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Review of the doctoral thesis entitled
"State transformations in quantum resource theories"
by Tulja Varun Kondra

The dissertation is centered around mathematical developments of selected topics of current interest in so-called quantum resource theories. Physically, a resource theory is a mathematical model that aims to characterize how and when certain tasks can be performed subject to various restrictions that in particular are motivated either by fundamental or practical reasons. A resource theory identifies two main components, a subset of all possible states — free states and free, i.e. allowed, operations, as well as resources — precisely states that cannot be obtained by acting on free states with the allowed operations. One of the main problems then in any resource theory is how resourceful states can be transformed into one another via allowed operations. This, in particular, may allow for a partial or complete ordering of resources (as in there is more resource in one state than another, for example) and can help operationally characterize resources (what is the amount of resource in a state, what is a resource useful for) and limitations on their use. While this topic seems rather abstract it has in various contexts proven to help structure and progress on solutions in a broad range of problems and phenomena in particular, but not only in, thermodynamic or information processing settings with technological implications.

In quantum information theory, the resource structure is easily recognized in various contexts and appears most familiarly in the problem of characterization of quantum entanglement under the so-called Local Operations and Classical Communication (LOCC) paradigm (also called the distant lab paradigm). Under appropriate constraints, many other features and phenomena in quantum mechanics can be qualitatively and quantitatively described as resource theories — coherence, non-gaussianity, nonlocality, imaginarity, asymmetry etc. This has resulted in quantum resource theories becoming an important part of current research on quantum information. The work of M.Sc. Tulja Varun Kondra focusses principally on the characterization of state transformations in general quantum resource theories with applications to various particular kinds of resources.

Before moving on to the main concrete scientific developments of the dissertation, it is perhaps worth summarizing the publication record of the Candidate. The reviewed thesis is based on 9 original research articles co-authored by the Candidate, 3 of which have been published in the prestigious journal Physical Review Letters, with two other articles published in the leading journals of the field: Physical Review A and New Journal of Physics. The remaining articles are preprints available at the preprint server arxiv.org and were unpublished at thesis submission time. The Candidate is the first author in 3 of these papers, including one in Physical Review Letters indicating strongly his leading role in obtaining results therein. Two more articles are listed in the dissertation as works not included in the thesis, one of which has already been published in Physical Review A. Although not listed in the thesis, he is additionally co-author of a review article published in 2023 in Reports on Progress in Physics.

The thesis itself consists of 5 chapters, a bibliography and a helpful table summarizing notations used. Below, I summarise the contents of the chapters with emphasis on the most important results.

Chapter 1 provides an introduction to the topic of general quantum resource theories in particular summarizing key concepts and mathematical definitions associated with resource theories and state transformations: deterministic, probabilistic and approximate single and many copy state transformations, and catalytic transformations in analogy to similar concepts in the theory of entanglement.

In Chapter 2, the Author describes results on *general* resource theories exploiting the connections between quantification of resources via so-called resource monotones and deterministic state transformations between two states under free operations. These are based on article [DGKS23] of the thesis. In general, there is not a unique quantifier or measure that captures the amount of a resource completely (as is well known, *e.g.*, in the theory of entanglement). Often one considers measures that do not increase under the action of free operations to capture the idea that resources cannot be produced by such operations. Such measures are called resource monotones. It is known that deterministic state convertibility is related to resource monotones in that one state can be converted into another in a resource theory if and only if no monotone increases during the state transformation. It is not clear, *a priori,* if a finite complete set of monotones exists in a particular resource theory that characterises all possible state transformations. The first main result of this chapter concerns this problem and is solved for a large class of resource theories (those that contain free pure states) via an elegantly proved theorem (Theorem 2.1) stating essentially that finite sets of continuous and faithful monotones cannot be complete. This theorem by virtue of inclusion of free pure states in the theories of entanglement, coherence, and imaginarity as well as asymmetry, applies to each of them.
It is next discussed how this limitation can be surpassed when allowing for discontinuous monotones with examples from the resource theory of coherence, imaginarity and asymmetry for single qubits. The author also provides an explicit construction of an infinite set of monotones for any resource theory that is complete.
Another way of surpassing the theorem is shown to be via so-called quantum catalysis. Strict catalytic transformations consist of adding additional quantum systems into the fray which when transformed together with the principal system can yield the sought output state of the system while leaving the catalyst state decoupled and unchanged by the end of the procedure. The Author considers more general so-called approximate catalysis, where system-catalyst correlations can build up that can be made arbitrarily small. Remarkably, in this scenario, it turns out that a single monotone (the relative entropy of coherence) completely characterises such catalytic transformations in the resource theory of coherence (Theorem 2.2).
Finally, the Author discusses aspects of simple resource theories which have a single complete continuous resource monotone. The main result here concerns the question: when can a resource theory be characterised by total ordering of resources? Total order means that given two states, a free transformation exists in at least one direction between the two states. All states can then be unambiguously ordered in terms of their resource content. The answer to this question is provided by proving a theorem (Theorem 2.3) stating that a theory is totally ordered if and only it has a single complete continuous monotone. This elegantly leads to the conclusion that in such simple theories, all pure states are equivalent, i.e. any two pure states can be transformed into one another via free operations. These results are then used to characterise all totally ordered single qubit resource theories.

Chapter 3 presents a variety of results and solutions to some open problems on probabilistic yet approximate single copy state conversion both in general resource theories as well as specifically in entanglement theory and the resource theory of imaginarity. This chapter is related to results in

articles [KDS22, WKR+21a,WKR+21b, KDS23]. The main goal here is to study the tradeoff between probability of success and fidelity that can be achieved under the class of state transformations considered. The Author studies free transformations as well as more general transformations which can produce a small amount of resources from resource free states. In particular, bounds for such transformations in *general* resource theories are proved for the optimal fidelity for state conversion with at least some probability, and optimal transformation probability of state conversion with at least some set fidelity value (Theorem 3.1). These bounds are then shown to imply constraints on asymptotic transformations leading to upper bounds on the achievable rate of conversion of n-copies of a given state in *general* resource theories (Theorem 3.2). Returning to the single copy bounds, in general, they can often be posed as convex optimisation problems and as such generically do not seem to take analytical forms. Moreover, these bounds are not, in general, tight. However, tighter bounds can be found for specific resource theories. The Author, in particular, proves a tighter bound for the optimal fidelity of the canonical problem of transforming a two qubit state into a Bell state by providing an optimal state transformation protocol.
A series of very interesting results is obtained for the mentioned class of single copy state transformations in entanglement theory. It turns out that the analytical expressions for optimal fidelity and optimal probability can be obtained, yielding thus a *complete solution* to the problem of stochastic state conversion between *pure* bipartite states (Theorem 3.3). Moreover, a *complete solution* is provided for states of 2-qubits in the case when the initial state is pure while the final state is mixed (Theorem 3.4).
Next, the Author presents results on the resource theory of imaginarity, where free operations are real valued transformations (in some chosen fixed basis). Here again, upon developing various mathematical tools and results, a *complete analytical solution* is presented to the problem of stochastic state conversion using real valued transformations when the starting state is pure (Theorem 3,6, 3.7).

Chapter 4 presents a derivation of a number of mathematical properties and results on catalytic transformations in the contexts of manipulation of bipartite entanglement and quantum communication over noisy channels. This chapter is based on results contained in articles [KDS21,GKS23,DKMS22b]. Firstly, the Author derives the result that if asymptotic state conversion is possible, this implies that approximate catalysis is also possible under LOCC operations. Furthermore, he proves that an entanglement measure, the so called squashed entanglement, of a bipartite system does not increase under approximate catalysis. These facts are then used to completely solve the open problem of when one bipartite *pure* state can be transformed into another bipartite *pure* state with the help of catalysts when arbitrarily small correlations between system and catalyst are allowed to be produced during the process. The result is that the entanglement content as measured by entanglement entropy completely characterises such approximate catalytic transformations. Catalytic transformations turn out to be possible if and only if the bipartite entanglement entropy in an input state is not less than that of the target state (Theorem 4.1).
The Author generalizes the above results to the case of mixed states that are distillable. In particular, the connection between asymptotic transformations of entanglement and entanglement catalysis is studied more deeply, specifically to answer the open question if these two commonly used ways of transforming states are equivalent. A framework is introduced, appropriately generalizing the notions of approximate catalysis and asymptotic transformations to correlated catalysis and marginal asymptotic transformations, respectively, which allow for correlations to be produced within the transformation protocols to study this problem. The Author, via a series of intricate steps, proves that these two seemingly disparate classes of transformations are in fact equivalent for the class of distillable entangled states (Theorem 4.2). This immediately implies, e.g., that these two classes of transformations are fully equivalent for all 2-qubit states, since the latter are always distillable when entangled.

Using these results, the Author studies whether catalysis can be beneficial for asymptotic transformation protocols. Proving a number of lemmas and theorems, the Author establishes the result that (correlated) catalysis does not increase the distillable entanglement of a distillable state (Theorem 4.3). This nicely leads to a generalization of the result for pure state to pure state catalytic transformations of Theorem 4.1 — the distillable entanglement completely characterises approximate catalytic transformations for a pure target state and initial distillable state for single copy transformations.
Finally, the Author presents results on the characterisation of noisy channels for reliable quantum communication in an unconventional setup. Communicating parties are assumed to access the channel only once, and are allowed use of unlimited rounds of classical communication and a quantum catalyst. Tools for estimating the catalyst assisted quantum communication capacity are developed allowing for the determination of upper and lower bounds.

The final chapter contains a short summary of the main results as well as some open questions.

Before moving on to the final conclusions of the report, as is conventional, I provide a few comments on the style of presentation. The structure of the thesis is without doubt quite appropriate. The thesis essentially is a collection of mathematical results and proofs and is presented as such. A small criticism is that the presentation is quite dry in places and more comments on the physical implications of various results, as well as figures explaining certain concepts, would have been welcome. However, this comment is marginal and does not in any way impact my very high evaluation of this thesis.

In summary, the thesis contains a large body of original mathematical theorems and important results on the manipulation and transformations of entanglement, as well as the characterization of general resource theories with further specific applications to the theories of coherence, asymmetry and imaginarity. In particular, the Author has solved a number of important open problems in quantum information theory including, but not restricted to: (i) a complete characterization of catalytic transformations of pure state bipartite entanglement, (ii) determination of equivalence between catalytic and asymptotic transformations for fully distillable (i.e. in general mixed) entangled states, (iii) characterization of complete sets of monotones in general resource theories. Moreover, significant progress has been achieved in the understanding of transformations that have both a finite probability of failure and are approximate, in various resource theories.

I have no doubt that the thesis by Tulja Varun Kondra satisfies all regulatory and customary conditions required of a PhD dissertation, and he therefore should be admitted to the further stages of the doctoral procedure. In line with the previous paragraph, I also believe that the originality and depth of results obtained fully justifies awarding the thesis a distinction.

R. W. Chhajlany

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