragraph 4.1.3). In this case, Newman–Penrose formalism enables one to recover Maxwell two-form from a reduced data which is adapted to spheroidal foliation. A new reduced data for electromagnetism, called Magic Hopfion, is obtained by an arbitrary complex shift in spacetime direction (section 4.2.1) which is applied to the fundamental solution (4.48). Next, Magic Hopfion solution is given explicitly as Maxwell two-form (4.107). Properties of Magic Hopfions are presented in paragraph 4.3.

1.2 Overview of presented results

In this section we give a brief description of presented material in the thesis. In particular, we highlight the own results obtained by the author.

In chapter 2, the survey on conformal Yano–Killing (CYK) two forms 2.1 is based on the author's supervisor results. In particular, we followed [38] and the references within. The section 2.2 is based on joint work of the author and his supervisor [37]. The construction is original. The description of Plebański–Demiański spacetime in 2.3.1 is build on the Plebański–Demiański original paper [50] and further analysis done by Griffiths and Podolský [27]. The most general solutions of CYK two forms for Plebański–Demiański generalized black hole, presented in 2.3.2, are obtained by Kubizňák and Krtouš [43]. However, the analysis of Weyl endomorphism in the space of two-forms for Plebański–Demiański black hole 2.3.2 has been done by the author. Fackerell and Ipser reduced Maxwell's equations on Kerr background to a single second order partial differential equation for a middle Newman–Penrose electromagnetic scalar[1] ϕ_0, see [22]. In [37], we have given an alternative proof of Fackerell–Ipser equation, which has been further generalized on Plebański–Demiański background. Generalized wave equation for electromagnetism 2.3.3 is one of the author's key result.

In chapter 3, introduction to Hopf fibration and description of Hopfion (sections 3.1.1-3.1.3) are based on classical papers, see review paper [4] and references within. The analysis of electromagnetic and gravitational Hopfions in terms of reduced data (paragraphs 3.2-3.5) is based on the joint paper of the author and his supervisor [58]. In particular, the gravitational analog of helicity, proposed in paragraph 3.4.5, has been obtained independently. The result has been further generalized to Helicity array for spin-2 field with the help of duality symmetry by Andersson's group [2].

The sections about Magic Field in chapter 4, 4.1.1 and 4.1.2, follow Lynden–Bell's papers [44, 45]. Description of Newman–Penrose formalism is based on Chandrasekhar book [14]. Non-mentioned content of the chapter 4 has been obtained by the author.

[1]Definition of ϕ_0 is given in section 4.1.3.

1.3 Notation and conventions

Our starting point will be a four-dimensional (except section 2.1) spacetime manifold M, equipped with a metric tensor of Lorentzian signature $g_{\mu\nu}$. In section 2.1, an arbitrary metric with a signature $(-+++)$ is assumed. In section 2.3, we use various forms of Plebański–Demiański metric which are described directly in the text. In chapter 3, Minkowski spacetime is the background with the metric $g = -\mathrm{d}t^2 + \delta_{kl}\mathrm{d}x^k\mathrm{d}x^l$. The three-dimensional spatial metric is denoted by δ_{ab}. In chapter 4, Minkowski metric in oblate spheroidal coordinates is used (4.6). Together with t = const surfaces with induced Euclidean metric in cylindrical coordinates $\delta_{kl}\mathrm{d}x^k\mathrm{d}x^l = \mathrm{d}z^2 + \mathrm{d}\widehat{R}^2 + \widehat{R}^2\mathrm{d}\varphi^2$.

We will denote by $T_{\ldots(\mu\nu)\ldots}$ the symmetric part and by $T_{\ldots[\mu\nu]\ldots}$ the antisymmetric part of tensor $T_{\ldots\mu\nu\ldots}$ with respect to indices μ and ν (analogous symbols will be used for more indices). In the thesis we use geometric units $c = G = 1$.

We assume existence of two nested levels of foliation in our spacetime manifold, a $1+1+2$ dimensional splitting. We work in an adapted set of coordinates — each foliation consist of level sets of an appropriate coordinate. This simplify the structure of equations and facilitate the distinction between objects belonging to different geometries. The first foliation will be the splitting of $\mathcal{M}$ into a family of spatial hypersurfaces, on which the temporal coordinate x^0 (interchangeably denoted as t) is constant. The hypersurface of constant t is denoted by Σ_t. Bold letters means three-dimensional spatial vectors on Σ_t and '$\cdot$' is a three-dimensional scalar product. For example, $\mathbf{E}\cdot\mathbf{C} = E^kC^l\delta_{kl}$. On these "slices of constant time" Σ_t, we distinguish the second foliation which splits each Σ_t into a collection of two-dimensional (2D) submanifolds, labelled by coordinate x^3. Its leaves will be topological spheres. x^3 is interchangeably denoted by R. In chapter 3, we consider only spherical foliations, where R parametrizes standard spheres of radius R.

For convenience we use index notation with Einstein summation convention. The range of indices for tensor objects is denoted through an indexing convention:

- Full spacetime dimension is denoted by small Greek indices $(\alpha, \beta, \gamma, \cdots)$, except θ and φ, which run over all coordinates on M — $(0, 1, 2, 3)$.
- Coordinates on Σ are denoted by small Latin indices $(a, b, c, \cdots)$, except t and r, which run over $(1, 2, 3)$.
- Coordinates on the leaves of two-dimensional foliation (spheres) are denoted by big Latin indices $(A, B, C, \cdots)$, except R, and take values $(1, 2)$.

This choice of coordinates means that projecting an object with only covariant indices to a lower dimension is performed simply by restricting the range of indices. In chapter 4, Newman–Penrose tetrad is introduced. We denote

$$\mathbf{e}_{\{a\}}{}^{\mu}\partial_\mu \quad (a = 1, 2, 3, 4)\,,$$

enclosure in curly brackets distinguishes the tetrad indices from the tensor indices.

In chapters 3 and 4, we use diagonal metrics only which do not require to distinguish between the inverse metric tensors on foliations. The covariant derivatives in dimensions four, three and two will be denoted by: ";", "|", and "||" respectively. Interchangeably, the symbol ∇ is used for the four-dimensional covariant derivative. All the covariant derivatives used in the manuscript are those coming from the appropriate Levi–Civita connection. When discussing linearized gravity (mainly in section 3.4) those will be derived from the background metric. ‘,’ denotes the partial derivative ∂. We use the symbol $\Box$ for the d'Alembert operator $\Box F = \nabla_\mu \nabla^\mu F = F_{;\mu}{}^{;\mu}$. By $\triangle$, we denote a three-dimensional Laplace operator $\triangle F = F_{|k}{}^{|k}$. The 2D Laplace–Beltrami operator on a unit sphere is denoted by $\mathbf{\Delta}$. Let us note that for a sphere of radius R we have $g^{AB} F_{||AB} = 1/R^2 \mathbf{\Delta} F$. The two-dimensional trace is denoted by $\overset{(2)}{X} = g^{CD} X_{CD}$, except $\overset{(2)}{\Phi}$, and the two-dimensional traceless part is given by $\overset{\circ}{X}_{AB} = X_{AB} - \frac{1}{2} g_{AB} \overset{(2)}{X}$.

Spherical harmonics on a unit sphere

Spherical harmonics are eigenfunctions of the Laplace–Beltrami operator on a unit 2D-sphere:

$$\mathbf{\Delta}\, Y_l(\theta, \phi) = -l(l+1) Y_l(\theta, \phi). \tag{1.1}$$

This equation, for a given integer $l \geq 0$, possesses a $(2l+1)$-dimensional space of solutions. However, the distinction between different solutions for the same eigenvalue will not be important in our considerations and “$Y_l(\theta, \phi)$” will be treated simply as a symbolic representation of the whole space of solutions for l. See (3.35) and the comments below for details.

Chapter 2

Conformal Yano–Killing two-form and its application in electrodynamics on generalized black hole

2.1 Survey on Conformal Yano–Killing two-forms

We present a review of classical results about CYK two-forms. The section is based on [38] and classical references within.

2.1.1 Fundamental properties

In this section, we assume M be an n-dimensional ($n > 1$) manifold.

Let $Q_{\mu\nu}$ be a skew-symmetric tensor field (two-form) on M. By $\mathcal{Q}_{\lambda\kappa\sigma}$ we denote a (three-index) tensor which is defined as follows:

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q, g) := Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{n-1}\left(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right) . \tag{2.1}$$

$\mathcal{Q}$ has the following algebraic properties:

$$\mathcal{Q}_{\lambda\kappa\mu}g^{\lambda\mu} = 0 = \mathcal{Q}_{\lambda\kappa\mu}g^{\lambda\kappa} , \quad \mathcal{Q}_{\lambda\kappa\mu} = \mathcal{Q}_{\mu\kappa\lambda} , \tag{2.2}$$

i.e. it is traceless and partially symmetric.

Definition 2.1.1. A skew-symmetric tensor $Q_{\mu\nu}$ is a conformal Yano–Killing tensor (or simply CYK tensor) for the metric g iff $\mathcal{Q}_{\lambda\kappa\sigma}(Q, g) = 0$.

In other words, $Q_{\mu\nu}$ is a conformal Yano–Killing tensor if it fulfils the following equation:

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1}\left(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu}\right) , \tag{2.3}$$

(first proposed by Tachibana and Kashiwada, cf. [63, 64]).

A more abstract way of describing a CYK two-form with the help of twistor theory can be found in [6], [47], [56] or [59]. CYK two-form is defined as the element of a kernel of a twistor operator $Q \to \mathcal{T}_{wist}Q$ defined[1] as follows:

$$\forall X \;\; \mathcal{T}_{wist}Q(X) := \nabla_X Q - \frac{1}{p+1} X \lrcorner \mathrm{d}Q + \frac{1}{n-p+1} g(X) \wedge \mathrm{d}^* Q \,.$$

However, to simplify the exposition, we prefer abstract index notation which also seems to be more popular.

Equation (2.3) may be transformed into the following equivalent form:

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \frac{3}{n-1} g_{\sigma[\lambda} Q_{\kappa]}{}^{\delta}{}_{;\delta} = 0 \,, \tag{2.4}$$

and this is a generalization of the equation

$$Q_{\lambda(\kappa;\sigma)} - Q_{\kappa(\lambda;\sigma)} + \eta_{\sigma[\lambda} Q_{\kappa]}{}^{\delta}{}_{;\delta} = 0 \,, \tag{2.5}$$

which appeared in Penrose and Rindler book [49, p. 396] as the equation for skew-symmetric tensor field $Q_{\mu\nu}$ in Minkowski spacetime with the metric $\eta_{\mu\nu}$.

Using the following symbol:

$$\xi_\mu := Q^{\nu}{}_{\mu;\nu} \,, \tag{2.6}$$

we can rewrite equation (2.3) in the form:

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1} \left(g_{\sigma\lambda}\xi_\kappa - g_{\kappa(\lambda}\xi_{\sigma)} \right) . \tag{2.7}$$

Let us notice that if $\xi_\mu = 0$, then $Q_{\mu\nu}$ fulfils the equation:

$$Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = 0 \;. \tag{2.8}$$

Skew-symmetric tensors fulfiling equation (2.8) are called in the literature Yano tensors (or Yano–Killing tensors; see [25], [42], [70]). It is obvious that a two-form $Q_{\mu\nu}$ is a Yano tensor iff $Q_{\mu\nu;\lambda}$ is totally skew-symmetric in all indices. So, if $Q_{\mu\nu}$ fulfils (2.8), then $\xi_\mu = g^{\kappa\lambda} Q_{\kappa\mu;\lambda} = 0$ (because $g^{\kappa\lambda}$ is symmetric in its indices). That means that each Yano tensor is a conformal Yano–Killing tensor, but not the other way around. The necessary and sufficient condition for a CYK tensor to be a Yano tensor is the vanishing of ξ_μ.

CYK tensors are the conformal generalization of Yano tensors. More precisely, for any positive function Ω on M the tensor $\mathcal{Q}$ transforms with respect to the conformal rescaling as follows:

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q,g) = \Omega^{-3} \mathcal{Q}_{\lambda\kappa\sigma}(\Omega^3 Q, \Omega^2 g) \,, \tag{2.9}$$

which implies the following

[1] Obviously $\mathcal{T}_{wist}Q$ corresponds to tensor $\mathcal{Q}(Q,g)$ (in abstract index notation). Here X is a vector field, Q is a p-form, $g : TM \to T^*M$ is a Riemannian metric, d^* denotes coderivative etc. Conformal Killing p-forms are defined with the help of natural differential operators on Riemannian manifolds. We know from the representation theory of the orthogonal group, that the space of p-form valued one-forms ($T^*M \bigotimes \bigwedge^p T^*M$) decomposes into the orthogonal and irreducible sum of forms of degree $p+1$ (which gives the exterior differential d), the forms of degree $p-1$ (defined by the coderivative d^*) and the trace-free part of the partial symmetrization (the corresponding first order operator is denoted by $\mathcal{T}_{wist}$).

Theorem 2.1.1. *If $Q_{\mu\nu}$ is a CYK tensor for the metric $g_{\mu\nu}$, then $\Omega^3 Q_{\mu\nu}$ is a CYK tensor for the conformally rescaled metric $\Omega^2 g_{\mu\nu}$.*

Proof of the formula (2.9) is given in [38, Appendix A]. The form of equation (2.8) does not remain unchanged under conformal rescaling. But that equation is a particular case of equation (2.3) whose form remains unchanged under such a transformation. That means that if Q is a Yano tensor of the metric g, then although in general $\Omega^3 Q$ is not a Yano tensor of the metric $\Omega^2 g$, it is a CYK tensor. In this sense equation (2.3) is a conformal generalization of equation (2.8).

Wave equation satisfied by CYK tensor

By a direct calculation, one can prove the following equality. If $Q_{\lambda\sigma}$, $R^{\sigma}{}_{\lambda\nu\mu}$, and $R_{\mu\nu}$ are a CYK tensor, a Riemann tensor and a Ricci tensor respectively then the following equality holds:

$$\nabla_\mu \nabla^\mu Q_{\lambda\kappa} = R^{\sigma}{}_{\kappa\lambda\nu} Q_{\sigma}{}^{\nu} - R_{\sigma[\kappa} Q_{\lambda]}{}^{\sigma} \,, \tag{2.10}$$

The above equality was proved in [38].

2.1.2 The connection between CYK tensors and Killing tensors

It is a known fact that the "square" of a Yano tensor is a Killing tensor. It turns out that in the same way CYK tensors are connected with conformal Killing tensors. In the following definitions we will restrict ourselves to the Killing tensors and conformal Killing tensors of rank 2, although one can consider tensors of any rank ([17], [42]).

Definition 2.1.2. A symmetric tensor $A_{\mu\nu}$ is a Killing tensor iff it fulfils the equation:

$$A_{(\mu\nu;\kappa)} = 0. \tag{2.11}$$

Definition 2.1.3. A symmetric tensor $A_{\mu\nu}$ is a conformal Killing tensor iff it fulfils the equation:

$$A_{(\mu\nu;\kappa)} = g_{(\mu\nu} A_{\kappa)} \,, \tag{2.12}$$

for a certain covector A_κ.

It is obvious that equation (2.11) is a particular case of (2.12). It is easy to see that the covector A_κ is unambiguously determined by equation (2.12) (it can be shown e.g. by contracting the equation with $g^{\mu\nu}$). To be more precise:

$$A_\kappa = \frac{1}{n+2}(2A^{\mu}{}_{\kappa;\mu} + A^{\mu}{}_{\mu;\kappa}).$$

From the above definitions one can easily see that a (conformal) Killing tensor is a generalization of a (conformal) Killing vector (cf. [12], [29], [54], [67], [69]).

Obviously, if $Q_{\mu\nu}$ is a skew-symmetric tensor, then $A_{\mu\nu}$ defined by the formula

$$A_{\mu\nu} = Q_{\mu\lambda} Q^{\lambda}{}_{\nu} \,, \tag{2.13}$$

is a symmetric tensor. It turns out that if a skew-symmetric tensor $Q_{\mu\nu}$ fulfils equation (2.3) (it is a CYK tensor), then $A_{\mu\nu}$ defined by (2.13) fulfils equation (2.12) with

$$A_\kappa = \frac{2}{n-1} Q_\kappa{}^\lambda Q_\lambda{}^\delta{}_{;\delta} \,, \tag{2.14}$$

(therefore – since $A_{\mu\nu}$ is symmetric – it is a conformal Killing tensor). If $Q_{\mu\nu}$ is a Yano tensor, then A_κ defined by formula (2.14) vanishes, thus $A_{\mu\nu}$ (defined by (2.13)) is a Killing tensor. That enables one to formulate the following (cf. Prop. 5.1. in [25] or 35.44 in [42])

Theorem 2.1.2. *If $Q_{\mu\nu}$ and $P_{\mu\nu}$ are (conformal) Yano–Killing tensors, then the symmetrized product $A_{\mu\nu} := Q_{\lambda(\mu}P_{\nu)}{}^\lambda$ is a (conformal) Killing tensor.*

Proof. Let $Q_{\mu\nu}$ and $P_{\mu\nu}$ be conformal Yano-Killing tensors. We have then

$$Q_{\kappa\lambda;\sigma} + Q_{\sigma\lambda;\kappa} = \frac{2}{n-1}\left(g_{\sigma\kappa}\xi_\lambda - g_{\lambda(\kappa}\xi_{\sigma)}\right) \,,$$

and

$$P_{\kappa\lambda;\sigma} + P_{\sigma\lambda;\kappa} = \frac{2}{n-1}\left(g_{\sigma\kappa}\zeta_\lambda - g_{\lambda(\kappa}\zeta_{\sigma)}\right) ,$$

where $\xi_\mu = Q^\nu{}_{\mu;\nu}$ and $\zeta_\mu = P^\nu{}_{\mu;\nu}$. Contracting the first of the above equations with $P_\nu{}^\lambda$, we get

$$Q_{\kappa\lambda;\sigma}P_\nu{}^\lambda + Q_{\sigma\lambda;\kappa}P_\nu{}^\lambda = \frac{2}{n-1}\left(g_{\sigma\kappa}\xi_\lambda P_\nu{}^\lambda - \frac{1}{2}P_{\nu\kappa}\xi_\sigma - \frac{1}{2}P_{\nu\sigma}\xi_\kappa\right) .$$

Symmetrizing this equation in κ, σ and ν, we get (since P is skew-symmetric)

$$Q_{\lambda(\kappa;\sigma}P_{\nu)}{}^\lambda = -\frac{1}{n-1} g_{(\kappa\sigma}P_{\nu)}{}^\lambda \xi_\lambda \,.$$

Analogously we get

$$Q_{\lambda(\kappa}P_\nu{}^\lambda{}_{;\sigma)} = -\frac{1}{n-1} g_{(\kappa\sigma}Q_{\nu)}{}^\lambda \zeta_\lambda \,.$$

Finally, if $A_{\kappa\nu} := Q_{\lambda(\kappa}P_{\nu)}{}^\lambda$, then

$$\begin{aligned} A_{(\kappa\nu;\sigma)} &= \left(Q_{\lambda(\kappa}P_\nu{}^\lambda\right)_{;\sigma)} = Q_{\lambda(\kappa;\sigma}P_{\nu)}{}^\lambda + Q_{\lambda(\kappa}P_\nu{}^\lambda{}_{;\sigma)} \\ &= -\frac{1}{n-1}\left(g_{(\kappa\sigma}P_{\nu)}{}^\lambda\xi_\lambda + g_{(\kappa\sigma}Q_{\nu)}{}^\lambda\zeta_\lambda\right) = g_{(\kappa\nu}A_{\sigma)} \,, \end{aligned}$$

where

$$A_\nu := \frac{1}{n-1}\left(P^\lambda{}_\nu\xi_\lambda + Q^\lambda{}_\nu\zeta_\lambda\right) \,.$$

This means that $A_{\kappa\nu}$ is a conformal Killing tensor. If now Q and P are Yano tensors, then $\xi_\mu = \zeta_\mu = 0$, which implies $A_\mu = 0$. In that case $A_{\kappa\nu}$ is a Killing tensor.

□

2.1.3 The connection between CYK tensors and Killing vectors

Let us denote by $R^{\sigma}{}_{\kappa\lambda\mu}$ the Riemann tensor describing the curvature of the manifold (M, g). We use now an integrability condition

$$2Q_{\lambda\kappa;\nu\mu} = \frac{2}{n-1}\left(g_{\lambda\mu}\xi_{\kappa;\nu} + g_{\nu\lambda}\xi_{\kappa;\mu} - g_{\mu\nu}\xi_{\kappa;\lambda} - g_{\kappa(\lambda}\xi_{\mu);\nu} + g_{\kappa(\mu}\xi_{\nu);\lambda} - g_{\kappa(\nu}\xi_{\lambda);\mu}\right)$$
$$+Q_{\sigma\lambda}R^{\sigma}{}_{\kappa\mu\nu} + Q_{\sigma\mu}R^{\sigma}{}_{\kappa\lambda\nu} + Q_{\sigma\nu}R^{\sigma}{}_{\kappa\lambda\mu} + 2Q_{\sigma\kappa}R^{\sigma}{}_{\mu\nu\lambda}\,, \tag{2.15}$$

which is proven [38, Appendix B].

A contraction in indices κ and ν gives us:

$$g_{\mu\lambda}\xi^{\sigma}{}_{;\sigma} + (n-2)\xi_{(\mu;\lambda)} = (n-1)R_{\sigma(\mu}Q_{\lambda)}{}^{\sigma}\,, \tag{2.16}$$

where by $R_{\mu\nu}$ we denote the Ricci tensor of the metric $g_{\mu\nu}$. Taking the trace of (2.16) we obtain:

$$(2n-2)\xi^{\sigma}{}_{;\sigma} = (n-1)R_{\sigma\mu}Q^{\mu\sigma} = 0\,,$$

where the last equality results from the fact that $R_{\sigma\mu}$ is a symmetric tensor and $Q^{\mu\sigma}$ is a skew-symmetric one. Therefore, we have $\xi^{\sigma}{}_{;\sigma} = 0$ which, for $n > 2$, implies

$$\xi_{(\mu;\lambda)} = \frac{n-1}{n-2}R_{\sigma(\mu}Q_{\lambda)}{}^{\sigma}\,. \tag{2.17}$$

If M is an Einstein manifold, i.e. its Ricci tensor $R_{\mu\nu}$ is proportional to its metric $g_{\mu\nu}$, then using equation (2.17) we obtain

$$\xi_{(\mu;\nu)} = -\frac{n-1}{n-2}\Lambda g_{\sigma(\mu}Q_{\nu)}{}^{\sigma} = -\frac{n-1}{n-2}\Lambda Q_{(\nu\mu)} = 0\,.$$

Here $R_{\mu\nu} = -\Lambda g_{\mu\nu}$ and by Λ we denote a cosmological constant. The condition $\xi_{(\mu;\lambda)} = 0$ means that ξ^{μ} is a Killing vector field of the metric $g_{\mu\nu}$. That enables one to formulate the following

Theorem 2.1.3. *If $g_{\mu\nu}$ is a solution of the vacuum Einstein equations with cosmological constant and $Q_{\mu\nu}$ is its CYK tensor, then $\xi^{\mu} = \nabla_{\nu}Q^{\nu\mu}$ is a Killing vector field of the metric $g_{\mu\nu}$.*

Let us notice that this fact reduces the number of Einstein metrics possessing a non-trivial CYK tensor. The existence of a solution of equation (2.3) which is *not* a Yano tensor implies that our manifold M has at least one symmetry. In the case of a Yano tensor that does not have to be true.

Hodge duality

In the space of differential forms on an oriented manifold one can define a mapping called the Hodge duality (Hodge star). It assigns to every p-form an $(n-p)$-form (where n is the dimension of the manifold). We consider the case of $n = 4$ and $p = 2$. The Hodge star then becomes a mapping which assigns to a two-form ω a two-form $*\omega$. We can express this mapping in the following way:

$$*\omega_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta}{}^{\mu\nu}\omega_{\mu\nu}\,, \tag{2.18}$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is the antisymmetric Levi-Civita tensor[2] determining orientation of the manifold ($\frac{1}{4!}\varepsilon_{\alpha\beta\gamma\delta}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta$ is the volume form of the manifold M). For the Lorentzian metric we have $**\omega = -\omega$. Due to CYK tensor being a two-form, it is reasonable to ask what are the properties of its dual. Let Q be a CYK tensor and $*Q$ its dual. Moreover, let us introduce the following covector $\chi_\mu := \nabla^\nu * Q_{\nu\mu}$. It was proved in [38], that

$$*Q_{\lambda\mu;\nu} + *Q_{\nu\mu;\lambda} = \frac{2}{3}\left(g_{\nu\lambda}\chi_\mu + g_{\mu(\lambda}\chi_{\nu)}\right)\,. \tag{2.19}$$

It is not hard to recognize that this is Eq. (2.3) for the tensor $*Q$. It proves the following theorem:

Theorem 2.1.4. *Let $g_{\mu\nu}$ be a metric tensor on a four-dimensional differential manifold M. An antisymmetric tensor $Q_{\mu\nu}$ is a CYK tensor of the metric $g_{\mu\nu}$ if and only if its dual $*Q_{\mu\nu}$ is also a CYK tensor of this metric.*

The above theorem implies that for every four-dimensional manifold, solutions of Eq. (2.3) exist in pairs – to each solution we can assign the dual solution (in the Hodge duality sense).

2.2 Description of electrodynamics with the help of CYK two-form

We use vacuum Maxwell equation in terms of Maxwell field. Maxwell field $F_{\mu\nu}$ is a two-form (antisymmetric tensor) field. The vacuum Maxwell equation in terms of Maxwell field takes the form

$$\left\{\begin{array}{r} \mathrm{d}F = 0\,, \\ \mathrm{d}*F = 0\,, \end{array}\right. \iff \left\{\begin{array}{r} F_{[\mu\nu;\lambda]} = 0\,, \\ F_{\mu\nu}{}^{;\mu} = 0\,, \end{array}\right. \tag{2.20}$$

where $*$ denotes Hodge duality (2.18). We consider real Maxwell fields. Taking a divergence of the first Maxwell equation (2.20) and combining with (2.22), we can easily transform the d'Alembertian of Maxwell field:

$$\begin{aligned} \Box F_{\mu\nu} &= F_{\mu\nu;\lambda}{}^{;\lambda} \\ &= F_{\lambda\nu;\mu}{}^{;\lambda} - F_{\lambda\mu;\nu}{}^{;\lambda} \end{aligned}$$

[2]It can be defined by the formula $\varepsilon_{\alpha\beta\gamma\delta} = \sqrt{-\det g_{\mu\nu}}\epsilon_{\alpha\beta\gamma\delta}$, where

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ is an even permutation of } 0,\,1,\,2,\,3 \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ is an odd permutation of } 0,\,1,\,2,\,3 \\ 0 & \text{in any other case} \end{cases}$$

$$
\begin{aligned}
&= \underbrace{F_{\lambda\nu}{}^{;\lambda}{}_{\mu}}_{0} - g^{\rho\lambda}(R^{\alpha}{}_{\lambda\rho\mu}F_{\alpha\nu} + R^{\alpha}{}_{\nu\rho\mu}F_{\lambda\alpha}) \\
&\quad - \underbrace{F_{\lambda\mu}{}^{;\lambda}{}_{\nu}}_{0} + g^{\rho\lambda}(R^{\alpha}{}_{\lambda\rho\nu}F_{\alpha\mu} + R^{\alpha}{}_{\mu\rho\nu}F_{\lambda\alpha}) \\
&= g^{\rho\lambda}\left[(R^{\alpha}{}_{\lambda\rho\nu}F_{\alpha\mu} - R^{\alpha}{}_{\lambda\rho\mu}F_{\alpha\nu}) + (R^{\alpha}{}_{\mu\rho\nu}F_{\lambda\alpha} - R^{\alpha}{}_{\nu\rho\mu}F_{\lambda\alpha})\right] \\
&= (R^{\lambda}{}_{\alpha\lambda\nu}F_{\mu}{}^{\alpha} - R^{\lambda}{}_{\alpha\lambda\mu}F_{\nu}{}^{\alpha}) + (R^{\alpha}{}_{\nu\mu\lambda}F^{\lambda}{}_{\alpha} - R^{\alpha}{}_{\mu\nu\lambda}F^{\lambda}{}_{\alpha}),
\end{aligned}
$$

and we obtain

$$
\Box F_{\mu\nu} = -2R^{\lambda}{}_{\alpha\lambda[\mu}F_{\nu]}{}^{\alpha} - 2R^{\alpha}{}_{[\mu\nu]\lambda}F^{\lambda}{}_{\alpha}\,, \tag{2.21}
$$

where, according to our conventions, the commutator of covariant derivatives for any tensor field $T_{\mu\nu}$ on M reads:

$$
T_{\lambda\kappa;\nu\mu} - T_{\lambda\kappa;\mu\nu} = T_{\sigma\kappa}R^{\sigma}{}_{\lambda\nu\mu} + T_{\lambda\sigma}R^{\sigma}{}_{\kappa\nu\mu}\,. \tag{2.22}
$$

The above identity follows directly from the Riemann tensor definition.

Combining Maxwell equation (2.20) with CYK equation (2.3), we can check that the term $\nabla^{\lambda}F_{\mu\nu}\nabla_{\lambda}Q^{\mu\nu}$ is vanishing. More precisely,

$$
\begin{aligned}
0 &= \overbrace{F_{[\mu\nu;\lambda]}}^{0}Q^{\mu\nu;\lambda} \\
&= F_{\mu\nu;\lambda}Q^{\mu\nu;\lambda} + 2F_{\lambda\mu;\nu}Q^{\mu\nu;\lambda} \\
&= 3F_{\mu\nu;\lambda}Q^{\mu\nu;\lambda} + 2F_{\lambda\mu;\nu}(Q^{\mu\nu;\lambda} + Q^{\mu\lambda;\nu}) \\
&= 2F_{\lambda\mu;\nu}\underbrace{\left[Q^{\mu\nu;\lambda} + Q^{\mu\lambda;\nu} + \frac{2}{3}(g^{\nu\lambda}Q^{\rho\mu}{}_{;\rho} + g^{\mu(\lambda}Q^{\nu)\rho}{}_{;\rho})\right]}_{\mathcal{Q}^{\mu\nu\lambda}=0} \\
&\quad + 3F_{\mu\nu;\lambda}Q^{\mu\nu;\lambda} - 2\underbrace{F^{\lambda}{}_{\mu;\lambda}}_{0}Q^{\rho\mu}{}_{;\rho} \\
&= 3F_{\mu\nu;\lambda}Q^{\mu\nu;\lambda}\,.
\end{aligned} \tag{2.23}
$$

Finally, the d'Alembertian of Maxwell–CYK contraction takes the following form:

$$
\begin{aligned}
\Box(F_{\mu\nu}Q^{\mu\nu}) &= Q^{\mu\nu}\Box F_{\mu\nu} + F^{\mu\nu}\Box Q_{\mu\nu} + 2F_{\mu\nu;\lambda}Q^{\mu\nu;\lambda} \\
&= Q^{\mu\nu}\Box F_{\mu\nu} + F^{\mu\nu}\Box Q_{\mu\nu}\,.
\end{aligned} \tag{2.24}
$$

The last equality is implied by (2.23). The above considerations lead to the following

Theorem 2.2.1. *Let $F_{\mu\nu}$, $Q_{\mu\nu}$ and $R^{\sigma}{}_{\lambda\nu\mu}$ be respectively a Maxwell field, a CYK tensor and the Riemann tensor corresponding to the metric $g_{\mu\nu}$. Then*

$$
\Box(F_{\mu\nu}Q^{\mu\nu}) + \frac{1}{2}F^{\sigma\lambda}R_{\sigma\lambda\mu\nu}Q^{\mu\nu} + Q^{\mu\nu}R_{\sigma\mu}F_{\nu}{}^{\sigma} = 0\,. \tag{2.25}
$$

Proof. Making use of equations (2.10), (2.21) and (2.24), we can transform Maxwell–CYK contraction in the following way:

$$
\Box(F_{\mu\nu}Q^{\mu\nu}) = -2Q^{\mu\nu}\left(R^{\lambda}{}_{\sigma\lambda\mu}F_{\nu}{}^{\sigma} + R^{\sigma}{}_{\mu\nu\lambda}F^{\lambda}{}_{\sigma}\right) + F^{\mu\nu}\left(R^{\sigma}{}_{\nu\mu\lambda}Q_{\sigma}{}^{\lambda} + R_{\sigma\mu}Q_{\nu}{}^{\sigma}\right)
$$

$$
\begin{aligned}
&= -Q^{\mu\nu}R_{\sigma\mu}F_{\nu}{}^{\sigma} + R_{\sigma\mu\nu\lambda}(2Q^{\mu\nu}F^{\sigma\lambda} - F^{\mu\nu}Q^{\sigma\lambda}) \\
&= -\frac{1}{2}F^{\sigma\lambda}R_{\sigma\lambda\mu\nu}Q^{\mu\nu} - Q^{\mu\nu}R_{\sigma\mu}F_{\nu}{}^{\sigma}\,,
\end{aligned}
$$

(the last equality uses Bianchi identity $R^{\sigma}{}_{[\mu\nu\lambda]} = 0$). □

It is convenient to split the Riemann tensor into Weyl tensor $C_{\sigma\lambda\mu\nu}$, Ricci tensor $R_{\mu\nu}$ and curvature scalar R, and rewrite Eq. (2.25) in the equivalent form

$$
\left(\Box - \frac{1}{6}R\right)(F_{\mu\nu}Q^{\mu\nu}) + \frac{1}{2}F^{\sigma\lambda}C_{\sigma\lambda\mu\nu}Q^{\mu\nu} = 0\,. \tag{2.26}
$$

We may note here that the above equation (2.26) is crucial for our further investigation. For flat spacetime Eq. (2.26) reduces to the wave equation $\Box\phi = 0$, where $\phi := F_{\mu\nu}Q^{\mu\nu}$ is a scalar function. We will show in the next section that for Plebański–Demiański generalized black hole (2.45) we can reduce Eq. (2.26) to wave equation with potential. Let us remind: a Weyl tensor $C^{\sigma}{}_{\lambda\mu\nu}$ remain unchanged under a conformal rescaling $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ for any positive function Ω on M. Obviously, $F_{\mu.\nu}$ does not depend on g. Hence, it is conformally invariant. Moreover, tensor $\mathcal{Q}_{\lambda\mu\nu}$ (see (2.1)) transforms under the conformal rescaling in the following way:

$$
\mathcal{Q}_{\lambda\mu\nu}(Q, g) = \Omega^{-3}\mathcal{Q}_{\lambda\mu\nu}(\Omega^3 Q, \Omega^2 g)\,,
$$

which implies

Proposition 2.2.1. *If $Q_{\mu\nu}$ is a CYK tensor for the metric $g_{\mu\nu}$, then $\Omega^3 Q_{\mu\nu}$ is a CYK tensor for the conformally rescaled metric $\Omega^2 g_{\mu\nu}$.*

Moreover, the (upper index) tensor $Q^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}Q_{\mu\nu}$ rescales by Ω^{-1}. In addition to this, let ϕ be a scalar function on n-dimensional manifold. If ϕ rescales conformally $\tilde{\phi} \to \Omega^k\phi$, where $k = \frac{2-n}{2}$, then the operator presented below transforms under conformal change of a metric ($\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$) in the following way:

$$
\left(\widetilde{\Box} - \frac{1}{4}\frac{n-2}{n-1}\widetilde{R}\right)\widetilde{\phi} = \Omega^{k-2}\left(\Box - \frac{1}{4}\frac{n-2}{n-1}R\right)\phi\,, \tag{2.27}
$$

where R is a curvature scalar.
The above facts lead to a proposition presented below.

Proposition 2.2.2. *The equation (2.26) remains unchanged under conformal transformation of the metric: $g \to \widetilde{g} = \Omega^2 g$.*

2.3 Electrodynamics on the Plebański–Demiański generalized black hole

2.3.1 Plebański–Demiański generalized black hole

Plebański–Demiański original metric

The complete family of type D spacetimes in four dimensions, including the black hole spacetimes like the Kerr metric, the metrics describing the accelerating sources as the

C-metric, or the nonexpanding Kundt's class type D solutions, can be represented by the general seven-parameter metric discovered by Plebański and Demiański [50]. Griffiths and Podolský [27] put this metric into a new form which enabled a better physical interpretation of parameters and simplified a procedure how to derive all special cases.

The original form of the Plebański–Demiański metric [50] is given by

$$g = \overline{\Omega}^2 \left[\frac{\overline{P}\left(\mathrm{d}\tau + r^2 \mathrm{d}\sigma\right)^2}{r^2+p^2} - \frac{\overline{Q}\left(\mathrm{d}\tau - p^2 \mathrm{d}\sigma\right)^2}{r^2+p^2} + \frac{r^2+p^2}{\overline{P}}\mathrm{d}p^2 + \frac{r^2+p^2}{\overline{Q}}\mathrm{d}r^2 \right] . \tag{2.28}$$

This metric fulfills the Einstein-Maxwell equations with the electric and magnetic charges e and g and the cosmological constant Λ when functions $\overline{P} = \overline{P}(p)$ and $\overline{Q} = \overline{Q}(r)$ take the particular form

$$\begin{aligned} \overline{Q} &= k + e^2 + \mathrm{g}^2 - 2mr + \epsilon r^2 - 2nr^3 - (k + \Lambda/3) r^4 \,, \\ \overline{P} &= k + 2np - \epsilon p^2 + 2mp^3 - \left(k + e^2 + \mathrm{g}^2 + \Lambda/3\right) p^4 . \end{aligned} \tag{2.29}$$

The conformal factor is

$$\overline{\Omega}^{-1} = 1 - pr \,, \tag{2.30}$$

and the vector potential reads

$$\overline{A}_\mu \mathrm{d}x^\mu = -\frac{1}{r^2+p^2}\left[er\left(\mathrm{d}\tau - p^2\mathrm{d}\sigma\right) + \mathrm{g}p\left(\mathrm{d}\tau + r^2 \mathrm{d}\sigma\right)\right] . \tag{2.31}$$

One can perform the transformations of coordinates and parameters to obtain the complete family of type D spacetimes. Following [27] we introduce two new continuous parameters α (the acceleration) and ω (the "twist") by the rescaling

$$p \to \sqrt{\alpha\omega}\, p, \quad r \to \sqrt{\frac{\alpha}{\omega}}\, r, \quad \sigma \to \sqrt{\frac{\omega}{\alpha^3}}\, \sigma, \quad \tau \to \sqrt{\frac{\omega}{\alpha}}\, \tau \,, \tag{2.32}$$

and relabel the other parameters as

$$\begin{aligned} m \to \left(\tfrac{\alpha}{\omega}\right)^{3/2} m, \quad n \to \left(\tfrac{\alpha}{\omega}\right)^{3/2} n, \quad e \to \tfrac{\alpha}{\omega} e, \\ \mathrm{g} \to \tfrac{\alpha}{\omega}\mathrm{g}, \qquad \epsilon \to \tfrac{\alpha}{\omega}\epsilon, \quad k \to \alpha^2 k \,. \end{aligned} \tag{2.33}$$

Then the metric (2.28) and the vector potential 2.31 take the form

$$\begin{aligned} g \;&=\; \widehat{\Omega}^2 \left[\frac{\widehat{P}\left(\omega\mathrm{d}\tau + r^2\mathrm{d}\sigma\right)^2}{r^2+\omega^2p^2} - \frac{\widehat{Q}\left(\mathrm{d}\tau - \omega p^2\mathrm{d}\sigma\right)^2}{r^2+\omega^2p^2} \right. \\ &\qquad \left. + \frac{r^2+\omega^2p^2}{\widehat{P}}\mathrm{d}p^2 + \frac{r^2+\omega^2p^2}{\widehat{Q}}\mathrm{d}r^2 \right] , \end{aligned} \tag{2.34}$$

$$\widehat{A}_\mu \mathrm{d}x^\mu = -\frac{1}{r^2+\omega^2p^2}\left[er\left(\mathrm{d}\tau - \omega p^2\mathrm{d}\sigma\right) + \mathrm{g}p\left(\omega\mathrm{d}\tau + r^2\mathrm{d}\sigma\right)\right] , \tag{2.35}$$

with

$$\widehat{\Omega}^{-1} = 1 - \alpha pr \,, \tag{2.36}$$

and

$$\widehat{Q} = \omega^2 k + e^2 + \mathrm{g}^2 - 2mr + \epsilon r^2 - \frac{2\alpha n}{\omega} r^3 - \left(\alpha^2 k + \frac{\Lambda}{3}\right) r^4 , \tag{2.37}$$

$$\widehat{P} = k + \frac{2n}{\omega} p - \epsilon p^2 + 2\alpha m p^3 - \left(\alpha^2 \left(\omega^2 k + e^2 + \mathrm{g}^2\right) + \omega^2 \frac{\Lambda}{3}\right) p^4 . \tag{2.38}$$

When $\Lambda = 0$, the line element (2.34) already contains the Kerr–Newman solution for a charged rotating black hole. It also contains the charged C-metric for accelerating black holes. However, it does not include the type D non-singular NUT solution [48]. To cover all these cases and their generalizations, it is necessary to introduce a specific shift in the coordinate p. In fact, this procedure is essential to obtain the correct metric for accelerating and rotating black holes. We therefore start with the metric (2.34) with (2.37)-(2.38), and perform the coordinate transformation

$$p = \frac{l}{\omega} + \frac{a}{\omega} \tilde{p} , \qquad \tau = t - \frac{(l+a)^2}{a} \varphi , \qquad \sigma = -\frac{\omega}{a} \varphi , \tag{2.39}$$

where a and l are new arbitrary parameters. By this procedure, we obtain the metric

$$\begin{aligned} g = \frac{1}{\Omega^2} \Bigg\{ & \frac{Q}{\rho^2} \left[\mathrm{d}t - \left(a(1-\tilde{p}^2) + 2l(1-\tilde{p})\right) \mathrm{d}\varphi\right]^2 - \frac{\rho^2}{Q} \mathrm{d}r^2 \\ & - \frac{\tilde{P}}{\rho^2} \Big[a \mathrm{d}t - \Big(r^2 + (l+a)^2\Big) \mathrm{d}\varphi\Big]^2 - \frac{\rho^2}{\tilde{P}} \mathrm{d}\tilde{p}^2 \Bigg\} , \end{aligned} \tag{2.40}$$

where

$$\begin{aligned} &\Omega = 1 - \frac{\alpha}{\omega}(l + a\tilde{p}) r , \\ &\rho^2 = r^2 + (l + a\tilde{p})^2 , \\ &\tilde{P} = a_0 + a_1 \tilde{p} + a_2 \tilde{p}^2 + a_3 \tilde{p}^3 + a_4 \tilde{p}^4 , \\ &Q = (\omega^2 k + e^2 + \mathrm{g}^2) - 2mr + \epsilon r^2 - 2\alpha \frac{n}{\omega} r^3 - \left(\alpha^2 k + \frac{\Lambda}{3}\right) r^4 , \end{aligned}$$

and we have put

$$\begin{aligned} a_0 &= \frac{1}{a^2} \left(\omega^2 k + 2nl - \epsilon l^2 + 2\alpha \frac{l^3}{\omega} m - \left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + \mathrm{g}^2) + \frac{\Lambda}{3}\right] l^4\right) , \\ a_1 &= \frac{2}{a} \left(n - \epsilon l + 3\alpha \frac{l^2}{\omega} m - 2\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + \mathrm{g}^2) + \frac{\Lambda}{3}\right] l^3\right) , \\ a_2 &= -\epsilon + 6\alpha \frac{l}{\omega} m - 6\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + \mathrm{g}^2) + \frac{\Lambda}{3}\right] l^2 , \\ a_3 &= 2\alpha \frac{a}{\omega} m - 4\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + \mathrm{g}^2) + \frac{\Lambda}{3}\right] al , \\ a_4 &= -\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + \mathrm{g}^2) + \frac{\Lambda}{3}\right] a^2 . \end{aligned}$$

These solutions generally have seven essential parameters m, n, e, g, α, ω and Λ. They also have two parameters k and ϵ which can be scaled to any convenient values.

In addition, we have the further parameters a and l which can be chosen arbitrarily. In practice, it is convenient to choose a and l to satisfy certain conditions which simplify the form of the metric, and then to re-express n and ω in terms of these parameters.

Let us highlight that the properties of the solutions in this family depend significantly on the character of the function $\tilde{P}(\tilde{p})$. In fact, as an arbitrary quartic, $\tilde{P}$ can have up to four distinct roots, and Lorentzian space-times only occur for ranges of $\tilde{p}$ for which $\tilde{P} > 0$. When more than one such range exists, the different possibilities correspond to distinct space-times which have different physical interpretations.

When $\tilde{P}$ has no roots, this function can only be positive and $\tilde{p} \in (-\infty, \infty)$.

For the cases in which $\tilde{P}$ has at least one root, without loss of generality we can choose the parameters a and l so that such a root occurs at $\tilde{p} = 1$. The metric (2.40) is then regular at $\tilde{p} = 1$ which corresponds to a coordinate pole on an axis, and it is then appropriate to take φ as a periodic coordinate.

When another distinct root of $\tilde{P}$ exists, it is always possible to exhaust the freedom in a and l to set the second root at $\tilde{p} = -1$. The metric component $a(1-\tilde{p}^2)$ is then regular at this second pole while the component $2l(1-\tilde{p})$ is not. Thus, the metric is regular at $\tilde{p} = 1$, but a singularity of some kind occurs at $\tilde{p} = -1$. (In fact, unless $l = 0$, the region near $\tilde{p} = -1$ contains closed timelike lines.)

Plebański–Demiański generalized black hole

Next, we will concentrate on the physically most relevant particular case of the line element (2.40) for which $\tilde{P}$ has at least two distinct roots and $a_0 > 0$, so that we can set $a_0 = 1$. In this case, the surfaces spanned by $\tilde{p}$ and φ have positive curvature. With this choice, and for the positive curvature case, both poles are located on a continuous axis. We have now introduced through (2.39) a shift and scaling of p such that, if $\tilde{P}$ has at least two roots, then it admits the two factors $(1-\tilde{p})$ and $(1+\tilde{p})$. Thus

$$\tilde{P} = (1-\tilde{p}^2)(1 - a_3\tilde{p} - a_4\tilde{p}^2)\,,$$

which implies that the above coefficients must satisfy the conditions

$$a_1 + a_3 = 0\,, \qquad a_0 + a_2 + a_4 = 0\,. \tag{2.41}$$

These conditions provide two linear equations which specify the two parameters ϵ and n in terms of a and l. Moreover, the condition $a_0 = 1$ enable one to give a condition for the parameter k. We have

$$\epsilon = \frac{\omega^2 k}{a^2 - l^2} + 4\alpha\frac{l}{\omega}\,m - (a^2+3l^2)\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + \mathrm{g}^2) + \frac{\Lambda}{3}\right], \tag{2.42}$$

$$n = \frac{\omega^2 k\, l}{a^2 - l^2} - \alpha\frac{(a^2-l^2)}{\omega}\,m + (a^2-l^2)l\left[\frac{\alpha^2}{\omega^2}(\omega^2 k + e^2 + \mathrm{g}^2) + \frac{\Lambda}{3}\right], \tag{2.43}$$

$$\left(\frac{\omega^2}{a^2-l^2} + 3\alpha^2 l^2\right) k = a_0 + 2\alpha\frac{l}{\omega}\,m - 3\alpha^2\frac{l^2}{\omega^2}(e^2+\mathrm{g}^2) - l^2\Lambda\,. \tag{2.44}$$

If we additionally assume that $\tilde{p}$ is taken to cover the range between the roots $\tilde{p} = \pm 1$ and it is natural to put $\tilde{p} = \cos\theta$, where $\theta \in [0, \pi]$. In this case, the metric (2.40) becomes

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{\Omega^2} \Bigg\{ & \frac{Q}{\Sigma} \left[\mathrm{d}t - \left(a \sin^2\theta + 4l \sin^2 \tfrac{\theta}{2}\right) \mathrm{d}\varphi\right]^2 - \frac{\Sigma}{Q}\, \mathrm{d}r^2 \\ & - \frac{\tilde{P}}{\Sigma} \Big[a \mathrm{d}t - \Big(r^2 + (a+l)^2\Big) \mathrm{d}\varphi\Big]^2 - \frac{\Sigma}{\tilde{P}} \sin^2\theta \, \mathrm{d}\theta^2 \Bigg\} \,, \end{aligned} \tag{2.45}$$

where

$$\begin{aligned} \Omega &= 1 - \frac{\alpha}{\omega}(l + a\cos\theta)\, r \,, \\ \Sigma &= r^2 + (l + a\cos\theta)^2 \,, \\ \tilde{P} &= \sin^2\theta \,(1 - a_3 \cos\theta - a_4 \cos^2\theta) \,, \\ Q &= (\omega^2 k + e^2 + \mathrm{g}^2) - 2mr + \epsilon r^2 - 2\alpha \frac{n}{\omega} r^3 - \left(\alpha^2 k + \frac{\Lambda}{3}\right) r^4 \,, \end{aligned} \tag{2.46}$$

and

$$\begin{aligned} a_3 &= 2\alpha \frac{a}{\omega} m - 4\alpha^2 \frac{al}{\omega^2} (\omega^2 k + e^2 + \mathrm{g}^2) - 4\frac{\Lambda}{3} al \,, \\ a_4 &= -\alpha^2 \frac{a^2}{\omega^2} (\omega^2 k + e^2 + \mathrm{g}^2) - \frac{\Lambda}{3} a^2 \,, \end{aligned} \tag{2.47}$$

with ϵ, n and k given by (2.42)–(2.44), setting $a_0 = 1$. It is also assumed that $|a_3|$ and $|a_4|$ are sufficiently small that $\tilde{P}$ has no additional roots with $\theta \in [0, \pi]$. This solution contains eight arbitrary parameters m, e, g, a, l, α, Λ and ω. Of these, the first seven can be varied independently, and ω can be set to any convenient value if a or l are not both zero. We present below a common physical interpretation of the parameters. Simultaneously, let us note there are some subtleties related with the freedom of choice of ω. The interpretation is blured when a few parameters are interfering. A common physical interpretation of the parameters reads

m – mass of the black hole,

a – Kerr rotation parameter,

Λ – cosmological constant,

l – NUT parameter,

e – electric charge of the black hole,

g – magnetic charge of the black hole,

α – uniform acceleration of the black hole.

It was shown in [26] that, when $\Lambda = 0$, the metric (2.45) represents an accelerating and rotating charged black hole with a generally non-zero NUT parameter. However, an arbitrary cosmological constant is now included so that the background is either Minkowski, de Sitter or anti-de Sitter space-time. The special cases are generally well-known and will not be discussed here. In the next parts of the thesis, we will restrict ourselves to the generalized black hole spacetime with the particular choice of the parameter

$$\omega = a \,. \tag{2.48}$$

2.3.2 CYK tensors for Plebański–Demiański spacetime

Known solutions

In four dimensions the integrability conditions for the existence of nondegenerate Killing-Yano tensor restricts the Petrov type of spacetime to type D (see [20]). Demiański and Francaviglia [18] demonstrated that from the known type D solutions only spacetimes without acceleration of sources actually admit this tensor. Further generalization and corrections done by Kubizňák and Krtouš [43] explicitly demonstrated that for the general Plebański–Demiański metric exist a pair of CYK two-forms.

In [43], Kubizňák and Krtouš claimed that the general Plebański-Demiański metric (2.28) admits two linearly independent CYK tensor solutions. The first one reads

$$\overline{K}_{\mu\nu} dx^{\mu} \wedge dx^{\nu} = \overline{\Omega}^{3} \left[p \mathrm{d}r \wedge \left(\mathrm{d}\tau - p^2 \mathrm{d}\sigma \right) + r \mathrm{d}p \wedge \left(\mathrm{d}\tau + r^2 \mathrm{d}\sigma \right) \right] . \tag{2.49}$$

One can check that it fulfills the CYK equation (2.3). The second solution can be easily obtained from the theorem 2.1.4

$$\overline{H} = *\overline{K} \, , \tag{2.50}$$

or explicitly

$$\overline{H} = \overline{\Omega}^{3} \left[r \mathrm{d}r \wedge \left(p^2 \mathrm{d}\sigma - \mathrm{d}\tau \right) + p \mathrm{d}p \wedge \left(r^2 \mathrm{d}\sigma + \mathrm{d}\tau \right) \right] . \tag{2.51}$$

Additionally, it turns out that

$$\overline{H} = \overline{\Omega}^{3} \mathrm{d}\overline{B} \, , \tag{2.52}$$

where

$$2\overline{B} = \left(p^2 - r^2 \right) \mathrm{d}\tau + p^2 r^2 \mathrm{d}\sigma \, . \tag{2.53}$$

From the perspective of the next sections, CYK solutions for the generalized black hole metric (2.45), with additional condition (2.48), are especially useful. The pair of solutions of CYK equation has a form

$$\begin{aligned} Y \;=\; & \frac{1}{\Omega^3} \Big\{ (l + a\cos\theta)\, \mathrm{d}r \wedge \left\{ \mathrm{d}t + \mathrm{d}\varphi \left[2l(\cos\theta - 1) - a \sin^2\theta \right] \right\} \\ & - r \sin\theta \, \mathrm{d}\theta \wedge \left\{ a \mathrm{d}t - \mathrm{d}\varphi \left[(l+a)^2 + r^2 \right] \right\} \Big\} \, , \end{aligned} \tag{2.54}$$

the theorem 2.1.4 assures the dual companion of Y is also CYK

$$\begin{aligned} *Y \;=\; & \frac{1}{\Omega^3} \Big\{ r \mathrm{d}r \wedge \left\{ \mathrm{d}\varphi \left[2l(1 - \cos\theta) + a \sin^2\theta \right] - \mathrm{d}t \right\} + \\ & \sin\theta (l + a\cos\theta) \mathrm{d}\theta \wedge \left\{ [r^2 + (l+a)^2] \mathrm{d}\varphi - a \mathrm{d}t \right\} \Big\} \, . \end{aligned} \tag{2.55}$$

Weyl endomorphism in the space of two-forms

Weyl curvature tensor has two pairs of antisymmetric indices. The algebraic structure of the Weyl tensor $C_{\mu\nu}{}^{\lambda\kappa}$ allows it to be treated as an endomorphism in the space of two-forms at each point $p \in M$:

$$C : \bigwedge^{2} \mathrm{T}_p^* M \to \bigwedge^{2} \mathrm{T}_p^* M \, .$$

In the six-dimensional space $\bigwedge^2 \mathrm{T}_p^*\mathrm{M}$ we can distinguish a two-dimensional subspace $\mathbb{V}$ which is spanned by Y and $*Y$. $\mathbb{V}$ turns out to be an invariant subspace of the endomorphism C. More precisely,

$$C_{\mu\nu}{}^{\lambda\kappa}\left(Y_{\lambda\kappa} + \imath * Y_{\lambda\kappa}\right) = 2V\left(Y_{\mu\nu} + \imath *Y_{\mu\nu}\right)\,, \tag{2.56}$$

where $C_{\mu\nu}{}^{\lambda\kappa}$ is the Weyl tensor for the generalized black hole metric (2.45), with additional condition (2.48). The eigenfunction is

$$\begin{aligned} V \;&= V_{el}(e,\mathrm{g}) + \frac{2\Omega^3 a^2}{(a^2+3\alpha^2a^2l^2-3\alpha^2l^4)[r+\imath(l+a\cos\theta)]^3}\left\{\frac{4}{3}\imath l\Lambda(a^2-4l^2)\right. \\ &+ \left.\left(1+\frac{\imath\alpha l^2}{a}-\imath\alpha a\right)\left[m\left(1+\frac{\imath\alpha l^2}{a}+\imath\alpha a\right)\left(1+\frac{\imath\alpha l^2}{a}-\imath\alpha l\right)+\imath l\left(1+\frac{\imath\alpha l^2}{a}-\imath\alpha a\right)\right]\right\}\,, \end{aligned} \tag{2.57}$$

$V_{el}(e,g)$ has the form $(e^2+g^2)\cdot G$. G is a complicated rational function of the Plebański–Demiański parameters $\{m, a, \Lambda, l, \alpha\}$.

2.3.3 Generalized wave equation for Plebański–Demiański black hole

Now we return to the equation (2.26) and rewrite it for $Y_{\mu\nu}$ and its dual $*Y_{\mu\nu}$ multiplied by $\imath$:

$$\left\{\begin{array}{rcl} \left(\Box - \frac{1}{6}R\right)\left(F_{\mu\nu}Y^{\mu\nu}\right) + \frac{1}{2}F^{\sigma\lambda}C_{\sigma\lambda\mu\nu}Y^{\mu\nu} &=& 0\,, \\ \left(\Box - \frac{1}{6}R\right)\left(\imath F_{\mu\nu}(*Y^{\mu\nu})\right) + \frac{\imath}{2}F^{\sigma\lambda}C_{\sigma\lambda\mu\nu}(*Y^{\mu\nu}) &=& 0\,. \end{array}\right. \tag{2.58}$$

Adding both sides and using (2.56), we obtain:

$$\Box\left[F_{\mu\nu}\left(Y^{\mu\nu}+\imath(*Y^{\mu\nu})\right)\right] + VF^{\mu\nu}\left(Y_{\mu\nu}+\imath(*Y_{\mu\nu})\right) = 0\,. \tag{2.59}$$

Introducing $\Phi = \frac{\imath}{2}F^{\mu\nu}\left[Y_{\mu\nu}+\imath(*Y_{\mu\nu})\right]$, we obtain a scalar electromagnetic wave equation:

$$\Box\Phi + \left(V - \frac{1}{6}R\right)\Phi = 0\,. \tag{2.60}$$

The above calculations prove the following

Theorem 2.3.1. *Dynamics of a Maxwell field in the Plebański–Demiański spacetime can be reduced to the scalar wave equation:*

$$\Box\Phi + \left(V - \frac{1}{6}R\right)\Phi = 0\,, \tag{2.61}$$

where $\Phi = \frac{\imath}{2}F^{\mu\nu}\left[Y_{\mu\nu}+\imath(*Y_{\mu\nu})\right]$*, R is a 4-dimensional curvature scalar and V is given by (2.57).*

The equation (2.61) has a complex potential, so it can not be treated as a two independent real equations. The equation (2.61) does not obviously arise as the Euler-Lagrange equation for any real-valued Lagrangian, which means that we cannot use Noether's theorem to construct conserved quantities from a symmetries for the solutions of (2.61).

If we restrict ourselves to the Kerr spacetime, the equation (2.61) reduces to the one which was discovered for the first time by Fackerell and Ipser [22] in 1972. For this reason we call it Fackerell–Ipser (F–I) equation. With the help of F–I equation, Andersson and Blue [3] have proven the boundedness of a positive definite energy of electromagnetic field on each hypersurface of constant time for a slowly rotated Kerr black hole.

2.3.4 Relations with a null tetrad

We introduce a null tetrad below as a convenient tool for some aspects of analysis of algebraically special solutions of electromagnetism on Plebański–Demiański background. However, the full description of electromagnetisms in Newman–Penrose formalism is given in the section 4.1.3. The tetrad is build with two real and two complex vector fields: the first two vector fields, $L^\mu\partial_\mu$ and $N^\mu\partial_\mu$ are a pair of real null vectors. The complex vector fields in tetrad are given by a complex vector field $M^\mu\partial_\mu$ and its complex conjugate $\bar{M}^\mu\partial_\mu$. We use the following conventions

$$L^\mu N_\mu = -1\,, \quad \bar{M}^\mu M_\mu = 1\,. \tag{2.62}$$

with the all other scalar products being zero.

For Plebański–Demiański generalized black hole, we use symmetric tetrad for the metric (2.45) with the condition (2.48). It has the form

$$\begin{aligned}
L^\mu\partial_\mu &= \frac{\Omega}{\sqrt{2\Sigma}}\left\{\frac{1}{\sqrt{Q}}\left[r^2+(a+l)^2\right]\partial_t+\sqrt{Q}\partial_r+\frac{a}{\sqrt{Q}}\partial_\varphi\right\}\,,\\
N^\mu\partial_\mu &= \frac{\Omega}{\sqrt{2\Sigma}}\left\{\frac{1}{\sqrt{Q}}\left[r^2+(a+l)^2\right]\partial_t-\sqrt{Q}\partial_r+\frac{a}{\sqrt{Q}}\partial_\varphi\right\}\,,\\
M^\mu\partial_\mu &= \frac{\Omega}{\sqrt{2\Sigma}}\left\{\frac{\imath\sin^2\theta}{\sqrt{\widetilde{P}}}\left(a+\frac{2l}{1+\cos\theta}\right)\partial_t+\frac{\sqrt{\widetilde{P}}}{\sin\theta}\partial_\theta+\frac{\imath}{\sqrt{\widetilde{P}}}\partial_\varphi\right\}\,,
\end{aligned} \tag{2.63}$$

and the corresponding cotetrad

$$\begin{aligned}
L_\mu\mathrm{d}x^\mu &= \frac{1}{\Omega\sqrt{2\Sigma}}\left\{\sqrt{Q}\mathrm{d}t-\frac{\Sigma}{\sqrt{Q}}\mathrm{d}r-\sqrt{Q}\left[2l(1-\cos\theta)+a\sin^2\theta\right]\mathrm{d}\varphi\right\}\,,\\
N_\mu\mathrm{d}x^\mu &= \frac{1}{\Omega\sqrt{2\Sigma}}\left\{\sqrt{Q}\mathrm{d}t+\frac{\Sigma}{\sqrt{Q}}\mathrm{d}r-\sqrt{Q}\left[2l(1-\cos\theta)+a\sin^2\theta\right]\mathrm{d}\varphi\right\}\,,\\
M_\mu\mathrm{d}x^\mu &= \frac{1}{\Omega\sqrt{2\Sigma}}\left\{\imath a\sqrt{\widetilde{P}}\mathrm{d}t-\frac{\Sigma\sin\theta}{\sqrt{\widetilde{P}}}\mathrm{d}\theta-\imath\left[r^2+(a+l)^2\right]\sqrt{\widetilde{P}}\mathrm{d}\varphi\right\}\,.
\end{aligned} \tag{2.64}$$

The above tetrad in the Kerr limit become symmetric Carter tetrad.
The following relation between null cotetrad (2.64) and CYK two-forms (2.54)-(2.55) holds

$$\imath\,(Y-\imath * Y) = 2\frac{r-\imath(l+a\cos\theta)}{\Omega}\left[N\wedge L+\bar{M}\wedge M\right]\,. \tag{2.65}$$

2.3.5 Special solutions of Maxwell equations

In this section we present a singular family of complex Maxwell fields on Kerr background. The solution Φ of generalized F–I equation (2.60) constructed from any representant of this family is equal to zero.

All the information about an electromagnetic field can be encoded in a single, complex two-form

$$\mathcal{F} = F + \imath * F\,. \tag{2.66}$$

Recall that the Hodge star operator Eq. (2.18) for Kerr metric satisfies $*^2 = -\mathrm{id}$. A two-form p is anti-self dual in Hodge sense if

$$*p = -\imath p\,. \tag{2.67}$$

Note that $\mathcal{F}$ is anti-self dual. Maxwell equations in terms of $\mathcal{F}$ take a simple form

$$\mathrm{d}\mathcal{F} = 0\,. \tag{2.68}$$

We will call a two-form q algebraically special if it fulfils

$$q \wedge q = 0\,. \tag{2.69}$$

Robinson and Trautman have constructed a singular, anti-self dual and algebraically special Maxwell field for optical geometry metrics (see [55]). We have constructed such Maxwell field on Plebański–Demiański generalized black hole background. We will denoted it by $\mathcal{F}$. $\mathcal{F}$ is built from two principal null covectors (2.64) and it takes the following form:

$$\mathcal{F} = \frac{f(\xi,\eta)}{\sqrt{Q}\sin\theta} N \wedge \bar{M}\,, \tag{2.70}$$

where

$$\mathrm{d}\xi = \mathrm{d}t + \frac{a^2 + 2al + l^2 + r^2}{Q}\,\mathrm{d}r + \frac{\imath \sin\theta(1-\cos\theta)(a\cos\theta + a + 2l)}{\widetilde{P}}\,\mathrm{d}\theta\,, \tag{2.71}$$

and

$$\mathrm{d}\eta = \mathrm{d}\varphi + \frac{a}{Q}\mathrm{d}r + \frac{\imath\sin\theta}{\widetilde{P}}\mathrm{d}\theta\,. \tag{2.72}$$

The functions $Q = Q(r)$ and $\widetilde{P} = \widetilde{P}(\theta)$ are defined by the equations (2.46). We have also found another family of solutions which satisfies conditions (2.67)–(2.69). We denote it by $\mathcal{H}$

$$\mathcal{H} = \frac{h(\kappa,\sigma)}{\sqrt{Q}\sin\theta} L \wedge M\,, \tag{2.73}$$

where

$$\mathrm{d}\kappa = \mathrm{d}t - \frac{a^2 + 2al + l^2 + r^2}{Q}\mathrm{d}r - \frac{\imath \sin\theta(1-\cos\theta)(a\cos\theta + a + 2l)}{\widetilde{P}}\mathrm{d}\theta\,, \tag{2.74}$$

and

$$\mathrm{d}\sigma = \mathrm{d}\varphi - \frac{a}{Q}\mathrm{d}r - \frac{\imath\sin\theta}{\widetilde{P}}\mathrm{d}\theta\,. \tag{2.75}$$

Chapter 3

Hopfions

3.1 Introduction and historical review

In this section we restrict our considerations to electrodynamics and linearized gravitation separately on the Minkowski background. We analyze fields with non-trivial topological structure. The concept of field lines whose tangents are the electric or magnetic field is typically used to visualize static solutions of Maxwell's equations. Propagating solutions often have simple field-line structures and so are not usually described in terms of field lines. In the present chapter, we study a propagating fields whose defining and most striking property is the topological structure of its electric and magnetic field lines. Part of the analysis can be further generalized for weak gravitational field.
Hopfion or Hopf soliton is a 'solitonary' solution of spin-N field which has rich topological structure related to Hopf fibration. The Hopf fibration is the simplest non-trivial fibration of three-dimensional sphere. We will study electromagnetic and gravitational solutions based on the Hopf projection which is a surjective map sending great circles on $\mathbb{S}^3$ to points on $\mathbb{S}^2$. These circles weave nested toroidal surfaces and each is linked with every other circle exactly once, creating the characteristic Hopf fibration. The characteristic structure of Hopfion can be easily seen on the integration curves of the vector field (see [30]). The structure of closed, linked field lines of Hopfions propagates without intersections along the light cone. We begin discussion about Hopfion with brief survey about Hopf fibration.

3.1.1 Hopf fibration

The Hopf fibration (also known as the Hopf bundle or Hopf map) is non-trivial[1] fibration of three-dimensional sphere $\mathbb{S}^3$ over two-dimensional sphere $\mathbb{S}^2$. The Hopf bundle has been discovered by Heinz Hopf [28] in 1931, it is an influential early example of a fiber bundle. In this case, $\mathbb{S}^3$ is composed of fibers, where each fiber is a circle. The projection map (many-to-one continuous function) of Hopf bundle we denote by h. The structure of Hopf fibration of $\mathbb{S}^3$ is given on the figure 3.1. The inverse image of any point in $\mathbb{S}^2$ under h is a circle in $\mathbb{S}^3$. Each such circle is linked with any other fiber exactly once. There are no crossings between fibers. The pre-image of a 'circle of latitude' forms a tori from linked circles. Each 'circle of latitude' has a corresponding tori in $\mathbb{S}^3$. The toruses are nested each other — the last picture in the left column contains fibres from toruses of three various sizes. As the torus grows in size, the hole in its center becomes smaller and smaller. There are two 'special' toruses: the circle of unit radius that corresponds to the infinitely thin torus, and the straight line, or circle of infinite radius, that corresponds to an infinitely large torus. That infinitely large torus is a so called horn torus — torus without hole in the center.

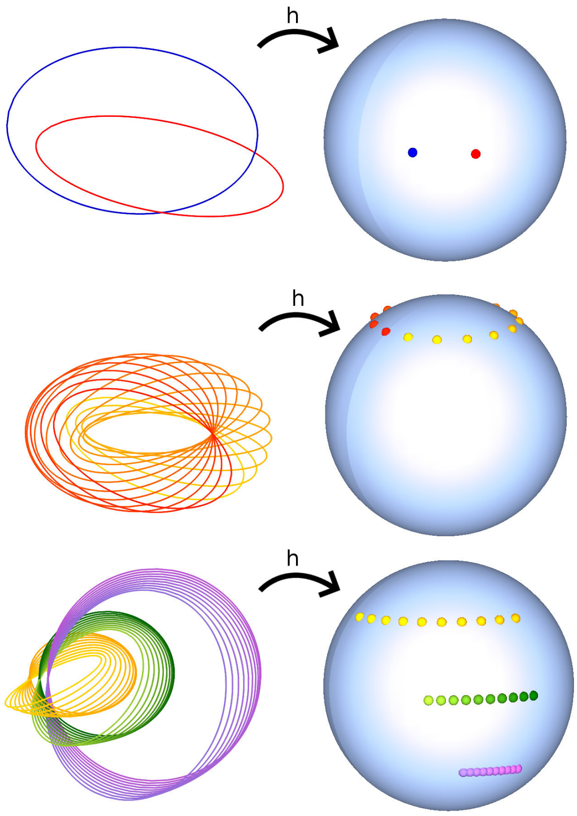

Figure 3.1: The structure of Hopf fibration of $\mathbb{S}^3$. A torus can be constructed out of circles (fibres) in such a way that no two circles cross and each circle is linked to every other one. In the left picture in the second row, each circle in such a configuration wraps once around each circumference of the torus. The last picture in the left column contains parts of nesting such tori into one another, the whole of three dimensional space, including the point at $R = \infty$ $\left(\mathbb{R}^3 \cup \infty \sim \mathbb{S}^3\right)$ can be filled with linked circles. The right column contains corresponding images of fibres from left column under Hopf projection.

We wish to define precisely the Hopf projection h as a map

$$h : \mathbb{S}^3 \sim \mathbb{R}^3 \cup \{\infty\} \to \mathbb{S}^2 \sim \mathbb{C} \cup \{\infty\}\,. \tag{3.1}$$

Let us introduce locally on $\mathbb{S}^3$ a Cartesian coordinate system (x, y, z). With the help of stereographic projection, two dimensional sphere can be parametrized (without one point) by a complex variable. The Hopf projection can in turn be expressed explicitly as a complex function in $\mathbb{R}^3$ whose lines of constant amplitude and phase are circles, and surfaces of constant amplitude are nested

[1]By non-trivial fibration we mean a fibration which is not globally a Cartesian product of two topological spaces. Classical example of trivial fibration is a foliation of $\mathbb{S}^3$ by a one-parameter family of two-dimensional spheres.

tori

$$h(x,y,z) = \frac{x + \imath z}{-y + \imath\left(\tilde{A} - 1\right)}\,, \tag{3.2}$$

where $\imath^2 = -1$ and $\tilde{A} = \frac{1}{2}(x^2 + y^2 + z^2 + 1)$.
There are numerous generalizations of the Hopf fibration. The unit sphere in complex coordinate space $\mathbb{C}^{n+1}$ fibers naturally over the complex projective space $\mathbb{CP}^n$ with circles as fibers, and there are also real, quaternionic, and octonionic versions of these fibrations. In particular, the Hopf fibration belongs to a family of four fiber bundles in which the total space, base space, and fiber space are all spheres. However, we do not discuss such generalizations in the thesis.

3.1.2 Classical electromagnetic Hopfion

Let us consider a classical electromagnetic hopfions obtained by Rañada [51] in 1989. The construction was cast in terms of differential forms, which provide a natural way to map fields between spaces of differing dimensions. To highlight a direct relation between electromagnetic fields and Hopf projection we use Euler potentials (see appendix C.1). In terms of Euler potentials the resulting electric and magnetic fields have simple expressions:

$$\mathbf{B} = \frac{\kappa}{2\pi i}\frac{\nabla\eta\times\nabla\bar{\eta}}{(1+\bar{\eta}\eta)^2}\,,\quad \mathbf{E} = \frac{\kappa}{2\pi i}\frac{\nabla\zeta\times\nabla\bar{\zeta}}{(1+\bar{\zeta}\zeta)^2}\,, \tag{3.3}$$

with

$$\zeta(x,y,z,t) = \frac{(Ax+ty)+i(Az+t(A-1))}{(tx-Ay)+i(A(A-1)-tz)}\,, \tag{3.4}$$

$$\eta(x,y,z,t) = \frac{(Az+t(A-1))+i(tx-Ay)}{(Ax+ty)+i(A(A-1)-tz)}\,, \tag{3.5}$$

where $A = \frac{1}{2}\left(x^2+y^2+z^2-t^2+1\right)$. κ is a constant introduced so that the magnetic and electric fields have correct dimensions. The electromagnetic fields fulfills vacuum Maxwell equations (B.1)-(B.5). Since both $\nabla\eta$ and $\nabla\bar{\eta}$ are perpendicular to lines of constant η, the magnetic field is tangential to lines of constant η. A similar argument holds for the electric field and ζ. We have a direct relation between Euler potentials (3.4)-(3.5) at $t=0$ and Hopf projection (3.2)

$$\zeta(x,y,z,0) = h(x,y,z)\,, \tag{3.6}$$

$$\eta(x,y,z,0) = h(z,x,-y)\,. \tag{3.7}$$

The field lines evolves in time without intersections. In the figures 3.2 and 3.3 we present integral curves of magnetic field (3.3) for $t=0$ and for $t=1$ respectively. To investigate topological properties of electromagnetic field lines we need to define appropriate charges, like helicity, which are introduced in the next section.

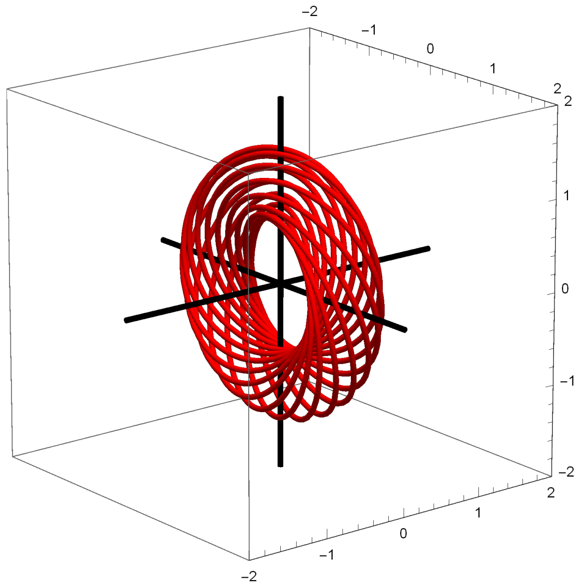

Figure 3.2: Integral curves of magnetic field at $t = 0$.

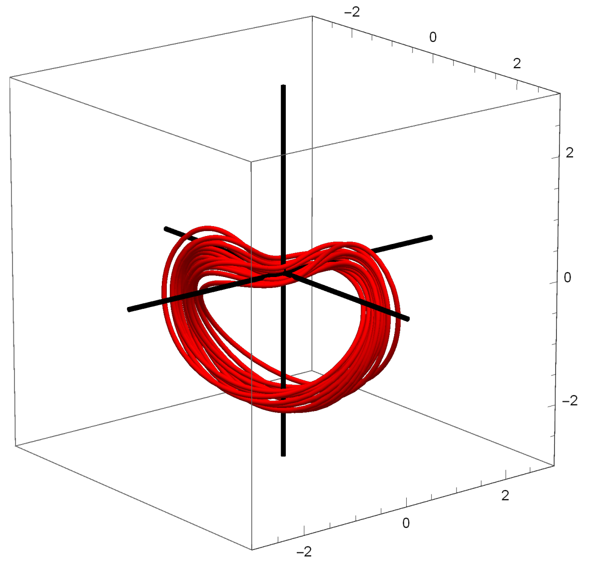

Figure 3.3: Integral curves of magnetic field at $t = 1$.

3.1.3 Topological charges

An intriguing configuration for field lines is to be linked and/or knotted. One of topological charges – helicity – is strictly related with the number of linkedness and knotness of closed integral curves of electromagnetic fields. Intuitively, two curves are linked if one curve winds around the other at least once. A curve is knotted if it can not be transformed by continuous deformation (without cutting) into a circle. Mathematically, such two curves belongs to a different homotopy classes. In other words, knottedness can be treated as a self-linkedness of a curve. An analytic description of linkedness and knotness arises from Gauss linking integral.

Gauss linking integral

Let γ_1, γ_2 be two smooth, disjoint, closed, oriented curves in $\mathbb{S}^3$, and $\mathbf{r}_1(t_1), \mathbf{r}_2(t_2)$ their parametrizations, with $\{t_1, t_2\} \in [0, 2\pi]$. To each pair $(Q_1, Q_2) \in \gamma_1 \times \gamma_2$ there corresponds a point (t_1, t_2) on the torus $\mathbb{T}$. The Gauss map $\psi : \mathbb{T} \to \mathbb{S}^2 \subset \mathbb{S}^3$, associates to each point (t_1, t_2) the unit vector

$$\mathbf{n}(t_1, t_2) = \frac{\mathbf{r}_1(t_1) - \mathbf{r}_2(t_2)}{|\mathbf{r}_1(t_1) - \mathbf{r}_2(t_2)|}\,. \tag{3.8}$$

The original formula of Gauss gives the following definition of linking integral:

Definition 3.1.1. The linking number $L(\gamma_1, \gamma_2)$ of γ_1 and γ_2 is defined by

$$L(\gamma_1, \gamma_2) := \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \det\left(\mathbf{n}, \frac{\partial \mathbf{n}}{\partial t_1}, \frac{\partial \mathbf{n}}{\partial t_2}\right) dt_1 dt_2\,. \tag{3.9}$$

For a single field line $\mathbf{c}(\tau)$ the self-linking number, $L(\mathbf{c},\mathbf{c})$, is a measure of knottedness. The linking integral L can also be computed visually by projecting the field lines onto a plane and subsequently counting the crossings in an oriented way. Any two unlinked curves have linking number zero. However, two curves with linking number zero may still be linked (e.g. the Whitehead link). The linking number depends on orientations of curves – reversing the orientation of either of the curves negates the linking number, while reversing the orientation of both curves leaves it unchanged.
Gauss linking integral is a basis for one of the most important quantity in algebraic topology — Hopf index. In the next sections we wish to discuss Hopf index and its physical analog.

Hopf index

Consider a map $f : \mathbb{S}^3 \to \mathbb{S}^2$. According to the Hopf theorem [28], if the map f is smooth, the inverse image of any two distinct points ζ_1 and ζ_2 of $\mathbb{S}^2, f^{-1}(\zeta_1)$ and $f^{-1}(\zeta_2)$, are two disjoint closed curves in $\mathbb{S}^3$ (or $R^3 \cup \{\infty\}$). The linking number (3.9) of the curves $f^{-1}(\zeta_1)$ and $f^{-1}(\zeta_2)$ does not depend on the particular pair of points, since by moving continuously from (ζ_1, ζ_2) to (ζ_1', ζ_2') the inverse images can neither untie nor tie any further to one another[2]. That means the linking number (3.9) specifies a topological invariant for any smooth mapping $f : \mathbb{S}^3 \to \mathbb{S}^2$

$$H(f) = L\left(f^{-1}(\zeta_1), f^{-1}(\zeta_2)\right), \forall \zeta_1 \neq \zeta_2 \in \mathbb{S}^2, \tag{3.10}$$

which is called the Hopf index. The Hopf index remains unchanged for a continuous evolution (in any continuous parameter, e. g. time) of the map f. Thus the Hopf index enables one to introduce equivalence relation between maps. It means that the class of all smooth maps $\mathbb{S}^3 \to \mathbb{S}^2$ can be classified in homotopy classes, each one labeled by an integer number $H(f)$, called the Hopf index.

An equivalent definition of the Hopf index is the following. Let us take the inverse images of two distinct points ζ_1 and ζ_2 of $\mathbb{S}^2$. According to Hopf theorem [28], they are two disjoint closed curves, their linking number (the Hopf index) is equal to the number of times that one of them, say $f^{-1}(\zeta_2)$, cuts a surface S_1 spanned by the closed curve $f^{-1}(\zeta_1)$. This allowed Whitehead [68] to write the Hopf index as an integral in $\mathbb{S}^3$. We discuss it briefly below.
Let us denote the area two-form in $\mathbb{S}^2$ by σ, so that it is normalized to unity,

$$\int_{\mathbb{S}^2} \sigma = 1. \tag{3.11}$$

Let us consider the pull-back of the area two-form by the map $f : \mathbb{S}^3 \to \mathbb{S}^2$, denoted by $f^*\sigma$. The pull-back of the area two-form in $\mathbb{S}^2$ is closed, since

$$d(f^*\sigma) = f^*(d\sigma) = 0, \tag{3.12}$$

[2]Since, for this to happen, they should have at a certain moment a common point with two different images by f.

area two-form is a form of maximal degree, so $d\sigma = 0$. The cohomological properties of $\mathbb{S}^3$ imply then that $f^*\sigma$ is also an exact form (Poincaré lemma), what means that there is a one-form g in $\mathbb{S}^3$ such that

$$f^*\sigma = dg\,. \tag{3.13}$$

Whitehead showed [68] that the Hopf index $H(f)$ can be written as the integral

$$H(f) = \int_{\mathbb{S}^3} g \wedge f^*\sigma\,. \tag{3.14}$$

In terms of vector fields the Hopf index can be defined with the help of Whitehead vector and its vector potential. Whitehead vector is defined as follows

$$\mathbf{B}_W = \frac{1}{2}\varepsilon^{ijk}\big(f^*\sigma\big)_{jk}\partial_i\,. \tag{3.15}$$

$f^*\sigma$ is an exact form means that there exist a vectorial potential for Whitehead vector

$$\mathbf{B}_W = \nabla \times \mathbf{A}_W\,, \tag{3.16}$$

where A_W is a vector field in R^3 related to the one-form g in (3.13). The Hopf index as the Whitehead integral (3.14) can be reformulated as

$$H(f) = \int_{\mathbb{S}^3} d^3r\mathbf{A}_W \cdot \mathbf{B}_W\,. \tag{3.17}$$

The vector potential $\mathbf{A}_W$ is not defined uniquely, nevertheless the above integral is well-defined. The above formula for Hopf index is very similar to magnetic helicity (3.21) — physical analogue of topological charge.
Let us discuss (3.17) for a particular choice of coordinates. $\mathbb{S}^2$ via stereograpic projection can be parametrized by a complex variable z. The area two-form is given by

$$\sigma = \frac{1}{2\pi i}\frac{dz \wedge d\bar{z}}{(1+\bar{z}z)^2}\,. \tag{3.18}$$

The pull-back $f^*\sigma$ reads

$$f^*\sigma = \frac{1}{2\pi i}\frac{df \wedge d\bar{f}}{(1+\bar{f}f)^2}\,. \tag{3.19}$$

The Whitehead integral (3.17) can be finally represent as

$$H(f) = \frac{1}{4\pi i}\int_{\mathbb{S}^3} \mathrm{d}^3r\,\frac{g_i\varepsilon^{ijk}\partial_j f\partial_k\bar{f}}{(1+\bar{f}f)^2}\,, \tag{3.20}$$

where $g = g_i dx^i$ is one-form defined by (3.13).

Electromagnetic helicities

In the case of magnetic or electric fields, averaging the linking integral over all field-line pairs together with the self-linking number over all field lines gives rise to the magnetic and electric helicities:

$$h_{\mathrm{m}} = \int \mathrm{d}^3 r \mathbf{A} \cdot \mathbf{B} \,, \tag{3.21}$$

$$h_{\mathrm{e}} = \int \mathrm{d}^3 r \, \mathbf{C} \cdot \mathbf{E} \,, \tag{3.22}$$

where $\mathbf{B}$ and $\mathbf{E}$ are magnetic and electric fields respectively. Moreover, A is a vector potential for magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. Analogically, iff the electric field is sourceless we can introduce vector potential for electric field $\mathbf{E} := \nabla \times \mathbf{C}$.
We wish to discuss relations between Hopf index and electromagnetic helicities. We begin by considering two complex scalar fields: ζ given by (3.4) and η defined by (3.5). Both complex scalar functions can be interpreted, at each moment of time, as maps $\mathbb{S}^3 \to \mathbb{S}^2$. $\zeta(x, y, z, t)$ is a smooth function, so according to Hopf theorem (section 3.1.3), the linking number of two inverse images $\zeta^{-1}(\zeta_0)$ and $\zeta^{-1}(\zeta_1)$ is the same for all the pairs of points $\zeta_0, \zeta_1 \in \mathbb{S}^2$. This linking number is equal to the Hopf index $H(\zeta)$ of the map ζ. $H(\zeta)$ is equal to $H(\zeta)$ at $t = 0$

$$H(\zeta) = H(\zeta_{t=0}) = H(h) = 1 \,. \tag{3.23}$$

where we have used (3.2), (3.6) and (3.20). The result agrees with the number of linkedness for Hopf fibration (see figure 3.1). Let us now analyze corresponding electric helicity h_e for the electric field (3.3). Using (3.22) and (3.3), we have

$$\begin{aligned} h_{\mathrm{e}} &= \int_{\mathbb{S}^3} \mathrm{d}^3 r \, \mathbf{C} \cdot \mathbf{E} \\ &= \frac{\kappa}{4\pi i} \int_{\mathbb{S}^3} \mathrm{d}^3 r \frac{C_i \varepsilon^{ijk} \partial_j \zeta \partial_k \bar{\zeta}}{(1 + \bar{\zeta}\zeta)^2} \,. \end{aligned} \tag{3.24}$$

We can evaluate electric helicity at $t = 0$, so (3.6) hold. The vector potential $\mathbf{C}$ at $t = 0$,called $\overset{0}{\mathbf{C}}$ can be chosen so that

$$\overset{0}{\mathbf{C}} = \kappa G^{-1}(g_h) \,, \tag{3.25}$$

where $G^{-1} : T^*\mathbb{S}^3 \to T\mathbb{S}^3$ is the inverse metric isomorphism on $\mathbb{S}^3$ and g_h is the potential form defined by (3.13) for Hopf projection ($f := h$). Comparing (3.3), (3.6) and $\mathbf{E} = \nabla \times \mathbf{C}$ with (3.13) and (3.19) for Hopf projection ($f := h$) one can check that (3.25) is justified. Continuing (3.24), electric helicity for classical Hopfion at $t = 0$ reads

$$\begin{aligned} h_{\mathrm{e}} &= \frac{\kappa}{4\pi i} \int_{\mathbb{S}^3} \mathrm{d}^3 r \frac{\overset{0}{C_i} \, \varepsilon^{ijk} \partial_j h \partial_k \bar{h}}{(1 + \bar{h}h)^2} \\ &= \frac{\kappa^2}{4\pi i} \int_{\mathbb{S}^3} \mathrm{d}^3 r \frac{[G^{-1}(g_h)]_i \varepsilon^{ijk} \partial_j h \partial_k \bar{h}}{(1 + \bar{h}h)^2} \end{aligned}$$

$$= \kappa^2 H(h)$$
$$= \kappa^2 . \tag{3.26}$$

The above relation clearly shows correspondence between electric helicity and Hopf index. Similar considerations can be done for magnetic helicity.

3.2 Reduced data for electromagnetism and linerized gravity

In [32] Professor Jezierski proposed a kind of reduced data for weak gravitational field. Such reduced data which are represented as complex scalar field Ψ enables one to obtain quasi-locally a full gravito-electric (magnetic) tensor for linearized gravity. Similar construction can be obtained also for electromagnetic field. We would like to highlight two advantages of the presented approach:

- Considerations (recovery procedure of full gravito-electromagnetic field, investigation of integral quantities, gauge-invariance, etc.) in our framework simplify significantly when the reduced data Ψ for linearized gravity have one multipole structure. That happens in the case of hopfions. It holds also in electromagnetic case. See the comments nearby the equations (3.33) and (3.40) for electromagnetic case. Analogically, (3.66) for linearized gravity.
- Presented approach is consistent with non-local nature of gravitational field. Non local physical quantities, like energy (see section 3.4.2) or topological charge (see section 3.4.5), can be easily represented in terms of our reduced data and its derivatives.

Our framework can be easily generalized to curved spacetimes which possess spherical symmetry. According to chapter 2, Conformal Yano–Kiling tensors enables one togeneralize the approach for type D spacetimes.
The section has two main parts: the first is related to electromagnetic generalization of hopfion, the second one presents the linearized gravity case.

In the first part, we briefly present a description of electromagnetic hopfion-like solution with the help of a complex scalar field[3] Φ. Its particular application to hopfions drastically simplifies the description and enables one to generalize this notion easily. We demonstrate a simple parametrization of such class of generalized hopfions by scalar wave function Φ.
The constructed scalar represents true degrees of freedom of the field which carries a gauge independent information of the field. The description of E-M field in terms of Φ is presented in the appendix B. The reconstruction of electromagnetic field from such function is presented. Next, we show the condition for conservation of topological charge – electric (magnetic) helicity in time in terms of Φ.

[3] The E-M field can be equivalently represented by the complex scalar field $\Phi = (\mathbf{E} + \imath\mathbf{B}) \cdot \mathbf{r}$, where $\mathbf{E}$ – electric vector field, $\mathbf{B}$ – magnetic vector field, $\mathbf{r}$ – position vector field and $\imath^2 = -1$. Cf. equation (3.27) and the appendix B.

In the second part, the description of linearized gravity hopfions in terms of the complex scalar field Ψ is presented. This approach is also an original idea introduced by one of us. Ψ plays analogical role to Φ in electromagnetism. The constructed scalar represents true degrees of freedom of the weak gravitational field which carries a gauge independent information of the field. Gravitational hopfions in terms of Ψ have a simple form and they can be easily generalized to a class of solutions (3.66). The reconstruction of a gravitational hopfion from such complex, scalar function is performed. We propose a new definition of a topological charge for spin-2 field in analogy to the electromagnetic case. Hamiltonian energy for linearized gravity is discussed. To indicate the difference between the super-energy of spin-2 field and the Hamiltonian energy which is a physical energy of gravitational field we present a few quasi-local (super-)energy densities in terms of the complex scalar Ψ. We compare such quasi-local (super-)energy densities for gravitational hopfion.

To clarify the exposition, a full explanation of a complex, scalar framework for electromagnetism/linearized gravity has been placed in the appendix.

3.3 Generalized hopfions in electrodynamics

3.3.1 Class of generalized hopfions

Consider a class of complex functions on the Minkowski background which are harmonic:

$$\Box\Phi = 0\,, \tag{3.27}$$

where $\Box$ is the d'Alembert operator in Minkowski spacetime.
There exists a bijection between electromagnetic solutions and such complex scalar fields. For a given Riemann-Silberstein vector $\mathbf{Z} := \mathbf{E} + \imath\mathbf{B}$, complex combination of electric vector field $\mathbf{E}$ and magnetic vector field $\mathbf{B}$, we simply define $\Phi := \mathbf{Z}\cdot\mathbf{r}$ i.e. Φ is the scalar product of Riemann–Silberstein vector and position vector (c.f. equation (B.7) and comments nearby). Let us note that (3.27) covers with (2.61) in the Minkowski spacetime limit. Hodge duality enables one to have

$$\Phi = \frac{\imath}{2}F^{\mu\nu}\left[Y_{\mu\nu} + \imath(*Y_{\mu\nu})\right] = \left[F^{\mu\nu} + \imath(*F^{\mu\nu})\right]\widetilde{Y}_{\mu\nu}\,, \tag{3.28}$$

where we we have used CYK two-form solution for Minkowski space $\widetilde{Y} = Rdt\wedge dR$, to obtain $\Phi = \mathbf{Z}\cdot\mathbf{r}$.
To check the inverse mapping we need to show the reconstruction of the full EM data $\mathbf{Z}$ from a wave function Φ.
From now, we restrict ourselves to use (t,Θ,φ,R) coordinates[4]. On each $t = \text{const.}$ slice we have a metric $\delta_{ab}\mathrm{d}x^a\mathrm{d}x^b = \mathrm{d}R^2 + R^2\left(\mathrm{d}\Theta^2 + \sin^2\Theta\mathrm{d}\varphi^2\right)$. The procedure presented below describes how to recover Riemann–Silberstein vector field Z from Φ. We would like to stress that the presented procedure can be used for any smooth solution of (3.27).

[4] t and R denote respectively time and radial coordinate. Θ and φ parametrizes the two-sphere.

The definition of Φ and some of the vacuum Maxwell equations (B.9) and (B.10) in terms of scalar Φ (in index notation) take the form

$$\begin{aligned} \partial_R (R\Phi) &= -R^2 Z^A{}_{||A} \,, & (3.29) \\ \partial_t \Phi &= \imath\varepsilon^{AB} Z_{A||B} \,, & (3.30) \\ \Phi &= R Z^R \,, & (3.31) \end{aligned}$$

where ε^{AB} is a Levi-Civita tensor[5] on a sphere t =const., R =const. Hence, quasi-locally the above formulae enable one to reconstruct $\mathbf{Z}$. More precisely, according to Hodge–Kodaira theory applied to differential forms on a sphere[6], $Z_A \mathrm{d}x^A$ can be decomposed into a gradient and co-gradient of some functions α and β

$$Z_A = \alpha_{,A} + \varepsilon_A{}^B \beta_{,B} \,. \tag{3.32}$$

The equations (3.29)-(3.32) allow to obtain $\boldsymbol{\Delta}\alpha$ and $\boldsymbol{\Delta}\beta$. The two-dimensional Laplace operator $\boldsymbol{\Delta}$ on the unit sphere can be quasi-locally inverted with the help of methods which are presented in appendix A. From now, we restrict ourselves to the function Φ which is a l-pole like in the formula (3.40). For convenience, we define the time-radius part ϕ of Φ:

$$\Phi = \phi(t, R) Y_l(\Theta, \varphi) \,, \tag{3.33}$$

where Y_l is the l-th spherical harmonics – eigenfunction of the two-dimensional Laplace operator on the unit sphere, i.e.

$$\boldsymbol{\Delta} Y_l = -l(l+1) Y_l \,, \tag{3.34}$$

where Y_l is a spherical harmonic of l-th degree. The construction presented in the thesis, except section 3.4.4, requires only two properties of spherical harmonics:

- Spherical harmonics are eigenfunctions of the two-dimensional Laplace operator:

$$\boldsymbol{\Delta} Y_l = -l(l+1) Y_l \,. \tag{3.35}$$

- There are two distinguished cases with specified order m of spherical harmonic: axially symmetrical harmonic (order of multipole $m = 0$) and a harmonic with a maximal order $m = \pm l$.

For convenience of the reader, we choose the following representation of spherical harmonics:

$$Y_{lm} = P_{lm}(\cos\Theta) \mathrm{e}^{\imath m \varphi} \,, \tag{3.36}$$

[5]It can be defined by the formula $\varepsilon_{\alpha\beta\gamma\delta} = R^2 \sin\Theta \epsilon_{\alpha\beta\gamma\delta}$, where

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ is an even permutation of } \{t, \Theta, \varphi, R\} \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ is an odd permutation of } \{t, \Theta, \varphi, R\} \\ 0 & \text{in any other case} \end{cases}$$

For lower dimensional case, we have $\varepsilon_{abc} = \varepsilon_{tabc}$ and $\varepsilon_{AB} = \varepsilon_{RAB}$.

[6]See appendix A.3. There are no harmonic one-forms on a two-sphere.

where P_{lm} are the associated Legendre polynomials. Often the order m of a spherical harmonic does not have to be specified. In that case, Y_l denotes a linear combination of spherical harmonics with distinguished degree l and any order m.
We can split α and β into multipoles. We highlight that the multipole decomposition is convenient to use in the examined case but the reconstruction procedure does not require multipole splitting in general. (3.33) suggests that only one l-pole will be non-vanishing in the expansion

$$Z^A = a(t,R)(Y_l)^{,A} + b(t,R)\varepsilon^{AB}(Y_l)_{,B}\,. \tag{3.37}$$

Combining (3.29) with (3.37), the direct formula for complex scalar function $a(t,r)$ is obtained:

$$a(t,R) = \frac{\partial_R\left(R\phi(t,R)\right)}{l(l+1)}\,. \tag{3.38}$$

Analogically, using (3.30) and (3.37), we obtain the function $b(t,R)$:

$$b(t,R) = -\imath\frac{R}{l(l+1)}\partial_t\phi(t,R)\,. \tag{3.39}$$

We reconstruct the two-dimensional part of $\mathbf{Z}$. The radial component of $\mathbf{Z}$ is algebraically related with $\Phi = Z^R R$. We recover the full form of $\mathbf{Z}$ in that way.
In the context of hopfions, the interesting set of solutions of (3.27) is

$$\Phi_H = \frac{R^l Y_l}{\left[R^2-(t-\imath)^2\right]^{l+1}}\,. \tag{3.40}$$

The dipole solution from (3.40) is related to Hopfion solution from [65], so we call (3.40) generalized hopfions. The properties of solutions (3.40) are discussed in the sequel at the end of section 3.3.

3.3.2 Chandrasekhar–Kendall vector potential

A vector potential is defined up to a gradient of some function by the formula $\mathbf{Z} = \operatorname{curl}\mathbf{V}$ – cf. appendix B, equation (B.11). The field $\mathbf{Z}$ for presented class of generalized hopfions (3.40) has simple multipole structure. It leads to a similar form of $\mathbf{V}$. We propose for $\mathbf{V}$ following *ansatz*:

$$V^R = s(t,R)Y_l\,, \tag{3.41}$$
$$V^A = p(t,R)(Y_l)^{,A} + q(t,R)\varepsilon^{AB}(Y_l)_{,B}\,. \tag{3.42}$$

The above formulae and the Maxwell equation (B.11) imply

$$R\phi(t,R) = l(l+1)q(t,R)\,, \tag{3.43}$$
$$a(t,R) = \partial_R[q(t,R)]\,, \tag{3.44}$$
$$b(t,R) = s(t,R) - \partial_R[p(t,R)]\,. \tag{3.45}$$

For solutions (3.40), freedom of choice of $\mathbf{V}$ enables one to construct vector potential in Chandrasekhar–Kendall (C–K) form[7]. C–K potential is an eigenvector of the curl operator

$$\mathbf{Z} = \lambda(t, R)\mathbf{V}\,, \tag{3.46}$$

where $\lambda(t, R)$ is a complex, scalar function. It leads to an overdetermined system of equations

$$\phi(t, R) = \lambda s(t, R)\,, \tag{3.47}$$
$$a(t, R) = \lambda p(t, R)\,, \tag{3.48}$$
$$b(t, R) = \lambda q(t, R)\,. \tag{3.49}$$

It turns out that the equations (3.43-3.45) and (3.47-3.49) for solutions (3.40) are self-consistent. For (3.40), we introduce the time-radius part (3.33) denoted by $\phi_H(t, r)$. The solutions are the following functions

$$s(t, R) = \frac{\imath\phi_H(t, R)^2}{\partial_t\phi_H(t, R)}\,, \tag{3.50}$$
$$p(t, R) = \frac{\imath\phi_H(t, R)\partial_R(R\phi_H(t, R))}{l(l+1)\partial_t\phi_H(t, R)}\,, \tag{3.51}$$
$$q(t, R) = \frac{R\phi_H(t, R)}{l(l+1)}\,, \tag{3.52}$$

which represent eigenvector of (3.46) with the following eigenvalue:

$$\lambda(t, R) = -\imath\partial_t \ln(\phi_H(t, R))\,. \tag{3.53}$$

3.3.3 Conservation of topological charge in time

For electric and magnetic field fulfilling constraints one can introduce vector potentials:

$$\mathbf{E} = \operatorname{curl}\mathbf{C}\,, \quad \mathbf{B} = \operatorname{curl}\mathbf{A}\,, \quad \mathbf{V} := \mathbf{C} + \imath\mathbf{A}\,, \quad \mathbf{Z} = \operatorname{curl}\mathbf{V}\,.$$

See appendix B for details. Helicity integrals measure topological properties of field lines. For electromagnetic field, electric helicity

$$h_E = \int_\Sigma \mathbf{C}\cdot\mathbf{E}\,, \tag{3.54}$$

and magnetic helicity

$$h_M = \int_\Sigma \mathbf{A}\cdot\mathbf{B}\,, \tag{3.55}$$

are quantities which are related to a number of linkedness and knotness of the integral curves of the electric $\mathbf{E}$ (magnetic $\mathbf{B}$) vector field. $\mathbf{C}$ and $\mathbf{A}$ are vector potentials for $\mathbf{E}$ and $\mathbf{B}$ respectively (see appendix B for details). Σ means the whole spatial space on a

[7] Chandrasekhar–Kendall potential is part of a family of fields known as force-free fields and is of broad importance in plasma physics and fluid dynamics. See [30] and the citations within.

slice $\{t = \text{const.}\}$ and $\cdot$ is a scalar product.
Helicities quantifies various aspects of field structure. Examples of fields which poses non-vanishing helicity include twisted, linked, knotted or kinked flux tubes, sheared layers of flux, and force-free fields.The origins of helicity integrals are related with Gauss linking integral. See [7] for detailed review.
It is convenient to present the helicities in terms of Riemann-Silberstein vector field $\mathbf{Z}$ and its vector potential $\mathbf{V}$:

$$h_E + h_M = \int_\Sigma \Re\left(\mathbf{Z}\cdot\bar{\mathbf{V}}\right) = \Re\int_\Sigma Z^a\bar{V}^b\delta_{ab}\,\mathrm{d}^3x\,, \tag{3.56}$$

$$h_E - h_M = \int_\Sigma \Re\left(\mathbf{Z}\cdot\mathbf{V}\right) = \Re\int_\Sigma Z^a V^b\delta_{ab}\,\mathrm{d}^3x\,, \tag{3.57}$$

where $\bar{\mathbf{V}}$ is the complex conjugate of $\mathbf{V}$ and $\Re$ denotes the real part. Using the scalar description of E-M fields (appendix B) we can express total helicity (3.56) in terms of Φ:

$$h_E + h_M = \int_\Sigma \Re\left[\imath\left(\Phi\boldsymbol{\Delta}^{-1}\partial_t\bar{\Phi} - \bar{\Phi}\boldsymbol{\Delta}^{-1}\partial_t\Phi\right)\right]\,, \tag{3.58}$$

where $\imath^2 = -1$ and $\boldsymbol{\Delta}^{-1}$ is an inverse operator to the two-dimensional Laplace operator on the unit sphere (see appendix A).
The equations (3.58) and (3.27) imply conservation law for total helicity:

$$\begin{aligned}\partial_t\left(h_E + h_M\right) &= \lim_{R\to\infty}\int_{B(0,R)} \Re\left[\imath\left(\Phi\boldsymbol{\Delta}^{-1}\partial_t^2\bar{\Phi} - \bar{\Phi}\boldsymbol{\Delta}^{-1}\partial_t^2\Phi\right)\right] \\ &= \lim_{R\to\infty}\int_{\partial B(0,R)} \Re\left[\imath\left(\Phi\boldsymbol{\Delta}^{-1}R^2\partial_R\bar{\Phi} - R^2\partial_r\Phi\boldsymbol{\Delta}^{-1}\bar{\Phi}\right)\right]\,,\end{aligned} \tag{3.59}$$

where $B(0,R) = \{x\in\Sigma : ||x|| \le R\}$. We assume the E-M fields are localized,[8] hence the boundary terms at infinity can be neglected. In general, for the quantity $h_E - h_M$ (3.57) we have no time dependence. However, in terms of Φ we have

$$h_E - h_M = -2\int_\Sigma \Re\left(\imath\Phi\boldsymbol{\Delta}^{-1}\partial_t\Phi\right)\,, \tag{3.60}$$

and

$$\partial_t\left(h_E - h_M\right) = -2\int_\Sigma \Re\partial_t\left(\imath\Phi\boldsymbol{\Delta}^{-1}\partial_t\Phi\right)\,, \tag{3.61}$$

which lead to the following

Proposition 3.3.1. *For localized fields, the helicities (3.54) and (3.55) are preserved in time if and only if*

$$\int_\Sigma \Re\partial_t\left(\imath\Phi\boldsymbol{\Delta}^{-1}\partial_t\Phi\right) = 0\,. \tag{3.62}$$

[8] By localized we mean compactly supported or with fall off sufficiently fast which enables one to neglect boundary terms.

The following quasi-local equality

$$\int_{\partial B(0,R)} \mathbf{Z}\cdot\mathbf{Z} = \int_{\partial B(0,R)} \partial_t \left(\Phi \mathbf{\Delta}^{-1} \partial_t \Phi\right) , \tag{3.63}$$

gives equivalence to the Rañada result in [51]. We highlight that $\cdot$ denotes scalar product without complex conjugate.

3.3.4 Discussion

The conservation of topological charge imposes an additional condition (3.62) for solutions (3.40). The integral in (3.62) for solutions (3.40) contains an integral of a square of a single multipole Y_l over a two-dimensional sphere. $\int_0^\pi \mathrm{d}\Theta \int_0^{2\pi} \mathrm{d}\varphi (Y_l)^2$ is equal to zero for non-zero order m of multipole. We denote $Y_l = Y_{lm}$ where l and m are respectively a degree and an order[9] of multipole. Hence these values of m for each l lead to an E-M field which preserves the topological charge. Such E-M solution is a generalization of the null hopfion. For $l = 1$ our solutions with the maximal order are equal (up to a constant) to the null hopfion described in [65]. The case $l = 1, m = 0$ corresponds to non-null hopfion from [65].

3.4 Spin-2 field and generalized gravitational hopfions

Consider a weak gravitational field on the Minkowski background. The used complex scalar framework is related to the linearized Weyl tensor splitted into a tidal (gravito-electric) part E_{kl} and frame-dragging (gravito-magnetic) part B_{kl} (see appendix D.2). Both E_{kl} and B_{kl} are symmetric and traceless. With the help of the constraint equations,

$$E^{kl}{}_{|l} = 0, \tag{3.64}$$

$$B^{kl}{}_{|l} = 0, \tag{3.65}$$

we can quasi-locally describe the field in the terms of complex scalar field Ψ. See the appendix D.3 for precise formulation and details. The used notation and denotings are presented in appendix D.

3.4.1 Reconstruction for linearized gravity field

The reconstruction for linearized gravity field is a generalization of the procedure for electromagnetic field described in the section 3.3.1. Constraint equations and the Hodge–Kodaira decomposition for two-dimensional tensors on a sphere (see appendices A.4 and D.3) enable one to encode quasi-locally a spin-2 field into a complex scalar field. For

[9] Physicists usually use the naming convention which is associated with quantum mechanics. The degree of multipole is related to spin number and the order of multipole corresponds to magnetic spin number.

a given l-pole field the reconstruction is simplified because the inverse operator to the two-dimensional Laplacian has a simple form. In the context of hopfions, we consider a class of complex scalar fields in the following form

$$\Psi_H = \frac{R^l Y_l}{\left[R^2 - (t - \imath)^2\right]^{l+1}} , \tag{3.66}$$

for $l \geq 2$. For convenience, we define

$$\psi_H = \frac{R^l}{\left[R^2 - (t - \imath)^2\right]^{l+1}} . \tag{3.67}$$

$\Psi_H = \psi_H Y_l$ is the same function as Φ_H (3.40) for the set of generalized electromagnetic hopfions. For $l = 2$, the solution (3.66) is related to gravitational hopfion[10], so we call the set of solutions (3.40) generalized gravitational hopfions. Ψ_H fulfills wave equation $\Box\Psi_H = 0$ and represents gauge-invariant reduced data for linearized vacuum Einstein equation. For given l-pole field (3.66) the structure of reconstructed gravito-electromagnetic tensor Z_{kl} is as follows:

$$Z^{RR} = a_g(t,R) Y_l , \tag{3.68}$$
$$Z^{RA} = b_g(t,R)(Y_l)^{||A} + c_g(t,R)\varepsilon^{RAB}(Y_l)_{||B} , \tag{3.69}$$
$$\overset{(2)}{Z} = -a_g(t,R) Y_l , \tag{3.70}$$
$$\overset{\circ}{Z}_{AB} = d_g(t,R)\left((Y_l)_{||AB} - \frac{1}{2}\frac{g_{AB}}{R^2}\right) + e_g(t,R)(Y_l)_{||C(A}\varepsilon_{B)}{}^C , \tag{3.71}$$

where

$$a_g(t,R) = \frac{\psi_H}{R^2} , \tag{3.72}$$
$$b_g(t,R) = \frac{\partial_R(R\psi_H)}{l(l+1)R} , \tag{3.73}$$
$$c_g(t,R) = \frac{\partial_t\psi_H}{l(l+1)} , \tag{3.74}$$
$$d_g(t,R) = \frac{\partial_R\left(R\partial_R\left(R\psi_H\right)\right) - \frac{1}{2}l(l+1)\psi_H}{l\,(l+1)\left[l\,(l+1) - 2\right]} , \tag{3.75}$$
$$e_g(t,R) = \frac{\partial_R\left(R^2\partial_t\psi_H\right)}{l\,(l+1)\left[l\,(l+1) - 2\right]} , \tag{3.76}$$

which are similar to (3.38)-(3.39) for electromagnetic case.

3.4.2 Hamiltonian energy for linearized gravity

Energy of a gravitational field is an issue which is problematic in various contexts. The ambiguity of energy of gravitational system can be observed even in Newtonian theory.

[10] The type N hopfion from [65] covers (up to a constant) with the solution from class (3.66) for $l = 2$ and for the spherical harmonic with maximum spin number ($m = l = 2$).

Consider Newtonian gravitational potential ϕ on the flat three-dimensional space with a gravitational force given by one-form $\mathrm{d}\phi = \phi_{|k}\mathrm{d}x^k$. According to Galileo–Eötvös experiment, i.e., the principle of equivalence, there is an ambiguity in the gravitational force: It is determined only up to an additive constant covector field ω_k, and hence by an appropriate transformation $\phi_{|k} \to \phi_{|k} + \omega_k$ the gravitational force $\phi_{|k}$ at a given point $p \in \Sigma$ can be made zero. Thus, at this point both the gravitational energy density and the spatial stress have been made vanishing. On the other hand, they can be made vanishing on an open subset $\mathcal{O} \subset \Sigma$ only if the tidal force, $\phi_{|kl}$, is vanishing on $\mathcal{O}$. Therefore, the gravitational energy and the spatial stress cannot be localized to a point, i.e., they suffer from the ambiguity in the gravitational force above. For a more detailed discussion of the energy in the (relativistically corrected) Newtonian theory, see [23].
In the case of General Relativity theory the issue is more complicated and well-known. Brill and Deser have published a series of classical papers [10, 9, 11] in which the issue of ambiguity and positivity of energy is discussed.
Sections 3.4.2–3.4.4 have been written to point out that the real (Hamiltonian) energy of weak gravitational field can not be localized. We compare such quasi-local Hamiltonian energy density with chosen well-known (super-) energy densities using hopfions as an example.

In [35] (see also [31] and [32]) Professor Jezierski proposed energy functional $\mathcal{H}$ which takes the following form in Minkowski spacetime:

$$\begin{aligned}\mathcal{H} &= \frac{1}{32\pi}\int_\Sigma \Big[(R\dot{x})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(R\dot{x}) + (R\dot{y})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(R\dot{y}) \\ &\quad + (Rx)_{,R}\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(Rx)_{,R} - x(\boldsymbol{\Delta}+2)^{-1}x \\ &\quad + (Ry)_{,R}\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(Ry)_{,R} - y(\boldsymbol{\Delta}+2)^{-1}y\Big]\mathrm{d}R\sin\Theta\mathrm{d}\Theta\mathrm{d}\varphi\,,\end{aligned} \tag{3.77}$$

where x and y are defined by relation $\Psi = x + \imath y$. $\mathcal{H}$ have simpler form in terms of Ψ:

$$\begin{aligned}\mathcal{H} &= \frac{1}{32\pi}\int_\Sigma \Big[(R\partial_t\Psi)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(R\partial_t\bar{\Psi}) \\ &\quad + (R\Psi)_{,R}\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(R\bar{\Psi})_{,R} - \Psi(\boldsymbol{\Delta}+2)^{-1}\bar{\Psi}\Big]\mathrm{d}R\sin\Theta\mathrm{d}\Theta\mathrm{d}\varphi\,.\end{aligned} \tag{3.78}$$

The formula has its origins in the canonical (Hamiltonian) formulation of the linearized theory of gravity. In this sense it describes a true energy of linearized gravitational field. In section 3.4.4 we remind another two super-energy functionals[11] Θ_0 (3.86) (see also [33]) and super-energy (3.93) which arises for spin-2 field in a natural way. In particular, the integrals (3.86) and (3.78) differ by the operator $(\boldsymbol{\Delta}+2)^{-1}$, hence for each spherical mode (i.e. after spherical harmonics decomposition) they are proportional to each other. Hamiltonians for whom functions in multipole expansions differ by a constant multiplicative factor lead to the same dynamics. We will discuss in a separate paper [36] how the functional $\mathcal{H}$ is related to the following expression:

$$16\pi\overline{\mathcal{H}} = \int_\Sigma \big(E^{ab}(-\triangle)^{-1}E_{ab} + B^{ab}(-\triangle)^{-1}B_{ab}\big)$$

[11] We remind a quasi-local densities of such energy functionals. The diffrence is only to integrate over the radial coordinate i.e. $\Theta_0 = \int_0^\infty U_{\Theta_0}\mathrm{d}r$

$$= \iint_{\Sigma\times\Sigma} \left[\frac{E^{ab}(\mathbf{r}')\, E_{ab}(\mathbf{r}'')}{4\pi\|\mathbf{r}'-\mathbf{r}''\|} + \frac{B^{ab}(\mathbf{r}')\, B_{ab}(\mathbf{r}'')}{4\pi\|\mathbf{r}'-\mathbf{r}''\|} \right] \mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}'' \tag{3.79}$$

$$= \int_\Sigma \left(Z^{ab}(-\triangle)^{-1}\bar{Z}_{ab} \right)$$

$$= \iint_{\Sigma\times\Sigma} \left[\frac{Z^{ab}(\mathbf{r}')\, \bar{Z}_{ab}(\mathbf{r}'')}{4\pi\|\mathbf{r}'-\mathbf{r}''\|} \right] \mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}'' \,, \tag{3.80}$$

which is proposed by I. Bialynicki-Birula [8] and has a nice property – it is manifestly covariant with respect to the Euclidean group. In the future we also plan to incorporate boundary terms because we want to generalize the above formulae to finite region with boundary.

Let us consider localized initial data on Σ, i.e. compactly supported or with fall off sufficiently fast which enables one to neglect boundary terms. The following theorem (to be presented in detail in [36])

Theorem 3.4.1. *For localized data* $\mathcal{H} = \overline{\mathcal{H}}$.

can be checked as follows:

Proof. Let us observe that $x = 2x^k x^l E_{kl}$, $y = 2x^k x^l B_{kl}$. If we introduce transverse–traceless potentials[12] $\mathtt{e}$ and $\mathtt{h}$:

$$-\triangle \mathtt{e}_{kl} = E_{kl} \,, \quad -\triangle \mathtt{h}_{kl} = B_{kl} \,,$$

where $\triangle$ is the three-dimensional Laplacian[13], then for $a := 2x^k x^l \mathtt{e}_{kl}$, $b := 2x^k x^l \mathtt{h}_{kl}$ we get

$$-\triangle a = x \,, \quad -\triangle b = y \,,$$

Moreover, for finite region $V \subset \Sigma$

$$16\pi\overline{\mathcal{H}}_V := \int_V \left(\mathtt{e}_{kl}E^{kl} + \mathtt{h}_{kl}B^{kl} \right) \mathrm{d}^3x \tag{3.81}$$

$$= \frac{1}{2}\int_V \frac{1}{R^2}\Big((R\dot{a})(-\boldsymbol{\Delta})^{-1}(R\dot{x}) + \partial_R(Ra)(-\boldsymbol{\Delta})^{-1}\partial_R(Rx) + \frac{1}{2}ax$$
$$+ (R\dot{b})(-\boldsymbol{\Delta})^{-1}(R\dot{y}) + \partial_R(Rb)(-\boldsymbol{\Delta})^{-1}\partial_R(Ry) + \frac{1}{2}by \Big) \mathrm{d}R \sin\Theta \mathrm{d}\Theta \mathrm{d}\varphi$$
$$+ \frac{1}{2}\int_V \frac{1}{R^2}\Big[\partial_R(R^2\dot{a})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_R(R^2\dot{x}) + \frac{1}{4}ax \tag{3.82}$$
$$+ \big(\partial_R[R\partial_R(Ra)] + \frac{1}{2}\boldsymbol{\Delta}a \big)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\big(\partial_R[R\partial_R(Rx)] + \frac{1}{2}\boldsymbol{\Delta}x \big)$$
$$+ \big(\partial_R[R\partial_R(Rb)] + \frac{1}{2}\boldsymbol{\Delta}b \big)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\big(\partial_R[R\partial_R(Ry)] + \frac{1}{2}\boldsymbol{\Delta}y \big)$$
$$+ \partial_R(R^2\dot{b})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_R(R^2\dot{y}) + \frac{1}{4}by \Big] \mathrm{d}R \sin\Theta \mathrm{d}\Theta \mathrm{d}\varphi \,.$$

[12] Transverse–traceless symmetric tensor-field h_{kl} means $h_{kl}\delta^{kl} = 0$ and ${h^{kl}}_{|l} = 0$.

[13] In Cartesian coordinates it is simply $\triangle = \sum_{i=1}^{3} \left(\frac{\partial}{\partial x^i} \right)^2$.

Now, we have to integrate by parts many times and finally we obtain energy (3.78) up to boundary terms

$$
\begin{aligned}
16\pi\overline{\mathcal{H}}_V &= \frac{1}{2}\int_V \Big[(-R^2\triangle\dot{a})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\dot{x} + \partial_R(-R\triangle a)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_R(Rx) \\
&+\triangle a(\boldsymbol{\Delta}+2)^{-1}x + \partial_R(-R\triangle b)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_R(Ry) \\
&+(-R^2\triangle\dot{b})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\dot{y} + \triangle b(\boldsymbol{\Delta}+2)^{-1}y\Big]\mathrm{d}R\sin\Theta\mathrm{d}\Theta\mathrm{d}\varphi \\
&+\frac{1}{2}\int_{\partial V}\Big[\partial_R(R^2\dot{a})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\dot{x} - \frac{1}{2R^2}\partial_R(R^2a)(\boldsymbol{\Delta}+2)^{-1}x \qquad (3.83)\\
&+\Big(R\triangle a + \frac{1}{R}\partial_R(Ra) - \frac{1}{2R}\boldsymbol{\Delta}a\Big)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_R(Rx) \\
&+\partial_R(R^2\dot{b})\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\dot{y} - \frac{1}{2R^2}\partial_R(R^2b)(\boldsymbol{\Delta}+2)^{-1}y \\
&+\Big(R\triangle b + \frac{1}{R}\partial_R(Rb) - \frac{1}{2R}\boldsymbol{\Delta}b\Big)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_R(Ry)\Big]\sin\Theta\mathrm{d}\Theta\mathrm{d}\varphi\,.
\end{aligned}
$$

More precisely, the volume term in the above formula equals $16\pi\mathcal{H}$ given by (3.78). □

3.4.3 Quasi-local (super-)energy density for spin-2 field

We present quasi-local (q-l) energy and super-energy densities for spin-2 field and linearized gravity. By q-l density we mean a functional which is an integral over a two-dimensional topological sphere. In this section, we calculate q-l densities over $\{t = \text{const.}, R = \text{const.}\}$ surface. A few of analyzed energies (for example $\mathcal{H}$ (3.78)) are defined with the help of q-l integral operator. They do not have density which can be calculated locally at point. The q-l (super-)energy densities listed below are presented in general form – they are valid for every localized weak gravitational field represented as a complex harmonic function Ψ. The compared q-l (super-)energy densities can be organized as follows:

1. Related to the canonical (Hamiltonian) theory:

   (a) The q-l energy density of hamiltonian energy $\mathcal{H}$ (3.78) derived from the canonical formulation of linearized theory of gravity.

   $$
   \begin{aligned}
   U_{\mathcal{H}} &= \frac{1}{32\pi}\int_{S(t,R)} \sin\Theta\Big[(R\partial_t\Psi)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(R\partial_t\bar{\Psi}) \qquad (3.84)\\
   &\quad + (R\Psi)_{,R}\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}(R\bar{\Psi})_{,R} - \Psi(\boldsymbol{\Delta}+2)^{-1}\bar{\Psi}\Big]\,,
   \end{aligned}
   $$

   where $S(t,R)$ denotes $\{t = \text{const.}, R = \text{const.}\}$ surface.

   (b) Θ_0 functional is obtained with the help of Conformal Yano–Killing (CYK) tensors. The contraction of CYK tensor $Q^{\mu\nu}$ with Weyl tensor $W_{\mu\nu\alpha\beta}$ is a two-form $F^{(Q)}_{\alpha\beta} = Q^{\mu\nu}W_{\mu\nu\alpha\beta}$, where $Q^{\mu\nu}\partial_{x^\mu}\wedge\partial_{x^\nu} = \mathcal{D}\wedge\partial_t$ is a CYK tensor for Minkowski spacetime and $\mathcal{D} = x^\nu\partial_\nu$ is a generator of dilatations in Minkowski spacetime. $F^{(Q)}_{\alpha\beta}$ fulfills vacuum Maxwell equations. Θ_0 is an electromagnetic energy calculated for $F^{(Q)}_{\alpha\beta}$ from stress-energy tensor

$$T^{\rm EM}_{\mu\nu}(F) := \frac{1}{2}\left(F_{\mu\sigma}F_{\nu}{}^{\sigma} + F^{*}{}_{\mu\sigma}F^{*}{}_{\nu}{}^{\sigma}\right) , \tag{3.85}$$

where $F^{*\mu\lambda} = \frac{1}{2}\varepsilon^{\mu\lambda\rho\sigma}F_{\rho\sigma}$. See [33] for details. The q-l density of Θ_0 is

$$\begin{aligned} 4\pi U_{\Theta_0} &= \int_{S(t,R)} T^{EM}\big(\partial_t, \partial_t, F(W, \mathcal{D}\wedge\mathcal{T}_t)\big)R^2\sin\Theta{\rm d}\Theta{\rm d}\varphi \\ &= \frac{1}{2}\int_{S(t,R)} R^2\left(E_{kR}E^{kR} + B_{kR}B^{kR}\right)R^2\sin\Theta{\rm d}\Theta{\rm d}\varphi \\ &= \frac{1}{4}\int_{S(t,R)} \big[\partial_t(R\Psi)(-\boldsymbol{\Delta})^{-1}\partial_t(R\bar{\Psi}) \\ &\quad +\partial_R(R\Psi)(-\boldsymbol{\Delta})^{-1}\partial_R(R\bar{\Psi}) + \Psi\bar{\Psi}\big]\sin\Theta{\rm d}\Theta{\rm d}\varphi\,. \end{aligned} \tag{3.86}$$

(c) We compare q-l energy densities for linearized gravity with q-l electromagnetic energy densities for the corresponding electromagnetic solution (compare (3.99) and (3.100)). Let us define

$$\begin{aligned} F_1(\Phi) &:= \Big[(R\partial_t\Phi)(-\boldsymbol{\Delta}^{-1})(R\partial_t\bar{\Phi}) \\ &\quad +(R\Phi)_{,R}(-\boldsymbol{\Delta}^{-1})(R\bar{\Phi})_{,R} + \Phi\bar{\Phi}\Big]\sin\Theta\,, \end{aligned} \tag{3.87}$$

$$F_2(\Phi) := \left[\partial_R(R\Phi)\boldsymbol{\Delta}^{-1}R\partial_t\bar{\Phi} + \partial_R(R\bar{\Phi})\boldsymbol{\Delta}^{-1}R\partial_t\Phi\right]\sin\Theta\,. \tag{3.88}$$

The electromagnetic q-l energy density in terms of electromagnetic scalar Φ is equal to

$$\begin{aligned} 4\pi U_{EM} &= \int_{S(t,R)} T^{EM}\big(\partial_t, \partial_t, \Phi\big)R^2\sin\Theta{\rm d}\Theta{\rm d}\varphi \\ &= \frac{1}{4}\int_{S(t,R)} F_1(\Phi){\rm d}\Theta{\rm d}\varphi\,. \end{aligned} \tag{3.89}$$

The electromagnetic q-l energy density for the conformal field

$$\mathcal{K} = 2Rt\partial_R + \left(t^2 + R^2\right)\partial_t\,, \tag{3.90}$$

is the following

$$\begin{aligned} 4\pi U_{CEM} &= \int_{S(t,R)} T^{EM}(\mathcal{K}, \partial_t, \Phi)R^2\sin\Theta{\rm d}\Theta{\rm d}\varphi \\ &= \frac{1}{4}\int_{S(t,R)} \left[\left(R^2 + t^2\right)F_1(\Phi) + 2RtF_2(\Phi)\right]{\rm d}\Theta{\rm d}\varphi\,. \end{aligned} \tag{3.91}$$

2. Associated to Bel–Robinson tensor. The Bel–Robinson tensor has the structure

$$T^{BR}_{\mu\nu\kappa\lambda} := W_{\mu\rho\kappa\sigma}W_{\nu}{}^{\rho}{}_{\lambda}{}^{\sigma} + W^{*}{}_{\mu\rho\kappa\sigma}W^{*}{}_{\nu}{}^{\rho}{}_{\lambda}{}^{\sigma}\,, \tag{3.92}$$

where $(W^*)_{\alpha\beta\gamma\delta} = \frac{1}{2}W_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\mu\nu\gamma\delta}$. The spin-2 field equations (D.16) and (D.17) remain invariant under the global $U(1)$ transformation $Z^{kl} \to e^{i\alpha}Z^{kl}$. The duality invariance[14] is a property of Bel–Robinson tensor. The q-l density of super-energy fulfills

$$\begin{aligned} 4\pi U_S &= \int_{S(t,R)} \frac{1}{2}T^{BR}(\partial_t,\partial_t,\partial_t,\partial_t,\Psi)R^2\sin\Theta \mathrm{d}\Theta\mathrm{d}\varphi \\ &= \int_{S(t,R)} u_S R^2 \sin\Theta\mathrm{d}\Theta\mathrm{d}\varphi \\ &= \frac{1}{4}\int_{S(t,R)} F_3(\Psi)\,, \end{aligned} \tag{3.93}$$

where $F_3(\Psi)$ is given by (3.94). Let us introduce

$$\begin{aligned} F_3(\Psi) &:= \frac{1}{R^2}\Big\{(R\partial_t\Psi)(-\boldsymbol{\Delta})^{-1}(R\partial_t\bar\Psi) + \partial_R(R\Psi)(-\boldsymbol{\Delta})^{-1}\partial_R(R\bar\Psi) + \frac{1}{2}\Psi\bar\Psi \\ &+\big(\partial_R[R\partial_R(R\Psi)] + \frac{1}{2}\boldsymbol{\Delta}\Psi\big)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\big(\partial_R[R\partial_R(R\bar\Psi)] + \frac{1}{2}\boldsymbol{\Delta}\bar\Psi\big) \\ &+\partial_R(R^2\partial_t\Psi)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_R(R^2\partial_t\bar\Psi) + \frac{1}{4}\Psi\bar\Psi\Big\}\sin\Theta\,, \end{aligned} \tag{3.94}$$

$$F_4(\Psi) := \Big\{\frac{1}{2}\left[\partial_R(R\Psi)(-\boldsymbol{\Delta})^{-1}(R\partial_t\bar\Psi) + \partial_R(R\bar\Psi)(-\boldsymbol{\Delta})^{-1}(R\partial_t\Psi)\right] \tag{3.95}$$

$$\begin{aligned} &+\big(\partial_R[R\partial_R(R\Psi)] + \frac{1}{2}\boldsymbol{\Delta}\Psi\big)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_R(R^2\partial_t\bar\Psi) \\ &+\big(\partial_R[R\partial_R(R\bar\Psi)]\big)\boldsymbol{\Delta}^{-1}(\boldsymbol{\Delta}+2)^{-1}\partial_R(R^2\partial_t\Psi)\Big\}\sin\Theta\,. \end{aligned} \tag{3.96}$$

The Bel–Robinson charge for a conformal field is as follows

$$\begin{aligned} 4\pi U_{CS} &= \int_{S(t,R)} \frac{1}{2}T^{BR}(\mathcal{K},\partial_t,\partial_t,\partial_t,\Psi)R^2\sin\Theta\mathrm{d}\Theta\mathrm{d}\varphi \\ &= \frac{1}{4}\int_{S(t,R)} \left[\left(t^2+R^2\right)F_3(\Psi) + 2RtF_4(\Psi)\right]\mathrm{d}\Theta\mathrm{d}\varphi\,, \end{aligned} \tag{3.97}$$

where conformal field $\mathcal{K}$ is defined by (3.90).

3.4.4 Comparison of the energies for hopfions

In [65], the following super-energy density

$$u_S = \frac{E_{ab}E^{ab} + B_{ab}B^{ab}}{2}\,, \tag{3.98}$$

[14] Introducing $\mathcal{W}_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + \imath^* W_{\alpha\beta\gamma\delta}$, Bell-Robinson tensor (3.92) has the form $T^{BR}_{\mu\nu\kappa\lambda} := \mathcal{W}_{\mu\rho\kappa\sigma}\bar{\mathcal{W}}_{\nu}{}^{\rho}{}_{\lambda}{}^{\sigma}$. All components of $\mathcal{W}_{\mu\rho\kappa\sigma}$ depends of Z_{kl} without complex conjugate $\bar Z_{kl}$. It means that the components of Bell-Robinson tensor are proportional to "$Z\bar Z$" which are invariant under $Z^{kl} \to e^{i\alpha}Z^{kl}$ transformation.

has been calculated for gravitational type N hopfion. We highlight that type N hopfion overlap (up to a constant) with the solution from our class (3.66) for $l = 2$ and for the spherical harmonic with maximal order ($m = l = 2$). For such quadrupole solution

$$\Psi_q := \frac{R^2 Y_{22}}{\left[R^2 - (t - \imath)^2\right]^3}\,, \tag{3.99}$$

where Y_{22} is given by (3.36). We analyze q-l (super-)energy densities for linearized gravity (3.84), (3.86), (3.93) and (3.97) which are presented in the previous section. We compare them with the electromagnetic q-l energy densities (3.89) and (3.91) for the corresponding to Ψ_q (3.99) electromagnetic quadrupole solution

$$\Phi_q := \frac{R^2 Y_{22}}{\left[R^2 - (t - \imath)^2\right]^3}\,. \tag{3.100}$$

Let us define

$$\xi(t,R) := \frac{R^4}{\left((R+t)^2+1\right)^4\left((R-t)^2+1\right)^4}\left[t^4 + (\frac{14}{5}R^2+2)t^2 + (R^2+1)^2\right], \tag{3.101}$$

$$\kappa(t,R) := \frac{R^5 t}{\left((R+t)^2+1\right)^4\left((R-t)^2+1\right)^4}\left[R^2+t^2+1\right], \tag{3.102}$$

$$\begin{aligned}\eta(t,R) := {} & \frac{R^2}{\left((R+t)^2+1\right)^5\left((R-t)^2+1\right)^5}\Big[R^8 + (12t^2+4)R^6 \\ & + \left(\frac{126}{5}t^4 + 28t^2 + 6\right)R^4 + (12t^6 + 28t^4 + 20t^2 + 4)R^2 + (t^2+1)^4\Big]\,,\end{aligned} \tag{3.103}$$

$$\tau(t,R) := \frac{tR^3\left(R^2+t^2+1\right)}{\left((R+t)^2+1\right)^5\left((R-t)^2+1\right)^5}\left[t^4 + \left(\frac{22}{5}R^2+2\right)t^2 + (R^2+1)^2\right]. \tag{3.104}$$

The results for quadrupole hopfion are the following:

$$U_{\mathcal{H}}(\Psi_q) = \frac{1}{24}\xi(t,R)\,, \tag{3.105}$$

$$U_{\Theta_0}(\Psi_q) = \frac{1}{3}\xi(t,R)\,, \tag{3.106}$$

$$U_{EM}(\Phi_q) = \frac{1}{3}\xi(t,R)\,, \tag{3.107}$$

$$U_S(\Psi_q) = \frac{1}{2}\eta(t,R)\,, \tag{3.108}$$

$$U_{CEM}(\Phi_q) = \frac{4}{15}\left[\frac{5}{4}\left(t^2+R^2\right)\xi(t,R) - 6Rt\kappa(t,R)\right], \tag{3.109}$$

$$U_{CS}(\Psi_q) = \frac{1}{2}\left[\left(t^2+R^2\right)\eta(t,R) + 2Rt\tau(t,R)\right]. \tag{3.110}$$

One can observe the following:

1. The q-l energy densities can be divided into two sets:

$$X_1 = \{U_{\mathcal{H}}, U_{\Theta_0}, U_{EM}; U_S\}\,, \tag{3.111}$$

$$X_2 = \{U_{CEM}, U_{CS}\}\,. \qquad (3.112)$$

Functions in each set have similar properties. It means:

(a) In the set X_1 we can distinguish a subset $\{U_{\mathcal{H}}, U_{\Theta_0}, U_{EM}\}$. Q-l (super-) energy densities in the subset differ by a multiplicative constant. Simple, single-multipole structure of the solutions (3.99) and (3.100) is responsible for proportionality of q-l (super-)energy densities in the subset. For solutions with the richer multipole structure relations between the densities will be more complicated.

(b) The set X_2 contains q-l energy densities for the conformal field $\mathcal{K}$ (3.90). For $t = 0$, the conformal q-l densities are proportional to theirs counterparts for ∂_t field. R^2 is the proportional factor

$$\begin{aligned} U_{CEM}(\Phi_q, t=0) &= R^2 U_{EM}(\Phi_q, t=0)\,, \\ U_{CS}(\Psi_q, t=0) &= R^2 U_{EM}(\Psi_q, t=0)\,. \end{aligned}$$

2. All the above presented q-l (super-)energy densities are localized on light cones for large t and r.

3.4.5 Topological charge

Consider the following non-local objects:

$$h_{GE} = \int_\Sigma E^{ab}(-\triangle^{-1})S_{ab} = \iint_{\Sigma\times\Sigma} \frac{E^{ab}(\mathbf{r}')\,S_{ab}(\mathbf{r}'')}{4\pi\|\mathbf{r}'-\mathbf{r}''\|}\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}''\,, \qquad (3.113)$$

$$h_{GB} = \int_\Sigma B^{ab}(-\triangle^{-1})P_{ab} = \iint_{\Sigma\times\Sigma} \frac{B^{ab}(\mathbf{r}')\,P_{ab}(\mathbf{r}'')}{4\pi\|\mathbf{r}'-\mathbf{r}''\|}\mathrm{d}\mathbf{r}'\mathrm{d}\mathbf{r}''\,, \qquad (3.114)$$

where $\triangle^{-1}$ is an inverse operator to the three-dimensional Laplacian $\triangle$ (details in appendix A). P_{ab} and S_{ab} are respectively ADM momentum and its dual counterpart discussed nearby (D.10) and (D.11). For convenience we work with complex objects[15]

$$h_G = h_{GE} - h_{GB} = \Re \int_\Sigma Z^{ab}(-\triangle^{-1})V_{ab}\,, \qquad (3.115)$$

$$\widetilde{h}_G = h_{GE} + h_{GB} = \Re \int_\Sigma Z^{ab}(-\triangle^{-1})\bar{V}_{ab}\,. \qquad (3.116)$$

We list properties of the above quantities (3.113) and (3.114):

- (3.113) and (3.114) are well-defined and gauge invariant[16].
- Similarities with the electromagnetic case:

[15] We use $Z^{ab} = E^{ab} + \imath B^{ab}$ and $V^{ab} = S^{ab} + \imath P^{ab}$. See appendices D.2 and D.3 for more details.

[16] Gauge invariance of h_{GE} and h_{GB} can be easily deduced from equations (3.117)-(3.118). h_{GE} and h_{GB} in terms of h_G and $\widetilde{h}_G$ can be expressed by gauge invariant Ψ and its derivatives.

– Analogy to the electromagnetic helicity – the quantity (3.115) in terms of complex scalar field (3.117) is similar to (3.60).

– Analogy to the conservation law – (3.115) is conserved in time if (3.121) is fulfilled. It is analogous to (3.62).

- (3.115) is conserved in time for an example of gravitational type N hopfion described in [65].

- Structure comparable to other quantities defined for linearized gravity field, for example the energy (3.80).

- Further results obtained by Aghapour et al. [1, 2] shows that both topological charges (helicities) for electromagnetism and linearized gravity are associated with duality symmetry.

To highlight the analogy with electromagnetic field we express equation (3.115) in terms of complex scalar field. Using the reduction presented in appendices A.4 and D.3 the result is as follows:

$$h_G = -\Re \int_\Sigma \imath \Psi \mathbf{\Delta}^{-1} (\mathbf{\Delta}+2)^{-1} \partial_t \Psi \,, \tag{3.117}$$

$$\widetilde{h}_G = \frac{1}{2}\int_\Sigma \Re \left[\imath \left(\Psi \mathbf{\Delta}^{-1} (\mathbf{\Delta}+2)^{-1} \partial_t \bar{\Psi} - \bar{\Psi} \mathbf{\Delta}^{-1} (\mathbf{\Delta}+2)^{-1} \partial_t \Psi\right)\right] . \tag{3.118}$$

If we compare $\partial_t h_G$ and the real part of $\int_\Sigma Z^{kl} \triangle^{-1} Z_{kl}$ in terms of the scalar Ψ then turns out that they are equal up to the factor 2

$$\partial_t h_G = -2\Re \int_\Sigma \imath Z^{kl}(-\triangle^{-1}) Z_{kl} = -\Re \int_\Sigma \imath \partial_t \left(\Psi \mathbf{\Delta}^{-1} (\mathbf{\Delta}+2)^{-1} \partial_t \Psi\right) . \tag{3.119}$$

The gravitational helicity $\widetilde{h}_G$ is preserved in time for all Ψ which fulfill wave equation

$$\partial_t \widetilde{h}_G = \frac{1}{2} \lim_{R\to\infty} \int_{\partial B(0,R)} \Re \left[\imath \left(\Psi \mathbf{\Delta}^{-1} (\mathbf{\Delta}+2)^{-1} R^2 \partial_R \bar{\Psi} - R^2 \partial_R \Psi \mathbf{\Delta}^{-1} (\mathbf{\Delta}+2)^{-1} \bar{\Psi}\right)\right] , \tag{3.120}$$

where $B(0,R) = \{x \in \Sigma : ||x|| \leq R\}$. We assume the linearized gravity fields are localized[17]. It implies that (3.120) vanishes. Only (3.119) leads to a condition for conservancy of topological charges. The results (3.119) and (3.120) give

Proposition 3.4.1. *For localized fields, the objects h_{GE} (3.113) and h_{GB} (3.114) are preserved in time if and only if*

$$2\Re \int_\Sigma Z^{kl}(-\triangle^{-1}) Z_{kl} = \Re \int_\Sigma \partial_t \left(\Psi \mathbf{\Delta}^{-1} (\mathbf{\Delta}+2)^{-1} \partial_t \Psi\right) = 0 . \tag{3.121}$$

[17] By localized we mean compactly supported or with fall off sufficiently fast which enables one to neglect boundary terms.

The above theorem corresponds to proposition 3.3.1 in electrodynamics. Analyzing the results obtained by Aghapour et al. [1, 2], we have realized that the gravitational helicities (3.113) and (3.114) have been previously obtained by Barnett [5]. Barnett's approach is based on duality symmetric formulation, where the analog of helicity for linearized gravity was derived as the Noether current for the action of duality symmetry. It is worth mentioning at this point that helicity and duality symmetry for Maxwell theory and linearized gravity have previously been studied in terms of the standard formulation, and from a Hamiltonian point of view by Deser and Teitelboim [19]. In [1], the duality symmetric formulation of linearized gravity is used to derive generalizations of the helicity, spin, and infra-zilch conservation laws, and a generalization of the helicity array for linearized gravity on Minkowski space.

3.5 Comparison of structures

The electromagnetic hopfions are described in terms of the complex scalar Φ which contains the full information about Maxwell field — two unconstrained degrees of freedom. The scalar Φ formalism for electrodynamics is presented in appendix B. We generalize the electromagnetic hopfions by the natural generalization[18] of Φ to the higher multipole solution (3.40). The physical quantities, like energy or helicity, are expressed in terms of the scalar.
The electromagnetic case can be treated as a "toy-model" for the linearized gravity. Next, the scalar Ψ description of gravito-electromagnetic formulation of linearized gravity is presented (appendix D.3). In analogy to electromagnetism, we generalize the gravitational hopfion to higher multipole solution (3.66). We propose the notion of helicity for linearized gravity h_{GE} (3.113) and h_{GB} (3.113). The properties of gravitational helicities in terms of the scalar Ψ are similar to electromagnetic ones. We compare gravitational quasi-local densities for quadrupole solution (3.99). The results are presented and discussed in section 3.4.4.
The structure of the theory for electromagnetism and linearized gravity can be illustrated on the diagram 3.1:

We would like to point out the following:

1. Spin-2 field theory (see appendix C.2) starts with linearized Weyl tensor as a primary object and Bianchi identities play a role of evolution equations. Theory of linearized gravity has a richer structure. It contains "potentials" for curvature tensors: metric, momenta and their dual counterparts. See rhs of diagram 3.1. That simple observation has consequences for (non-)locality of densities of energy and helicity.

2. The energy functional for Maxwell theory is constructed from electromagnetic vector fields **E** and **B**. The energy density is local at a point in terms of **E** and **B**. In the case of linearized gravity the Hamiltonian energy density (see (3.80)) becomes local

[18] The scalar Φ for hopfions is equal to $\frac{RY_1}{(R^2-(t-\imath)^2)^2}$. The type of hopfion, namely null or non-null (see [65]) is related to the order of the dipole.

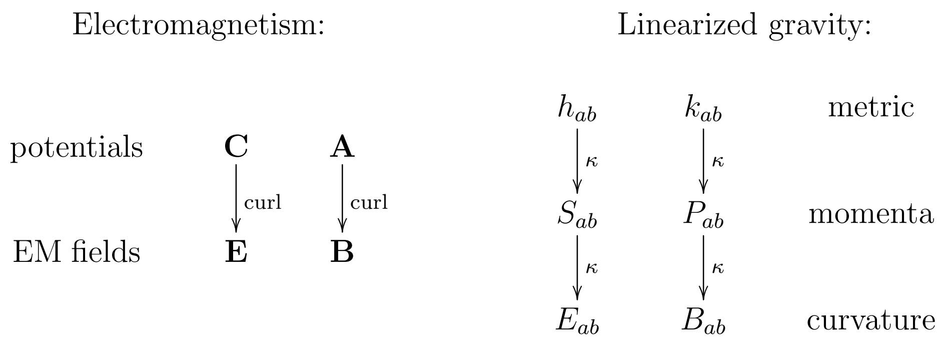

Diagram 3.1: Comparison of the structures of electromagnetism and linearized gravity. Where h_{ab} and k_{ab} are respectively the linearized metric and its dual companion. κ is the first order differential operator. For transverse-traceless gauge, κ is simply the symmetric curl operator for symmetric tensors.

as a combination of the metric and curvature. However, in terms of spin-2 field, the energy functional (3.80) contains non-local $Z^{kl}(-\triangle)^{-1}Z_{kl}$ term. More precisely, the object $(-\triangle)^{-1}Z_{kl} = (\kappa^{-1})^2 Z_{kl}$ is locally related to a combination of metrics h_{ab} and k_{ab} (see rhs of diagram 3.1). Another form of the energy functional ($(\kappa^{-1}Z_{kl})^2$ – square of momenta) contains non-local integral operator κ^{-1} which is responsible for the non-locality of energy density described by (3.80).

3. For helicity of linearized gravity the similar problems occur like for energy. The natural objects for helicity functional to be local are metric and momenta.

The precise description of κ operator and the structure of linearized gravity is presented in [36].

Chapter 4

Magic Hopfions

4.1 Introduction and historical review

4.1.1 Origins of Magic Field: Kerr–Newman spacetime

Kerr–Newman spacetime is a solution of Einstein–Maxwell equations, sub-case of Plebański–Demiański generalized black hole, see (2.45). It describes a spacetime region in the neighborhood of massive, charged and rotating body. In Boyer–Lindquist coordinates, the spacetime metric is given by

$$\begin{aligned} g_{KN} &= \Sigma\left(\frac{1}{\widetilde{\Delta}}\mathrm{d}r^2 + \mathrm{d}\theta^2\right) + \frac{\sin^2\theta}{\Sigma}\left(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\varphi\right)^2 \\ &\quad -\frac{\widetilde{\Delta}}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi\right)^2 , \end{aligned} \tag{4.1}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta \,, \tag{4.2}$$

$$\widetilde{\Delta} = (r^2+a^2) - 2mr + e^2 \,, \tag{4.3}$$

with $t \in \mathbb{R}$, $r \in \mathbb{R}$, and θ, φ being the standard coordinates parameterizing a two-dimensional spheroid. We will keep away from zeros of Σ and Δ, and ignore the coordinate singularities $\sin\theta = 0$. This metric describes a rotating object of mass m, charge e and angular momentum $J = ma$. The advantage of the above coordinates is that for r much grater than m and a the metric becomes asymptotically flat, i.e. $g \approx -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)$.

Electromagnetic four-potential associated with Kerr–Newman solution reads

$$A_{KN} = \frac{q\,r}{\Sigma}\left(dt - a\,\sin^2\theta\,d\varphi\right). \tag{4.4}$$

Magic field is the limit of Kerr-Newman electromagnetic field with the mass parameter M goes to zero. In this case the limit is trivial

$$A_{MF} = A_{KN} = \frac{q\,r}{\Sigma}\left(dt - a\,\sin^2\theta\,d\varphi\right). \tag{4.5}$$

Simultaneously, we neglect the terms proportional to the square of electric charge in the Kerr-Newman metric and then we pass to $m \to 0$ limit. In this way, Minkowski metric in oblate spheroidal coordinates is obtained

$$_{\mathrm{S}}g = -dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \Delta \sin^2\theta d\varphi^2 \,, \tag{4.6}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta \,, \tag{4.7}$$

$$\Delta = (r^2 + a^2) \,. \tag{4.8}$$

A transformation between oblate spheroidal coordinates (t, r, θ, φ) and spherical ones (t, R, Θ, φ) is two-dimensional between (r, θ) and (R, Θ). t and φ remains unchanged. The coordinate surfaces of constant r, θ are surfaces of revolution (see figure 4.1). For r-constant, we receive oblate spheroids. Hyperboloids of one sheet are obtained for θ-constant.

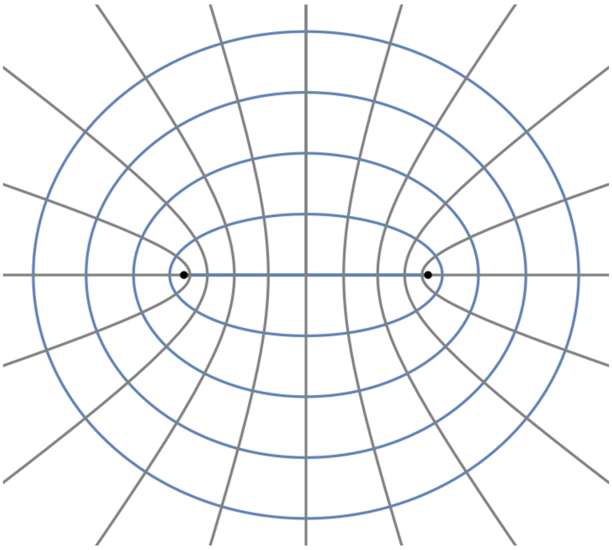

Figure 4.1: The oblate spheroidal coordinate surfaces intersected with $x = 0$ plane. r-const (blue) and θ-const (gray).

An explicit form of coordinate transformation is given in appendix A.5. From now, we consider Minkowski spacetime only.

4.1.2 Properties of Magic Field

The electromagnetic potential 1-form for Magic Field (4.5)

$$A_{MF} = \frac{qr}{\Sigma}(dt - a\sin^2\theta d\varphi) \,,$$

satisfies $A^0_{,0} = A^k_{,k} = 0$ gauge. Maxwell field two-form $F_{MF} = dA_{MF}$ reads

$$\begin{aligned} F_{MF} &= \frac{q}{\Sigma^2}\big[(r^2 - a^2\cos^2\theta)\, dt \wedge dr - a^2 r \sin 2\theta \, dt \wedge d\theta \\ &\quad + a(r^2 - a^2\cos^2\theta)\sin^2\theta dr \wedge d\varphi - ar(r^2 + a^2)\sin 2\theta d\theta \wedge d\varphi\big] \,, \end{aligned} \tag{4.9}$$

and its Hodge dual 2-form

$$\begin{aligned} *F_{MF} &= -\frac{q}{\Sigma^2}\big[2a^2 r \sin^2\theta\cos\theta dr \wedge d\varphi + a(r^2 - a^2\cos^2\theta)\sin\theta\, dt \wedge d\theta \\ &\quad + 2ar\cos\theta\, dt \wedge dr + (r^2 + a^2)(r^2 - a^2\cos^2\theta)\sin\theta d\theta \wedge d\varphi\big] \,. \end{aligned} \tag{4.10}$$

These fields are, except disc at the origin, static solutions of vacuum Maxwell equations $dF_{MF} = 0\,,\; d*F_{MF} = 0\,.$ Magic Field has a compact source[1] on the disc $\{r = 0\}$. Apart of that, we can introduce a potential one-form for $*F_{MF}$, which fulfills $*F_{MF} = dC_{MF}$

$$C_{MF} = \frac{q}{\Sigma}(-a\cos\theta\, dt + (r^2 + a^2)\cos\theta\, d\varphi). \tag{4.11}$$

The potential obeys $C^0_{,0} = C^k_{,k} = 0$ gauge.

Consider a foliation of $\{t = \text{const}\}$ hypersurfaces. For each leaf, vector field $n^\mu \partial_\mu = \partial_t$ is normal to the spacial hypersurfaces and electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields are

$$\mathbf{E}_{MF} \;=\; n\lrcorner F = \frac{q}{\Sigma^2}\left((r^2 - a^2\cos^2\theta)\, dr - a^2 r \sin 2\theta\, d\theta\right)\,, \tag{4.12}$$

$$\mathbf{B}_{MF} \;=\; n\lrcorner(*F) = \frac{-q}{\Sigma^2}\left(2ar\cos\theta\, dr + a(r^2 - a^2\cos^2\theta)\sin\theta\, d\theta\right)\,, \tag{4.13}$$

where $\lrcorner$ is the interior product. The analysis below mainly follow Lynden-Bell [44, 45]. For convenience, we use a complex representation of electromagnetic field with the help of Riemann–Silberstein vector

$$\mathbf{Z}_{MF} = \mathbf{E}_{MF} + \imath\mathbf{B}_{MF}\,. \tag{4.14}$$

Alternatively, both the electrostatic potential, κ, and the magnetostatic potential, χ, can be introduced. A complex representation $\Xi = \Phi + \imath\chi$ for Magic Field fulfills

$$\mathbf{Z}_{MF} = -\boldsymbol{\nabla}\Xi_{MF}\,. \tag{4.15}$$

It is convenient to perform further analysis in cylindrical coordinates $\{\widehat{R}, \varphi, z\}$. Away from charges and currents, the exact form of Ξ_{MF} reads

$$\Xi_{MF} = \frac{q}{\sqrt{\widehat{R}^2 + (z - \imath a)^2}}\,, \tag{4.16}$$

which is harmonic (except at singularities and branch points)

$$\triangle\Xi = 0\,. \tag{4.17}$$

The singularities lie at $\widehat{R} = a$ and $z = 0$. The complex function (4.16) is multi-valued in general. The cut on the disk $z = 0, \widehat{R} \le a$ make it well-defined. On the cut itself we have for $\widehat{R} < a$ and $z \to 0^+$

$$\mathbf{Z}_{MF}|_{z\to 0^+} = -\frac{q}{\left(a^2 - \widehat{R}^2\right)^{3/2}}\left(a\partial_z + i\widehat{R}\partial_{\widehat{R}}\right)\,. \tag{4.18}$$

Notice that $\mathbf{E}$ is orthogonal to the disk so the disk is an equipotential and indeed its potential is zero (earthed) as may be seen by taking the real part of (4.16) on $z = 0$ with

[1] See (4.19) and the comments below. Precise, formal approach to the source in terms of Functional Analysis (theory of distributions) is given in [39].

$\widehat{R}^2 < a^2$. The magnetic field lies parallel to the radius below the disk and antiparallel above as though the disk has a Meissner effect[2]. It does not cross the disk except at the singular ring. For $\widehat{R} > a$ and $z = 0, \mathbf{B}$ points downwards everywhere. Thus every field line returns to the upper hemisphere through the singular ring. From the discontinuous change of electric field, one can obtain charge density on the symmetry plane

$$\sigma_{MF} = -\frac{qa}{2\pi\left(a^2 - \widehat{R}^2\right)^{3/2}} \quad \text{for} \quad \widehat{R} < a\,. \tag{4.19}$$

This charge density gives a divergent total charge but that divergence is "compensated" by a ring singularity of opposite charge. The formal analysis of the source with the help of Functional Analysis (theory of distributions) has been done by Gerald Kaiser in [39]. The total charge in a cylinder with a radius less than $\widehat{R}$ is

$$Q_{MF}(\widehat{R}) = -q\left(\frac{a - \sqrt{a^2 - \widehat{R}^2}}{\sqrt{a^2 - \widehat{R}^2}}\right) \quad \text{for} \quad \widehat{R} < a\,. \tag{4.20}$$

However, the Gauss law for closed two-dimensional surface which surrounds the disc gives

$$Q_{MF}^{Total} = q\,. \tag{4.21}$$

From the discontinuity in the $\mathbf{B}$ field across the cut we find that the φ-component of the surface current is non-vanishing

$$J_{MF}^{\varphi} = -\frac{q}{2\pi}\frac{\widehat{R}}{\left(a^2 - \widehat{R}^2\right)^{3/2}}\,. \tag{4.22}$$

This corresponds to the charge density given above rotating with angular velocity $\Omega = 1/a$, reaching the velocity of light at the singularity. The total current within $\widehat{R} < a$ is $Q(\widehat{R})\Omega/(2\pi)$. Again the magnetic effects of this current are overwhelmed by the current around the singular ring which is of opposite sign. The fields are illustrated in Figures 4.2 and 4.3, generated together with Sajad Aghapour.

We now list other properties of this Magic electromagnetic field. (For proofs see Lynden-Bell [44].)

1. Relativistic Invariants are

$$\mathbf{Z}_{MF}^2 = \mathbf{E}_{MF}^2 - \mathbf{B}_{MF}^2 + 2\imath\mathbf{E}_{MF}\cdot\mathbf{B}_{MF} = \frac{q^2}{\widehat{R}(\widehat{R}^2 + (z - \imath a)^2)^2}\,. \tag{4.23}$$

2. $\mathbf{E}^2 = \mathbf{B}^2$ only on two spheres of radius $\sqrt{2}a$ centered on $z = \pm a$. They intersect on the singular ring.

3. $\mathbf{E}\cdot\mathbf{B} = 0$ on the sphere $\widehat{R} = a$ and also on the plane $z = 0$.

[2]The Meissner effect (or Meissner–Ochsenfeld effect) is the expulsion of a magnetic field from a body. Usually observed in a superconductors.

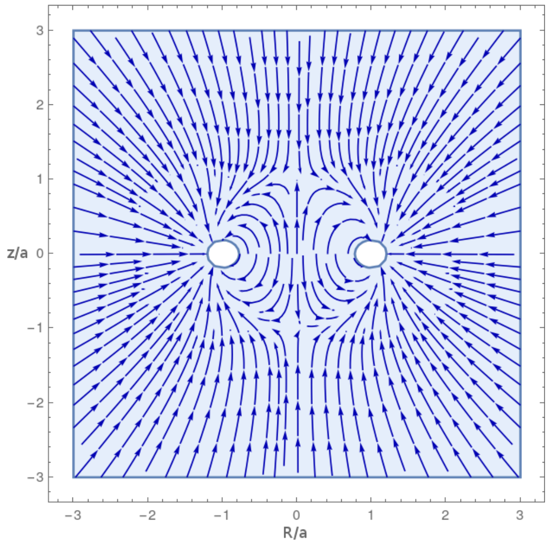

Figure 4.2: Electric field lines on $x = 0$ surface for $q < 0, a > 0$.

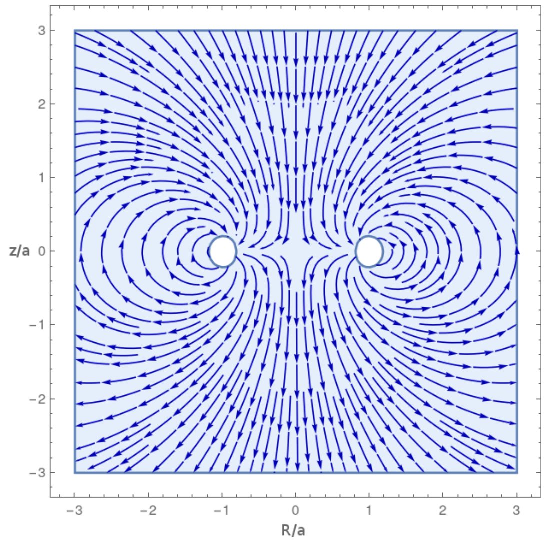

Figure 4.3: Magnetic field lines on $x = 0$ surface for $q < 0, a > 0$.

4. The field energy density reads

$$\mathcal{E}_{MF} = \frac{1}{8\pi}\mathbf{Z}_{MF} \cdot \bar{\mathbf{Z}}_{MF} = \frac{q^2}{8\pi}\frac{\widehat{R}^2 + a^2}{\left[\widehat{R}^2 + (z - \imath a)^2\right]^3} . \tag{4.24}$$

This diverges like $(\widehat{R} - a)^{-3}$ when $z = 0$ near $\widehat{R} = a$.

5. The Poynting vector is given as

$$\mathbf{P}_{MF} = \frac{1}{2\imath}\bar{\mathbf{Z}}_{MF} \times \mathbf{Z}_{MF} = \frac{q^2 a}{\left[\widehat{R}^2 + (z - \imath a)^2\right]^3}\partial_z , \tag{4.25}$$

likewise diverges.

6. The total field energy and the total angular momentum, both diverge due to their divergence at the singular ring.

4.1.3 Description of electromagnetism in Newmann-Penrose formalism

Since the inception, in 1915, of Einstein equations for General Relativity, there has been a variety of different (physically and mathematically equivalent) ways of describing them. They include the standard coordinate-basis version using the metric tensor components as the basic variable and the Christoffel symbols for the connection, the methods of Cartan using differential forms, the space-time (orthonormal) tetrad version and the spin-coefficient, Newman-Penrose version. See [14] and references therein. Though all versions have significant domains of useful applicability, one of the most significant is the tetrad formalism proposed by Ezra Ted Newman and Roger Penrose. The Newman-Penrose (N–P) formalism is a tetrad formalism with a special choice of the basis null-vectors. The novelty of the formalism, when it was first proposed by Newman and Penrose in 1962, was precisely in their choice of a null basis: it was a departure from the choice of an orthonormal basis which was customary till then. The underlying motivation for the choice of a null basis was Penrose's strong belief that the essential element of a space-time is its light-cone structure which makes possible the introduction of a spinor basis. We mainly follow the introduction to N–P formalism from Chandrasekhar book [14].

Properties of a null base The choice that is made is a tetrad of null vectors $L^\mu\partial_\mu$, $N^\mu\partial_\mu$, $M^\mu\partial_\mu$, and $\bar{M}^\mu\partial_\mu$ of which $L^\mu\partial_\mu$ and $N^\mu\partial_\mu$ are real and $M^\mu\partial_\mu$ and $\bar{M}^\mu\partial_\mu$ are complex conjugates of one another. We use the following conventions

$$L^\mu N_\mu = -1\,, \quad \bar{M}^\mu M_\mu = 1\,. \tag{4.26}$$

with the all other scalar products being zero. In other words, the metric can be written as

$$g = \left[-2L_{(\mu}N_{\nu)} + 2M_{(\mu}\bar{M}_{\nu)}\right] dx^\mu dx^\nu\,. \tag{4.27}$$

Connection coefficients in a non-holonomic basis Covariant derivative can be defined by specifying how it acts on base vectors. In particular, Christoffel symbols of the second kind are defined as the unique coefficients such that

$$\nabla_\mu\partial_\nu = \Gamma^\kappa_{\mu\nu}\partial_\kappa\,. \tag{4.28}$$

Analog of Christoffel symbols in non-holonomic basis are the Ricci rotation coefficients

$$\omega_{\{cab\}} = \mathbf{e}_{\{c\}}{}^\kappa \nabla_\nu \mathbf{e}_{\{a\}}{}^\mu \mathbf{e}_{\{b\}}{}^\nu g_{\kappa\mu} \tag{4.29}$$

where enclosure in curly brackets distinguishes the tetrad indices from the tensor indices. We have also assumed that at each point of space-time a basis of four tetrad vectors is given

$$\mathbf{e}_{\{a\}}{}^\mu\partial_\mu \quad (a = 1, 2, 3, 4)\,, \tag{4.30}$$

and

$$g_{\mu\nu}\mathbf{e}_{\{a\}}{}^\mu\mathbf{e}_{\{b\}}{}^\nu = \eta_{\{ab\}} \tag{4.31}$$

where $\eta_{\{ab\}}$ is a symmetric matrix, defined by a particular choice of tetrad structure. In the case of N–P tetrad, we have

$$\left[\eta_{\{ab\}}\right] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\,. \tag{4.32}$$

For N–P formalism, Ricci rotation coefficients (4.29) are called spin coefficients $\{\kappa, \kappa', \sigma, \sigma', \tau, \tau', \rho, \rho', \epsilon, \epsilon', \beta, \beta'\}$, defined by

$$\begin{aligned} M^\nu \nabla_\mu L_\nu &= \tau L_\mu + \kappa N_\mu - \rho M_\mu - \sigma \bar{M}_\mu \,, && (4.33) \\ \frac{1}{2}\left(N^\nu \nabla_\mu L_\nu - \bar{M}^\nu \nabla_\mu M_\nu\right) &= -\epsilon' L_\mu + \epsilon N_\mu + \beta' M_\mu - \beta \bar{M}_\mu \,, && (4.34) \\ -\bar{M}^\nu \nabla_\mu N_\nu &= -\kappa' L_\mu - \tau' N_\mu + \sigma' M_\mu + \rho' \bar{M}_\mu \,. && (4.35) \end{aligned}$$

The intrinsic derivative of $X = X^{\{a\}}\mathbf{e}_{\{a\}}$ in the direction $\boldsymbol{e}_{\{b\}}$ has the form

$$X_{\{a\ddagger b\}} = \mathbf{e}^\mu_{\{a\}} \nabla_\nu X_\mu \mathbf{e}^\nu_{\{b\}} \,. \tag{4.36}$$

Maxwell equations In the Newman-Penrose formalism, the antisymmetric Maxwell-tensor, $F_{\mu\nu}$, is replaced by the three complex scalars

$$\begin{aligned} \phi_0 &= F_{\mu\nu} L^\mu M^\nu \,, && (4.37) \\ \phi_1 &= = \frac{1}{2} F_{\mu\nu}\left(L^\mu N^\nu + \bar{M}^\mu M^\nu\right) \,, && (4.38) \\ \phi_2 &= F_{\mu\nu} \bar{M}^\mu N^\nu && (4.39) \end{aligned}$$

Tensorial vacuum Maxwell equations (2.20) in terms of tetrad components and intrinsic derivatives (4.36) read

$$F_{\{ab\ddagger c\}} + F_{\{ca\ddagger b\}} + F_{\{bc\ddagger a\}} = 0 \,, \quad \eta^{\{nm\}} F_{\{an\ddagger m\}} = 0 \,. \tag{4.40}$$

The explicit forms of (4.40) in terms of N-P coefficients read

$$\begin{aligned} D\phi_1 - \delta'\phi_0 &= (\kappa + 2\beta')\phi_0 + 2\rho\phi_1 - \kappa\phi_2 \,, && (4.41) \\ D\phi_2 - \delta'\phi_1 &= \sigma'\phi_0 - 2\tau'\phi_1 + (\rho - 2\varepsilon)\phi_2 \,, && (4.42) \\ \delta\phi_1 - D'\phi_0 &= (2\epsilon' - \rho')\phi_0 + 2\tau\phi_1 - \sigma\phi_2 \,, && (4.43) \\ \delta\phi_2 - D'\phi_1 &= \kappa'\phi_0 - 2\rho'\phi_1 + (\tau - 2\beta)\phi_2 \,, && (4.44) \end{aligned}$$

where

$$D = L^\mu \nabla_\mu \,, \quad D' = N^\nu \nabla_\nu \,, \quad \delta = M^\mu \nabla_\mu \,, \quad \delta' = \bar{M}^\nu \nabla_\nu \,. \tag{4.45}$$

4.2 Origins of Magic Hopfions: complex shift in spacetime direction

In this section, we generalize the idea of imaginary shift in time for fundamental solution from Appendix B in our paper [58].

According to Theorem 2.3.1, see (2.61), electromagnetic field can be encoded with the help of CYK tensors in a single scalar field Φ. As we analyzed in section 3.3.1, the reduced data Φ in the case of Minkowski spacetime fulfills

$$\Box \Phi = 0 \,. \tag{4.46}$$

We wish to present that the imaginary shift in time direction is essential to generate Hopfions. Thus, let us consider four-dimensional Laplace equation in the four-dimensional Euclidean space

$$\overset{(4)}{\triangle} f(\overset{(4)}{\mathbf{x}}) = -4\pi^2\delta(\overset{(4)}{\mathbf{x}}) \,, \tag{4.47}$$

where $\delta(x)$ is the four-dimensional Dirac delta. We focus on the following solution of (4.47) given in the Cartesian coordinates:

$$f(\overset{(4)}{\mathbf{x}}) = \frac{1}{(x^0)^2 + ||\mathbf{r}||^2} \,, \tag{4.48}$$

where $||\mathbf{r}|| = r = \sqrt{\sum_{i=1}^{3} x_i^2}$. It is called the fundamental solution for four-dimensional Laplacian. The solution (4.48) can be extended analytically on Minkowski spacetime by the transformation

$$x^0 = \imath t \,. \tag{4.49}$$

We receive

$$\widetilde{f}(t, R) = \frac{1}{R^2 - (t)^2} \,, \tag{4.50}$$

which fulfills wave equation on Minkowski background. The pre-Hopfion solution, called Synge function, is a representative of Hopfion class of solutions Φ_H (3.40) for $l = 0$,

$$\Phi_0 = \frac{1}{R^2 - (t - \imath)^2} \,, \tag{4.51}$$

is obtained by imaginary shift in time-like direction

$$t \to t - \imath \,, \tag{4.52}$$

applied for (4.50). The imaginary shift has been successfully used many times. One of the most significant result obtained in this way is the Kerr–Newman metric. Kerr–Newman black hole is generated from Reissner–Nordström metric with the help of Newman–Janis algorithm [21]. Analogically, Magic Field solution, described in section 4.1.2 (see potential (4.16)), can be obtained by an imaginary shift in spatial direction $z \to z - \imath a$, applied to Columb field potential

$$V_c(\overset{(3)}{\mathbf{x}}) = \frac{1}{||\overset{(3)}{\mathbf{x}}||} \,. \tag{4.53}$$

Note that The Culomb field potential is the fundamental solution of three-dimensional Laplace equation in Euclidean space

$$\overset{(3)}{\triangle} V_c(\overset{(3)}{\mathbf{x}}) = -4\pi\delta(\overset{(3)}{\mathbf{x}}) \,, \tag{4.54}$$

where $\delta(x)$ is the three-dimensional Dirac delta.

Generation of Hopfions with the help of $\mathcal{S}$ operator

The aim of this section is to find the first order operator $\mathcal{S}$ which enables one to generate Hopfion family of solutions from pre-Hopfion solution (4.51).

In section 3.3.1, equation (3.40), class of Hopfion solutions is given. For clarity of exposition, we highlight l-pole Hopfion solution by the following denoting

$$ {}_{\text{H}}\Phi^{l} = \frac{R^{l} Y_{l}}{\left[(R^{2} - (t + \imath\alpha)^{2})\right]^{l+1}} . \tag{4.55} $$

where Y^l is a multipole of l-th order and maximal degree $m = \pm l$.

Moreover, ${}_{\text{H}}\Phi^{l}$ has a separable structure, namely

$$ {}_{\text{H}}\Phi^{l} = {}_{\text{H}}\phi^{l}(t, r) Y^{l} . \tag{4.56} $$

It is trivial from the point of view of multipole decomposition on the sphere. Only one l-pole remains in such decomposition. It leads to conclusions that the generating operator $\mathcal{S}$ has the following properties

1. $\mathcal{S}$ maps a multipole of l-th order and maximal degree $m = \pm l$ into a multipole of (l+1)-th order. In particular for Hopfion class of solutions, we have

$$ \mathcal{S}\, {}_{\text{H}}\Phi^{l} = {}_{\text{H}}\Phi^{l+1} . \tag{4.57} $$

2. $\mathcal{S}$ commutes with d'Alembert operator for any arbitrary smooth function $F(t, R, \theta, \varphi)$

$$ (\Box\, \mathcal{S} - \mathcal{S}\, \Box) F(t, r, \theta, \varphi) = 0 \tag{4.58} $$

Let $S_{\Box}(M)$ is the space of solutions of wave equation (4.46) on Minkowski spacetime. The second condition means that $\mathcal{S}$ is an endomorphism in $S_{\Box}(M)$. It guarantees that $\mathcal{S}\Phi$ is a good reduced data ($\mathcal{S}\Phi \in S_{\Box}(M)$) of Maxwell field if $\Phi \in S_{\Box}(M)$.

We have found a particular example of $\mathcal{S}$ with the help of the following observation. Consider l-pole Hopfion solution with a particular choice of spherical harmonics representation in Cartesian coordinates

$$ {}_{\text{H}}\Phi^{l} = \frac{(x + \imath y)^{l}}{(x^{2} + y^{2} + z^{2} - (t + \imath\alpha)^{2})^{l+1}} . \tag{4.59} $$

Using (4.57) and (4.59), we have

$$ \mathcal{S}\left(\ln {}_{\text{H}}\Phi^{l}\right) = \frac{{}_{\text{H}}\Phi^{l+1}}{{}_{\text{H}}\Phi^{l}} = \frac{(x + \imath y)}{(x^{2} + y^{2} + z^{2} - (t + \imath\alpha)^{2})} . \tag{4.60} $$

Let us observe that

$$ \mathcal{S}\left(\ln {}_{\text{H}}\Phi^{l}\right) = \mathcal{S}\left[l\left(\ln {}_{\text{H}}\Phi^{1} - \ln {}_{\text{H}}\Phi^{0}\right) + \ln {}_{\text{H}}\Phi^{0}\right] = \mathcal{S} \ln {}_{\text{H}}\Phi^{0} , \tag{4.61} $$

and

$$\frac{(x+\imath y)}{(x^2+y^2+z^2-(t+\imath\alpha)^2)} = \frac{1}{2}\left(\partial_x + \imath\partial_y\right)\ln\frac{1}{{}_{\text{H}}\Phi^0}\,. \tag{4.62}$$

Comparing (4.60), (4.61) and (4.62), we find

$$\left[\mathcal{S} + \frac{1}{2}\left(\partial_x + \imath\partial_y\right)\right]\ln\,{}_{\text{H}}\Phi^0 = 0\,. \tag{4.63}$$

That means

$$\mathcal{S} = -\frac{1}{2}\left(\partial_x + \imath\partial_y\right)\,. \tag{4.64}$$

In spherical coordinates, it has a form

$$\mathcal{S} = -\frac{1}{2}e^{\imath\varphi}\left[\sin\Theta\partial_R + \frac{\cos\Theta}{R}\partial_\Theta + \frac{\imath}{R\sin\Theta}\partial_\varphi\right]\,. \tag{4.65}$$

In appendix A.5, $\mathcal{S}$ in oblate spheroidal coordinates is given. Acting $\mathcal{S}$ l-times on pre-Hopfion (4.51), we have

$${}_{\text{H}}\Phi^l = \mathcal{S}^l\,{}_{\text{H}}\Phi^0\,, \tag{4.66}$$

where ${}_{\text{H}}\Phi^l$ from the above equation agrees with (4.55).

4.2.1 Pre-Magic Hopfion: fundamental solution with complex spacetime shift

Hopfions can be generated from fundamental solution of wave equation (4.50) via complex shift in time-like direction. We wish to generalize the construction by implementing a complex shift in arbitrary spacetime direction.

Without loss of generality, we can set a coordinate system in such a way, that the spatial part of the shift is performed in z direction. In oder words, an arbitrary complex shift in spacetime direction is given in Cartesian coordinates as

$$t \;\rightarrow\; t + \imath\alpha \tag{4.67}$$

$$z \;\rightarrow\; z + \imath a \tag{4.68}$$

Applying the above transformation into fundamental solution of wave equation (4.50), in Cartesian coordinates, we receive unified generalization of Magic Field and Hopfions

$${}_{\text{MH}}\Phi^0 = \frac{1}{x^2+y^2+(z+\imath a)^2-(t+\imath\alpha)^2}\,. \tag{4.69}$$

It is convenient to analyze the shift in oblate spheroidal coordinates (t, r, θ, φ), see appendix A.5,

$${}_{\text{MH}}\Phi^0 = \frac{1}{(r+\imath a\cos\theta)^2-(t+\imath\alpha)^2}\,. \tag{4.70}$$

We call it pre-Magic Hopfion solution. Acting generating operator $\mathcal{S}$, see (A.16), l times we find l-th order Magic Hopfion

$${}_{\text{MH}}\Phi^l = \frac{(a^2+r^2)^{\frac{l}{2}}\sin^l\theta e^{\imath l\varphi}}{\left((r+\imath a\cos\theta)^2-(t+\imath\alpha)^2\right)^{l+1}}\,. \tag{4.71}$$

For $a \rightarrow 0$, one receives l-pole hopfion solution from (4.71).

4.2.2 Reconstruction of Maxwell tensor from Magic Hopfion reduced data

Reconstruction method of electromagnetic field, described in section 3.3 together with appendices A and B, is effective for reduced data which has simple multipole structure. For pre-Magic Hopfion reduced data (4.71), we use an alternative method of reconstruction whic is based on N–P formalism. Φ is closely related with Newman- Penrose electromagnetic scalar ϕ_1, see (4.94).

Let us chose the following tetrad[3] in oblate spheroidal coordinates:

$$L^\mu \partial_\mu = \left[1, 1, 0, \frac{a}{\Delta}\right] , \tag{4.72}$$

$$N^\mu \partial_\mu = \frac{1}{2\Sigma} [\Delta, -\Delta, 0, a] , \tag{4.73}$$

$$M^\mu \partial_\mu = \frac{1}{\sqrt{2}\varrho} \left[\imath a \sin\theta, 0, 1, \frac{\mathrm{i}}{\sin\theta}\right] , \tag{4.74}$$

where $\varrho := r - \imath a \cos\theta$, $\Delta = r^2 + a^2$ and $\Sigma = r^2 + a^2 \cos^2\theta$. The nonzero N–P spin coefficients are listed below

$$\tau' = -\frac{i}{\sqrt{2}} \frac{a \sin\theta}{\varrho^2} , \tag{4.75}$$

$$\rho' = \frac{1}{2} \frac{\Delta}{\Sigma\varrho} , \tag{4.76}$$

$$\tau = -\frac{i}{\sqrt{2}} \frac{a \sin\theta}{\Sigma} , \tag{4.77}$$

$$\rho = -\frac{1}{\varrho} , \tag{4.78}$$

$$\epsilon' = \rho' - \frac{1}{2} \frac{r}{\Sigma} , \tag{4.79}$$

$$\beta = \frac{1}{2\sqrt{2}} \frac{\cot\theta}{\bar{\varrho}} , \tag{4.80}$$

$$\beta' = \tau' + \bar{\beta} . \tag{4.81}$$

Maxwell equations in N–P formalism (4.41)-(4.44), appropriate for Magic Hopfion geometry, in oblate spheroidal coordinates take the form

$$\frac{1}{\varrho\sqrt{2}} \left(\mathscr{L}_1 - \frac{ia \sin\theta}{\varrho}\right) \phi_0 = + \left(\mathscr{D}_0 + \frac{2}{\varrho}\right) \phi_1, \tag{4.82}$$

$$\frac{1}{\varrho\sqrt{2}} \left(\mathscr{L}_0 + \frac{2ia \sin\theta}{\varrho}\right) \phi_1 = + \left(\mathscr{D}_0 + \frac{1}{\varrho}\right) \phi_2, \tag{4.83}$$

$$\frac{1}{\bar{\varrho}\sqrt{2}} \left(\mathscr{L}_1^\dagger + \frac{ia \sin\theta}{\varrho}\right) \phi_2 = -\frac{\Delta}{2\rho^2} \left(\mathscr{D}_0^\dagger + \frac{2}{\varrho}\right) \phi_1, \tag{4.84}$$

$$\frac{1}{\bar{\varrho}\sqrt{2}} \left(\mathscr{L}_0^\dagger + \frac{2ia \sin\theta}{\varrho}\right) \phi_1 = -\frac{\Delta}{2\rho^2} \left(\mathscr{D}_1^\dagger - \frac{1}{\varrho}\right) \phi_0 , \tag{4.85}$$

where

$$\mathscr{D}_n = \partial_t + \partial_r + \frac{a}{\Delta} \partial_\varphi + 2n \frac{r}{\Delta} , \tag{4.86}$$

[3] This is the Kinnersley's tetrad for Kerr spacetime in Boyer-Lindquist coordinates with the limit the Kerr mass parameter $m \to 0$.

$$\mathscr{D}_n^\dagger = -\partial_t + \partial_r - \frac{a}{\Delta}\partial_\varphi + 2n\frac{r}{\Delta}\,, \tag{4.87}$$
$$\mathscr{L}_n = \partial_\theta - \imath a\sin\theta\partial_t - \imath a\csc\theta\partial_\varphi + n\cot\theta\,, \tag{4.88}$$
$$\mathscr{L}_n^\dagger = \partial_\theta + \imath a\sin\theta\partial_t + \imath a\csc\theta\partial_\varphi + n\cot\theta\,. \tag{4.89}$$

For scalars, the relations between the above operators and tetrad directional derivatives (4.45) are

$$D = L^\mu\partial_\mu = \mathscr{D}_0\,, \tag{4.90}$$
$$D' = N^\nu\partial_\nu = -\frac{\Delta}{2\Sigma}\mathscr{D}_0^\dagger\,, \tag{4.91}$$
$$\delta = M^\mu\partial_\mu = \frac{1}{\sqrt{2}\bar\varrho}\mathscr{L}_0^\dagger\,, \tag{4.92}$$
$$\delta' = \bar M^\nu\partial_\nu = \frac{1}{\sqrt{2}\varrho}\mathscr{L}_0\,. \tag{4.93}$$

The reduced data Φ, see the comment above (4.46), is proportional to the middle Newman-Penrose electromagnetic scalar

$$\bar\Phi = \sqrt{2}\varrho\phi_1\,. \tag{4.94}$$

For convenience, we use modified Newman-Penrose scalars

$$\overset{(0)}{\Phi} = \phi_0,\quad \overset{(1)}{\Phi} = \bar\Phi,\quad \overset{(2)}{\Phi} = 2\varrho^2\phi_2\,. \tag{4.95}$$

To avoid notational misunderstandings, we consequently use Φ instead of $\overset{(1)}{\Phi}$. Maxwell equations in terms of modified N-P scalars take more symmetric form

$$\left(\mathscr{L}_1 - \frac{ia\sin\theta}{\varrho}\right)\overset{(0)}{\Phi} = \left(\mathscr{D}_0 + \frac{1}{\varrho}\right)\bar\Phi\,, \tag{4.96}$$
$$\left(\mathscr{L}_0 + \frac{ia\sin\theta}{\varrho}\right)\bar\Phi = \left(\mathscr{D}_0 - \frac{1}{\varrho}\right)\overset{(2)}{\Phi}\,, \tag{4.97}$$
$$\left(\mathscr{L}_1^\dagger - \frac{ia\sin\theta}{\varrho}\right)\overset{(2)}{\Phi} = -\Delta\left(\mathscr{D}_0^\dagger + \frac{1}{\varrho}\right)\bar\Phi\,, \tag{4.98}$$
$$\left(\mathscr{L}_0^\dagger + \frac{ia\sin\theta}{\varrho}\right)\bar\Phi = -\Delta\left(\mathscr{D}_1^\dagger - \frac{1}{\varrho}\right)\overset{(0)}{\Phi}\,. \tag{4.99}$$

Finding a full of N-P scalars for Magic Hopfions is technically challenging. We restrict ourselves to the case $l = 1$ in (4.71). For convenience, we introduce

$$\widetilde{u} = r - \imath a\cos\theta + t - \imath\alpha\,, \tag{4.100}$$
$$\widetilde{v} = r - \imath a\cos\theta - t + \imath\alpha\,. \tag{4.101}$$

The full set of N–P modified scalars reads

$${}_{\text{MH}}\overset{(0)}{\Phi}{}^1 = \frac{(r-\imath a)(1+\cos\theta)e^{-\imath\varphi}}{\sqrt{a^2+r^2}\widetilde{u}^3\widetilde{v}} + c_1{}_{\text{K}}\overset{(0)}{\Phi}\,, \tag{4.102}$$

$$\underset{\text{MH}}{}\Phi^1 = \bar{\Phi} = \frac{\sqrt{a^2+r^2}\sin\theta e^{-\imath\varphi}}{\widetilde{u}^2\widetilde{v}^2}, \tag{4.103}$$

$$\underset{\text{MH}}{}\overset{(2)}{\Phi}{}^1 = \frac{\sqrt{a^2+r^2}(r+\imath a)(1-\cos\theta)e^{-\imath\varphi}}{\widetilde{u}\widetilde{v}^3} + c_2 \underset{\text{K}}{}\overset{(2)}{\Phi}. \tag{4.104}$$

It turns out that $\underset{\text{MH}}{}\overset{(0)}{\Phi}{}^1$ and $\underset{\text{MH}}{}\overset{(2)}{\Phi}{}^1$ are defined up to singular solutions $\underset{\text{K}}{}\overset{(0)}{\Phi}$, $\underset{\text{K}}{}\overset{(2)}{\Phi}$ respectively (see appendix C.2). We set $c_1 = c_2 = 0$ to obtain clasical Hopfions for $a = 0$.

An unique form of Maxwell field is encoded in a triple of N–P electromagnetic scalars (ϕ_0, ϕ_1, ϕ_2)

$$\begin{aligned} F_{\mu\nu} &= 2\left(\phi_1+\bar{\phi}_1\right)N_{[\mu}L_{\nu]} - 2\phi_2 L_{[\mu}M_{\nu]} - 2\bar{\phi}_2 L_{[\mu}\bar{M}_{\nu]} \\ &\quad +2\phi_0\bar{M}_{[\mu}N_{\nu]} + 2\bar{\phi}_0 M_{[\mu}N_{\nu]} + 2\left(\phi_1-\bar{\phi}_1\right)M_{[\mu}\bar{M}_{\nu]}. \end{aligned} \tag{4.105}$$

Using (4.105), we obtain Maxwell two-form associated with the N–P electromagnetic scalars (4.102)-(4.104). *Anti-self dual* form of Maxwell two-form is defined as

$$\mathcal{F} = F + \imath * F. \tag{4.106}$$

Anti-self-dual Maxwell two form for Magic Hopfion reads

$$\begin{aligned} \underset{\text{MH}}{}\mathcal{F} &= \frac{\sqrt{2}\sqrt{r^2+a^2}e^{\imath\varphi}}{\bar{\widetilde{u}}^3\bar{\widetilde{v}}^3}\Big[\Big(-\zeta r + 2\left(a^2+r^2\right)\left(r-(t-\imath(a+\alpha))\cos\theta\right)\Big) \\ &\times\Big(\imath\sin^2\theta \mathrm{d}\theta\wedge\mathrm{d}\varphi - \frac{\sin\theta}{a^2+r^2}\mathrm{d}t\wedge\mathrm{d}r\Big) \\ &+\Big(\zeta\cos\theta - 2\left((t-\imath(a+\alpha))r + a^2\cos\theta\right)\sin^2\theta\Big)\left(\mathrm{d}t\wedge\mathrm{d}\theta + \imath\sin\theta\mathrm{d}r\wedge\mathrm{d}\varphi\right) \\ &+\zeta\Big(\imath\sin\theta\mathrm{d}t\wedge\mathrm{d}\varphi + \frac{r^2+a^2\cos^2\theta}{r^2+a^2}\mathrm{d}r\wedge\mathrm{d}\theta\Big)\Big], \end{aligned} \tag{4.107}$$

where

$$\zeta = r^2 + t^2 - a^2\cos^2\theta - 2rt\cos\theta - \alpha^2 - 2a\alpha + 2\imath(a+\alpha)(r\cos\theta - t). \tag{4.108}$$

4.2.3 $3+1$ decomposition of electromagnetism

Consider a one-parameter family of Cauchy surfaces Σ_t parametrized by time t embedded in Minkowski spacetime. The unit normal vector to the surfaces is denoted by

$$n = \partial_t, \tag{4.109}$$

Magic Hopfions on each Σ_t is described by Riemann–Silberstein vector

$$\underset{\text{MH}}{}\mathbf{Z} = G^{-1}\left(n \lrcorner \underset{\text{MH}}{}\mathcal{F}\right), \tag{4.110}$$

where G is a metric isomorphism related with three-dimensional metric on Σ_t. From (4.107) one obtains

$$\underset{\text{MH}}{}\mathbf{Z} = \frac{\sqrt{2}\sqrt{r^2+a^2}e^{\imath\varphi}}{\bar{\widetilde{u}}^3\bar{\widetilde{v}}^3}\Big[$$

$$
\frac{\big(\zeta r - 2\left(a^2+r^2\right)\left(r-(t-\imath(a+\alpha))\cos\theta\right)\big)\sin\theta}{r^2+a^2\cos^2\theta}\partial_r
+\frac{\zeta\cos\theta - 2\left((t-\imath(a+\alpha))r + a^2\cos\theta\right)\sin^2\theta}{r^2+a^2\cos^2\theta}\partial_\theta
+\frac{\imath\zeta}{(a^2+r^2)\sin\theta}\partial_\varphi\Big] . \tag{4.111}
$$

3-D vector potential in Chandrasekhar–Kendall form (*curl* eigenvector)

Vector potential ${}_{\text{CK}}\mathbf{V}$ fulfills simultaneously the following two conditions

$$
{}_{\text{MH}}\mathbf{Z} = \operatorname{curl} {}_{\text{CK}}\mathbf{V} , \tag{4.112}
$$

$$
{}_{\text{MH}}\mathbf{Z} = {}_{\text{CK}}\lambda \, {}_{\text{CK}}\mathbf{V} . \tag{4.113}
$$

The above equations are an overdetermined system of conditions in general. We check that ${}_{\text{CK}}\mathbf{V}$ can be given in Chandrasekhar–Kendall form. Indeed, the system of six scalar equations is linearly dependent. It can be reduced to three independent equations. We obtain

$$
\frac{1}{{}_{\text{CK}}\lambda} = \Xi(t,r,\theta)\Big[\imath(r\cos\theta + \imath a + \imath\alpha - t)\big(r^2 + 2\imath\cos\theta a r + \imath\cos\theta r\alpha
- rt\cos\theta - a^2\cos\theta^2 - \imath a t - a\alpha\big) - \mu(t)\bar{\bar{u}}^2\bar{\bar{v}}^2\Big] , \tag{4.114}
$$

where a function Ξ is equal to

$$
\Xi(t,r,\theta) = \frac{\bar{\bar{u}}\,\bar{\bar{v}}}{\left(r^2+t^2-\alpha^2-a^2\cos^2\theta - 2rt\cos\theta - 2a\alpha + 2\imath(a+\alpha)(r\cos\theta - t)\right)^2} .
$$

Part of the vector ${}_{\text{MH}}\mathbf{Z}/{}_{\text{CK}}\lambda$, which is proportional to $\mu(t)$, see (4.114), belongs to a kernel of curl operator. Setting

$$
\mu(t) = \frac{1}{4\imath t + 4\alpha} , \tag{4.115}
$$

in (4.114) we finally obtain

$$
{}_{\text{CK}}\lambda = -\frac{4\imath(t-\imath\alpha)}{\bar{\bar{u}}\,\bar{\bar{v}}} . \tag{4.116}
$$

Let us note that (4.116) is related to (4.71) with $l=1$ by

$$
{}_{\text{CK}}\lambda = i\partial_t \ln\left({}_{\text{MH}}\bar{\Phi}^1\right) . \tag{4.117}
$$

Potential in transverse gauge

We transform the three-dimensional vector potential, given by Equations (4.111), (4.113) and (4.116), into potential in transverse (Coulomb) gauge ${}_{Q}\mathbf{V}$, defined as

$$
\operatorname{div} {}_{Q}\mathbf{V} = 0 . \tag{4.118}
$$

The potentials are related by the gauge transformation

$$\underset{Q}{\mathbf{V}} = \underset{\text{CK}}{\mathbf{V}} + \xi^{|k}\partial_k \,, \tag{4.119}$$

where ξ is a gauge function. Chandrasekhar–Kendall potential fulfills

$$\begin{aligned} 0 &= \underset{\text{MH}}{\mathbf{Z}}^k{}_{|k} \\ &= \left(\underset{\text{CK}}{\mathbf{V}}^k \underset{\text{CK}}{\lambda}\right)_{|k} \\ &= \underset{\text{CK}}{\lambda}\underset{\text{CK}}{\mathbf{V}}^k{}_{|k} + \underset{\text{CK}}{\mathbf{V}}^k \underset{\text{CK}}{\lambda}_{|k} \,, \end{aligned} \tag{4.120}$$

where we have used Maxwell equations and (4.113). It leads to

$$\underset{\text{CK}}{\mathbf{V}}^k{}_{|k} = \underset{\text{MH}}{\mathbf{Z}}^k \underset{\text{CK}}{\lambda}^{-1}_{|k} \tag{4.121}$$

Equations (4.111) and (4.116) enables one to obtain from (4.121)

$$\underset{\text{CK}}{\mathbf{V}}^k{}_{|k} = -\frac{\sqrt{2}\imath}{2(t+\imath\alpha)}\underset{\text{MH}}{\bar{\Phi}}{}^1 \tag{4.122}$$

Using (4.118), (4.119) and (4.122), the condition for gauge function (ξ) reads

$$\frac{\sqrt{2}\imath}{2(t+\imath\alpha)}\underset{\text{MH}}{\bar{\Phi}}{}^1 + \triangle(\xi) = 0 \tag{4.123}$$

where $\triangle$ is a three-dimensional Laplace operator. Let us recall that $\underset{\text{MH}}{\bar{\Phi}}{}^1$ fulfills wave equation, we have

$$-\partial_t^2 \underset{\text{MH}}{\bar{\Phi}}{}^1 + \triangle \underset{\text{MH}}{\bar{\Phi}}{}^1 = 0 \tag{4.124}$$

we denote $\underset{\text{MH}}{\bar{\Phi}}{}^1 = \underset{\text{MH}}{\bar{\Phi}}{}^1(t, x^k)$. Integrating the above equation with respect to time parameter two times, we obtain up to integration constants

$$-\underset{\text{MH}}{\bar{\Phi}}{}^1 + \triangle \int_0^t \mathrm{d}\tau \left(\int_0^\tau \underset{\text{MH}}{\bar{\Phi}}{}^1(s, x^k)\,\mathrm{d}s \right) = 0 \tag{4.125}$$

Comparing the above equation with (4.123), one finds

$$\xi = \frac{\sqrt{2}\imath}{2(t+\imath\alpha)} \int_0^t \mathrm{d}\tau \left(\int_0^\tau \underset{\text{MH}}{\bar{\Phi}}{}^1(s, x^k)\,\mathrm{d}s \right) \tag{4.126}$$

The explicit form of ξ reads

$$\begin{aligned} \xi \;=\; & -\frac{\imath\sqrt{2}\sqrt{a^2+r^2}\mathrm{e}^{\imath\varphi}\sin\theta}{8(t+\imath\alpha)(r+\imath a\cos\theta)^3}\Big[\\ & (t-\imath\alpha)\ln\left(\frac{t-\imath\alpha+r+\imath a\cos\theta}{t-\imath\alpha-r-\imath a\cos\theta}\right) - 2(r+\imath a\cos\theta)\Big] \end{aligned} \tag{4.127}$$

Gauge function ξ with (4.119), (4.111) and (4.116) enables one to obtain vector potential in transverse (Coulomb) gauge.

4.3 Basic properties of Magic Hopfions

Helicity

Helicity integrals measure topological properties of field lines. We wish to analyze helicities as functions of parameters (t, α, a). We recall definitions given in section 3.3.3. In particular, we have

$$h_E + h_M = \Re \int_\Sigma \left({}_{\text{MH}}\mathbf{Z} \cdot {}_{\text{CK}}\bar{\mathbf{V}} \right) , \tag{4.128}$$

$$h_E - h_M = \Re \int_\Sigma \left({}_{\text{MH}}\mathbf{Z} \cdot {}_{\text{CK}}\mathbf{V} \right) . \tag{4.129}$$

The second integral contains an integrand $e^{2\imath\varphi}F(t, r, \theta)$, which leads to

$$h_E = h_M . \tag{4.130}$$

The first one, toghether with the above result, gives

$$h_E = h_M = \frac{\pi^2}{8(\alpha - a)^3(a + \alpha)} . \tag{4.131}$$

Both helicities do not depend on time which means field lines do not intersect each other during time evolution. The helicities are not defined for $\alpha = \pm a$. Note that, if α, a are non-zero, the rescaling in t and r enables one to rescale one of the parameters to ± 1[4]. In other words, the topological structure of the Magic Hopfion solution depends on that the parameters α, a are equal to zero or not and on theirs ratio α/a.

Electric charge

For Magic Hopfions Riemann-Silberstein vector has a form

$${}_{\text{MH}}\mathbf{Z}^k \partial_k = e^{\imath\varphi} Z^k(t, r, \theta)\partial_k , \tag{4.132}$$

from which we immediately receive

$$Q(r) = \int_{S_r} {}_{\text{MH}}\mathbf{Z}^k dS_k = 0 , \tag{4.133}$$

for any $r =$ const spheroid. We conclude the charge in the whole space is zero. Magic Hopfion is a purely wave solution which propagates on the light cone.

Energy

Electromagnetic energy density of Magic Hopfion reads

$$\mathcal{E} = \mathcal{E}(t, r, \theta, \alpha, a) = {}_{\text{MH}}\mathbf{Z}^k \, {}_{\text{MH}}\bar{\mathbf{Z}}_k , \tag{4.134}$$

[4]The most convenient way for such analysis is to observe how rescaling is acting on the reduced data (4.71).

which is explicitly equal to

$$\mathcal{E} = \frac{4\left(2a^2 + 2a\alpha + \alpha^2 + r^2 + t^2 - a^2\cos^2\theta - 2rt\cos\theta\right)^2}{\left(a^2\cos^2\theta + 2a\alpha\cos\theta + \alpha^2 + (r-t)^2\right)^3\left(a^2\cos^2\theta - 2a\alpha\cos\theta + \alpha^2 + (r+t)^2\right)^3}\,. \tag{4.135}$$

The energy density is finite at each point of spacetime for $\alpha > a$. If we set $\mathcal{E}(t, r, \theta, 1, 0)$, we obtain the energy density for classical Hopfion (see energy density[5] in [61], p. 3).

4.4 Epilogue

In this chapter, we have obtained an electromagnetic solution, called Magic Hopfion, by a generalization of imaginary shift for any space-time direction, see (4.71), and investigate its basic properties. Preliminary analysis of integral curves, see figures 4.4 and 4.5 do not

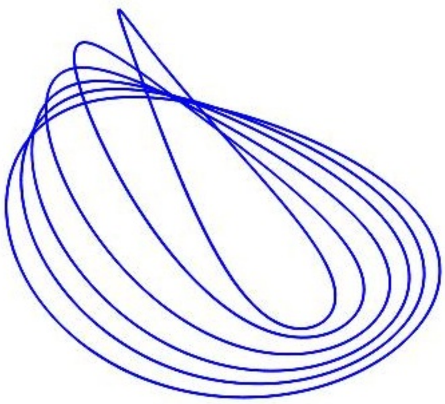

Figure 4.4: Magic Hopfion ($a = 0.1, \alpha = 1, t = 0$)

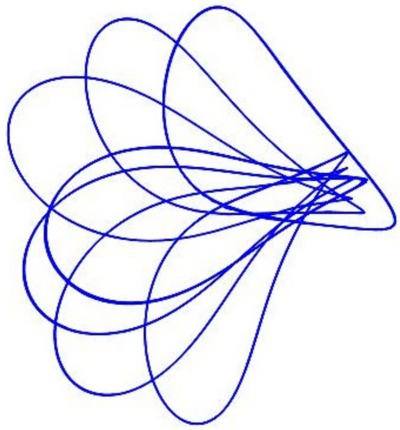

Figure 4.5: Magic Hopfion ($a = 0.5, \alpha = 1, t = 0$)

show significant qualitative differences, comparing to Hopfions, for $a < \alpha$. The research does not end with the end of this chapter. In the future, we plan to continue task from the list below at the University of Warsaw and in friendly research groups abroad. The interesting topics are:

1. Detailed analysis of Magic Hopfions for $a \geq \alpha$. In particular, the case $\alpha = \pm a$ is the most intriguing.

2. Magic Hopfion solution can be also generalized for spin 2 field — weak gravitational wave. In particular, the results in the section 3.4 can be extended.

3. For further analysis of topological structures, the whole helicity array, see [2], for the (Magic) Hopfions solutions may be investigated.

[5]In [61], the solution is rotated around x axis $x \to x, y \to -z, z \to y$. The formulas are equal after applying rotation operator.

Appendix

A Mathematical supplement

A.1 Three-dimensional Laplace operator and its inverse

Consider Laplace equation

$$\triangle G(\mathbf{r}, \mathbf{r}') = -\delta^{(3)}(\mathbf{r} - \mathbf{r}') \,, \tag{A.1}$$

with a solution on an open set without boundary. $\delta^{(3)}(\mathbf{r} - \mathbf{r}')$ is a three-dimensional Dirac delta. $G(\mathbf{r}, \mathbf{r}')$ is the following Green function of (A.1)

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi||\mathbf{r} - \mathbf{r}'||} \,. \tag{A.2}$$

The solution of Poisson equation

$$\triangle u(\mathbf{r}) = -f(\mathbf{r}) \,, \tag{A.3}$$

is the convolution of $f(\mathbf{r})$ and Green function

$$u(\mathbf{r}) = \int_{\Sigma} f(\mathbf{r}')G(\mathbf{r}, \mathbf{r}')\mathrm{d}\mathbf{r}' = \int_{\Sigma} \frac{f(\mathbf{r}')}{4\pi||\mathbf{r} - \mathbf{r}'||}\mathrm{d}\mathbf{r}' \,. \tag{A.4}$$

A.2 Two-dimensional Laplace operator and its inverse

Consider two-dimensional unit sphere in $\mathbb{R}^3$, parameterized by a unit position vector $\mathbf{n}$. One of the main differences is that the domain of the solutions is the compact surface without boundary. The conclusions of the Stokes theorem ($\int_{\mathbb{S}^2} \mathbf{\Delta} u(\mathbf{n}) = 0$) require a modified problem to be examined than in the three-dimensional case. Consider the following two-dimensional Laplace equation with an additional condition

$$\begin{aligned} \mathbf{\Delta G}(\mathbf{n}, \mathbf{n}') &= 1 - \delta^{(2)}(\mathbf{n} - \mathbf{n}') \,, && \text{(A.5)} \\ \int_{\mathbb{S}^2} \sigma \mathbf{G}(\mathbf{n}, \mathbf{n}')\mathrm{d}\mathbf{n}' &= 0 \,, && \text{(A.6)} \end{aligned}$$

where σ is area element on $\mathbb{S}^2$. We have the solution

$$\mathbf{G}(\mathbf{n}, \mathbf{n}') = -\frac{1}{4\pi}\left(\ln\left(\frac{1 - \mathbf{n}\cdot\mathbf{n}'}{2}\right) + 1\right) \,, \tag{A.7}$$

where '$\cdot$' is a scalar product of the position vectors[6]. The solution of the Poisson equation

$$\mathbf{\Delta} s(\mathbf{n}) = -f(\mathbf{n}), \tag{A.8}$$

$$\int_{\mathbb{S}^2} \sigma f(\mathbf{n}) \mathrm{d}\mathbf{n} = 0, \tag{A.9}$$

is the convolution of $f(\mathbf{n}')$ and the Green function

$$u(\mathbf{n}) = \int_{\mathbb{S}^2} f(\mathbf{n}')\mathbf{G}(\mathbf{n},\mathbf{n}')\mathrm{d}\mathbf{n}' = -\int_{\mathbb{S}^2} \frac{1}{4\pi}\left(\ln(\frac{1-\mathbf{n}\cdot\mathbf{n}'}{2}) + 1\right) f(\mathbf{n}')\mathrm{d}\mathbf{n}', \tag{A.10}$$

See [16] and [34] for detailed view, [62] is a specialized literature on the subject.

A.3 Operations on the sphere

Let us denote by $\mathbf{\Delta}$ the Laplace–Beltrami operator associated with the standard metric h_{AB} on S^2. Let SH^l denote the space of spherical harmonics of degree l ($g \in SH^l \iff \mathbf{\Delta} g = -l(l+1)g$). Consider the following sequence

$$V^0 \oplus V^0 \xrightarrow{i_{01}} V^1 \xrightarrow{i_{12}} V^2 \xrightarrow{i_{21}} V^1 \xrightarrow{i_{10}} V^0 \oplus V^0 .$$

Here V^0 is the space of, say, smooth functions on S^2, V^1 – that of smooth covectors on S^2, and V^2 – that of symmetric traceless tensors on S^2. The various mappings above are defined as follows:

$$i_{01}(f,g) = f_{||a} + \varepsilon_a{}^b g_{||b},$$
$$i_{12}(v) = v_{a||b} + v_{b||a} - h_{ab}v^c_{||c},$$
$$i_{21}(\chi) = \chi_a{}^b{}_{||b},$$
$$i_{10}(v) = \left(v^a{}_{||a}, \varepsilon^{ab}v_{a||b}\right),$$

where $||$ is used to denote the covariant derivative with respect to the Levi–Civita connection of the standard metric h_{AB} on S^2. For more details see appendix E in [34].

A.4 Identities on the sphere

We have used the following identities on a sphere

$$-\int_{S(r)} \pi^A v_A = \int_{S(r)} (r\pi^A{}_{||A})\mathbf{\Delta}^{-1}(rv^A{}_{||A}) + \int_{S(r)} (r\pi^{A||B}\varepsilon_{AB})\mathbf{\Delta}^{-1}(rv_{A||B}\varepsilon^{AB}), \tag{A.11}$$

and similarly for the traceless tensors we have

$$\begin{aligned}\int_{S(r)} \mathring{\pi}^{AB}\, \mathring{v}_{AB} &= 2\int_{S(r)} (r^2\varepsilon^{AC}\, \mathring{\pi}_A{}^B{}_{||BC})\mathbf{\Delta}^{-1}(\mathbf{\Delta}+2)^{-1}(r^2\varepsilon^{AC}\, \mathring{v}_A{}^B{}_{||BC}) \\ &+2\int_{S(r)} (r^2\, \mathring{\pi}^{AB}{}_{||AB})\mathbf{\Delta}^{-1}(\mathbf{\Delta}+2)^{-1}(r^2\, \mathring{v}^{AB}{}_{||AB}).\end{aligned} \tag{A.12}$$

[6]For a given point (θ,φ) in spherical coordinates on the unit sphere, the three-dimensional position vector in the Cartesian embedding is $n = \sin\theta\cos\varphi\partial_x + \sin\theta\sin\varphi\partial_y + \cos\theta\partial_z$. Then we use scalar product with Euclidean metric.

A.5 Minkowski in spheroidal coordinates

The spheroidal coordinates (t, r, θ, φ) (Kerr with $M = 0$) are related with standard spherical coordinates (t, R, Θ, φ) by two-dimensional change of coordinates (t, φ remains unchanged)

$$\begin{aligned} R &= \sqrt{r^2 + a^2 \sin^2 \theta} \\ \sin \Theta &= \frac{\sin \theta \sqrt{1 + \frac{a^2}{r^2}}}{\sqrt{1 + \frac{a^2}{r^2} \sin^2 \theta}} \\ \cos \Theta &= \frac{\cos \theta}{\sqrt{1 + \frac{a^2}{r^2} \sin^2 \theta}} \end{aligned} \tag{A.13}$$

Inverse transformation

$$r = \frac{\sqrt{2R^2 - 2a^2 + 2\sqrt{-4a^2 R^2 \sin^2 \Theta + R^4 + 2a^2 R^2 + a^4}}}{2} \tag{A.14}$$

$$\sin \theta = \frac{\sqrt{2R^2 + 2a^2 - 2\sqrt{-4a^2 R^2 \sin^2 \Theta + R^4 + 2a^2 R^2 + a^4}}}{2a} \tag{A.15}$$

Using (A.13), we obtain $\mathcal{S}$ in spheroidal coordinates

$$\mathcal{S} = \frac{r\sqrt{a^2 + r^2} e^{\imath\varphi} \sin \theta}{r^2 + a^2 \cos^2 \theta} \partial_r + \frac{\sqrt{a^2 + r^2} e^{\imath\varphi} \cos \theta}{r^2 + a^2 \cos^2 \theta} \partial_\theta + \frac{\imath e^{\imath\varphi}}{\sqrt{a^2 + r^2} \sin \theta} \partial_\varphi \,. \tag{A.16}$$

B Scalar representation of electromagnetic field

Let us consider an electromagnetic field on Minkowski background. We present how to describe electromagnetism in terms of complex scalar function Φ. The section is organized as follows: we start with a description of standard electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields with help of complex Riemann–Silberstein vector $\mathbf{Z} = \mathbf{E} + \imath\mathbf{B}$. Then, we decompose $\mathbf{Z}$, in the spherical coordinate system, into radial and angular part. We show that the radial part is sufficient to recover quasi-locally the whole $\mathbf{Z}$ vector.
The vacuum Maxwell equations for electric field vector $\mathbf{E}$, magnetic field $\mathbf{B}$, and vector potential $\mathbf{A}$ are

$$\operatorname{div} \mathbf{E} = 0 \,, \tag{B.1}$$

$$\operatorname{div} \mathbf{B} = 0 \,, \tag{B.2}$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \,, \tag{B.3}$$

$$\operatorname{curl} \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \,, \tag{B.4}$$

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \,. \tag{B.5}$$

If electric field $\mathbf{E}$ is sourceless then additional vector potential can be introduced[7]. It is defined up to a gradient of a function in the following way

$$\mathbf{E} = \operatorname{curl} \mathbf{C} . \tag{B.6}$$

It is convenient to use one complex electromagnetic vector field $\mathbf{Z}$, called Riemann–Silberstein vector, instead of $\mathbf{E}$ and $\mathbf{B}$. $\mathbf{Z}$ is defined as follows

$$\mathbf{Z} = \mathbf{E} + \imath \mathbf{B} , \tag{B.7}$$

where $\imath^2 = -1$. For sake of simplicity, we will use complex vector potential $\mathbf{V}$ instead of $\mathbf{C}$ and $\mathbf{A}$:

$$\mathbf{V} = \mathbf{C} + \imath \mathbf{A} . \tag{B.8}$$

The vacuum Maxwell equations (B.1–B.5) with vector potential $\mathbf{C}$ (B.6) can be written in the form of three complex, differential equations for vector fields

$$\operatorname{div} \mathbf{Z} = 0 , \tag{B.9}$$
$$\operatorname{curl} \mathbf{Z} = \imath \frac{\partial \mathbf{Z}}{\partial t} , \tag{B.10}$$
$$\mathbf{Z} = \operatorname{curl} \mathbf{V} . \tag{B.11}$$

In the next part of the section, we will use spherical coordinate system. Each vector $w = (w^R, w^A)$ can be decomposed into its radial part w^R and two-dimensional angular part w^A. The capital letter index runs angular coordinates.
We split two-dimensional vector into its longitudinal ${w^A}_{||A}$ and transversal part $\varepsilon_{RAB} w^{A||B}$. The Maxwell equations in terms of the decomposition have the form:

$$R Z^R = \Phi , \tag{B.12}$$
$$R^2 Z^A_{||A} = -\partial_R (R\Phi) , \tag{B.13}$$
$$R \varepsilon_{RAB} Z^{A||B} = -\imath \partial_t \Phi , \tag{B.14}$$
$$\mathbf{\Delta} V_R - {V_{C,R}}^{||C} = -\imath \partial_t \Phi , \tag{B.15}$$
$$R \varepsilon_{RAB} V^{A||B} = -\Phi . \tag{B.16}$$

C Alternative descriptions of electromagnetic field

C.1 Clebsch representation and Euler potentials of electromagnetic fields

Clebsch representation is based on canonical form of two-form. If ω is a non-degenerate two-form in a four-dimensional vector space, given in terms of a basis of one-forms $f^a (a = 1, \ldots, 4)$ by

$$\omega = \frac{1}{2} \omega_{ab} f^a \wedge f^b ,$$

[7]We remark that the description of electromagnetism with the help of complex scalar function holds also without the additional potential.

then there is symplectic basis of one-forms $g^a(a = 1, \dots, 4)$, such that ω can be written as

$$\omega = g^1 \wedge g^2 + g^3 \wedge g^4 \,. \tag{C.1}$$

The symplectic basis is not defined uniquely. We wish to present an example of transformation for symplectic basis. Let us consider a Maxwell two-form in the following form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \mathbf{E}_i dx^0 \wedge dx^i - \varepsilon_{ijk} \mathbf{B}_i dx^j \wedge dx^k \,. \tag{C.2}$$

Assuming[8] $\mathbf{E}_1 \neq 0$, one can write equivalently ,

$$F = \left(\mathbf{E}_1 dx^0 + \mathbf{B}_3 dx^2 - \mathbf{B}_2 dx^3\right) \wedge \left(dx^1 + \frac{\mathbf{E}_2}{\mathbf{E}_1} dx^2 + \frac{\mathbf{E}_3}{\mathbf{E}_1} dx^3\right) - \left(\frac{\mathbf{E}\cdot\mathbf{B}}{\mathbf{E}_1}\right) dx^2 \wedge dx^3 \,.$$

It means a good choice of basis one-forms to be

$$g^0 = \mathbf{E}_1 dx^0 + \mathbf{B}_3 dx^2 - \mathbf{B}_2 dx^3 \,, \tag{C.3}$$

$$g^1 = dx^1 + \frac{\mathbf{E}_2}{\mathbf{E}_1} dx^2 + \frac{\mathbf{E}_3}{\mathbf{E}_1} dx^3 \,, \tag{C.4}$$

$$g^2 = -\left(\frac{\mathbf{E}\cdot\mathbf{B}}{\mathbf{E}_1}\right) dx^2 \,, \tag{C.5}$$

$$g^3 = dx^3 \,. \tag{C.6}$$

Hence, F takes the form (C.1). Moreover, if the determinant of the matrix F

$$\det F_{\mu\nu} = (\mathbf{E}\cdot\mathbf{B})^2 \,, \tag{C.7}$$

vanishes, then F is degenerate and becomes a decomposable two-form, i.e. there is a pair of one-forms (a, b) such that

$$F = a \wedge b \,. \tag{C.8}$$

For electromagnetic field, we have two Lorentz invariants

$$-\frac{1}{4} F_{\mu v} F^{\mu\nu} = \frac{\mathbf{E}^2 - \mathbf{B}^2}{2} \,, \tag{C.9}$$

$$-\frac{1}{4} F_{\mu v} * F^{\mu\nu} = \mathbf{E}\cdot\mathbf{B} \,. \tag{C.10}$$

Decomposable two-form means obviously that the Lorentz invariant (C.10) is zero ($g^2 = 0$). Moreover, $*$F is also decomposable. It turns out that a decomposable electromagnetic field can be written locally in terms of canonical variables (q, p) and (v, u) as

$$F = dq \wedge dp \,, \tag{C.11}$$

$$*F = dv \wedge du \,. \tag{C.12}$$

For details see [4], page 9 and references therein. Let us suppose that a decomposable electromagnetic field is given globally by (C.11)-(C.12). Then the magnetic field $\mathbf{B}$ and

[8] If $\mathbf{E}_1 = 0$, one can choose any $\mathbf{E}_i \neq 0$.

the electric field $\mathbf{E}$ can be expressed by the same functions (q,p) and (v,u) in all points of space-time,

$$\mathbf{B} = \nabla p \times \nabla q\,, \tag{C.13}$$
$$\mathbf{E} = \nabla v \times \nabla u\,. \tag{C.14}$$

We wish to analyze the magnetic helicity (electric helicity can be considered in analogous way). If $F = dq \wedge dp$ globally, then the magnetic field can be written as $\mathbf{B} = \nabla p \times \nabla q$. If we additionally assume that the functions p and q are single-valued, a well-defined vector potential $\mathbf{A}$ is given

$$\mathbf{A} = p\nabla q\,. \tag{C.15}$$

Using (C.13) and (C.15), one can check easily that $\mathbf{A}\cdot\mathbf{B} = 0$, so the magnetic helicity is zero. We wish to highlight that contribution to non-zero helicity can be obtained only from points where one of the vector fields, ∇p or ∇q, diverge.

Euler potentials

The Euler potentials [60] of the magnetic field are given by two real functions $\alpha_1(\mathbf{r},t) \in \mathbb{R}$ and $\alpha_2(\mathbf{r},t) \in \mathbb{R}$ such that

$$\mathbf{B} = \nabla\alpha_1 \times \nabla\alpha_2\,. \tag{C.16}$$

By comparing these equations with (C.13), one can see that the Euler potentials are canonical variables of the magnetic field. They provide a Clebsch representation of this field.
Analogically, we can introduce real-valued Euler potentials $\beta_1(\mathbf{r},t)$ and $\beta_2(\mathbf{r},t)$ by the equation

$$\mathbf{E} = \nabla\beta_2 \times \nabla\beta_1\,. \tag{C.17}$$

Euler potentials for electric field can be introduced only in vacuum — the condition $\nabla\cdot\mathbf{E} = 0$ is required.
For convenience, Euler potentials can be encoded in a single scalar function. Since $\zeta(\mathbf{r},t)$ is a complex field, one can define Euler potential for magnetic field as

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{2\pi i}\frac{\nabla\zeta \times \nabla\bar{\zeta}}{(1+\bar{\zeta}\zeta)^2} = \nabla\left(\frac{1}{1+|\zeta|^2}\right) \times \nabla\left(\frac{\arg(\zeta)}{2\pi}\right)\,. \tag{C.18}$$

where modulus of ζ and argument of ζ are given respectively by $|\zeta| = \sqrt{(\Re(\zeta))^2 + (\Im(\zeta))^2}$, and $\arg(\zeta) = \arctan(\Im(\zeta)/\Re(\zeta))$. By comparing with (C.16), we have

$$\alpha_1(\mathbf{r},t) = \frac{1}{1+|\zeta|^2}\,, \tag{C.19}$$
$$\alpha_2(\mathbf{r},t) = \frac{\arg(\zeta)}{2\pi}\,. \tag{C.20}$$

Analogically, for sourceless electric field one can obtain

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi i}\frac{\nabla\bar{\xi} \times \nabla\xi}{(1+\bar{\xi}\xi)^2} = -\nabla\left(\frac{1}{1+|\xi|^2}\right) \times \nabla\left(\frac{\arg(\xi)}{2\pi}\right)\,. \tag{C.21}$$

The relations with real Euler potentials are

$$\beta_1(\mathbf{r},t) = \frac{1}{1+|\xi|^2}, \tag{C.22}$$

$$\beta_2(\mathbf{r},t) = \frac{\arg(\xi)}{2\pi}. \tag{C.23}$$

Moreover, Euler potentials are convenient to describe electromagnetic field lines.

The magnetic lines are given by the equations $\alpha_1 = k_1$ and $\alpha_2 = k_2$. From (C.19)-(C.20), these two real equations can be written as the complex equation

$$\zeta(\mathbf{r},t) = \zeta_0\,, \tag{C.24}$$

so that the complex scalar field ζ gives all the magnetic lines. Analogously, the equation

$$\xi(\mathbf{r},t) = \xi_0\,,$$

gives all the electric lines when the constant ξ_0 is set to take values in the complex plane. Consequently, if an electromagnetic field can be written as functions of the two complex scalar fields ζ and ξ, it allows to study directly the magnetic and electric lines for every time, and moreover define canonical variables and Euler potentials for the magnetic and electric field.
We wish to highlight that the complex scalar fields η and ζ in the equations[9] (3.3) defines Euler potentials (and canonical variables) of the magnetic and the electric field for Hopfions.

C.2 Ambiguities in reconstruction N–P electromagnetic scalars from $\overset{(1)}{\Phi}$

For Maxwell equations (4.96)-(4.99), we investigate well-definiteness of $\overset{(0)}{\Phi}$ and $\overset{(2)}{\Phi}$ when Φ is given. Let us consider a reformulated problem: Consider a solutions of Maxwell equations, $\overset{(0)}{\Phi}$ and $\overset{(2)}{\Phi}$, in the case when $\Phi \equiv 0$ for all points of spacetime. In other words, we solve Maxwell equations (4.96)-(4.99) for $\Phi \equiv 0$. The equations reads

$$\left(\mathscr{L}_1 - \frac{ia\sin\theta}{\varrho}\right) {}_{\kappa}\overset{(0)}{\Phi} = 0\,, \tag{C.25}$$

$$\left(\mathscr{D}_1^\dagger - \frac{1}{\varrho}\right) {}_{\kappa}\overset{(0)}{\Phi} = 0\,, \tag{C.26}$$

$$\left(\mathscr{L}_1^\dagger - \frac{ia\sin\theta}{\varrho}\right) {}_{\kappa}\overset{(2)}{\Phi} = 0\,, \tag{C.27}$$

$$\left(\mathscr{D}_0 - \frac{1}{\varrho}\right) {}_{\kappa}\overset{(2)}{\Phi} = 0\,. \tag{C.28}$$

[9] Compare with (C.18) and (C.18).

Subsystems (C.25), (C.26) and (C.27), (C.28) can be analyzed separately. The family of solutions for a subsystem (C.25), (C.26) is equal to

$$\underset{\kappa}{\overset{(0)}{\Phi}} = \frac{\varrho\beta\left(t+\varrho, \varphi + \operatorname{atan}\left(\frac{r}{a}\right) - \imath \operatorname{atanh}(\cos\theta)\right)}{(a^2+r^2)\sin\theta} \,. \tag{C.29}$$

where $\beta(\cdot,\cdot)$ is an arbitrary differentiable function of two variables. C_1 is a constant. Analogically, the solutions of (C.27), (C.28) read

$$\underset{\kappa}{\overset{(2)}{\Phi}} = \frac{\varrho\gamma\left(t-\varrho, \varphi - \operatorname{atan}\left(\frac{r}{a}\right) + \imath \operatorname{atanh}(\cos\theta)\right)}{\sin\theta} \,. \tag{C.30}$$

where $\gamma(\cdot,\cdot)$ is an arbitrary differentiable function of two variables. C_2 is a constant.

D Scalar description of linearized gravity

D.1 Equivalent definitions of spin-2 field

Let us start with the standard formulation of a spin-2 field $W_{\mu\alpha\nu\beta}$ in the Minkowski spacetime equipped with a flat metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. We consider vacuum case. The field W can be also interpreted as a Weyl tensor for linearized gravity (see [15], [31], [34]).

The following algebraic properties:

$$W_{\mu\alpha\nu\beta} = W_{\nu\beta\mu\alpha} = W_{[\mu\alpha][\nu\beta]} \;,\;\; W_{\mu[\alpha\nu\beta]} = 0 \;,\;\; g^{\mu\nu} W_{\mu\alpha\nu\beta} = 0 \,, \tag{D.1}$$

and Bianchi identities which play a role of field equations

$$\overset{(4)}{\nabla}_{[\lambda} W_{\mu\nu]\alpha\beta} = 0 \,, \tag{D.2}$$

can be used as a definition of spin-2 field W. The $*$–operation defined as

$$({}^*W)_{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu} W^{\mu\nu}{}_{\gamma\delta} \,, \quad (W^*)_{\alpha\beta\gamma\delta} = \frac{1}{2} W_{\alpha\beta}{}^{\mu\nu}\varepsilon_{\mu\nu\gamma\delta} \,,$$

has the following properties:

$$({}^*W^*)_{\alpha\beta\gamma\delta} = \frac{1}{4}\varepsilon_{\alpha\beta\mu\nu} W^{\mu\nu\rho\sigma}\varepsilon_{\rho\sigma\gamma\delta} \,, \quad {}^*W = W^* \;,\;\; {}^*({}^*W) = {}^*W^* = -W \,,$$

where $\varepsilon_{\mu\nu\gamma\delta}$ is a Levi–Civita skew-symmetric tensor[10] and ${}^*W$ is called dual spin-2 field. The above formulae are also valid for general Lorentzian metrics.

[10] Defined in footnote 5.

D.2 Gravito-electric and gravito-magnetic formulation

Following Maartens [46], spin-2 field can be equivalently described in terms of gravito-electric and gravito-magnetic tensors. We perform a $(3+1)$–decomposition of the Weyl tensor. The ten independent components of W split into two three-dimensional symmetric, traceless tensors: the electric part

$$E(X,Y) := W(X,\partial_t,\partial_t,Y)\,, \tag{D.3}$$

and the magnetic part

$$B(X,Y) := {}^*W(X,\partial_t,\partial_t,Y)\,. \tag{D.4}$$

The following relations between W and the three-dimensional tensors hold:

$$W_{0kl0} = E_{kl}\,, \quad W_{0kij} = B_{kl}\varepsilon^l{}_{ij}\,, \quad W_{klmn} = \varepsilon^i{}_{kl}\varepsilon^j{}_{mn}E_{ij}\,. \tag{D.5}$$

The classical formulation of gravito-electromagnetism uses the constraint equations

$$E^{kl}{}_{|l} = 0\,, \tag{D.6}$$
$$B^{kl}{}_{|l} = 0\,, \tag{D.7}$$

and the dynamical equations

$$\partial_t E^{kl} = \varepsilon^{pq(k}B^{l)}{}_{q|p}\,, \tag{D.8}$$
$$\partial_t B^{kl} = -\varepsilon^{pq(k}E^{l)}{}_{q|p}\,, \tag{D.9}$$

where $[\mathrm{curl}\,X]_{ab} := \varepsilon_{cd(a}X_{b)}{}^{d|c}$, is the symmetric curl operator for tensors.

ADM momentum P and the dual counterpart S as "potentials" for Weyl tensor. Analogically to electromagnetic case we introduce potentials for Weyl tensor in gravito-electromagnetic formulation. The potential for gravito-magnetic part is the ADM momentum P. It fulfills

$$B_{ab} = \varepsilon_{cd(a}P_{b)}{}^{d|c}\,. \tag{D.10}$$

The second potential can be introduced for gravito-electrical part

$$E_{ab} = \varepsilon_{cd(a}S_{b)}{}^{d|c}\,. \tag{D.11}$$

The potentials fulfill constraint equations

$$P^{kl}{}_{|l} = 0\,, \tag{D.12}$$
$$S^{kl}{}_{|l} = 0\,. \tag{D.13}$$

It is convenient to use a complex combination of E_{kl} and B_{kl} as follows

$$Z_{kl} := E_{kl} + \imath B_{kl}\,, \tag{D.14}$$

and its potentials P_{kl} and S_{kl}

$$V_{kl} = S_{kl} + \imath P_{kl}\,. \tag{D.15}$$

The equations (D.6)-(D.13) in terms of complex objects are

$$\begin{aligned} Z^{kl}{}_{|l} &= 0\,, && \text{(D.16)}\\ \dot{Z}^{kl} &= -\imath\varepsilon^{pq(k}Z^{l)}{}_{q|p}\,, && \text{(D.17)}\\ Z_{ab} &= \varepsilon_{cd(a}V_{b)}{}^{d|c}\,, && \text{(D.18)}\\ V^{kl}{}_{|l} &= 0\,. && \text{(D.19)}\end{aligned}$$

D.3 Scalar representation of spin-2 field

Spin-2 field can be represented as a complex, scalar function defined analogically to the electromagnetic case[11].
In the spherical coordinates it has the form

$$\Psi = 2Z_{kl}x^kx^l = 2Z_{RR}R^2\,. \tag{D.20}$$

A counterpart of gravito-electromagnetic equations (D.6)-(D.9) for Ψ is

$$\Box\Psi = 0\,, \tag{D.21}$$

where $\Box$ is a d'Alembert operator for Minkowski background. The recovery procedure of the Z_{kl} field from Ψ uses the constraint equations for linearized Weyl tensor and the dynamical equations. The $(2+1)$–splitting of the constraint (D.16):

$$\begin{aligned} \partial_R(R^3Z^{RR}) + R^3Z^{RA}{}_{||A} &= 0\,, && \text{(D.22)}\\ \partial_R(R^4Z^{RA}{}_{||A}) + R^4\overset{\circ}{Z}{}^{AB}{}_{||AB} - \frac{1}{2}R^2\mathbf{\Delta}Z^{RR} &= 0\,, && \text{(D.23)}\\ \partial_R(R^4Z^R{}_{A||B}\varepsilon^{AB}) + R^4\overset{\circ}{Z}{}_A{}^B{}_{||BC}\varepsilon^{AC} &= 0\,, && \text{(D.24)}\end{aligned}$$

and the $(2+1)$–decomposition of the dynamical equations (D.17) :

$$\begin{aligned} \partial_tZ^{RR} &= -R^2\varepsilon^{RAB}Z_{RA||B}\,, && \text{(D.25)}\\ \partial_R(R^2\partial_tZ^{RR}) &= -\imath R^4\overset{\circ}{Z}{}_A{}^B{}_{||BC}\varepsilon^{RAC}\,, && \text{(D.26)}\end{aligned}$$

enables one to express explicitly all electromagnetic components of the Weyl tensor in terms of Ψ and $\partial_t\Psi$:

$$\begin{aligned} R^2Z^{RR} &= \frac{1}{2}\Psi\,, && \text{(D.27)}\\ R^2Z_{RA||B}\varepsilon^{RAB} &= -\frac{1}{2}\imath\partial_t\Psi\,, && \text{(D.28)}\\ R^3Z^{RA}{}_{||A} &= -\frac{1}{2}\partial_R(R\Psi)\,, && \text{(D.29)}\\ R^2\overset{(2)}{Z} &= -\frac{1}{2}\Psi\,, && \text{(D.30)}\end{aligned}$$

[11] See the equation (3.27) and the comments below.

$$R^4 \overset{\circ}{Z}{}^{AB}{}_{||AB} = \frac{1}{2}\partial_R \left(R\partial_R(R\Psi)\right) + \frac{1}{4}\mathbf{\Delta}\Psi \,, \tag{D.31}$$

$$R^4 \overset{\circ}{Z}{}_A{}^B{}_{||BC}\varepsilon^{RAC} = \frac{1}{2}\imath\partial_R(R^2\partial_t\Psi) \,, \tag{D.32}$$

where $\overset{(2)}{Z} = g_{AB}Z^{AB}$, and $\overset{\circ}{Z}_{AB} = Z_{AB} - \frac{1}{2}g_{AB} \overset{(2)}{Z}$. The scalar is related to a gauge-independent part of the potential V_{ab}. The $(2+1)$ – splitting of (D.18), (D.19) and use of (D.27–D.32) gives

$$\mathbf{\Delta}(\mathbf{\Delta}+2)V^R{}_R = -\Big(2\imath\partial_t\Psi + 2\imath\,(R\Pi)_{,R} + \imath(\mathbf{\Delta}+2)\Pi\Big) \,, \tag{D.33}$$

$$(\mathbf{\Delta}+2)V^{RA}{}_{||A} = \imath\frac{\partial_t\Psi + (R\Pi)_{,R}}{R} \,, \tag{D.34}$$

$$2R^2V^{RA||B}\varepsilon_{RAB} = -\Psi \,, \tag{D.35}$$

$$R^2 \overset{(2)}{V} = \Psi \,, \tag{D.36}$$

$$2R^2 \overset{\circ}{V}{}^{AB}{}_{||AB} = -\imath\left(\partial_t\Psi - \Pi\right) \,, \tag{D.37}$$

$$2R^4 \overset{\circ}{V}{}^C{}_{A||CB}\varepsilon^{RAB} = \left(R^2\Psi\right)_{,R} \,, \tag{D.38}$$

where $\Pi = 2RV^{RA}{}_{||A} + \mathbf{\Delta}V^R{}_R$, is a gauge dependent part.
The presented formulation of linearized gravity also holds in the case of sources.
Presented decomposition of linearized Einstein equation can be repeated with non-vanishing stress-energy tensor. Monopole and dipole part of reduced data Ψ are related to stationary[12] fields in that case. See [32] for details.

[12]Precisely, monopole and dipole part of reduced data are related to fields which are stationary or linear in time.

E Index of symbols

The following list contains most of the notation used throughout the manuscript. Symbols appearing only in a single instance, in the immediate vicinity of their definition, have been generally omitted.

Symbol	Meaning
M	A 4-dimensional spacetime manifold, equipped with a metric $g_{\mu\nu}$ satisfying the Einstein equation.
Σ_t	A hypersurface in $\mathcal{M}$, defined by $x^0 = t$. Generally assumed to be spatial. The subscript t is usually omitted.
$\mathbb{S}^2$, $S(R)$	Notations for a 2D sphere, used interchangeably.
$B(x, r)$	Ball of radius r, with its center at point x.
(x^0, x^1, x^2, x^3)	Coordinates in four-dimensional spacetime. Usually assumed to be adapted to the foliation scheme: x^0 is constant on hypersurfaces and x^3 labels the 2D leaves of foliation within the hypersurfaces.
$\alpha, \beta, \gamma, \cdots$	Indices running over all spacetime coordinates: $(0, 1, 2, 3)$.
$a, b, c, \cdots$	Indices running over hypersurface coordinates: $(1, 2, 3)$.
$A, B, C, \cdots$	Indices running over coordinates in 2D foliation leaves: $(1, 2)$.
(x, y, z)	Cartesian coordinates.
(R, Θ, φ)	Spherical coordinates.
$(\widehat{R}, \varphi, z)$	Cylindrical coordinates.
(t, r, θ, φ)	In chapter 2, coordinate system for Plebański–Demiański black hole. In chapter 4, oblate spheroidal coordinates.
$g_{\mu\nu}$	4-dimensional metric of Lorentzian signature.
g	Determinant of the four dimensional metric $g_{\mu\nu}$.
$\varepsilon_{\alpha\beta\gamma\delta}, \varepsilon_{klm}, \varepsilon_{AB}$	Levi–Civita tensors associated with $g_{\mu\nu}$. For lower dimensional case, we have $\varepsilon_{klm} = \varepsilon_{tklm}$ and $\varepsilon_{AB} = \varepsilon_{RAB}$.
δ_{kl}	Three dimensional spatial metric on Σ.
$;$, $\vert$, $\Vert$	Covariant derivatives, associated with the metric, in dimensions 4, 3, and 2, respectively. Note that in approximated theory the covariant derivatives are calculated with respect to the background metric.
∇	Covariant derivative associated with the metric $g_{\mu\nu}$.
curl	Rotation operator, three dimensional differential operator which acts on vectors.
$\Box$	D'Alembert operator.
$\boldsymbol{\Delta}$	Laplace–Beltrami operator on a unit sphere.

Symbol	Meaning
$R_{\mu\nu\rho\sigma}$	Riemann tensor.
$C_{\mu\nu\rho\sigma}$	Weyl tensor.
$R_{\mu\nu}$	Ricci tensor.
∂	Partial derivative.
$\pounds_K$	Lie derivative with respect to the vector field K.
$\mathbb{R}$	The set of real numbers.
$O(r^{-k})$	"Big O notation". $h(r) = O(r^{-k})$ means that, for r sufficiently large and some positive constant M, $\|h(r)\| \leq Mr^{-k}$.
$\|\|\mathbf{v}\|\|$	Length of vector $\mathbf{v}$.
$\langle \mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_k} \rangle$	Vector space spanned by the vectors $\mathbf{v_1}$ to $\mathbf{v_k}$.
$\langle A \rangle$	Averaged value of function A.
$\mathbf{r}$	Three-dimensional position vector.
Y_l	Spherical harmonic on a unit sphere, corresponding to the Laplace–Beltrami operator eigenvalue of $-l(l+1)$.
$\overset{(2)}{X} = g^{CD} X_{CD}$	The two-dimensional trace (except $\overset{(2)}{\Phi}$).
$\mathring{X}_{AB}$	The two-dimensional traceless part $\mathring{X}_{AB} = X_{AB} - \frac{1}{2} g_{AB} \overset{(2)}{X}$.
$\imath$	imaginary unit $\imath^2 = -1$.
$*$	Hodge dual mapping for forms, i.e. $*F$.
$\mathbf{e}_{\{a\}}{}^{\mu} \partial_\mu$	representative of tetrad basis. Enclosure in curly brackets distinguishes the tetrad indices from the tensor indices, $a \in (1, 2, 3, 4)$.
$\{L, N, M, \bar{M}\}$	Newman–Penrose tetrad.
F, $*F$	Maxwell two-form and its dual.
$\mathcal{F}_{\mu\nu}$	Anti-Self dual Maxwell two-form $\mathcal{F} = F + \imath * F$
$Q_{\mu\nu}$	conformal Yano-Killing two-form.
Y, $*Y$	conformal Yano-Killing two-form solutions for Plebański–Demiański metric (2.45).
Λ	Cosmological constant.
m	Mass parameter.
Φ	Reduced electromagnetic data in terms of complex scalar field. On Plebański–Demiański background, it is defined in section 2.3.3. For Hopfions, see equation (3.27) and comments nearby. The paragraphs nearby the equations (4.46) and (4.94) clarify the application of Φ for Magic Hopfions.
h	Hopf map, see section 3.1.1.

Symbol	Meaning
$\mathbf{E}\,,\mathbf{B}$	Electric (magnetic) vector field.
$\mathbf{Z}$	Riemann–Silberstein complex vector field $\mathbf{Z} = \mathbf{E} + \imath\mathbf{B}$.
$\mathbf{A}$	Vector potential for magnetic field $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$.
$\mathbf{C}$	Vector potential for electric field $\mathbf{E} = \mathrm{curl}\,\mathbf{C}$.
$\mathbf{V}$	Complex vector potential $\mathbf{Z} = \mathrm{curl}\,\mathbf{V}$.
h_E, h_M	Electric (magnetic) helicity.
Ψ	Reduced data for spin-2 field, see appendix D.3.
E^{kl}, B^{kl}	gravitoelectric (gravitomagnetic) symmetric, traceless tensor.
Z^{kl}	gravitoelectromagnetic tensor $Z^{kl} = E^{kl} + \imath B^{kl}$.
P^{kl}	Linearized ADM momentum, potential for B^{kl}, see (D.10) and comments nearby.
S^{kl}	Counterpart of linearized ADM momentum, potential for E^{kl}, see (D.11) and comments nearby.
V^{kl}	complex potential for gravitoelectromagnetic tensor Z^{kl}, defined by (D.15).
$h_{GE}\,, h_{GB}$	Helicity analogs for spin-2 field
(ϕ_0, ϕ_1, ϕ_2)	Newman–Penrose electromagnetic scalars.
$(\overset{(0)}{\Phi}\,, \overset{(1)}{\Phi}\,, \overset{(2)}{\Phi})$	Modified Newman–Penrose electromagnetic scalars, defined by (4.95).
${}_{\text{MH}}\mathcal{F}$	Anti-self dual Maxwell two-form for Magic Hopfion.
${}_{\text{MH}}\mathbf{Z}$	Riemann-Silberstein vector for Magic Hopfion.
${}_{\text{CK}}\mathbf{V}$	Complex vector potential for Magic Hopfion in Chandrasekhar–Kendall form (*curl* eigenvector).

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