A-polynomials, symmetries and permutohedra for quivers and knots

PhD thesis summary

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String theory was put forward in the 1970s as a candidate for the theory unifying quantum mechanics and general relativity. In the meantime, string theory has also found its place as a fertile soil in various areas of mathematics. One of the strengths of string theory is that it predicts a large set of new dualities between physical theories. Thanks to these dualities, we have been able to gain a deeper understanding of non-perturbative effects of various physical theories in elegant geometric terms. The concept of duality is so vast and rich that to this day physicists and mathematicians devote their entire carriers to the study of dualities in string theory.

One of such dualities is between the 3d Chern-Simons (CS) theory [1–3] and the 3d $\mathcal{N} = 2$ supersymmetric (SUSY) gauge theory [4–7]. At first sight, these are two completely unrelated theories. On one hand, CS is a topological field theory. The observables in CS theory are Wilson loops, which can be non-trivially knotted, and their vacuum expectation values (vev) produce knot invariants. On the other hand, 3d $\mathcal{N} = 2$ is a SUSY gauge theory with many interacting fields (scalar, vector, spinor). One of the relevant observables of this theory is the SUSY index[1], which is a protected quantity (independent of certain coupling constants) with simple behavior under the flow of the renormalization group and can be computed by means of supersymmetric localization. Because the SUSY index is a protected quantity, it provides information on

[1]An index is the vev of the operator $(-1)^F$, where F counts the number of fermions in a given state. Given that in SUSY theories almost all states have a fermionic pair, the states cancel each other out, and the only contribution to the index is from the states that do not have a fermionic pair. The states without fermionic pairs are called BPS states since they saturate the BPS bound [8].

non-perturbative effects of the theory; in this case, it counts the number of Bogomol'nyi-Prasad-Sommerfeld (BPS) states/particles with a given spin.

A spectacular manifestation of the aforementioned duality is that the generating function of vevs of a Wilson loop colored by all symmetric representations in CS can be rewritten as a BPS index of 3d $\mathcal{N}=2$ theory. More concretely, the vev of a Wilson loop in CS theory with $SU(2)$ gauge group in its symmetric representations (colors) yields the famous colored Jones polynomial [9, 10]. The colored Jones polynomial can be generalized to the colored HOMFLYPT polynomial (Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki and Traczyk) [11, 12]. We can construct the generating function of these polynomials by simply adding a source term to the CS action. The result is a q-series identical to that of the 3d $\mathcal{N}=2$ SUSY index after a suitable change of variables.

The relation between these physical theories goes further. We can perform a Wentzel–Kramers–Brillouin (WKB) expansion of the generating function for Wilson loops at large color of the gauge group. Then, the leading contribution of the WKB expansion is encoded in an algebraic curve known as the A-polynomial. The A-polynomial is another knot invariant, which for $SU(2)$ gauge group is defined as an $SL_2(\mathbb{C})$ characteristic variety of the knot complement [13]. Analogously to the previous knot polynomials, the A-polynomial can also be generalized to the $SU(N)$ case with an extra parameter encoding the N dependence [14]. Thanks to the duality, the A-polynomial of a knot, for the 3d SUSY gauge theory, plays a similar role to the Seiberg-Witten curve for 4d $\mathcal{N}=2$ theories. It is also a generating function that counts the classical BPS particles[2].

The above duality was first discovered by Ooguri and Vafa [15] in the context of topological string theory[3]. Thanks to a mechanism known as a geometric transition, they conjectured that the open topological string amplitudes on the deformed conifold are related to the amplitudes on a different background geometry, named the resolved conifold. This had a major consequence for the quantum field theories describing the effective low-energy modes of certain strings and branes configurations. On one hand, the free energy of the topological string partition function in the deformed conifold is described by the non-abelian 3d CS theory. On the other hand, 3d $\mathcal{N}=2$ supersymmetric gauge theories reproduce the free energy of topological strings in the resolved conifold.

The above duality has tremendous implications that go beyond those two physical theories. For instance, it allows to assign the BPS spectrum of the 3d $\mathcal{N}=2$ SUSY theory as new knot invariants called Labastida-Mariño-Ooguri-Vafa (LMOV) invariants. Considering that the SUSY index counts BPS states LMOV conjectured, their invariants should also be nonnegative integers.

[2]Where classical means that we sum over different values of the spin of the BPS particles.

[3]Topological string theory is a simplified version of the original string theory. In this simplified version, the string amplitude has no contribution from the local propagating degrees of freedom, but only the global topological degrees of freedom are relevant. This makes topological string theory an integrable theory in which various quantities can be computed exactly, and thus very useful for analysing dualities originating from string theory.

A very promising approach to prove the integrality of LMOV invariants that has already proved the integrality for several infinite families of knots is known as the knots-quivers correspondence [16–20]. A quiver is a directed graph, and a quiver representation is when we associate a vector space to each node in the graph and a homeomorphism across the vector spaces to each arrow in the graph. The Donaldson-Thomas (DT) invariants count the number of equivalent quiver representations up to isomorphisms for a given quiver. Surprisingly, the generating function of DT invariants, also referred to as the quiver series, can be identified with the index of the 3d SUSY theory. The previous relation leads to the interpretation of DT (and LMOV) invariants as counts of BPS particles. Ultimately, the DT invariants were also conjectured to be integer by Kontsevich and Soibelman [21] and later proven by Efimov [22] using quantum Hall algebras. This reduces the proof of the LMOV conjecture to the proof of the knots-quivers correspondence, which so far has been rigorously proven for all arborescent knots.

As we have laid out so far, the overlap between knots, quivers, 3d SUSY, and topological strings is an enormous territory, in which there is still much to investigate. In this thesis we begin by answering the question, can distinct 3d SUSY theories be associated to the same CS theory? We find that the answer is yes. This may also be interpreted as a set of equivalent quivers related to the same knot. That leads us to uncover an intricate network of dualities across 3d $\mathcal{N} = 2$ SUSY theories and possibly new knot invariants. The second question we address is, can we derive the properties of the A-polynomial of a knot from a quiver, and do they extend to any quiver, even if not associated with a knot? Again, the answer is yes. Here, we introduce the concept of a quiver A-polynomial along with various properties and provide the first classification of them in terms of their genus.

More specifically, the objective of this thesis is twofold:

The first one is to explore a web of dualities between 3d $\mathcal{N} = 2$ SUSY gauge theories in the IR sector. This is based on my published work with my collaborators

- J. Jankowski, P. Kucharski, H. Larraguível, D. Noshchenko and P. Sułkowski, *Permutohedra for knots and quivers*. Phys.Rev.D 104 (2021).

We see this duality from the fact that we may assign two or multiple quivers to the same 3d supersymmetric theory, as well as the same knot. Therefore, we treat those quivers as equivalent and we show that there is a simple operation relating them. We begin by examining these operations for certain small quivers and then apply them to quivers that produce knot invariants. Quite unexpectedly, except for the first two simplest knots, the rest of knots enjoy a gigantic number of equivalent quivers. To visualise the intricate web of dualities of equivalent quivers, we encode it in a graph structure, where a vertex represents an equivalent quiver and an edge represents the operation relating those two equivalent quivers. Amazingly, we recognised that those graphs could be constructed out of permutohedra. A permutohedron is a graph that represents the action of the permutation group. It assigns nodes to each permutation and edges to permutations related by a transposition of neighbouring elements [23]. As we mentioned, the knots-quivers correspondence has only been proven for arborescent knots [24, 25]; if true for all knots, these permutohedra graphs might constitute new knot invariants themselves.

The second objective of this thesis is to derive and classify several families of A-polynomial curves for simple 3d supersymmetric gauge theories. This part is based on the second work with my collaborators published in

- H. Larraguivel, D. Noshchenko, M. Panfil and P. Sułkowski, *Nahm sums, quiver A-polynomials and topological recursion.* J. High Energ. Phys. 2020, 151 (2020).

From the recursion relations satisfied by the quiver series, we rewrite them as linear operators acting on the quiver series and setting it to zero (annihilating it)[4]. We show that these operators generate the ideal of all operators that annihilate the quiver series. Due to that observation, we are able to construct the quantum A-polynomial of a quiver using non-commutative elimination[5]. After taking the classical limit, we recover the classical mirror curve, or the (classical) A-polynomial of the quiver. We then classify the quiver A-polynomials of quivers with two nodes according to the genus of the A-polynomial. Moreover, taking advantage of known operations for the knot A-polynomials which do not change their genus, we further simplify the equivalence classes of quiver A-polynomials. Finally, we obtain just a few infinite families of quivers with genus zero A-polynomials, and finitely many quivers whose A-polynomial has a given genus higher than zero.

We expect our results to be a stepping stone to proving the knots-quivers correspondence in its entirety, implying the LMOV conjecture. If the knots-quivers correspondence holds for all knots, then the next direction would be to prove that the permutohedra graphs are indeed knot invariants. This could also lead to unforeseen geometrical implications for topological string theory, 3d $\mathcal{N} = 2$ SUSY theory, and quiver representation theory. The study of the A-polynomials we present here may help to understand the asymptotics or growth of the DT invariants, as well as a classification of them in terms of algebraic geometry. We further hope that this thesis also serves as a friendly introduction and a concise overview, especially for researchers new to the field and who wish to learn these wonderful tools.

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