Isolated horizons in spacetimes with cosmological constant

Denis Dobkowski-Ryłko

Dissertation for the degree of

Doctor of Philosophy

under the supervision of
prof. dr hab. Jerzy Lewandowski

University of Warsaw
Faculty of Physics
June 11, 2022

Abstract

The theory of isolated horizons is considered in the context of generic black holes and gravitational radiation in spacetimes with cosmological constant Λ. Intrinsic geometry of such horizons consists of the induced degenerate metric tensor and covariant derivative. Assuming embeddability in Λ-vacuum Einstein's equations and imposing a condition on stationarity to the second order allow to generically determine the spacetime Weyl tensor on the horizon, and based on its property provide a complete classification of the Petrov types of isolated horizons. Condition distinguishing the type D horizon is derived and takes a form of a second order differential equation on a complex invariant constructed from a Gaussian curvature and rotation scalar. It is referred to as the Petrov type D equation with cosmological constant and its general properties as well as solutions considering various structures of the horizons constitute a significant part of this work.

To start with, axisymmetric horizons of the topology: $\mathcal{S}_2 \times \mathbb{R}$ are investigated. All the solutions to type D equation with cosmological constant are derived and found to be embeddable in the Schwarzschild/Kerr-(anti) de Sitter spacetimes or the near extremal horizon spacetimes obtained by the Horowitz limit form the extremal Schwarzschild/Kerr-(anti) de Sitter metric. The family of solutions may be parametrized by two parameters, the area and angular momentum, which provides the foundation for formulating the local version of the no-hair theorem. It is a generalization of the earlier result valid for the vanishing cosmological constant. Furthermore, the isolated horizons with spacelike cross-section of genus>0 are considered. The only solutions are those with constant Gaussian curvature and no rotation. Consequently, a quasi-local argument is provided for the rotating black holes in 4-dimensional spacetimes to have a cross section of a topological 2-sphere. Finally, we consider the Petrov type D equation with cosmological constant on horizons generated by null curves that form nontrivial $U(1)$-bundles. The type D equation couples the $U(1)$ connection, 2-metric of the base manifold and surface gravity in a nontrivial form. We derive all axisymmetric solutions which set a 4-dimensional family of isolated horizons and discuss the issue of their embeddability.

The type D equation with cosmological constant is also satisfied by extremal isolated horizons regardless of the Petrov type of the Weyl tensor on the horizon. Therefore, our results may be applied to the near horizon geometry equation for the extremal horizons. Consequently, we found all solutions to the near horizon geometry equation on the 2-surfaces of genus>0.

The notion of isolated horizons is also applied in context of gravitational radiation, where the isolated horizon serves as a generalization of the conformal boundary in

Minkowski case for spacetimes with positive cosmological constant. We consider a time changing matter source in de Sitter spacetime emitting gravitational radiation. The formula for the energy flux passing through a cosmological horizon is derived and expressed in terms of the mass and pressure quadruple moments. It is written explicitly up to the first order in Hubble parameter $H := \sqrt{\Lambda/3}$. We found that the zeroth order term coincides with the Einstein's quadruple formula for the perturbed Minkowski spacetime, whereas the first order term is a new correction.

Acknowledgements

First and foremost, I would like to thank my supervisor, Prof. Jerzy Lewandowski, for giving me a chance to become a member of the relativity group in Warsaw. His guidance and support, in and outside of my academic life, has been invaluable. I am grateful for every enlightening discussion that we had, for the encouragement and motivation, thanks to which my PhD studies were not only a highly enriching but also a very challenging experience. Prof. Lewandowski is someone who I will always look up to and most importantly who I consider to be a true friend of mine.

I would also like to extend my sincere thanks to a friend and colleague Adam Szereszewski, who I could count on regardless of the problem I was facing. He was always available and eager to help, whether it was an advice regarding my research, academic problems or teaching duties.

I shared some of the most unforgettable moments with my colleagues Cong, Gioele, Ilkka, Mehdi and Maciej who accompanied me during the studies. I am sure that our team would have been a real threat to *The Big bang theory* TV series if only were given a chance. Going back to the times that we spent together will always put a smile on my face, which I am very thankful for.

I could have not undertaken the journey of PhD studies if it wasn't for my parents. Not only they have always believed in me but also provided with an unconditional love and support at every stage of my life. I can only hope that one day I will be at least half the parent they were to me.

Last but not least, I would like to thank my beloved Klaudia, who has been there for me since the day we have first met. She sacrificed her own ambitions and professional career to take the best care of me. Thanks to her I was able to fully commit to science, especially at the final stages of writing this thesis. I will always be extremely grateful for that.

Contents

Introduction

On the eventful history of the cosmological constant

In 1917 Albert Einstein modified his field equations of general relativity by adding a new term with the so-called cosmological constant Λ, which is compatible with the energy-momentum law $\nabla_\nu T^{\mu\nu} = 0$ for matter, to obtain:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}.$$

Indeed, in the 1972 Lovelock [6] shown that in 4-dimensions, the linear combination of $G_{\alpha\beta}$ and $g_{\alpha\beta}$ is the most general tensor constructed from the metric tensor and its derivatives up to the second order that does not violate the basic properties of Einstein's equations. The reason for this modification was Einstein's discontent with the prediction of the former equations of a dynamic universe. The new constant represented a repulsive force to gravity, thanks to which Einstein's equations finally admitted the static solution. Einstein explained his motivation for the addition of the cosmological constant in *Cosmological considerations on the general theory of relativity* [7], where he wrote:

At any rate, this view is logically consistent, and from the standpoint of the general theory of relativity lies nearest at hand; whether, from the standpoint of present astronomical knowledge, it is tenable, will not here be discussed. In order to arrive at this consistent view, we admittedly had to introduce the extension of the field equations of gravitation which is not justified by our actual knowledge of gravitation. It has to be emphasized, however, that a positive curvature of space is given by our results, even if the supplementary term is not introduced. That term is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars.

At that time the astronomical observations were limited to the stars of our galaxy, the relative velocities of the stars indeed seemed really small, therefore the conviction regarding the static universe was quite reasonable. Despite the independent works of Friedman [8] and Lemaître [9] on the expanding universe, such models were not highly popular nor respected by the scientific society. Indeed, Einstein admitted, that their papers were mathematically correct, however he saw no physical significance in them. It was only Hubble's redshift observations in 1929 and a successful explanation by Lemaître that lead to the sensational discovery of the universe expansion and changed the viewpoint of most researchers in the field [10]. As a consequence, Einstein erased the cosmological constant term from the field equations. His action was not that surprising since already in 1923, in the letter to Weyl, Einstein expressed his feelings regarding the constant:

If there is no quasi-static world, then away with the cosmological term.

It is worth noting that the redshifts of radiation emitted by distant nebulae were interpreted by Hubble in the framework of de Sitter model discovered in 1917, which is a static cosmological model that contains no matter and incorporates the cosmological constant Λ.

The concept of cosmological constant no longer seemed to be necessary and therefore since the early 30's has been abandoned in the majority of the research. However, in 1998 two independent groups of researchers, the Supernova Cosmology Project [11] and High-Z Supernova Search Team [12], observed the redshift of supernovae and in consequence came to the conclusion that the expansion of the universe is in fact accelerating. This fascinating discovery was a big surprise to many relativists and cosmologists as it was previously expected that the expansion is actually decelerating. It turned out that cosmological models with vanishing cosmological constant were not compatible with the observed acceleration. After all, Einstein was right to introduce the cosmological constant, despite the fact that his motivation was incorrect. An interesting observation was made by Lemaître[1]:

The history of science provides many instances of discoveries which have been made for reasons which are no longer considered satisfactory. It may be that the discovery of the cosmological constant is such a case.

To the best of our knowledge, the cosmological constant should be included in Einstein's equations, moreover there is evidence that its value is small but yet positive. One may naively conclude that, because of its small value, the cosmological constant does not significantly complicate the generalization of the known theories nor of the theorems derived from the Einstein's field equations without the constant. On the contrary, it required considerable effort and this very thesis constitutes a perfect example as we consider isolated horizons and gravitational radiation in the context of spacetimes with the non-vanishing cosmological constant.

Isolated Horizons

The theory of the isolated horizons has been often used as a quasi-local approach [14,15] to describe black hole horizons [16, 17]. The *locality* is emphasized by applying the notion of null surfaces of co-dimension 1 in their description, which admit similar properties to black hole horizons. Such null surfaces are assumed to have compact spacelike cross-sections of co-dimension 2 which makes this framework *quasi*-local. The quasi-

[1] This citation was taken from the article titled *The Cosmological Constant* published in a book *Albert Einstein: Philosopher-Scientist* in 1949 [13]. It is worth noting, that back then Lemaître could not have been fully aware of how accurate his statement was.

local nature distinguishes the isolated horizons from event and Killing horizons [17, 18]. Specifically, the presence of a Killing vector field is not required in their neighborhood.

Isolated horizons may be applied to black holes at equilibrium in possibly dynamical spacetime, in particular, to model approximate event horizons at late stages of the gravitational collapse or black hole mergers whenever back-scattered radiation falling into the black hole becomes negligible [19]. They were first introduced by Pajerski and Newman in 1971 [20], but really gain popularity in the late 90's and early 2000's [14, 15, 21–23], where they were used to reformulate and generalize the laws of black hole mechanics and were applied to problems involving gravitational radiation in the exterior region of spacetime. There exist many analogies between the properties of isolated horizons and standard black hole physics. Among them are the following: mechanics of isolated horizons corresponding to black hole thermodynamics [23], the rigidity theorem [24], and also the uniqueness theorems [25, 26]. The intrinsic geometry of isolated horizons consists of the induced metric tensor and induced covariant derivative. It provides physically relevant features of the geometry of the stationary black hole horizons.

The difference between isolated horizons and black holes comes to the degrees of freedom. That is, the degrees of freedom of stationary black holes are finite-dimensional [27, 28] which is not the case for generic isolated horizons. Indeed, the theory may be applied to cosmological horizons, the null boundaries of the conformally compactified asymptotically flat spacetimes [29], as well as black hole holograph construction of spacetimes about Killing horizons [30, 31]. Moreover, it found its application in extracting physics from numerical simulations of black holes at early times when they are distant from each other and late times after reaching equilibrium [19, 32], as a starting point for determining statistical mechanical entropy [33] and also physical models of hairy black holes for background independent quantum gravity [34, 35].

The isolated horizons are expansion and shear free and may be referred to as the Killing horizons to the zeroth order. By implementing further assumptions they convert to the Killing horizons to the second order, which we call the isolated horizons stationary to the second order. The spacetime Weyl tensor on the non-extremal isolated horizon, satisfying the assumption on the stationarity to the second order, is determined via the vacuum Einstein's equations with cosmological constant and Bianchi identities by the intrinsic geometry of the horizon, that is a degenerate metric tensor and a torsion-free covariant derivative. Therefore, the Petrov type, despite the fact that it normally characterizes the entire spacetime, may be associated with the isolated horizon. We provide a thorough discussion on the possible Petrov types of the horizon as well as find all horizons of the type 0. Significant part of this work is devoted to the Petrov type D isolated horizons since the black hole solutions known from the literature are exactly of this type. A condition for the type D of the horizon takes a form of second order differential equation imposed on a Riemannian metric g_{AB} and a covector ω_A defined on a 2-dimensional cross-section of the isolated horizon. Geometrically, the Petrov type D equation with cosmological constant is a new equation in mathematical physics, which could lead to the new structures in a 2-dimensional Riemannian geometry.

As already mentioned, the notion of isolated horizons has a variety of applications and is not limited just to the analysis of the generalized black holes. It may be applied to a problem of gravitational radiation in spacetimes with positive cosmological constant, which will be a subject of the second part of the thesis. The theory of gravitational radiation requires a development of an adequate generalization of the boundary of asymptotically flat spacetime that would be valid in case of the non-vanishing cosmological constant. A possible generalization is the system consisting of two transversal null surfaces; the radiating region is bounded from the past by a weakly isolated horizon, whereas the radiation itself is encoded in the future part of the generalized boundary which is also a null surface. A natural choice for the generalization of Minkowski spacetime when considering gravitational radiation in case of the non-vanishing cosmological constant is the de Sitter spacetime. It is a maximally symmetric solution to vacuum Einstein's equations with cosmological constant Λ. The cosmological horizon, which is a special case of an isolated horizon, may serve as a generalization of the conformal boundary from the Minkowski case [36, 37].

The key results of the thesis

Geometric properties of the type D isolated horizons (Chapter 1)

The spacetime Weyl tensor may be determined on the isolated horizon by the intrinsic geometry of the horizon consisting of the induced degenerate metric tensor and induced covariant derivative via Λ-vacuum Einstein's equations and assumption on stationarity to the second order of the horizon. We provide a complete classification of the Weyl tensor on the isolated horizon and find that it has to be either of the type 0, II or D, whereas the types III and N appear only as measure zero subsets of the horizon. We found all geometries of the type 0 and show that they are characterized by the Gaussian curvature of the spatial cross-section equal to $\Lambda/3$ and zero rotation. The condition for the type D isolated horizon is derived and takes a form of the second order differential equation on a certain complex valued invariant constructed form Gaussian curvature and the rotation scalar. We write the equation in various forms, in particular covariant one or in complex variables, analyze it in detail and find that it is also an integrability condition for the near horizon geometry equation [38], which is an equation that was first discovered in the context of extremal isolated horizons [14, 26]. The geometric consequence of the appearance of the near horizon equation is a non-twisting of the second principal null direction of the Weyl tensor.

Solution to the Petrov type D equation on a spacelike cross-section diffeomorphic to a 2-sphere $\mathcal{S}_2$ of the isolated horizon admitting axial symmetry (Chapter 2)

We solve the type D equation with cosmological constant on the isolated horizons of the topology $\mathcal{S}_2 \times \mathbb{R}$ assuming axial symmetry. General solutions set a 2-dimensional (for every value of the cosmological constant) family and may be parametrized by the area and angular momentum. These horizons can be identified with either the outer or inner horizon of the non-extremal Schwarzschild/Kerr-(anti) de Sitter spacetime or the horizon of the near horizon geometry obtained as the Horowitz-Bardeen limit for the extremal Schwarzschild/Kerr-(anti) de Sitter metric [39]. We find the relation between our parameters and the Kerr and mass parameters of the Kerr-(anti) de Sitter spacetime as well as determine their constraints. Finally, a local version of the no-hair theorem is formulated.

General solutions to the Petrov type D equations on cross-section with genus>0 of the isolated horizons (Chapter 3)

The isolated horizons of the topology $S \times \mathbb{R}$, where S is a compact 2-surface of genus higher than 0 are studied. We find the family of all solutions to type D equation with cosmological constant for those horizons by first considering 2-dimensional torus and then higher genus compact surfaces. Furthermore, we show that they are characterized by the constant Gauss curvature and vanishing rotation. As a consequence, we provide an argument that the rotating black holes in 4-dimensional spacetime need to have a cross-section of a topology of a 2-sphere. Moreover, if the metric tensor g_{AB} together with the 1-form ω_A are a solution to the near horizon geometry equation then the Gaussian curvature is constant and the 1-form is closed. Combining this result with the previous works on this subject, yields that for negative cosmological constant Λ the 1-form ω_A has to vanish. Therefore, any solution to the near horizon geometry equation consists of constant Gauss curvature and the vanishing 1-form ω_A.

Solution to the Petrov type D equation on horizons of the nontrivial $U(1)$-bundle topology (Chapter 4)

Finally, we consider the type D equation on isolated horizons of nontrivial $U(1)$-bundle structure, that is a Hopf bundle or more general the Dirac monopole bundle. In such case, despite the fact that it does not admit a global spacelike cross-section, it may be considered on the space of null generators that is topologically a 2-sphere. The example of spacetime admitting such horizon structure is the family of the Taub-NUT spacetimes. We find all axisymmetric solutions to the type D equation with cosmological constant and show that, in generic case, they set a 3-dimensional family for every value of the cosmological constant. We provide a full classification of the solutions and show that in the non-rotating case they are embeddable in Taub-NUT-(anti) de Sitter spacetimes. However, our generic solution turns out not to be embeddable in the Kerr-NUT-(anti) de Sitter.

A generalization of the Einstein's quadruple formula (Chapter 5)

We investigate gravitational radiation coming from the time changing compact source (e.g. isolated star, coalescence of binary system) in spacetime with positive cosmological constant. As a generalization of the conformal boundary of Minkowski spacetime, the cosmological Killing horizon, which is a special case of an isolated horizon, is chosen. Assuming the cosmological constant is small, the cosmological horizon is very far from the source. By applying Wald-Zoupas [40] and Chandrasekaran-Flanagan-Prabhu [41] formalism for the energy passing through the null surfaces we derive an expression for the flux of the radiated energy by the source. We expand our formula for small values of the cosmological constant and write it explicitly up to the first order in Hubble parameter H that is a square root of one third of the cosmological constant. The zeroth order term is consistent with the standard Einstein's quadrupole formula [17], whereas the first order term in Hubble parameter is a new correction - a consequence of the emergence of the cosmological constant.

Author's contribution

This thesis is based on the results published in the following articles:

[1] D. Dobkowski-Ryłko, J. Lewandowski and T. Pawłowski, *The Petrov type D isolated null surfaces*, *Class. Quant. Grav.* **35** (2018) 17, [arXiv:1803.03203].

[2] D. Dobkowski-Ryłko, J. Lewandowski and T. Pawłowski, *Local version of the no-hair theorem*, *Phys. Rev.* **D98** (2018) 2, 024008, [arXiv:1803.05463].

[3] D. Dobkowski-Ryłko, W. Kamiński, J. Lewandowski and A. Szereszewski, *The Petrov type D equation on genus* > 0 *sections of isolated horizons*, *Phys. Lett.* **B783**, (2018), 415-420, [arXiv:1804.09614].

[4] D. Dobkowski-Ryłko, J. Lewandowski and I. Rácz, *Petrov type D equation on horizons of nontrivial bundle topology*, *Phys. Rev.* **D100** (2019) 8, 084058, [arXiv:1907.03484].

[5] D. Dobkowski-Ryłko and J. Lewandowski, *A generalization of the quadruple formula for the energy of gravitational radiation in de Sitter spacetime*, to be published in *Phys. Rev.* **D**, [arXiv:2205.09050].

The above numbering is consistent with the numbering of the references, listed at the end of the thesis, and of the chapters in which the results of the corresponding articles are presented.

Notation and conventions

Throughout this thesis we consider a 4-dimensional spacetime that consists of a manifold $\mathcal{M}$ and a metric tensor $g_{\mu\nu}$ of the signature $-+++$. The geometrized unit system is applied. We use the following (abstract) index notation [17]:

- Indices of the spacetime tensors are denoted by lower Greek letters: $\alpha, \beta, \gamma, ... = 1, 2, 3, 4$.
- Tensors defined on the (perturbed) horizon $\mathcal{H}$ are denoted by lower Latin letters: $a, b, c, ... = 1, 2, 3$.
- Tensors defined on the cosmological slices $\eta = \text{const}$ carry indices denoted by lower Latin letters: $i, j, k, ... = 1, 2, 3$.
- Capital Latin letters $A, B, C, ... = 1, 2$ are used as the indices of tensors defined on the 2-dimensional space $\mathcal{S}$ of the null curves in $\mathcal{H}$.

We denote by ∇_μ the torsion free covariant derivative in $\mathcal{M}$ that corresponds to $g_{\mu\nu}$ via:

$$\nabla_\alpha g_{\mu\nu} = 0.$$

The curvature tensors are defined as:

$$2\nabla_{[\alpha}\nabla_{\beta]}k_\mu = R_{\alpha\beta\mu}{}^\nu k_\nu, \qquad R_{\alpha\beta} = R_{\alpha\nu\beta}{}^\nu, \qquad R = R_{\mu\nu}g^{\mu\nu}.$$

Both, the covariant framework and Newman-Penrose formalism [42] will be used for the isolated horizons in 4-dimensional spacetime. In some parts of the thesis the adapted null frames will be used for particular derivations. The necessary elements of Newman-Penrose formalism will be introduced explicitly. We follow the notation of [42] in exception of vectors k and ℓ defined in [42] which correspond to vectors ℓ and n, respectively, in our work.

I shall conduct the reader over the road that I have myself travelled, rather a rough and winding road, because otherwise I cannot hope that he will take much interest in the result at the end of the journey.

Albert Einstein

Chapter 1

Isolated horizons and the Petrov type D equation

In this chapter we introduce the notion and discuss the properties of the non-expanding horizons (NEHs), weakly isolated horizons (WIHs) and isolated horizons (IHs). We present the null frame adapted to the isolated horizon and use the Newman-Penrose formalism for our calculations. A list of all possible Petrov types of the Weyl tensor on the horizon is discussed. All of the geometries of type 0 are found, whereas the condition for the type D is derived. It takes a form of the second order differential equation on the complex invariant constructed from the Gaussian curvature and rotation scalar. We provide different forms of this equation and study its properties. In particular, we find that it is an integrability condition of the near horizon geometry equation. If the geometry of the non-extremal IH satisfies both equations then the second principal null direction of the Weyl tensor is non-twisting. We assume that the metric tensor $g_{\mu\nu}$ satisfies the vacuum Einstein's equations with the cosmological constant:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.1}$$

where $G_{\mu\nu}$ is the Einstein tensor.

1.1 Geometry of isolated horizons

In this section we consider the 3-dimensional null surfaces[1] and gradually apply to them additional constrains that result in definitions of non-expanding horizons, weakly isolated horizons and isolated horizons. Then we introduce the elements of the intrinsic geometry of the NEHs and study them further for the case of IHs.

1.1.1 Non-expanding horizons

Consider a 3-dimensional null surface in 4-dimensional manifold $\mathcal{M}$:

$$\mathcal{H} \subset \mathcal{M}. \tag{1.2}$$

[1] Isolated horizons were also studied in higher dimensions by Lewandowski, Pawłowski and Korzyński [43, 44]

Assume that it contains a slice that intersects each of the null curves in $\mathcal{H}$ exactly once, in other words that $\mathcal{H}$ is of the topology:

$$\mathcal{H} = \mathcal{S} \times \mathbb{R}, \tag{1.3}$$

such that $\mathcal{S}$ is a 2-dimensional connected space of the null curves in $\mathcal{H}$. The spacetime metric tensor $g_{\mu\nu}$ determines the degenerate metric of the null surface $\mathcal{H}$, therefore at each point $x \in \mathcal{H}$ there exists a vector $\ell \in T_x\mathcal{H}$ satisfying:

$$\ell^a g_{ab} = 0. \tag{1.4}$$

The distribution of the degenerate direction forms a 1-dimensional subbundle $L \subset T(\mathcal{H})$, which every section $\ell \in \Gamma(L)$ is a nontrivial vector field that is tangent and orthogonal to $\mathcal{H}$ at every point. The null generators of $\mathcal{H}$ are the integral curves tangent to L. They are null, future directed and foliate $\mathcal{H}$, moreover each of them is a geodesic in the spacetime $\mathcal{M}$.

Furthermore, we assume that the spacetime covariant derivative ∇_μ preserves the tangent bundle $T(\mathcal{H})$ and in consequence, via the restriction, endows $\mathcal{H}$ with a covariant derivative ∇_a, which could also be stated as follows:

Assumption 1.1.1. *Let X and Y be vector fields tangent to $\mathcal{H}$. Then the spacetime vector field $X^\alpha \nabla_\alpha Y^\mu$ is also tangent to $\mathcal{H}$:*

$$X, Y \in \Gamma(T(\mathcal{H})) \Rightarrow \nabla_X Y \in \Gamma(T(\mathcal{H})). \tag{1.5}$$

Equivalently, one could write this assumption as the vanishing of the extrinsic curvature of surface $\mathcal{H}$, that is:

$$X^\alpha Y^\beta \nabla_\alpha \ell_\beta = 0, \tag{1.6}$$

for all vector fields $X, Y \in \Gamma(T(\mathcal{H}))$ and $\ell \in \Gamma(L)$.

Definition 1.1.1. *The pair (g_{ab}, ∇_a) is called the intrinsic geometry of the null surface $\mathcal{H}$.*

It is straightforward to show that derivative ∇_a in $\mathcal{H}$ is torsion free and satisfies the pseudo metricity condition:

$$\nabla_c g_{ab} = 0. \tag{1.7}$$

From the properties of the covariant derivative ∇_a follows that the degenerate metric g_{ab} is Lie dragged along the null vector field ℓ:

$$\mathcal{L}_\ell g_{ab} = 0, \tag{1.8}$$

which is yet another equivalent expression of the Assumption 1.1.1. For any general null surface its degenerate metric tensor g_{ab} can be uniquely decomposed into expansion θ and shear σ_{ab} in the following manner:

$$\tfrac{1}{2}\mathcal{L}_\ell g_{ab} = \tfrac{1}{2}\theta g_{ab} + \sigma_{ab}, \tag{1.9}$$

where σ_{ab} is traceless in every 2-dimensional fiber of $T(\mathcal{H})/L$. From eq. (1.8) follows that the lefthand side of the above decomposition vanishes, and in the consequence both, the expansion and shear tensor, vanish simultaneously, which makes $\mathcal{H}$ a non-expanding and shear-free surface.

We conclude this section with the following definition [14]:

Definition 1.1.2. *Surface $\mathcal{H}$ is called a non-expanding horizon if:*

- *$\mathcal{H}$ is diffeomorphic to the product $\mathcal{S} \times \mathbb{R}$ where $\mathcal{S}$ is a 2-surface and the fibers of the projection:*

$$\Pi : \mathcal{S} \times \mathbb{R} \to \mathcal{S} \tag{1.10}$$

  are null generators of $\mathcal{H}$;

- *the expansion θ of any null vector field ℓ normal to $\mathcal{H}$ vanishes;*

- *Einstein's equations hold on $\mathcal{H}$.*

Remark. Assuming Einstein's equations (1.1) hold and applying Raychaudhuri equation yields that the vanishing of the expansion θ implies the vanishing of the shear tensor σ_{ab}. Therefore, any NEH is also a shear-free surface.

1.1.2 The intrinsic geometry of the non-expanding horizons

In this subsection we examine the intrinsic geometry of the NEHs which plays a significant role in the further chapters. We obtain the degenerate metric g_{ab} with the signature $0++$ via pullback of the spacetime metric $g_{\mu\nu}$. Moreover, since ℓ is a null vector field on $\mathcal{H}$ and satisfies eq. (1.8) it follows, that one can uniquely define the Riemannian metric tensor g_{AB}. Indeed, the degenerate metric g_{ab} is the pullback Π^* of g_{AB}, that is:

$$g_{ab} = {\Pi^*}_{ab}{}^{AB} g_{AB}. \tag{1.11}$$

Action of the intrinsic derivative operator ∇_a on vector fields is determined by eq. (1.6). One can extend ∇_a to the covectors W_a, which are the section of the dual bundle $T^*(\mathcal{H})$, using the Leibniz rule:

$$Y^b \nabla_a W_b = \nabla_a (W_b Y^b) - W_b (\nabla_a Y^b), \tag{1.12}$$

for any $Y \in \Gamma(T(\mathcal{H}))$. Moreover, for W_a ortogonal to ℓ, that is:

$$W_a \ell^a = 0, \tag{1.13}$$

the action of the covariant derivative is such that:

$$\nabla_a W_b = \partial_{[a} W_{b]} + \tfrac{1}{2} \mathcal{L}_{\hat{W}} g_{ab}, \tag{1.14}$$

where the vector field $\hat{W}^a$ on $\mathcal{H}$ is defined via:

$$\hat{W}^a g_{ab} = W_b. \tag{1.15}$$

The covariant derivative ∇_a can be split into two parts. One that coincides with covariant derivative ∇_A on $\mathcal{H}$, and another, independent of g_{ab}, which we define by introducing a tensor:

$$S_{ab} := \nabla_a n_b, \tag{1.16}$$

where the covector $n \in \Gamma(T^*(\mathcal{H}))$ is nowhere orthogonal to L. It is convenient to make a choice of n_a such that:

$$n_a := -\nabla_a v, \tag{1.17}$$

where the function $v : \mathcal{H} \to \mathbb{R}$ satisfies:

$$\ell^a \partial_a v = 1 \tag{1.18}$$

for a given non-vanishing vector field $\ell \in \Gamma(L)$. It follows that S_{ab} is symmetric:

$$S_{ab} = S_{(ab)} \tag{1.19}$$

and:

$$\ell^a n_a = -1, \qquad\qquad \mathcal{L}_\ell n_a = 0. \tag{1.20}$$

Furthermore, for every vector field $\ell \in \Gamma(L)$ there exists a 1-form:

$$\omega_a^{(\ell)} := \ell^b S_{ab}, \tag{1.21}$$

such that:

$$\nabla_a \ell^b = \omega_a^{(\ell)} \ell^b. \tag{1.22}$$

In particular:

$$\ell^a \nabla_a \ell^b = \kappa^{(\ell)} \ell^b, \tag{1.23}$$

where function $\kappa^{(\ell)}$ defined as:

$$\kappa^{(\ell)} := \omega_a^{(\ell)} \ell^a \tag{1.24}$$

is a self-acceleration of the vector field ℓ and will be referred to as the surface gravity. Furthermore, the potential 1-form $\omega_a^{(\ell)}$ transforms with rescaling of ℓ in the following way:

$$\omega_a^{(f\ell)} = \omega_a^{(\ell)} + \partial_a \ln f. \tag{1.25}$$

The external derivative of the 1-form $\omega_a^{(\ell)}$ reads:

$$\Omega_{ab} = \partial_a \omega_b^{(\ell)} - \partial_b \omega_a^{(\ell)} \tag{1.26}$$

and we call it a rotation[2] 2-form invariant of the intrinsic geometry (g_{ab}, ∇_a).

[2]As long as the angular momentum of the non-expanding horizon is well-defined, one can calculate it via an integral of an expression proportional to Ω_{ab}, hence the name *rotation* 2-form.

Next, we consider the constraints on the intrinsic geometry (g_{ab}, ∇_a) of the non-expanding horizon $\mathcal{H}$ imposed by the vacuum Einstein's equation (1.1) with cosmological constant Λ. For a complete derivation of the constraints see [14]. The first constrain is the Zeroth Law, which says that for every null vector field $\ell \in \Gamma(L)$ the corresponding rotation 1-form potential $\omega_a^{(\ell)}$ and the surface gravity $\kappa^{(\ell)}$ satisfy:

$$\partial_a \kappa^{(\ell)} = \mathcal{L}_\ell \omega_a^{(\ell)}. \tag{1.27}$$

The Zeroth Law has a direct implication on the rotation 2-form Ω_{ab}. The 2-form is invariant of the intrinsic geometry of $\mathcal{H}$, therefore to calculate it, one may choose any ℓ, in particular $\ell_{_0} \in \Gamma(L)$ such that:

$$\ell_{_0}^a \nabla_a \ell_{_0}^b = 0. \tag{1.28}$$

Then from (1.27) follows that:

$$0 = \partial_a \kappa^{(\ell_0)} = \mathcal{L}_{\ell_{_0}} \omega_a^{(\ell_0)}. \tag{1.29}$$

It is straightforward to see that equations:

$$\ell^a \Omega_{ab} = 0 = \mathcal{L}_\ell \Omega_{ab} \tag{1.30}$$

hold not just for $\ell = \ell_{_0}$ but also for any other $\ell \in \Gamma(L)$.

Moreover, the Zeroth Law motivates the definition of the weakly isolated horizon. As previously discussed, there is a large freedom in the choice of the null vector field ℓ as it can be rescaled by an arbitrary positive function f:

$$\ell \mapsto \ell' = f\ell. \tag{1.31}$$

One could equip the horizon $\mathcal{H}$ with the specific choice of $\ell \in \Gamma(L)$, namely:

Definition 1.1.3. *A non-expanding horizon $\mathcal{H}$ is called a weakly isolated horizon $(\mathcal{H}, \ell)$ if the flow of ℓ preserves the rotation 1-form potential $\omega_a^{(\ell)}$:*

$$\mathcal{L}_\ell \omega_a^{(\ell)} = 0. \tag{1.32}$$

The second constraint on the intrinsic geometry of $\mathcal{H}$ is formulated with respect to a given null vector field $\ell \in \Gamma(L)$ and the function $v : \mathcal{H} \to \mathbb{R}$. A section of (1.10) is defined naturally for every value v_1 of the function v:

$$s_{v_1} : \mathcal{S} \to \mathcal{H}, \tag{1.33}$$

for every value v_1 of v and might be used to pullback the covectors defined on $\mathcal{H}$ to $\mathcal{S}$. Notice that the pullbacks:

$$s^*_{v_1 AB}{}^{ab}\, g_{ab} = g_{AB}, \qquad\qquad s^*_{v_1 AB}{}^{ab}\, \Omega_{ab} = \Omega_{AB} \tag{1.34}$$

are independent of the value v_1 of the function v, the function itself and the null vector field $\ell \in \Gamma(L)$. Any $\ell \in \Gamma(L)$ can be rescaled without the lack of generality so that the surface gravity $\kappa^{(\ell)}$ is constant. Therefore, such rescaling is chosen and the following pullback is considered:

$$s^{*}_{v_1 A}{}^{a}\omega^{(\ell)}_{a} = \omega^{(\ell)}_{A}. \tag{1.35}$$

In contrast to (1.34) the result depends on the choice of the null vector field ℓ and the choice of a function v, with an exception for the case when $\kappa^{(\ell)} = 0$. However, given ℓ and v, we find that ω_A is independent of the chosen section defined by the value v_1. Moreover, the relation between the pullbacks ω_A and Ω_{AB} reads:

$$d\omega_{AB} = \Omega_{AB}, \tag{1.36}$$

Finally, we consider the following pullback:

$$S_{AB}(v) := s^{*ab}_{v\,AB} S_{ab} \tag{1.37}$$

where we have omitted index 1 of v_1. The pullback S_{AB} depends on v that is:

$$\tfrac{d}{dv} S_{AB}(v) = -\kappa^{(\ell)} S_{AB}(v) + \nabla_{(A}\omega_{B)} + \omega_A\omega_B - \tfrac{1}{2}R_{AB} + \tfrac{1}{2}\Lambda g_{AB}, \tag{1.38}$$

where R_{AB} is the Ricci tensor of the metric g_{AB}.

To sum up, the intrinsic geometry of the NEH in a vacuum spacetime with cosmological constant Λ consisting of the degenerate metric tensor g_{ab} and covariant derivative ∇_a may be determined by choosing (i) a null vector field $\ell \in \Gamma(L)$ such that the surface gravity $\kappa^{(\ell)}$ is constant, (ii) a section:

$$s_{v_1} : \mathcal{S} \to \mathcal{H}, \tag{1.39}$$

and (iii) the pullbacks g_{AB}, ω_A (1.35) and S_{AB} (1.37) by s^*_{v1} onto $\mathcal{S}$ via the constraints (1.4), (1.8), the Zeroth Law (1.27) and (1.38).

1.1.3 Isolated horizons

In this subsection we finally come to the definition of the isolated horizons by adding yet another assumption on the geometric structure of $\mathcal{H}$. That is, we consider a non-expanding horizon $\mathcal{H}$ with intrinsic geometry (g_{ab}, ∇_a) and equipped with a non-vanishing vector field $\ell \in \Gamma(L)$ satisfying:

$$[\mathcal{L}_\ell, \nabla_a] = 0. \tag{1.40}$$

Notice that the above condition is preserved only by such transformations $\ell^a \to f\ell^a$ that $f = f_0 = \text{const.}$, in contrast to (1.8) which is invariant for any function f. The consequence of the condition (1.40) and definition of $\omega^{(\ell)}_a$ (1.22) is that:

$$\mathcal{L}_\ell \omega^{(\ell)}_a = 0. \tag{1.41}$$

Moreover, the Zeroth Law implies that:

$$\partial_a \kappa^{(\ell)} = 0, \tag{1.42}$$

in other words, the surface gravity $\kappa^{(\ell)}$ is constant along $\mathcal{H}$. Since the rescaling of ℓ by a constant f_0 implies the rescaling of the surface gravity $\kappa^{(\ell)}$:

$$\kappa^{(f_0\ell)} = f_0\kappa^{(\ell)}, \tag{1.43}$$

it follows that there exist two distinct cases[3], one for vanishing $\kappa^{(\ell)}$ and the other for a constant but non-zero $\kappa^{(\ell)}$.

Definition 1.1.4. *A null surface $\mathcal{H}$ is called an isolated horizon if it is non-expanding (and shear-free) and its intrinsic geometry (g_{ab}, ∇_a) admits $\ell \in \Gamma(L)$, which is non-vanishing on $\mathcal{H}$ and satisfying:*

$$[\mathcal{L}_\ell, \nabla_a] = 0. \tag{1.44}$$

An isolated horizon $\mathcal{H}$ is non-extremal if the surface gravity $\kappa^{(\ell)}$ does not vanish, and extremal if $\kappa^{(\ell)}$ vanishes.

In this thesis we focus mainly on the non-extremal case. In such case the constraint on the pullback S_{AB} (1.38) allows one to express it in terms of the metric g_{AB} and ω_A. From (1.40) and the definition of S_{ab} (1.16) follows that the left-hand side of (1.38) vanishes:

$$\tfrac{d}{dv} S_{AB} = 0. \tag{1.45}$$

Consequently, the right-hand side of (1.38) needs to vanish as well which results in the following expression:

$$S_{AB} = \tfrac{1}{\kappa^{(\ell)}}\Big(\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \tfrac{1}{2}R_{AB} + \tfrac{1}{2}\Lambda g_{AB}\Big). \tag{1.46}$$

We conclude with the following statement. For the non-extremal isolated horizon $\mathcal{H}$ and the infinitesimal symmetry generator $\ell \in \Gamma(L)$, the intrinsic geometry (g_{ab}, ∇_a) is determined by the degenerate metric g_{ab}, the rotation 1-form potential $\omega_a^{(\ell)}$ and surface gravity $\kappa^{(\ell)}$. This data is free modulo the constrains (1.8), (1.41), (1.42) and the signature ++ of the restriction of g_{ab} to the 2-surface $\mathcal{S}$.

Moreover, one can reconstruct the rotation 1-form potential $\omega_a^{(\ell)}$ and the degenerate metric tensor g_{ab} on the non-extremal isolated horizon $\mathcal{H}$ via a given section $s_{v_1} : \mathcal{S} \to \mathcal{H}$ and pullbacks g_{AB} and ω_A. As already stated, the 1-form potential ω_A depends on the section. A transformation of v by an arbitrary function $f : \mathcal{S} \to H$ satisfying $\ell^a\partial_a f = 0$:

$$v' = v - f \tag{1.47}$$

results in the following change of ω_A:

$$\omega'_A = \omega_A + \kappa^{(\ell)}\partial_A f. \tag{1.48}$$

[3]There exist known exceptions, when the isolated horizon $\mathcal{H}$ admits two or more dimensional null symmetry group [14]. In such case, $\mathcal{H}$ is non-extremal with respect to one $\ell \in \Gamma(\ell)$ and extremal with respect to another $\ell_0 \in \Gamma(L)$. In consequence, in the context of isolated horizons one needs to specify (up to the constant rescaling) the null vector field ℓ generating the null symmetry (1.40).

Notice, that in the extremal case the situation is quite different. The symmetry constraint (1.40) for the isolated horizon together with eq. (1.38) imply [14]:

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \tfrac{1}{2}R_{AB} + \tfrac{1}{2}\Lambda g_{AB} = 0, \tag{1.49}$$

where the pullback ω_A (1.35) is independent of the section s_{v_1}.

Definition 1.1.5. *Given a 2-dimensional surface $\mathcal{S}$ endowed with a Riemannian metric tensor g_{AB} and a 1-form potential ω_A the eq. (1.49) is called a vacuum extremal isolated horizon equation with a cosmological constant Λ.*

Although, we will not discuss the extremal isolated horizons in detail, it is worth mentioning, that the extremal equation (1.49) has attracted interest of many mathematical relativists [14, 26, 45, 46]. Firstly, the equation provided a classification of the extremal Killing horizons. Secondly, its solution determines an exact solution to the Einstein vacuum equations with cosmological constant Λ foliated by a bifurcated Killing horizon which common part is an extremal Killing horizon [47, 48]. Such spacetimes are called the Near Horizon Geometries and describe neighborhoods of the extremal isolated Killing horizons [38, 49, 50]. For that reason the extremal equation (1.49) is often referred to as the near horizon geometry equation.

1.2 Weyl tensor of the non-extremal isolated horizon

Consider a 3-dimensional, non-extremal isolated horizon $\mathcal{H}$, endowed with the intrinsic geometry (g_{ab}, ∇_a) and the null symmetry generator $\ell \in \Gamma(L)$, contained in spacetime $\mathcal{M}$. The metric tensor $g_{\mu\nu}$ of $\mathcal{M}$ satisfies the vacuum Einstein's equations (1.1) with cosmological constant Λ.

In this section we attribute the spacetime Weyl tensor to the intrinsic geometry of the non-extremal isolated horizon $\mathcal{H}$. First, however, we provide an extension of the null vector field ℓ^a to a vector field t^μ defined in a neighborhood of the isolated horizon $\mathcal{H}$ in spacetime $\mathcal{M}$, such that:

$$t|_{\mathcal{H}} = \ell \qquad \text{and} \qquad \mathcal{L}_t g_{\mu\nu}|_{\mathcal{H}} = 0. \tag{1.50}$$

Therefore, we assume that every such t is also a symmetry of the spacetime Weyl tensor at the horizon $\mathcal{H}$.

Assumption 1.2.1. *For very vector field t satisfying (1.50) the following holds:*

$$\mathcal{L}_t C^{\alpha}{}_{\beta\gamma\delta}|_{\mathcal{H}} = 0. \tag{1.51}$$

We refer to the above assumption as the assumption of the stationarity to the second order.

1.2.1 The complex invariant of the non-extremal isolated horizon

Consider the Riemannian metric tensor g_{AB} and a differential 1-form ω_A defined on the 2-dimensional space $\mathcal{S}$ of the null geodesics in $\mathcal{H}$. One could construct two scalar invariants from this data: the Gaussian curvature[4] K and rotation scalar, denoted by $\mathcal{O}$, that is related to the rotation 2-form Ω_{AB} and the area 2-form ϵ_{AB} in the following way:

$$\Omega_{AB} =: \mathcal{O}\epsilon_{AB}. \tag{1.52}$$

A specific complex valued combination of the real valued invariants K and $\mathcal{O}$ will play a major role in this and the following chapters of the thesis.

Definition 1.2.1. *The complex valued function Ψ, defined on the space $\mathcal{S}$ of the null geodesics in $\mathcal{H}$, of the form*

$$\Psi := -\tfrac{1}{2}(K + i\mathcal{O}) \tag{1.53}$$

is called the complex invariant of the isolated horizon $\mathcal{H}$.

The Gussian curvature K may be derived from the given metric tensor g_{AB} whereas the rotation scalar $\mathcal{O}$ can be determined by the 1-form ω_A. In particular, the rotation scalar $\mathcal{O}$ may be expressed in terms of an *a priori* unconstrained function $U : \mathcal{S} \to \mathbb{R}$:

$$\mathcal{O} = -\nabla^A \nabla_A U. \tag{1.54}$$

1.2.2 Spacetime null frame adapted to the isolated horizon

At this point it is convenient to use the Newman-Penrose formalism and decompose the Weyl tensor in terms of the spacetime null frame $(e_1{}^\mu, e_2{}^\mu = \overline{e_1}{}^\mu, e_3{}^\mu, e_4{}^\mu)$ and it's dual $(e^1{}_\mu, e^2{}_\mu = \overline{e^1}{}_\mu, e^3{}_\mu, e^4{}_\mu)$. The spacetime metric tensor in the neighborhood of $\mathcal{H}$ reads:

$$g_{\mu\nu} = e^1{}_\mu e^2{}_\nu + e^1{}_\nu e^2{}_\mu - e^3{}_\mu e^4{}_\nu - e^3{}_\nu e^4{}_\mu. \tag{1.55}$$

Next, we adapt the above null frame to the isolated horizon $\mathcal{H}$. We start by setting:

$$e_4{}^a|_{\mathcal{H}} = \ell^a. \tag{1.56}$$

Therefore, the null frame is well-defined at every point $x \in \mathcal{H}$. It follows directly from the definition of the null frame, that the complex valued vector field e_1 has to be orthogonal to ℓ on the horizon $\mathcal{H}$, and therefore it is tangent to $\mathcal{H}$. Due to (1.8) one can choose

$$e_1{}^a|_{\mathcal{H}} =: m^a, \tag{1.57}$$

such that

$$[\ell, m] = 0. \tag{1.58}$$

A vector field $e_2{}^\mu$ is a complex conjugate of $e_1{}^\mu$, consequently we have:

$$e_2{}^a|_{\mathcal{H}} =: \overline{m}^a. \tag{1.59}$$

[4]which equal to half of the Ricci scalar.

Moreover, a vector field:

$$e_3{}^\mu =: n^\mu \tag{1.60}$$

is transversal to $\mathcal{H}$. Additionally, we assume that a pullback $e^4{}_a$ of $e^4{}_\mu$ coincides with n_a defined by (1.17):

$$e^4{}_a|_{\mathcal{H}} = n_a. \tag{1.61}$$

This restricts the ambiguities in the vector fields m^a and n^μ. The introduced frames together with the Newman-Penrose formalism[5] will be applied to the notion of the isolated horizons introduced in the previous section.

Definition 1.2.2. *A null frame is called adapted to the isolated horizon $\mathcal{H}$ and to the null symmetry generator ℓ if it satisfies conditions (1.56)-(1.61).*

The spacetime null 4-frame provides a 3-frame tangent to the isolated horizon $\mathcal{H}$, that is $(m^a, \overline{m}^a, \ell^a)$, whereas the corresponding dual 3-coframe $(\overline{m}_a, m_a, n_a)$ coincides with the pullback of $(e^1{}_\mu, e^2{}_\mu, e^4{}_\mu)$ to $\mathcal{H}$. From (1.58) it follows that the co-frame is Lie dragged by ℓ:

$$\mathcal{L}_\ell m_a = 0 = \mathcal{L}_\ell n_a. \tag{1.62}$$

The degenerate metric tensor in this frame reads:

$$g_{ab} = m_a \overline{m}_b + \overline{m}_a m_b. \tag{1.63}$$

One could express the components of the intrinsic covariant derivative ∇_a on the isolated horizon $\mathcal{H}$ in terms of the Newman-Penrose coefficients corresponding to the null frame:

$$\nabla_a \ell^b = \big((\alpha + \overline{\beta}) m_a + (\bar{\alpha} + \beta)\overline{m}_a - (\varepsilon + \overline{\varepsilon}) n_a\big)\ell^b, \tag{1.64}$$

$$\overline{m}^b \nabla_a n_b = \lambda m_a + \mu \overline{m}_a - \pi n_a, \tag{1.65}$$

$$m^b \nabla_a \overline{m}_b = -(\alpha - \overline{\beta}) m_a + (\overline{\alpha} - \beta)\overline{m}_a + (\varepsilon - \overline{\varepsilon}) n_a = -\overline{m}^b \nabla_a m_b. \tag{1.66}$$

On the isolated horizon $\mathcal{H}$ several equalities are satisfied [14]:

- from (1.58) it follows that:

$$\varepsilon = \overline{\varepsilon}, \tag{1.67}$$

$$\pi = \alpha + \overline{\beta}, \tag{1.68}$$

$$\kappa = 0, \tag{1.69}$$

- contacting both sides of (1.64) with ℓ^a yields:

$$\kappa^{(\ell)} = \varepsilon + \overline{\varepsilon} = \text{const}, \tag{1.70}$$

[5] Notice that vectors k and ℓ introduced in [42] correspond to vectors ℓ and n, respectively, in our work.

- from the symmetry of S_{ab} (1.19) we find that:

$$\mu = \overline{\mu}. \tag{1.71}$$

- due to the symmetry (1.40) all of the coefficient are constant along the null geodesics in $\mathcal{H}$:

$$D\alpha = D\beta = D\lambda = D\mu = D\varepsilon = 0, \tag{1.72}$$

  where $D := \ell^a \partial_a$.

The rotation 1-form potential $\omega_a^{(\ell)}$ written in the introduced co-frame takes a form:

$$\omega_a^{(\ell)} = (\alpha + \overline{\beta})m_a + (\overline{\alpha} + \beta)\overline{m}_a - \kappa^{(\ell)} n_a. \tag{1.73}$$

Furthermore, the push forward $\Pi_* m$ onto $\mathcal{S}$ is a uniquely defined vector field m^A on $\mathcal{S}$ and it's neighborhood:

$$m^A := \Pi_*{}^A{}_a m^a. \tag{1.74}$$

Consequently, $\mathcal{S}$ is equipped with a null frame for the metric g_{AB}:

$$g_{AB} = m_A \overline{m}_B + \overline{m}_A m_B. \tag{1.75}$$

One can calculate the expressions for the pullbacks (1.35) and (1.37) in terms of the Newman-Penrose coefficients expressed in the dual 2-coframe, that is:

$$\omega_A = (\alpha + \overline{\beta})m_A + (\overline{\alpha} + \beta)\overline{m}_A, \tag{1.76}$$

$$S_{AB} = \mu(m_A \overline{m}_B + \overline{m}_A m_B) + \lambda m_A m_B + \overline{\lambda}\overline{m}_A \overline{m}_B. \tag{1.77}$$

Recall, the above formula for S_{AB} is determined solely by g_{AB} and ω_A. Contracting both sides of equations (1.77) and (1.46) with $\overline{m}^A \overline{m}^B$ and comparing the results yields the expression for the function λ, whereas contracting those two equations with g^{AB} gives the coefficient μ. They read:

$$\lambda = \tfrac{1}{\kappa^{(\ell)}}\Big(\overline{\delta}\pi + \pi(\pi + \alpha - \overline{\beta})\Big), \tag{1.78}$$

$$\mu = \tfrac{1}{2\kappa^{(\ell)}}\Big(\nabla^A \omega_A + 2\pi\overline{\pi} - K + \Lambda\Big), \tag{1.79}$$

where the differential operators are defined as $\delta := m^A \partial_A$, $\overline{\delta} := \overline{m}^A \partial_A$. The divergence of the 1-form ω_A in the Newman-Penrose formalism yields:

$$\nabla^A \omega_A = \delta\pi + \overline{\delta}\overline{\pi} - (\alpha - \overline{\beta})\overline{\pi} - (\overline{\alpha} - \beta)\pi. \tag{1.80}$$

1.2.3 The Weyl tensor in Newman-Penrose formalism

The spacetime Weyl tensor $C^\mu{}_{\alpha\beta\gamma}$ may be expressed in the Newman-Penrose formalism via five complex valued components $\Psi_0, ..., \Psi_4$, that is:

$$\Psi_0 = C_{4141}, \qquad \Psi_1 = C_{4341}, \qquad \Psi_2 = C_{4123}, \qquad \Psi_3 = C_{3432}, \qquad \Psi_4 = C_{3232}. \tag{1.81}$$

When considering the components of the Weyl tensor in the null frame adapted to the isolated horizon $\mathcal{H}$ certain conditions arise. In particular, all but one component of the Weyl tensor are constant along the null generators of $\mathcal{H}$:

$$D\Psi_I = 0, \qquad\qquad I = 0, 1, 2, 3. \tag{1.82}$$

It is a consequence of the fact that on $\mathcal{H}$ the components $\Psi_0, ..., \Psi_3$ are determined by the intrinsic geometry (g_{ab}, ∇_a), of which the null vector field ℓ is an infinitesimal symmetry. The expressions for $\Psi_0, ..., \Psi_3$ will be derived, but first consider the remaining component Ψ_4. It satisfies evolution equation along the null generators of $\mathcal{H}$ with an arbitrary initial value set at a fixed transversal section of $v = v_1$. The evolution equation is one of the tensorial Bianchi identities, namely:

$$\nabla_\alpha C^\alpha{}_{\beta\gamma\delta} = 0, \tag{1.83}$$

and in the Newman-Penrose formalism reads:

$$0 = D\Psi_4 - \bar{\delta}\Psi_3 + 3\lambda\Psi_2 - 2(2\pi + \alpha)\Psi_3 + 2\kappa^{(\ell)}\Psi_4. \tag{1.84}$$

If we make an additional assumption that Ψ_4 is also constant along the null generators of the IH:

$$D\Psi_4 = 0, \tag{1.85}$$

then for the non-extremal case, that is $\kappa^{(\ell)} \neq 0$, the component Ψ_4 of the Weyl tensor can be determined by the intrinsic geometry (g_{ab}, ∇_a) of $\mathcal{H}$.

Assumption 1.2.2. *The component Ψ_4 of the Weyl tensor in the null frame adapted to the isolated horizon $\mathcal{H}$ is assumed to be constant along the null geodesics of $\mathcal{H}$.*

The Assumption 1.2.2 is of a technical nature and is implied by Assumption 1.2.1. To demonstrate it consider the vector field n^μ on $\mathcal{H}$ that is the e_3^μ element of the adapted null frame. It satisfies the normalization condition $\ell^\mu n_\mu = -1$ and is orthogonal to the space-like foliation of $\mathcal{H}$ defined by the surfaces $v = \text{const}$. We extend n^μ to a neighborhood of $\mathcal{H}$ by the following assumption:

$$n^\mu \nabla_\mu n^\nu = 0. \tag{1.86}$$

Furthermore, we define a vector field t^μ in the neighborhood of $\mathcal{H}$ satisfying:

$$t^\mu|_{\mathcal{H}} = \ell^a, \qquad \text{and} \qquad \mathcal{L}_n t = 0. \tag{1.87}$$

From the construction of the vector field t, it follows that:

$$\mathcal{L}_t e_1{}^\mu|_{\mathcal{H}} = \mathcal{L}_t e_2{}^\mu|_{\mathcal{H}} = \mathcal{L}_t e_3{}^\mu|_{\mathcal{H}} = \mathcal{L}_t e_4{}^\mu|_{\mathcal{H}} = 0 \tag{1.88}$$

and in consequence:

$$\mathcal{L}_t g_{\mu\nu}|_{\mathcal{H}} = 0. \tag{1.89}$$

From the Assumption 1.2.1 we have:

$$\mathcal{L}_t C^{\alpha}{}_{\beta\gamma\delta}|_{\mathcal{H}} = 0 \tag{1.90}$$

which in particular implies that:

$$D\Psi_4 = 0, \tag{1.91}$$

and therefore the Assumption 1.2.2 is satisfied. One could show that the two assumptions are equivalent.

As a consequence of the Assumption 1.2.2 all components of the Weyl tensor can be expressed solely by g_{ab}, ∇_a and their derivatives with respect to m^a and $\overline{m}^a$. Using the Ricci identities, often referred to as Newman-Penrose equations, and the Bianchi identities we derive the components $\Psi_0, ..., \Psi_4$ and discuss their properties.

First notice, that the components Ψ_0 and Ψ_1 vanish identically on $\mathcal{H}$:

$$\Psi_0 = -(\rho + \overline{\rho})\sigma - (3\epsilon - \overline{\epsilon})\rho + (\tau - \overline{\pi} + \overline{\alpha} + 3\beta)\kappa = 0 \tag{1.92}$$

$$\begin{aligned}\Psi_1 &= -(\alpha + \pi)\sigma - (\overline{\rho} - \overline{\epsilon})\beta + (\mu + \gamma)\kappa + (\overline{\alpha} - \overline{\pi})\epsilon \\ &= \epsilon(\beta + \overline{\alpha} - \overline{\pi}) = 0\end{aligned} \tag{1.93}$$

due to the vanishing of the shear σ and expansion ρ of the vector field ℓ together with the coefficient κ as well as the identities (1.68), (1.67). The component Ψ_2 of the Weyl tensor reads:

$$\Psi_2 = \overline{\delta}\beta - \delta\alpha + \alpha\overline{\alpha} + \beta\overline{\beta} - 2\alpha\beta + \Lambda/6. \tag{1.94}$$

Notice that it is invariant with respect to the allowed transformations of the adapted null frame and is closely related to the invariant (1.53), that is:

$$\Psi_2 = \Psi + \Lambda/6, \tag{1.95}$$

since the Gaussian curvature K and rotation scalar $\mathcal{O}$ expressed in Newman-Penrose formalism read[6]:

$$K = \delta(\alpha - \overline{\beta}) + \overline{\delta}(\overline{\alpha} - \beta) - 2(\alpha - \overline{\beta})(\overline{\alpha} - \beta), \tag{1.96}$$

$$\mathcal{O} = -\Big(\overline{\delta}\delta + \delta\overline{\delta} + (\beta - \overline{\alpha})\overline{\delta} + (\overline{\beta} - \alpha)\delta\Big)U. \tag{1.97}$$

The general expression for the component Ψ_3 is of the form:

$$\Psi_3 = \overline{\delta}\mu - \delta\lambda + \mu(\alpha + \overline{\beta}) + \lambda(\overline{\alpha} - 3\overline{\beta}). \tag{1.98}$$

The identities (1.78), (1.79) and (1.80) as well as the Bianchi identities allow one to simplify the above expression for Ψ_3, and write it as:

$$\Psi_3 = \tfrac{1}{\kappa^{(\ell)}}\big(\overline{\delta} + 3\alpha + 3\overline{\beta}\big)\Psi_2. \tag{1.99}$$

Moreover, from Bianchi identities we also find that:

$$0 = D\Psi_3 - \overline{\delta}\Psi_2 + \kappa^{(\ell)}\Psi_3 - 3\pi\Psi_2. \tag{1.100}$$

Due to the Assumption 1.2.2 the final component Ψ_4 of the Weyl tensor on the isolated horizon $\mathcal{H}$ takes the form:

$$\Psi_4 = \tfrac{1}{2\kappa^{(\ell)}}\big(\overline{\Psi}_3 - 3\lambda\Psi_2 + 2(2\pi + \alpha)\Psi_3\big). \tag{1.101}$$

[6]We will elaborate more on function U in Chapter 2.

1.2.4 The possible Petrov types of the Weyl tensor

In the previous subsection, we have shown that by assuming the vacuum Einstein's equations and stationarity to the second order (Assumptions 1.2.1 or 1.2.2) one can express the components of the Weyl tensor in terms of the intrinsic geometry (g_{ab}, ∇_a). It follows that the Petrov type of the Weyl tensor at a point $x \in \mathcal{H}$ may be attributed to the intrinsic geometry of $\mathcal{H}$. The vanishing of Ψ_0 (1.92) and Ψ_1 (1.93) implies that the null vector field ℓ is parallel to a double principal null direction of the Weyl tensor. Therefore, the possible Petrov types are 0, II, D, III or N, but not I. Type III or N require the vanishing of the component Ψ_2. However, the formulae (1.99) and (1.101) imply that for every open subset of $\mathcal{H}$ the following is true:

$$\Psi_2 = 0 \Rightarrow \Psi_3 = \Psi_4 = 0. \tag{1.102}$$

Therefore, if the Petrov type of the Weyl tensor is constant on the horizon $\mathcal{H}$, then the only possible types are 0, II or D. On the other hand, if the component Ψ_2 vanishes on the isolated horizon $\mathcal{H}$, then the Petrov type is necessarily 0, which implies:

$$K = \tfrac{1}{3}\Lambda, \qquad \text{and} \qquad \Omega_{AB} = 0. \tag{1.103}$$

Theorem 1.2.1. *Suppose $\mathcal{H}$ is a 3-dimensional non-extremal isolated horizon in a 4-dimensional spacetime such that the vacuum Einstein's equation (1.1) with cosmological constant Λ and the assumption on stationarity to the second order hold. If the Petrov type of the spacetime Weyl tensor is constant on $\mathcal{H}$, then the possible types are* 0*, II or D. In particular, the necessary and sufficient conditions for the type* 0 *on all $\mathcal{H}$ are spelled in (1.103).*

If Ψ_2 is non vanishing at some point $x \in \mathcal{H}$, then the Weyl tensor at x is either of the type II if the following is satisfied:

$$2\Psi_3^2(x) - 3\Psi_2(x)\Psi_4(x) \neq 0, \tag{1.104}$$

or of the Petrov type D otherwise, that is if

$$2\Psi_3^2(x) - 3\Psi_2(x)\Psi_4(x) = 0. \tag{1.105}$$

Notice that the expression on the left-hand side is independent of the ambiguities remaining in our choice of the null frame.

1.3 The Petrov type D isolated horizons

In this section we will consider specifically the non-extremal IHs on which the Weyl tensor is of the type D, namely the Petrov type D non-extremal isolated horizons. The explicit examples of such horizons are provided by the exact solutions to the vacuum Einstein's equations (1.1) with cosmological constant that contain non-extremal Killing horizons. Among the well known families of solutions are the non-extremal vacuum black hole solutions, namely the Schwarzschild and Kerr for the vanishing cosmological constant, Schwarzschild-(anti) de Sitter and Kerr-(anti) de Sitter for the non-vanishing cosmological constant. The characterization of the Kerr spacetime providing the uniqueness properties was studied by Mars in [51, 52].

Moreover, another class of the known type D solutions containing non-extremal IHs are the axisymmetric spacetimes that are foliated by the two, transversal to each other, families of the Killing horizons [47, 48]. Such class is also known as the near horizon geometries [38]. The Killing horizons in such spacetimes are simultaneously non-extremal and extremal, and of the type D. Every axisymmetric Petrov type D non-extremal IH in case of the vanishing cosmological constant is contained in one of the solutions listed above [25]. In the next chapter we will generalize this theorem to the $\Lambda \neq 0$, whereas in Chapter 3 we consider a 2-dimensional cross-section of the type D isolated horizon $\mathcal{H}$ which is of a higher genus.

Furthermore, there also exist IHs of type D that do not admit a global spacelike cross-sections and such will be a subject of Chapter 4. The known solutions to the Einstein equations containing such type of the horizon are the Schwarzschild/Kerr-NUT and Schwarzschild/Kerr-NUT-(anti) de Sitter spacetimes.

1.3.1 Derivation of the Petrov type D equation with cosmological constant

We now examine the conditions for the Petrov type D isolated horizon formulated in the previous section. They are imposed on the freely defined data on a cross-section $\mathcal{S}$ of the isolated horizon $\mathcal{H}$, namely:

- a Riemannian metric tensor g_{AB};
- an exact rotation 2-form Ω_{AB}, represented by a function U (1.54);
- an arbitrary value of the cosmological constant Λ.

Suppose, the surface gravity $\kappa^{(\ell)}$ takes an arbitrarily fixed nonzero value. The Petrov type D equation will be a condition on the already introduced invariant Ψ (1.53):

$$\Psi = \tfrac{1}{2}(-K + i\Delta U), \tag{1.106}$$

where $\Delta := \nabla^A \nabla_A$. We consider a metric tensor represented by a null 2-coframe $(\overline{m}_A dx^A, m_A dx^A)$:

$$g_{AB} = m_A \overline{m}_B + \overline{m}_A m_B, \tag{1.107}$$

whereas the rotation 1-form potential ω_A and rotation 2-form Ω_{AB} take a form:

$$\omega_A = (\alpha + \overline{\beta}) m_A + (\overline{\alpha} + \beta)\overline{m}_A, \tag{1.108}$$

$$\Omega_{AB} = \partial_A \omega_B - \partial_B \omega_A. \tag{1.109}$$

The coefficients α and β are defined via:

$$\alpha := \tfrac{1}{2}(n_{a;b}\ell^a \overline{m}^b - \overline{m}_{a;b} m^a \overline{m}^b) \tag{1.110}$$

$$\beta := -\tfrac{1}{2}(\ell_{a;b} n^a \overline{m}^b - m_{a;b}\overline{m}^a m^b) \tag{1.111}$$

and are completed by the formula for the commutator of the tangent 2-frame $(m^A, \overline{m}^A)$:

$$[\delta, \overline{\delta}] = (\overline{\beta} - \alpha)\delta - (\beta - \overline{\alpha})\overline{\delta}, \tag{1.112}$$

where the vector field m^A is identified with the differential operator δ:

$$\delta = m^A \partial_A. \tag{1.113}$$

Next, we write the components Ψ_2, Ψ_3 and Ψ_4 of the Weyl tensor defined on $\mathcal{S}$ in terms of the metric tensor g_{AB} and rotation 2-form Ω_{AB}:

$$\Psi_2 = \Psi + \tfrac{1}{6}\Lambda, \tag{1.114}$$
$$\Psi_3 = \tfrac{1}{\kappa^{(\ell)}}\big(\overline{\delta} + 3(\alpha + \overline{\beta})\big)\Psi_2, \tag{1.115}$$
$$\Psi_4 = \tfrac{2}{\kappa^{(\ell)}}\big(\overline{\delta}\Psi_3 - 3\lambda\Psi_2 + 2(3\alpha + 2\overline{\beta})\Psi_3\big). \tag{1.116}$$

To calculate λ we use (1.78) together with (1.68). The type D condition reads:

$$2\Psi_3^2(x) = 3\Psi_2(x)\Psi_4(x), \tag{1.117}$$

and

$$\Psi_2(x) \neq 0. \tag{1.118}$$

The formulae (1.115) and (1.116) allow one to express the type D condition (1.117) as a constraint solely on Ψ_2 that reads:

$$4(\overline{\delta}\Psi_2)^2 - 3(\alpha - \overline{\beta})\Psi_2\overline{\delta}\Psi_2 - 3\Psi_2\overline{\delta}\overline{\delta}\Psi_2 = 0. \tag{1.119}$$

The equation can be simplified and written in a compact form:

$$(\overline{\delta} + \alpha - \overline{\beta})\overline{\delta}(\Psi_2(x))^{-\frac{1}{3}} = 0. \tag{1.120}$$

One could use expression (1.114) and write the above in terms of the complex invariant Ψ:

$$(\overline{\delta} + \alpha - \overline{\beta})\overline{\delta}(\Psi(x) + \tfrac{1}{6}\Lambda)^{-\frac{1}{3}} = 0, \tag{1.121}$$

which we will refer to as the *Petrov type D equation with cosmological constant.*

The differential operators acting on Ψ_2 in (1.120) are known from the GHP formalism [42]. Introducing the so-called *edth'* operator yields:

$$\eth\eth\big(\Psi_2(x)\big)^{-\frac{1}{3}} = 0. \tag{1.122}$$

Notice, that the dependence of the rotation 1-form potential ω_A is hidden in the function ΔU present in Ψ, which is determined only by the rotation 2-form Ω_{AB}. The 2-frame vector field m^A and the corresponding differential operator δ are defined modulo the local rotations, that is:

$$m'^A = e^{i\phi} m^A, \tag{1.123}$$

where ϕ is a real valued function. Consequently, the equation (1.121) must be invariant with respect to such transformations. It is manifest after writing the type D equation in terms of the covariant derivative ∇_A of the metric tensor g_{AB}:

$$\left(\overline{\delta\delta} - (\overline{m}^B \nabla_B \overline{m}^A)\partial_A\right)\left(\Psi(x) + \tfrac{1}{6}\Lambda\right)^{-\frac{1}{3}} = 0. \tag{1.124}$$

It can be written in an even more suggestive way, which is:

$$\overline{m}^A \overline{m}^B \nabla_A \nabla_B \left(\Psi(x) + \tfrac{1}{6}\Lambda\right)^{-\frac{1}{3}} = 0 \tag{1.125}$$

which is explicitly invariant with respect to the transformations (1.123).

We have proved the following theorem:

Theorem 1.3.1. *Suppose $\mathcal{H}$ is a 3-dimensional non-extremal IH in a 4-dimensional spacetime such that the vacuum Einstein's equations (1.1) with the cosmological constant hold and the Assumption 1.2.1 on the stationarity to the second order is satisfied. Then, the necessary and sufficient condition for the spacetime Weyl tensor to be of the Petrov type D at each point $x \in \mathcal{S}$ of the null geodesic is that the invariant Ψ (1.53) satisfies the following two conditions:*

$$\Psi(x) \neq -\tfrac{1}{6}\Lambda, \tag{1.126}$$

and

$$\overline{m}^A \overline{m}^B \nabla_A \nabla_B \left(\Psi(x) + \tfrac{1}{6}\Lambda\right)^{-\frac{1}{3}} = 0. \tag{1.127}$$

Remark. One could interpret the Petrov type D equation without using a specific null frame but by a concept of the anti-holomorphic covariant derivative. For every tensor field T defined on $\mathcal{S}$, we define the following operation:

$$\nabla^{(0,1)} T := \nabla_{\overline{m}} T \otimes m_A dx^A, \tag{1.128}$$

which turns T into a new tensor. That is how the anti-holomorphic covariant derivative action on tensors works. It is invariant with respect to the local rotations (1.123), and gives yet another form of the type D equation with cosmological constant:

$$\left(\nabla^{(0,1)}\right)^2 \left(\Psi(x) + \tfrac{1}{6}\Lambda\right)^{-\frac{1}{3}} = 0. \tag{1.129}$$

1.3.2 The Petrov type D equation as an integrability condition for the near horizon geometry equation

Consider a 2-dimensional manifold $\mathcal{S}$ endowed with a metric tensor g_{AB} and a rotation 1-form ω_A together with a cosmological constant Λ. Suppose that these data satisfy the vacuum extremal isolated horizon equation with cosmological constant Λ:

$$\nabla_{(A}\omega_{B)} + \omega_A \omega_B - \tfrac{1}{2}R_{AB} + \tfrac{1}{2}\Lambda g_{AB} = 0. \tag{1.130}$$

The equation is also known as the near horizon geometry equation [38] and has been studied in particular by Ashtekar, Beetle, Lewandowski and Pawłowski in [14, 26]. We now prove, that the corresponding 2-form Ω_{AB}, the Riemannian metric tensor g_{AB} and the cosmological constant Λ simultaneously satisfy the Petrov type D equation (1.125) of the Subsection 1.3.1. The given manifold $\mathcal{S}$ together with the metric g_{AB}, 1-form ω_A and the cosmological constant Λ allow to construct on $\mathcal{S} \times \mathbb{R}$ the intrinsic geometry of a vacuum non-extremal isolated horizon $\mathcal{H}$. Since the near horizon geometry equation is satisfied, tensor S_{AB} (1.37) vanishes identically. Indeed, for the non-extremal case, that is non-vanishing $\kappa^{(\ell)}$, from expression (1.46) we see that:

$$S_{AB} = \tfrac{1}{\kappa^{(\ell)}}\Big(\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \tfrac{1}{2}R_{AB} + \tfrac{1}{2}\Lambda g_{AB}\Big) = 0. \tag{1.131}$$

Then, in expression (1.77) for the same tensor S_{AB} but written in Newman-Penrose formalism it is manifest that both coefficients λ and μ also vanish and that leads to the vanishing of the components Ψ_3 and Ψ_4, that is:

$$\Psi_3 = \bar{\delta}\mu - \delta\lambda + \mu(\alpha + \bar{\beta}) + \lambda(\bar{\alpha} - 3\bar{\beta}) = 0, \tag{1.132}$$

$$\Psi_4 = \frac{1}{2\kappa^{(\ell)}}\big(\bar{\delta}\Psi_3 - 3\lambda\Psi_2 + 2(3\alpha + 2\bar{\beta})\Psi_3\big) = 0. \tag{1.133}$$

Therefore, one of the Petrov type D conditions (1.117) is satisfied. On the other hand, the second condition (1.118) does not have to be necessarily true. In general, the invariant Ψ (1.53) of (g_{AB}, ω_A) can take any value, in particular:

$$\Psi(x) = -\tfrac{1}{6}\Lambda \tag{1.134}$$

at some $x \in \mathcal{S}$. Hence, we have just proved the following theorem:

Theorem 1.3.2. *Supposed a differential 1-form ω_A and a Riemannian metric tensor:*

$$g_{AB} = m_A\overline{m}_B + \overline{m}_A m_B, \tag{1.135}$$

both defined on a 2-dimensional manifold $\mathcal{S}$ satisfy the near horizon geometry equation with a cosmological constant:

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \tfrac{1}{2}R_{AB} + \tfrac{1}{2}\Lambda g_{AB} = 0, \tag{1.136}$$

then the invariant Ψ defined by (1.53) with

$$\Omega_{AB} = \partial_A\omega_B - \partial_B\omega_A \tag{1.137}$$

satisfies the Petrov type D equation with cosmological constant:

$$\overline{m}^A\overline{m}^B\nabla_A\nabla_B\Big(\Psi(x) + \tfrac{1}{6}\Lambda\Big)^{-\frac{1}{3}} = 0, \tag{1.138}$$

at every $x \in \mathcal{S}$ such that:

$$\Psi(x) \neq -\tfrac{1}{6}\Lambda. \tag{1.139}$$

Remark. If the 2-manifold $\mathcal{S}$ is a topological 2-sphere and the data on $\mathcal{S}$, namely the metric tensor g_{AB} and 1-form ω_A satisfy the near horizon geometry equation (1.130) then by the global argument given in [26] for the vanishing cosmological constant[7] the inequality (1.139) is satisfied at every $x \in \mathcal{S}$, unless $\Psi + \frac{1}{6}\Lambda$ vanish identically on $\mathcal{S}$. It could also be generalized with a bit more effort to the arbitrary orientable compact 2-manifold $\mathcal{S}$ [53].

To conclude, Theorem 1.3.2 may be also considered as an integrability condition for the near horizon geometry equation to investigate the space of solutions.

Remark. We observed that for every extremal isolated horizon $\mathcal{H}$ in a vacuum spacetime, the corresponding metric g_{AB} and 1-form ω_A defined on the cross-section $\mathcal{S}$ of $\mathcal{H}$ satisfy the hypothesis of the theorem 1.3.2. Therefore, the conclusion also holds, meaning the elements of the geometry also satisfy the type D equation with cosmological constant as long as (1.139) is true. Nevertheless, the Petrov type of the spacetime Weyl tensor on $\mathcal{H}$ is not necessarily (and generically is not) D. Notice that Theorem 1.3.1 holds for non-extremal isolated horizons. Recall that for the extremal isolated horizon $\kappa^{(\ell)}$ vanishes, hence the tensor S_{AB} is independent of variable v, decouples from the metric g_{AB} and 1-form ω_A, and is arbitrary on a given cross-section of $\mathcal{S}$ (1.38). The same occurs for the component Ψ_4. Moreover, the arbitrariness of S_{AB} on $\mathcal{S}$ passes the freedom to the component Ψ_3 which is manifestly seen in equations (1.77) and (1.98).

1.3.3 Non-twisting of the second principle null direction of the Weyl tensor

There exist two double principal null directions of the Weyl tensor at each point of a Petrov type D non-extremal isolated horizon $\mathcal{H}$. We already mentioned the first one, that coincides with null symmetry generator ℓ and is orthogonal to $\mathcal{H}$. The second one is generically twisting and is a subject of this subsection. Recall, that we have assumed that both null vectors e_3 and e_4 of the adapted null frame are non-twisting on $\mathcal{H}$. Hence, in the generic case one cannot choose e_3 in our frame to be pointing in the second double-principal direction. There is, however, a special case when a second principal null direction is also hyper-surface orthogonal due to the extremal isolated horizon equation (1.130) (the near horizon equation).

Consider a non-extremal isolated horizon $\mathcal{H}$ admitting a section

$$s : \mathcal{S} \to \mathcal{H} \tag{1.140}$$

such that the rotation 1-form

$$\omega_A = s^* \omega^{(\ell)}_a \tag{1.141}$$

satisfies the near horizon geometry equation (1.130). As we already pointed out, it follows that the tensor S_{AB} vanishes and so do the Newman-Penrose coefficients μ and λ. They correspond to the expansion and shear of the vector field n^μ. Hence, the null

[7] This can be easily generalized to the case when $\Lambda \neq 0$.

vector field n^μ, transversal to $\mathcal{H}$ and orthogonal to $s(\mathcal{S}) \subset \mathcal{H}$, is non-expanding and shear-free. It may be normalized as follows:

$$n_\mu \ell^\mu = -1. \tag{1.142}$$

Notice that if ℓ was vanishing on $s(\mathcal{S})$, this normalization would not be possible. Consequently, one may choose a spacetime null frame adapted to the isolated horizon $\mathcal{H}$, such that:

$$e_3|_H = n^\mu. \tag{1.143}$$

In the considered frame, as shown in the Subsection 1.3.2, the components Ψ_3 and Ψ_4 of the Weyl tensor satisfy:

$$\Psi_3|_\mathcal{H} = 0 = \Psi_4|_\mathcal{H}, \tag{1.144}$$

and, therefore, the isolated horizon $\mathcal{H}$ is of type D at section $s(\mathcal{S})$. Moreover, in such case the principal null direction of the Weyl tensor transversal to $\mathcal{H}$ is orthogonal to the 2-surface $s(\mathcal{S}) \subset \mathcal{H}$, hence it is hyper-surface orthogonal. Applying the symmetry of $\mathcal{H}$ generated by the null vector field ℓ to the slice $s(\mathcal{S})$, results in a foliation of $\mathcal{H}$ whose leaves are the 2-sections of equal properties as the original one, which means all $\mathcal{H}$ is of the Petrov type D.

Implication in the opposite direction is also true, although it is slightly more complicated. Consider the Petrov type D isolated horizon $\mathcal{H}$ such that the transversal to $\mathcal{H}$ principal null direction of the Weyl tensor is hyper-surface orthogonal. It follows that the hyper-surfaces are 2-dimensional, space-like and foliate $\mathcal{H}$. Next, suppose a section

$$s : \mathcal{S} \to \mathcal{H} \tag{1.145}$$

is such that $s(\mathcal{S})$ is the leaf of the foliation. We introduce a null frame $(e_1{}^\mu, e_2{}^\mu, e_3{}^\mu, e_4{}^\mu)$ on $\mathcal{H}$, such that $e_3{}^\mu$ is orthogonal to $s(\mathcal{S})$. Consequently, from the definition of a double principal null direction of the Weyl tensor follows that:

$$\Psi_3 = 0 = \Psi_4. \tag{1.146}$$

The vanishing of those components on $s(\mathcal{S})$ implies via (1.101) that:

$$0 = \Psi_4 = -\tfrac{3}{2\kappa^{(\ell)}}\lambda\Psi_2. \tag{1.147}$$

Since we assumed that $\mathcal{H}$ is of the Petrov type D it follows that Ψ_2 is non-vanishing and therefore it is the shear of n^ν that has to vanish:

$$\lambda = 0. \tag{1.148}$$

Consequently, the traceless part of equation (1.130) is satisfied. Next, from equation (1.98) and vanishing of the shear of n^μ we find that:

$$0 = \Psi_3 = \overline{\delta}\mu + \mu(\alpha + \overline{\beta}). \tag{1.149}$$

Therefore, there are two possibilities, either:

$$\mu = 0 \tag{1.150}$$

which implies that the trace of (1.130) vanishes, or

$$\alpha + \overline{\beta} = \frac{\overline{\delta}\mu}{\mu}, \tag{1.151}$$

from which we see that the 1-form ω_A becomes a pure gradient:

$$\omega_A = \partial_A \ln \mu. \tag{1.152}$$

Hence, the invariant rotation 2-form vanishes:

$$\Omega_{AB} = 0. \tag{1.153}$$

Remark. There is subtlety regarding the section $s(S)$ orthogonal to the assumed principal null direction of the Weyl tensor. In case of not simply connected S it may happen, that the section is not continuous. On the other hand, the pullback $s^*\omega^{(\ell)}$ is continuous and differentiable, which follows from the invariance of the foliation and of the rotation 1-form $\omega_a^{(\ell)}$ on the flow of the null vector field ℓ^a.

The following conclusions arise:

Theorem 1.3.3. *Suppose $\mathcal{H}$ is a 3-dimensional non-extremal isolated horizon in a 4-dimensional spacetime such that the vacuum Einstein's equations (1.1) with cosmological constant Λ and the Assumption 1.2.1 on stationarity to the second order (or equivalently Assumption 1.2.2) are satisfied. Let ℓ^a be the generator of the null symmetry of $\mathcal{H}$, and $s : \mathcal{S} \to \mathcal{H}$ be a section of (1.10). Then,*

(i) the null direction transversal to $\mathcal{H}$ and orthogonal to $s(\mathcal{S})$ is non-expanding and shear-free if and only if the 1-form

$$\omega_A := s^*\omega_a^{(\ell)} \tag{1.154}$$

satisfies the near horizon geometry equation (1.130).

(ii) if the null direction transversal to $\mathcal{H}$ and orthogonal to $s(\mathcal{S})$ is non-expanding and shear-free, then at every $x \in s(\mathcal{S})$ it is a double principal direction of the Weyl tensor or the Weyl tensor vanishes at x.

(iii) suppose the rotation scalar 2-form Ω_{AB} does not identically vanish on $\mathcal{S}$. If the null direction orthogonal to $s(\mathcal{S})$ and transversal to $\mathcal{H}$ is a double principal null direction of the Weyl tensor, then it is non-expanding and shear free.

(iv) the symmetry of $\mathcal{H}$ generated by ℓ^a spreads the slice $s(\mathcal{S})$ to a foliation of $\mathcal{H}$ by the slices of the same geometric properties.

Notice, that incidentally in the non-rotating case the condition (1.152) on μ, which is already determined by the Bianchi identities by ω_A, becomes a constraint. Nevertheless, there exist non-trivial solutions such as spherically symmetric section of the Schwarzschild horizon.

1.3.4 Simultaneously non-extremal and extremal isolated horizons

Now we further study the case of isolated horizons admitting two null symmetry generators: non-extremal and extremal, which was mentioned in the previous subsection. We argue that it is necessarily of the Petrov type D.

Consider a non-expanding isolated horizon $\mathcal{H}$ of a symmetry generator ℓ and intrinsic geometry (g_{ab}, ∇_a). Suppose, that another null vector ℓ tangent to $\mathcal{H}$ exist. We call it ℓ_o and assume it is non-vanishing on a danse subset of $\mathcal{H}$ and the following equalities hold:

$$\kappa^{(\ell_o)} = 0, \qquad \text{and} \qquad [\mathcal{L}_{\ell_o}, \nabla_a] = 0. \tag{1.155}$$

The relation between the two symmetry generators reads:

$$\ell = f_1 \ell_o, \qquad \text{where} \qquad df_1 \neq 0. \tag{1.156}$$

Consequently, on $\mathcal{H}$ we have not one, but two rotation 1-form potentials, that is $\omega_a^{(\ell)}$ and $\omega_a^{(\ell_o)}$ which are related via (1.25), namely:

$$\omega_a^{(\ell)} = \omega_a^{(\ell_o)} + \partial_a \ln f_1. \tag{1.157}$$

It follows that for every section $s : \mathcal{S} \to \mathcal{H}$, there are two pullback 1-forms:

$$\omega_A = s^* \omega_a^{(\ell)}, \tag{1.158}$$

$$\omega_A^{(o)} = s^* \omega_a^{(\ell_o)} = \omega_A - \partial_A \ln s^* f_1. \tag{1.159}$$

Recall, that because of the vanishing of the surface gravity $\kappa^{(\ell_o)}$, 1-form $\omega^{(\ell_o)}$ is independent of the choice of section s and satisfies the near horizon geometry equation:

$$\nabla_{(A} \omega_{B)}^{(o)} + \omega_A^{(o)} \omega_B^{(o)} - \tfrac{1}{2} R_{AB} + \tfrac{1}{2} \Lambda g_{AB} = 0. \tag{1.160}$$

Moreover, there exist a local section s such that the two null symmetry generators coincide [14]:

$$\ell|_{s(\mathcal{S})} = \ell_o, \tag{1.161}$$

where $\tilde{\mathcal{S}}$ is the corresponding cross-section of $\mathcal{H}$. Indeed, there is a function $u : \mathcal{H} \to \mathbb{R}$ satisfying:

$$Du = \kappa^{(\ell)} u, \qquad \ell_o^a \partial_a u = f_2, \qquad Df_2 = 0, \tag{1.162}$$

where $D = \ell^a \partial_a$. The sought section s is defined by the following condition:

$$u|_{s(\mathcal{S})} = \frac{f_2}{\kappa^{(\ell)}}. \tag{1.163}$$

For such section the pullbacks of $\omega_a^{(\ell_o)}$ and $\omega_a^{(\ell)}$ are equal:

$$\omega_A^{(o)} = \omega_A, \tag{1.164}$$

which implies that extremal isolated horizon equation (1.160) also holds for ω_A. It follows from the Theorem 1.3.2, that the Weyl tensor on $\mathcal{H}$ is of the Petrov type D.

Next, we show that the opposite is also true. Consider a non-extremal isolated horizon $\mathcal{H}$ of the symmetry generator ℓ, and suppose there exist a section $s : \mathcal{S} \to \mathcal{H}$, such that:

$$\omega_A = s^* \omega_a^{(\ell)} \tag{1.165}$$

satisfies the extremal isolated horizon equation (1.130). There is a null vector field ℓ_o tangent to $\mathcal{H}$ uniquely defined by:

$$\kappa^{(\ell_o)} = 0, \qquad \text{and} \qquad \ell_o|_{s(\mathcal{S})} = \ell. \tag{1.166}$$

It is nowhere vanishing and from (1.130) follows that [48]:

$$[\mathcal{L}_{\ell_o}, \nabla_a] = 0 \tag{1.167}$$

everywhere on the isolated horizon $\mathcal{H}$. Therefore, $\mathcal{H}$ is an extremal isolated horizon, with two null symmetries: extremal ℓ_o and non-extremal ℓ. We sum up with the following theorem:

Theorem 1.3.4. *Suppose $\mathcal{H}$ is a 3-dimensional non-extremal isolated horizon in a 4-dimensional spacetime such that the vacuum Einstein's equations (1.1) with cosmological constant Λ and assumption 1.2.1 on stationarity to the second order (or equivalently assumption 1.2.2) are satisfied. Let ℓ^a be the generator of the null symmetry of $\mathcal{H}$.*

(i) Suppose there exists a null vector field ℓ_o tangent to $\mathcal{H}$ and nowhere vanishing on $\mathcal{H}$, and satisfying:

$$\kappa^{(\ell_o)} = 0, \qquad \textit{and} \qquad [\mathcal{L}_{\ell_o}, \nabla_a] = 0, \tag{1.168}$$

then at every point $x \in \mathcal{H}$ the spacetime Weyl tensor is either of the Petrov type D or 0.

(ii) Suppose there is a section $s : \mathcal{S} \to \mathcal{H}$ such that the corresponding ω_A (1.35) satisfies the near horizon geometry equation (1.130). Then there exists a nowhere vanishing function f defined on $\mathcal{H}$ such that:

$$\ell_o := f\ell \tag{1.169}$$

satisfies

$$\tag{1.170}$$

$$\kappa^{(\ell_o)} = 0 \qquad \textit{and} \qquad [\mathcal{L}_{\ell_o}, \nabla_a] = 0. \tag{1.171}$$

Chapter 2

Petrov type D equation on an axisymmetric 2-sphere

This chapter is a continuation of our study on the non-extremal isolated horizon $\mathcal{H}$ embeddable in 4-dimensional spacetimes satisfying the vacuum Einstein's equations (1.1) with cosmological constant Λ. We assume the stationarity to the second order (1.2.1) and type D of the spacetime Weyl tensor on the IH. We find all axisymmetric solutions to the type D equation (1.125) with cosmological constant on topological 2-sphere and discuss their properties and embeddability in the known type D spacetimes. Last but not least, we formulate a local non-hair theorem for the type D axisymmetric isolated horizons with topologically spherical cross-sections.

2.1 Axial symmetry of the isolated horizon $\mathcal{H}$

We consider an axisymmetric[1] $\mathcal{H}$ such that the space $\mathcal{S}$ of the null generators is diffeomorphic to a 2-dimensional sphere.

2.1.1 The generator of the 1-dimensional group of rotations of isolated horizon

Suppose $\Phi^a \in \Gamma(T(\mathcal{H}))$ is the generator of the 1-dimensional group of rotations of $\mathcal{H}$ preserving the intrinsic geometry (g_{ab}, ∇_a) invariant. That is:

$$\mathcal{L}_\Phi g_{ab} = 0, \qquad \text{and} \qquad [\mathcal{L}_\Phi, \nabla_a] = 0. \tag{2.1}$$

As a consequence, the vector field Φ^a Lie drags the rotation 2-form invariant Ω_{ab}:

$$\mathcal{L}_\Phi \Omega_{ab} = 0. \tag{2.2}$$

The push forward of the vector filed Φ^a, via the projection (1.10) $\Pi : \mathcal{H} \to \mathcal{S}$ uniquely defines a vector field Φ^A on $\mathcal{S}$, that is $\Phi^A \in \Gamma(T(\mathcal{S}))$. It becomes a Killing vector field

[1] The assumption of axial symmetry is not needed in case of the bifurcated horizon with both components of the Petrov type D. It follows that the geometry of the horizon necessarily admits axial symmetry [54]. It was proven locally regardless of rigidity theorem.

of the metric tensor g_{AB} and Lie drags the pullback Ω_{AB}:

$$\mathcal{L}_\Phi g_{AB} = 0 \qquad \text{and} \qquad \mathcal{L}_\Phi \Omega AB = 0. \tag{2.3}$$

Hence, by solving the Petrov type D equation (1.125) on a 2-sphere assuming the axial symmetry of the unknowns: metric g_{AB} and an exact 2-form Ω_{AB}, we find all the Petrov type D isolated horizons in this class.

2.1.2 Coordinates adapted to axial symmetry

To solve the Petrov type D on the axisymmetric 2-sphere it is convenient to introduce the coordinates adapted to the symmetry. First, however, notice that the 1-form ω_A (1.76) may be expressed via Hodge decomposition in terms of the rotation potential U and function B encoding its exact and coexact part respectively:

$$\omega_A^{(\ell)} = d\ln B_A + \star dU_A. \tag{2.4}$$

The coexact part is always invariant whereas the exact one is a pure gauge provided non-vanishing surface gravity $\kappa^{(\ell)}$. In particular $d\ln B$ can be set to zero by an appropriate choice of v. This gauge condition will be used from now on.

Let the coordinate v compatible with ℓ be chosen in a way that the orbits of axial symmetry lie entirely on the constancy surfaces of v. This condition is compatible with the requirement discussed above, namely the vanishing of the exact part of the rotation 1-form ω_A. Furthermore, one can choose such frame vectors e_1 and e_2 which are preserved by the axial symmetry.

The system adapted to the axial symmetry will be defined via coordinate system (x, φ), which are constructed in the following way. Consider a 2-sphere metric tensor with a conformal factor Σ, independent of φ due to the symmetry, in the standard angular coordinates (θ, φ):

$$g_{AB}dx^A dx^B = \Sigma^2(\theta)(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.5}$$

The area element of such metric tensor is of the form:

$$\epsilon = \Sigma^2 \sin\theta \; d\theta \wedge d\varphi. \tag{2.6}$$

We introduce a coordinate x via tranformation:

$$dx = \frac{\Sigma^2 \sin\theta}{R^2} d\theta, \tag{2.7}$$

where R^2 is defined to be the area radius satisfying:

$$A = 4\pi R^2. \tag{2.8}$$

Next, we introduce a function P:

$$P^2 = \tfrac{\Sigma^2 \sin^2\theta}{R^2}, \tag{2.9}$$

which we refer to as a frame coefficient. From calculating the area of the transformed metric g_{AB}:

$$A = \int \epsilon = 2\pi R^2 (x_1 - x_0), \tag{2.10}$$

follows that the difference of x_1 and x_0 has to be equal to 2. Since x has been defined up to an additive constant, we can set $x_1 = 1$ and $x_0 = -1$. The coordinate φ is related to the infinitesimal axial symmetry generator Φ via:

$$\Phi = \partial_\varphi \tag{2.11}$$

and the curves $\varphi = \text{const}$ are orthogonal to the infinitesimal symmetry of ℓ. Consequently, we write the metric tensor g_{AB} in the following form:

$$g_{AB} dx^A dx^B = R^2 \left(\frac{1}{P^2(x)} dx^2 + P^2(x) d\varphi^2 \right), \tag{2.12}$$

while the 2-frame vector e_1 and its dual read:

$$e_1 = m^A \partial_A = \frac{1}{R\sqrt{2}} \left(P(x) \partial_x + i \frac{1}{P(x)} \partial_\varphi \right), \tag{2.13}$$

$$e^1 = \overline{m}_A dx^A = \frac{R}{\sqrt{2}} \left(\frac{1}{P(x)} dx - i P(x) d\varphi \right). \tag{2.14}$$

Next, we study the conditions for the regularity of the metric tensor. First, from definition of the frame coefficient P (2.9) follows that it should vanish at the poles (corresponding to $(\theta = \pi, 0)$) of the 2-sphere:

$$P(x = \pm 1) = 0. \tag{2.15}$$

Moreover, the condition for the axisymmetric scalar function f defined globally on a 2-sphere $\mathcal{S}_2$ for the lack of the conical singularity (differentiability at the poles) reads

$$\partial_\theta f|_{\theta=0,\pi} = 0, \tag{2.16}$$

and its equivalent to

$$P \partial_x f|_{x=\pm 1} = 0. \tag{2.17}$$

We impose this condition on the Ψ_2 component of the Weyl tensor, and it passes to the Gaussian curvature K and rotation scalar $\mathcal{O}$. For the metric tensor g_{AB} (2.12) to be at least once differentiable at the poles, it has to satisfy condition (2.15), as well as:

$$\begin{aligned}
\partial_x P^2(x)\Big|_{x=\pm 1} &= 2 \left(P(\theta) \frac{d\theta}{dx} \partial_\theta P(\theta) \right) \Big|_{\theta=0,\pi} \\
&= 2 \left(\frac{\Sigma \sin\theta}{R} \cdot \frac{R^2}{\Sigma^2 \sin\theta} \left(\Sigma_{,\theta} \frac{\sin\theta}{R} + \Sigma \frac{\cos\theta}{R} \right) \right) \Big|_{\theta=0,\pi} \\
&= 2 \left(\Sigma_{,\theta} \frac{\sin\theta}{\Sigma} + \cos\theta \right) \Big|_{\theta=0,\pi} \\
&= \mp 2,
\end{aligned} \tag{2.18}$$

where in the last equality we used the fact that $\Sigma_{,\theta}$ has to vanish on the poles in accordance with eq. (2.16). An analogous assumption on the lack of the conical singularity is imposed on the rotation potential U, which results with the following constraint[2]:

$$P\partial_x U\big|_{x=\pm 1} = 0. \tag{2.19}$$

Definition 2.1.1. The coordinates (x, φ) are called the coordinates adapted to the axial symmetry.

Remark. In Appendix A we show that conditions (2.15) and (2.18) are necessary and sufficient for the metric tensor g_{AB} (2.12) to be continuous and differentiable at the poles.

2.1.3 The type D equation in the coordinates adapted to the axial symmetry

Consider the type D equation with cosmological constant written in terms of the Newman-Penrose coefficients:

$$\left(\overline{\delta} + \alpha - \overline{\beta}\right)\overline{\delta}\Psi_2^{-\frac{1}{3}} = 0, \tag{2.20}$$

where $\delta = m^A\partial_A$. The component Ψ_2 of the Weyl tensor may be expressed by the Gaussian curvature K and rotation potential function U, that is:

$$\Psi_2 = \tfrac{1}{2}(-K + i\Delta U) + \tfrac{1}{6}\Lambda. \tag{2.21}$$

In the introduced coordinate system we calculate the connection component $\alpha - \overline{\beta}$, Gaussian curvature K and Laplacian $\nabla^A\nabla_A$ acting on scalar functions:

$$\alpha - \overline{\beta} = \overline{m}^A\delta\overline{m}_A - m^A\overline{\delta}\overline{m}_A = -\tfrac{1}{R\sqrt{2}}\partial_x P, \tag{2.22}$$

$$K = \delta(\alpha - \overline{\beta}) + \overline{\delta}(\overline{\alpha} - \beta) - 2(\alpha - \overline{\beta})(\overline{\alpha} - \beta) = -\tfrac{1}{2R^2}\partial_x^2 P^2, \tag{2.23}$$

$$\nabla^A\nabla_A = \overline{\delta}\delta + \delta\overline{\delta} + (\beta - \overline{\alpha})\overline{\delta} + (\overline{\beta} - \alpha)\delta = \tfrac{1}{R^2}\partial_x(P^2\partial_x). \tag{2.24}$$

Consequently, the type D equation (2.21) becomes the following complex ordinary differential equation for Ψ_2:

$$0 = \left(\tfrac{1}{R\sqrt{2}}P\partial_x - \tfrac{1}{R\sqrt{2}}\partial_x P\right)\tfrac{1}{R\sqrt{2}}P\partial_x\Psi_2^{-\frac{1}{3}} \tag{2.25}$$

$$= \tfrac{P^2}{2R^2}\partial_x^2\Psi_2^{-\frac{1}{3}}. \tag{2.26}$$

Integrating the above twice yields:

$$\Psi_2 = (c_1 x + c_2)^{-3}, \tag{2.27}$$

[2]See expression for the Laplacian $\nabla^A\nabla_A$ acting on scalar functions in the coordinates adapted to the axial symmetry written explicitly in eq. (2.24).

where c_1 and c_2 are complex constants. Next, we combine the two expressions, that is (2.21) and (4.24), for the component Ψ_2 to obtain:

$$\frac{4R^2}{(c_1x+c_2)^3} = \partial_x^2 P^2 + 2i\partial_x(P^2\partial_x U) + \tfrac{2}{3}R^2\Lambda. \tag{2.28}$$

Remark. A geometric relation between the symmetry generator Φ and the component Ψ_2 as a consequence of (4.24) arises[3]:

$$d\left(\Psi_2^{-\frac{1}{3}}\right) = -\frac{c_1}{R^2}\Phi \lrcorner \epsilon. \tag{2.29}$$

2.2 Solution to the type D equation on a 2-sphere for the axisymmetric isolated horizon and its embeddability

In this section we find solutions to the type D equation with cosmological constant on a 2-sphere using the coordinates adapted to the axial symmetry introduced in the previous section. We will now consider two cases, first for the vanishing constant c_1 resulting in the non-rotating solution and second, generic one, for arbitrary constants c_1 and c_2 admitting rotating horizons.

Since the cosmological constant Λ usually appears in the equations of this section with a coefficient $\frac{1}{6}$, it is convenient to introduce a rescaled cosmological constant, namely:

$$\Lambda' := \tfrac{1}{6}\Lambda. \tag{2.30}$$

2.2.1 The non-rotating solution

Suppose the integration complex constant c_1 vanishes:

$$c_1 = 0. \tag{2.31}$$

Hence, now the equation (2.28) reads:

$$\frac{4R^2}{c_2^3} = \partial_x^2 P^2 + 2i\partial_x(P^2\partial_x U) + 4R^2\Lambda'. \tag{2.32}$$

We may integrate it and apply the regularity conditions (2.15), (2.18) and (2.19) to obtain a relation between the area radius R^2 and c_2, that is:

$$R^2 = \frac{-c_2^3}{2(1-c_2^3\Lambda')}. \tag{2.33}$$

Since R^2 has to be positive, the constant c_2 has to be real and the following must hold:

$$c_2^3 \in \Big(-\infty, 0\Big) \vee \Big(\frac{1}{\Lambda'}, \infty\Big), \qquad \text{for } \Lambda > 0; \tag{2.34}$$

$$c_2^3 \in \Big(-\infty, \tfrac{1}{\Lambda'}\Big), \qquad \text{for } \Lambda < 0. \tag{2.35}$$

[3]It is also known in the $\Lambda = 0$ case [25].

Next, we input expression (2.33) for R^2 to eq. (2.32) and obtain:

$$-2 = \partial_x^2 P^2 + 2iP^2\partial_x U. \tag{2.36}$$

Performing yet another integration, however only of the real part of the above, results in the expression for the frame coefficient:

$$P^2 = 1 - x^2. \tag{2.37}$$

Notice that one by substituting P^2 into equation (2.36) obtains an equation on the rotation potential U:

$$0 = (1 - x^2)\partial_x U, \tag{2.38}$$

that must hold for all $x \in [-1, 1]$. Therefore, the only possible solution is:

$$U = \text{const}. \tag{2.39}$$

Moreover, it follows that the Ψ_2 component of the Weyl tensor is constant on the entire isolated horizon $\mathcal{H}$:

$$\Psi_2 = \text{const}. \tag{2.40}$$

Summarizing, the general solution to the equation (2.28) with vanishing integration constant c_1 is given by the function $P^2 : [-1, 1] \to \mathbb{R}$ (2.37), constant function U and arbitrary constant:

$$R > 0. \tag{2.41}$$

They give rise to the metric g_{AB} (2.12), whereas the rotation 1-form potential ω_A vanishes:

$$\omega_A = 0. \tag{2.42}$$

The Gaussian curvature K is constant:

$$K = -\tfrac{1}{2R^2}\partial_x^2 P^2 = 2\Big(\Lambda - \tfrac{1}{c_2^3}\Big), \tag{2.43}$$

therefore, g_{AB} is a round sphere metric tensor of the radius R.

Furthermore, there exists a coordinate $\theta(x)$ such that the metric tensor on $\mathcal{S}$ is of the canonical form, that is:

$$g_{AB}dx^A dx^B = R^2\left(\tfrac{1}{P^2}dx^2 + P^2 d\varphi^2\right) = R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{2.44}$$

By the comparison of coefficients in (2.44) and the fact that in both cases the infinitesimal axial symmetry ∂_φ is suitably normalized we obtain the relation between the coordinates θ and x that reads:

$$1 - x^2 = \sin^2\theta. \tag{2.45}$$

From the above relation and the second equation in (2.44) follows:

$$\frac{R^2}{1-x^2}dx^2 = R^2 d\theta^2$$
$$x = -\cos\theta. \tag{2.46}$$

Since, conversely, every spherically symmetric solution to the type D equation is of this form, we conclude that the family of spherically symmetric type D isolated horizons that satisfy vacuum Einstein's equations (1.1) with cosmological constant Λ is 1-dimensional and may be parametrized by the area radius R.

2.2.2 Embeddability of the non-rotating 1-dimensional family of spherically symmetric isolated horizons

In this subsection we will study the embeddability of the obtained 1-dimensional spherically symmetric non-extremal IHs of the type D in the generalized Schwarzschild-(anti) de Sitter spacetime.

Such spacetime is a static, spherically symmetric black hole solution to Einstein's equations (1.1) with the cosmological constant Λ as a source term. It is characterized by the parameters: M - the black hole mass and Λ - the cosmological constant. Both of those parameters have to be positive in order to obtain the Schwarzschild-de Sitter spacetime. In case of the negative cosmological constant we get the Schwarzschild-anti de Sitter spacetime. In static coordinates the metric takes a form:

$$ds^2 = -\left(1 - \tfrac{2M}{r} - \tfrac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.47}$$

Furthermore, every spacetime in this family is of the Petrov type D.

The vanishing of expansion of the outgoing null geodesics indicates the position of the event horizons[4] in this spacetime. The expansion is proportional to g^{rr}, therefore we obtain a truncated cubic equation:

$$g^{rr} = 1 - \tfrac{2M}{r} - \tfrac{\Lambda r^2}{3} = 0, \tag{2.48}$$

which in general has three complex roots. All of the roots are real if the following condition is satisfied:

$$9M^2\Lambda < 1. \tag{2.49}$$

However, even in such case, only two of them are positive [56]. The smaller one corresponds to the Schwarzschild black hole horizon and the larger to the cosmological horizon. Every horizon obtained from (2.48), as long as it is non-extremal, is a type D isolated horizon introduced in Section 1.3.

For every value of the area radius R from our solution, such that R is a root of the horizon equation (2.48), the parameter M in (2.47) can be determined. Due to (2.48) we observe that:

$$M = \tfrac{R}{2}\Big(1 - \tfrac{\Lambda}{3}R^2\Big). \tag{2.50}$$

[4]To be specific, this technique will provide the locations of apparent horizons [55], however, since the spacetime is static the location of apparent and event horizon is the same.

For a positive cosmological constant Λ and

$$R = \sqrt{\tfrac{3}{\Lambda}} \tag{2.51}$$

we obtain de Sitter metric. There exists also a nontrivial case, namely:

$$\Lambda > 0, \qquad \text{and} \qquad R > \tfrac{3}{\Lambda}. \tag{2.52}$$

that results in the constraint on the mass parameter:

$$M < 0, \tag{2.53}$$

and a nonphysical metric tensor which still contains a non-extremal type D isolated horizon.

Recall, that if an isolated horizon is non-extremal the tensor:

$$\begin{aligned} &\mu'(m_A\overline{m}_B + m_B\overline{m}_A) + \lambda' m_A m_B + \overline{\lambda}'\overline{m}_A\overline{m}_B \\ &\quad = \tfrac{1}{\kappa^{(\ell)}}\left(\nabla_{(A}(\omega_{B)} + f_{,B)}) + (\omega_A + f_{,A})(\omega_B + f_{,B}) - \tfrac{1}{2}Kq_{AB} + \tfrac{1}{2}\Lambda q_{AB}\right) \end{aligned} \tag{2.54}$$

for every function f does not vanish. Otherwise it is extremal. In case of the spherically symmetric intrinsic geometry of the horizon it is sufficient to consider such f that:

$$\omega_A + f_{,A} = 0. \tag{2.55}$$

Consequently, the right hand side of the equation (2.54) significantly simplifies, namely:

$$\begin{aligned} &\tfrac{1}{\kappa^{(\ell)}}\left(\nabla_{(A}(\omega_{B)} + f_{,B)}) + (\omega_A + f_{,A})(\omega_B + f_{,B}) - \tfrac{1}{2}Kq_{AB} + \tfrac{1}{2}\Lambda q_{AB}\right) \\ &\quad = \tfrac{1}{2}\Big(- \tfrac{1}{R^2} + \Lambda\Big) q_{AB}. \end{aligned} \tag{2.56}$$

Therefore, the condition for the non-extremal Killing horizon in spacetime (2.47) becomes:

$$R^2 \neq \tfrac{1}{\Lambda}. \tag{2.57}$$

There is yet another extremal solution that arises from the condition for a repeated root in equation (2.48) which yields:

$$c_2^3 = -\tfrac{3}{\Lambda} \tag{2.58}$$

or eqiuvalently:

$$R^2 = \tfrac{1}{\Lambda}. \tag{2.59}$$

This solution is embeddable in the near horizon geometry spacetime that is obtained by the near horizon limit from spacetime (2.47), [38]. It is an example of an isolated horizon that is both, extremal and non-extremal [14, 48].

2.2.3 The rotating solution

In an analogous way we proceed with the study of the general case when:

$$c_1 \neq 0. \tag{2.60}$$

Going back to eq. (2.28), integrating it and applying regularity conditions (2.15) and (2.18) for the frame coefficient yields:

$$\frac{-2R^2}{c_1(c_1x+c_2)^2}\Bigg|_{x=-1}^{x=1} = \left(\partial_x P^2 + 2iP^2\partial_x U + 4R^2\Lambda' x\right)\Bigg|_{x=-1}^{x=1} \tag{2.61}$$

from which it follows that:

$$\frac{-2R^2}{c_1(c_1+c_2)^2} + \frac{2R^2}{c_1(-c_1+c_2)^2} = -4 + 8R^2\Lambda'. \tag{2.62}$$

From the above equality we find the relation between R^2 and parameters c_1, c_2:

$$R^2 = \tfrac{(c_2^2-c_1^2)^2}{-2c_2+2\Lambda'(c_2^2-c_1^2)^2}. \tag{2.63}$$

Radius R is real and therefore we we obtain an additional constrain on the constants c_1 and c_2:

$$\mathrm{Im}\left[\frac{c_2}{-c_2+\Lambda'(c_2^2-c_1^2)^2}\right] = 0. \tag{2.64}$$

Introducing a new real parameter γ defined via:

$$\gamma = \tfrac{(c_2^2-c_1^2)^2}{c_2}, \tag{2.65}$$

where we assumed that the constant c_2 is non-zero[5], simplifies the expression for the area radius R:

$$R^2 = \frac{1}{2}\frac{\gamma}{\Lambda'\gamma - 1}. \tag{2.66}$$

The regularity conditions on the poles (2.15), (2.18) and (2.19) fix the additive constant freedom:

$$\begin{aligned}\partial_x P^2 + 2iP^2\partial_x U = &- \frac{(c_2^2-c_1^2)^2}{c_1(c_1x+c_2)^2\left(\left(c_2^2-c_1^2\right)^2\Lambda' - c_2\right)} - 2\frac{\Lambda'(c_2^2-c_1^2)^2 x}{\Lambda'(c_2^2-c_1^2)^2 - c_2} \\ &+ \frac{c_2^2+c_1^2}{c_1\left(\left(c_2^2-c_1^2\right)^2\Lambda' - c_2\right)}.\end{aligned} \tag{2.67}$$

Integrating a real part of the above and evaluating the result at the poles yields:

$$\mathrm{Re}\left[\frac{c_2}{(c_2^2-c_1^2)^2\Lambda' - c_2}\right] = 0. \tag{2.68}$$

[5] Recall that for vanishing constant c_2 the geometry of the isolated horizon is not well-defined at $x=0$. Such case will be excluded from our considerations.

Finally, owing to eq. (2.64), the right-hand side of eq. (2.67) simplifies to:

$$\partial_x P^2 + 2iP^2\partial_x U = -\,2x + \frac{1}{1-\gamma\Lambda'}\frac{1}{\zeta}\left[\frac{(1-\zeta^2)^2}{(x+\zeta)^2} - \zeta^2 - 1 + 2\zeta x\right], \tag{2.69}$$

where a purely imaginary parameter $\zeta = c_2/c_1$ has been introduced.

In order to obtain the expression for the frame coefficient P^2 we integrate the real part of the above equation and use yet another parameter $\eta = -i\zeta$, which yields:

$$P^2 = 1 - x^2 + \frac{1}{1-\gamma\Lambda'}\frac{(x^2-1)^2}{x^2+\eta^2}. \tag{2.70}$$

Combining eq. (2.67) and (2.70) allows one to find:

$$\partial_x U = \frac{1}{2\eta}\frac{3\eta^4 - x^2 + \eta^2(x^2+1)}{(x^2+\eta^2)(\eta^2+1-\gamma\Lambda'(x^2+\eta^2))}. \tag{2.71}$$

It is clear that condition (2.19) is satisfied since the following are true:

$$\begin{aligned} P\Big|_{x=\pm 1} &= 0, \\ \partial_x U\Big|_{x=\pm 1} &= \frac{1}{2\eta}\frac{3\eta^4 + 2\eta^2 - 1}{(1-\gamma\Lambda')(1+\eta^2)^2}, \\ 1-\gamma\Lambda' &\neq 0, \\ \eta &\neq 0. \end{aligned}$$

From the expression (2.71) it is straightforward to calculate the rotation 1-form ω_A via (2.4), while keeping in mind that we use such gauge fixing for which $d\ln B$ vanishes:

$$\begin{aligned} \omega_A dx^A &= \epsilon_{\varphi x} g^{xx} dU_x d\varphi = -P^2\partial_x U d\varphi \\ &= \frac{(1-x^2)\left(3\eta^4+\eta^2+x^2(\eta^2-1)\right)}{2\eta(\gamma\Lambda'-1)(x^2+\eta^2)^2} d\varphi. \end{aligned} \tag{2.72}$$

The component Ψ_2 of the spacetime Weyl tensor on the cross-section $\mathcal{S}$ of the horizon written in terms of parameters η and γ is of the form[6]:

$$\Psi_2 = \frac{(\eta^2+1)^2}{i\gamma\eta(x+i\eta)^3}. \tag{2.73}$$

In summary, the general solution to the type D equation with cosmological constant (2.28) on the axisymmetric IH for the non-vanishing integration constant c_1 is given by the function $P^2 : [-1,1] \to \mathbb{R}$ (2.70), arbitrary radius R^2 (2.66) and the rotation 1-form ω_A (2.72), that are expressed in terms of the real parameters η, γ and cosmological constant Λ.

Next, we study the conditions, that must be satisfied by the parameters η and γ in order for the metric tensor g_{AB} to be well-defined. One has to assure that the square of the area radius R^2 is positive:

$$R^2 = \frac{\gamma}{\Lambda'\gamma - 1} > 0 \Leftrightarrow \Lambda' > \frac{1}{\gamma}. \tag{2.74}$$

[6] Recall, that the function Ψ_2 as a solution to eq. (2.25) is defined up to the constant factor.

The same goes for the frame coefficient P^2 on its domain $(-1,1)$:

$$
\begin{aligned}
P^2 = \left(1-x^2\right)\left(1+\frac{1}{1-\Lambda'\gamma}\frac{1-x^2}{x^2+\eta^2}\right) > 0 \\
\Leftrightarrow x^2+\eta^2+\frac{1}{1-\Lambda'\gamma}\left(1-x^2\right) > 0
\end{aligned}
\tag{2.75}
$$

Therefore, one of the following must hold:

(i) $\gamma < 0$, or

(ii) $\gamma > 0 \wedge \eta^2 > \frac{1}{\Lambda'\gamma-1}$.

To conclude, for the cosmological constant $\Lambda > 0$, γ is either negative, or positive while satisfying the following inequalities:

$$
\Lambda' > \frac{1}{\gamma} \qquad \text{and} \qquad \eta^2 > \frac{1}{\Lambda'\gamma - 1}. \tag{2.76}
$$

For the negative values of Λ, the parameter γ has to be negative and the following must hold:

$$
\Lambda' > \frac{1}{\gamma}. \tag{2.77}
$$

Those constraints play a significant role in finding the range of values for the area and angular momentum, as well as the study of embeddability of the isolated horizon $\mathcal{H}$ of the topology $\mathcal{S}_2 \times \mathbb{R}$, which both are the subject of the next two subsections.

Remark. We have shown that the frame coefficient P^2 is continuous and at least once differentiable if eq. (2.15) and (2.18) are satisfied. We assume that Ψ_2 is well-defined and therefore the complex constant c_2 cannot vanish. Otherwise there would be a singularity at $x=0$. Therefore, since Ψ_2 is well-defined, from the form of the type D equation (2.28), follows that the second derivative of P^2 also exists. For the non-vanishing complex constant c_1 expression for Ψ_2 (4.24) maybe written in the following form:

$$
\Psi_2 = \frac{1}{(c_1 x + c_2)^3} = \frac{1}{c_1^3 (x + \frac{c_2}{c_1})^3}, \tag{2.78}
$$

where c_2/c_1, as we previously noticed, is a purely imaginary constant. It follows that the above function is of the class C^∞ on $[-1,1]$ as long as $c_2 \neq 0$ and odd derivatives at the poles vanish. Therefore, the real and imaginary part of the right-hand side of equation (2.28) must be simultaneously infinitely differentiable. For the vanishing constant c_1 the left-hand side of (2.28) is constant and the same result holds trivially.

2.2.4 The area and angular momentum of the axially symmetric isolated horizon of topology $\mathcal{S}_2 \times \mathbb{R}$

We now have all of the elements required to calculate the area A and angular momentum J in terms of the rescaled cosmological constant Λ' and parameters η, γ. First, we find the area via eq. (2.8), that is:

$$A = 4\pi R^2 = 2\pi \frac{\gamma}{\Lambda'\gamma - 1}. \tag{2.79}$$

The imaginary part of the component Ψ_2 of the Weyl tensor is used to calculate the angular momentum J, namely:

$$J = -\frac{1}{4\pi} \int_{\mathcal{S}_2} \epsilon \phi \mathrm{Im}\Psi_2, \tag{2.80}$$

where the function ϕ is defined up to an additive constant as the generator of the vector field Φ:

$$\phi_{,B} := \Phi^A \epsilon_{AB}. \tag{2.81}$$

Integrating eq. (2.80) by parts and using the constraint (2.19) yields:

$$\begin{aligned} J &= \frac{1}{8\pi} \int_0^{2\pi} d\varphi \int_{-1}^{1} dx P^2 \partial_x U \partial_x \phi \\ &= \frac{1}{4(\Lambda'\gamma - 1)^2} \frac{\gamma}{\eta}. \end{aligned} \tag{2.82}$$

Furthermore, we check weather the obtained solution could be parametrized by the area A and angular momentum J. Suppose the data consisting of the area and angular momentum (A_i, J_i) with the corresponding parameters (γ_i, η_i) satisfies the equations (2.79) and (2.82). It follows that:

$$A_1 = A_2 \Leftrightarrow \gamma_1 = \gamma_2; \tag{2.83}$$

$$J_1 = J_2 \Leftrightarrow \eta_1 = \eta_2. \tag{2.84}$$

Next, from the parameters' constraints (2.75), in case of the positive (rescaled) cosmological constant Λ' we find that for negative parameter γ the following must hold:

$$A \in \left(0, \tfrac{2\pi}{\Lambda'}\right) \qquad \text{and} \qquad J \in (-\infty, 0) \vee (0, \infty), \tag{2.85}$$

whereas for the positive γ we have:

$$A \in (\tfrac{2\pi}{\Lambda'}, \infty) \qquad \text{and} \qquad |J| \in \left(0, \tfrac{A}{8\pi}\sqrt{\tfrac{\Lambda' A}{2\pi} - 1}\right). \tag{2.86}$$

For the negative cosmological constant, on the other hand, γ can be only negative, and we obtain that:

$$A \in (0, \infty) \qquad \text{and} \qquad J \in (-\infty, 0) \vee (0, \infty). \tag{2.87}$$

To sum up, from the above considerations follows that there is a one-to-one correspondence between pair consisting of the area and angular momentum (A, J) and the parameters (η, γ). In other words, the specific values of A and J uniquely determine the type D isolated horizon.

2.2.5 Embeddability in the generalized Kerr-(anti) de Sitter spacetimes

Analogously to Subsection 2.2.2, we now study the embeddability of the obtained solution, this time in the (generalized) Kerr-(anti) de Sitter spacetimes. It is a 2-dimensional family of spacetimes of axisymmetric solutions to vacuum Einstein's equations (1.1) with cosmological constant Λ, which are describing a black hole with Kerr (rotational) and mass parameters: a and M respectively. The solution was found by Carter and published in the 1973 *Les Houches* lectures [57]. Just like (2.47) it is contained in the Plebański-Demiański family of solutions [58], which is the most general solution of the Petrov type D. In the Boyer-Lindquist like coordinates the Kerr-(anti) de Sitter metric takes the following form:

$$\begin{aligned} ds^2 =& \varrho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) + \frac{\Delta_\theta \sin^2\theta}{\varrho^2} \left(a\frac{dt}{\Xi} - (r^2+a^2)\frac{d\phi}{\Xi} \right)^2 \\ &- \frac{\Delta_r}{\varrho^2} \left(\frac{dt}{\Xi} - a\sin^2\theta \frac{d\phi}{\Xi} \right)^2 \end{aligned} \tag{2.88}$$

where:

$$\Delta_r = (r^2+a^2)(1-\tfrac{1}{3}r^2) - 2Mr, \tag{2.89}$$

$$\Delta_\theta = 1 + \tfrac{1}{3}\Lambda a^2 \cos^2\theta, \tag{2.90}$$

$$\varrho^2 = r^2 + a^2\cos^2\theta, \tag{2.91}$$

$$\Xi = 1 + \tfrac{1}{3}\Lambda a^2. \tag{2.92}$$

The factor Ξ was introduced by Carter so that there are no conical singularities at $\theta \in \{0, \pi\}$.

There are four roots r_0 of the following equation:

$$\left(r_0^2 + a^2\right)\left(1 - \tfrac{1}{3}\Lambda r_0^2\right) - 2Mr_0 = 0 \tag{2.93}$$

and every one of them defines a Killing horizon in a suitably extended spacetime. Two of them correspond to the Kerr black hole horizons, another one is the cosmological horizon. Each of them is of the type D and as long as it is non-extremal, it is one of our solutions derived in Subsection 2.2.3.

Given a solution consisting of the frame coefficient P^2 (2.70), potential U (2.71) and area radius R (2.66) we find the corresponding Killing horizon in spacetime (2.88) by matching mass and rotational parameters M and a, respectively. To start with, compare the metric tensors of the 2-dimensional cross-sections of the IHs, that is:

$$R^2 \left(\frac{1}{P^2}dx^2 + P^2 d\varphi^2 \right) = \frac{\varrho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta (r_0^2+a^2)^2}{\Xi^2 \varrho^2}\sin^2\theta d\phi^2, \tag{2.94}$$

where the left-hand side is given. First, compare the total areas:

$$R^2 = \frac{r_0^2 + a^2}{\Xi}. \tag{2.95}$$

Then, due to the fact that the infinitesimal axial symmetry ∂_φ corresponds to ∂_ϕ and that they are both suitably normalized we obtain:

$$\frac{\Delta_\theta (r_0^2 + a^2)^2}{\Xi^2 \varrho^2} \sin^2\theta = R^2 P^2. \tag{2.96}$$

Combining eq.(2.95) and (2.96) results in the following expression for the frame coefficient P^2:

$$P^2 = \frac{\Delta_\theta}{\varrho^2} R^2 \sin^2\theta. \tag{2.97}$$

Finally, using the above we find the relation of x and θ:

$$\frac{R^2}{P^2} dx^2 = \frac{\varrho^2}{\Delta_\theta} d\theta^2 \tag{2.98}$$

$$\frac{\varrho^2}{\Delta_\theta \sin^2\theta} dx^2 = \frac{\varrho^2}{\Delta_\theta} d\theta^2 \tag{2.99}$$

$$x = -\cos\theta. \tag{2.100}$$

In order to find the relation between parameters (γ, η) and (a, r_0) we take the above expression and use it to express eq. (2.97) solely in terms of x, that is:

$$\frac{1-x^2}{1-\gamma\Lambda/6} \left(\frac{(1-\gamma\Lambda/6)(x^2+\eta^2) + 1 - x^2}{x^2+\eta^2} \right) = \frac{\left(1-x^2\right)\left(1+\frac{1}{3}\Lambda a^2 x^2\right) R^2}{r_0^2 + a^2 x^2}. \tag{2.101}$$

Now, one only needs to compare coefficients standing by different powers of x to find:

$$\eta^2 = \frac{r_0^2}{a^2} \qquad \text{and} \qquad \gamma = \frac{6(a^2 + r_0^2)}{r_0^2 \Lambda - 3}. \tag{2.102}$$

Equivalently, parameters η and γ maybe also written in terms of the area radius R and angular momentum J, namely:

$$\eta^2 = \frac{R^4}{4J^2}\left(1 - \tfrac{1}{3}\Lambda R^2\right) \qquad \text{and} \qquad \gamma = \frac{2R^2}{\frac{1}{3}\Lambda R^2 - 1}. \tag{2.103}$$

The Kerr-(anti) de Sitter parameters r_0 and a may be expressed in terms of parameters η and γ:

$$r_0^2 = \frac{3\gamma}{\Lambda\gamma - 6(\frac{1}{\eta^2} + 1)} \qquad \text{and} \qquad a^2 = \frac{3\gamma}{\Lambda\gamma\eta^2 - 6(\eta^2+1)}. \tag{2.104}$$

Next, we check whether different pairs of the parameters η and γ could admit the same values of parameters r_0 and a. In other words, we check if there is a one-to-one correspondence of these two pairs of parameters, that is:

$$\begin{aligned}
r_{01}^2 = r_{02}^2 \quad &\Leftrightarrow \quad \frac{3\gamma_1}{\Lambda\gamma_1 - 6(\frac{1}{\eta_1^2}+1)} = \frac{3\gamma_2}{\Lambda\gamma_2 - 6(\frac{1}{\eta_2^2}+1)} \quad \Leftrightarrow \quad \gamma_2 = \gamma_1 \frac{\eta_1^2(1+\eta_2^2)}{\eta_2^2(1+\eta_1^2)}, \\
a_1^2 = a_2^2 \quad &\Leftrightarrow \quad \frac{3\gamma_1}{\Lambda\gamma_1\eta_1^2 - 6(\eta_1^2+1)} = \frac{3\gamma_2}{\Lambda\gamma_2\eta_2^2 - 6(\eta_2^2+1)} \\
&\Leftrightarrow \quad \gamma_1(1+\eta_2^2)(\eta_2^2 - \eta_1^2)\left(\frac{\Lambda\gamma_1\eta_1^2}{1+\eta_1^2} - 6\right) = 0 \\
&\Leftrightarrow \quad \eta_1^2 = \eta_2^2 \quad \vee \quad \eta_1^2 = \frac{1}{\Lambda'\gamma_1 - 1}.
\end{aligned}$$

The above calculations show that there exist two scenarios, in which from the same two pairs of (r_0, a) we get different values of η and that would also mean that the two parameters γ_1 and γ_2 also do not match. Notice, however, that the first case:

$$\eta^2 = \tfrac{1}{\Lambda'\gamma - 1} \tag{2.105}$$

does not satisfy the constraint (2.75) for η. The second scenario happens when:

$$\eta_1 = \eta_2, \tag{2.106}$$

however it results of the opposite direction of rotation ($a_1 = -a_2$). Moreover, notice that Kerr-(anti) de Sitter horizon parameters r_0^2 and a^2 can take arbitrary positive values, therefore there is a one-to-one correspondence between them and the parameters γ and η characterizing the considered type D isolated horizon. Consequently, we conclude that the obtained solutions are embeddable in the Kerr-(anti) de Sitter spacetime provided that the corresponding horizon is non-extremal.

Every pair of parameters (γ, η) satisfying the constraints (2.74) and (2.75), determines the mass parameter M in (2.88) via horizon equation (2.93), namely:

$$M = \frac{(1+\eta^2)^2}{2\sqrt{2}\eta^2}\sqrt{\frac{-\gamma\eta^2}{\left(1+\eta^2(1-\frac{1}{6}\Lambda\gamma)\right)^3}}. \tag{2.107}$$

Even though r may take negative value, using transformation:

$$r_0 \to -r_0 \qquad\qquad M \to -M, \tag{2.108}$$

one can keep r_0 or M positive. The Killing horizon determined by $r = r_0$ in the Kerr-(anti) de Sitter spacetime defined by the established values of a and M is the type D isolated horizon provided it is non-extremal. Notice, that in such case, condition (2.54) is significantly more difficult to verify. On the other hand, one could study the extremality by analysis of dependence of the roots of coefficient Δ_r and parameters M, a, and Λ. Problem is soluble, although the solution does not provide us with the explicit conditions on those parameters. Moreover, in the extremal case, the corresponding Killing horizon in spacetime with metric tensor (2.88) is not one of our non-extremal horizons. In such case, our non-extremal type D isolated horizon is embeddable in the near horizon limit spacetime [38, 39].

2.2.6 No hair theorem for the Petrov type D axisymmetric isolated horizons

We conclude chapter 2 with the formulation of the local no hair theorem for the Petrov type D axisymmetric isolated horizon of sections diffeomorphic to 2-spheres:

Theorem 2.2.1 (No hair). *Every axisymmetric solution to the Petrov type D equation (2.20) with cosmological constant Λ on a topological 2-sphere, consisting of the metric tensor g_{AB} and rotation 2-form Ω_{AB}, is uniquely determined by a pair of real parameters: the area A and the angular momentum J. The range of parameters (A, J) corresponding to the cosmological constant, that is*

(i) for the positive cosmological constant $\Lambda > 0$:
$J \in (-\infty, \infty)$ *for* $A \in \left(0, \frac{12\pi}{\Lambda}\right)$ *or* $|\, J \,| \in \left[0, \frac{A}{8\pi}\sqrt{\frac{\Lambda A}{12\pi} - 1}\right)$ *for* $A \in \left(\frac{12\pi}{\Lambda}, \infty\right)$;

(ii) for the non-positive cosmological constant $\Lambda \leq 0$:
$J \in (-\infty, \infty)$ *and* $A \in (0, \infty)$.

Each solution determines a type D isolated horizon whose intrinsic geometry consisting of the degenerate metric tensor g_{ab} and covariant derivative ∇_a coincides with the intrinsic geometry of a non-extremal Killing horizon contained in one of the following (locally defined) spacetimes:

(i) the generalized Kerr-(anti) de Sitter spacetime (2.88);

(ii) the generalized Schwarzschild-(anti) de Sitter spacetime (2.47);

(iii) the near horizon limit spacetime near an extremal horizon contained in either the generalized Kerr-(anti) de Sitter spacetime or in the generalized Schwarzschild-(anti) de Sitter spacetime [39]. Similar result has been previously derived for the vanishing cosmological constant [25].

Chapter 3

Petrov type D equation on genus>0 cross-sections of isolated horizons

In Chapter 1 we introduced the notion of the type D isolated horizons and discussed their geometrical properties. In Chapter 2, we derived all solutions the type D equation with cosmological constant imposed on the 2-metric tensor g_{AB} and the rotation scalar on a cross-section of the horizon which we assumed to be an axisymmetric topological 2-sphere. The type D equation was used to uniquely distinguish the Schwarzschild-(anti) de Sitter and Kerr-(anti) de Sitter spacetimes. In this chapter we continue our study of that equation by considering a closed 2-dimensional surface of genus greater than zero as a cross-section of the IH.

Embeddability in the 4-dimensional spacetime satisfying vacuum Einstein's equations (1.1) with cosmological constant Λ is assumed. We derive the solutions to the type D equation with cosmological constant and prove that they are all characterized by the constant Gaussian curvature K and vanishing rotation. Finally, we present a quasi-local argument for the rotating black hole in 4-dimensional spacetime to have a cross-section of a topological 2-sphere.

3.1 The Petrov type D equation on a 2-dimensional torus

Suppose the cross-section $\mathcal{S}$ is a 2-dimensional torus T_2, that is:

$$S = T_2 = S_1 \times S_1, \tag{3.1}$$

where S_1 is a circle. On the first copy S_1 we introduce coordinate $\phi \in [0, 2\pi)$ whereas on the second one the coordinate $\psi \in [0, 2\pi)$. Therefore, the set of coordinates on T_2 reads:

$$(x^A) = (\phi, \psi). \tag{3.2}$$

The coordinates are globally defined on $\mathcal{S}$, although they are not continuous at $\phi, \psi = 0$. The tangent frame ∂_ϕ and ∂_ψ and its cotangent dual $d\phi$ and $d\psi$ are globally defined, continuous and smooth everywhere. This property will play an important role in the forthcoming considerations. Every flat metric tensor g^{flat}_{AB}, modulo diffeomorphism $\Phi : \mathcal{S} \to \mathcal{S}$, that is:

$$(g, \omega) \to (\Phi^* g, \Phi^* \omega), \tag{3.3}$$

can be expressed in the following form:

$$g^{\text{flat}}_{AB} dx^A dx^B = \frac{1}{Q_0^2} \left(a^2 d\phi^2 + 2ab \, d\phi d\psi + (1 + b^2) d\psi^2 \right), \tag{3.4}$$

where constants a, b, Q_0 are real and also a, Q_0 are non-vanishing [59]. Moreover, every general 2-metric tensor g_{AB} of a manifold $\mathcal{S}$ is conformally equivalent to the flat one [60], that is g^{flat}_{AB}. It follows that there exist a non-vanishing function Q in $\mathcal{S}$ such that the metric tensor is of the form:

$$g_{AB} dx^A dx^B = \frac{1}{Q^2} \left(a^2 d\phi^2 + 2ab \, d\phi d\psi + (1 + b^2) d\psi^2 \right). \tag{3.5}$$

We make an assumption that g_{AB} and consequently Q are at least 4 times differentiable in order to be compatible with the Petrov type D equation (1.125).

Consider a transformation to complex coordinates $(z, \bar{z})$, defined by:

$$z = \frac{a\phi + b\psi + i\psi}{\sqrt{2}}. \tag{3.6}$$

Notice, that the new set of coordinates is linear in ϕ and ψ, consequently they are globally defined on $\mathcal{S}$ carrying the same type of non-continuity as ϕ and ψ. Simple calculation yields, that in coordinates $(z, \bar{z})$, the metric tensor g_{AB} takes a form:

$$g_{AB} dx^A dx^B = \frac{2}{Q^2} dz d\bar{z}, \tag{3.7}$$

while the area 2-form reads:

$$\epsilon = i \frac{1}{Q^2} dz \wedge d\bar{z}. \tag{3.8}$$

It follows that the null frame and co-frame are of the following form:

$$m^A \partial_A = Q \partial_z, \qquad \text{and} \qquad m_A dx^A = \frac{1}{Q} d\bar{z}. \tag{3.9}$$

An analogy to the coordinate system ϕ, ψ occurs. That is, the tangent frame $(m^A, \overline{m}^A)$ and its cotangent dual $(\overline{m}_A, m_A)$, the complex valued vector fields ∂_z, $\partial_{\bar{z}}$ together with complex valued 1-forms dz, $d\bar{z}$ are all globally defined on $\mathcal{S}$, despite of the discontinuity of the coordinates $(z, \bar{z})$.

Recall, that one may express the component Ψ_2 in terms of the Gaussian curvature K and rotation scalar $\mathcal{O}$, then the Petrov type D equation with cosmological constant is of the following form:

$$\overline{m}^A \overline{m}^B D_A D_B \left(K - \tfrac{1}{3}\Lambda + i\mathcal{O} \right)^{-\frac{1}{3}} = 0, \tag{3.10}$$

where

$$K - \tfrac{1}{3}\Lambda + i\mathcal{O} \neq 0 \tag{3.11}$$

at every point of $\mathcal{S}$. Before we move on to solving equation (3.10), there is some subtlety regarding the existence and uniqueness of the cubic root, that requires our attention. Suppose, that for every point in a contractible open neighborhood exist a function, which is as continuous and of the same differentiability class as $K + i\mathcal{O}$ which satisfies:

$$f^3 = K - \tfrac{1}{3}\Lambda + i\mathcal{O}. \tag{3.12}$$

Notice, that it is defined up to a constant factor, namely:

$$f \to e^{\frac{2\pi m}{3}i} f, \tag{3.13}$$

where m is an integer. Consider a case when the function $K - \frac{1}{3}\Lambda + i\mathcal{O}$, defined by the metric tensor g_{AB} and 1-form ω_A, does not have a unique cubic root continuous on the entire 2-manifold $\mathcal{S}$. Then, introduce a 3-fold covering 2-manifold $\tilde{\mathcal{S}}$, such that the metric $\tilde{g}_{AB}$ and 1-form $\tilde{\omega}_A$, that are the pullbacks of g_{AB} and ω_A respectively, define a function $\tilde{K} - \frac{1}{3}\Lambda + i\tilde{\mathcal{O}}$ which admits a globally defined continuous cubic root on $\tilde{\mathcal{S}}$. Therefore, different points of a fiber of the covering correspond to different roots on the base space $\mathcal{S}$. The covering 2-manifold $\tilde{\mathcal{S}}$ is connected, orientable and closed, whereas its Euler characteristic $\chi(\tilde{\mathcal{S}})$ is three times the one of $\mathcal{S}$, that is:

$$\chi(\tilde{\mathcal{S}}) = 3\chi(\mathcal{S}). \tag{3.14}$$

Consequently, from the Gauss-Bonnet theorem we find that:

$$\tilde{g} = 3g - 2, \tag{3.15}$$

hence its greater or equal to the genus of $\mathcal{S}$. If $\mathcal{S}$ is a topological torus then so is the covering $\tilde{\mathcal{S}}$.

Next, we express the differential operator of the type D equation (3.10) in the coordinates $(z, \bar{z})$:

$$\begin{aligned}\overline{m}^A\overline{m}^B\nabla_A\nabla_B f &= \overline{m}^A\overline{m}^B(f_{,AB} - \Gamma^C{}_{AB}f_{,C}) \\ &= Q^2\partial_{\bar{z}}^2 f + 2QQ_{,\bar{z}}\partial_{\bar{z}} f \\ &= \partial_{\bar{z}}(Q^2\partial_{\bar{z}})f.\end{aligned} \tag{3.16}$$

Consequently, the type D equation (3.10) takes the following form:

$$\partial_{\bar{z}}(Q^2\partial_{\bar{z}})f = 0, \tag{3.17}$$

where

$$f = (K - \tfrac{1}{3}\Lambda + i\mathcal{O})^{-\frac{1}{3}}. \tag{3.18}$$

As already stated, a suitable extension $(\tilde{\mathcal{S}}, \tilde{g}, \tilde{\omega})$ may be used to ensure the continuity of the cubic root. However, since the covering is topologically a torus, we will just drop tilde in our notation.

From the global properties of the frame coefficient P^2, vector field $\partial_{\bar{z}}$ and eq. (3.17) follows that $P^2\partial_{\bar{z}}f$ is an entire holomorphic function on whole $\mathcal{S}$. Therefore, since $\mathcal{S}$ is compact we find that:

$$Q^2\partial_{\bar{z}}f = F_0 = \text{const}, \tag{3.19}$$

or equivalently

$$\partial_{\bar{z}}f = \frac{F_0}{Q^2}. \tag{3.20}$$

Another conclusion following from the compactness of $\mathcal{S}$ maybe used to find that F_0 is not only constant but also has to vanish:

$$F_0 \int_{\mathcal{S}} \epsilon = i \int_{\mathcal{S}} \partial_{\bar{z}} f dz \wedge d\bar{z} = -i \int_{\mathcal{S}} d(f dz) = 0. \tag{3.21}$$

Moreover, the fact that:

$$F_0 = 0 \tag{3.22}$$

implies that eq. (3.20) now reads:

$$\partial_{\bar{z}}f = 0. \tag{3.23}$$

Therefore, we conclude that f is the entire holomorphic function on all $\mathcal{S}$, therefore it has to be constant:

$$f = \text{const}, \tag{3.24}$$

which is a general solution to eq. (3.17). Furthermore, since the function f consist of the Gaussian curvature K and rotation scalar $\mathcal{O}$ (eq. (3.18)) we see that:

$$K = K_0 = \text{const}, \qquad \text{and} \qquad \mathcal{O} = \mathcal{O}_0 = \text{const}. \tag{3.25}$$

Now, applying the Gauss-Bonnet theorem, and using the fact that Euler characteristic vanishes for the torus yields:

$$K_0 \int_{\mathcal{S}} \epsilon = \int_{\mathcal{S}} K\epsilon = 0, \tag{3.26}$$

which means

$$K_0 = 0. \tag{3.27}$$

In a similar fashion, we apply the integral law to the rotation scalar $\mathcal{O}$ to find that

$$\mathcal{O}_0 \int_{\mathcal{S}} \epsilon = \int_{\mathcal{S}} \mathcal{O}\epsilon = \int_{\mathcal{S}} d\omega = 0, \tag{3.28}$$

where we have used eq. (1.52) and (1.109). It follows that $\mathcal{O}_0$ is also vanishing.

Finally, we recover the tilde labeling that we ignored before and conclude that for the non-vanishing cosmological constant Λ the extension $(\tilde{g}_{AB}, \tilde{\omega}_A)$ to the Petrov type D equation (3.10) on the 2-dimensional torus:

$$\mathcal{S} = T_2, \tag{3.29}$$

consist of the flat metric tensor $\tilde{g}_{AB}$ and closed 1-form $\tilde{\omega}_A$. As a consequence, the metric g_{AB} and 1-form ω_A of the original T_2 are also flat and closed, respectively, meaning the extension is trivial. For the vanishing cosmological constant Λ, on the other hand, there are no solutions due to the condition (3.11).

Notice, that from the point of view of the reconstruction of the stationary to the second order isolated horizon from the data (g_{AB}, ω_A), the 1-form ω_A is meaningful only modulo gauge transformations:

$$\omega'_A = \omega_A + \kappa D_A g \tag{3.30}$$

where g is a function on $\mathcal{S}'$, that is another section of $\mathcal{H}$.

In conclusion, the general solution modulo gauge transformations:

$$\omega \to \omega + dh, \qquad \text{where} \qquad h \in C^n(\mathcal{S}) \tag{3.31}$$

and diffeomorphisms (3.3) is of the form:

$$g_{AB}dx^A dx^B = \frac{1}{Q_0^2}\left(a^2 d\phi^2 + 2ab d\phi d\psi + (1+b^2)d\psi^2\right), \tag{3.32}$$

$$\omega_A dx^A = A d\phi + B d\psi, \tag{3.33}$$

where $P_0 > 0$, $a > 0$, b, A, B are arbitrary real constants.

3.2 The Petrov type D equation on the surfaces of higher genus

In this section we suppose that $\mathcal{S}$ is an orientable and closed 2-manifold of a genus $g > 1$. Just as in Section 3.1, it is endowed with a 2-metric tensor g_{AB} and 1-form ω_A. Again, if the function f is not continuous on the entire $\mathcal{S}$, we consider extension $\tilde{\mathcal{S}}$ and omit tilde labels.

We cover 2-manifold $\mathcal{S}$ with charts, so that on each of them the complex coordinates read:

$$g_{AB}dx^A dx^B = \frac{2}{Q^2} dz d\bar{z}, \tag{3.34}$$

whereas the equation

$$\overline{m}^A \overline{m}^B D_A D_B f = 0, \tag{3.35}$$

for the arbitrary function f takes the following form:

$$\partial_{\bar{z}}(Q^2 \partial_{\bar{z}} f) = 0. \tag{3.36}$$

The expression under the parentheses can be regarded as a component of a complex vector field, namely

$$(P^2\partial_{\bar{z}}f)\partial_z = (g^{zA}\partial_A f)\partial_z. \tag{3.37}$$

Notice, that this vector field is globally defined on the section $\mathcal{S}$. It is constructed from the gradient of the function f, that is

$$(g^{AB}\partial_A f)\partial_B = (P^2\partial_{\bar{z}}f)\partial_z + (P^2\partial_z f)\partial_{\bar{z}} \tag{3.38}$$

via a globally defined decomposition of complexified tangent space at each point

$$Y = Y^z\partial_z + Y^{\bar{z}}\partial_{\bar{z}} \to Y^z\partial_z. \tag{3.39}$$

This decomposition uses locally defined coordinates $(z, \bar{z})$. Nevertheless, the most general coordinate transformation, $z \mapsto z'(z)$, preserving the metric tensor (3.34) satisfies the following condition

$$\frac{\partial z'}{\partial \bar{z}} = 0. \tag{3.40}$$

Such transformation also preserves the decomposition (3.39).

We introduce yet another operation, which is invariant with respect to the holomorphic coordinate transformation (3.40), that is the anti-holomorphic derivative $\bar{\partial}$. It acts in the vector space of all the vector fields:

$$X = X^z\partial_z \tag{3.41}$$

with an arbitrary component X^z in the following way

$$\bar{\partial}X = \partial_{\bar{z}}X^z\partial_z \otimes d\bar{z}. \tag{3.42}$$

Solutions to the equation:

$$\bar{\partial}X = 0 \tag{3.43}$$

are called the holomorphic vector fields. The dimension of space of the holomorphic vector fields is known for every compact orientable 2-surface $\mathcal{S}$ [61]. In particular, for genus $g > 1$ the dimension is equal to zero:

$$X = 0. \tag{3.44}$$

Therefore, eq. (3.36) implies that:

$$\partial_{\bar{z}}f = 0. \tag{3.45}$$

Moreover, from the compactness of $\mathcal{S}$ it follows that:

$$f = \text{const}. \tag{3.46}$$

Hence, similarly as for the torus, eq. (3.18) implies that the Gaussian curvature and rotation scalar are constant:

$$K = \text{const} \qquad\qquad \text{and} \qquad\qquad \mathcal{O} = \text{const}. \tag{3.47}$$

On every compact 2-surface $\mathcal{S}$ the rotation scalar $\mathcal{O}$ satisfies:

$$\int_{\mathcal{S}} \mathcal{O}\epsilon = 0. \tag{3.48}$$

It follows that $\mathcal{O}$ vanishes:

$$\mathcal{O} = 0, \tag{3.49}$$

which is the same conclusion as for the torus. Notice that gaussian curvature K is proportional to the inverse of the area A via the Gauss-Bonnet theorem, namely:

$$K = \frac{4\pi(1 - \mathrm{g})}{A} \tag{3.50}$$

unless:

$$K = \frac{1}{3}\Lambda, \tag{3.51}$$

which does not satisfy the constraint (3.11).

Remark. Notice that we could have used the same argument on the holomorphic vector field to the 2-dimensional torus discussed in the previous section. Every torus admits a 1-complex-dimensional space of the holomorphic vector fields. It immediately implies eq. (3.19). However, it requires proving that the constant F_0 is vanishing for the vector field $(g^{zA}\partial_A f)\partial_z$ in a similar fashion as in the section 3.1.

3.3 Summary and the consequences of the obtained results for IH with cross-section of genus> 0

In sections 3.1 and 3.2 we derived all solutions on a compact, ortientable 2-surface $\mathcal{S}$ of genus > 0 to the Petrov type D equation (3.10) consisting of the metric tensor g_{AB} and 1-form ω_A. Now, we comment on the results and list them in Theorems 3.3.1 and 3.3.2 and corollaries 3.3.1 and 3.3.2 as well as provide some implications of the derived solutions.

Theorem 3.3.1. *A pair (g_{AB}, ω_A) is a solution to the Petrov type D equation (3.10) with cosmological constant Λ on a compact, orientable 2-surface of genus $g > 0$ if and only if g_{AB} has constant Gaussian curvature (Ricci scalar)*

$$K = const \neq \tfrac{1}{3}\Lambda \tag{3.52}$$

and ω_A is closed

$$d\omega = 0. \tag{3.53}$$

Assumption on type D could be relaxed to possible type 0 in some degenerate subsets of $\mathcal{S}$. In result, more solutions could possibly exist.

The obtained family of solutions is more than zero-dimensional. Notice, that for $\mathcal{S} = T_2$ the family of solutions is 5-dimensional. Moreover, the corresponding isolated horizons stationary to the second order are non-rotating so their angular momentum J vanishes. We conclude with the following theorem:

Theorem 3.3.2. *Every rotating Petrov type D isolated horizon stationary to the second order and contained in a 4-dimensional spacetime satisfying the vacuum Einstein's equations (1.1) with cosmological constant Λ has a spacelike cross-section of the topology of a 2-sphere.*

Therefore, for a spacelike cross-sections of genus higher than zero there are no rotating Petrov type D isolated horizons stationary to the second order contained in a 4-dimensional spacetime, which satisfies the vacuum Einstein's equations (1.1) with cosmological constant Λ. Notice that since the rotation scalar $\mathcal{O}$ vanishes, each of the solutions also satisfies the conjugate Petrov type D equation, which is of the form

$$m^A m^B D_A D_B \left(K - \tfrac{1}{3}\Lambda + i\mathcal{O}\right)^{-\frac{1}{3}} = 0. \tag{3.54}$$

Consequently, using the black hole holograph technique introduced in [30, 31, 62], from $\mathcal{S}$ and (g_{AB}, ω_A) one constructs a spacetime:

$$\mathcal{M} = \mathcal{S} \times \mathbb{R} \times \mathbb{R}, \tag{3.55}$$

which contains a bifurcated horizon on a bifurcated surface $\mathcal{S}$ and the spacetime Weyl tensor is of the Petrov type D on the horizon [54].

Recall, that the Petrov type D equation is an integrability condition for the near horizon geometry equation:

$$\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \tfrac{1}{2}K g_{AB} + \tfrac{1}{2}\Lambda g_{AB} = 0 \tag{3.56}$$

which is valid for the non-vanishing component Ψ_2 of the Weyl tensor, that is

$$\Psi_2 = -\tfrac{1}{2}\left(K - \tfrac{1}{3}\Lambda + i\mathcal{O}\right) \neq 0. \tag{3.57}$$

It follows that every solution to the near horizon geometry equation on a 2-surface of genus $g > 0$ such that Ψ_2 is non vanishing (3.57) at every point of $\mathcal{S}$ is the solution described in theorem 3.3.2.

Consequently, we formulate the following corollary:

Corollary 3.3.1. *If (g_{AB}, ω_A) is a solution to the near horizon geometry equation (3.56) on a connected, orientable, compact 2-manifold $\mathcal{S}$ of genus $g > 0$, such that (3.57) holds everywhere on $\mathcal{S}$, then it is static and the Gaussian curvature is constant, that is*

$$d\omega = 0, \qquad \textit{and} \qquad K = \textit{const.} \tag{3.58}$$

Next, we combine the obtained results with a prior work on the near horizon geometry for higher genus. First consider the case when cosmological constant satisfies the following inequality:

$$\Lambda \geq 0. \tag{3.59}$$

Then, the topological constraint derived in [47] indicates that the only genus>0 compact solution is the trivial solution with the vanishing cosmological constant $\Lambda = 0$, cross-section of a 2-dimensional torus $\mathcal{S} = T_2$ with the flat metric tensor g_{AB} and vanishing rotation 1-form $\omega_a = 0$ (see also [63,64] and Theorem 3.1 in [38]). Therefore, in the case of non-negative Λ, we find that the issue of the equation on the higher genus surfaces is solved in the literature meaning that the new integrability condition is not needed.

Now consider the negative cosmological constant case

$$\Lambda < 0 \tag{3.60}$$

which weakens the topological constraint and therefore one may expect solutions of genus $g > 0$. Notice, however, that the axially symmetric solutions with $\mathcal{S} = T_2$ have been excluded (for the prove see [65]). On the other hand, the static near horizon geometries with any compact 2-manifold $\mathcal{S}$ have to satisfy:

$$\omega_A = 0 \qquad \text{and} \qquad K = \text{const}, \tag{3.61}$$

which was proven in [66].

Our Corollary 3.57 together with the latter result provides a general solution to the near horizon geometry equation (3.56) on the surfaces of higher genus, such that the condition (3.57) is satisfied:

Corollary 3.3.2. *The only solutions (g_{AB}, ω_A) to the near horizon geometry equation (3.56) on a connected, orientable, compact 2-manifold $\mathcal{S}$ of genus $g > 0$ such that condition (3.57) holds, satisfy:*

$$\omega_A = 0 \qquad \textit{and} \qquad K = \frac{4\pi(1-g)}{A} = \Lambda, \tag{3.62}$$

where A is the area of the surface $\mathcal{S}$.

Finally, from the literature we know, that for every solution to the near horizon geometry equation (3.56) on $\mathcal{S} = \mathcal{S}_2$, the component Ψ_2 of the Weyl tensor is either identically or nowhere zero[1] [26,46]. With a little bit of work one can show, that Ψ_2 is either identically or nowhere zero on any compact, orientable 2-manifold $\mathcal{S}$ [53]. This result seems to complete the issue of the near horizon geometry equation on genus $g > 0$ surfaces.

[1] Notice that the argument used there for $\Lambda = 0$ case may be easily generalized to the arbitrary Λ.

Chapter 4

Petrov type D equation on horizons of nontrivial bundle topology

We have already considered isolated horizon $\mathcal{H} = \mathcal{S} \times \mathbb{R}$, where $\mathcal{S}$ is an axisymmetric 2-sphere (Chapter 2) or a compact 2-surface of a higher genus (Chapter 3). Now we move on from the IHs that admit a global space-like cross-section $\mathcal{S}$ to the horizons of nontrivial bundle topology. We consider IHs that are generated by the null curves forming nontrivial $U(1)$-bundles. An example of a spacetime with such horizon is the Taub-NUT spacetime. A natural interplay of the geometries of the IH and the $U(1)$-bundle is found. The condition of the Petrov type D of the spacetime Weyl tensor on the horizon is imposed. It couples the $U(1)$ connection, a metric tensor g_{AB} defined on the base manifold and surface gravity $\kappa^{(\ell)}$ in a nontrivial way. In particular, we are interested in the $U(1)$-bundles over 2-dimensional manifolds that are diffeomorphic to a 2-sphere. In this chapter we derive all of the axisymmetric solutions to the Petrov type D equation with cosmological constant and find that they set a 3-dimensional family of isolated horizons for a chosen value of the cosmological constant Λ. A new parameter emerges, that is the topological charge times surface gravity. There exist a 4-dimensional (including the cosmological constant Λ) family of the Kerr-NUT-(anti) de Sitter spacetimes in the literature, but surprisingly, in the generic case our horizons do not correspond to those spacetimes. However, the results obtained here were followed by the work of Lewandowski and Ossowski [67], where they found the necessary conditions for those two families of type D solutions to agree. For the completeness, we will briefly discuss their result in Subsection 4.2.3.

4.1 Type D isolated horizons of nontrivial $U(1)$-bundle topology

We begin this section by introducing general definitions and properties of the 3-dimensional IHs which null generators are of a nontrivial fibration structure. Even though the IHs under consideration are 3-surfaces in 4-dimensional spacetimes, their intrinsic geometry can be considered independently of the embedding. We will discuss this in Subsection 4.1.1. Whereas in Subsection 4.1.2 we focus on the IHs embedded in 4-dimensional spacetime, the assumed symmetries as well as Einstein's constraints and the Petrov type D equation. Notice, that derivation of the type D equation is local (see Subsection 1.3.1) and may be applied to the isolated horizons of the nontrivial topology considered in this chapter.

4.1.1 Isolated horizon's structure on a $U(1)$ bundle

We now introduce a nontrivial bundle structure which is a new element in the notion of IHs. Consider a principal fiber bundle with the structure group $U(1)$:

$$\Pi : \mathcal{H} \to \mathcal{S}. \tag{4.1}$$

The flow of the fundamental null vector field ℓ coincides with the action $U(1)$ on the horizon $\mathcal{H}$. Vector field ℓ is normalized in such a way that the parameter of its flow takes values in the interval $[0, 2\pi]$. Similarly as for the IHs considered in Chapter 2 and 3 the geometry compatible with the bundle structure consists of:

(i) degenerate metric tensor g_{ab} of the signature $0++$, satisfying:

$$\ell^a g_{ab} = 0 = \mathcal{L}_\ell g_{ab}, \tag{4.2}$$

where from the second equality we conclude that g_{ab} is invariant with respect to the action of the $U(1)$ group on the isolated horizon $\mathcal{H}$.

(ii) covariant derivative ∇_a on $T(\mathcal{H})$ which is torsion free, satisfies the pseudo metricity condition (1.7) and:

$$[\mathcal{L}_\ell, \nabla_a] = 0, \tag{4.3}$$

which also means that ∇_a is invariant with respect to the action of the $U(1)$ group on $\mathcal{H}$.

We assume that the surface gravity satisfies:

$$\kappa^{(\ell)} = \text{const} \neq 0, \tag{4.4}$$

meaning that $\mathcal{H}$ is a non-extremal IH. The significant role plays the rotation 1-from potential ω_a defined via:

$$\nabla_a \ell^b = \omega_a^{(\ell)} \ell^b \tag{4.5}$$

and satisfying:

$$\mathcal{L}_\ell \omega_a^{(\ell)} = 0. \tag{4.6}$$

Since the surface gravity $\kappa^{(\ell)}$ is constant and non-vanishing (4.4), the 1-form defined as:

$$\tilde{\omega} := \frac{1}{\kappa^{(\ell)}} \omega^{(\ell)} \tag{4.7}$$

is a connection 1-form on the $U(1)$-bundle (4.1). Indeed, from the definition of ω_a (4.5) and (4.3) it follows that the 1-form $\tilde{\omega}_a$ satisfies:

$$\ell^a \tilde{\omega}_a = 1 \qquad \text{and} \qquad \mathcal{L}_\ell \tilde{\omega}_a = 0. \tag{4.8}$$

The metric tensor g_{ab} induces on the space of null generators $\mathcal{S}$ a genuine metric tensor g_{AB} via:

$$g_{ab} = {\Pi^*}_{ab}{}^{AB} g_{AB}. \tag{4.9}$$

The corresponding area 2-form ϵ_{AB} defined on $\mathcal{S}$ can be also pulled back to $\mathcal{H}$:

$$\epsilon_{ab} = {\Pi^*}_{ab}{}^{AB} \epsilon_{AB}. \tag{4.10}$$

A rotation scalar $\mathcal{O}$ satisfies:

$$\mathcal{O}\epsilon_{ab} = \kappa^{(\ell)} d\tilde{\omega}_{ab}, \tag{4.11}$$

and since:

$$\ell^a \mathcal{O}_{,a} = 0, \tag{4.12}$$

it is considered to be a function on $\mathcal{S}$. The 1-form ω_a may be represented by the locally defined in a neighborhood of every point $x \in \mathcal{S}$, 1-forms ω_A satisfying:

$$d\omega_{AB} = \mathcal{O}\epsilon_{AB}, \tag{4.13}$$

where $\mathcal{O}$ is a globally defined scalar function regular on the entire 2-manifold $\mathcal{S}$.

4.1.2 Embedded isolated horizons and the type D equation

We assume that vacuum Einstein's equations (1.1) with cosmological constant hold together with the assumption on the stationarity to the second order (1.2.1). Consequently, all of the components of the spacetime Weyl tensor on the IHs are determined by the intrinsic geometry, that consists of the degenerate metric tensor g_{ab} and 1-form ω_a.

We will use the complex null frame (1.75) introduced in Subsection 1.2.2, in which the area 2-form ϵ_{AB} reads

$$\epsilon_{AB} = i(\overline{m}_A m_B - \overline{m}_B m_A). \tag{4.14}$$

We consider the type D equation with cosmological constant derived in Subsection 1.3.1 on the base manifold $\mathcal{S}$ assuming that it is diffeomorphic to a 2-sphere:

$$\mathcal{S} = \mathcal{S}_2. \tag{4.15}$$

The integers number all of the $U(1)$ bundles. Therefore we introduce an integer m corresponding to the isolated horizon $\mathcal{H}$, which may be calculated from the curvature of the $U(1)$-connection 1-form $\tilde{\omega}_A$. As a consequence, we obtain a condition on the rotation scalar $\mathcal{O}$:

$$\int_{\mathcal{S}_2} \mathcal{O}\epsilon_{AB} = 2\pi m\kappa^{(\ell)} =: 2\pi n. \tag{4.16}$$

Moreover, for each rotation scalar $\mathcal{O}$ there exist a 1-form ω^+_A and ω^-_A on the surface $\mathcal{S}_2$ in exception of the southern and northern pole, respectively. Therefore, we have:

$$d\omega^{\pm}_{AB} = \mathcal{O}\epsilon_{AB}. \tag{4.17}$$

From the mathematical perspective, the most interesting case is when surface gravity $\kappa^{(\ell)} = 1$, however we do not see any particular reason implied by general relativity to distinguish such case.

The 2-metric tensor g_{AB} and scalar $\mathcal{O}$ defined on $\mathcal{S}_2$ are assumed to admit axial symmetry and therefore we will use the coordinates adapted to the symmetry that were introduced in Subsection 2.1.2. Recall, that the metric tensor g_{AB} is of the form:

$$g_{AB}dx^A dx^B = R^2\left(\frac{1}{P^2(x)}dx^2 + P^2(x)d\varphi^2\right), \tag{4.18}$$

where R is the area radius parameter and the domains for the coordinates x and φ are:

$$x \in [-1, 1], \qquad\qquad \varphi \in [0, 2\pi). \tag{4.19}$$

The 2-frame vector m^A and its dual are of the form (2.13) and (2.14). For the metric tensor to be twice differentiable the frame coefficient P^2 has to satisfy conditions (derived in Subsection 2.1.2):

$$P^2|_{x=\pm 1} = 0, \tag{4.20}$$

$$\partial_x P^2|_{x=\pm 1} = \mp 2, \tag{4.21}$$

whereas for the rotation scalar $\mathcal{O}$ the following needs to hold:

$$\int_{-1}^{1} dx R^2 \mathcal{O} dx = m\kappa^{(\ell)} = n. \tag{4.22}$$

The type D equation in coordinates adapted to the axial symmetry is of the form:

$$\partial_x^2 \Psi_2 = 0, \tag{4.23}$$

whereas its solution reads:

$$\Psi_2 = (c_1 x + c_2)^{-\frac{1}{3}}, \tag{4.24}$$

where c_1 and c_2 are complex constants as in Chapter 2. Writing eq. (1.95) explicitly yields[1]:

$$\Psi_2 = -\tfrac{1}{2}(K + i\mathcal{O}) + \Lambda'. \tag{4.25}$$

Compering it with eq. (4.24) while expressing Gaussian curvature K in terms in the coordinates adapted to axial symmetry wie find:

$$\frac{1}{(c_1 x + c_2)^3} = \frac{1}{4R^2}\partial_x^2 P^2 - \frac{1}{2}i\mathcal{O} + \Lambda'. \tag{4.26}$$

The above equation may be solved for the complex constants c_1 and c_2 which satisfy solvability conditions.

[1] Recall that it is convenient to use rescaled cosmological constant $\Lambda' = \Lambda/6$.

4.2 The solution to the Petrov type D equation on the nontrivial bundle topology

Our goal in this section is to find all solutions to the Petrov type D equation with cosmological constant (4.26) on the IH of nontrivial bundle topology. We use a similar approach as the one in Chapter 2, where we solved the type D equation on trivial bundle, that is for $n = 0$. Therefore, we split the derivation into two cases. First one for the vanishing constant c_1 and the second, more general, for non-vanishing c_1. Notice, that the complex constant c_2 cannot vanish, otherwise the geometry is not well-defined.

4.2.1 The solution for the vanishing constant c_1 and its embeddability in the Taub-NUT spacetime

Setting $c_1 = 0$ in the eq. (4.26) yields:

$$\frac{4R^2}{c_2^3} = \partial_x^2 P^2 - 2iR^2\mathcal{O} + 4R^2\Lambda'. \tag{4.27}$$

Then, we integrate both sides of the above and apply the differentiability conditions (4.21) and (4.22) to find the relation between the complex constant c_2 and parameters R and Λ':

$$c_2^3 = \frac{4R^2}{-2 - in + 4\Lambda' R^2}. \tag{4.28}$$

It follows that the solution to eq. (4.27) is expressed in terms of the frame coefficient which is of the form:

$$P^2 = 1 - x^2, \tag{4.29}$$

and the rotation scalar that reads:

$$\mathcal{O} = \frac{n}{2R^2}. \tag{4.30}$$

Using the above expression for $\mathcal{O}$ we find the rotation 1-form potential $\omega_A^{\pm}$ via eq. (4.17). Notice, that $\omega_A^{\pm}$ also has to satisfy the regularity conditions at the poles, namely:

$$\omega_A^{+}|_{x=1} = 0 = \omega_A^{-}|_{x=-1}, \tag{4.31}$$

consequently we find:

$$\omega_A^{\pm} dx^A = \frac{n}{2}(x \mp 1)d\varphi. \tag{4.32}$$

The obtained solution to the Petrov type D equation (4.26) for the vanishing c_1 is therefore parametrized be the area radius R and parameter n. In particular, for the vanishing n, the 1-form $\omega_A^{\pm}$ also vanishes. The function Ψ_2 is constant on the entire horizon $\mathcal{H}$:

$$\Psi_2 = \text{const.} \tag{4.33}$$

Next we study the embeddability of the found solution in Taub-NUT-(anti) de Sitter spacetime, which is of the Petrov type D and is defined by the static spacetime metric satisfying the vacuum Einstein's equations (1.1) with the cosmological constant Λ. Its metric tensor may be written in the following form [68]:

$$ds^2 = -\frac{Q}{\rho^2}\left[dt - 4l\sin^2(\tfrac{1}{2}\theta)d\phi\right]^2 + \frac{\rho^2}{Q}dr^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.34}$$

where

$$\rho^2 = r^2 + l^2, \tag{4.35}$$

$$Q = r^2 - 2Mr - l^2 - \Lambda\left(-l^4 + 2l^2r^2 + \tfrac{1}{3}\right). \tag{4.36}$$

Its extension contains Killing horizons. Each of them is parametrized by the roots of the horizon equation:

$$r_{\mathcal{H}}^2 - 2Mr_{\mathcal{H}} - l^2 - \Lambda\left(-l^4 + 2l^2r_{\mathcal{H}}^2 + \tfrac{1}{3}r_{\mathcal{H}}^4\right) = 0. \tag{4.37}$$

Among the horizons corresponding to the roots of the above equation, the non-extremal ones are of the typ D. The 2-metric tensor g_{AB} on the space of the null generators of the Killing horizon admits spherical symmetry, that is:

$$ds_2^2 = \rho^2\left(d\theta^2 + \sin^2\theta d\phi\right), \tag{4.38}$$

and the coordinates x, φ are related to θ, ϕ via:

$$x(\theta) = -\cos\theta, \qquad \text{and} \qquad \varphi(\phi) = \phi. \tag{4.39}$$

Parameters R and n may be expressed in terms of the parameters of Taub-NUT horizon: r and l. We compare the areas of the 2-metrics of the horizons to find:

$$R^2 = r_{\mathcal{H}}^2 + l^2. \tag{4.40}$$

The Killing vector field defining our generator of the null symmetry on the horizon $\mathcal{H}$ is of the form:

$$\xi = M\frac{\partial}{\partial t}, \tag{4.41}$$

where the factor M makes it dimensionless. On the horizon the following holds:

$$\ell = \xi|_{\mathcal{H}}. \tag{4.42}$$

The surface gravity on $\mathcal{H}$ may be calculated using the formula:

$$(\xi^\mu\xi_\mu)_{;\nu}|_{\mathcal{H}} = -2\kappa^{(\ell)}\xi_\nu. \tag{4.43}$$

The result reads:

$$\kappa^{(\ell)} = -\frac{M}{2r_{\mathcal{H}}}\left(\frac{-\Lambda r_{\mathcal{H}}^4 + (1-\Lambda l^2)r_{\mathcal{H}}^2 + (1-\Lambda l^2)l^2}{r_{\mathcal{H}}^2 + l^2}\right). \tag{4.44}$$

Notice, that the 1-form $\tilde{\omega}_A^-$ for the Taub-NUT-(anti) de Sitter spacetime is of the form:

$$\tilde{\omega}_A^- dx^A = -\frac{4l}{M}\sin^2\left(\tfrac{1}{2}\theta\right) d\phi. \tag{4.45}$$

Using the expression for the surface gravity $\kappa^{(\ell)}$ (4.44) and the one for 1-form $\tilde{\omega}_A^-$ (4.45) and substituting them into eq. (4.7), (4.16) and (4.17) allows one to calculate the relation between parameter n and parameters r and l, that is:

$$n = -\frac{4l\kappa^{(\ell)}}{M} \tag{4.46}$$

$$= \frac{2l}{r_{\mathcal{H}}}\left(\tfrac{-\Lambda r_{\mathcal{H}}^4 + (1-2\Lambda l^2) r_{\mathcal{H}}^2 + (1-\Lambda l^2) l^2}{r_{\mathcal{H}}^2 + l^2}\right). \tag{4.47}$$

To sum up, the obtained horizon for $c_1 = 0$ is embeddable in the quotient of Taub-NUT-(anti) de-Sitter spacetime by the symmetry $t \to t + 2\pi M$, whereas eq. (4.40) and (4.46) describe the correspondence of our parameters and those of the Taub-NUT-(anti) de Sitter horizons. Its clear that the embedding is not unique and depends on the choice of symmetry.

4.2.2 The solution for the non-vanishing constant c_1

Now we move on to the more general case, that is when:

$$c_1 \neq 0 \qquad \text{and} \qquad c_2 \neq 0. \tag{4.48}$$

We integrate eq. (4.26) twice to find the expression for the frame coefficient, that is:

$$P^2 = 2R^2 \mathrm{Re}\left[\frac{1}{c_1^2(c_1 x + c_2)}\right] - 2R^2\Lambda' x^2 + Cx + D, \tag{4.49}$$

where C and D are real constants. The boundary conditions (4.20) and (4.21) provide us with the explicit expressions for the two constants:

$$C = -2 + 4R^2\Lambda' + 2R^2 \mathrm{Re}\left[\frac{1}{c_1(c_1+c_2)^2}\right]$$
$$= -2R^2 \mathrm{Re}\left[\frac{1}{c_1(c_1^2 - c_2^2)}\right], \tag{4.50}$$

$$D = 2R^2 \mathrm{Re}\left[\frac{2c_1 + c_2}{c_1^2(c_1+c_2)^2}\right] + 2R^2\Lambda' - 2. \tag{4.51}$$

The area radius R may be expressed in terms of Λ', n and the constants c_1 and c_2 by integrating eq. (4.26) and applying conditions (4.21) and (4.22), which yields

$$R^2 = \frac{1}{4}\frac{-2 - in}{\frac{c_2}{(c_1^2 - c_2^2)} - \Lambda'}. \tag{4.52}$$

For R^2 to be real it is necessary that the following equation holds:

$$\mathrm{Im}\left[\frac{\frac{c_2}{(c_1^2 - c_2^2)} - \Lambda'}{2 + in}\right] = 0. \tag{4.53}$$

Consider the following parametrization

$$\frac{c_2}{\left(c_1^2 - c_2^2\right)^2} = \frac{1}{\gamma} - i\frac{n}{2}\left(\Lambda' - \frac{1}{\gamma}\right), \tag{4.54}$$

where parameter $\gamma \in \mathbb{R}$. The last equality in (4.50) yields:

$$\frac{1}{2R^2} - \Lambda' = \mathrm{Re}\left[\left(\frac{c_1}{c_2} - 1\right)\frac{c_2}{\left(c_1^2 - c_2^2\right)^2}\right], \tag{4.55}$$

which gives rise to yet another parameter $\eta \in \mathbb{R}$. It is defined by:

$$\frac{c_2}{c_1} = \frac{\eta n}{4\Lambda' R^2 - 2} + i\eta = \frac{1}{2}\eta n(\Lambda'\gamma - 1) + i\eta, \tag{4.56}$$

where we assumed:

$$1 - 2\Lambda' R^2 \neq 0. \tag{4.57}$$

The expression for the frame coefficient P^2 (4.49) written in terms of the introduced parameters takes the form:

$$P^2 = \frac{\left(1 - x^2\right)\left(\left(x - \frac{1}{2}\eta n\left(1 - \Lambda'\gamma\right)\right)^2 + \eta^2 + \frac{1-x^2}{1-\Lambda'\gamma}\right)}{\left(x - \frac{1}{2}\eta n\left(1 - \Lambda'\gamma\right)\right)^2 + \eta^2}. \tag{4.58}$$

Notice, that for vanishing parameter n the above equation reduces to (2.70), which is the expression for the frame coefficient P^2 of the isolated horizon of the trivial topology: $\mathcal{H} = \mathcal{S}_2 \times \mathbb{R}$, admitting axial symmetry. The similar is true for the component Ψ_2 of the Weyl tensor on $\mathcal{H}$ which expressed in terms of η, γ, n and Λ' reads:

$$\Psi_2 = \frac{\left(\left(\frac{1}{2}\eta n(\Lambda'\gamma + 1) + i\eta\right)^2 - 1\right)^2}{i\eta\gamma\left(x + i\eta + \frac{1}{2}\eta n(\Lambda'\gamma - 1)\right)}. \tag{4.59}$$

Next, we calculate the rotation 1-form potential $\omega_A^{\pm}$ in a similar fashion as for the case of the vanishing constant c_1. Consider the imaginary part of the type D equation (4.26) which yields:

$$\mathcal{O} = \mathrm{Im}\left[\frac{-2}{c_1^3\left(x + \frac{c_2}{c_1}\right)^3}\right] = \mathrm{Im}\left[\frac{2i\left(1 - \eta^2\left(\frac{1}{2}n\left(\Lambda'\gamma - 1\right) + i\right)^2\right)}{\eta\gamma\left(x + \frac{1}{2}\eta n\left(\Lambda'\gamma - 1\right) + i\eta\right)^3}\right]. \tag{4.60}$$

It follows that:

$$\omega_A^{\pm} dx^A = \mathrm{Im}\left[\frac{i\left(1 - \eta^2\left(\frac{n}{2}\left(\Lambda'\gamma - 1\right) + i\right)^2\right)^2}{2\eta\left(1 - \Lambda'\gamma\right)\left(x + \eta\left(\frac{n}{2}\left(\Lambda'\gamma - 1\right) + i\right)\right)^2} + iC^{\pm}\right] d\phi. \tag{4.61}$$

Recall, that on the poles $\omega_A^{\pm}$ has to satisfy conditions (4.31), which allow us to find the expression for the integration constant $C^{\pm}$:

$$C^{\pm} = \frac{1}{2\eta(1 - \gamma\Lambda')}\left[1 - \eta^2 + \tfrac{1}{4}n^2\eta^2(1 - \gamma\Lambda')^2 \mp n\eta(1 - \gamma\Lambda')\right]. \tag{4.62}$$

Consequently, the family of solutions to type D equation with cosmological constant (4.26) may be expressed in terms of cosmological constant Λ and parameters: γ, η and n.

We now go back to the omitted case when eq. (4.57) is not satisfied, therefore consider:

$$1 - 2\Lambda' R^2 = 0. \tag{4.63}$$

We may introduce the parametrization:

$$\frac{c_2}{(c_1^2 - c_2^2)^2} = -\frac{1}{2} in\Lambda' \qquad\qquad \frac{c_2}{c_1} = -\frac{n\Lambda'}{2\alpha}. \tag{4.64}$$

where $\alpha \in \mathbb{R}$. Notice, that for $n = 0$, the right-hand sides of the above expressions vanish and that leads to $c_2 = 0$ which we do not allow. It is consistent with the result of Subsection 2.2.3 where we excluded $R^2 = 1/2\Lambda'$ so that the geometry is well-defined. On the other hand, for the non-vanishing n we find that the frame coefficient reads

$$P^2 = 1 - x^2, \tag{4.65}$$

whereas the rotation scalar is of the form

$$\mathcal{O} = -\frac{2\alpha\left(1 - (\frac{n\Lambda'}{2\alpha})^2\right)^2}{\left(x - \frac{n\Lambda'}{2\alpha}\right)^3}. \tag{4.66}$$

Finally, we find the rotation 1-form potential

$$\omega_A^{\pm} dx^A = \left[\frac{\alpha\left(1 - \left(\frac{n\Lambda'}{2\alpha}\right)^2\right)^2}{2\Lambda'\left(x - \frac{n\Lambda'}{2\alpha}\right)^2} + C^{\pm}\right] d\varphi, \tag{4.67}$$

where the integration constant reads

$$C^{\pm} = -\frac{\alpha}{2\Lambda'}\left(1 \pm \frac{n\Lambda'}{2\alpha}\right)^2, \tag{4.68}$$

and function Ψ_2 is of the form:

$$\Psi_2 = \frac{i\alpha\left(\frac{n^2\Lambda'^2}{4\alpha^2} - 1\right)^2}{x - \frac{n\Lambda'}{2\alpha}}. \tag{4.69}$$

4.2.3 Embeddability of the solution for the non-vanishing complex constant c_1

In the previous subsections of this chapter we have derived a general solution to the type D equation on the isolated horizons of nontrivial $U(1)$-bundle over $\mathcal{S}_2$ assuming axial symmetry. The next obvious step would be to study its embeddability in the Kerr-NUT-(anti) de Sitter spacetimes. Surprisingly the obtained solution generically is not embeddable in such spacetimes. However, for the case of the positive cosmological constant Λ satisfying certain condition, the correspondence between our solution and the Kerr-NUT-de Sitter spacetimes have been be found by Lewandowski and Ossowski in [67]. For the completeness of the discussion on horizons of nontrivial bundle topology, we will now briefly present their results.

Consider, the Kerr-NUT-(anti) de Sitter spacetime defined by a metric tensor [68]:

$$ds^2 = -\frac{\mathcal{Q}}{\Sigma}(dt - Ad\phi)^2 + \frac{\Sigma}{\mathcal{Q}}dr^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(adr - \rho d\phi)^2, \tag{4.70}$$

where

$$\Sigma = r^2 + (l + a\cos\theta)^2, \tag{4.71}$$
$$A = a\sin^2\theta + 4l\sin^2(\tfrac{1}{2}\theta), \tag{4.72}$$
$$\rho = r^2 + (l + a)^2 \tag{4.73}$$
$$\mathcal{Q} = (a^2 - l^2) - 2mr + r^2 - \Lambda\big((a^2 - l^2)l^2 + (\tfrac{1}{3}a^2 + 2l^2)r^2 + \tfrac{1}{3}r^4\big), \tag{4.74}$$
$$P = 1 + \tfrac{4}{3}\Lambda al\cos\theta + \tfrac{1}{3}\Lambda a^2\cos^2\theta. \tag{4.75}$$

Its extension contains Killing horizons that are parametrized by the roots of the horizon equation:

$$(a^2 - l^2) - 2mr_{\mathcal{H}} + r_{\mathcal{H}}^2 - \Lambda\big((a^2 - l^2)l^2 + (\tfrac{1}{3}a^2 + 2l^2)r_{\mathcal{H}}^2 + \tfrac{1}{3}r_{\mathcal{H}}^4\big) = 0. \tag{4.76}$$

Lewandowski and Ossowski [67] provide the geometry of the null generators of $\mathcal{H}$ and investigate the necessary conditions to remove the singularity specific for the Kerr-NUT-(anti) de Sitter horizon [69]. For the non-vanishing values of the parameters a, l and Λ they found that the following must hold:

$$\Lambda = \frac{3}{a^2 + 2l^2 + 2r_{\mathcal{H}}^2}. \tag{4.77}$$

Notice, that due to the above constraint negative values of the cosmological constant Λ must be excluded form the consideration. The expression (4.77) for the cosmological constant implies that the area radius R has a specific value dependent on Λ, that is:

$$R^2 = \frac{3}{2\Lambda}. \tag{4.78}$$

Consequently, it was proven that the type D isolated horizon of the nontrivial $U(1)$-bundle topology is embeddable in the Kerr-NUT-de Sitter spacetime as long as the condition (4.77) is satisfied. The result has been generalized by considering the accelerated Kerr-NUT-(anti) de Sitter spacetimes,, that are also of the nontrivial bundle topology over S_2 and contain a 4-parameter family of type D isolated horizons. Details on the embeddability of our solution in the (accelerated)-Kerr-NUT-(anti) de Sitter spacetimes are provided in [67] whereas the study of non-singular Kerr-NUT-de Sitter spacetimes may be found in [70, 71].

4.3 Classification of the type D isolated horizons of nontrivial $U(1)$-bundle topology

A 3-dimensional non-extremal IHs generated by the null curves forming nontrivial $U(1)$-bundles were considered. For the non-extremal case, it happens that the rotation 1-form potential ω_a divided by the surface gravity $\kappa^{(\ell)}$ corresponds to the connection on the bundle. Therefore, we have a natural interplay of IH's geometry and the geometry of the $U(1)$-bundle. From the point of view of the 4-dimensional spacetime, solutions to the Petrov type D equation define IHs embeddable in vacuum spacetimes satisfying Einstein equation's (1.1) with cosmological constant Λ as Killing horizons to the second order such that the spacetime Weyl tensor on the horizon is of the Petrov type D. From the perspective of the $U(1)$-bundle structure, the type D equation couples $U(1)$-connection with the metric tensor g_{AB} of the base manifold $\mathcal{S}_2$ and surface gravity $\kappa^{(\ell)}$. We have specifically focused on the IHs of the structure of the $U(1)$-bundles over 2-manifolds diffeomorphic to $\mathcal{S}_2$. Such bundle is characterized by the integer corresponding to the topological charge and is mathematically equivalent to the Dirac monopole. Notice however, that in our case, the electromagnetic vector potential of the Dirac monopole is replaced by the rotation 1-form potential $\omega^{\pm}_A$ divided by the surface gravity $\kappa^{(\ell)}$.

We have derived all axisymmetric solutions to the type D equation with cosmological constant and discussed their embeddability. We found that the solutions consisting of the frame coefficient P^2 and area radius R^2, together with the cosmological constant Λ, determine the metric tensor g_{AB} (4.18) and the rotation scalar $\mathcal{O}$. Derivative ∇_a can be reconstructed from the 2-metric g_{AB} and scalar $\mathcal{O}$. The topological charge m, which is an integer number, of the $U(1)$-bundle structure of $\mathcal{H}$ and the surface gravity $\kappa^{(\ell)}$ determine the parameter n. Now we present a classification (see table 4.1) and summary of our results.

We start with the first class that is characterized by metric tensor g_{AB} of the constant Gaussian curvature:

$$K = \frac{1}{R^2}, \tag{4.79}$$

and the constant rotation scalar $\mathcal{O}$ expressed in terms of n and R^2 (see table 4.1). There are no constraints on the area radius R^2 which can take any values in $\mathbb{R}^+$. This class is embeddable in the Taub-NUT-(anti) de Sitter spacetime. Such spacetime contains a horizon of the nontrivial bundle structure. Specifically its has a structure of the Hopf fibration of S_3 over S_2 and is of the Petrov type D, which is also the case for the spacetime itself. Notice that the cosmological constant is arbitrary, therefore we parametrize this class by three real parameters R^2, n and Λ'.

The second class is constrained by the specific relation of the radius R and rescaled cosmological constant Λ', that is:

$$R^2 = \frac{1}{2\Lambda'} \tag{4.80}$$

and the condition on rotation scalar $\mathcal{O}$:

$$\partial_A \mathcal{O} \neq 0. \tag{4.81}$$

It is parametrized by three real parameters, that is n, α and Λ'. There are however some constrains, namely the cosmological constant has to be positive, that is:

$$\Lambda' > 0, \tag{4.82}$$

Possible solutions to the Petrov type D equation		
Class	Condition on area radius R^2	Components of the geometry
I	$R^2 > 0$	$P^2 = 1 - x^2$ $\mathcal{O} = \frac{n}{2R^2}$
II	$R^2 = \frac{1}{2\Lambda'}$ and $\Lambda' > 0$	$P^2 = 1 - x^2$ $\mathcal{O} = -\frac{2\alpha\left(1-\left(\frac{n\Lambda'}{2\alpha}\right)^2\right)^2}{\left(x-\frac{n\Lambda'}{2\alpha}\right)^3}$
III	$R^2 \neq \frac{1}{2\Lambda'}$	$P^2 = \frac{(1-x^2)\left(\left(x-\frac{1}{2}\eta n(1-\Lambda'\gamma)\right)^2+\eta^2+\frac{1-x^2}{1-\Lambda'\gamma}\right)}{\left(x-\frac{1}{2}\eta n(1-\Lambda'\gamma)\right)^2+\eta^2}$ $\mathcal{O} = \mathrm{Im}\left[\frac{2i\left(1-\eta^2\left(\frac{1}{2}n(\Lambda'\gamma-1)+i\right)^2\right)}{\eta\gamma\left(x+\frac{1}{2}\eta n(\Lambda'\gamma-1)+i\eta\right)^3}\right]$

Table 4.1: Solutions to the type D equation with cosmological constant on horizons of non-trivial bundle topology divided into three classes.

otherwise the area radius would be negative which we do not allow. Moreover, the frame coefficient P^2 is of the form (4.65) and its manifest that it is non-negative on the entire domain of x. One has to be attentive in regard to the behavior of the component Ψ_2 (4.24) of the Weyl tensor on the domain $x \in [-1, 1]$. For it to be well-defined, the following inequality must hold:

$$\left|\frac{n\Lambda'}{2\alpha}\right| > 1. \tag{4.83}$$

Finally, the most generic class III is parametrized by four real parameters η, γ, n and Λ'. One has to study domains of those parameters for the metric tensor g_{AB} to be well-defined and differentiable, also at the poles $x = \pm 1$. First constraint that we get is due to the area radius R^2 being positive:

$$R^2 = \frac{\gamma}{2(\gamma\Lambda' - 1)} > 0 \quad \Leftrightarrow \quad \Lambda' > \frac{1}{\gamma}. \tag{4.84}$$

Additionally, the frame coefficient P^2 has to be positive on the interval $(-1,1)$, that is:

$$P^2 > 0 \Leftrightarrow \left(x - \frac{1}{2}\eta n\left(1-\Lambda'\gamma\right)\right)^2 + \eta^2 + \frac{1-x^2}{1-\Lambda'\gamma} > 0 \tag{4.85}$$

The above inequality is satisfied if one of the following cases is true:

(i) $\gamma < 0$;

(ii) $\left(\gamma > 0\right) \wedge \left(\eta^2 > \frac{-\Lambda'\gamma}{(1-\Lambda'\gamma)\left((1-\Lambda'\gamma)^2\frac{n^2}{4}+\Lambda'\gamma\right)}\right)$;

(iii) $\left(\gamma > 0\right) \wedge \left(\eta^2 < \frac{-\Lambda'\gamma}{(1-\Lambda'\gamma)\left((1-\Lambda'\gamma)^2\frac{n^2}{4}+\Lambda'\gamma\right)}\right)$
$\wedge \left(\left|\eta n\right| < \frac{\Lambda'\gamma+\sqrt{((1-\Lambda'\gamma)\eta^2+1)\Lambda'\gamma+\frac{1}{4}\eta^2 n^2(1-\Lambda'\gamma)^3}}{\frac{1}{2}(1-\Lambda'\gamma)}\right)$.

Recall, that the complex constant c_2 cannot vanish, otherwise Ψ_2 is ill-defined at $x=0$. Consequently, the parameter η has to be non-vanishing for all three cases. This solution is embeddable in the Kerr-NUT-de Sitter spacetime for a specific values of the area radius (4.78) and cosmological constant (4.77). In a general case, our isolated horizon is embeddable in the accelerated Kerr-NUTT-(anti) de Sitter spacetime [67].

Theorem 4.3.1. *The axisymmetric solution to the Petrov type D equation with cosmological constant on the non-extremal isolated horizon generated by the null curves forming $U(1)$-bundles over $\mathcal{S}_2$ is of one of the three classes displayed in Table 4.1 and in the most generic case (class III) depends on 3 parameters: η, γ and n, for a given value of the cosmological constant Λ.*

Remark. A $U(1)$-bundle and the IH structure may be reconstructed from the data provided in Table 4.1. Suppose the arbitrary topological charge m is fixed,

$$m \neq 0 \tag{4.86}$$

together with the corresponding $U(1)$-bundle:

$$\Pi : \mathcal{H} \to \mathcal{S}_2. \tag{4.87}$$

Consequently, for all data from Table 4.1 such that the parameter n is non-vanishing, we calculate the surface gravity, that is:

$$\kappa^{(\ell)} = \frac{n}{m}, \tag{4.88}$$

and reconstruct a unique isolated horizon structure g_{ab} and ∇_a, modulo automorphisms of $\mathcal{H}$. On the other hand, in case of:

$$n = 0 \tag{4.89}$$

data from Table 4.1 reduces to the axisymmetric solutions to the Petrov type D equation for IH of the topology:

$$\mathcal{H} = \mathbb{R} \times S_2, \tag{4.90}$$

which we found in Chapter 2. Such type D isolated horizons can be defined by the subgroup of $\mathbb{R}$ and, in consequence, become the trivial $U(1)$-bundle: $U(1) \times \mathcal{S}_2$.

Chapter 5

The gravitational radiation in the de Sitter background

The beginning of the theory of gravitational waves dates back to the 1950's, when a young Polish physicist, Andrzej Trautman, of the Polish Academy of Science, published the results of his work on the novel formalism of gravitational radiation [72–74]. He was then followed by Pirani [75], Bondi [76–78], Sachs [79], and others in a development of the theory of gravitational radiation in spacetimes with the vanishing cosmological constant. It was only in 2015, when gravitational waves were finally observed for the very first time by LIGO detector in the USA. The detection was a great accomplishment itself, awarded with a Nobel Prize in 2017. In case of the vanishing cosmological constant Λ, to calculate the energy of gravitational radiation emitted by a compact source one would use Einstein's quadruple formula obtained via weak field approximation around Minkowski spacetime [80, 81]. Now, however, there is a strong evidence that the cosmological constant is positive and therefore a generalization of the famous quadruple formula is needed and will be a subject of this chapter.

5.1 Our approach to calculating the generalized quadruple formula

We study the linearized Einstein's equations with the positive cosmological constant to calculate the energy carried away by gravitational radiation emitted by time-changing compact source. In case of the vanishing cosmological constant, we usually study small perturbations of Minkowski spacetime and use a suitable notion, that is the conformal boundary called *scri plus* ($\mathcal{I}^+$). A natural way to approach the problem of gravitational waves in $\Lambda > 0$ spacetime is the analysis of the perturbed de Sitter metric. A generalization of the framework of Minkowski spacetime $\mathcal{I}^+$ to de Sitter spacetime is, however, ambiguous and depends on the properties that one wants to preserve. Notice, that the conformal boundary $\mathcal{I}^+$ for the de Sitter spacetime is a spacelike surface instead of a null surface as in Minkowski spacetime. Ashtekar et al. [36, 37] proposed that a role of the generalized $\mathcal{I}^+$ in de Sitter spacetime may be served by a cosmological horizon (which satisfies the definition of the isolated horizon). Such choice for $\mathcal{I}^+$ would preserve its null character. We will follow this proposal and investigate the problem of gravitational waves in de Sitter spacetime.

Consider the gravitational radiation emitted by a time changing compact source in slow motion approximation in the background of de Sitter spacetime. We make several assumption on the nature of the matter source. First, assume that the system is stationary at distant past and future and its physical size is bounded by D_0, which is significantly smaller than the inverse of the Hubble parameter H. We also assume that the cosmological constant is very small, which makes the cosmological Killing horizon very distant from the considered source. Additionally, suppose that the system is isolated in a sense that there is no incoming radiation passing through the past boundary of the Poincaré patch. The Wald-Zoupas [40] and Chandrasekaran-Flanagan-Prabhu [41] notion of radiation through a null surfaces is applied, however, a suitable gauge needs to be employed so that the cosmological horizon remains null with respect to the perturbed geometry. We find an appropriate gauge transformation and calculate the energy flux, with respect to the time translation Killing vector field generating the horizon, to the second order in the perturbation and first order in the Hubble parameter $H := \sqrt{\Lambda/3}$. We find that the zeroth order term in H coincides with the standard Einstein's quadruple formula [81] for the perturbed Minkowski spacetime. The first order term in Hubble parameter H, on the other hand, is the new correction coming from the non-vanishing cosmological constant Λ.

A generalization of the quadruple formula for the spacelike conformal boundary of de Sitter spacetime was derived and studied in [82]. Even though our formula for the energy flux is derived for a different surface, the total integrals along each surface should coincide due to the symplectic theory on the space of solutions to Einstein's equations [83]. The notion of the radiation through a general NEH was also studied by Ashtekar et al. in [84,85]. Although motivated by an example of a black hole as the final product of a binary coalescence, authors assumed that the expansion of the perturbed horizon vanishes asymptotically in the future. Consequently, the expansion in the linear order, given a suitable gauge fixing (see [86]), vanishes on the entire horizon. In our approach, on the other hand, the argument from [86] is not applicable, as the suitable MOTS operator is not invertible.

We will begin our consideration by introducing some of the results obtained by Ashtekar et al. in [82]. Among them are the solution to the linearized Einstein's equations with the cosmological constant Λ and definitions of the mass and pressure quadruple moments. Moreover, we take the purely spatial components of the retarded solution expressed in terms of the quadruple moments derived in [82] and using gauge conditions find the rest of the components of the solution.

5.2 A time-changing matter source in de Sitter spacetime emitting gravitational waves

In this section we study a time-changing matter source, such that its size is uniformly bounded in time. We investigate the radiation emitted by the source in the background of de Sitter spacetime, provided the solution to the linearized Einstein's equations with cosmological constant Λ.

5.2.1 The future Poincaré patch

A time changing matter source in de Sitter spacetime producing gravitational radiation is considered. The spatial size of the source is uniformly bounded in time, whereas its causal structure covers the so-called future Poincaré patch which is depicted in Figure 5.1. Next, consider the background de Sitter metric tensor $\bar{g}_{\alpha\beta}$ of the following form:

$$\bar{g}_{\alpha\beta}dx^\alpha dx^\beta = \tfrac{1}{H^2\eta^2}\mathring{g}_{\alpha\beta}dx^\alpha dx^\beta = \tfrac{1}{H^2\eta^2}(-d\eta^2 + dx^2 + dy^2 + dz^2). \tag{5.1}$$

We focus on the future Poincaré patch where the co-moving spatial coordinates (x, y, z) take arbitrary real values while the domain of conformal time η is $(-\infty, 0]$.

To derive the formula for energy radiated by an isolated system in the presence of cosmological constant $\Lambda > 0$ consider first order perturbations of the de Sitter spacetime. Furthermore, we use the solution to the linearized Einstein's equations provided in [82, 87], where the perturbed metric tensor $g_{\alpha\beta}$ takes a form:

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \epsilon\gamma_{\alpha\beta}, \tag{5.2}$$

and ϵ denotes a smallness parameter. To solve linearized Einstein's equation it is convenient to introduce the trace-reversed metric perturbation, namely:

$$\bar{\gamma}_{\alpha\beta} := \gamma_{\alpha\beta} - \tfrac{1}{2}\bar{g}_{\alpha\beta}\gamma, \tag{5.3}$$

and consequently write the linearized field equations in the presence of a first order linearized source $T_{\alpha\beta}$ as:

$$\bar{\Box}\bar{\gamma}_{\alpha\beta} - 2\bar{\nabla}_{(\alpha}\bar{\nabla}^{\mu}\bar{\gamma}_{\beta)\mu} + \bar{g}_{\alpha\beta}\bar{\nabla}^{\mu}\bar{\nabla}^{\nu}\bar{\gamma}_{\mu\nu} - \tfrac{2}{3}\Lambda(\bar{\gamma}_{\alpha\beta} - \bar{g}_{\alpha\beta}\bar{\gamma}) = -16\pi T_{\alpha\beta}, \tag{5.4}$$

where $\bar{\nabla}$ and $\bar{\Box}$ are the derivative and d'Alembertian operators, respectively, corresponding to the de Sitter metric $\bar{g}_{\alpha\beta}$. The solution to (5.4) has been studied thoroughly in particular in [82, 87]. In the next subsection we will provide the main steps of its derivation following [82].

5.2.2 The retarded solution to the linearized Einstein's equations

Ashtekar et al. [82] introduce the trace-reversed, rescaled metric perturbation in order to solve the linearized Einstein's equations (5.4). Here we follow the derivation of the solution to (5.4) and use their generalized definitions of the mass and quadruple moments for non-vanishing cosmological constant, which will play an important role in the remaining part of this chapter. Not only we use the totally spatial part of the retarded solution expressed in terms of the quadruple moments, but also derive the $\eta\eta$-component via the gauge condition.

First, consider a vector field denoted by η^α, that is normal to the $\eta = \text{const}$ hypersurface and such that:

$$\eta^\alpha\nabla_\alpha\eta = 1. \tag{5.5}$$

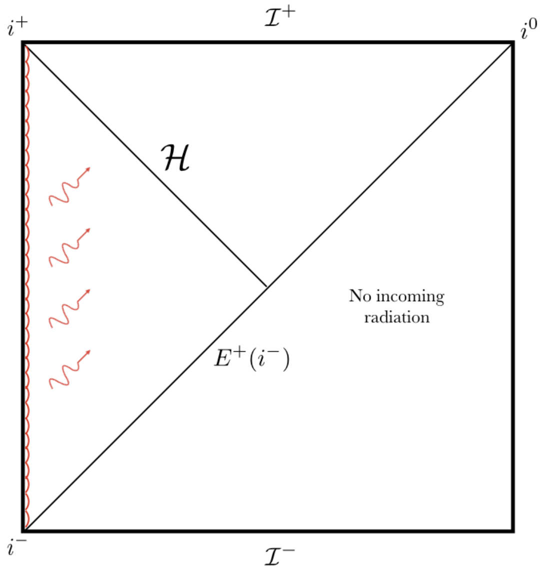

Figure 5.1: A time-changing matter source emitting gravitational waves in the background of de Sitter spacetime. The spatial size of the system is uniformly bounded in time, whereas its causal future covers the future Poincaré patch (the upper triangle with vertices at i^-, i^0 and i^+). We assume no incoming radiation across the past boundary $E^+(i^-)$, which is the future event horizon of i^- [82]. The cosmological horizon is denoted by $\mathcal{H}$.

In the introduced coordinates 5.1 it is of the form:

$$\eta^\alpha \partial_\alpha = \partial_\eta. \tag{5.6}$$

Define n^α, that is the future pointing unit normal to cosmological slices, namely:

$$n^\alpha := -H\eta\eta^\alpha. \tag{5.7}$$

In [82] the following gauge condition is chosen:

$$\bar{\nabla}^\alpha \bar{\gamma}_{\alpha\beta} = 2Hn^\alpha \bar{\gamma}_{\alpha\beta}, \tag{5.8}$$

which simplifies solving the linearized Einstein's equations (5.4). Next consider:

$$\bar{\chi}_{\alpha\beta} := H^2\eta^2 \bar{\gamma}_{\alpha\beta}, \tag{5.9}$$

as well as the following decomposition:

$$\tilde{\chi} := (\eta^\alpha\eta^\beta + \mathring{q}^{\alpha\beta})\bar{\chi}_{\alpha\beta}, \qquad \chi_\alpha := \eta^\sigma \mathring{q}_\alpha{}^\beta \bar{\chi}_{\beta\sigma}, \qquad \chi_{\alpha\beta} := \mathring{q}_\alpha{}^\mu \mathring{q}_\beta{}^\nu \bar{\chi}_{\mu\nu}, \tag{5.10}$$

$$\tilde{\mathcal{T}} := (\eta^\alpha\eta^\beta + \mathring{q}^{\alpha\beta})T_{\alpha\beta}, \qquad \mathcal{T}_\alpha := \eta^\sigma \mathring{q}_\alpha{}^\beta T_{\beta\sigma}, \qquad \mathcal{T}_{\alpha\beta} := \mathring{q}_\alpha{}^\mu \mathring{q}_\beta{}^\nu T_{\mu\nu}. \tag{5.11}$$

where $\mathring{q}^{\alpha\beta}$ is a contravariant spatial metric tensor on a cosmological slice $\eta = \text{const}$ induced by the flat metric tensor $\mathring{g}_{\alpha\beta}$,. The gauge conditions (5.8) via decomposition (5.10) take the following form:

$$\mathring{D}^i \chi_{ij} = \partial_\eta \chi_j - \tfrac{2}{\eta}\chi_j, \qquad \mathring{D}^i \chi_i = \partial_\eta(\tilde{\chi} - \chi) - \tfrac{1}{\eta}\tilde{\chi}, \tag{5.12}$$

where $\mathring{D}$ denotes the derivative operator with respect to the metric tensor $\mathring{q}_{ij}$. Consequently, the linearized Einstein's equations (5.4) break into a set of three equations:

$$\mathring{\Box}\Big(\tfrac{1}{\eta}\tilde{\chi}\Big) = -\tfrac{16\pi}{\eta}\tilde{\mathcal{T}}, \tag{5.13}$$

$$\mathring{\Box}\Big(\tfrac{1}{\eta}\chi_i\Big) = -\tfrac{16\pi}{\eta}\mathcal{T}_i, \tag{5.14}$$

$$\Big(\mathring{\Box} + \tfrac{2}{\eta}\partial_\eta\Big)\chi_{ij} = -16\pi\mathcal{T}_{ij}. \tag{5.15}$$

We are particularly interested in the totally spatial projection χ_{ab}. Via gauge conditions it will provide the expression for the $\eta\eta$-components of the trace-reversed, rescaled perturbation $\bar{\chi}_{\alpha\beta}$. Assuming no incoming radiation across the past boundary of the future Poincaré patch the solutions to the above equations read [82]:

$$\tilde{\chi}(\eta, \vec{x}) = 4\eta \int \frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}\frac{1}{\eta_{\text{Ret}}}\tilde{\mathcal{T}}(\eta_{\text{ret}}, \vec{x}') \tag{5.16}$$

$$\chi_i(\eta, \vec{x}) = 4\eta \int \frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}\frac{1}{\eta_{\text{Ret}}}\mathcal{T}_i(\eta_{\text{ret}}, \vec{x}') \tag{5.17}$$

$$\chi_{ij}(\eta, \vec{x}) = 4 \int \frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}\mathcal{T}_{ij}(\eta_{\text{Ret}}, \vec{x}') + 4\int d^3\vec{x}' \int_{-\infty}^{\eta_{\text{Ret}}} d\eta' \frac{1}{\eta'}\partial_{\eta'}\mathcal{T}_{ij}(\eta', \vec{x}') \tag{5.18}$$

where $\eta_{\text{Ret}} := \eta - |\vec{x}-\vec{x}'|$. Moreover, suppose that the physical size of the isolated system is uniformly bounded by D_0 on all cosmological slices $\eta = \text{const}$ such that:

$$D_0 \ll 1/H. \tag{5.19}$$

Therefore, the source will remain compact regardless of the expansion of the universe.

Next, we assume that the system is stationary at distance past and future, which means that the following occurs:

$$\mathcal{L}_T T_{\alpha\beta} = 0 \tag{5.20}$$

outside some finite time, and where T is the Killing vector field:

$$T = -H\Big(\eta\partial_\eta + x\partial_x + y\partial_y + z\partial_z\Big). \tag{5.21}$$

In [37] the vector field T is referred to as the time translation vector field since it is equal to the limit of the time translation Killing vector field for the Schwarzschild-de Sitter spacetime for the vanishing mass whereas its limit for $\Lambda \to 0$ reduces to a time translation in Minkowski spacetime.

Finally, the slow motion approximation is used, in other words, the velocity of the source is small $v \ll 1$ in $c = 1$ units. Consequently, the solution (5.18) may be simplified to [82]:

$$\chi_{ij}(\eta, \vec{x}) = \frac{4}{r}\int d^3\vec{x}'\mathcal{T}_{ij}(\eta_{\text{ret}}, \vec{x}') + 4\int d^3\vec{x}' \int_{-\infty}^{\eta_{\text{ret}}} d\eta' \frac{1}{\eta'}\partial_{\eta'}\mathcal{T}_{ij}(\eta', \vec{x}'), \tag{5.22}$$

where $\eta_{\text{ret}} = \eta - r$.

5.2.3 The retarded solution expressed in terms of the quadruple moments

In this subsection we follow the derivation in [82] to provide the expression for retarded solution (5.22) written in terms of the mass and pressure quadruple moments. They are defined in the following way:

$$Q_{ij}^{(\rho)}(\eta) := \int_\Sigma d^3V \rho(\eta)\bar{x}_i\bar{x}_j, \tag{5.23}$$

$$Q_{ij}^{(p)}(\eta) := \int_\Sigma d^3V \big(p_1(\eta)+p_2(\eta)+p_3(\eta)\big)\bar{x}_i\bar{x}_j, \tag{5.24}$$

where Σ is any cosmological surface $\eta = \text{const}$, whereas d^3V is the proper volume element and $\bar{x}_i := -\frac{1}{H\eta}x_i$. We define matter density ρ and pressure p_i in a usual fashion[1]

$$\rho = T_{\alpha\beta}n^\alpha n^\beta, \tag{5.25}$$

$$p_i = T^{\alpha\beta}\partial_\alpha x_i \partial_\beta x_i. \tag{5.26}$$

Similar strategy, as the one in Minkowski spacetime, is used to determine the expression for the traced-reversed, rescaled perturbation χ_{ij}. Both terms in (5.22) contain the integral of the totally spatial components of the stress-energy tensor $\mathcal{T}_{ij}$ of the source. Conservation of the stress-energy tensor:

$$\bar{\nabla}^\alpha T_{\alpha\beta} = 0 \tag{5.27}$$

allows one to first write the integrals from (5.22) in terms of the second time derivative of the matter density ρ and then using definitions (5.23) and (5.24) calculate[2]:

$$\begin{aligned}\chi_{ij} = &-\tfrac{2H}{r}\eta_{\text{ret}}[\ddot{Q}_{ij}^{(\rho)} + 2H\dot{Q}_{ij}^{(\rho)} + H\dot{Q}_{ij}^{(p)} + 2H^2 Q_{ij}^{(p)}](\eta_{\text{ret}}) \\ &- 2H\int_{-\infty}^{\eta_{\text{ret}}} \frac{d\eta'}{\eta'}\partial'_\eta\Big(\eta'[\ddot{Q}_{ij}^{(\rho)} + 2H\dot{Q}_{ij}^{(\rho)} + H\dot{Q}_{ij}^{(p)} + 2H^2 Q_{ij}^{(p)}](\eta')\Big)\end{aligned} \tag{5.28}$$

where the dot denotes the Lie derivative with respect to the time translation Killing vector field T.

Our goal is to find the first order corrections in Hubble parameter H to the quadruple formula for the energy of gravitational waves passing through the cosmological Killing horizon. The formula should coincide with Einstein's quadruple formula for the vanishing H (or equivalently vanishing cosmological constant Λ). Notice however, that the introduced coordinate system (η, x, y, z) is not well defined for $H \to 0$, as the denominator in the conformal factor in (5.1) is ill-defined for $H = 0$. Therefore, we introduce coordinates (t, r, θ, φ) such that:

$$\eta = -\tfrac{1}{H}e^{-Ht}, \qquad x = r\cos\theta, \qquad y = r\sin\theta\cos\varphi, \qquad z = r\sin\theta\sin\varphi, \tag{5.29}$$

[1] Notice that there is no summation over index i in the definition of pressure p_i (5.26).

[2] A full derivation of the totally spatial components of the traced-reversed perturbation χ_{ij} was presented [82].

in which the metric $\bar{g}_{\alpha\beta}$ reads:

$$\bar{g}_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + e^{2Ht}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2). \tag{5.30}$$

It is clear that the above metric is well-defined for the vanishing Hubble parameter H and recovers the Minkowski metric tensor. At times it will be convenient to also use the coordinates $(t, \vec{x})$. The time translation vector field T takes a form:

$$T = (\partial_t - Hr\partial_r)|_{\mathcal{H}} = \partial_t - e^{-Ht}\partial_r, \tag{5.31}$$

where the last equality is satisfied on the cosmological horizon $\mathcal{H}$, which is defined by:

$$r = \tfrac{1}{H}e^{-Ht}. \tag{5.32}$$

Next, transform the trace-reversed, rescaled perturbation to the new coordinates, which yields:

$$\begin{aligned}\chi_{ij} = {} & \tfrac{2}{r}e^{-Ht_{\rm ret}}[\ddot{Q}^{(\rho)}_{ij} + 2H\dot{Q}^{(\rho)}_{ij} + H\dot{Q}^{(p)}_{ij} + 2H^2 Q^{(p)}_{ij}](t_{\rm ret}) \\ & - 2H\int_{-\infty}^{t_{\rm ret}} dt'[\dddot{Q}^{(\rho)}_{ij} + 3H\ddot{Q}^{(\rho)}_{ij} + 2H^2\dot{Q}^{(\rho)}_{ij} + H\ddot{Q}^{(p)}_{ij} + 3H^2\dot{Q}^{(p)}_{ij} + 2H^3 Q^{(p)}_{ij}]\end{aligned} \tag{5.33}$$

where the dot denotes the Lie derivative with respect to the time translation vector field T and $t_{\rm ret}$ is the retarded time, that is:

$$\dot{Q}_{ij}(t) := \mathcal{L}_T Q_{ij}(t) = \partial_t Q_{ij}(t) - 2HQ_{ij}(t), \tag{5.34}$$

$$t_{\rm ret} := -\tfrac{1}{H}\ln(e^{-Ht} + Hr)). \tag{5.35}$$

Next, we express $\eta\eta$-component of the trace-reversed, rescaled perturbation in terms of the mass and pressure quadruple moments. To start with, consider gauge conditions (5.12), that lead to expressions for χ_i and $\bar{\chi}_{\eta\eta}$, that is:

$$\chi_i = e^{-2Ht}\int^t e^{Ht'}D^j\chi_{ij}dt' + e^{-2Ht}c_i \tag{5.36}$$

$$\bar{\chi}_{\eta\eta} = e^{-Ht}\int^t (D^i\chi_j - He^{Ht'}\chi)dt' + e^{-Ht}d. \tag{5.37}$$

Notice, that from the form of (5.17) follows the vanishing of χ_i on the past boundary $E^+(i^-)$ of the future Poincaré patch. Therefore, the integration constant c_i also vanishes in (5.36). From the definition of $\tilde{\chi}$ (5.10) follows the identity:

$$\bar{\chi}_{\eta\eta} = \tilde{\chi} - \chi, \tag{5.38}$$

where $\chi := \mathring{q}^{ij}\chi_{ij}$. Due to the vanishing of $\tilde{\chi}$ and χ on $E^+(i^-)$, the integration constant in (5.37) also has to vanish:

$$d = 0. \tag{5.39}$$

Therefore, we find that both constants, c_i and d, are equal to zero which is consistent with no-incoming radiation assumption. Consequently, substituting the expression (5.36) for χ_i in (5.37) yields:

$$\bar{\chi}_{\eta\eta} = e^{-Ht}\int^t \left(e^{-2Ht'}\int^{t'} e^{Ht''}D^iD^j\chi_{ij}dt'' - He^{Ht'}\chi\right)dt'. \tag{5.40}$$

To find the expression for $\chi_{\eta\eta}$ in terms of the quadruple moments, it is convenient to first perform integration in (5.33) and separate the higher order terms $\mathcal{O}(H^3)$, that is:

$$\begin{aligned}\chi_{ij} &= \tfrac{2}{r}e^{-Ht_{\text{ret}}}[\ddot{Q}^{(\rho)}_{ij} + 2H\dot{Q}^{(\rho)}_{ij} + H\dot{Q}^{(p)}_{ab} + 2H^2 Q^{(p)}_{ij}](t_{\text{ret}}) \\ &\quad - 2H\int_{-\infty}^{t_{\text{ret}}} dt'[\dddot{Q}^{(\rho)}_{ij} + 3H\ddot{Q}^{(\rho)}_{ij} + 2H^2\dot{Q}^{(\rho)}_{ij} + H\ddot{Q}^{(p)}_{ij} + 3H^2\dot{Q}^{(p)}_{ab} + 2H^3 Q^{(p)}_{ij}] \\ &= \tfrac{2}{r}e^{-Ht}[\partial_t^2 Q^{(\rho)}_{ij} - 2H\partial_t Q^{(\rho)}_{ij} + H\partial_t Q^{(p)}_{ij}](t_{\text{ret}}) + 2H^2[\partial_t Q^{(\rho)}_{ij}](t_{\text{ret}}) + \mathcal{O}(H^3). \quad (5.41)\end{aligned}$$

Next, combining the above with (5.40) and performing some tedious calculations leads to an expression for $\chi_{\eta\eta}$ in terms of the quadruple moments, namely:

$$\begin{aligned}\bar{\chi}_{\eta\eta} &= 2H(H+\tfrac{1}{r})\tfrac{\tilde{x}^i\tilde{x}^j}{(e^{-Ht}+Hr)}[\partial_t Q^{(p)}_{ij}](t_{\text{ret}}) + 2(\tfrac{1}{r^2} - H^2)\tfrac{\tilde{x}^i\tilde{x}^j}{e^{-Ht}+Hr}[\partial_t Q^{(\rho)}_{ij}](t_{\text{ret}}) \\ &\quad - 2Ht\tfrac{\tilde{x}^i\tilde{x}^j}{r(e^{-Ht}+Hr)}[\partial_t^2 Q^{(\rho)}_{ij}](t_{\text{ret}}) + 2(H+\tfrac{1}{r})e^{-Ht}\tfrac{\tilde{x}^i\tilde{x}^j}{e^{-Ht}+Hr}[\partial_t^2 Q^{(\rho)}_{ij}](t_{\text{ret}}) \\ &\quad - 2(H+\tfrac{1}{r})^2[\partial_t Q^{(\rho)}](t_{\text{ret}}) + \tfrac{2}{r}(H+\tfrac{2}{r})\tilde{x}^i\tilde{x}^j[\partial_t Q^{(\rho)}_{ij}](t_{\text{ret}}) + \mathcal{O}(H^3). \quad (5.42)\end{aligned}$$

where we have introduced $\tilde{x}_i$ defined as:

$$\tilde{x}_i := \frac{x_i}{r}. \quad (5.43)$$

Finally, on the horizon we that the following is true:

$$\begin{aligned}T^\mu\partial_\mu\bar{\chi}_{\eta\eta} &= -8H^2[\partial_t^2 Q^{(\rho)}](t_{\text{ret}}) + 6H^2\tilde{x}^i\tilde{x}^j[\partial_t^2 Q^{(\rho)}_{ij}](t_{\text{ret}}) + 2H\tilde{x}^i\tilde{x}^j[\partial_t^3 Q^{(\rho)}_{ij}](t_{\text{ret}}) \\ &\quad + 2H^2\tilde{x}^i\tilde{x}^j[\partial_t^2 Q^{(p)}_{ij}](t_{\text{ret}}), \quad (5.44)\end{aligned}$$

where we omitted higher order terms $\mathcal{O}(H^3)$ since they will not contribute to the leading order corrections to Einstein's quadruple formula[3].

5.3 The energy flux passing through a null surface

In this section, we provide the formula for the energy flux passing through any null surface, which was introduced by Chandrashekaran et al. [41]. It requires some gauge fixing, as our perturbed Killing horizon needs to remain null with respect to the vector field T, and that, generally, is not the case. Eventually, we find the expression for the energy flux passing through the perturbed cosmological horizon in terms of the trace-reversed, rescaled perturbation χ_{ij}.

5.3.1 The formula for the energy flux and the gauge fixing

Energy flux passing through a null surface $\Delta\mathcal{N}$ with the proper volume form d^3V can be calculated using the formula provided in [41]:

$$E_\ell = \tfrac{1}{8\pi}\int_{\Delta\mathcal{N}} d^3V\left(\sigma_{AB}\sigma^{AB} - \tfrac{1}{2}\theta^2\right), \quad (5.45)$$

[3]We provide a full derivation of expression (5.44) in Appendix B.

where shear and expansion of the time translation ℓ are denoted by σ_{AB} and θ, respectively. Notice, that the perturbed surface $\mathcal{H}$ in a generic case does not have to be null and therefore, eq. (5.45) would not be applicable. For the perturbed horizon $\mathcal{H}$ to sustain its null character with respect to the perturbed geometry we introduce a suitable gauge:

$$T^{\mu}\tilde{g}_{\mu a} = 0 \tag{5.46}$$

where the metric tensor $\tilde{g}_{\mu\nu}$ reads:

$$\tilde{g}_{\mu\nu} := g_{\mu\nu} + \mathcal{L}_{\xi} g_{\mu\nu}. \tag{5.47}$$

It can be also interpreted as a procedure of deforming $\mathcal{H}$ in such a way that given the original perturbation of spacetime it remains null. When considering gauge eq. (5.46) it is convenient to use coordinates adapted to the horizon $\mathcal{H}$, which are defined as:

$$v := \eta + r, \tag{5.48}$$

$$u := \eta - r. \tag{5.49}$$

Considering such transformation, we obtain the following form of the de Sitter metric:

$$\bar{g}_{\mu\nu} dx^{\mu} dx^{\nu} = \tfrac{1}{H^2(u+v)^2}\Big(- 4dudv + (v-u)^2(d\theta^2 + \sin^2\theta d\varphi^2)\Big). \tag{5.50}$$

The surface defined by setting $v = 0$ corresponds to the cosmological horizon $\mathcal{H}$. In coordinates (u, v, θ, φ) the gauge condition (5.46) may be written explicitly as a set of three equations, that have to be satisfied on the horizon:

$$\gamma_{uu} + \mathcal{L}_{\xi}\bar{g}_{uu} = 0, \tag{5.51}$$

$$\gamma_{u\theta} + \mathcal{L}_{\xi}\bar{g}_{u\theta} = 0, \tag{5.52}$$

$$\gamma_{u\varphi} + \mathcal{L}_{\xi}\bar{g}_{u\varphi} = 0. \tag{5.53}$$

Calculating the Lie derivatives and solving for ξ^a yields:

$$\xi^v = \tfrac{1}{4}H^2 \int^u u'^2 \gamma_{uu} du' \tag{5.54}$$

$$\xi^\theta = \int^u \Big(2\tfrac{1}{u'^2}\partial_\theta \xi^v - H^2\gamma_{u\theta}\Big) du' \tag{5.55}$$

$$\xi^\varphi = \tfrac{1}{\sin^2\theta} \int^u \Big(2\tfrac{1}{u'^2}\partial_\varphi \xi^v - H^2\gamma_{u\varphi}\Big) du', \tag{5.56}$$

which in the (t, r, θ, φ) chart reads:

$$\xi = \tfrac{1}{2}e^{Ht}\xi^v \partial_t + \tfrac{1}{2}\xi^v\partial_r + \xi^\theta\partial_\theta + \xi^\varphi\partial_\varphi. \tag{5.57}$$

To calculate the energy flux (5.45) we will need the totally angular components of the perturbation, namely:

$$\tilde{\gamma}_{\theta\theta} = \gamma_{\theta\theta} + \mathcal{L}_{\xi}\bar{g}_{\theta\theta}, \tag{5.58}$$

$$\tilde{\gamma}_{\theta\varphi} = \gamma_{\theta\varphi} + \mathcal{L}_\xi \bar{g}_{\theta\varphi}, \tag{5.59}$$

$$\tilde{\gamma}_{\varphi\varphi} = \gamma_{\varphi\varphi} + \mathcal{L}_\xi \bar{g}_{\varphi\varphi}. \tag{5.60}$$

which written in terms of the vector field ξ are of the form:

$$\tilde{\gamma}_{\theta\theta} = \gamma_{\theta\theta} + 2He^{2Ht}r^2\xi^t + 2e^{2Ht}r\xi^r + 2e^{2Ht}r^2\partial_\theta\xi^\theta \tag{5.61}$$

$$\tilde{\gamma}_{\theta\varphi} = \gamma_{\theta\varphi} + e^{2Ht}r^2\partial_\varphi\xi^\theta + e^{2Ht}r^2\sin^2\theta\partial_\theta\xi^\varphi \tag{5.62}$$

$$\begin{aligned}\tilde{\gamma}_{\varphi\varphi} = \gamma_{\varphi\varphi} &+ 2He^{2Ht}r^2\sin^2\theta\xi^t + 2e^{2Ht}r\sin^2\theta\xi^r + 2e^{2Ht}r^2\sin\theta\cos\theta\xi^\theta \\ &+ 2e^{2Ht}r^2\sin^2\theta\partial_\varphi\xi^\varphi\end{aligned} \tag{5.63}$$

Once we made sure that the formula (5.45) is applicable to our system by a suitable gauge fixing, we may then consider the second fundamental form of $\mathcal{H}$, that is:

$$K_{ab} = \tfrac{1}{2}\mathcal{L}_T q_{ab}, \tag{5.64}$$

where q_{ab} is the degenerate metric tensor on the surface $\mathcal{H}$ defined by eq. (5.32). Due to:

$$\ell^a K_{ab} = 0 \tag{5.65}$$

we write it as K_{AB} and uniquely decompose it into traceless shear σ_{AB} and expansion θ, that is:

$$K_{AB} = \tfrac{1}{2}\theta q_{AB} + \sigma_{AB}, \tag{5.66}$$

where q_{AB} is the metric tensor on the space-like cross-section $\mathcal{S}$ of the horizon $\mathcal{H}$. A straightforward calculation yields:

$$\sigma_{AB} = \tfrac{1}{2}T^\mu\partial_\mu\tilde{\gamma}_{AB} - \tfrac{1}{4}q_{AB}q^{CD}T^\mu\partial_\mu\tilde{\gamma}_{CD}, \tag{5.67}$$

$$\theta = \tfrac{1}{2}q^{AB}T^\mu\partial_\mu\tilde{\gamma}_{AB}. \tag{5.68}$$

Consequently, using the above expressions we write the energy flux formula (5.45) in terms of the perturbation $\tilde{\gamma}_{AB}$:

$$E_T = \tfrac{1}{16\pi}\int_{\mathcal{H}} d^3V q^{\theta\theta}q^{\varphi\varphi}\Big((T^\mu\partial_\mu\tilde{\gamma}_{\theta\varphi})^2 - T^\nu\partial_\nu\tilde{\gamma}_{\theta\theta}T^\mu\partial_\mu\tilde{\gamma}_{\varphi\varphi}\Big). \tag{5.69}$$

Additionally, using eq. (5.58), (5.59) and (5.60) we write the above in terms of the original perturbation γ_{AB} and split it into E_T^1 (which comes from the new gauge fixing) and E_T^0:

$$\begin{aligned}
E_T &= \\
&= \tfrac{1}{16\pi}\int_{\mathcal{H}} d^3V q^{\theta\theta}q^{\varphi\varphi}\Big((T^\mu\partial_\mu(\gamma_{\theta\varphi} + \mathcal{L}_\xi\bar{g}_{\theta\varphi}))^2 - T^\nu\partial_\nu(\gamma_{\theta\theta} + \mathcal{L}_\xi\bar{g}_{\theta\theta})T^\mu\partial_\mu(\gamma_{\varphi\varphi} + \mathcal{L}_\xi\bar{g}_{\varphi\varphi})\Big) \\
&= \tfrac{1}{16\pi}\int_{\mathcal{H}} d^3V q^{\theta\theta}q^{\varphi\varphi}\Big(2T^\mu\partial_\mu\gamma_{\theta\varphi}T^\mu\partial_\mu\mathcal{L}_\xi g_{\theta\varphi} - T^\nu\partial_\nu\gamma_{\theta\theta}T^\mu\partial_\mu\mathcal{L}_\xi g_{\varphi\varphi} - T^\mu\partial_\mu\mathcal{L}_\xi g_{\theta\theta}T^\mu\partial_\mu\gamma_{\varphi\varphi}\Big) \\
&+ \tfrac{1}{16\pi}\int_{\mathcal{H}} d^3V q^{\theta\theta}q^{\varphi\varphi}\Big((T^\mu\partial_\mu\gamma_{\theta\varphi})^2 - T^\nu\partial_\nu\gamma_{\theta\theta}T^\mu\partial_\mu\gamma_{\varphi\varphi}\Big) + \mathcal{O}(H^2)
\end{aligned}$$

$$=: E_T^1 + E_T^0 + \mathcal{O}(H^2). \tag{5.70}$$

Notice, that E_T^0 consists of the zeroth and higher order terms in H, whereas E_T^1 is at least linear in H. We write the terms with Lie derivatives explicitly, that is:

$$\begin{aligned}
T^\mu\partial_\mu\mathcal{L}_\xi\bar{g}_{\theta\theta} &= \tfrac{8}{H}\sin^2\theta\chi_{xx} + \tfrac{8}{H}\cos^2\theta\sin^2\varphi\chi_{zz} + \tfrac{8}{H}\cos^2\theta\cos^2\varphi\chi_{yy} \\
&\quad - \tfrac{16}{H}\cos\theta\sin\theta\cos\varphi\chi_{xy} - \tfrac{16}{H}\cos\theta\sin\theta\sin\varphi\chi_{xz} \\
&\quad + \tfrac{16}{H}\cos^2\theta\sin\varphi\cos\varphi\chi_{yz}, && (5.71) \\
T^\mu\partial_\mu\mathcal{L}_\xi\bar{g}_{\theta\varphi} &= \tfrac{8}{H}\sin\theta\cos\theta\sin\varphi\cos\varphi\chi_{zz} - 8\tfrac{1}{H}\sin\theta\cos\theta\sin\varphi\cos\varphi\chi_{yy} \\
&\quad + \tfrac{8}{H}\chi_{xy}\sin^2\theta\sin\varphi + \tfrac{8}{H}\chi_{yz}\sin\theta\cos\theta(\cos^2\varphi - \sin^2\varphi)) \\
&\quad - \tfrac{8}{H}\chi_{xz}\sin^2\theta\cos\varphi && (5.72) \\
T^\mu\partial_\mu\mathcal{L}_\xi\bar{g}_{\varphi\varphi} &= -\tfrac{16}{H}\sin^2\theta\sin\varphi\cos\varphi\chi_{yz} + \tfrac{8}{H}\chi_{zz}\sin^2\theta\cos^2\varphi \\
&\quad + \tfrac{8}{H}\chi_{yy}\sin^2\theta\sin^2\varphi. && (5.73)
\end{aligned}$$

To express the energy flux (5.70) in terms of the quadruple moments, we first write the perturbation $\gamma_{\alpha\beta}$ in (t,r,θ,φ) chart in terms of retarded solution $\chi_{\alpha\beta}$ in (η,x,y,z) chart, that is:

$$\begin{aligned}
\gamma_{\theta\theta} &= \tfrac{r^2}{2}e^{2Ht}\Big[\bar{\chi}_{\eta\eta} + \chi_{xx}(\sin^2\theta - \cos^2\theta) + \chi_{yy}(-\sin^2\theta + \cos^2\theta(\cos^2\varphi - \sin^2\varphi)) \\
&\quad + \chi_{zz}(-\sin^2\theta - \cos^2\theta(\cos^2\varphi - \sin^2\varphi)) - 4\bar{\chi}_{xy}\sin\theta\cos\theta\cos\varphi \\
&\quad - 4\chi_{xz}\sin\theta\cos\theta\sin\varphi + 4\chi_{yz}\cos\theta\sin\varphi\cos\theta\cos\varphi\Big], \\
\gamma_{\varphi\varphi} &= \tfrac{r^2}{2}e^{2Ht}\sin^2\theta\Big[\bar{\chi}_{\eta\eta} - \chi_{xx} + (\bar{\chi}_{yy} - \chi_{zz})(\sin^2\varphi - \cos^2\varphi) - 4\chi_{yz}\sin\varphi\cos\varphi\Big], \\
\gamma_{\theta\varphi} &= r^2e^{2Ht}\Big[(-\chi_{yy} + \chi_{zz})\sin\theta\cos\theta\sin\varphi\cos\varphi + \chi_{xy}\sin^2\theta\sin\varphi - \chi_{xz}\sin^2\theta\cos\varphi \\
&\quad + \chi_{yz}\sin\theta\cos\theta(-\sin^2\varphi + \cos^2\varphi)\Big].
\end{aligned}$$

Using the above transformation together with formula (5.70) and integrating over the angles where possible, yields:

$$\begin{aligned}
E_T^0 &= \tfrac{1}{48H^2}\int dt\Big((T^\mu\partial_\mu\chi_{xx})^2 + (T^\mu\partial_\mu\chi_{yy})^2 + (T^\mu\partial_\mu\chi_{zz})^2 + 4(T^\mu\partial_\mu\chi_{xy})^2 + 4(T^\mu\partial_\mu\chi_{yz})^2 \\
&\quad + 4(T^\mu\partial_\mu\chi_{xz})^2 - 2T^\mu\partial_\mu\chi_{xx}T^\nu\partial_\nu\chi_{yy} - 2T^\mu\partial_\mu\chi_{yy}T^\nu\partial_\nu\chi_{zz} - 2T^\mu\partial_\mu\chi_{xx}T^\nu\partial_\nu\chi_{zz}\Big) \\
&\quad - \tfrac{1}{64\pi H^2}\int dt\int_0^\pi d\theta\int_0^{2\pi} d\varphi\sin\theta T^\nu\partial_\nu\bar{\chi}_{\eta\eta}\Big(T^\mu\partial_\mu\bar{\chi}_{\eta\eta} - \tfrac{2}{r^2}x^ax^bT^\mu\partial_\mu\chi_{ab}\Big), && (5.74) \\
E_T^1 &= \tfrac{4}{15H}\int dt\Big(2\chi_{xx}T^\mu\partial_\mu\chi_{xx} + 2\chi_{yy}T^\mu\partial_\mu\chi_{yy} + 2\chi_{zz}T^\mu\partial_\mu\chi_{zz} + 6\chi_{xy}T^\mu\partial_\mu\chi_{xy} \\
&\quad + 6\chi_{xz}T^\mu\partial_\mu\chi_{xz} + 6\chi_{yz}T^\mu\partial_\mu\chi_{yz} - \chi_{zz}T^\mu\partial_\mu\chi_{xx} - \chi_{yy}T^\mu\partial_\mu\chi_{xx} \\
&\quad - \chi_{xx}T^\mu\partial_\mu\chi_{yy} - \chi_{zz}T^\mu\partial_\mu\chi_{yy} - \chi_{xx}T^\mu\partial_\mu\chi_{zz} - \chi_{yy}T^\mu\partial_\mu\chi_{zz}\Big). && (5.75)
\end{aligned}$$

Now it is manifest, that the overall factor in E^1_T is one order higher in H than in E^0_T as previously stated. Our goal is to find the leading order corrections in H to the Einstein's quadruple formula, consequently for E^0_T we only consider the terms up to the cubic order in H whereas for E^1_T up to the quadratic order in H.

5.3.2 A generalization of the Einstein's quadruple formula

The quadruple nature of the gravitational radiation is apparent in both terms (5.74) and (5.75) of the energy flux formula. Recall, that the totally spatial components of the trace-reversed, rescaled perturbation can be expressed by the mass and pressure quadruple moments (5.41), [82]. We already have all the ingredients needed to write the generalized quadrupole formula:

$$
\begin{aligned}
E_T = \tfrac{2}{15}\int dt \Big[& (\partial_t^3 Q^{(\rho)}_{xx})^2 + (\partial_t^3 Q^{(\rho)}_{yy})^2 + (\partial_t^3 Q^{(\rho)}_{zz})^2 + 3(\partial_t^3 Q^{(\rho)}_{xy})^2 + 3(\partial_t^3 Q^{(\rho)}_{xz})^2 \\
& + 3(\partial_t^3 Q^{(\rho)}_{yz})^2 - (\partial_t^3 Q^{(\rho)}_{xx})(\partial_t^3 Q^{(\rho)}_{yy}) - (\partial_t^3 Q^{(\rho)}_{xx})(\partial_t^3 Q^{(\rho)}_{zz}) - (\partial_t^3 Q^{(\rho)}_{yy})(\partial_t^3 Q^{(\rho)}_{zz}) \\
& + 2H\partial_t^3 Q^{(\rho)}_{xx}\left(\partial_t^2 Q^{(p)}_{xx} + 7\partial_t^2 Q^{(\rho)}_{xx}\right) + 2H\partial_t^3 Q^{(\rho)}_{yy}\left(\partial_t^2 Q^{(p)}_{yy} + 7\partial_t^2 Q^{(\rho)}_{yy}\right) \\
& + 2H\partial_t^3 Q^{(\rho)}_{zz}\left(\partial_t^2 Q^{(p)}_{zz} + 7\partial_t^2 Q^{(\rho)}_{zz}\right) + 6H\partial_t^3 Q^{(\rho)}_{xy}\left(\partial_t^2 Q^{(p)}_{xy} + 7\partial_t^2 Q^{(\rho)}_{xy}\right) \\
& + 6H\partial_t^3 Q^{(\rho)}_{xz}\left(\partial_t^2 Q^{(p)}_{xz} + 7\partial_t^2 Q^{(\rho)}_{xz}\right) + 6H\partial_t^3 Q^{(\rho)}_{yz}\left(\partial_t^2 Q^{(p)}_{yz} + 7\partial_t^2 Q^{(\rho)}_{yz}\right) \\
& - H\partial_t^3 Q^{(\rho)}_{xx}\left(\partial_t^2 Q^{(p)}_{yy} + 7\partial_t^2 Q^{(\rho)}_{yy}\right) - H\partial_t^3 Q^{(\rho)}_{yy}\left(\partial_t^2 Q^{(p)}_{xx} + 7\partial_t^2 Q^{(\rho)}_{xx}\right) \\
& - H\partial_t^3 Q^{(\rho)}_{xx}\left(\partial_t^2 Q^{(p)}_{zz} + 7\partial_t^2 Q^{(\rho)}_{zz}\right) - H\partial_t^3 Q^{(\rho)}_{zz}\left(\partial_t^2 Q^{(p)}_{xx} + 7\partial_t^2 Q^{(\rho)}_{xx}\right) \\
& - H\partial_t^3 Q^{(\rho)}_{yy}\left(\partial_t^2 Q^{(p)}_{zz} + 7\partial_t^2 Q^{(\rho)}_{zz}\right) - H\partial_t^3 Q^{(\rho)}_{zz}\left(\partial_t^2 Q^{(p)}_{yy} + 7\partial_t^2 Q^{(\rho)}_{yy}\right)\Big](t_{\mathrm{ret}}) \\
& + \mathcal{O}(H^2).
\end{aligned} \tag{5.76}
$$

At first the formula may seem quite complicated, however by introducing traceless quadruple moments:

$$
q^{(I)}_{ij} := Q^{(I)}_{ij} - \tfrac{1}{3}\mathring{q}_{ij} Q^{(I)}, \tag{5.77}
$$

where I stands for ρ (mass quadruple moment) or p (pressure quadruple moment) and $Q^{(I)} = \mathring{q}^{ij} Q^{(I)}_{ij}$, it may be written in a more compact and readable form, that is:

$$
E_T = \tfrac{1}{5}\int dt \sum_{i,j=1}^{3}\left[\left(\frac{d^3 q^{(\rho)}_{ij}}{dt^3}\right)^2 + 2H\left(\frac{d^3 q^{(\rho)}_{ij}}{dt^3}\left(\frac{d^2 q^{(p)}_{ij}}{dt^2} + 7\frac{d^2 q^{(\rho)}_{ij}}{dt^2}\right)\right)\right](t_{\mathrm{ret}}) + \mathcal{O}(H^2). \tag{5.78}
$$

The generalized quadruple formula (5.78) for the flux of energy carried away by gravitational radiation passing through the cosmological horizon $\mathcal{H}$ in the background of de Sitter spacetime consists of the third time derivatives of the mass quadruple moment in the zeroth order, just like for Minkowski case, but also terms linear in Hubble parameter $H = \sqrt{\Lambda/3}$ with the second time derivatives of the mass and pressure quadruple moments. It is evaluated at retarded time:

$$
t_{\mathrm{ret}} = (-\ln(e^{-Ht} + Hr)/H\ )|_{\mathcal{H}} = \ 2t. \tag{5.79}
$$

We wrote the generalized quadruple formula (5.78) in the $(t, \vec{x})$ chart, therefore taking a limit for vanishing H (or equivalently $\Lambda = 0$) is straightforward and requires only setting $H = 0$, that is:

$$E_T = \frac{1}{5} \int dt \sum_{i,j=1}^{3} \left[\left(\frac{d^3 q_{ij}^{(\rho)}}{dt^3} \right)^2 \right] (t_{\mathrm{ret}}). \tag{5.80}$$

The result is Einstein's quadruple formula obtained in perturbed Minkowski spacetime[4] [17, 81]. We conclude with the following theorem:

Theorem 5.3.1. *The generalized Einstein's quadruple formula for the flux of the energy carried away by gravitational waves passing through the cosmological Killing horizon $\mathcal{H}$ with respect to the time translation vector field T in the de Sitter background written explicitly to the first order in Hubble parameter $H = \sqrt{\Lambda/3}$ reads:*

$$E_T = \frac{1}{5} \int dt \sum_{i,j=1}^{3} \left[\left(\frac{d^3 q_{ij}^{(\rho)}}{dt^3} \right)^2 + 2H \left(\frac{d^3 q_{ij}^{(\rho)}}{dt^3} \left(\frac{d^2 q_{ij}^{(p)}}{dt^2} + 7 \frac{d^2 q_{ij}^{(\rho)}}{dt^2} \right) \right) \right] (t_{ret}) + \mathcal{O}(H^2). \tag{5.81}$$

[4] Note that in [17] the quadruple moments are defined as $Q_{ab} = 3 \int T_{00} x_a x_b$, therefore the overall factor in (5.80), considering such definition, is not $1/5$ but $1/45$.

Summary

In this thesis we have presented a series of results obtained within the theory of isolated horizons and gravitational radiation for spacetimes with the non-vanishing cosmological constant Λ, based on which several theorems and corollaries were formulated. We assumed embeddability in Λ-vacuum Einstein's equations and stationarity to the second order of the isolated horizon $\mathcal{H}$, which allow to determine the spacetime Weyl tensor by the intrinsic geometry of $\mathcal{H}$ consisting of the degenerate metric tensor g_{ab} and covariant derivative ∇_a. Consequently, the properties of the Weyl tensor pass to the intrinsic geometry of $\mathcal{H}$. We found that the Petrov type of the Weyl tensor on the horizon has to be either II, D or 0, the types III and N appear only as measure zero subsets of the isolated horizon (Theorem 1.2.1). Type 0 is not particularly interesting, as it corresponds to the non-rotating isolated horizons of gaussian curvature of the spatial cross-section of $\mathcal{H}$ equal to $\Lambda/3$. We derived the complex differential equation on the complex invariant Ψ, constructed from the Gauss curvature and rotation scalar, which is a condition for the Petrov type D of the spacetime Weyl tensor on $\mathcal{H}$ (Theorem 1.3.1). Several interesting results related to the near horizon geometry were obtained. First, we found that the Petrov type D equation is its integrability condition (Theorem 1.3.2). Moreover, the transversal double principal null direction of the Weyl tensor is generally twisting. If the second null direction is orthogonal to the spacelike leaves of foliation of $\mathcal{H}$, then the pullback of the rotation 1-form potential, ω_A, satisfies the near horizon equation (Theorem 1.3.3). Whenever the data on the type D isolated horizon $\mathcal{H}$ satisfy the near horizon geometry equation (1.130), $\mathcal{H}$ is both, extremal and non-extremal (Theorem 1.3.4).

The general solution to the Petrov type D equation with cosmological constant on the unknown metric tensor g_{AB} and the rotation scalar $\mathcal{O}$ on the axisymmetric 2-sphere has been found. In the process, we introduced the coordinate system adapted to the axial symmetry. The found family of solutions is 2-dimensional and may be parametrized by the area (that takes positive real values) and angular momentum (taking any real values), which we regard as the no-hair theorem (Theorem 2.2.1) for the axisymmetric isolated horizons of the Petrov type D. We found that the solutions are embeddable in the generalized Schwarzschild/Kerr-(anti) de Sitter spacetime. The only exception are the horizons which admit another null symmetry that is extremal, then they are embeddable in the near extremal horizon spacetimes [38] constructed from the generalized Schwarzschild/Kerr-(anti) de Sitter metrics. The result is a generalization of the previous work [25], where the axisymmetric Petrov type D IHs were studied in the $\Lambda = 0$ case. We calculated the correspondence between our parameters (area A and

angular momentum J (or equivalently η and γ) and parameters on the horizon slice of the generalized Kerr-(anti) de Sitter metric in Boyer-Lindquist coordinates. The obtained solutions to the Petrov type D equation with cosmological constant provide a local characteristic for the spacetime, which distinguishes Schwarzschild/Kerr-(anti) de Sitter solutions. It could be used for the black hole holograph construction of spacetimes about Killing horizons [30,31], which requires the same data, namely the 2-metric tensor g_{AB} and rotational 2-form invariant Ω_{AB} on a 2-surface, as a starting point.

Next, the Petrov type D equation with cosmological constant was considered on a closed 2-surface of genus greater than zero. All of the solutions were derived assuming the embeddability in 4-dimensional spacetime satisfying vacuum Einstein equations with cosmological constant Λ. All of them are characterized by constant Gaussian curvature and closed 1-form ω_A, meaning they are non-rotating (Theorem 3.3.1), that is $J = 0$. We conclude that every rotating Petrov type D isolated horizon stationary to the second order embeddable in a 4-dimensional spacetime satisfying the vacuum Einstein's equations with cosmological constant has a spacelike cross-section of a topological 2-sphere (Theorem 3.3.2). Here we have also considered the near horizon geometry equation on a compact 2-manifold $\mathcal{S}$. We found that if the data consisting of the metric tensor g_{AB} and 1-form ω_A are a solution to this equation then Gauss curvature is constant, whereas the 1-form is closed (Corollary 3.3.1). Combining it with the previous results, we find that in case of a negative cosmological constant Λ the solution is the vanishing 1-form ω_A. Therefore, any solution to the near horizon geometry equation has to be of a constant Gauss curvature and vanishing 1-form ω_A (Corollary 3.3.2).

The type D isolated horizons stationary to the second order of the nontrivial bundle topology were also analyzed. Such horizons do not admit a global spacelike cross-section $\mathcal{S}$, instead they are generated by the null curves forming nontrivial $U(1)$ bundles. A natural interplay of the geometry of the horizon $\mathcal{H}$ and the $U(1)$ bundle was found. The type D condition couples $U(1)$ connection, metric tensor g_{AB} defined on the base manifold and surface gravity in a nontrivial way. All the axisymmetric solutions to the Petrov type D equation with cosmological constant on the space of null generators diffeomorphic to a 2-sphere were derived and classified in Table 4.1. A new parameter related to the topological charge and surface gravity emerged. The found family of solutions is 3-dimensional for a fixed value of the cosmological constant Λ. We investigated the issue of embeddability of the obtained solutions in the known type D black hole spacetimes and for a non-rotating case, found that they are embeddable in the Taub-NUT-(anti) de Sitter spacetime. The embeddability of the general solution in the Kerr-Taub-NUT-(anti) de Sitter has been studied with no positive results. However, Lewandowski-Ossowski in [67] showed, that a special condition on cosmological constant Λ provides the embeddability of our solution in the Kerr-NUT-de Sitter spacetime.

Last but not least, a special case of an IH, that is the cosmological horizon, was considered and used as a generalization of the conformal boundary from the Minkowski spacetime. In this context, we studied the first order time changing matter source producing gravitational waves in de Sitter spacetime. An example of such source is an isolated star or a coalescence of a binary system. We recalled the retarded solution to the linearized Einstein's equations on the background de Sitter spacetime calculated in [82]. Moreover, we have calculated the $\eta\eta$-component of the trace reversed, rescaled perturbation and expressed it in terms of the totally spacelike components χ_{ab}. The general formula for the energy flux on a null surface [41] was used after introducing a suitable gauge ensuring that the horizon remains a null surface with respect to the perturbed geometry and also that the Killing vector field T stays null. We calculated the energy flux expression and wrote it in terms of the perturbation $\bar{\chi}_{\alpha\beta}$. Finally, we expressed it in terms of the mass and pressure quadruple moments and wrote it explicitly to the first order. Although the found result is quite lengthy, the introduced trace-free quadruple moments $q_{ab}^{(i)}$ help to write it in an elegant form (Theorem 5.3.1). It consists of the zeroth and first order terms in Hubble parameter H. The zeroth order term recovers the famous Einstein's formula for the perturbed Minkowski spacetime, whereas the first order term in H is a new correction.

Appendix A

Differentiability condition for the 2-metric g_{AB} in coordinates adapted to axial symmetry

We already showed that condition (2.15) and (2.18) provide continuity and differentiability of the metric tensor g_{AB} (2.12) at the poles $x = \pm 1$. Now, we prove that these two conditions are also sufficient. Recall, that:

$$P^2 = \frac{\Sigma^2 \sin^2\theta}{R^2}. \tag{A.1}$$

It follows that:

$$\Sigma_{,\theta} = \partial_\theta \left(\frac{PR}{\sin\theta} \right) = \frac{RP_{,\theta} - \Sigma\cos\theta}{\sin\theta} = \frac{\frac{\Sigma^2 \sin\theta}{R} P_{,x} - \Sigma\cos\theta}{\sin\theta} = \Sigma \frac{PP_{,x} - \cos\theta}{\sin\theta}. \tag{A.2}$$

Taking a limit as θ approaches 0 (or π) and using the L'Hospital's rule we find:

$$\Sigma_{,\theta}\big|_{\theta=0,\pi} = \lim_{\theta\to 0,\pi} \frac{RP(1 + \frac{1}{2}\frac{P^2}{\sin^2\theta}\partial_x^2 P^2)}{\cos\theta}. \tag{A.3}$$

Therefore, as long as the expression $\frac{P}{\sin\theta}$ has a finite limit at the poles, the right hand side of (A.2) will vanish. Use the general expression for the frame coefficient P^2 with the topological parameter n, that is[1]:

$$P^2 = \frac{(1-x^2)\left((x - \frac{1}{2}\eta n(1-\Lambda'\gamma))^2 + \eta^2 + \frac{1-x^2}{1-\Lambda'\gamma}\right)}{(x - \frac{1}{2}\eta n(1-\Lambda'\gamma))^2 + \eta^2} \tag{A.4}$$

and input it into the differential relation of x and θ:

$$\frac{1}{P^2}dx = \frac{1}{\sin\theta}d\theta. \tag{A.5}$$

[1]Recall that for $n = 0$ the expression for P^2 reduces to the case where the cross-section of $\mathcal{H}$ is a topological 2-sphere

Introduce parameters: $b := -\frac{1}{2}\eta n(1-\Lambda'\gamma)$, $g = \frac{1}{1-\Lambda'\gamma}$ and simplify $\frac{1}{P^2}$, namely:

$$\frac{1}{P^2} = \frac{1}{1-x^2} - \frac{g}{x^2(1-g)+2bx+\eta^2+b^2+g}. \tag{A.6}$$

Then, integrating the left hand side of eq. (A.5) yields:

$$\begin{aligned}
L &= \int \left(\frac{1}{1-x^2} - \frac{g}{x^2(1-g)+2bx+\eta^2+b^2+g} \right) dx \\
&= \int \left(\frac{1}{1-x^2} - \frac{g}{(1-g)} \frac{1}{x^2+2x\frac{b}{1-g}+\frac{\eta^2+b^2+g}{1-g}} \right) dx \\
&= \log\left(\sqrt{\frac{x+1}{1-x}}\right) - \frac{2G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}}\right) + C \\
&= \log\left(C'\sqrt{\frac{x+1}{1-x}}\right) - \frac{2G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}}\right),
\end{aligned} \tag{A.7}$$

where:

$$\begin{aligned}
G &= \frac{g}{1-g}; \\
A &= \frac{\eta^2+b^2+g}{1-g}; \\
B &= \frac{2b}{1-g};
\end{aligned}$$

and we assumed that $4A-B^2>0$. Otherwise the term under square root would take the form: $-4A+B^2$ and the sign in front of the arctan function would change. Furthermore, integrate the right hand side of eq. (A.5) to find:

$$R = \int \frac{1}{\sin\theta} d\theta = -\log(\cot\theta + \frac{1}{\sin\theta}) + D = \log\left(\frac{\sin\theta}{\cos\theta+1}\right) + D. \tag{A.8}$$

From (A.7) and (A.8) we obtain the expression for θ as a function of x:

$$\theta(x) = 2\arctan\left(C''\sqrt{\frac{x+1}{1-x}} \exp\left(\frac{-2G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}}\right)\right)\right). \tag{A.9}$$

Next, write $\sin^2\theta$ in terms of x, that is:

$$\begin{aligned}
\sin^2\theta = 4\Bigg(& C''^2 \frac{x+1}{1-x} \exp\left(\frac{-4G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}}\right)\right) \\
& + \frac{1}{C''^2}\frac{1-x}{x+1} \exp\left(\frac{4G}{\sqrt{4A-B^2}} \arctan\left(\frac{B+2x}{\sqrt{4A-B^2}}\right)\right) + 2\Bigg)^{-1}.
\end{aligned} \tag{A.10}$$

Finally, we use expressions (2.70) and (A.10) to find:

$$
\begin{aligned}
&\frac{P^2}{\sin^2\theta} = \qquad\qquad \text{(A.11)}\\
&= \frac{(x+a)^2+\eta^2+g(1-x^2)}{4((x+a)^2+\eta^2)}\left(C''^2(x+1)^2\exp\left(\frac{-4G}{\sqrt{4A-B^2}}\arctan\left(\frac{B+2x}{\sqrt{4A-B^2}}\right)\right)\right.\\
&\left.+\frac{1}{C''^2}(1-x)^2\exp\left(\frac{4G}{\sqrt{4A-B^2}}\arctan\left(\frac{B+2x}{\sqrt{4A-B^2}}\right)\right)^{-1}+2(1-x^2)\right)\\
&= \frac{(x+\frac{1}{2}B(1+G)^{-1})^2+(A(1+G)-G(1+G)-\frac{1}{4}B^2)(1+G)^{-2}+G(1+G)^{-1}(1-x^2)}{4((x+\frac{1}{2}B(1+G)^{-1})^2+\eta^2)}\\
&\times\left(C''^2(x+1)^2\exp\left(\frac{-4G}{\sqrt{4A-B^2}}\arctan\left(\frac{B+2x}{\sqrt{4A-B^2}}\right)\right)\right.\\
&\left.+\frac{1}{C''^2}(1-x)^2\exp\left(\frac{4G}{\sqrt{4A-B^2}}\arctan\left(\frac{B+2x}{\sqrt{4A-B^2}}\right)\right)^{-1}+2(1-x^2)\right)
\end{aligned}
$$

from which it follows that the term $\frac{P^2}{\sin\theta}$ is finite at the poles and in consequence the right hand side of eq. (A.2) vanishes[2].

[2]It is easy to see that for $P^2 = 1-x^2$ same conclusion is true.

Appendix B

Expression for $\eta\alpha$-components of the trace-reversed, rescaled perturbation in terms of quadruple moments

We calculate $\bar{\chi}_{\eta\eta}$ and $\bar{\chi}_{\eta a}$ and their derivatives with respect to the time translation T, then express them in terms of χ_{ab} and finally in terms of the mass and pressure quadruple moments. It is crucial for finding leading order corrections to Einstein's quadruple formula. Consider the expression for the $\eta\eta$-component of the perturbation, that is:

$$\bar{\chi}_{\eta\eta} = e^{-Ht}\int^{t}\left(e^{-2Ht'}D^aD^b\int^{t'}e^{Ht''}\chi_{ab}dt'' - He^{Ht'}\chi\right)dt'. \tag{B.1}$$

It is convenient to write down the partial derivatives with respect to t, r and x^a of the quadruple moments, that only depend on the retarded time $t_{\rm ret}$, in terms of partial derivatives with respect to t calculated at $t_{\rm ret}$:

$$\partial_t[Q^{(I)}_{ab}(t_{\rm ret})] = [\partial_t Q^{(I)}_{ab}](t_{\rm ret})/(Hre^{Ht}+1), \tag{B.2}$$

$$\partial_r[Q^{(I)}_{ab}(t_{\rm ret})] = -[\partial_t Q^{(I)}_{ab}](t_{\rm ret})/(Hr+e^{-Ht}), \tag{B.3}$$

$$D^a[Q^{(I)}_{ab}(t_{\rm ret})] = -\tfrac{x^a}{r(e^{-Ht}+Hr)}[\partial_t Q^{(I)}_{ab}](t_{\rm ret}), \tag{B.4}$$

where I stands for ρ or p and:

$$t_{\rm ret} := -\tfrac{1}{H}\ln(e^{-Ht}+Hr)). \tag{B.5}$$

We do not need to to write the expression for $\chi_{\eta\eta}$ explicitly, but only consider its derivative with respect to the vector field T, that is:

$$\begin{aligned}T^\mu\partial_\mu\chi_{\eta\eta} &= (\partial_t - e^{-Ht}\partial_r)\left[e^{-Ht}\int^{t}\left(e^{-2Ht'}\int^{t'}e^{Ht''}D^aD^b\chi_{ab}dt'' - He^{Ht'}\chi\right)dt'\right]\\ &= -He^{-Ht}\int^{t}\left(e^{-2Ht'}\int^{t'}e^{Ht''}D^aD^b\chi_{ab}dt'' - He^{Ht'}\chi\right)dt'\end{aligned}$$

$$+ e^{-3Ht}\int^{t'} e^{Ht''} D^a D^b \chi_{ab} dt'' - H\chi$$
$$- e^{-2Ht}\int^{t}\left(e^{-2Ht'}\int^{t'} e^{Ht''}\partial_r D^a D^b \chi_{ab} dt'' - He^{Ht'}\chi\right)dt'. \tag{B.6}$$

Each term is calculated separately, we begin with:

$$-He^{-Ht}\int^{t}\left(e^{-2Ht'}\int^{t'} e^{Ht''} D^a D^b \chi_{ab} dt'' - He^{Ht'}\chi\right)dt', \tag{B.7}$$

where we neglected higher order terms ($\mathcal{O}(H^3)$). The factor in front of a second term in the above expression is of order H^2 whereas χ itself is linear in H. Therefore, this term is at least cubic in H and will be neglected. In consequence, we obtain:

$$\begin{aligned}-He^{-Ht}&\int^{t}\left(e^{-2Ht'}\int^{t'} e^{Ht''} D^a D^b \chi_{ab} dt'' - He^{Ht'}\chi\right)dt' \\ &= -He^{-Ht}\int^{t} e^{-2Ht'}\int^{t'} e^{Ht''} D^a D^b \chi_{ab} dt'' dt' + \mathcal{O}(H^3).\end{aligned} \tag{B.8}$$

Recall, that:

$$\begin{aligned}\chi_{ab} &= \tfrac{2}{r} e^{-Ht_{\mathrm{ret}}}[\ddot{Q}^{(\rho)}_{ab} + 2H\dot{Q}^{(\rho)}_{ab} + H\dot{Q}^{(p)}_{ab} + 2H^2 Q^{(p)}_{ab}](t_{\mathrm{ret}}) \\ &- 2H\int_{-\infty}^{t_{\mathrm{ret}}} dt'[\dddot{Q}^{(\rho)}_{ab} + 3H\ddot{Q}^{(\rho)}_{ab} + 2H^2\dot{Q}^{(\rho)}_{ab} + H\ddot{Q}^{(p)}_{ab} + 3H^2\dot{Q}^{(p)}_{ab} + 2H^3 Q^{(p)}_{ab}] \\ &= \tfrac{2}{r} e^{-Ht}[\partial_t^2 Q^{(\rho)}_{ab} - 2H\partial_t Q^{(\rho)}_{ab} + H\partial_t Q^{(p)}_{ab}](t_{\mathrm{ret}}) \\ &+ 2H[H\partial_t Q^{(\rho)}_{ab} - 2H^2 Q^{(\rho)}_{ab} + H^2 Q^{(p)}_{ab}](t_{\mathrm{ret}}).\end{aligned} \tag{B.9}$$

Since the last two terms are of the overall factor of H^3 we omit them. When acting with D^a on expression (B.9) we get:

$$\begin{aligned}D^a\chi_{ab} &= -\tfrac{2x^a}{r^3} e^{-Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) - 2\tfrac{x^a}{r^2(1+Hre^{Ht})}[\partial_t^3 Q^{(\rho)}_{ab} - 2H\partial_t^2 Q^{(\rho)}_{ab} + H\partial_t^2 Q^{(p)}_{ab}](t_{\mathrm{ret}}) \\ &- 2H^2 e^{Ht}\tfrac{x^a}{r(1+Hre^{Ht})}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) + \mathcal{O}(H^3).\end{aligned} \tag{B.10}$$

Next, we act with yet another derivative D^b on χ_{ab} to obtain:

$$\begin{aligned}D^b D^a\chi_{ab} &= -2\tfrac{1}{r^2(1+Hre^{Ht})}[\partial_t^3 Q^{(\rho)}](t_{\mathrm{ret}}) + 2\tfrac{x^a x^b(3Hre^{Ht}+2)}{r^4(1+Hre^{Ht})^2}[\partial_t^3 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\ &+ 2\tfrac{x^a x^b}{r^4(1+Hre^{Ht})}[\partial_t^3 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) + 2H^2 e^{2Ht}\tfrac{x^a x^b}{r^2(1+Hre^{Ht})^2}[\partial_t^3 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\ &+ 2\tfrac{x^a x^b}{r^3(1+Hre^{Ht})^2} e^{Ht}[\partial_t^4 Q^{(\rho)}_{ab} - 2H\partial_t^3 Q^{(\rho)}_{ab} + H\partial_t^3 Q^{(p)}_{ab}](t_{\mathrm{ret}}) + \mathcal{O}(H^3).\end{aligned} \tag{B.11}$$

Now, we integrate the above to find:

$$\begin{aligned}\int^{t} e^{Ht'} D^b D^a\chi_{ab} &= -2\tfrac{1}{r^2(1+Hre^{Ht})}[\partial_t^3 Q^{(\rho)}](t_{\mathrm{ret}}) + 2\tfrac{x^a x^b(3Hre^{Ht}+2)}{r^4(1+Hre^{Ht})} e^{Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\ &+ 2\tfrac{x^a x^b}{r^4} e^{Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) + 2H^2 e^{3Ht}\tfrac{x^a x^b}{r^2(1+Hre^{Ht})}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\ &+ 2\tfrac{x^a x^b}{r^3(1+Hre^{Ht})} e^{2Ht}[\partial_t^3 Q^{(\rho)}_{ab} - 2H\partial_t^2 Q^{(\rho)}_{ab} + H\partial_{t'}^2 Q^{(p)}_{ab}](t_{\mathrm{ret}})\end{aligned}$$

$$
\begin{aligned}
&+2\int^t \tfrac{1}{r^2(1+Hre^{Ht'})}[\partial_{t'}^3 Q^{(\rho)}](t'_{\rm ret})dt' - 2\int^t \partial_{t'}\left(\tfrac{x^a x^b}{r^4}e^{Ht'}\right)[\partial_{t'}^2 Q^{(\rho)}_{ab}](t'_{\rm ret})dt'\\
&-\int^t \partial_{t'}\left(2\tfrac{x^a x^b(3Hre^{Ht'}+2)}{r^4(1+Hre^{Ht'})}e^{Ht'}\right)[\partial_{t'}^2 Q^{(\rho)}_{ab}](t'_{\rm ret})dt'\\
&-2\int^t \partial_{t'}\left(\tfrac{x^a x^b}{r^3(1+Hre^{Ht'})}e^{2Ht'}\right)[\partial_t'^3 Q^{(\rho)}_{ab} - 2H\partial_{t'}^2 Q^{(\rho)}_{ab} + H\partial_{t'}^2 Q^{(p)}_{ab}](t'_{\rm ret})dt'\\
&-2H^2\int^t \partial_{t'}\left(e^{3Ht'}\tfrac{x^a x^b}{r^2(1+Hre^{Ht'})}\right)[\partial_{t'}^2 Q^{(\rho)}_{ab}](t'_{\rm ret})dt' + \mathcal{O}(H^3),
\end{aligned}
\tag{B.12}
$$

where the integration by parts was used. After some calculations, keeping in mind that the cosmological horizon is defined by $r = e^{-Ht}/H$, we find that all but one of those integrands consists of the terms of order H^2 or lower:

$$
\begin{aligned}
&-2\int^t \partial_{t'}\left(\tfrac{x^a x^b}{r^3(1+Hre^{Ht'})}e^{2Ht'}\right)[\partial_t'^3 Q^{(\rho)}_{ab}](t'_{\rm ret})dt'\\
&\quad = -2H\int^t \tfrac{x^a x^b(Hre^{Ht'}+2)}{r^3(1+Hre^{Ht'})^2}e^{2Ht'}[\partial_t'^3 Q^{(\rho)}_{ab}](t'_{\rm ret})dt'\\
&\quad = -2H\int^t \tfrac{x^a x^b(Hre^{Ht'}+2)}{r^3(1+Hre^{Ht'})}e^{2Ht'}\partial_{t'}[\partial_t'^2 Q^{(\rho)}_{ab}](t'_{\rm ret})dt'\\
&\quad = -2H\tfrac{x^a x^b(Hre^{Ht}+2)}{r^3(1+Hre^{Ht})}e^{2Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\rm ret})dt + \mathcal{O}(H^3),
\end{aligned}
\tag{B.13}
$$

where in the second equality the identity (B.2) was used, whereas in the last one we once again integrated by parts and neglected higher order terms. That leads to:

$$
\begin{aligned}
\int^t e^{Ht}D^b D^a \chi_{ab} &= -2\tfrac{1}{r^2}[\partial_t^2 Q^{(\rho)}](t_{\rm ret}) + 2\tfrac{x^a x^b(3Hre^{Ht}+2)}{r^4(1+Hre^{Ht})}e^{Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\rm ret})\\
&+2\tfrac{x^a x^b}{r^4}e^{Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\rm ret}) + 2H^2 e^{3Ht}\tfrac{x^a x^b}{r^2(1+Hre^{Ht})}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\rm ret})\\
&+2\tfrac{x^a x^b}{r^3(1+Hre^{Ht})}e^{2Ht}[\partial_t^3 Q^{(\rho)}_{ab} - 2H\partial_t^2 Q^{(\rho)}_{ab} + H\partial_t^2 Q^{(p)}_{ab}](t_{\rm ret})\\
&-2H\tfrac{x^a x^b(Hre^{Ht}+2)}{r^3(1+Hre^{Ht})}e^{2Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\rm ret}) + \mathcal{O}(H^3).
\end{aligned}
\tag{B.14}
$$

Next, we calculate the second integrant of the above and find that on the horizon it yields:

$$
\begin{aligned}
&-He^{-Ht}\int^t\left(e^{-2Ht'}\int^{t'} e^{Ht''}D^a D^b \chi_{ab}dt'' - He^{Ht'}\chi\right)dt'\\
&\qquad = -2H^2\tilde{x}^a\tilde{x}^b[\partial_t^2 Q^{(\rho)}_{ab}](t_{\rm ret}) + \mathcal{O}(H^3).
\end{aligned}
\tag{B.15}
$$

Calculating the second line in the last equality of (B.6) and taking the trace of (B.9), while using the expression (B.14) gives:

$$
\begin{aligned}
&e^{-3Ht}\int^{t'} e^{Ht''}D^a D^b \chi_{ab}dt'' - H\chi\\
&\quad = -2\tfrac{e^{-3Ht}}{r^2}[\partial_t^2 Q^{(\rho)}](t_{\rm ret}) + 2\tfrac{x^a x^b(3Hre^{Ht}+2)}{r^4(1+Hre^{Ht})}e^{-2Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\rm ret})\\
&\quad +2\tfrac{x^a x^b}{r^4}e^{-2Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\rm ret}) + 2H^2\tfrac{x^a x^b}{r^2(1+Hre^{Ht})}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\rm ret})\\
&\quad +2\tfrac{x^a x^b}{r^3(1+Hre^{Ht})}e^{-Ht}[\partial_t^3 Q^{(\rho)}_{ab} - 2H\partial_t^2 Q^{(\rho)}_{ab} + H\partial_t^2 Q^{(p)}_{ab}](t_{\rm ret})
\end{aligned}
$$

$$
-2H\tfrac{x^a x^b(Hre^{Ht}+2)}{r^3(1+Hre^{Ht})}e^{-Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) - \tfrac{2H}{r}e^{-Ht}[\partial_t^2 Q^{(\rho)}](t_{\mathrm{ret}}) + \mathcal{O}(H^3). \tag{B.16}
$$

In the final step, we calculate the last term of (B.6):

$$
\begin{aligned}
&-e^{-2Ht}\int^t \left(e^{-2Ht'}\int^{t'} e^{Ht''}\partial_r D^a D^b\chi_{ab}dt'' - He^{Ht'}\partial_r\chi\right)dt' \\
&\quad= -2\tfrac{e^{-2Ht}}{r^2}[\partial_t^2 Q^{(\rho)}](t_{\mathrm{ret}}) + 2\tfrac{\tilde{x}^a\tilde{x}^b(3Hre^{Ht}+2)}{r^2(1+Hre^{Ht})}e^{-2Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\
&\quad+ 2\tfrac{\tilde{x}^a\tilde{x}^b}{r(1+Hre^{Ht})}e^{-Ht}[\partial_t^3 Q^{(\rho)}_{ab} - 2H\partial_t^2 Q^{(\rho)}_{ab} + H\partial_t^2 Q^{(p)}_{ab}](t_{\mathrm{ret}}) \\
&\quad+ 2H^2 e^{2Ht}\tfrac{\tilde{x}^a\tilde{x}^b}{(1+Hre^{Ht})}e^{-2Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) + 2\tfrac{\tilde{x}^a\tilde{x}^b(3Hre^{Ht}+1)}{r^2(1+Hre^{Ht})}e^{-2Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\
&\quad- 2H\tfrac{\tilde{x}^a\tilde{x}^b(Hre^{Ht}+3)}{r(1+Hre^{Ht})}e^{-Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) - 2H\tfrac{\tilde{x}^a\tilde{x}^b}{r(1+Hre^{Ht})}e^{-Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\
&\quad+ 2\tfrac{\tilde{x}^a\tilde{x}^b}{r^2}e^{-2Ht}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) - \tfrac{2H}{r}e^{-Ht}[\partial_t^2 Q^{(\rho)}](t_{\mathrm{ret}}) + \mathcal{O}(H^3).
\end{aligned} \tag{B.17}
$$

Taking it all into consideration and setting $r = e^{-Ht}/H$ we find that on the cosmological horizon the following holds:

$$
\begin{aligned}
T^\mu\partial_\mu\chi_{\eta\eta}|_{\mathcal{H}} &= -8H^2[\partial_t^2 Q^{(\rho)}](t_{\mathrm{ret}}) + 6H^2\tilde{x}^a\tilde{x}^b[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\
&\quad+ 2H\tilde{x}^a\tilde{x}^b[\partial_t^3 Q^{(\rho)}_{ab} + H\partial_t^2 Q^{(p)}_{ab}](t_{\mathrm{ret}}).
\end{aligned} \tag{B.18}
$$

One could also use gauge condition (5.12)to express the ηa-component of the retarded solution in terms of χ_{ab} and consequently quadruple moments:

$$
\begin{aligned}
\bar{\chi}_{\eta b} = \chi_b &= e^{-2Ht}\int^t e^{Ht'}D^a\chi_{ab}dt' \\
&= -2H^2 t\tfrac{x^a}{r}\partial_t[\partial_t Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) - 2(H+\tfrac{1}{r})(1-Ht)\tfrac{x^a}{r}\partial_t[\partial_t Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\
&\quad- 2H(H+\tfrac{1}{r})\tfrac{x^a}{r}\partial_t[-2Q^{(\rho)}_{ab} + Q^{(p)}_{ab}](t_{\mathrm{ret}}) - 2(H+\tfrac{1}{r})\tfrac{x^a}{r^2}\partial_t[Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\
&= -2H^2 t\tfrac{x^a}{r(Hre^{Ht}+1)}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) - 2(H+\tfrac{1}{r})(1-Ht)\tfrac{x^a}{r(Hre^{Ht}+1)}[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) \\
&\quad- 2H(H+\tfrac{1}{r})\tfrac{x^a}{r(Hre^{Ht}+1)}[-2\partial_t Q^{(\rho)}_{ab} + \partial_t Q^{(p)}_{ab}](t_{\mathrm{ret}}) \\
&\quad- 2(H+\tfrac{1}{r})\tfrac{x^a}{r^2(Hre^{Ht}+1)}[\partial_t Q^{(\rho)}_{ab}](t_{\mathrm{ret}})
\end{aligned} \tag{B.19}
$$

where we used (B.10) and focused on terms linear in H. On the horizon it yields:

$$
\bar{\chi}_{\eta b}|_{\mathcal{H}} = -2H\tilde{x}^a[\partial_t^2 Q^{(\rho)}_{ab}](t_{\mathrm{ret}}) + \mathcal{O}(H^2) \tag{B.20}
$$

References

[1] D. Dobkowski-Ryłko, J. Lewandowski, and T. Pawłowski. The Petrov type D isolated null surfaces. *Class. Quant. Grav.*, 35(17):175016, 2018.

[2] D. Dobkowski-Ryłko, J. Lewandowski, and T. Pawłowski. Local version of the no-hair theorem. *Phys. Rev. D*, 98(2):024008, 2018.

[3] D. Dobkowski-Ryłko, W. Kamiński, J. Lewandowski, and A. Szereszewski. The Petrov type D equation on genus >0 sections of isolated horizons. *Phys. Lett. B*, 783:415–420, 2018.

[4] D. Dobkowski-Ryłko, J. Lewandowski, and I. Rácz. Petrov type D equation on horizons of nontrivial bundle topology. *Phys. Rev. D*, 100(8):084058, 2019.

[5] D. Dobkowski-Ryłko and J. Lewandowski. A generalization of the quadruple formula for the energy of gravitational radiation in de Sitter spacetime. 2022. to be published in *Phys. Rev. D*.

[6] D. Lovelock. The Four-Dimensionality of Space and the Einstein Tensor. *J. Math. Phys.*, 13:874, 1972.

[7] A. Einstein. Cosmological considerations on the general theory of relativity. *Philosophical Problems in Science*, (63):183–204, 2017.

[8] A. Friedman. On the Curvature of space. *Z. Phys.*, 10:377–386, 1922.

[9] G. Lemaître. Un univers homogène de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques. In *Annales de la Société scientifique de Bruxelles*, volume 47, pages 49–59, 1927.

[10] N. Straumann. The history of the cosmological constant problem. XVIIIth IAP Colloquium: Observational and theoretical results on the accelerating universe, 2002.

[11] A. G. Riess, A. V. Filippenko, P. Challis, A. Clocchiatti, A. Diercks, P. M. Garnavich, R. L. Gilliland, C. J. Hogan, S. Jha, R. P. Kirshner, B. Leibundgut, M. M. Phillips, D. Reiss, B. P. Schmidt, R. A. Schommer, R. C. Smith, J. Spyromilio, C. Stubbs, N. B. Suntzeff, and J. Tonry. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *AJ*, 116(3):1009–1038, 1998.

[12] B. P. Schmidt, N. B. Suntzeff, M. M. Phillips, R. A. Schommer, A. Clocchiatti, R. P. Kirshner, P. Garnavich, P. Challis, B. Leibundgut, J. Spyromilio, A. G. Riess, A. V. Filippenko, M. Hamuy, R. C. Smith, C. Hogan, C. Stubbs, A. Diercks, D. Reiss, R. Gilliland, J. Tonry, J. Maza, A. Dressler, J. Walsh, and R. Ciardullo. The high-z supernova search: Measuring cosmic deceleration and global curvature of the universe using type ia supernovae. *ApJ*, 507(1):46–63, 1998.

[13] G. E. Lemaître. The cosmological constant. In P. A. Schilpp, editor, *Albert Einstein: Philosopher-Scientist*. The Library of Living Philosophers, Illinois, 19494.

[14] A. Ashtekar, C. Beetle, and J. Lewandowski. Geometry of generic isolated horizons. *Class. Quant. Grav.*, 19:1195–1225, 2002.

[15] J. Lewandowski. Spacetimes admitting isolated horizons. *Class. Quant. Grav.*, 17(4):L53–L59, 2000.

[16] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1973.

[17] R. M. Wald. *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984.

[18] I. Racz and R. M. Wald. Extensions of spacetimes with killing horizons. *Class. Quant. Grav.*, 9(12):2643–2656, 1992.

[19] A. Ashtekar and B. Krishnan. Isolated and dynamical horizons and their applications. *Living Rev. Rel.*, 7:10, 2004.

[20] D. W. Pajerski and E. T. Newman. Trapped Surfaces and the Development of Singularities. *J. Math. Phys.*, 12(9):1929, 1971.

[21] A Ashtekar, C. Beetle, and S. Fairhurst. Isolated horizons: a generalization of black hole mechanics. *Class. Quant. Grav.*, 16(2):L1–L7, 1999.

[22] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski, and J. Wiśniewski. Generic isolated horizons and their applications. *Phys. Rev. Lett.*, 85:3564–3567, 2000.

[23] A. Ashtekar, C. Beetle, and J. Lewandowski. Mechanics of rotating isolated horizons. *Phys. Rev.*, D64:044016, 2001.

[24] J. Lewandowski and T. Pawłowski. Symmetric non-expanding horizons. *Class. Quant. Grav.*, 23:6031–6058, 2006.

[25] J. Lewandowski and T. Pawłowski. Geometric characterizations of the Kerr isolated horizon. *Int. J. Mod. Phys.*, D11:739–746, 2002.

[26] J. Lewandowski and T. Pawłowski. Extremal isolated horizons: A Local uniqueness theorem. *Class. Quant. Grav.*, 20:587–606, 2003.

[27] R. P. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, 11:237–238, 1963.

[28] M. Demianski. Some new solutions of the einstein equations of astrophysical interest. *Acta Astron., v. 23, no. 3, pp. 197-232*, 1973.

[29] A. Ashtekar and A. Magnon-Ashtekar. On the Symplectic Structure o f General Relativity. *Commun. Math. Phys.*, 86:55, 1982.

[30] I. Racz. Stationary Black Holes as Holographs. *Class. Quant. Grav.*, 24:5541–5571, 2007.

[31] I. Racz. Stationary Black Holes as Holographs II. *Class. Quant. Grav.*, 31:035006, 2014.

[32] O. Dreyer, B. Krishnan, D. Shoemaker, and E. Schnetter. Introduction to isolated horizons in numerical relativity. *Phys. Rev. D*, 67:024018, 2003.

[33] A. Ghosh and A. Perez. Black hole entropy and isolated horizons thermodynamics. *Phys. Rev. Lett.*, 107:241301, 2011.

[34] A. Ashtekar, A. Corichi, and D. Sudarsky. Hairy black holes, horizon mass and solitons. *Class. Quant. Grav.*, 18(5):919–940, 2001.

[35] A. Ashtekar and A. Corichi. Non-minimal couplings, quantum geometry and black-hole entropy. *Class. Quant. Grav.*, 20(20):4473–4484, 2003.

[36] A. Ashtekar, B. Bonga, and A. Kesavan. Asymptotics with a positive cosmological constant: I. basic framework. *Class. Quant. Grav.*, 32(2):025004, 2014.

[37] A. Ashtekar, B. Bonga, and A. Kesavan. Asymptotics with a positive cosmological constant. II. Linear fields on de Sitter spacetime. *Phys. Rev. D*, 92(4):044011, 2015.

[38] H. K. Kunduri and J. Lucietti. Classification of near-horizon geometries of extremal black holes. *Living Rev. Rel.*, 16:8, 2013.

[39] J. M. Bardeen and G. T. Horowitz. The Extreme Kerr throat geometry: A Vacuum analog of AdS(2)$\times S^2$. *Phys. Rev. D*, 60:104030, 1999.

[40] R. M. Wald and A. Zoupas. General definition of "conserved quantities" in general relativity and other theories of gravity. *Phys. Rev. D*, 61:084027, 2000.

[41] V. Chandrasekaran, É. É. Flanagan, and K. Prabhu. Symmetries and charges of general relativity at null boundaries. *J. High Energ. Phys.*, 125, 2018.

[42] H. Stephani, D. Kramer, M. A. H. MacCallum, Cornelius Hoenselaers, and Eduard Herlt. *Exact solutions of Einstein's field equations.* Cambridge University Press, 2004.

[43] J. Lewandowski and T. Pawłowski. Quasi-local rotating black holes in higher dimension: Geometry. *Class. Quant. Grav.*, 22:1573–1598, 2005.

[44] M. Korzyński, J. Lewandowski, and T. Pawłowski. Mechanics of multidimensional isolated horizons. *Class. Quant. Grav.*, 22(11):2001–2016, 2005.

[45] J. Jezierski and B. Kaminski. Towards uniqueness of degenerate axially symmetric Killing horizon. *Gen. Rel. Grav.*, 45:987–1004, 2013.

[46] P. T. Chruściel, S. J. Szybka, and P. Tod. Towards a classification of vacuum near-horizons geometries. *Class. Quant. Grav.*, 35(1):015002, 2018.

[47] T. Pawłowski, J. Lewandowski, and J. Jezierski. Space-times foliated by Killing horizons. *Class. Quant. Grav.*, 21:1237–1252, 2004.

[48] J. Lewandowski, A. Szereszewski, and P. Waluk. Spacetimes foliated by nonexpanding and Killing horizons: Higher dimension. *Phys. Rev.*, D94(6):064018, 2016.

[49] J. Podolský and M. Žofka. General kundt spacetimes in higher dimensions. *Class. Quant. Grav.*, 26(10):105008, 2009.

[50] J. Podolský and R. Švarc. Physical interpretation of kundt spacetimes using geodesic deviation. *Class. Quant. Grav.*, 30(20):205016, 2013.

[51] M. Mars. Uniqueness properties of the Kerr metric. *Class. Quant. Grav.*, 17:3353, 2000.

[52] M. Mars. A Space-time characterization of the Kerr metric. *Class. Quant. Grav.*, 16:2507–2523, 1999.

[53] D. Dobkowski-Ryłko, W. Kamiński, J. Lewandowski, and A. Szereszewski. The Near Horizon Geometry equation on compact 2-manifolds including the general solution for g $>$ 0. *Phys. Lett. B*, 785:381–385, 2018.

[54] Jerzy Lewandowski and Adam Szereszewski. Axial symmetry of kerr spacetime without the rigidity theorem. *Phys. Rev. D*, 97:124067, 2018.

[55] S. A. Hayward. General laws of black-hole dynamics. *Phys. Rev. D*, 49:6467–6474, 1994.

[56] S. Akcay and R. Matzner. The Kerr-de Sitter Universe. *Class. Quant. Grav.*, 28:085012, 2011.

[57] *Black holes. Les Astres occlus. Edited by C. DeWitt [and] B. S. DeWitt.* Gordon and Breach, New York, 1973.

[58] J. F. Plebański and M. Demiański. Rotating, charged, and uniformly accelerating mass in general relativity. *Annals Phys.*, 98:98–127, 1976.

[59] E. Kiritsis. Introduction to superstring theory. 1997.

[60] H. Weyl. *The Concept of a Riemann Surface.* Dover Publications, Inc., 2009.

[61] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* John Wiley & Sons, 1978.

[62] J. Lewandowski, I. Racz, and A. Szereszewski. Near horizon geometries and black hole holograph. *Phys. Rev. D*, 96:044001, 2017.

[63] J. Jezierski. On the existence of kundts metrics and degenerate (or extremal) killing horizons. *Class. Quant. Grav.*, 26(3):035011, 2009.

[64] H. K. Kunduri and J. Lucietti. A classification of near-horizon geometries of extremal vacuum black holes. *J. Math. Phys.*, 50(8):082502, 2009.

[65] C. Li and J. Lucietti. Uniqueness of extreme horizons in einstein–yang–mills theory. *Class. Quant. Grav.*, 30(9):095017, 2013.

[66] P. T. Chruściel, H. S. Reall, and P. Tod. On non-existence of static vacuum black holes with degenerate components of the event horizon. *Class. Quant. Grav.*, 23(2):549–554, 2005.

[67] J. Lewandowski and M. Ossowski. Non-singular kerr-NUT-de sitter spacetimes. *Class. Quant. Grav.*, 37(20):205007, 2020.

[68] J. B. Griffiths and J. Podolsky. *Exact Space-Times in Einstein's General Relativity.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2009.

[69] C. W. Misner. The flatter regions of newman, unti, and tamburino's generalized schwarzschild space. *J. Math. Phys.*, 4(7):924–937, 1963.

[70] J. Lewandowski and M. Ossowski. Projectively nonsingular horizons in kerr-nut–de sitter spacetimes. *Phys. Rev. D*, 102:124055, 2020.

[71] J. Lewandowski and M. Ossowski. Nonsingular extension of the kerr-nut–(anti–)de sitter spacetimes. *Phys. Rev. D*, 104:024022, 2021.

[72] A. Trautman. Boundary conditions at infinity for physical thories. *Bull. Acad. Polon. Sci., sér. sci. math., astr. et phys.*, 6:403–406, 1958.

[73] A. Trautman. Radiation and boundary conditions in the theory of gravitation. *Bull. Acad. Polon. Sci., sér. sci. math., astr. et phys.*, 6:407–412, 1958.

[74] A. Trautman. On gravitational radiation damping. *Bull. Acad. Polon. Sci., sér. sci. math., astr. et phys.*, 6:627–633, 1958.

[75] F. A. E. Pirani. Invariant formulation of gravitational radiation theory. *Phys. Rev.*, 105:1089–1099, 1957.

[76] H. Bondi. *Proceedings on theory of gravitation.* by L. Infeld (Gauthier-Villars, Paris, PWN Editions Scientific de Pologne), Warszawa and Pergamon Press, Oxford, 1964.

[77] H. Bondi. Gravitational waves in general relativity. *Nature*, 186(4724):535, 1960.

[78] R. Penrose and H. Bondi. Zero rest-mass fields including gravitation: asymptotic behaviour. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 284(1397):159–203, 1965.

[79] R. K. Sachs. Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times. *Proc. Roy. Soc. Lond. A*, 270:103–126, 1962.

[80] A. Einstein. Approximative Integration of the Field Equations of Gravitation. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1916:688–696, 1916.

[81] A. Einstein. Über Gravitationswellen. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1918:154–167, 1918.

[82] A. Ashtekar, B. Bonga, and A. Kesavan. Asymptotics with a positive cosmological constant: III. The quadrupole formula. *Phys. Rev. D*, 92(10):104032, 2015.

[83] S. J. Hoque and A. Virmani. On propagation of energy flux in de sitter spacetime. *Gen. Relativ. Gravitat.*, 50(4):40, 2018.

[84] A. Ashtekar, N. Khera, M. Kolanowski, and J. Lewandowski. Non-expanding horizons: multipoles and the symmetry group. *J. High Energ. Phys.*, 2022(1):28, 2022.

[85] A. Ashtekar, N. Khera, M. Kolanowski, and J. Lewandowski. Charges and fluxes on (perturbed) non-expanding horizons. *J. High Energ. Phys.*, 2022(2):66, 2022.

[86] S. Hollands and R. M. Wald. Stability of black holes and black branes. *Commun. Math. Phys.*, 321(3):629–680, 2013.

[87] H. J. de Vega, J. Ramirez, and N. Sanchez. Generation of gravitational waves by generic sources in de sitter space-time. *Phys. Rev. D*, 60:044007, 1999.