State transformations in quantum resource theories

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DOCTORAL THESIS

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Abstract

Quantum theory provides us with a mathematical framework to understand nature at the microscopic level. It explains phenomena that classical physics could not account for. A deeper understanding of these non-classical phenomena (e.g., entanglement and quantum coherence) is not only of a foundational interest, but also paves the way towards potential technological advancements. Developments in the field of quantum information showed that such quantum features help us to perform operational tasks which would be impossible in the framework of classical physics. Therefore, these quantum phenomena can be viewed as resources which help us to go beyond the restrictions imposed by classical physics. Inspired by this, the mathematical framework of quantum resource theories was developed. Each resource theory singles out a particular quantum resource by setting the states which contain no resource as free states. Furthermore, the set of free operations are defined such that they do not generate resource states out of free states. An important problem in any resource theory is to understand how different resources transform into each other under the action of free operations. This problem of state transformation will be the main focus of this thesis. We will start by studying the notion of deterministic transformations, where one aims to achieve a target state without a chance of failure. We will explore the connection between deterministic transformations and the problem of quantification of resources. We will then go further and concentrate on transformations which allow for a probability of failure, and study the fundamental restrictions imposed on the achievable error and the probability of success. Finally we will explore the idea of catalytic transformations, where we allow for an ancillary quantum system which increases the transformation power while remaining invariant in the process. For the resource theory of entanglement, we will give a full solution for this problem when the target state is pure and the initial state is distillable. Finally, we discuss potential applications of catalysis to the problem of sending quantum information through noisy channels.

Streszczenie

Teoria kwantowa zapewnia nam matematyczne ramy do zrozumienia natury na poziomie mikroskopowym. Wyjaśnia ona zjawiska, których fizyka klasyczna nie była w stanie wyjaśnić. Głebsze zrozumienie tych nieklasycznych zjawisk (np. splatania i spójności kwantowej) ma nie tylko fundamentalne znaczenie, ale także toruje droge do potencjalnych postepów technologicznych. Rozwój w dziedzinie informacji kwantowej pokazał, że takie cechy kwantowe pomagaja nam wykonywać zadania, które byłyby niemożliwe do wykonania w ramach fizyki klasycznej. Dlatego też zjawiska kwantowe można postrzegać jako zasoby, które pomagaja nam wyjść poza ograniczenia narzucone przez fizyke klasyczna. Inspirujac sie tym, opracowano matematyczne ramy kwantowych teorii zasobów. Każda teoria zasobów wyodrebnia konkretny zasób kwantowy, ustawiajac stany, które nie zawieraja żadnego zasobu, jako stany wolne. Co wiecej, zbiór wolnych operacji jest zdefiniowany w taki sposób, że nie generuja one stanów zawierajacych zasoby ze stanów wolnych. Ważnym problemem w każdej teorii zasobów jest zrozumienie, w jaki sposób różne zasoby przekształcaja sie w siebie nawzajem pod wpływem operacji swobodnych. Ten problem transformacji stanów bedzie głównym tematem niniejszej rozprawy. Zaczniemy od zbadania pojecia transformacji deterministycznych, w których daży sie do osiagniecia stanu docelowego bez możliwości niepowodzenia. Zbadamy zwiazek miedzy transformacjami deterministycznymi a problemem kwantyfikacji zasobów. Nastepnie skoncentrujemy sie na transformacjach, które dopuszczaja możliwość niepowodzenia, i zbadamy podstawowe ograniczenia nałożone na osiagalny błąd i prawdopodobieństwo sukcesu. Na koniec zbadamy idee transformacji katalitycznych, w których dopuszczamy pomocniczy system kwantowy, który zwieksza moc transformacji, pozostajac niezmiennym w procesie. Dla zasobowej teorii splatania podamy pełne rozwiazanie tego problemu, gdy stan docelowy jest czysty, a stan poczatkowy można destylować. Na koniec omówimy potencjalne zastosowania katalizy w problemie przesyłania informacji kwantowych przez zaszumione kanały.

Publications

Many of the results contained in this thesis have been published in the following articles and preprints:

- (WKR$^+$21a): Operational resource theory of imaginarity

  Kang-Da Wu, Tulja Varun Kondra, Swapan Rana, Carlo Maria Scandolo, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo and Alexander Streltsov

  Phys. Rev. Lett. **126**, 090401 — Published 1 March 2021

- (WKR$^+$21b): Resource theory of imaginarity: Quantification and state conversion

  Kang-Da Wu, Tulja Varun Kondra, Swapan Rana, Carlo Maria Scandolo, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo and Alexander Streltsov

  Phys. Rev. A **103**, 032401 — Published 1 March 2021

- (KDS21): Catalytic Transformations of Pure Entangled States

  Tulja Varun Kondra, Chandan Datta and Alexander Streltsov

  Phys. Rev. Lett. **127**, 150503 — Published 5 October 2021

- (KDS22): Stochastic approximate state conversion for entanglement and general quantum resource theories.

  Tulja Varun Kondra, Chandan Datta and Alexander Streltsov

  arXiv:2111.12646

- (DKMS22b): Entanglement catalysis for quantum states and noisy channels

  Chandan Datta, Tulja Varun Kondra, Marek Miller and Alexander Streltsov

  arXiv:2202.05228

- (KDS23): Real quantum operations and state transformations

  Tulja Varun Kondra, Chandan Datta and Alexander Streltsov

  New J. Phys. **25** 093043 — Published 25 September 2023

- (WKS⁺23): Resource theory of imaginarity: New distributed scenarios

  Kang-Da Wu, Tulja Varun Kondra, Carlo Maria Scandolo, Swapan Rana, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo, Alexander Streltsov

  arXiv:2301.04782

- (GKS23): Catalytic and asymptotic equivalence for quantum entanglement

  Ray Ganardi, Tulja Varun Kondra and Alexander Streltsov

  arXiv:2305.03488

- (DGKS23): Is There a Finite Complete Set of Monotones in Any Quantum Resource Theory?

  Chandan Datta, Ray Ganardi, Tulja Varun Kondra, and Alexander Streltsov

  Phys. Rev. Lett. **130**, 240204 — Published 16 June 2023

My other publications that are not included in this thesis

- (DGKS23): Entanglement and coherence in the Bernstein-Vazirani algorithm

  Moein Naseri, Tulja Varun Kondra, Suchetana Goswami, Marco Fellous-Asiani, Alexander Streltsov

  Phys. Rev. A **106**, 062429 — Published 21 December 2022

- (KGS23): Coherence manipulation in asymmetry and thermodynamics

  Tulja Varun Kondra, Ray Ganardi and Alexander Streltsov

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Notations

We shall use the following notation:

Symbol	Definition
$\mathcal{H}$	Hilbert space
$\mathcal{H}_d$	Hilbert space of dimension d
$\mathcal{D}$	set of density matrices
$\mathcal{D}_d$	set of density matrices of dimension d
$\mathcal{F}$	set of free states
$\mathcal{F}_s$	set of separable states
$\mathcal{F}_r$	set of real states
$\mathcal{O}$	set of free operations
X^T	transposition of operator X in the computational basis
X^*	complex conjugation of operator X in the computational basis
$X^\dagger$	complex conjugation and transpose of operator X in the computational basis
X^{T_A}	partial transposition of operator X in the computational basis of subsystem A
$\mathbb{I}$	Identity matrix
$\mathbb{I}_d$	Identity matrix of dimension d
$\mathbb{I}(\cdot)$	Identity operator
$\otimes$	Tensor product
H	Von Neumann entropy

Table 1: Table of notation

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Chapter 1

Introduction

The first quantum revolution was a time in the early twentieth century when scientists began to formulate the basic concepts of quantum mechanics. It was an important turning point in our understanding of the behavior of matter and energy at the microscopic level (DM02). Werner Heisenberg's formulation of matrix mechanics in 1925 (Hei25), Erwin Schrodinger's discovery of wave mechanics (Sch26) and Heisenberg's uncertainty principle (Hei27) established a new way of comprehending the probabilistic character of our nature at a microscopic level.

The ongoing wave of advancements and applications in quantum science and technology that has developed during the late twentieth century is referred to as the second quantum revolution (DM02). While the first quantum revolution laid the theoretical groundwork for quantum mechanics, the second quantum revolution focuses on applying those principles in real-world applications. Quantum computing (Val05), quantum communication (Che21), quantum cryptography (GRTZ02), quantum sensing (DRC17), and quantum metrology (GLM11) are all associated with the second quantum revolution. It entails manipulating and controlling quantum systems in order to take advantage of their distinctive features, such as superposition and entanglement, for a variety of applications.

Quantum entanglement (HHHH09), for example, is a quantum phenomenon where two or more sub-systems become correlated in such a way that the state of individual systems cannot be considered independently of the other systems. It has been found that quantum entanglement becomes necessary for various tasks like quantum teleportation (BBC+93a), super-dense coding (BW92), and quantum cryptography (GRTZ02). This inspired the development of a mathematical framework to understand the ultimate limitations on the manipulation of quantum resources. The notion of a resource theory was formalized, providing a rigorous framework to study the limitations and possibilities under specific constraints (CG19a). Over time, this framework expanded to include other resources beyond entanglement such as coherence (SAP17), asymmetry (MS13), and more, as it became clear that many quantum technological tasks are based on such resources.

Each resource theory focuses on a particular quantum property (or *quantum resource*) and defines a set of free operations that can be performed without consuming the considered resource. Quantum sates which can be prepared via free operations, without consuming the given resource, are called free states. All the other states which are called *resource states* or simply resources. Precisely speaking, a quantum resource theory is a tuple of free states and free operations, with the constraint that the set of free states is closed under the action of free operations. This captures the intuition that resources cannot be created for free. Quantum resource theories also provide a rigorous and operationally meaningful way to quantify the amount of resource present in a quantum state (CG19a). However, in general, there is no unique *resource quantifier* which encompasses all aspects of a resource theory. Depending on the problem under study, different quantifiers might be suitable (DGKS23).

Once a resource theory is defined, one of the important problems is to understand how various resource states can be transformed into each other via free operations. The motivation for looking into this problem comes from the fact that different operational tasks require different resource states for optimal performance. As an example, tasks like quantum teleportation and quantum dense coding require maximally entangled states for optimal performance (BBC+93a; BW92). In the ideal case one would like to obtain the target state exactly. But this often impossible to achieve. Various ways have been considered, through which one can overcome such limitations. At a single copy level (i.e., when one has access to only one copy of the initial state and aims to transform it into a single copy of the target state), one tries to achieve the target state probabilistically aiming to maximise the probability of success (Reg21). Alternatively one can also try to maximise the achievable fidelity (a measure of closeness between quantum states) to the target state such that there is no probability of failure (KDS22).

Another approach to the problem of state transformation is to consider multiple copies of the same state. By taking into account many copies of the state, one can overcome the constraints existing at single copy level (CG19a). In case of asymptotic transformations one considers infinitely many copies of the initial state and tries to optimize the rate of achieving the target state per copy of the initial state (via free operations) (CG19a; BBPS96a; BBP+96). This asymptotic regime allows for an understanding of the manipulations of quantum resources in scenarios where large numbers of copies are involved.

Catalytic state conversion is yet another way to overcome some of the limitations imposed by free operations at the single copy level. Inspired by *catalysis* in chemistry, quantum catalysis was first proposed by (JP99) in the resource theory entanglement. In chemistry, the term catalyst refers to a necessary substance which remains unchanged in the chemical process. Without this additional substance, the reaction would not be possible (Ber35; vSvLMA99). Analogous to chemical catalysts, the authors in (JP99) showed that an additional entangled system, which remains unchanged in the procedure can be used to accomplish state transformations which would not be possible otherwise. Later this notion of catalytic transformations has been studied in various other resource theories (DKMS22a; LBWN23).

1.1 Entanglement theory: a prototype resource theory

Our classical understanding of nature leads us to believe that certain characteristics of *entangled* quantum systems appear to be at odds with our intuition (HHHH09). Even Einstein found some of the effects of entanglement perplexing, coming to the conclusion that quantum theory cannot be complete (EPR35). Entangled quantum systems are currently being extensively investigated as a key component of the forthcoming quantum technologies (HHHH09). This includes establishing a provably secure key for communication between distant parties (quantum key distribution) (Eke91) and quantum teleportation (BBC⁺93b), which uses shared entanglement and classical communication to convey the state of a quantum system to a distant partner.

This motivated the study of entanglement as a resource (HHHH09), thus leading to the development of resource theory of entanglement. It has been noted that, when Alice and Bob are spatially separated and are only allowed for local quantum operations and classical communication (LOCC), they cannot create entangled states. Entangled states so become a useful resource, enabling the distant parties to carry out tasks that would not be possible without them. Formally speaking, a quantum state ρ_s^{AB} does not contain entanglement (separable state) if it can be written in the following way (HHHH09)

$$\rho_s^{AB} = \sum_i p_i \sigma_i^A \otimes \tau_i^B \text{ where } p_i \geq 0 \ \forall i \text{ and } \sum_i p_i = 1. \tag{1.1}$$

Here, σ_i^A and τ_i^B are quantum states acting on the Hilbert spaces of Alice (A) and Bob (B) respectively. In order to rigorously introduce LOCC maps, let us first look at 1-way LOCC maps. A CPTP map[1] $\Lambda_{AB}^{\rightarrow}$ is a 1-way LOCC map from Alice to Bob if it can be expressed as

$$\Lambda_{AB}^{\rightarrow}(\cdot) = \sum_i M_A^i \otimes N_B^i(\cdot), \tag{1.2}$$

where M_A^i are completely positive and trace non-increasing maps (corresponding to outcomes of a quantum measurement) and N_B^i are CPTP maps (deterministic quantum operations). Note that we additionally require $\sum_i M_A^i$ to be a CPTP map. Physically, this corresponds to

- Alice performs a quantum measurement locally, whose measurement outcomes are denoted by i.
- Alice then communicates the outcome of the measurement to Bob via classical channel.
- Bob then performs local deterministic quantum operation (N_B^i) conditioned on the outcome of Alice.

Similarly, one can also define a 1-way LOCC map from Bob to Alice ($\Lambda_{AB}^{\leftarrow}$). A LOCC map is a composition of finite number (n) of 1-way LOCC maps

$$\Lambda_{AB}^{\leftrightarrow}(\cdot) = \Lambda_{AB}^{(n)\rightarrow}(\cdot) \circ \Lambda_{AB}^{(n-1)\leftarrow}(\cdot) \circ \cdots \cdot \circ \Lambda_{AB}^{(2)\leftarrow}(\cdot) \circ \Lambda_{AB}^{(1)\rightarrow}(\cdot). \tag{1.3}$$

[1]Here, CPTP refers to completely positive and trace preserving. All the maps in this thesis are linear

It is easy to see that, a 1-way LOCC map (see Eq. (1.2)) acting on a separable state produces another separable state. Since any LOCC map is a finite composition of 1-way LOCC maps, it follows that LOCC operations cannot create entanglement out of separable states. One can also generalise LOCC operations to include quantum measurements by defining *stochastic* LOCC (SLOCC). This can be done by performing a partial sum in Eq. (1.2) (this defines a 1-SLOCC map from Alice to Bob) and then concatenating such 1-SLOCC maps in the same way as in Eq. (1.3). One can again easily see that such SLOCC operations map separable states to separable states, up to normalisation. This central observation is key to the idea of quantum resource theories, which will be described in the section below.

1.2 Quantum resource theories

It was observed that many quantum mechanical tasks rely on other quantum resources such as coherence (SAP17) and contextuality (KS67; BCG+21). This motivated the development of general (quantum) resource theories (CG19b). Following the same intuition as earlier, one defines the set of free states $\mathcal{F}$ and free operations $\mathcal{O}$, which correspond to quantum states and quantum transformations that are simple to establish or carry out in a given setting. An important feature of free operations is the fact that they do not create resource states from free states, i.e.,

$$\Lambda[\rho] \in \mathcal{F} \tag{1.4}$$

for any free state $\rho \in \mathcal{F}$ and free operation $\Lambda \in \mathcal{O}$ (CG19b). Note that the condition in Eq. (1.4), is not a sufficient condition. One can define a resource theory by defining the set of free operations as a strict subset of operations defines by Eq. (1.4). For example, in the resource theory of entanglement, certain quantum processes *do not create entanglement* (satisfy Eq.(1.4)) but cannot be carried out using LOCC (CCL12). Making Eq. (1.4) the single prerequisite for a free operation results in the maximal set of free operations i.e, any operation outside of this set will necessarily convert some free state into a resource state. This set of free operations is called *resource non generating operations* (RNG) (CG19b). Note that, such a set of free operations might create resources when acted upon a part of free state. In order to avoid such scenarios, one can define *completely resource non generating operations* (CG19b). A quantum operation Λ_c^S, is said to be completely resource non generating operation iff for every free state $\rho^{S'S}$ the following holds

$$\mathbb{I}^{S'} \otimes \Lambda_c^S[\rho^{S'S}] \in \mathcal{F}. \tag{1.5}$$

Here, $\rho^{S'S}$ is a state of the system $S' \otimes S$ and $\mathbb{I}^{S'}$ is an identity map acting on the system S'. One can also extend Eq. (1.4), to probabilistic (stochastic) quantum operations. A stochastic free operation ($\mathcal{E}$) necessarily satisfies the following condition (Reg22a)

$$\mathcal{E}[\rho]/\operatorname{Tr}\mathcal{E}[\rho] \in \mathcal{F} \;\forall \rho \in \mathcal{F}. \tag{1.6}$$

where $\mathcal{E}[\rho] = \sum_j K_j \rho K_j^\dagger$ with a (possibly incomplete) set of Kraus operators $\{K_j\}$. The transformation probability is then given by $p = \operatorname{Tr}\mathcal{E}[\rho] > 0$. This make sure that $\mathcal{E}$ does not create resources out of free states, with any non-zero probability.

1.3 State transformations in quantum resource theories

It is well known that, many quantum information tasks require maximally entangled states, to achieve their optimal performance (Eke91; BBC+93b). Such an observation has also been made in other resources like coherence (NKG+22), imaginarity (WKR+21a) etc. Therefore, it is important to develop optimal protocols to convert a less useful state into one which is potentially more useful. Knowing if a given state ρ can be converted into σ via free operations, turns out to be one of the fundamental problems in any resource theory. There are various settings in which such a problem can be studied. We now start with the setting of single copy transformations.

1.3.1 Single copy transformations

Here, one aims to convert a single copy of a quantum state ρ into one copy of target state σ via deterministic free operations. One of the first results in this direction was presented in (Nie99), where the authors completely characterise bipartite pure state transformations via LOCC. In general resource theories, one define the *transformation fidelity* between ρ and σ, in the following way:

$$F(\rho \to \sigma) = \sup_{\Lambda} F(\Lambda[\rho], \sigma), \tag{1.7}$$

where the supremum is taken over all free operations Λ and $F(\rho, \sigma) = [\mathrm{Tr}(\sqrt{\rho}\sigma\sqrt{\rho})^{1/2}]^2$ is the Uhlmann fidelity between ρ and σ. A quantum state ρ can be transformed into σ via deterministic free operations iff

$$F(\rho \to \sigma) = \sup_{\Lambda} F(\Lambda[\rho], \sigma) = 1. \tag{1.8}$$

Note that, an equivalent definition of deterministic operations is as follows. A quantum state ρ can be deterministically transformed into σ via free operations iff for every $\varepsilon > 0$, there exists a free operation Λ_f s.t

$$\left\|\Lambda_f(\rho) - \sigma\right\|_1 \leq \varepsilon. \tag{1.9}$$

This is due to the fact that, for a pair of states ρ and σ, the following inequalities hold (Fuchs-van de Graaf inequalities) (FvdG99)

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}. \tag{1.10}$$

In fact, pairs of states always exists in any nontrivial resource theory, between which no deterministic free conversion is feasible. In such cases, the transformation fidelity serves as an operational metric for the extent to which a particular transformation can be achieved (HHH99). Closed expressions for the transformation fidelity have been noted for various resource theories, for examples, entanglement (VJN00a), coherence (LS22) etc.

Practical protocols often use measurement-based techniques that are fundamentally probabilistic, due to the difficulties in realising deterministic transformations (HHHH09; CTV17). Therefore it becomes an important problem to extend resource theoretic approach to characterise probabilistic (non-deterministic) transformations (Reg22a). Here, the state ρ is transformed to σ with some nonzero probability. In such cases, the maximum probability for such a probabilistic conversion is given by

$$P(\rho \to \sigma) = \sup_{\mathcal{E}} \left\{ \mathrm{Tr}\left(\mathcal{E}[\rho]\right) : \frac{\mathcal{E}[\rho]}{\mathrm{Tr}\left(\mathcal{E}[\rho]\right)} = \sigma \right\}, \tag{1.11}$$

and the maximum is taken over all free probabilistic transformations $\mathcal{E}$. Even though determining the best conversion probability is difficult in general, in a number of quantum resource theories closed expression for certain settings have been found (Reg22a; Vid00a; Vid99a).

Note that the probabilistic conversions (in Eq. (1.11)) aim to transform states with zero error, in contrast to approximate conversions (in Eq. (1.7)), which transforms a state with unit probability. Both represent specific instances of a more general transformation. In order to study the intermediate regime, one can define *fidelity for stochastic approximate state conversion*, which quantifies the optimal fidelity for the transformation from ρ to σ with a conversion probability at least p:

$$F_p(\rho \to \sigma) = \sup_{\mathcal{E}} \left\{ F\left(\frac{\mathcal{E}[\rho]}{\mathrm{Tr}(\mathcal{E}[\rho])}, \sigma\right) : \mathrm{Tr}\left(\mathcal{E}[\rho]\right) \geq p \right\}. \tag{1.12}$$

Here, the supremum is taken over all stochastic free operations $\mathcal{E}$. In the same way, one can also define *probability for stochastic approximate state conversion*, capturing the optimal transformation probability for a transformation with fidelity at least f:

$$P_f(\rho \to \sigma) = \sup_{\mathcal{E}} \left\{ \mathrm{Tr}\left(\mathcal{E}[\rho]\right) : F\left(\frac{\mathcal{E}[\rho]}{\mathrm{Tr}\left(\mathcal{E}[\rho]\right)}, \sigma\right) \geq f \right\}. \tag{1.13}$$

Again the supremum is taken over all free probabilistic transformations $\mathcal{E}$. Closed expression for these quantities have be been found for various scenarios in (KDS22; Reg22b).

1.3.2 Many copy transformations

In this subsection, we review several kinds of state transformations involving multiple copies of both initial and final states. We start with the notion of multi-copy exact transformations. We say that ρ can be converted into σ exactly at a rate r, if for any $\delta > 0$ there exist integers m, n, deterministic free operation Λ such that

$$\Lambda\left(\rho^{\otimes n}\right) = \sigma^{\otimes m}, \tag{1.14a}$$

$$\frac{m}{n} + \delta > r. \tag{1.14b}$$

The supremum over all such rates will be called *exact transformation rate* $R^0(\rho \to \sigma)$. Such transformations have been studied in entanglement (WW23; APE03; BRS02; DFY05), purity (GMN+15) etc.

One can also define multi copy transformations, which are not exact. In other words, the final state in not exactly the target state. Therefore, there is an error in the transformation. But this error can be made arbitrarily low by choosing large enough copies of the state. These transformations are called *asymptotic transformations*. We say that a *asymptotic transformation* from ρ to σ is possible with rate r, if for any $\varepsilon > 0$ and any $\delta > 0$ there exist natural numbers m, n and a deterministic free operation Λ such that

$$\Lambda\left(\rho^{\otimes n}\right) = \mu^{S_1 \ldots S_m}, \tag{1.15a}$$

$$\left\|\mu^{S_1 \ldots S_m} - \sigma^{\otimes m}\right\|_1 < \varepsilon, \tag{1.15b}$$

$$\frac{m}{n} + \delta > r. \tag{1.15c}$$

Here, $||A||_1 = \mathrm{Tr}\sqrt{A^\dagger A}$ is the trace norm of a linear operator A and $\mu^{S_1 \ldots S_m}$ is a state of the system $S_1 \otimes S_2 \otimes \cdots \otimes S_m$. Note that each S_i is a copy of the system S . The supremum of r fulfilling these properties will be called *asymptotic transformation rate* $R(\rho \to \sigma)$. Asymptotic transformations have been first studied in entanglement theory (BBP+96; LP99) and then have been extensively studied in various other resource theories, for example see (Mar22; WY16; BaHO+13)

One can also define a variant of asymptotic transformations, called *marginal asymptotic transformations*. We say that a marginal asymptotic transformation from ρ to σ is possible with rate r, if for any $\varepsilon > 0$, $\delta > 0$ there exist integers m, n, and free operations Λ such that the following equations hold for all $i \leq m$:

$$\Lambda\left(\rho^{\otimes n}\right) = \tau^{S_1 \ldots S_m}, \tag{1.16a}$$

$$\left\|\tau^{S_i} - \sigma\right\|_1 < \varepsilon, \tag{1.16b}$$

$$\frac{m}{n} + \delta > r. \tag{1.16c}$$

Here, $\tau^{S_1 \ldots S_m}$ is a state of the system on the Hilbert space $S_1 \otimes S_2 \otimes \cdots \otimes S_m$, where each S_i is a copy of the system S. The supremum of r fulfilling these properties will be called *marginal transformation rate* $\tilde{R}(\rho \to \sigma)$. Note that the final state has m marginals and each marginal is ε-close to the target. It is worth noting the difference between asymptotic transformations and marginal asymptotic transformations. The main difference comes from Eq. (4.80b) and Eq. (1.16b). Eq. (4.80b) demands the final state to be ε-close to m-copies of σ as a whole. This condition is more stringent compared to Eq. (1.16b), as it requires each of the marginals to be ε-close to σ. These marginal asymptotic transformations have been studied in (FLTP23; GKS23).

1.4 Catalytic transformations

The phenomenon of *catalysis* was first observed in chemistry by Jacob Berzelius (Ber35). He noticed that several chemical reactions required an additional substance, which does not change in the chemical process. He defined this peculiar feature as catalysis, whose roots go back to Greek and can be translated as "loosen" (Ber35; vSvLMA99). In the current day and age, chemical catalysis is widely used in many industries. Quantum catalysis has a conceptual similarity to chemical catalysis. Here, instead of "chemical reaction", we have "quantum state transformations" and the quantum catalyst is an additional quantum state which enables an otherwise impossible state transformation. The phenomenon of *catalysis* in quantum physics was first introduced by (JP99) in the context of entanglement theory. As an example, the authors show that the following two states:

$$|\psi\rangle = \sqrt{0.4}\,|00\rangle + \sqrt{0.4}\,|11\rangle + \sqrt{0.1}\,|22\rangle + \sqrt{0.1}\,|33\rangle\,, \tag{1.17}$$

$$|\varphi\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.25}\,|11\rangle + \sqrt{0.25}\,|22\rangle\,. \tag{1.18}$$

cannot be converted from one to another via deterministic LOCC transformations. Surprisingly, if the two parties share an additional entangled state $|\eta\rangle = \sqrt{0.6}\,|00\rangle + \sqrt{0.4}\,|11\rangle$, then they can perform the transformation

$$|\psi\rangle \otimes |\eta\rangle \to |\varphi\rangle \otimes |\eta\rangle \tag{1.19}$$

with certainty via LOCC (Nie99; JP99). Note that, after the transformation the state $|\eta\rangle$ remains unchanged and can be used again for the same procedure. Therefore, the state $|\eta\rangle$ acts as a catalyst and enables a transformation that wouldn't be possible otherwise. Following this intuition, we now define *exact catalysis* (DKMS22a) for general resource theories as follows.

A quantum state ρ^S can be transformed into σ^S via *exact catalysis* iff there exists a free operation Λ and a catalyst state τ^C such that [2]

$$\Lambda[\rho^S \otimes \tau^C] = \sigma^S \otimes \tau^C. \tag{1.20}$$

Note that, from the above equation, we can see that the catalyst is not only unchanged but also remains uncorrelated by the end of the transformation.

Another notion of catalysis assumes that the catalyst remains unchanged but allows of correlations between the system and the catalyst. We call such transformations as *correlated catalysis* (DKMS22a). Formally, a quantum state ρ^S can be transformed into σ^S via *correlated catalysis* iff for every $\varepsilon > 0$, there exists a free operation Λ and a catalyst state τ^C such that

$$\| \mathrm{Tr}_C \left[\Lambda(\rho^S \otimes \tau^C)\right] - \sigma^S \|_1 \leq \varepsilon, \tag{1.21}$$

$$\mathrm{Tr}_S \left[\Lambda\left(\rho^S \otimes \tau^C\right)\right] = \tau^C, \tag{1.22}$$

where $\|A\|_1 = \mathrm{Tr}\,\sqrt{A^\dagger A}$ is the trace norm of a linear operator A. Here the condition $\mathrm{Tr}_S\left[\Lambda(\rho^S \otimes \tau^C)\right] = \tau^C$ makes sure that the catalyst is unchanged and $\| \mathrm{Tr}_C\left[\Lambda(\rho^S \otimes \tau^C)\right] - \sigma^S \|_1 < \varepsilon$ ensures that the final state of the system is ε-close to σ^S. Recently, in (KDS21) the authors suggested the notion of *approximate catalysis*. This framework allows for correlations between the system and catalyst, requiring that the correlations

[2] Here, S denotes the system and C denotes the catalyst.

can be made arbitrarily small. Precisely, ρ^S can be transformed into σ^S via approximate catalysis iff for every $\varepsilon > 0$ there exists a catalyst τ^C and a free operation Λ such that

$$\left\|\Lambda\left(\rho^S \otimes \tau^C\right) - \sigma^S \otimes \tau^C\right\|_1 \leq \varepsilon, \tag{1.23}$$

$$\mathrm{Tr}_S\left[\Lambda\left(\rho^S \otimes \tau^C\right)\right] = \tau^C, \tag{1.24}$$

Note that, the above definition allows for correlations between the system and the catalyst in the final state. However, due to Eq. (1.23), these correlations can be made arbitrarily small (because the final state is ε-close to a product state), by choosing an appropriate catalyst τ^C and free operation Λ. Furthermore, the condition in Eq. (1.24) implies that the catalyst remains unchanged. An alternative definition for *approximate catalysis* has been proposed in (RW20; SS21; Mul18). Eq. (1.23) ensures that the correlations can be made arbitrarily small as quantified by the trace norm. In principle, it is possible to formulate such a condition by choosing a different correlation measure such as mutual information $I^{A:B}(\rho^{AB}) = H(\rho^A) + H(\rho^B) - H(\rho^{AB})$[3]. Using this, the *approximate catalysis* can be defined in the following way. A quantum state ρ^S can be transformed into σ^S via *approximate catalysis* iff there exists a free operation Λ and a catalyst state τ^C such that for every $\varepsilon > 0$ and $\delta > 0$

$$\left\|\mathrm{Tr}_C\left[\Lambda\left(\rho^S \otimes \tau^C\right)\right] - \sigma^S\right\|_1 \leq \varepsilon,\ \mathrm{Tr}_S\left[\Lambda\left(\rho^S \otimes \tau^C\right)\right] \ = \tau^C \text{ and } I^{S:C}\left(\Lambda\left(\rho^S \otimes \tau^C\right)\right) \leq \delta. \tag{1.25}$$

In (RT22a) the authors showed that both the above mentioned definitions for approximate catalysis are equivalent.

1.4.1 Catalytic embezzling

In various notions of catalysis defined in the previous subsection, we always imposed the constraint that the catalyst is left unchanged at the end of the transformation. We now motivate this constraint by discussing a counter intuitive phenomenon called *catalytic embezzling*. The first observation of embezzling was in (vDH03), in which the authors studied entanglement catalysis, allowing for an arbitrarily small change in the final state of the catalyst. Using this, the authors in (vDH03) showed that for a family of pure entangled states $|\mu_n\rangle$ and any bipartite entangled state $|\psi\rangle$ there exists a LOCC operation Λ such that

$$\Lambda(|\mu_n\rangle\langle\mu_n|) = |\psi\rangle\langle\psi| \otimes |\omega_n\rangle\langle\omega_n| \text{ with } \||\mu_n\rangle\langle\mu_n| - |\omega_n\rangle\langle\omega_n|\|_1 \leq \varepsilon. \tag{1.26}$$

Here, $\varepsilon \to 0^+$ as $n \to \infty$. Note that from Eq. (1.26), it follows that any two bipartite entangled states $|\psi\rangle$ and $|\varphi\rangle$ can be transformed into each other via catalysis, if we allow for an arbitrarily small change in the catalyst. Therefore, this would allow us to extract unbounded number entangled states from the shared catalyst, making the problem of transforming states trivial.

[3] $H(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]$, is the von Neumann entropy of a quantum state ρ.

1.5 Summary of results

These notions of state transformations show us that the study of such transformations is incredibly rich and has a broad applicability. We will explore them in detail in the following chapters.

In Chapter 2, we discuss the idea of resource monotones. These are functions which assign non-negative real numbers to quantum states and they do not increase under free operations, thus providing necessary conditions on deterministic transformation of quantum states. We then address the question of complete set of monotones for deterministic transformations and show that there does not exist a finite set of resource monotones which completely determines all deterministic transformations. We also present the idea of totally ordered resource theories, in which every pair of quantum states is capable of undergoing a free transformation. Finally we demonstrate that totally ordered resource theories allow free transformations between all pure states and are equivalent to theories with a single resource monotone.

Chapter 3 deals with the intermediate regime between probabilistic and approximate transformations. We establish bounds on both the fidelity and the probability of state transitions, which are valid in all quantum resource theories. We then show how these bounds imply non-trivial constraints on the asymptotic rates for various classes of states. Finally, we close this chapter by completely solving the question of stochastic-approximate state transformations in various settings for the resource theory of entanglement (LOCC) and the resource theory of imaginarity.

Chapter 4 studies catalytic transformations, particularly focusing on correlated and approximate catalysis in bipartite entanglement theory (LOCC). We start by completely characterising pure state transformations under approximate catalysis. We then go ahead by showing the equivalence between correlated catalysis and marginal asymptotic transformations, for distillable states. Using this result, we show that using an entangled catalyst cannot increase the asymptotic singlet distillation rate of a distillable quantum state. We end this chapter by investigating the role of catalysis for quantum communication over noisy channels, providing tools to estimate the optimal communication rates in these settings.

Chapter 2

Deterministic transformations and complete set of monotones

2.1 Introduction

One of the fundamental problems in any quantum resource theory is to quantify the amount of resource in a quantum state (CG19a). *Resource measures* allow us to quantify the resource in a given state. A resource measure is a function f which maps quantum states to non negative real numbers ($f : \mathcal{D} \to \mathbb{R}_{\geq 0}$). Usually, depending on the setting of interest, one would like the resource measures to satisfy various other properties, some of which are defined below.

- *Monotonicity*: A function f is said to be a "monotone", if it does not increase under free operations.

$$f(\rho) \geq R(\Lambda_f(\rho)) \;\forall\; \Lambda_f \in \mathcal{O}. \tag{2.1}$$

  If all the free states are inter-convertible into each other via free operations, Eq. (2.1) implies

$$f(\rho_f) = f(\rho'_f) \;\forall\; \rho_f, \rho'_f \in \mathcal{F}. \tag{2.2}$$

- *Faithfulness*: A function f is faithful if the following holds

$$f(\rho) = 0 \iff \rho \in \mathcal{F}. \tag{2.3}$$

- *Strong-monotonicity*: "Strong-monotonous" functions do not increase on average under the action of stochastic free operations. Precisely, the following holds

$$f(\rho) \geq \sum_i \mathrm{Tr}[\Phi_i(\rho)] f(\frac{\Phi_i(\rho)}{\mathrm{Tr}[\Phi_i(\rho)]}). \tag{2.4}$$

  for all quantum states ρ and stochastic free operations $\{\Phi_i\}$. Where $\sum_i \Phi_i$ is completely positive and trace non increasing and $\mathrm{Tr}[\Phi_i(\rho)] > 0$.

- *Convexity*: A function f is convex if the following holds for every state ρ and every probability distribution $\{p_i\}$

$$f(\sum_i p_i\rho_i) \leq \sum_i p_i f(\rho_i). \tag{2.5}$$

- *Weak-additivity*: A function f is said to be weakly-additive if the following holds for any quantum state ρ

$$f(\rho^{\otimes n}) = n \cdot \rho. \tag{2.6}$$

- *Sub-additivity*: A function f is sub-additive if the following holds for any two states ρ and τ.

$$f(\rho \otimes \tau) \leq f(\rho) + f(\tau). \tag{2.7}$$

- *Additivity*: A function is said to be additive if the following holds for any two states ρ and τ

$$f(\rho \otimes \tau) = f(\rho) + f(\tau). \tag{2.8}$$

  One can easily see that this property implies weak-additivity 2.6.

- *Strong super-additivity*: A function f satisfies strong super-additivity if for any state $\mu^{S_1S_2}$ acting on the Hilbert space $S_1 \otimes S_2$, the following holds

$$f(\mu^{S_1S_2}) \geq f(\mu^{S_1}) + f(\mu^{S_1}). \tag{2.9}$$

  Here, $\mu^{S_1} = \mathrm{Tr}_{S_2}[\mu^{S_1S_2}]$ and $\mu^{S_2} = \mathrm{Tr}_{S_1}[\mu^{S_1S_2}]$.

- *Continuity*: Let ρ and σ be two states acting on a Hilbert space with dimension d. A function f is said to be continuous if $\forall\ \varepsilon > 0$ there exists a $\delta > 0$ s.t

$$\|\rho - \sigma\|_1 < \delta \implies |f(\rho) - f(\sigma)| < \varepsilon. \tag{2.10}$$

- *Asymptotic continuity*: A function f is asymptotically continuous if the following holds for any two states ρ and σ acting on a Hilbert space with dimension d.

$$|f(\rho) - f(\sigma)| \leq K\|\rho - \sigma\|_1 \log d + g(\|\rho - \sigma\|_1). \tag{2.11}$$

  Here, K is some constant and $g(x)$ is a continuous function such that $g(x)$ converges to zero as $x \to 0$.

From now on in this chapter, we focus on deterministic transformations between quantum states and show that the problem of *state transformations* and the problem of *resource quantification* are indeed interconnected. Let us note that, we say that ρ can be deterministically transformed into σ via free operations iff for every $\varepsilon > 0$ there exists a free operation Λ_f s.t

$$\left\|\Lambda_f(\rho) - \sigma\right\|_1 \leq \varepsilon. \tag{2.12}$$

We will assume that the resource measures satisfy monotonicity under free operations (see Eq. (2.1) for the definition of monotonicity). This is a well motivated assumption as it captures the intuition that resources cannot increase under free operations. Additionally we will assume that the resource measures are continuous (see Eq. (2.10) for the definition of continuity), to make sure they behave smoothly, without any abrupt jumps. In (TR19a), the authors show that a quantum state ρ can be transformed into σ via free operations iff $R(\rho) \geq R(\sigma)$ *for all* continuous monotones R. It is important to note that this set of all continuous monotones is not finite. If $R'(\rho) \geq R'(\sigma)$ holds for a certain continuous monotone R', there is no guarantee that ρ can be transformed into σ via free operations. It might still be true that there exists a finite set of continuous monotones $\{R_i\}$ which completely characterise all possible state transformations i.e,

$$\rho \rightarrow \sigma \text{ iff } R_i(\rho) \geq R_i(\sigma) \ \forall\ i. \tag{2.13}$$

Such a complete set of continuous monotones have been proposed for pure states in bipartite entanglement theory (Nie99). In fact, it has been shown that such a complete finite set of continuous monotones do not exist for bipartite entanglement theory when transformations between general mixed states are considered (Gou05).

2.2 Finite complete set of resource monotones

One of the main results of this chapter is that for a large class of resource theories, no finite complete set of continuous monotones exist. To prove this fact, we make use of an additional assumption that the continuous monotones are *faithful* (see Eq. 2.3). This are very common assumption fulfilled by a large number of resource measures (CG19a). Additionally, we also use the following standard assumptions:

- The set of free states is convex and closed.
- The identity operation is free.
- Any free state can be obtained from any state via free operations.

Note that, the set of free states ($\mathcal{F}$) is said to be convex if for any two free states ρ_1 and ρ_2 the following holds

$$p\rho_1 + (1-p)\rho_2 \in \mathcal{F} \tag{2.14}$$

for every $0 \leq p \leq 1$. The set free states ($\mathcal{F}$) is said to be closed if for every $\rho \notin \mathcal{F}$

$$\inf_{\rho_f \in \mathcal{F}} \left\|\rho_f - \rho\right\|_1 > 0. \tag{2.15}$$

Roughly speaking, Eq. (2.15) makes sure that no resource state can be approximated arbitrarily well by any sequence of free states. The last assumption implies that all resource monotones attain the minimal value and are constant for all free states (see Eq. (2.2)).

We now show that, it is not possible to transform a full rank state to a resourceful pure state via free operations, in any resource theory. This result was first shown in (FL20a; RBTL20)

Lemma 2.1. *(FL20a; RBTL20) In any resource theory, a full rank state ρ cannot be converted into a pure resource state $|\psi\rangle$ via free operations.*

Proof. We prove this by contradiction. Let's assume that there exists a free operation, transforming ρ into $|\psi\rangle$ i.e, for any $\varepsilon > 0$ there exists a free operation Λ_f such that

$$\left\|\Lambda_f(\rho) - |\psi\rangle\langle\psi|\right\|_1 \leq \varepsilon. \tag{2.16}$$

Let's recall Fuchs-van de Graaf inequalities, which state that for a pair of states ρ and σ, the following inequalities hold (FvdG99)

$$1 - \sqrt{F(\rho,\sigma)} \leq \frac{1}{2}\|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho,\sigma)}. \tag{2.17}$$

Here $F(\rho,\sigma) = [\mathrm{Tr}(\sqrt{\rho}\sigma\sqrt{\rho})^{1/2}]^2$ is the fidelity between quantum states ρ and σ. Using Eq. (2.16) and Eq. (2.17), we arrive at

$$F\left(\Lambda_f[\rho], |\psi\rangle\langle\psi|\right) = \langle\psi|\Lambda_f[\rho]|\psi\rangle \geq \left(1 - \frac{\varepsilon}{2}\right)^2. \tag{2.18}$$

Let $p_{\min}$ be the minimum eigenvalue of ρ. Therefore we know that,

$$\rho \geq p_{\min}\mathbb{I} \tag{2.19}$$

This implies the following holds,

$$\rho - p_{\min}\sigma \geq p_{\min}(\mathbb{I} - \sigma) \geq 0 \;\forall\; \sigma \in \mathcal{D}_d. \tag{2.20}$$

Here, $\mathcal{D}_d$ represents the set of quantum states of dimension d. Since $p_{\min} \leq 1/d < 1$, we can define the state

$$\sigma_1 = \frac{\rho - p_{\min}\sigma}{1 - p_{\min}}, \tag{2.21}$$

such that Eq. (2.20) can be equivalently expressed as

$$\rho = p_{\min}\sigma + (1 - p_{\min})\sigma_1, \tag{2.22}$$

where $p_{\min}$ is the smallest eigenvalue of ρ. From Eq. (2.22), it follows that

$$\begin{aligned}\langle\psi|\Lambda_f[\rho]|\psi\rangle &= p_{\min}\langle\psi|\Lambda_f[\sigma]|\psi\rangle + (1 - p_{\min})\langle\psi|\Lambda_f[\sigma_1]|\psi\rangle \\ &\leq 1 - \left(p_{\min}\left(1 - \langle\psi|\Lambda_f[\sigma]|\psi\rangle\right)\right).\end{aligned} \tag{2.23}$$

Together with Eq. (2.18) we obtain

$$\langle\psi|\Lambda_f[\sigma]|\psi\rangle \geq 1 - \frac{1 - \left(1 - \frac{\varepsilon}{2}\right)^2}{p_{\min}} \;\forall\; \sigma \in \mathcal{D}_d. \tag{2.24}$$

Note that here $\varepsilon > 0$ can be chosen arbitrarily small. Therefore for any state $\sigma \in \mathcal{D}_d$, can be converted arbitrarily close to $|\psi\rangle$. Since σ can also be chosen to be a free state, using the fact that the set of free states is closed (see Eq. (2.15)), we arrive at a contradiction. □

Using Lemma 2.1, we will now show that a finite complete set of continuous and faithful resource monotones do not exist for any non trivial resource theory containing free pure state.

Theorem 2.1. *(DGKS23) For any non trivial resource theory containing free pure states, a finite complete set of continuous and faithful resource monotones do not exist.*

Proof. Assume, there exists a complete set of resource monotones $\{R_i\}$ which are continuous and faithful. Let ρ be a full rank state which is not free. We will now construct $|\psi_\varepsilon\rangle$, a pure state given by

$$|\psi_\varepsilon\rangle = \sqrt{1-\varepsilon}\left|\varphi_f\right\rangle + \sqrt{\varepsilon}\,|\varphi\rangle\,. \tag{2.25}$$

Here, $\left|\varphi_f\right\rangle$ is a free pure state, $|\varphi\rangle$ is a resourceful (not free) pure state and $0 < \varepsilon < 1$. It is important to note that there always exists a resourceful pure state (like $|\varphi\rangle$), because if all the pure states are free, by convexity all quantum states would be free, making the theory trivial. By choosing $\left|\varphi_f\right\rangle$ to be on the boundary of the free states, we make sure $|\psi_\varepsilon\rangle$ is not free for all small $\varepsilon > 0$. From Eq. (2.25), one can show that

$$\left\| |\psi_\varepsilon\rangle\langle\psi_\varepsilon| - \left|\varphi_f\right\rangle\left\langle\varphi_f\right| \right\|_1 \leq \sqrt{\varepsilon}. \tag{2.26}$$

From continuity of $\{R_i\}$, one can choose a small enough ε such that

$$R_i(\rho) \geq R_i(|\psi_\varepsilon\rangle)\ \forall\ i. \tag{2.27}$$

Since, $\{R_i\}$ is a complete set of monotones, ρ can be converted into $|\psi_\varepsilon\rangle$ via free operations. Note that $|\psi_\varepsilon\rangle$ is a resourceful pure state and ρ is a full rank state. Therefore, from lemma 2.1, we know this is not possible, arriving at a contradiction. □

The previous theorem is applicable to the resource theory of entanglement in both bipartite and multipartite settings. Furthermore, the resource theories of coherence(WY16), asymmetry (MS13), and imaginarity (HG18) contain resource-free pure states, making our theorem applicable to these theories as well.

2.3 Surpassing the limitations

2.3.1 Discontinuous monotones

The result of theorem 2.1, does not take into account discontinuous monotones. By considering discontinuous monotones one can find complete set of finite monotones in various resource theories (which contain free pure states). Below we give some examples without going into the details of the underlying resource theories.

(i) For the resource theory of coherence, all qubit transformations are completely characterised by the robustness of coherence C_R and the Δ-robustness of coherence $C_{\Delta,R}$, which are given as (NBC+16; PCB+16a; CG16b; CG16a; CG17; SRBE17)

$$C_R(\rho) = \min_{\tau}\left\{s \geq 0 : \frac{\rho + s\tau}{1+s} \in \mathcal{I}\right\}, \tag{2.28}$$

$$C_{\Delta,R}(\rho) = \min_{\Delta[\sigma]=\Delta[\rho]}\left\{s \geq 0 : \frac{\rho + s\sigma}{1+s} \in \mathcal{I}\right\}, \tag{2.29}$$

where $\mathcal{I}$ is the set of incoherent states, i.e., states which are diagonal in a reference basis and $\Delta(\cdot)$ is a qubit channel which satisfies

$$\Delta(\rho) = \text{diag}(\rho)\ \forall\, \rho \in \mathcal{D}_2. \tag{2.30}$$

Here, diag(ρ) is a quantum state achieved by putting the off diagonal elements (in the reference basis) of ρ to zero and $\mathcal{D}_2$ represents the set of qubit states. For any qubit state ρ, both these measures have an analytical expression given by (NBC[+]16; CG16a; CG17)

$$C_R(\rho) = 2|\rho_{0,1}| \tag{2.31}$$

and

$$C_{\Delta,R}(\rho) = |\rho_{0,1}|/\sqrt{\rho_{0,0}\rho_{1,1}}. \tag{2.32}$$

Here, $\rho_{ij} = \langle i|\rho\,|j\rangle$. From Eq. (2.32), we see that $C_{\Delta,R}(\rho) = 1$ for all coherent pure states and $C_{\Delta,R}(\rho) = 0$ for all incoherent states. Therefore, $C_{\Delta,R}$ is not continuous.

(ii) In the resource theory of imaginarity, the complete set of monotones for qubit states are given in terms of bloch coordinates (r_x, r_y, r_z) as

$$I_1(\rho) = r_y^2, \tag{2.33}$$

$$I_2(\rho) = \frac{r_y^2}{1 - r_x^2 - r_z^2}. \tag{2.34}$$

In (WKR[+]21a; WKR[+]21b), it has been shown that both the above mentioned quantities do not increase under real quantum operations, therefore are monotones under real operations. Note that, from Eq. (2.34), one can see that $I_2(\rho) = 1$ for all imaginary pure states and $I_2(\rho) = 0$ for all real states. This shows that I_2 is not continuous.

(iii) Complete set of monotones for single qubit states have also been provided for the resource theory of asymmetry (GS08; MS14; LKJR15). For an initial state ρ, a target state σ is achievable iff,

$$|\sigma_{0,1}| \leq |\rho_{0,1}|\sqrt{\chi}, \tag{2.35}$$

where $\chi = \min\{\sigma_{0,0}/\rho_{0,0}, (1-\sigma_{0,0})/(1-\rho_{0,0})\}$. Using Eq. (2.35), one can construct complete set of monotones given by

$$A_1(\rho) = \frac{|\rho_{0,1}|}{\sqrt{\rho_{0,0}}}, \tag{2.36}$$

$$A_2(\rho) = \frac{|\rho_{0,1}|}{\sqrt{1-\rho_{0,0}}}. \tag{2.37}$$

One can easily check that both these (above mentioned) monotones are not continuous. In order to see this, consider a pure state with

$$\rho_{0,1} = \varepsilon\sqrt{1-\varepsilon^2} \text{ and } \rho_{0,0} = \varepsilon^2, \tag{2.38}$$

where $\varepsilon > 0$. Note that, $A_1(\rho)$ is equal to zero for free states whereas it converges to one in the limit $\varepsilon \to 0$. Therefore, it is a discontinuous monotone. In a similar way, by taking into account a pure state with

$$\rho_{0,1} = \varepsilon\sqrt{1-\varepsilon^2} \text{ and } \rho_{1,1} = \varepsilon^2, \tag{2.39}$$

one can check that $A_2(\rho)$ is discontinuous.

2.3.2 Infinite set of resource monotones

An alternative way to surpass the limitations set by theorem 2.1 is by considering infinite sets of continuous monotones. In fact, for any quantum resource theory, such complete set of continuous monotones can be obtained as follows. Consider the quantity, given by

$$R_\nu(\rho) = \inf_{\Lambda_f} \left\| \Lambda_f[\nu] - \rho \right\|_1, \tag{2.40}$$

where ν is a quantum state which serves as a parameter of the monotone R_ν. One can easily show that R_ν is a resource monotone. In order to prove this, consider $\tilde{\Lambda}_f$ to be a free operation such that $R_\nu(\rho) \geq \|\tilde{\Lambda}_f[\nu] - \rho\|_1 - \varepsilon$ for some $\varepsilon > 0$. Consider an arbitrary free operation Λ_f, we have

$$\begin{aligned} R_\nu(\rho) &\geq \left\| \tilde{\Lambda}_f[\nu] - \rho \right\|_1 - \varepsilon \geq \left\| \Lambda_f \circ \tilde{\Lambda}_f[\nu] - \Lambda_f[\rho] \right\|_1 - \varepsilon \\ &\geq R_\nu\left(\Lambda_f[\rho]\right) - \varepsilon, \end{aligned} \tag{2.41}$$

Here, we used the fact that trace norm does not increase under CPTP maps. Since the above equation holds for any $\varepsilon > 0$, it follows that

$$R_\nu(\rho) \geq R_\nu\left(\Lambda_f[\rho]\right) \tag{2.42}$$

for all ν and Λ_f. The continuity of R_ν trivially follows from the continuity of trace norm. Therefore in any resource theory, the set of all continuous monotones is *complete*. In other words, a state ρ can be transformed into σ iff $R(\rho) \geq R(\sigma)$ for all continuous monotones R. Note that, in (TR19b), authors provide an alternative complete set of monotones for any resource theory.

2.3.3 Catalytic transformations

Another interesting way to surpass the restrictions imposed by theorem 2.1, is to extend the notion of deterministic transformations to catalytic transformations (DKMS22a). As defined in the introductory chapter, a quantum catalyst is an ancillary quantum system which does not change in the procedure. Here we particularly focus on so-called *approximate* catalysis, where correlations are allowed to build up between system and catalyst assuming that these correlations can be made arbitrarily small. Recalling from the introductory chapter, ρ^S can be transformed into σ^S via approximate catalysis iff for every $\varepsilon > 0$ there exists a catalyst τ^C and a free operation Λ such that (KDS21; DKMS22b; RT22a; DKMS22a)

$$\left\| \Lambda_f\left(\rho^S \otimes \tau^C\right) - \sigma^S \otimes \tau^C \right\|_1 \leq \varepsilon, \tag{2.43}$$

$$\mathrm{Tr}_S\left[\Lambda_f\left(\rho^S \otimes \tau^C\right)\right] = \tau^C. \tag{2.44}$$

Surprisingly, (see theorem below) catalytic transformations in the resource theory of coherence are completely characterised by the relative entropy of coherence $C(\rho) = H(\Delta[\rho]) - H(\rho)$. Therefore, a single monotone completely characterises catalytic transformations in the resource theory of coherence. Before presenting the theorem, let us briefly describe the resource theory of coherence (SAP17). In order to formulate the resource theory of coherence, one has to fix a *reference basis*. For a d dimensional Hilbert space $\mathcal{H}_d$, let us denote the reference basis as $\{|i\rangle\}_{i=1,...,d}$. Quantum states diagonal in this basis are considered to be free (incoherent states). Therefore every free (incoherent) state of dimension d takes the following form

$$\rho_I = \sum_i^d p_i |i\rangle \langle i| . \tag{2.45}$$

Here, $\{p_i\}$ is a probability distribution. The dephasing operation Δ in the reference basis is given by

$$\Delta(\cdot) = \sum_i^d \langle i| \cdot |i\rangle |i\rangle \langle i| . \tag{2.46}$$

Note that, a quantum state ρ_I, is incoherent if

$$\Delta(\rho_I) = \rho_I. \tag{2.47}$$

A quantum operation Λ is said to be dephasing covariant if

$$\Lambda \circ \Delta(\rho) = \Delta \circ \Lambda(\rho) \;\forall \rho \in \mathcal{D}_d \tag{2.48}$$

From Eq. (2.48) and Eq. (2.47), one can see that dephasing covariant operations (DIO) map incoherent states to incoherent states. Therefore dephasing covariant operations can be considered as a possible set of free operations in the resource theory of coherence. Note that the largest set of (deterministic) quantum operations which preserve the set of incoherent states are known a maximally incoherent operations (MIO) (Abe06). It is known that DIO $\subset$ MIO (CG16c). With this background we go ahead and present our result about catalytic transformations in coherence theory.

Theorem 2.2. *(DGKS23) A quantum state ρ can be transformed into another quantum state σ via DIO and approximate catalysis iff*

$$C(\rho) \geq C(\sigma). \tag{2.49}$$

Proof. In Lemma 4.1 (from chapter 4), we will show that if ρ can be asymptotically converted into σ with a rate one (see Eq. (1.15)), then ρ can be converted into σ via approximate catalysis[1]. From (Chi18) we know that, using DIO, ρ can be asymptotically converted into σ with rate whenever Eq. (2.49) is satisfied. This implies that Eq. (2.49) is a sufficient condition for ρ to be transformed into σ via DIO with approximate catalysis.

We will now show that the following condition is necessary if ρ is converted into ν via DIO with approximate catalysis

$$C(\rho) \geq C(\sigma). \tag{2.50}$$

[1]Lemma 4.1 is presented for LOCC operations, but can be trivially extended to DIO (see the discussion below the proof of lemma 4.1).

Note that the relative entropy of coherence in bipartite systems fulfills strong super-additivity (XLF15)

$$C(\rho^{AB}) \geq C(\rho^{A}) + C(\rho^{B}). \tag{2.51}$$

Additionally equality holds whenever $\rho^{AB} = \rho^A \otimes \rho^B$. Here, $\rho^A = \mathrm{Tr}_B[\rho^{AB}]$ and $\rho^B = \mathrm{Tr}_A[\rho^{AB}]$ (we will use this notation throughout this thesis). From the definition of approximate catalysis, we know that for any $\varepsilon > 0$ there exists a catalyst state τ^C and a DIO Λ acting on system and the catalyst SC such that

$$\left\|\mathrm{Tr}_C\left[\Lambda(\rho^S \otimes \tau^C)\right] - \sigma^S\right\|_1 < \varepsilon, \tag{2.52}$$

$$\mathrm{Tr}_S\left[\Lambda(\rho^S \otimes \tau^C)\right] = \tau^C. \tag{2.53}$$

Now we use the properties of the relative entropy of coherence to obtain

$$C\left(\Lambda(\rho^S \otimes \tau^C)\right) \leq C\left(\rho^S\right) + C\left(\tau^C\right) \tag{2.54}$$

and also

$$C\left(\Lambda(\rho^S \otimes \tau^C)\right) \geq C\left(\mathrm{Tr}_C\left[\Lambda(\rho^S \otimes \tau^C)\right]\right) + C\left(\tau^C\right). \tag{2.55}$$

Eqs. (4.26) and (4.27) imply

$$C\left(\rho^S\right) \geq C\left(\mathrm{Tr}_C\left[\Lambda(\rho^S \otimes \tau^C)\right]\right). \tag{2.56}$$

Since $\|\mathrm{Tr}_C[\Lambda(\rho^S \otimes \tau^C)] - \sigma^S\|_1 \leq \varepsilon$ for every $\varepsilon > 0$, the continuity of the relative entropy of coherence (WY16) implies $C(\rho^S) \geq C(\sigma^S)$. This completes the proof. □

2.4 Single complete resource monotone and total order

In this section we will study resource theories having a single complete continuous monotone. This means for any two states ρ and σ, there exists a free operation transforming ρ into σ iff $R(\rho) \geq R(\sigma)$ for a single continuous monotone R.

Let us now define *total order*. A resource theory is said to have *total order* iff, for any two states ρ and σ, there always exists a free operation transforming either ρ into σ or σ into ρ. In the below theorem we connect both the above definitions.

Theorem 2.3. *(DGKS23) A resource theory has a total order if and only if it has a single complete continuous monotone.*

Proof. It is straightforward to see that if a resource theory has a single complete continuous monotone, then for any two states ρ and σ, there always exists a free operation transforming ρ into σ (if $R(\rho) \geq R(\sigma)$) or σ into ρ (if $R(\rho) < R(\sigma)$). Therefore, this induces a total order among the set of states.

In order to prove the converse, let us first define a continous monotone R given by

$$R(\rho) = \min_{\mu \in \mathcal{F}} \|\rho - \mu\|_1. \tag{2.57}$$

Here $\mathcal{F}$ denotes the set of free states. From the following argument, one can see that $R(\rho) \geq R(\Lambda[\rho])$, for all free operations Λ (monotonicity).

$$R(\rho) = \min_{\mu \in \mathcal{F}} \|\rho - \mu\|_1 = \|\rho - \nu\|_1 \geq \|\Lambda[\rho] - \Lambda[\nu]\|_1 \geq R(\Lambda[\rho]) \tag{2.58}$$

Here ν is considered to be the free state which achieves the minimisation for $R(\rho)$. The first inequality follows from the fact that trace norm does not increase under CPTP maps and the last inequality trivially follows from the definition of R. Additionally continuity of R follows from the continuity of trace norm. Therefore, we showed that the quantity R, defined in Eq. (2.58), is indeed a continuous monotone.

We will now show that R is a complete monotone if the theory has a total order i.e, we will show that, for any pair of states ρ and σ, there exists a free transformation from ρ to σ, whenever $R(\rho) \geq R(\sigma)$.

For any two states, ρ and σ, if $R(\rho) < R(\sigma)$, from monotonicity of R it follows that ρ cannot be transformed into σ via free operations. But since the resource theory has total order, a transformation from σ into ρ has to be possible. The remaining case is when $R(\rho) = R(\sigma)$. Now we define $\sigma_\varepsilon = (1-\varepsilon)\sigma + \varepsilon\mu_f$, where $\mu_f \in \mathcal{F}$ is a closest free state to σ (see Eq. (2.57)). We then have

$$R(\sigma_\varepsilon) \leq \left\|\sigma_\varepsilon - \mu_f\right\|_1 = (1-\varepsilon)\left\|\sigma - \mu_f\right\|_1 = (1-\varepsilon)R(\sigma). \tag{2.59}$$

Therefore, $R(\sigma_\varepsilon) < R(\rho)$ for all $\varepsilon > 0$. Since the theory has a total order, there exists a free transformation converting ρ into σ_ε for all $\varepsilon > 0$. Also note that, $\|\sigma_\varepsilon - \sigma\|_1$ can be made arbitrarily small by choosing a small enough ε. This completes the proof. □

The above theorem shows that the existence of a single complete continuous monotone is equivalent to the resource theory having a total order. Using this, we will show that in any totally ordered theory, all pure states have to be inter-convertible under free operations.

Lemma 2.2. *(DGKS23) In a resource theory with total order, any two pure states can be transformed into each other via free operations.*

Proof. Firstly, consider the resource monotone from Eq. (2.57). From Theorem 2.3, it follows that for a totally ordered resource theory, state transformations are determined by this monotone. Therefore, it suffices to prove that

$$R(|\psi\rangle) = R(|\varphi\rangle), \tag{2.60}$$

holds for all pure states $|\psi\rangle$ and $|\varphi\rangle$. Let us assume that there exist a pair of pure states such that $R(|\psi\rangle) > R(|\varphi\rangle) > 0$. Now, one can construct a full rank state

$$\rho_\varepsilon = (1-\varepsilon)\,|\psi\rangle\,\langle\psi| + \varepsilon\frac{\mathbb{I}}{d} \tag{2.61}$$

with $0 < \varepsilon < 1$. Continuity of R implies that, there exists a small enough $\varepsilon > 0$ such that $R(\rho_\varepsilon) > R(|\varphi\rangle)$. Since R is a complete monotone which fully determines all state transformations, there should exist a free transformation transforming ρ_ε into $|\varphi\rangle$. From lemma 2.1, we know that, a full rank state cannot be transformed into a pure resource state, arising at a contradiction. This shows that for any two pure state $|\psi\rangle$ and $|\varphi\rangle$,

$$R(|\psi\rangle) = R(|\varphi\rangle). \tag{2.62}$$

Since R is a complete monotone determining all state transformations, from Eq. (2.60) it follows that any pair of pure states can be transformed into each other via free operations. This completes the proof. □

Using these results, we will now completely characterise all totally ordered theories for $d = 2$ (qubits). In order to do this, we will first characterise the set of free states. For any two qubit states ρ and σ with corresponding bloch vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ respectively, it holds that

$$\|\rho - \sigma\|_1 = |\boldsymbol{r} - \boldsymbol{s}|. \tag{2.63}$$

Recalling $R(\rho) = \min_{\mu \in \mathcal{F}} \|\rho - \mu\|_1$, from Eq. (2.60) it follows that, all pure states have to be equidistant (in trace distance) from the set of free states. Therefore, the set of free states has to be a ball around the maximally mixed state ($\frac{\mathbb{I}_2}{2}$). If the radius of the ball is t, the set of free states can be given as:

$$\mathcal{F}_t = \left\{ \sigma : \left\| \sigma - \frac{\mathbb{I}_2}{2} \right\|_1 \leq t \right\}, \tag{2.64}$$

where $t \in [0, 1]$. Eq. (2.64), serves as a necessary condition for the set of free states, if the theory has a total order. We will now provide a necessary condition for the state transformations, in a totally ordered qubit theory. Note that, for a fixed t, the resource monotone R for any state ρ can be evaluated as:

$$R(\rho) = \max\{|\boldsymbol{r}| - t, 0\}. \tag{2.65}$$

Here, $\boldsymbol{r}$ is the bloch vector of ρ. Since R is a complete monotone, from Eq. (2.65) it follows that in a single qubit totally ordered resource theory all state transformations characterised by the Bloch vector i.e, for any pair of resource states ρ and σ with respective Bloch vectors given by $\boldsymbol{r}$ and $\boldsymbol{s}$, $\rho \to \sigma$ is possible via free transformations if and only if $|\boldsymbol{r}| \geq |\boldsymbol{s}|$. Additionally, when σ is a free state (i.e $|\boldsymbol{s}| \leq t$), the transformation $\rho \to \sigma$ is always possible whenever $|\boldsymbol{s}| \leq t$.

We will now see that both the above mentioned necessary conditions are also sufficient. Let us note that the resource theory of purity (HHO03; GMN+15) is an example of a totally ordered qubit resource theory (corresponds to the case when $t = 0$). In this theory, the free operations are unital operations (CPTP maps preserving $\frac{\mathbb{I}_2}{2}$). From (GMN+15), we know that a qubit state ρ can be transformed into another qubit state σ via unital operations if and only if $|\boldsymbol{r}| \geq |\boldsymbol{s}|$. We will now generalise this for any given $t \in [0, 1]$, by defining the set of free operations as follows

- All operations satisfying $\Lambda[\frac{\mathbb{I}_2}{2}] = \frac{\mathbb{I}_2}{2}$ (unital operations).
- All fixed-output operations satisfying $\Lambda[\rho] = \sigma$ with $\sigma \in \mathcal{F}_t$.

One can easily see that this set of free operations preserves the set of free states given by Eq. (2.64). Since, a qubit state ρ can be transformed into another qubit state σ via unital operations if and only if $|\boldsymbol{r}| \geq |\boldsymbol{s}|$ (GMN$^+$15), one can see that the free operations defined above give rise to a totally ordered resource theory with single complete monotone given in Eq.(2.65). Note that this construction is based the fact that unital operations induce a total order. However this is not true for higher dimensions ($d \geq 3$). Therefore, this construction cannot be generalised to dimensions more than 2.

2.5 Conclusions

In this chapter we have introduced various properties satisfied by commonly known resource measures and defined the notion of resource monotones. In generic quantum resource theories, we studied the idea of having a complete set of monotones. Using only the most basic assumptions, such as faithfulness, monotonicity and continuity, we have demonstrated that a complete finite set of monotones do not exist if a resource theory comprises free pure states. This result applies to the theories of entanglement in bipartite and multipartite situations. However, we show that such complete set of monotones can be found by allowing discontinuity, considering infinite sets and allowing for catalytic transformations. We then provided examples of such complete sets in several resource theories. We also looked at resource theories in which state transitions are dictated by a single continuous monotone. We demonstrated that any such theory must be totally ordered, with any pair of states allowing free transformation in (at least) one direction. We then showed that any totally ordered theory must allow for free transformations between any pair of pure states (in both directions). We then completely characterise all qubit resource theories with total order.

Chapter 3

Stochastic approximate state conversion

3.1 Introduction

In this chapter, we primarily deal with state transformations involving single copy of a quantum state. We focus on the regime in between probabilistic and approximate transformations. Very few results have been known so far for this regime (FL20b; RT21; Reg21; EW22). Here the goal is to convert a quantum state ρ into a target state σ with the optimal probability, while allowing for a small error in the transformation. This kind of transformations are motivated from various entanglement manipulation scenarios where a small probability of failure is allowed and the optimal achievable fidelity is considered in the case of success (DP22; RmkST+18; NFB14; BBP+96; DEJ+96; ZPZ01; YKI01). For example, in quantum networks, the trade off between fidelity and probability has been found to be very relevant (DLCZ01; BK05). As noted in the introductory chapter, one can define *fidelity for stochastic approximate state conversion*, which quantifies the optimal fidelity for the transformation from ρ to σ with a conversion probability at least p:

$$F_p(\rho \to \sigma) = \sup_{\mathcal{E}} \left\{ F\left(\frac{\mathcal{E}[\rho]}{\mathrm{Tr}(\mathcal{E}[\rho])}, \sigma\right) : \mathrm{Tr}\left(\mathcal{E}[\rho]\right) \geq p \right\}. \tag{3.1}$$

In the same way, one can also define *probability for stochastic approximate state conversion*, capturing the optimal transformation probability for a transformation with fidelity at least f:

$$P_f(\rho \to \sigma) = \sup_{\mathcal{E}} \left\{ \mathrm{Tr}\left(\mathcal{E}[\rho]\right) : F\left(\frac{\mathcal{E}[\rho]}{\mathrm{Tr}\left(\mathcal{E}[\rho]\right)}, \sigma\right) \geq f \right\}. \tag{3.2}$$

Both in Eq. (3.1) and Eq. (3.2) and the supremum is taken over all free probabilistic transformations $\mathcal{E}$. In this chapter we also consider a more general kind of state transformations, where one allows for maps (Λ_ε) which can generate at most ε amount of resource (according to some resource measure M). Therefore,

$$M\left(\frac{\Lambda_\varepsilon[\rho]}{\operatorname{Tr}\Lambda_\varepsilon[\rho]}\right) \leq \varepsilon \text{ for all free states } \rho. \tag{3.3}$$

We name these kind of operations as ε-resource generating operations. If the measure M is faithful, then free operations correspond to the case when $\varepsilon = 0$. One can generalise the quantities in Eqs. (3.1) and (3.2), to the case of ε-resource generating operations as follows

$$F_p^{M_\varepsilon}(\rho \to \sigma) = \max_{\Lambda_\varepsilon}\left\{F\left(\frac{\Lambda_\varepsilon[\rho]}{\operatorname{Tr}(\Lambda_\varepsilon[\rho])}, \sigma\right) : \operatorname{Tr}(\Lambda_\varepsilon[\rho]) \geq p\right\}. \tag{3.4}$$

and

$$P_f^{M_\varepsilon}(\rho \to \sigma) = \max_{\Lambda_\varepsilon}\left\{\operatorname{Tr}(\Lambda_\varepsilon[\rho]) : F\left(\frac{\Lambda_\varepsilon[\rho]}{\operatorname{Tr}(\Lambda_\varepsilon[\rho])}, \sigma\right) \geq f\right\}. \tag{3.5}$$

Let us now introduce three resource measures, which will be used in this chapter. These three resource measures will be geometric resource measure G, the generalised resource robustness R and the standard robustness S, given by

$$G(\rho) = 1 - \max_{\sigma \in \mathcal{F}} F(\rho, \sigma) \tag{3.6}$$

$$K(\rho) = \min_{\tau}\left\{s \geq 0 : \frac{\rho + s\tau}{1+s} \in \mathcal{F}\right\} \text{ and} \tag{3.7}$$

$$S(\rho) = \min_{\tau \in \mathcal{F}}\left\{s \geq 0 : \frac{\rho + s\tau_f}{1+s} \in \mathcal{F}\right\}. \tag{3.8}$$

Note that all these three quantities are non-negative, faithful, vanish for all free states and non-increasing under free operations(CG19a). G, R and S were first introduced in the context of resource theory of entanglement (VT99; Ste03; HN03; Shi95; BL01; WG03; SKB10) and later found applications in various other resource theories (TRB+19; TR19b; NBC+16; SSD+15). It's worthwhile to note that R and G can be alternatively written as

$$E_{1/2}(\rho) = -\log_2(1 - G(\rho)) = -\log_2 F_{\max}(\rho), \text{ and } E_{\max}(\rho) = \log_2(1 + K(\rho)) \tag{3.9}$$

where $E_{1/2}$ and $E_{\max}$ are given by

$$E_{1/2}(\rho) = \min_{\sigma \in \mathcal{F}} D_{1/2}(\rho\|\sigma), \; E_{\max}(\rho) = \min_{\sigma \in \mathcal{F}} D_{\max}(\rho\|\sigma). \tag{3.10}$$

Note that, $D_{1/2}(\rho\|\sigma) = D_{\alpha=1/2}(\rho\|\sigma)$ and $D_{\max}(\rho\|\sigma) = \lim_{\alpha\to\infty} D_\alpha(\rho\|\sigma)$, where D_α is the sandwiched Rényi relative entropy (see section 7.5 of (KW20) for extensive discussion on sandwiched Rényi entropies). From monotonicity property of sandwiched Rényi relative entropies, it follows that

$$D_\alpha(\rho\|\sigma) \leq D_{\alpha'}(\rho\|\sigma) \text{ for all } \alpha \leq \alpha'. \tag{3.11}$$

Additionally, sandwiched Rényi relative entropies satisfy the data processing inequality for $\alpha \in [\frac{1}{2}, \infty)$ i.e, for every CPTP map Λ and $\alpha \in [\frac{1}{2}, \infty)$

$$D_\alpha(\rho\|\sigma) \leq D_\alpha(\Lambda(\rho)\|\Lambda(\sigma)). \tag{3.12}$$

From Eq. (3.11), it follows that

$$D_{1/2}(\rho\|\sigma) \leq D_{\max}(\rho\|\sigma). \tag{3.13}$$

Assuming $D_{\max}(\rho\|\sigma') = \min_{\sigma\in\mathcal{F}} D_{\max}(\rho\|\sigma)$, we get the following inequalities

$$\min_{\sigma\in\mathcal{F}} D_{1/2}(\rho\|\sigma) \leq D_{1/2}(\rho\|\sigma') \leq D_{\max}(\rho\|\sigma') = \min_{\sigma\in\mathcal{F}} D_{\max}(\rho\|\sigma). \tag{3.14}$$

Therefore for any state ρ,

$$E_{1/2}(\rho) \leq E_{\max}(\rho). \tag{3.15}$$

The above inequality can be equivalently written as

$$K(\rho) \geq \frac{G(\rho)}{1 - G(\rho)}. \tag{3.16}$$

3.2 Single copy bounds for general resource theories

Using these tools, we will now provide bounds on $F_p^{M_\varepsilon}(\rho \to \sigma)$ and $P_f^{M_\varepsilon}(\rho \to \sigma)$, which hold for any quantum resource theory. We will present these bounds in the following theorem.

Theorem 3.1. *(KDS22) For any quantum resource theory and any two states ρ and σ the following inequalities hold:*

$$F_p^{M_\varepsilon}(\rho \to \sigma) \leq \min\left\{\frac{1}{p}\left[1 + K(\rho)\right] \times F_{\max}^{M_\varepsilon}(\sigma), 1\right\}, \tag{3.17}$$

$$P_f^{M_\varepsilon}(\rho \to \sigma) \leq \min\left\{\frac{1}{f}\left[1 + K(\rho)\right] \times F_{\max}^{M_\varepsilon}(\sigma), 1\right\}. \tag{3.18}$$

Here, $F_{\max}^{M_\varepsilon}(\sigma) = \max_{\sigma_\varepsilon : M(\sigma_\varepsilon)\leq\varepsilon} F(\sigma, \sigma_\varepsilon)$.

Proof. From the definition of generalised robustness, we know that there exists a state τ, such that

$$\frac{\rho + K(\rho)\tau}{1 + K(\rho)} = \rho_f. \tag{3.19}$$

Here, $\rho_f \in \mathcal{F}$. Let $\mathcal{E}$ be a stochastic ε-resource generating operation. Applying $\mathcal{E}$ both sides of the Eq. (3.19) gives

$$\frac{1}{1 + K(\rho)}\mathcal{E}(\rho) + \frac{K(\rho)}{1 + K(\rho)}\mathcal{E}(\tau) = \mathcal{E}(\rho_f). \tag{3.20}$$

Here, we assume $\mathrm{Tr}\,\mathcal{E}(\rho) > 0$. This is because we are interested in transformations with non zero probability. Therefore,

$$\frac{1}{1 + K(\rho)}\frac{\mathcal{E}(\rho)}{\mathrm{Tr}\,\mathcal{E}(\rho_f)} + \frac{K(\rho)}{1 + K(\rho)}\frac{\mathcal{E}(\tau)}{\mathrm{Tr}\,\mathcal{E}(\rho_f)} = \frac{\mathcal{E}(\rho_f)}{\mathrm{Tr}\,\mathcal{E}(\rho_f)}. \tag{3.21}$$

We now evaluate the fidelity between $\mathcal{E}(\rho_f)/\operatorname{Tr}[\mathcal{E}(\rho_f)]$ and σ:

$$F\left(\frac{\mathcal{E}(\rho_f)}{\operatorname{Tr}\mathcal{E}(\rho_f)},\sigma\right)=F\left(q\frac{\mathcal{E}(\rho)}{\operatorname{Tr}\mathcal{E}(\rho)}+(1-q)\frac{\mathcal{E}(\tau)}{\operatorname{Tr}\mathcal{E}(\tau)},\sigma\right),\tag{3.22}$$

where

$$q=\frac{\operatorname{Tr}\mathcal{E}(\rho)}{\operatorname{Tr}\mathcal{E}(\rho_f)[1+K(\rho)]}\tag{3.23}$$

and

$$1-q=\frac{\operatorname{Tr}\mathcal{E}(\tau)K(\rho)}{\operatorname{Tr}\mathcal{E}(\rho_f)[1+K(\rho)]}.\tag{3.24}$$

Using concavity of fidelity (KW20) in Eq. (3.22) we obtain

$$F\left(\frac{\mathcal{E}(\rho_f)}{\operatorname{Tr}\mathcal{E}(\rho_f)},\sigma\right)\geq qF\left(\frac{\mathcal{E}(\rho)}{\operatorname{Tr}\mathcal{E}(\rho)},\sigma\right).\tag{3.25}$$

Let us recall that,

$$F_{\max}^{M_\varepsilon}(\sigma)=\max_{\tau}F(\sigma,\tau)\ \text{s.t}\ M(\tau)\leq\varepsilon\tag{3.26}$$

and $\mathcal{E}$ is an ε-resource generating operation. Therefore,

$$qF\left(\frac{\mathcal{E}(\rho)}{\operatorname{Tr}\mathcal{E}(\rho)},\sigma\right)\leq F_{\max}^{M_\varepsilon}(\sigma)\tag{3.27}$$

Let us now substitute Eq. (3.23) and recall that $\operatorname{Tr}\mathcal{E}(\rho)=p$. Therefore,

$$\begin{aligned}[1+K(\rho)]\times F_{\max}^{M_\varepsilon}(\sigma)&\geq\frac{\operatorname{Tr}\mathcal{E}(\rho)}{\operatorname{Tr}\mathcal{E}(\rho_f)}F\left(\frac{\mathcal{E}(\rho)}{\operatorname{Tr}\mathcal{E}(\rho)},\sigma\right)\\&\geq pF\left(\frac{\mathcal{E}(\rho)}{\operatorname{Tr}\mathcal{E}(\rho)},\sigma\right).\end{aligned}\tag{3.28}$$

Here, also used $\operatorname{Tr}\mathcal{E}(\rho_f)\leq 1$. This completes the proof. □

Theorem 3.1 holds for general resource theories, providing upper bounds on the achievable fidelity and probabilities for stochastic approximate state conversion. For the special case of $\varepsilon=0$ (free operations), the bounds in theorem 3.1 can be written as

$$F_p(\rho\to\sigma)\quad\leq\tfrac{1}{p}\min\left\{2^{E_{\max}(\rho)-E_{1/2}(\sigma)},1\right\}\tag{3.29}$$

$$P_f(\rho\to\sigma)\quad\leq\tfrac{1}{f}\min\left\{2^{E_{\max}(\rho)-E_{1/2}(\sigma)},1\right\}.\tag{3.30}$$

For the cases of deterministic conversion with $p=1$ and exact probabilistic conversion with $f=1$, Eqs. (3.29) and (3.30) reduce to

$$F(\rho\to\sigma)\leq\min\left\{2^{E_{\max}(\rho)-E_{1/2}(\sigma)},1\right\},\tag{3.31}$$

$$P(\rho\to\sigma)\leq\min\left\{2^{E_{\max}(\rho)-E_{1/2}(\sigma)},1\right\}.\tag{3.32}$$

Note that, the above bounds are non-trivial iff $E_{\max}(\rho)<E_{1/2}(\sigma)$.

Whenever the set of free states is convex, $E_{\max}$ (generalised robustness) and $E_{1/2}$ (geometric measure) can be posed as convex optimization problems. Additionally, if the set of free states is SDP representable, which is the case for many resource theories like imaginarity, coherence, asymmetry (PCB$^+$16b; WKR$^+$21a) both $E_{\max}$ and $E_{1/2}$ can be posed as semidefinite programs (SDP).

Note that these bounds (see Eq. (3.31) and Eq. (3.32)) are not tight in general and for specific setups, tighter bounds can be obtained (see the proposition below).

Proposition 3.1. *(KDS22) For a two-qubit state ρ^{AB}, the optimal fidelity of achieving a singlet $\left(\left|\varphi_+^{AB}\right\rangle = \frac{\left|0^A0^B\right\rangle + \left|1^A1^B\right\rangle}{\sqrt{2}}\right)$ via SLOCC, for a given probability p is given by*

$$F_p\left(\rho^{AB} \to \left|\varphi_+^{AB}\right\rangle\right) \le \min\left\{\frac{K(\rho^{AB})}{2p} + \frac{1}{2}, 1\right\}. \tag{3.33}$$

Proof. The optimal fidelity of achieving a singlet (deterministically) via LOCC from a bipartite two qubit state ρ^{AB} is given by (VV03)

$$\max_{\Lambda \in \mathrm{LOCC}} F\left(\Lambda(\rho^{AB}), \left|\varphi_+^{AB}\right\rangle\right) = \frac{1 + K(\rho^{AB})}{2}. \tag{3.34}$$

Here, $K(\rho^{AB})$ is the robustness of entanglement of ρ^{AB}(VT99). Firstly, note that one can always make a trace-preserving LOCC operation out of a SLOCC operation, by preparing a separable state in the case of failure. The optimal LOCC protocol (see Eq. (3.34)) can be described as follows (VV03):
1. Alice and Bob perform a SLOCC protocol producing a state ρ_o^{AB}, with a probability of success p_o.
2. In the case of failure, Alice and Bob prepare a pure separable state $\left|0^A0^B\right\rangle\left\langle 0^A0^B\right|$.
Therefore,

$$\begin{aligned}\max_{\Lambda \in \mathrm{LOCC}} F\left(\Lambda(\rho^{AB}), \left|\varphi_+^{AB}\right\rangle\right) &= p_o \left\langle\varphi_+^{AB}\right| \rho_o^{AB} \left|\varphi_+^{AB}\right\rangle + \frac{1-p_o}{2}\left|\left\langle\varphi_+^{AB}|0^A0^B\right\rangle\right|^2 \\ &= \frac{1+K(\rho^{AB})}{2}.\end{aligned} \tag{3.35}$$

Let us now denote $F_o = F\left(\rho_o^{AB}, \left|\varphi_+^{AB}\right\rangle\right)$ and use the fact that $\left|\left\langle\varphi_+^{AB}|0^A0^B\right\rangle\right|^2 = \frac{1}{2}$. The above equation can be equivalently written as

$$p_o\left(F_o - \frac{1}{2}\right) + \frac{1}{2} = \frac{1 + K(\rho^{AB})}{2}. \tag{3.36}$$

This implies,
$$F_o = \frac{K(\rho)}{2p_o} + \frac{1}{2}. \tag{3.37}$$

If a SLOCC protocol has a success probability p, then the optimal achievable fidelity to singlet must satisfy the following inequality

$$F_p\left(\rho^{AB} \to \left|\varphi_+^{AB}\right\rangle\right) \le \min\left\{\frac{K(\rho^{AB})}{2p} + \frac{1}{2}, 1\right\}. \tag{3.38}$$

This is because, if a SLOCC protocol violates the above bound, then one can construct a LOCC protocol by preparing a (separable) state $\left|0^A0^B\right\rangle\!\left\langle 0^A0^B\right|$ in the case of failure. This LOCC protocol would violate the bound given in Eq. (3.34). Notice that for $p < 1$, the above bound (see Eq. (3.33) is tighter than the bound given in Eq. (3.29). □

3.3 Asymptotic bounds for general resource theories

In this section, we will study the asymptotic behaviour of the single copy bounds presented in theorem 3.1. We will show that these single copy bounds from theorem 3.1 imply upper bounds on the asymptotic rates of transformations. Let us first note that, the asymptotic rate for a transformation between ρ and σ can also be defined as

$$R(\rho \to \sigma) = \sup\{r : \lim_{n\to\infty} \inf_{\Lambda_f} \left\|\Lambda_f(\rho^{\otimes n}) - \sigma^{\otimes \lfloor rn \rfloor}\right\|_1 = 0\}. \tag{3.39}$$

Infimum in the above equation is over the set of deterministic free operations (Λ_f). Let us now generalise the above definition to the probabilistic case, allowing for a sub-exponential (in the number of copies) decay in the probability of success. Additionally, here we also allow for the generation of sub-exponential (in the number of copies of the state) amount of resource, quantified by M. In such a scenario, we can define asymptotic rates as follows

$$R_p^M(\rho \to \sigma) = \sup\{r : \lim_{n\to\infty} \inf_{\Lambda_{\varepsilon_n}} \left\| \frac{\Lambda_{\varepsilon_n}(\rho^{\otimes n})}{\operatorname{Tr} \Lambda_{\varepsilon_n}(\rho^{\otimes n})} - \sigma^{\otimes \lfloor rn \rfloor} \right\|_1 = 0\}. \tag{3.40}$$

Here, Λ_{ε_n} are ε_n-resource generating operations i.e, $M(\Lambda_{\varepsilon_n}(\rho_f)) \leq \varepsilon_n$ for all $\rho_f \in \mathcal{F}$, $\lim_{n\to\infty} \frac{\log \varepsilon_n}{n} = 0$ (sub-exponential resource generation) and $\lim_{n\to\infty} -\frac{\log \operatorname{Tr} \Lambda_{\varepsilon_n}(\rho^{\otimes n})}{n} = 0$ (sub-exponential decay of probability of success). We will now provide upper-bounds for these asymptotic rates for general resource theories.

Theorem 3.2. *(KDS22) For any quantum resource theory and any two states ρ and σ, such that $E_{1/2}(\sigma^{\otimes n}) = n \cdot E_{1/2}(\sigma)$, it holds that*

$$R_p^M(\rho \to \sigma) \leq \frac{E_{\max}(\rho)}{E_{1/2}(\sigma)}, \tag{3.41}$$

where M can be R (generalised robustness) or S (standard robustness).

Proof. When M is chosen to be generalised robustness (R) or standard robustness (S), the following inequality holds

$$F_{\max}^{M_\varepsilon}(\sigma) \leq F_{\max}(\sigma)(1 + \varepsilon). \tag{3.42}$$

This is because any state τ with robustness (generalised or standard) at most ε, can be expressed as follows

$$\frac{\tau + \varepsilon\tau'}{1 + \varepsilon} = \tau_{\text{free}} \tag{3.43}$$

When ε is standard robustness (S), τ' has to be a free state. Note that, τ' does not need to be a be free state if ε is generalised robustness. Now, using concavity of fidelity, one can show that

$$F(\sigma, \tau) \leq F(\sigma, \tau_{\text{free}})(1+\varepsilon) \leq F_{\max}(\sigma)(1+\varepsilon). \tag{3.44}$$

Eq. (3.42) follows from the fact that the above inequality holds for every τ such that Eq. (3.43) is true. Therefore the fidelity bound (from theorem 3.1) can be written as

$$F\left(\frac{\mathcal{E}(\rho)}{\operatorname{Tr}\mathcal{E}(\rho)}, \sigma\right) \leq (1+K(\rho))F_{\max}(\sigma)\frac{(1+\varepsilon)}{p}. \tag{3.45}$$

Here, $p = \operatorname{Tr}\mathcal{E}(\rho)$. Let us now use the following definitions to rewrite Eq. (3.45)

$$E_{\max}(\rho) = \min_{\rho_f} D_{\max}(\rho, \rho_f), \;\; E_{\max}(\rho) = \log_2(1+K(\rho)) \text{ and } E_{1/2}(\sigma) = -\log_2 F_{\max}(\sigma). \tag{3.46}$$

Rewriting Eq. (3.45), gives us

$$F\left(\frac{\mathcal{E}(\rho)}{\operatorname{Tr}\mathcal{E}(\rho)}, \sigma\right) \leq 2^{(E_{\max}(\rho)-E_{1/2}(\sigma))}\frac{(1+\varepsilon)}{p}. \tag{3.47}$$

Note that we assume

$$E_{1/2}(\sigma^{\otimes n}) = nE_{1/2}(\sigma). \tag{3.48}$$

Also note that $E_{\max}$ is sub-additive (Dat09) i.e,

$$E_{\max}(\rho^{\otimes n}) \leq nE_{\max}(\rho) \tag{3.49}$$

Therefore, from Eq. (3.47, 3.48 and 3.49) it follows that

$$F\left(\frac{\mathcal{E}(\rho^{\otimes n})}{\operatorname{Tr}\mathcal{E}(\rho^{\otimes n})}, \sigma^{\lfloor rn \rfloor}\right) \leq 2^{(nE_{\max}(\rho)-\lfloor rn \rfloor E_{1/2}(\sigma))}\frac{(1+\varepsilon_n)}{p_n} \tag{3.50}$$

$$\leq 2^{(nE_{\max}(\rho)-(rn-1)E_{1/2}(\sigma))}\frac{(1+\varepsilon_n)}{p_n} \tag{3.51}$$

$$= 2^{(nE_{\max}(\rho)-rnE_{1/2}(\sigma)+E_{1/2}(\sigma))}\frac{(1+\varepsilon_n)}{p_n} \tag{3.52}$$

Here $p_n = \operatorname{Tr}\mathcal{E}(\rho^{\otimes n})$. Second inequality is due to the fact that $\lfloor x \rfloor \geq x-1$. We will now choose $r = \frac{E_{\max}(\rho)}{E_{1/2}(\sigma)} + \delta$, where $\delta > 0$.

$$F\left(\frac{\mathcal{E}(\rho^{\otimes n})}{\operatorname{Tr}\mathcal{E}(\rho^{\otimes n})}, \sigma^{\lfloor rn \rfloor}\right) \leq 2^{(1-\delta\cdot n)E_{1/2}(\sigma)}\frac{(1+\varepsilon_n)}{p_n} \tag{3.53}$$

For every $\delta > 0$, $2^{(1-\delta\cdot n)E_{1/2}(\sigma)}$ goes to zero exponentially in n. Therefore (assuming $\delta > 0$), whenever ε_n does not increase exponentially and p_n does not decay exponentially, we can choose a large enough n such that the upper bound on the fidelity can be made arbitrarily small. This completes the proof. □

In fact, whenever we try to achieve any rate $r > \frac{E_{\max}(\rho)}{E_{1/2}(\sigma)}$, the fidelity of transformation exponentially goes down with n (this can be seen from Eq. (3.53)). Also note that the assumption $E_{1/2}(\sigma^{\otimes n}) = n\cdot E_{1/2}(\sigma)$ (additivity of $E_{1/2}$) is true for many classes of states, in various quantum resource theories. For example, all monotones E_α are additive in the resource theory of coherence (ZHC17) and in the resource theory of entanglement, additivity of E_α holds for all bipartite pure, GHZ, maximally correlated, Bell diagonal, isotropic, and generalized Dicke states (RT22b).

3.4 Resource theory of bipartite entanglement

3.4.1 Pure state transformations

The definition F_p and P_f in Eqs. (1.12) and (1.13) might suggest that an analytical expression for these quantities is out of reach in many settings. In this section, we will see that analytical expressions are indeed achievable in various interesting setups. Previously in the literature, bipartite state transformations (via SLOCC) for deterministic approximate (see Eq. (1.7)) (VJN00b) and probabilistic exact (see Eq. (1.11)) (Vid99b) settings have been presented. However, one can go beyond these settings and consider an interplay between achievable fidelity and probability. In the theorem below, we provide a complete solution for single-copy transformations between bipartite pure states. Before going into more details, let us first note that for every bipartite pure state $\left|\psi^{AB}\right\rangle$ (A and B are Alice's and Bob's systems respectively), there exist *Schmidt coefficients* ($\{\sqrt{\alpha_i}\}$) and orthonormal vectors $\{\left|\nu_i^A\right\rangle\}$ and $\{\left|\mu_i^B\right\rangle\}$ such that

$$\left|\psi^{AB}\right\rangle = \sum_i^{\mathrm{Sch}(\psi)} \sqrt{\alpha_i} \left|\nu_i^A\right\rangle \otimes \left|\mu_i^B\right\rangle. \tag{3.54}$$

Here, $\alpha_i > 0$ and $\mathrm{Sch}(\psi)$ is the *Schmidt number* of $\left|\psi^{AB}\right\rangle$. Without loss of generality, α_i's are assumed to ordered decreasingly. Let us consider another bipartite pure state $\left|\varphi^{A'B'}\right\rangle$ with Schmidt coefficients $\{\sqrt{\beta_i}\}$ and Schmidt number $\mathrm{Sch}(\varphi)$. We will now define $n = \max\{\mathrm{Sch}(\psi), \mathrm{Sch}(\varphi)\}$. $\{\alpha_i\}$ and $\{\beta_i\}$ (squared Schmidt coefficients in decreasing order) can both be treated as n-dimensional column vectors, by adding $n - \mathrm{Sch}(\psi)$ zeros to $\{\alpha_i\}$ if $\mathrm{Sch}(\psi) < n$ and by adding $n - \mathrm{Sch}(\varphi)$ zeros to $\{\beta_i\}$ if $\mathrm{Sch}(\varphi) < n$. For brevity, from now on we drop the labels of the systems of Alice and Bob, for example we will use $|\psi\rangle$ instead of $\left|\psi^{AB}\right\rangle$.

Theorem 3.3. *(KDS22) A bipartite pure state $|\psi\rangle$ can be converted into another state $|\varphi\rangle$ via SLOCC with a probability p and an optimal fidelity given by*

$$F_p(|\psi\rangle \to |\varphi\rangle) = \min_{l\in\{2,\cdots,n\}} \left\{1 - 4\left(E_l^{\varphi} - \frac{E_l^{\psi}}{p}\right)^2\right\}, \tag{3.55}$$

where $E_l^{\psi} = \sum_{i=l}^n \alpha_i$, $E_l^{\varphi} = \sum_{i=l}^n \beta_i$ and $\{\alpha_i\}$, $\{\beta_i\}$ are the squared Schmidt coefficients of $|\psi\rangle$ and $|\varphi\rangle$ in decreasing order.

Proof. Let us denote the set of states which can be achieved from $|\psi\rangle$ with at least a probability p as S_p. Therefore the following holds

$$F_p(|\psi\rangle \to |\varphi\rangle) = \max_{\rho\in S_p} \langle\varphi|\rho|\varphi\rangle. \tag{3.56}$$

We will now prove that the closest state (optimal fidelity) to $|\varphi\rangle$ in S_p can always be chosen to be a pure state. Let us first define

$$E_l^{\psi} = \sum_{i=l}^n \alpha_i \text{ and } E_l^{\varphi} = \sum_{i=l}^n \beta_i. \tag{3.57}$$

Here, $\{\alpha_i\}$, $\{\beta_i\}$ are the squared Schmidt coefficients of $|\psi\rangle$ and $|\varphi\rangle$ in decreasing order. $|\psi\rangle$ can be transformed into an ensemble of quantum states $\{q_j, \rho_j\}$ via LOCC if and only if there exists a pure state ensemble $\{q_k r_{k,j}, \left|\psi_{k,j}\right\rangle\}$, satisfying (VJN00b)

$$E_l^{\psi} \geq \sum_{k,j} q_k r_{k,j} E_l^{\psi_{k,j}} \text{ for all } l, \tag{3.58}$$

where $\rho_k = \sum_j r_{k,j} \left|\psi_{k,j}\right\rangle \left\langle\psi_{k,j}\right|$. Therefore, $|\psi\rangle$ can be transformed into ρ with a probability p iff the following inequality is satisfied for at least one pure state decomposition $\{p_k, |\mu_k\rangle\}$ of the state ρ:

$$E_l^{\psi} \geq p \sum_k p_k E_l^{\mu_k} \text{ for all } l. \tag{3.59}$$

Let the squared Schmidt coefficients (in decreasing order) of $|\mu_k\rangle$ be $\{\gamma_i^k\}$. We will prove that

$$f = \langle\varphi|\rho|\varphi\rangle \leq \sum_k p_k \left(\sum_{i=1}^{n} \sqrt{\gamma_i^k \beta_i}\right)^2 \leq \left(\sum_{i=1}^{n} \sqrt{\sum_k p_k \gamma_i^k \beta_i}\right)^2. \tag{3.60}$$

The first inequality follows from the fact that the fidelity between any two pure states (with fixed Schmidt coefficients) is maximum when they have the same Schmidt basis (VJN00b). Second inequality follows from the concavity of fidelity. In order to see this, we define diagonal matrices τ_φ, τ_{μ_k} and τ_χ, with diagonal elements as the squares of ordered Schmidt coefficients of $|\varphi\rangle$, $|\mu_k\rangle$ and $|\chi\rangle$ (will be defined below) respectively. We define τ_χ as

$$\tau_\chi = \sum_k p_k \tau_{\mu_k}. \tag{3.61}$$

Using concavity of fidelity,

$$\sum_k p_k F(\tau_{\mu_k}, \tau_\varphi) \leq F(\tau_\chi, \tau_\varphi). \tag{3.62}$$

We define a pure state $|\chi\rangle$, whose Schmidt coefficients are square-roots of the diagonal elements of τ_χ

$$|\chi\rangle = \sum_{i=1}^{n} \sqrt{\sum_k p_k \gamma_i^k}\, |i_A i_B\rangle. \tag{3.63}$$

Note that, $|i_A i_B\rangle$ are the same Schmidt basis as $|\varphi\rangle$. Therefore, one can see that

$$E_l^{\chi} = \sum_{i=l}^{n} \left(\sum_k p_k \gamma_i^k\right) = \sum_k p_k \sum_{i=l}^{n} \gamma_i^k = \sum_k p_k E_l^{\mu_k}. \tag{3.64}$$

Computing the inner product of $|\chi\rangle$ and $|\varphi\rangle$ we get,

$$|\langle\chi|\varphi\rangle|^2 = \left(\sum_{i=1}^{n} \sqrt{\sum_k p_k \gamma_i^k \beta_i}\right)^2. \tag{3.65}$$

The above quantity coincides with the upper bound on f (see Eq. (3.60)). From Eq. (3.59) and Eq. (3.64), it follows that

$$E_l^{\psi} \geq p \sum_i p_i E_l^{\mu_i} = p E_l^{\chi} \text{ for all } l. \tag{3.66}$$

This is because the Schmidt coefficients of $|\chi\rangle$ are square-roots of the diagonal elements of τ_χ. Therefore, from (Vid99b) we know that $|\psi\rangle$ can be transformed into $|\chi\rangle$ with a probability p via SLOCC. This proves that the closest state to $|\varphi\rangle$ can always be chosen to be pure. Below, we construct a pure state (in S_p) which optimises the fidelity with $|\varphi\rangle$.

Let us note that, a bipartite pure state $|\psi\rangle$ can be transformed into $|\varphi\rangle$ via LOCC iff the reduced states ψ^A and φ^A satisfy the following majorisation condition ($\psi^A \prec \varphi^A$)(Nie99).

$$\sum_{i=0}^{n} \alpha_i \leq \sum_{i=0}^{n} \beta_i \text{ for all } 0 \leq n \leq d-1, \tag{3.67}$$

Here, $\{\alpha_i\}$ and $\{\beta_i\}$ are the squared Schmidt coefficients (in decreasing order) of $|\psi\rangle$ and $|\varphi\rangle$ respectively and $d = \min\{d_A, d_B\}$, with d_A and d_B being the dimensions of the Hilbert spaces of Alice and Bob, respectively. For a probability distribution $\vec{\beta}$ (decreasing order and dimension n), one can define steepest δ-approximation of $\vec{\beta}$ as

$$\vec{\beta^\delta} = \begin{cases} \beta_1 + \frac{\delta}{2} & \text{for } \quad i = 1 \\ \beta_i & \text{for } \quad 1 < i < l^* + 1 \\ 1 - x & \text{for } \quad i = l^* + 1 \\ 0 & \text{for } \quad i > l^* + 1 \end{cases} \tag{3.68}$$

with $x = \beta_1 + \frac{\delta}{2} + \sum_{i=2}^{l^*} \beta_i$. From (HOS18), we know that $\vec{\beta^\delta}$ majorizes every other $\vec{\beta'}$ whenever $\left\|\vec{\beta} - \vec{\beta'}\right\|_1 \leq \delta > 0$ (δ-ball). Note that whenever $(1, 0, \ldots, 0)$ lies inside the δ-ball, $\vec{\beta^\delta} = (1, 0, \ldots, 0)$.

Using this along with Eq. (3.67), one can construct a pure state $\left|\varphi_{st}^{\delta}\right\rangle$ from $|\varphi\rangle$ as

$$\left|\varphi_{st}^{\delta}\right\rangle = \sum_i^n \sqrt{\beta_i^\delta}\, |i_A i_B\rangle, \tag{3.69}$$

where, $\delta = \sqrt{1 - f_1}$. By construction, the fidelity between $\left|\varphi_{st}^{\delta}\right\rangle$ and $|\varphi\rangle$ is f_1 and every pure state which has a fidelity at least f_1 (with $|\varphi\rangle$) can be converted (deterministically) to $\left|\varphi_{st}^{\delta}\right\rangle$ via LOCC. This implies, $F_p(|\psi\rangle \to |\varphi\rangle)$ equals to the maximum value of f_1, such that $\left|\varphi_{st}^{\sqrt{1-f_1}}\right\rangle$ can be achieved from $|\psi\rangle$ with a probability p. Equivalently, $F_p(|\psi\rangle \to |\varphi\rangle)$ is the maximum value of f_1 satisfying (Vid99b),

$$p \cdot \max\left\{\sum_{i=l}^{n} \beta_i - \frac{\sqrt{1-f_1}}{2}, 0\right\} \leq \sum_{i=l}^{n} \alpha_i \text{ for } l : 2, \cdots, n. \tag{3.70}$$

We can drop the “max” from the above inequality because α_i's are all non-negative. Therefore, we have

$$p(E_l^{\varphi} - \frac{\sqrt{1-f_1}}{2}) \leq E_l^{\psi} \text{ for } l : 2, \cdots, n. \tag{3.71}$$

Solving the above equation gives

$$f_1 \leq 1 - 4\left(E_l^{\varphi} - \frac{E_l^{\psi}}{p}\right)^2 \text{ for } l : 2, \cdots, n. \tag{3.72}$$

This implies

$$F_p(|\psi\rangle \to |\varphi\rangle) = \min_l \left\{1 - 4\left(E_l^{\varphi} - \frac{E_l^{\psi}}{p}\right)^2\right\} \text{ for } l : 2, \cdots, n. \tag{3.73}$$

Using the results from (Vid99b), one can also provide an expression for $P_f(|\psi\rangle \to |\varphi\rangle)$. In this case, the optimal transformation can be achieved by transforming $|\psi\rangle$ into $\left|\varphi_{st}^{\sqrt{1-f}}\right\rangle$.

$$P_f(|\psi\rangle \to |\varphi\rangle) = P\left(|\psi\rangle \to \left|\varphi_{st}^{\sqrt{1-f}}\right\rangle\right) = \min\left\{1, \min_l \left\{\frac{E_l^{\psi}}{\max\left\{E_l^{\varphi} - \frac{\sqrt{1-f}}{2}, 0\right\}}\right\}\right\}, \tag{3.74}$$

where l ranges from 2 to n. □

Theorem 3.3 provides a complete solution for the stochastic approximate transformations for bipartite pure states. As cases of special interest, by considering $p = 1$ in Eq. (3.55) one gets back the known result for deterministic approximate transformations given in (VJN00b) and by putting $f = 1$ in Eq. (3.74), we get back the result about probabilistic exact transformations given in (Vid99b) respectively.

3.4.2 Entangled states of two qubits

In this subsection, we consider a setting where the initial state is a pure bipartite state, but the final state is a two qubit mixed state. Even in this scenario, one can obtain analytical expressions for F_p and P_f. Before, we present the main theorem, we will now present tight continuity bounds for geometric measure of entanglement for two qubits. This will be used later in the derivation of F_p and P_f. Note that the geometric measure of entanglement G of a state ρ is given by

$$G(\rho) = 1 - \max_{\sigma \in \mathcal{F}_s} F(\rho, \sigma). \tag{3.75}$$

Here, $\mathcal{F}_s$ is the set of bipartite separable states. For a bipartite pure state $|\psi\rangle$,

$$G(|\psi\rangle) = \lambda_0, \tag{3.76}$$

where λ_0 is the square of the largest Schmidt coefficient of $|\psi\rangle$. This can be extended to any mixed state ρ in the following way (SKB10)

$$G(\rho) = \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i G(|\varphi_i\rangle\langle\varphi_i|). \tag{3.77}$$

Here the minimisation is performed over all the pure state decompositions of ρ. With this information, we present the continuity bounds for geometric measure of entanglement.

Lemma 3.1. *(KDS22) For a two-qubit state ρ consider a set of states $S_{\rho,f}$ such that $F(\rho,\rho') \geq f$ for all $\rho' \in S_{\rho,f}$. The minimal geometric entanglement in $S_{\rho,f}$ is given by*

$$\min_{\rho'\in S_{\rho,f}} G(\rho') = \sin^2\left(\max\left\{\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f}, 0\right\}\right). \tag{3.78}$$

For pure two-qubit states $|\psi\rangle$, the maximal geometric entanglement in $S_{\psi,f}$ is given by

$$\max_{\rho'\in S_{\psi,f}} G(\rho') = \sin^2\left(\min\left\{\sin^{-1}\sqrt{G(\psi)} + \cos^{-1}\sqrt{f}, \frac{\pi}{4}\right\}\right), \tag{3.79}$$

where $\psi = |\psi\rangle\langle\psi|$ denotes a projector onto a pure states $|\psi\rangle$.

Proof. In this proof we will make sure of the following distance measure called Bures angle

$$D(\rho,\sigma) = \cos^{-1}\left(\sqrt{F(\rho,\sigma)}\right), \tag{3.80}$$

It is important to note that Bures angle satisfies the triangle inequality (KW20). For any two-qubit state, the geometric measure of entanglement G is between 0 and 1/2. Therefore, for any two states ρ and $\rho' \in S_{\rho,f}$ it is possible to introduce $\{\alpha,\beta\} \in [0,\pi/4]$ and $k \in [0,\pi/2]$ such that the following equations hold.

$$G(\rho) = \sin^2\alpha, \tag{3.81a}$$

$$G(\rho') = \sin^2\beta, \tag{3.81b}$$

$$f = \cos^2 k. \tag{3.81c}$$

Let us assume ρ_s is the closest separable state to ρ and ρ_s' is the closest separable state to ρ' with respect to the Bures angle. Therefore, from Eq. (3.81a) one can see that

$$D(\rho,\rho_s) = \cos^{-1}\left(\sqrt{F(\rho,\rho_s)}\right) = \cos^{-1}\left(\sqrt{1-G(\rho)}\right) = \alpha. \tag{3.82}$$

Similarly,

$$D(\rho',\rho_s') = \beta, \tag{3.83}$$

$$D(\rho,\rho') \leq k. \tag{3.84}$$

Last inequality follows from the fact that $\rho' \in S_{\rho,f}$. We now use the triangle inequality and the fact that the closest separable state to ρ is ρ_s

$$\alpha = D(\rho,\rho_s) \leq D(\rho,\rho_s') \leq D(\rho,\rho') + D(\rho',\rho_s') \leq k + \beta. \tag{3.85}$$

This implies,

$$\beta \geq \max\{\alpha - k, 0\}. \tag{3.86}$$

Using Eqs. (3.81b) and (4.140), one can give a lower bound on $G(\rho')$ as

$$G(\rho') \geq \sin^2(\max\{\alpha - k, 0\}). \tag{3.87}$$

We will now find a lower bound to $G(\rho')$. Again using the triangle inequality and the fact the the closest separable state to ρ' is ρ'_s, we get

$$\begin{aligned}\beta = D(\rho', \rho'_s) \leq D(\rho', \rho_s) &\leq D(\rho, \rho') + D(\rho, \rho_s) \\ &\leq k + \alpha.\end{aligned} \tag{3.88}$$

This implies

$$\beta \leq \min\{\alpha + k, \pi/4\}. \tag{3.89}$$

Using Eqs. (3.81b) and (3.89), the upper bound on $G(\rho')$ can be give as follows

$$G(\rho') \leq \sin^2(\min\{\alpha + k, \pi/4\}). \tag{3.90}$$

We will now show that the lower bound in Eq. (3.87) is achievable. Firstly, let us note that for every two-qubit state ρ, there exists a pure state decomposition such that $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ and $G(\psi_i) = G(\rho)$ for all i (Woo98a; Vid00b; WG03). This implies that each of the states $|\psi_i\rangle$ from the pure state decomposition can be written as

$$|\psi_i\rangle \quad = \cos\alpha\, |a_i\rangle\, |b_i\rangle + \sin\alpha \left|a_i^\perp\right\rangle \left|b_i^\perp\right\rangle, \tag{3.91}$$

where $\left\langle a_i | a_i^\perp \right\rangle = \left\langle b_i | b_i^\perp \right\rangle = 0$. We will now define $\rho_{\min}$ as

$$\rho_{\min} = \sum_i q_i\, |\varphi_i\rangle\langle\varphi_i|\,. \tag{3.92}$$

with

$$|\varphi_i\rangle \quad = \cos\tilde{\beta}\, |a_i\rangle\, |b_i\rangle + \sin\tilde{\beta} \left|a_i^\perp\right\rangle \left|b_i^\perp\right\rangle. \tag{3.93}$$

Here, $\tilde{\beta} \in [0, \pi/4]$ and

$$q_i = \frac{p_i |\langle\psi_i|\varphi_i\rangle|^2}{\sum_k p_k |\langle\psi_k|\varphi_k\rangle|^2}. \tag{3.94}$$

Since geometric measure of entanglement is convex,

$$G(\rho_{\min}) \leq \sin^2\tilde{\beta}. \tag{3.95}$$

Using joint concavity of root-fidelity (Wil17), we obtain

$$\begin{aligned}\sqrt{F(\rho, \rho')} &\geq \sum_i \sqrt{p_i q_i} |\langle\psi_i|\varphi_i\rangle| \\ &= \sqrt{\sum_i p_i |\langle\psi_i|\varphi_i\rangle|^2} = |\cos(\alpha - \tilde{\beta})|.\end{aligned} \tag{3.96}$$

Therefore,

$$F(\rho, \rho_{\min}) \geq \cos^2(\alpha - \tilde{\beta}). \tag{3.97}$$

Let us now choose, $\tilde{\beta} = \max\{\alpha - k, 0\}$. For this choice Eq. (3.97) becomes

$$F(\rho, \rho_{\min}) \geq \cos^2(\min\{k, \alpha\}) \geq \cos^2 k. \tag{3.98}$$

The last inequality follows from the fact that, $\min\{k, \alpha\} \in [0, \pi/4]$ and $\cos^2$ is a monotonically decreasing function that interval. This shows that $\rho_{\min} \in S_{\rho,f}$. From Eq. (3.95) we obtain

$$G(\rho_{\min}) \leq \sin^2(\max\{\alpha - k, 0\}). \tag{3.99}$$

Since $\rho_{\min} \in S_{\rho,f}$, from Eq. (3.99) and Eq. (3.87) it follows that

$$G(\rho_{\min}) = \sin^2(\max\{\alpha - k, 0\}). \tag{3.100}$$

This shows that the lower bound in Eq. (3.87) is achievable. Hence, the minimum geometric measure within the set $S_{\rho,f}$ is given by

$$\begin{aligned} \min_{\rho' \in S_{\rho,f}} G(\rho') &= \sin^2(\max\{\alpha - k, 0\}) \qquad (3.101) \\ &= \sin^2\left(\max\left\{\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f}, 0\right\}\right). \end{aligned}$$

Let us now focus on the case where ρ is a pure state, i.e.,

$$|\psi\rangle = \cos\alpha\,|a\rangle\,|b\rangle + \sin\alpha\,\left|a^{\perp}\right\rangle\left|b^{\perp}\right\rangle. \tag{3.102}$$

We will now show that the upper bound given in Eq. (3.90) is achievable as well. Choose

$$\begin{aligned} |\psi_{\max}\rangle = &\cos(\min\{\alpha + k, \pi/4\})\,|a\rangle\,|b\rangle \qquad (3.103) \\ &+ \sin(\min\{\alpha + k, \pi/4\})\left|a^{\perp}\right\rangle\left|b^{\perp}\right\rangle. \end{aligned}$$

Note that

$$G(\psi_{\max}) = \sin^2(\min\{\alpha + k, \pi/4\}). \tag{3.104}$$

Also note that

$$F(\psi, \psi_{\max}) = \cos^2\left(\min\left\{k, \frac{\pi}{4} - \alpha\right\}\right) \geq \cos^2 k. \tag{3.105}$$

The above inequality holds because $\min\{k, \pi/4 - \alpha\} \in [0, \pi/4]$ and $\cos^2$ is a monotonically decreasing function in this interval. Therefore, $|\psi_{\max}\rangle \in S_{\psi,f}$ and has the maximum possible geometric entanglement

$$\begin{aligned} \max_{\rho' \in S_{\psi,f}} G(\rho') &= \sin^2(\min\{\alpha + k, \pi/4\}) \qquad (3.106) \\ &= \sin^2\left(\min\left\{\sin^{-1}\sqrt{G(\psi)} + \cos^{-1}\sqrt{f}, \pi/4\right\}\right). \end{aligned}$$

This completes the proof. □

Using lemma 3.1 we will now provide a complete solution for the stochastic approximate state conversion for two-qubit systems if the initial state is pure.

Theorem 3.4. *(KDS22) A pure state $|\psi\rangle$ can be transformed into a two-qubit state ρ via SLOCC with a fidelity f and a maximal probability given by*

$$P_f(|\psi\rangle \to \rho) = \begin{cases} 1 \;\text{ for }\; m_1 \geq 0 \\ \frac{G(\psi)}{\sin^2\left(\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f}\right)} \;\text{ otherwise,} \end{cases} \tag{3.107}$$

where $m_1 = \sin^{-1}\sqrt{G(\psi)} - \sin^{-1}\sqrt{G(\rho)} + \cos^{-1}\sqrt{f}$ and $\psi = |\psi\rangle\langle\psi|$ denotes a projector onto a pure states $|\psi\rangle$.

Proof. From (Vid00b), we know that the optimal probability of transforming $|\psi\rangle$ into ρ via SLOCC is given by

$$P(|\psi\rangle \to \rho) = \min\{\frac{G(\psi)}{G(\rho)}, 1\}. \tag{3.108}$$

Here, G is the geometric measure of entanglement. From Eq. (3.108) and the definitions of P_f and $S_{\rho,f}$ it follows that

$$P_f(|\psi\rangle \to \rho) = \min\{\frac{G(\psi)}{\min_{\rho' \in S_{\rho,f}} G(\rho')}, 1\}. \tag{3.109}$$

From lemma 3.1, we know that

$$\min_{\rho' \in S_{\rho,f}} G(\rho') = \sin^2\left(\max\left\{\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f}, 0\right\}\right). \tag{3.110}$$

Let us consider,

$$m_1 = \sin^{-1}\sqrt{G(\psi)} - \sin^{-1}\sqrt{G(\rho)} + \cos^{-1}\sqrt{f}. \tag{3.111}$$

Therefore $m_1 \geq 0$ is equivalent to

$$\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f} \leq \sin^{-1}\sqrt{G(\psi)}. \tag{3.112}$$

It holds that

$$\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f} \in [-\pi/2, \pi/4] \text{ and } \sin^{-1}\sqrt{G(\psi)} \in [0, \pi/4]. \tag{3.113}$$

Therefore, we have

$$\max\left\{\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f}, 0\right\} \leq \sin^{-1}\sqrt{G(\psi)}. \tag{3.114}$$

Using the above inequality, we obtain

$$\begin{aligned}\min_{\rho' \in S_{\rho,f}} G(\rho') &= \sin^2\left(\max\left\{\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f}, 0\right\}\right) \qquad (3.115)\\ &\leq \sin^2(\sin^{-1}\sqrt{G(\psi)}) = G(\psi).\end{aligned}$$

Therefore, whenever $G(\psi) > 0$, the following holds

$$\frac{G(\psi)}{\min_{\rho' \in S_{\rho,f}} G(\rho')} \geq 1. \tag{3.116}$$

This implies if $m_1 \geq 0$ then $P_f(|\psi\rangle \to \rho) = 1$. Now we consider the case $m_1 < 0$, equivalently expressed as

$$\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f} > \sin^{-1}\sqrt{G(\psi)} > 0. \tag{3.117}$$

Therefore, for this case ($m_1 < 0$) we have

$$P_f(|\psi\rangle \to \rho) = \frac{G(\psi)}{\sin^2(\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f})}. \tag{3.118}$$

This completes the proof. □

Theorem 3.4 provides a closed expression for P_f. In fact one can also obtain a closed expression for F_p. Note that, when $p \leq \frac{G(\psi)}{G(\rho)} < 1$, Eq. (3.108 implies $F_p(\psi \to \rho) = 1$. For the case when $1 \geq p > \frac{G(\psi)}{G(\rho)}$, the optimal fidelity can be obtained by solving Eq. (3.118) for f, which gives

$$F_p(|\psi\rangle \to \rho) = \cos^2\left[\sin^{-1}\sqrt{G(\rho)} - \sin^{-1}\sqrt{\frac{G(\psi)}{p}}\right]. \tag{3.119}$$

3.5 Resource theory of imaginarity

We will now use the techniques developed in proving theorem 3.4, to provide analytical expressions P_f and F_p in the resource theory of imaginarity (HG18), when the initial state is pure. Before we go into further details, let us first present the motivation behind imaginarity as a resource, then we will introduce the resource theory of imaginarity along with some background about relevant imaginarity measures and state transformations.

Complex numbers are widely used in various branches of classical physics, to allow for an elegant mathematical formulation of various phenomena. In fact, one can mathematically describe these phenomena just by using real numbers. Therefore, complex numbers are just used for the sake of mathematical simplicity and elegance, and are not necessary. Since the inception of quantum mechanics, complex numbers have also played an indispensable role in describing the properties of quantum systems. One can then ask the question, is the use of complex numbers necessary to describe quantum physics?

One way to approach this question is to use the standard laws of quantum physics, but to restrict all the states and measurement operators to be real matrices. Then one can aim to find physical phenomena which are possible in standard quantum physics but impossible in this restricted version of (real) quantum physics. Thus showing the necessity of complex numbers in the standard laws of quantum physics. We will describe one such phenomenon in the setting of *local state discrimination* (BCCL80; Woo90). Let us consider the following pair of bipartite real quantum states shared between Alice and Bob (WKR+21a)

$$\begin{aligned} \rho_1^{AB} &= \frac{1}{2}\left(|\varphi_-\rangle\langle\varphi_-| + |\psi_+\rangle\langle\psi_+|\right), \\ \rho_2^{AB} &= \frac{1}{2}\left(|\varphi_+\rangle\langle\varphi_+| + |\psi_-\rangle\langle\psi_-|\right), \end{aligned} \tag{3.120}$$

where $|\varphi_\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$, and $|\psi_\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$. Alice and Bob can perfectly distinguish these two states via LOCC. In order to see this, we rewrite the above states as follows:

$$\begin{aligned} \rho_1^{AB} &= \frac{1}{4}(\mathbb{I}_4 + \sigma_y \otimes \sigma_y), \\ \rho_2^{AB} &= \frac{1}{4}(\mathbb{I}_4 - \sigma_y \otimes \sigma_y). \end{aligned} \tag{3.121}$$

Here, $\mathbb{I}_4$ is an identity matrix of dimension 4 and σ_y is the Pauli-y matrix. Let us now consider a two outcome measurement with Kraus operators (each corresponding to an outcome) given by

$$\begin{aligned} M_1 &= |\hat{+}\rangle\langle\hat{+}| \otimes |\hat{+}\rangle\langle\hat{+}| + |\hat{-}\rangle\langle\hat{-}| \otimes |\hat{-}\rangle\langle\hat{-}|, \\ M_2 &= |\hat{+}\rangle\langle\hat{+}| \otimes |\hat{-}\rangle\langle\hat{-}| + |\hat{-}\rangle\langle\hat{-}| \otimes |\hat{+}\rangle\langle\hat{+}|. \end{aligned} \tag{3.122}$$

where $|\hat{+}\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$ and $|\hat{-}\rangle = (|0\rangle - i|1\rangle)/\sqrt{2}$. These Kraus operators can be implemented by an LOCC protocol as follows: Both Alice and Bob perform a local measurement in the $\{|\hat{+}\rangle, |\hat{-}\rangle\}$ basis and Alice shares her measurement outcome with Bob. If both their outcomes are correlated, then their shared state is ρ_1^{AB}. On the other hand, if their outcomes are anti-correlated, then their shared state is ρ_2^{AB}. This shows that the quantum states from Eq. (3.120) can be perfectly discriminated via LOCC. Now let us note that, $\mathrm{Tr}[S\sigma_y] = 0$ for any real symmetric 2×2 matrix S. This implies, for any POVM element $M_j = \sum_k A_{j,k} \otimes B_{j,k}$ such that $A_{j,k}$ are real symmetric matrices, the following holds

$$\mathrm{Tr}(M_j\rho_1^{AB}) = \mathrm{Tr}(M_j\rho_2^{AB}) = \frac{1}{4}\,\mathrm{Tr}(M_j). \tag{3.123}$$

Therefore, the bipartite states from Eqs (3.120) are indistinguishable when Alice and Bob are restricted to real quantum operations. This shows the necessity of complex numbers in local state discrimination and motivates the role of *imaginarity* as a resource (overcoming the limitations of real quantum mechanics). With this motivation, we will now present the basics of the resource theory of imaginarity.

In the resource theory of imaginarity (HG18), one defines the set of free states (*real states* $\mathcal{F}_r$) as quantum states with real matrix elements (in a reference basis $\{|m\rangle\}$)

$$\mathcal{F}_r = \{\rho : \langle m|\rho|n\rangle \in \mathbb{R}\}. \tag{3.124}$$

Note that a quantum state which is real in the reference basis is not necessarily real with respect to another basis. Therefore, similar to resource theory of coherence (SAP17), resource theory of imaginarity is a basis dependent theory. We consider the set of *real* *operations* as the set of free operations in this theory. A quantum operation Λ is said to be a real operation if it can be represented in the following way (HG18)

$$\Lambda[\rho] = \sum_i K_i \rho K_i^\dagger \ \forall \rho \in \mathcal{D}, \tag{3.125}$$

where K_i's (kraus operators) are real matrices (i.e, $\langle m|K_j|n\rangle \in \mathbb{R}$). In order to make sure that Λ is trace preserving, we additionally demand the following condition

$$\sum_i K_i^\dagger K_i = \mathbb{I}. \tag{3.126}$$

In (HG18), the authors show that the set of real operations coincides with the set of *completely non-imaginarity creating operations*. We say a quantum operation Λ^S is completely non-imaginarity creating if the following holds for every real state $\rho^{S'S}$

$$\mathbb{I}^{S'} \otimes \Lambda^S[\rho^{S'S}] \in \mathcal{F}_r. \tag{3.127}$$

Here, $\rho^{S'S}$ is a state of the system $S' \otimes S$ and $\mathbb{I}^{S'}$ is an identity map acting on the system S'. Moreover, in (HG18), the authors also show that a quantum operation Λ^S is a real operation if and only if it has a real dilation. Note that, a quantum operation Λ^S is said to have a real dilation if there exists a real orthogonal matrix O and a real state $\rho^{S'}$ such that

$$\Lambda^S(\cdot) = \mathrm{Tr}_{S'}[O\left(\cdot \otimes \rho^{S'}\right)O^T]. \tag{3.128}$$

Here, T denotes the transpose operator. One can define stochastic version of real operations by relaxing the condition in Eq.(3.126), in the following way

$$\sum_i K_i^\dagger K_i \leq \mathbb{I}. \tag{3.129}$$

Let us now define

$$L_0 = \sqrt{\mathbb{I} - \sum_i K_i^\dagger K_i}. \tag{3.130}$$

Since $K_i^\dagger$'s are real matrices, it follows that L_0 is also a real matrix satisfying

$$L_0^\dagger L_0 + \sum_i K_i^\dagger K_i = \mathbb{I}. \tag{3.131}$$

This shows that every set of incomplete real kraus operators (K_i) can be completed with real kraus operators (L_0).

One of the counter-intuitive features of the resource of imaginarity is existence of a *infinite resource state* $\left(\left|\hat{+}\right\rangle\right)$. Here, $\left|\hat{+}\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ is also called an *im-bit*. With one im-bit at hand, one can simulate arbitrary quantum operations or measurements via real operations. Let's say we wish to implement a quantum operation Λ (of dimension d) with Kraus operators $\{K_j\}$, satisfying $\sum_j K_j^\dagger K_j = P \leq \mathbb{I}_d$. Here, $\mathbb{I}_d$ is the identity matrix of dimension d. Let us now construct a real operation (Λ_r) with Kraus operators given by $\{K_j \otimes |\hat{+}\rangle\langle\hat{+}| + K_j^* \otimes |\hat{-}\rangle\langle\hat{-}|\}$. Here, $|\hat{-}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$. One can see that

$$\Lambda_r\left(\rho \otimes |\hat{+}\rangle\langle\hat{+}|\right) = \Lambda(\rho) \otimes |\hat{+}\rangle\langle\hat{+}| \tag{3.132}$$

and

$$\sum_j (K_j^\dagger \otimes |\hat{+}\rangle\langle\hat{+}| + K_j^T \otimes |\hat{-}\rangle\langle\hat{-}|)(K_j \otimes |\hat{+}\rangle\langle\hat{+}| + K_j^* \otimes |\hat{-}\rangle\langle\hat{-}|) = P \otimes |\hat{+}\rangle\langle\hat{+}| + P^T \otimes |\hat{-}\rangle\langle\hat{-}| \leq \mathbb{I}_2 \otimes \mathbb{I}_d. \tag{3.133}$$

The above inequality follows from the fact that $P \leq \mathbb{I}_d \iff P^T \leq \mathbb{I}_d$. This proves that one im-bit is sufficient to implement any quantum operation. Note that the im-bit is a qubit state, which can be used to implement any quantum operation of arbitrarily large dimension. In the lemma below, we show any pure state in this theory is equivalent to some qubit state.

Lemma 3.2. *(WKR+21b) There exists a real orthogonal matrix O_r for any pure state $|\psi\rangle$ such that*

$$O_r|\psi\rangle = \sqrt{\frac{1 + |\langle\psi^*|\psi\rangle|}{2}}|0\rangle + i\sqrt{\frac{1 - |\langle\psi^*|\psi\rangle|}{2}}|1\rangle. \tag{3.134}$$

Proof. Firstly, note that any pure state $|\psi\rangle$ can be written as

$$|\psi\rangle = a|\gamma_1\rangle + ib|\gamma_2\rangle. \tag{3.135}$$

Therefore,

$$|\psi^*\rangle = a|\gamma_1\rangle - ib|\gamma_2\rangle. \tag{3.136}$$

Here, $|\gamma_1\rangle$ and $|\gamma_2\rangle$ are real vectors and $\{a, b\}$ are real numbers. Also note that for any two real states $|\gamma_1\rangle$ and $|\gamma_2\rangle$, there exists a real orthogonal matrix O_r such that

$$O_r|\gamma_1\rangle = |0\rangle, \tag{3.137}$$

$$O_r|\gamma_2\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle, \tag{3.138}$$

where $\cos\theta = \langle\gamma_1|\gamma_2\rangle$. Now, we apply O_r to $|\psi\rangle$. This gives us

$$O_r|\psi\rangle = (a + ib\cos\theta)|0\rangle + ib\sin\theta|1\rangle. \tag{3.139}$$

This shows that $O_r|\psi\rangle$ is effectively a single-qubit state. Therefore we can associate a Bloch vector $\boldsymbol{r}$ with $O|\psi\rangle$, with coordinates given by

$$\begin{aligned} r_x &= b^2\sin(2\theta), \\ r_y &= 2ab\sin(\theta), \\ r_z &= a^2 + b^2\cos(2\theta). \end{aligned} \tag{3.140}$$

We now perform another real orthogonal transformation O'_r, such that $O'_r O_r|\psi\rangle$ lies in the positive y-z plane. The coordinates of $O'_r O_r|\psi\rangle$ ($\boldsymbol{s}$) can be given as follows:

$$s_x = 0, \quad s_y = |r_y|, \tag{3.141}$$

$$s_z = \sqrt{1 - r_y^2} = \sqrt{1 - 4a^2b^2 + 4a^2b^2\cos^2\theta}.$$

From Eq. (3.135) and Eq. (3.136), we obtain

$$a\,|\gamma_1\rangle = \frac{|\psi\rangle + |\psi^*\rangle}{2}, \tag{3.142a}$$

$$b\,|\gamma_2\rangle = \frac{|\psi\rangle - |\psi^*\rangle}{2i}. \tag{3.142b}$$

Therefore, we can express a^2 and b^2 as

$$a^2 = \left|\frac{|\psi\rangle + |\psi^*\rangle}{2}\right|^2 = \frac{1}{4}(2 + \langle\psi^*|\psi\rangle + \langle\psi|\psi^*\rangle), \tag{3.143}$$

$$b^2 = \left|\frac{|\psi\rangle - |\psi^*\rangle}{2i}\right|^2 = \frac{1}{4}(2 - \langle\psi^*|\psi\rangle - \langle\psi|\psi^*\rangle). \tag{3.144}$$

Recall that $\cos\theta = \langle\gamma_1|\gamma_2\rangle$. Using this we arrive at

$$ab\cos\theta = ab\,\langle\gamma_1|\gamma_2\rangle = \frac{1}{4i}(\langle\psi^*|\psi\rangle - \langle\psi|\psi^*\rangle). \tag{3.145}$$

Simplifying Eq. (3.141) using Eq. (3.143), Eq. (3.144) and Eq. (3.145) and gives us the following

$$s_x = 0, \quad s_y = \sqrt{1 - |\langle\psi^*|\psi\rangle|^2}, \quad s_z = |\langle\psi^*|\psi\rangle|. \tag{3.146}$$

The Bloch vector $\boldsymbol{s}$ corresponds to a pure state given by Eq. (3.134). This completes the proof. □

3.5.1 Geometric measure of imaginarity

For the resource theory of imaginarity, the geometric measure can be defined as

$$G(\rho) = 1 - \max_{\rho_r \in \mathcal{F}_r} F(\rho, \rho_r) \tag{3.147}$$

Here, $\mathcal{F}_r$ refers to the set of real states. This quantity will be later used to provide a complete solution for single-copy transformations, when the initial state is pure. We will now calculate this quantity for general pure states.

Lemma 3.3. *(WKR⁺21b) For a pure state $|\psi\rangle$ the geometric measure of imaginarity is given as*

$$G(|\psi\rangle) = \frac{1 - |\langle\psi^*|\psi\rangle|}{2}. \tag{3.148}$$

Proof. From lemma 3.2, it follows that

$$G(|\psi\rangle) = G\left(\sqrt{\frac{1 + |\langle\psi^*|\psi\rangle|}{2}}\,|0\rangle + i\sqrt{\frac{1 - |\langle\psi^*|\psi\rangle|}{2}}\,|1\rangle\right). \tag{3.149}$$

Consider any state of the form

$$|\mu\rangle = a_0\,|0\rangle + ia_1\,|1\rangle \tag{3.150}$$

with $a_0 \geq a_1 \geq 0$ and $a_0^2 + a_1^2 = 1$. Let $|\nu\rangle = \sum_j b_j |j\rangle$ be a arbitrary real state. Then we have

$$|\langle\nu|\mu\rangle|^2 = |a_0 b_0 + i a_1 b_1|^2 = a_0^2 b_0^2 + a_1^2 b_1^2 \leq a_0^2, \tag{3.151}$$

The inequality is due to the fact that $\sum_j b_j^2 = 1$. Since $|\langle 0|\mu\rangle|^2 = a_0^2$, it follows that

$$\max_{|\nu\rangle \in \mathcal{F}_r} |\langle\nu|\mu\rangle|^2 = a_0^2, \tag{3.152}$$

and thus $G(|\mu\rangle) = a_1^2$. This completes the proof. □

Since any real state can be expressed as a convex combination of pure real states, from (SKB10), it follows that

$$G(\rho) = 1 - \max_{\rho_r \in \mathcal{F}_r} F(\rho, \rho_r) = \min_{\{p_j, |\psi_j\rangle\}} \sum_j p_j G(|\psi_j\rangle). \tag{3.153}$$

Here that minimisation is performed over all pure state decompositions of ρ, i.e, $\{p_j, |\psi_j\rangle : \sum_j p_j |\psi_j\rangle\langle\psi_j| = \rho\}$. Note that, from Eq. (3.153), we see that G is convex:

$$G(\sum_j p_j \rho_j) \leq \sum_j p_j G(\rho_j). \tag{3.154}$$

In lemma (3.3), we gave analytical expression for G for arbitrary pure states. We will now extend this result to any arbitrary quantum state.

Theorem 3.5. *(KDS23) For a quantum state ρ, the geometric measure of imaginarity is given by*

$$G(\rho) = \frac{1 - \sqrt{F(\rho, \rho^T)}}{2}, \tag{3.155}$$

Proof. From lemma 3.3, we know that

$$G(|\psi\rangle) = \frac{1 - |\langle\psi^*|\psi\rangle|}{2} = \frac{1 - \sqrt{F(|\psi\rangle\langle\psi|, |\psi^*\rangle\langle\psi^*|)}}{2}. \tag{3.156}$$

Let us assume $\{p_j, |\psi_j\rangle\}$ be the ensemble of ρ, achieving the minimisation in Eq. (3.153). We will now use the joint concavity property of root-fidelity (Wil17) to find the following

$$\begin{aligned}
\sqrt{F(\rho, \rho^T)} &\geq \sum_j p_j \sqrt{F(|\psi_j\rangle\langle\psi_j|, |\psi_j^*\rangle\langle\psi_j^*|)} \\
\frac{1 - \sqrt{F(\rho, \rho^T)}}{2} &\leq \sum_j p_j \frac{1 - \sqrt{F(|\psi_j\rangle\langle\psi_j|, |\psi_j^*\rangle\langle\psi_j^*|)}}{2} \\
&= G(\rho),
\end{aligned} \tag{3.157}$$

This gives the following lower bound of geometric measure of imaginarity

$$G(\rho) \geq \frac{1-\sqrt{F(\rho,\rho^T)}}{2}. \tag{3.158}$$

From Eq. (3.153), we know

$$G(\rho) = \min_{\{p_j,|\psi_j\rangle\}} \sum_j p_j G(|\psi_j\rangle), \tag{3.159}$$

where the minimisation is over all the pure state ensembles of ρ. One can equivalently write Eq. (3.158) as

$$\max_{\{p_j,|\psi_j\rangle\}} \sum_j p_j |\langle\psi_j|\psi_j^*\rangle| \leq \sqrt{F(\rho,\rho^T)}. \tag{3.160}$$

We will now construct a pure state decomposition which saturates the above bound. Note that any other pure state ensemble of ρ (let's assume $\{q_i,|\varphi_i\rangle\}$) relates to the pure state ensemble $\{p_j,|\psi_j\rangle\}$ as follows

$$\sqrt{q_i}\,|\varphi_i\rangle = \sum_j U_{ij}^* \sqrt{p_j}\,|\psi_j\rangle. \tag{3.161}$$

Here U_{ji} correspond to the matrix elements of a unitary matrix U. Additionally, without loss of generality, we assume $\{q_j,|\varphi_j\rangle\}$ to be the eigenvalues and eigenstates of ρ. From, Eq. (3.161), it follows that

$$\sqrt{q_i q_j}\,\langle\varphi_i|\varphi_j^*\rangle = (UAU^T)_{ij}, \tag{3.162}$$

where $A_{ij} = \sqrt{p_i p_j}\,\langle\psi_i|\psi_j^*\rangle$. The singular values of A correspond to the square roots of eigenvalues of $AA^\dagger (= AA^*)$. Note that,

$$\begin{aligned}(AA^*)_{ij} &= \sum_k p_k \sqrt{p_i p_j}\,\langle\psi_i|\psi_k^*\rangle\langle\psi_k^*|\psi_j\rangle & (3.163)\\ &= \sqrt{p_i p_j}\,\langle\psi_i|\rho^T|\psi_j\rangle & (3.164)\\ &= \langle\psi_i|\sqrt{\rho}\rho^T\sqrt{\rho}|\psi_j\rangle. & (3.165)\end{aligned}$$

Hence, the matrix $(AA^*)_{ij}$ is $\sqrt{\rho}\rho^T\sqrt{\rho}$ expressed in the eigenbasis of ρ. Therefore, the singular values of $(A)_{ij}$ (matrix) are the eigenvalues of $\sqrt{\sqrt{\rho}\rho^T\sqrt{\rho}}$. Let us note that, for any symmetric matrix S, there exists a decomposition of the form (BGG88)

$$S = Q\Sigma Q^T, \tag{3.166}$$

where Σ is a positive diagonal matrix and Q is a unitary matrix. Eqs. (3.162) and (3.166) imply that there exists a pure state decomposition of ρ (say $\{\lambda_i,|\mu_i\rangle\}$) such that

$$\sqrt{\lambda_i\lambda_j}\,\langle\mu_i|\mu_j^*\rangle = \delta_{ij}D_j, \tag{3.167}$$

where D_j is the j^{th} eigenvalue of $\sqrt{\sqrt{\rho}\rho^T\sqrt{\rho}}$. This decomposition saturates the bound in 3.160. This completes the proof. □

We will now show that geometric measure of imaginarity is a strong imaginarity monotone. For the definition of strong monotonicity see Eq. (2.4). This strong monotonicity property helps us to provide upper bounds on transformation probabilities in general resource theories. Precisely, from (WTX+20), we know that for any resource quantifier R, which is convex and strongly monotonic under free operations, the following holds:

$$P(\rho \to \sigma) \leq \min \frac{R(\rho)}{R(\sigma)}, 1. \tag{3.168}$$

Here, $P(\rho \to \sigma)$ is the optimal probability of transforming ρ into σ via stochastic free operations.

Lemma 3.4. *(WKR+21b) Geometric measure of imaginarity (G) is a strong imaginarity monotone.*

Proof. Let us first prove strong monotonicity for pure states. From lemma 3.2, it is sufficient to prove it for states of the following form

$$|\alpha\rangle = \cos\alpha\,|0\rangle + i\sin\alpha\,|1\rangle\,, \tag{3.169}$$

with $\alpha \in [0, \pi/4]$. The geometric measure of imaginarity of this state is given by $G(|\alpha\rangle) = \sin^2\alpha$. Proving strong monotonicity for pure states reduces to proving the inequality

$$\sum_j \max_{|\varphi_j\rangle\in\mathcal{F}_r} |\left\langle\varphi_j|K_j|\alpha\right\rangle|^2 \geq \cos^2\alpha, \tag{3.170}$$

where $\{K_j\}$ is a set of real Kraus operators. Let us first note that

$$\sum_j \max_{|\varphi_j\rangle\in\mathcal{F}_r} |\left\langle\varphi_j|K_j|\alpha\right\rangle|^2 \geq \sum_j \frac{|\left\langle 0|K_j^T K_j|\alpha\right\rangle|^2}{s_j}. \tag{3.171}$$

Here we introduced

$$s_j = \left\langle 0|K_j^T K_j|0\right\rangle. \tag{3.172}$$

Let us recall that all the Kraus operators K_j are real. Using Eq. (3.169), we obtain

$$\begin{aligned} |\left\langle 0|K_j^T K_j|\alpha\right\rangle|^2 &= |\left\langle 0|K_j^T K_j|0\right\rangle|^2\cos^2\alpha \\ &\quad + |\left\langle 0|K_j^T K_j|1\right\rangle|^2\sin^2\alpha \\ &\geq |\left\langle 0|K_j^T K_j|0\right\rangle|^2\cos^2\alpha. \end{aligned} \tag{3.173}$$

Eq. (3.171) and Eq. (3.173) imply

$$\sum_j \max_{|\varphi_j\rangle\in\mathcal{F}_r} |\left\langle\varphi_j|K_j|\alpha\right\rangle|^2 \geq \sum_j \frac{|\left\langle 0|K_j^T K_j|0\right\rangle|^2}{s_j}\cos^2\alpha. \tag{3.174}$$

Definition of s_j (see Eq. (3.172)) along with the fact that $\sum_j K_j^T K_j = \mathbb{I}$, gives us the desired inequality (3.170).

The above arguments show that G satisfies strong monotonicity for all pure states. We consider an optimal decomposition of a mixed state $\rho = \sum_j p_j \left|\psi_j\right\rangle\!\left\langle\psi_j\right|$ to extend this result to mixed states.

$$G(\rho) = \sum_j p_j G\left(\left|\psi_j\right\rangle\right). \tag{3.175}$$

We will now introduce $s_{jk} = \left\langle\psi_k|K_j^T K_j|\psi_k\right\rangle$ to obtain

$$\begin{aligned}
\sum_j q_j G\left(\frac{K_j \rho K_j^T}{q_j}\right) &= \sum_j q_j G\left(\sum_k p_k \frac{K_j \left|\psi_k\right\rangle\!\left\langle\psi_k\right| K_j^T}{q_j}\right) \\
&= \sum_j q_j G\left(\sum_k \frac{p_k s_{jk}}{q_j} \times \frac{K_j \left|\psi_k\right\rangle\!\left\langle\psi_k\right| K_j^T}{s_{jk}}\right) \\
&\leq \sum_{j,k} p_k s_{jk} G\left(\frac{K_j \left|\psi_k\right\rangle\!\left\langle\psi_k\right| K_j^T}{s_{jk}}\right) \\
&\leq \sum_j p_j G(\left|\psi_j\right\rangle) = G(\rho),
\end{aligned} \tag{3.176}$$

where the first inequality follows from the fact that G is convex. This completes the proof of strong monotonicity of geometric measure of imaginarity for all mixed states. □

3.5.2 Stochastic approximate state conversion

Using the tools developed above, we will now provide a complete solution for single copy transformations via real operations, starting from a pure state. Below we will provide the optimal probability of transforming a pure state into another pure state via real operations. We will then generalise this result to arbitrary target states and then extend it to the regime of stochastic-approximate transformations.

Lemma 3.5. *(WKR[+]21b) The maximum probability for a pure state transformation $|\psi\rangle \to |\varphi\rangle$ via real operations is given by*

$$P(\,|\psi\rangle \to |\varphi\rangle\,) = \min\left\{\frac{1 - |\,\langle\psi^*\,|\psi\rangle\,|}{1 - |\,\langle\varphi^*\,|\varphi\rangle\,|}, 1\right\}. \tag{3.177}$$

Proof. Since G is a strong imaginarity monotone

$$P(\rho \to \sigma) \leq \min\{\frac{G(\rho)}{G(\sigma)}, 1\}. \tag{3.178}$$

When initial and target states are pure, proposition 3.3 implies

$$P(|\psi\rangle \to |\varphi\rangle) \leq \min\{\frac{1 - |\langle\psi^*|\psi\rangle|}{1 - |\langle\varphi^*|\varphi\rangle|}, 1\}. \tag{3.179}$$

We will consider the case when

$$|\langle\psi^*|\psi\rangle| \geq |\langle\varphi^*|\varphi\rangle|, \tag{3.180}$$

and show the existence of a real operation saturating the bound (3.179). In order to show this, we will first apply a real orthogonal transformation to $|\psi\rangle$ to bring it into the following form (see lemma 3.2)

$$\left|\psi'\right\rangle = \sqrt{\frac{1+|\langle\psi^*|\psi\rangle|}{2}}\,|0\rangle + i\sqrt{\frac{1-|\langle\psi^*|\psi\rangle|}{2}}\,|1\rangle, \tag{3.181}$$

Consider a real quantum operation with the following Kraus operators

$$K_0 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad K_1 = \sqrt{-K_0^2}, \tag{3.182}$$

where a is given by

$$a = \sqrt{\frac{1-|\langle\psi^*|\psi\rangle|}{1-|\langle\varphi^*|\varphi\rangle|} \times \frac{1+|\langle\varphi^*|\varphi\rangle|}{1+|\langle\psi^*|\psi\rangle|}}. \tag{3.183}$$

Eq. (3.180) implies that $a \leq 1$. By inspection, one can see that the Kraus operator K_0 transforms $|\psi'\rangle$ into the following state

$$\left|\varphi'\right\rangle = \sqrt{\frac{1+|\langle\varphi^*|\varphi\rangle|}{2}}\,|0\rangle + i\sqrt{\frac{1-|\langle\varphi^*|\varphi\rangle|}{2}}\,|1\rangle, \tag{3.184}$$

with a probability (p) given by

$$p = \frac{1-|\langle\psi^*|\psi\rangle|}{1-|\langle\varphi^*|\varphi\rangle|}. \tag{3.185}$$

Up to a real orthogonal transformation, $|\varphi'\rangle$ is equivalent to the desired state $|\varphi\rangle$ (see lemma 3.2). For the case when, $|\langle\psi^*|\psi\rangle| < |\langle\varphi^*|\varphi\rangle|$, the transformation $|\psi\rangle \to |\varphi\rangle$ can be achieved deterministically (unit probability) (HG18). □

Let us now generalise this result to arbitrary target states.

Theorem 3.6. *(KDS23) A pure state $|\psi\rangle$ can be transformed into a quantum state ρ via real operations, with a maximum probability given by*

$$P(|\psi\rangle \to \rho) = \min[\frac{G(|\psi\rangle)}{G(\rho)}, 1]. \tag{3.186}$$

Proof. From strong monotonicity of geometric measure, we know that the ratio of geometric measures gives the upper bound for the optimal achievable probability of transforming σ to ρ

$$P(\sigma \to \rho) \leq \min[\frac{G(\sigma)}{G(\rho)}, 1]. \tag{3.187}$$

From lemma 3.5, we know that this inequality is saturated for pure to pure transformations,

$$P(|\psi\rangle \to |\varphi\rangle) = \min[\frac{G(|\psi\rangle)}{G(|\varphi\rangle)}, 1] \tag{3.188}$$

Now we will show that Eq. (3.187) is saturated whenever the initial state is pure ($\sigma = |\psi\rangle\langle\psi|$) and for arbitrary target states. Let us assume $\{p'_j, \left|\psi'_j\right\rangle\}$ is the ensemble of the state $\rho = \sum_j p'_j \left|\psi'_j\right\rangle\left\langle\psi'_j\right|$, achieving the minimisation in Eq. (3.153). We will now choose a purification of ρ given by,

$$|\rho\rangle = \sum_j \sqrt{p'_j}\left|\psi'_j\right\rangle \otimes |j\rangle^A, \tag{3.189}$$

where A is an ancillary system. One can easily notice that

$$G(|\rho\rangle) = G(\rho). \tag{3.190}$$

Therefore from Eq. (3.188), we can say that

$$P(|\psi\rangle \to |\rho\rangle) = \min\{\frac{G(|\psi\rangle)}{G(|\rho\rangle)}, 1\} = \min\{\frac{G(|\psi\rangle)}{G(\rho)}, 1\}. \tag{3.191}$$

The desired transformation is achieved by first converting $|\psi\rangle$ to $|\rho\rangle$ with probability given in Eq. (3.191) and then discarding the ancilla. This completes the proof. □

The result of theorem 3.6 is analogous to the result in bipartite entanglement theory (see Eq. (3.108)), where the optimal probability of transforming a pure state into a 2-qubit mixed state via SLOCC is given by the ratio of geometric measure of entanglement of the initial and final states. Therefore using similar techniques used in section 3.4.2 we can extend the result of theorem 3.6 to stochastic-approximate regime. As we did in lemma 3.1, we first provide tight continuity bounds for the geometric measure of imaginarity. The proof of the following lemma is similar to the proof provided for lemma 3.1.

Lemma 3.6. *(KDS23) For a state ρ consider a set of states $S_{\rho,f}$ such that $F(\rho,\rho') \geq f$ for all $\rho' \in S_{\rho,f}$. The minimal geometric imaginarity in $S_{\rho,f}$ is given by*

$$\min_{\rho' \in S_{\rho,f}} G(\rho') = \sin^2\left(\max\left\{\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f}, 0\right\}\right). \tag{3.192}$$

For a pure state $|\psi\rangle$, the maximal geometric imaginarity in $S_{\psi,f}$ is given by

$$\max_{\rho' \in S_{\psi,f}} G(\rho') = \sin^2\left(\min\left\{\sin^{-1}\sqrt{G(|\psi\rangle)} + \cos^{-1}\sqrt{f}, \frac{\pi}{4}\right\}\right), \tag{3.193}$$

From theorem 3.6, we know that the optimal probability of transforming $|\psi\rangle$ into ρ via real operations is given by

$$P(|\psi\rangle \to \rho) = \min\{\frac{G(|\psi\rangle)}{G(\rho)}, 1\}. \tag{3.194}$$

Here, G is the geometric measure of imaginarity. From Eq. (3.194) and the definitions of P_f (see Eq. (3.2) and $S_{\rho,f}$ it follows that

$$P_f(|\psi\rangle \to \rho) = \min\{\frac{G(|\psi\rangle)}{\min_{\rho' \in S_{\rho,f}} G(\rho')}, 1\}. \tag{3.195}$$

This allows us provide an analytical expression P_f in the case of initial pure states. The proof of this theorem will again be analogous to the proof of theorem 3.4.

Theorem 3.7. *(KDS23) A pure state $|\psi\rangle$ can be transformed into a two-qubit state ρ via real operations with a fidelity f and a maximal probability given by*

$$P_f(|\psi\rangle \to \rho) = \begin{cases} 1 \text{ for } m_1 \geq 0 \\ \frac{G(\psi)}{\sin^2\left(\sin^{-1}\sqrt{G(\rho)} - \cos^{-1}\sqrt{f}\right)} \text{ otherwise,} \end{cases} \tag{3.196}$$

where $m_1 = \sin^{-1}\sqrt{G(|\psi\rangle)} - \sin^{-1}\sqrt{G(\rho)} + \cos^{-1}\sqrt{f}$.

For the special case of $f = 1$, we recover the result stated in Eq. (3.191). The optimal fidelity for a given probability can also be found easily:

$$F_p(|\psi\rangle \to \rho) = \begin{cases} 1 \text{ for } p \leq \frac{G(|\psi\rangle)}{G(\rho)}, \\ \cos^2\left[\sin^{-1}\sqrt{G(\rho)} - \sin^{-1}\sqrt{\frac{G(|\psi\rangle)}{p}}\right] \text{ otherwise.} \end{cases} \tag{3.197}$$

Theorem 3.7 provides a closed expression for P_f. In fact one can also obtain a closed expression for F_p. Note that, when $p \leq \frac{G(\psi)}{G(\rho)} < 1$, Eq. (3.191) implies $F_p(\psi \to \rho) = 1$. For the case when $1 \geq p > \frac{G(\psi)}{G(\rho)}$, the optimal fidelity can be obtained by solving Eq. (3.197) for f, which gives

$$F_p(|\psi\rangle \to \rho) = \cos^2\left[\sin^{-1}\sqrt{G(\rho)} - \sin^{-1}\sqrt{\frac{G(|\psi\rangle)}{p}}\right]. \tag{3.198}$$

3.6 Conclusions

In this chapter, we studied the regime in between probabilistic and approximate transformations. For general resource theories, we give upper bounds on the achievable probability and fidelity for single copy transformations in this regime. We show that these single-copy bounds imply upper bounds on asymptotic transformation rates, for various classes of quantum states. We then focus on bipartite entanglement theory, solving the question of stochastic-approximate state conversion via SLOCC for pure states of arbitrary dimension and when the target state is an arbitrary two-qubit state (initial state considered to be pure). We then move to resource theory of imaginarity, providing an analytical expression for geometric measure of imaginarity and proving some important properties of it. We use these results to provide complete solution to stochastic-approximate state conversion via real operations, when the initial state is pure.

Chapter 4

Catalytic transformations

4.1 Introduction

Catalysis is a phenomenon widely used in chemical processes (vSvLMA99). William Ostwald's definition of catalysis says "Catalysis is the acceleration of a slow chemical process by the presence of a foreign substance" (Ost94). For thousands of years, humans have been using catalysts in fermentation processes (vSvLMA99). Quantum catalysis, on the other hand, has some conceptual similarities to chemical catalysis, but differs in various aspects. A quantum catalyst is a quantum state which helps us realise an otherwise impossible state transformation, while remaining unchanged by the end of the transformation. The first example of quantum catalysis was provided in (JP99) in the context of bipartite entanglement theory. Before showing the example of quantum catalysis, let us again recall that a bipartite pure state $|\psi\rangle$ can be transformed into $|\varphi\rangle$ via LOCC iff the reduced states ψ^A and φ^A satisfy the following majorisation condition ($\psi^A \prec \varphi^A$)(Nie99).

$$\sum_{i=0}^{n} \alpha_i \leq \sum_{i=0}^{n} \beta_i \ \text{ for all } \ 0 \leq n \leq d-1, \tag{4.1}$$

Here, $\{\alpha_i\}$ and $\{\beta_i\}$ are the squared Schmidt coefficients (in decreasing order) of $|\psi\rangle$ and $|\varphi\rangle$ respectively and $d = \min\{d_A, d_B\}$, with d_A and d_B being the dimensions of the Hilbert spaces of Alice and Bob, respectively. See the discussion around Eq. (3.54) for details about Schmidt decomposition and Schmidt coefficients.

Now let us consider the following pair of states, in their respective Schmidt decomposition

$$|\psi\rangle = \sqrt{0.4}\,|00\rangle + \sqrt{0.4}\,|11\rangle + \sqrt{0.1}\,|22\rangle + \sqrt{0.1}\,|33\rangle\,, \tag{4.2}$$

$$|\varphi\rangle = \sqrt{0.5}\,|00\rangle + \sqrt{0.25}\,|11\rangle + \sqrt{0.25}\,|22\rangle\,. \tag{4.3}$$

From the family of inequalities given in Eq.(4.1), one can easily see that

$$\psi^A \not\prec \varphi^A \text{ and } \varphi^A \not\prec \psi^A \tag{4.4}$$

Therefore, neither $|\psi\rangle$ can be transformed into $|\varphi\rangle$ nor $|\varphi\rangle$ can be transformed into $|\psi\rangle$. Let us now consider a catalyst system in a bipartite pure state $|\eta\rangle = \sqrt{0.6}\,|00\rangle + \sqrt{0.4}\,|11\rangle$. Again using the majorisation inequalities (given in Eq. (4.1)), one can see that

$$|\psi\rangle \otimes |\eta\rangle \xrightarrow{LOCC} |\varphi\rangle \otimes |\eta\rangle\,. \tag{4.5}$$

Note that, the state of the catalyst ($|\eta\rangle$) is unchanged in the process and can be reused to perform a similar transformation. After this example of entanglement catalysis was provided, major efforts were made to characterize catalytic transformations between any two bipartite states $|\psi\rangle$ and $|\varphi\rangle$. When there exists a pure catalyst $|\eta\rangle$ enabling the transformation from $|\psi\rangle$ and $|\varphi\rangle$, we say "$|\psi\rangle$ is trumped by ($\prec_T$) $|\varphi\rangle$". In other words $|\psi\rangle \prec_T |\varphi\rangle$ iff there exists a finite dimensional pure entangled state $|\eta\rangle$ such that $|\psi\rangle\otimes|\eta\rangle \prec |\varphi\rangle\otimes|\eta\rangle$. In (Kli07; Tur07), the authors completely characterise the trumping relation between bipartite pure entangled states. Precisely, they show that $|\psi\rangle \prec_T |\varphi\rangle$ is equivalent to (Tur07; Kli07)

$$f_k(\vec{\alpha}) > f_k(\vec{\beta}) \;\forall\; k \in (-\infty, \infty), \tag{4.6}$$

where $\vec{\alpha}$ and $\vec{\beta}$ are the squared Schmidt coefficients of the states $|\psi\rangle$ and $|\varphi\rangle$ respectively and for a d-dimensional probability distribution ($\vec{x}$), $f_k(x)$ is given by

$$f_k(x) = \begin{cases} \ln \sum_{i=1}^{d} x_i^k & (k > 1); \\ \sum_{i=1}^{d} x_i \ln x_i & (k = 1); \\ -\ln \sum_{i=1}^{d} x_i^k & (0 < k < 1); \\ -\sum_{i=1}^{d} \ln x_i & (k = 0); \\ \ln \sum_{i=1}^{d} x_i^k & (k < 0). \end{cases} \tag{4.7}$$

Note, that this is not a complete characterisation of catalytic transformations between pure entangled states, as mixed state catalysts are not considered. Another problem with this solution is that for a given pair of pure states, in general, one needs to check infinitely many inequalities (4.7) to conclusively show the existence of trumping relation between them. In order to solve both these problems, we will introduce the idea of *approximate catalysis*, where we allow for arbitrarily small amount of correlation between system and the catalyst. Formally, a bipartite state ρ^S can be transformed into σ^S via approximate catalysis iff for every $\varepsilon > 0$ there exists a catalyst τ^C and a LOCC operation Λ such that

$$\left\|\Lambda\left(\rho^S \otimes \tau^C\right) - \sigma^S \otimes \tau^C\right\|_1 \leq \varepsilon, \tag{4.8}$$

$$\mathrm{Tr}_S\left[\Lambda\left(\rho^S \otimes \tau^C\right)\right] = \tau^C, \tag{4.9}$$

Note that Eq. (4.9) makes sure that the catalyst remains unchanged and Eq. (4.8) implies (because trace norm does not increase under partial trace)

$$\left\|\mathrm{Tr}_C\left[\Lambda\left(\rho^S \otimes \tau^C\right)\right] - \sigma^S\right\|_1 \leq \varepsilon. \tag{4.10}$$

This shows that the final state of the system is ε-close to σ^S. Additionally, Eq. (4.8) also says that the final state of the system and catalyst $\left(\Lambda\left(\rho^S \otimes \tau^C\right)\right)$ is ε-close to a product state. Therefore, the correlations between the system and the catalyst can be made arbitrarily small.

4.2 Catalytic transformations between pure entangled states

We will now use the above mentioned notion of approximate catalysis and completely characterise the catalytic transformations between pure entangled states. Let us first recall the notion of asymptotic transformations.

We say that a *asymptotic transformation* from ρ to σ is possible with rate r, if for any $\varepsilon > 0$ and any $\delta > 0$ there exist natural numbers m, n and a LOCC operation Λ such that

$$\Lambda\left(\rho^{\otimes n}\right) = \mu^{S_1 \ldots S_m}, \tag{4.11a}$$

$$\left\|\mu^{S_1 \ldots S_m} - \sigma^{\otimes m}\right\|_1 < \varepsilon, \tag{4.11b}$$

$$\frac{m}{n} + \delta > r. \tag{4.11c}$$

Here, $\mu^{S_1 \ldots S_m}$ is a state of the system $S_1 \otimes S_2 \otimes \cdots \otimes S_m$. Note that each S_i is a copy of the system S. The supremum of r fulfilling these properties will be called *asymptotic transformation rate* $R(\rho \to \sigma)$. We say that ρ can be asymptotically converted into σ if $R(\rho \to \sigma) \geq 1$. Inspired by techniques introduced within quantum thermodynamics (SS21), we will now connect this notion of asymptotic transformations to approximate catalysis.

Lemma 4.1. *(KDS21) If ρ^S can be asymptotically converted into σ^S then ρ^S can be converted into σ^S via approximate catalysis.*

Proof. Let a LOCC protocol Λ converts n copies of ρ into Γ

$$\Gamma = \Lambda\left[\rho^{\otimes n}\right]. \tag{4.12}$$

Here, Γ acts on the system $S_1 \otimes S_2 \otimes \cdots \otimes S_n$, where every S_i is a copy of S. We will denote Γ_i as the reduced state of Γ on $S_1 \otimes S_2 \otimes \cdots \otimes S_i$ and $\Gamma_0 = 1$. Additionally, $\Gamma_i^{(j)}$ denotes the reduced state of Γ_i on S_j for $j \leq i$.

From the definition of asymptotic transformations, we know that if ρ is asymptotically convertible into σ then for every $\varepsilon > 0$ and $\delta > 0$ there exist natural numbers $m \leq n$ such that

$$\left\|\Gamma_m - \sigma^{\otimes m}\right\|_1 < \varepsilon, \tag{4.13a}$$

$$\frac{m}{n} + \delta > 1 \tag{4.13b}$$

We will now construct a catalyst τ transforming ρ into σ:

$$\tau = \frac{1}{n}\sum_{k=1}^{n}\rho^{\otimes(k-1)} \otimes \Gamma_{n-k} \otimes |k\rangle\langle k| \,. \tag{4.14}$$

The catalyst C acts on $S_2 \dots S_n \otimes K$. Here K is an ancillary system held by Alice. K has a dimension of n with basis $\{|k\rangle \,, k \in [1, n]\}$

Let us now perform the following LOCC protocol on the system and the catalyst:

(i) Alice measures K in the basis $|k\rangle$ (rank one projective measurement) and communicates the measurement outcome to Bob. If the measurement outcome is n, Alice and Bob perform the LOCC protocol Λ (see Eq. (4.12)) on $S_1 \otimes S_2 \otimes \cdots \otimes S_n$. For all the other outcomes of Alice's measurement, both the parties do nothing.

(ii) Alice applies a unitary on K which converts $|n\rangle \to |1\rangle$ and $|i\rangle \to |i+1\rangle$.

(iii) A SWAP unitary is applied by both the parties on their parts of (S_i, S_{i+1}) and (S_1, S_n) to transform $S_i \to S_{i+1}$ and $S_n \to S_1$.

Note that the initial state of the system along with the catalyst is given by

$$\rho \otimes \tau = \frac{1}{n}\sum_{k=1}^{n}\rho^{\otimes k} \otimes \Gamma_{n-k} \otimes |k\rangle \langle k| \,. \tag{4.15}$$

After step (i), $\rho \otimes \tau$ transforms into

$$\mu^{i} = \frac{1}{n}\sum_{k=1}^{n-1}\rho^{\otimes k} \otimes \Gamma_{n-k} \otimes |k\rangle \langle k| + \frac{1}{n}\Gamma \otimes |n\rangle \langle n| \,. \tag{4.16}$$

Step (ii) transforms μ^{i} into μ^{ii}, where

$$\mu^{ii} = \frac{1}{n}\sum_{k=1}^{n}\rho^{\otimes k-1} \otimes \Gamma_{n+1-k} \otimes |k\rangle \langle k| \,. \tag{4.17}$$

If we trace out S_n from μ^{ii}, the resulting state is the initial state of the catalyst τ (see Eq. (4.14)). Therefore, step (iii) transforms μ^{ii} into μ^{SC}, satisfying $\mathrm{Tr}_S[\mu^{SC}] = \tau$. This shows that the state of the catalyst is invariant in this protocol.

We will now show that $\left\|\mu^{SC} - \sigma^{S} \otimes \tau^{C}\right\|_1$ can be made arbitrarily small. Note that μ^{SC} and μ^{ii} are equivalent up to a cyclic SWAP. Therefore,

$$||\mu^{SC} - \sigma^{S} \otimes \tau^{C}||_1 = ||\mu^{ii} - \gamma||_1, \tag{4.18}$$

where

$$\gamma = \frac{1}{n}\sum_{k=1}^{n}\rho^{\otimes k-1} \otimes \tilde{\Gamma}_{n+1-k} \otimes |k\rangle\langle k| \,. \tag{4.19}$$

Here $\tilde{\Gamma}_i$ is constructed from Γ_i by putting σ at S_i, without any correlations with Γ_{i-1}. That is, $\tilde{\Gamma}_i = (\mathrm{Tr}_{S_i}\Gamma_i) \otimes \sigma$ (up to the order of the components in the tensor product:

$S_1 \otimes S_2 \otimes \cdots \otimes S_i \otimes C$). Using this, we get

$$
\begin{aligned}
\|\mu^{ii} - \gamma\|_1 &= \frac{1}{n}\sum_{k=1}^{n} \|\Gamma_{n+1-k} - \tilde{\Gamma}_{n+1-k}\|_1 \\
&= \frac{1}{n}\sum_{k=1}^{n-m} \|\Gamma_{n+1-k} - \tilde{\Gamma}_{n+1-k}\|_1 + \frac{1}{n}\sum_{k=n-m+1}^{n} \|\Gamma_{n+1-k} - \tilde{\Gamma}_{n+1-k}\|_1 \\
&\leq 2\frac{(n-m)}{n} + \frac{1}{n}\sum_{k=n-m+1}^{n} \|\Gamma_{n+1-k} - \tilde{\Gamma}_{n+1-k}\|_1 \\
&< 2\delta + \frac{1}{n}\sum_{k=n-m+1}^{n} \|\Gamma_{n+1-k} - \sigma^{n+1-k}\|_1 + \frac{1}{n}\sum_{k=n-m+1}^{n} \|\sigma^{n+1-k} - \tilde{\Gamma}_{n+1-k}\|_1 \\
&< 2\delta + \varepsilon + \varepsilon = 2(\delta + \varepsilon). \qquad (4.20)
\end{aligned}
$$

Here, the first inequality follows from the fact that $\|\rho - \sigma\|_1 \leq 2$, for any two quantum states ρ and σ. Second inequality follows from Eq. (4.13b). Third inequality due to triangle inequality and in the last inequality we use Eq. (4.13a). Finally note that both ε and δ can be made arbitrarily small. This completes our proof. □

Note that, even though lemma 4.1 is in the context of LOCC, the same result holds for many other resource theories. One can easily see that, lemma 4.1 holds for any set of free operations, which include (TS22)

- Relabelling of a classical register
- free operations conditioned on the classical register.

We will also make use of an entanglement quantifier, called squashed entanglement (CW04). The squashed entanglement of a bipartite quantum states ρ^{AB} is defined as (CW04)

$$
E_{sq}\left(\rho^{AB}\right) = \inf\left\{\frac{1}{2}I(A;B|E) : \rho^{ABE} \text{ extension of } \rho^{AB}\right\}, \qquad (4.21)
$$

where

$$
I(A;B|E) = H\left(\rho^{AE}\right) + H\left(\rho^{BE}\right) - H\left(\rho^{ABE}\right) - H\left(\rho^{E}\right) \qquad (4.22)
$$

is the quantum conditional mutual information of ρ^{ABE} and the infimum is taken over all quantum states ρ^{ABE} with $\rho^{AB} = \mathrm{Tr}_E\left(\rho^{ABE}\right)$. Squashed entanglement satisfies the following properties (CW04)

- Non-increasing under LOCC operations (monotonicity)
- Strong-superadditivity $\left(E_{sq}\left(\rho^{AA'BB'}\right) \geq E_{sq}\left(\rho^{AB}\right) + E_{sq}\left(\rho^{A'B'}\right)\right)$
- Additive $\left(E_{sq}\left(\rho^{AB} \otimes \rho^{A'B'}\right) = E_{sq}\left(\rho^{AB}\right) + E_{sq}\left(\rho^{A'B'}\right)\right)$
- Continuity

- For a pure state $|\psi\rangle^{AB}$ squashed entanglement is equal to the entanglement entropy, i.e., the entropy of the reduced state: $E_{sq}\left(|\psi\rangle^{AB}\right) = H\left(\psi^A\right)$

Using these properties, we will now show that squashed entanglement does not increase under approximate catalysis.

Lemma 4.2. *(KDS21) If ρ^{AB} can be transformed into ν^{AB} via approximate catalysis, then the following holds*

$$E_{sq}\left(\rho^{AB}\right) \geq E_{sq}\left(\nu^{AB}\right). \tag{4.23}$$

Proof. If ρ^{AB} can be transformed into ν^{AB} via approximate catalysis, then for any $\varepsilon > 0$ there exists a catalyst $\tau^{A'B'}$ and an LOCC operation Λ s.t, the final state $\sigma^{AA'BB'} = \Lambda(\rho^{AB} \otimes \tau^{A'B'})$ satisfies

$$\left\|\mathrm{Tr}_{A'B'}\left[\sigma^{AA'BB'}\right], \nu^{AB}\right\|_1 < \varepsilon, \tag{4.24}$$

$$\mathrm{Tr}_{AB}\left[\sigma^{AA'BB'}\right] = \tau^{A'B'}. \tag{4.25}$$

Using monotonicity and additivity of squashed entanglement, we find

$$E_{sq}\left(\sigma^{AA'BB'}\right) \leq E_{sq}\left(\rho^{AB}\right) + E_{sq}\left(\tau^{A'B'}\right). \tag{4.26}$$

From strong-superadditivity of squashed entanglement, it follows

$$E_{sq}\left(\sigma^{AA'BB'}\right) \geq E_{sq}\left(\mathrm{Tr}_{A'B'}\left[\sigma^{AA'BB'}\right]\right) + E_{sq}\left(\tau^{A'B'}\right). \tag{4.27}$$

Combining Eqs. (4.26) and (4.27), we get

$$E_{sq}\left(\rho^{AB}\right) \geq E_{sq}\left(\mathrm{Tr}_{A'B'}\left[\sigma^{AA'BB'}\right]\right). \tag{4.28}$$

Note that, from definition of approximate catalysis, it follows that $\mathrm{Tr}_{A'B'}\left[\sigma^{AA'BB'}\right]$ can be made arbitrarily close to ν^{AB} in trace distance. Therefore from continuity of squashed entanglement (AF04), we get $E_{sq}\left(\rho^{AB}\right) \geq E_{sq}\left(\nu^{AB}\right)$. This completes the proof. □

Using lemma 4.1 and lemma 4.2, we present the following result, which completely characterises pure state transformations via approximate catalysis and LOCC.

Theorem 4.1. *(KDS21) $|\psi\rangle^{AB}$ can be transformed into $|\varphi\rangle^{AB}$ via approximate catalysis if and only if*

$$H\left(\psi^A\right) \geq H\left(\varphi^A\right). \tag{4.29}$$

Proof. Note that if $H\left(\psi^A\right) \geq H\left(\varphi^A\right)$, then $|\psi\rangle^{AB}$ can be asymptotically transformed into $|\varphi\rangle^{AB}$ via LOCC (BBPS96a). Therefore, from lemma 4.1, if follows that if $H\left(\psi^A\right) \geq H\left(\varphi^A\right)$, then $|\psi\rangle^{AB}$ can be transformed into $|\varphi\rangle^{AB}$ via approximate catalysis. In order to show the opposite direction, we use the fact that squashed entanglement does not increase under approximate catalysis (lemma 4.2). Therefore, $|\psi\rangle^{AB}$ can be transformed into $|\varphi\rangle^{AB}$ via approximate catalysis, then

$$E_{sq}\left(|\psi\rangle^{AB}\right) \geq E_{sq}\left(|\varphi\rangle^{AB}\right). \tag{4.30}$$

Note that for every bipartite pure state $|\psi\rangle^{AB}$, squashed entanglement is equal to the entanglement entropy $H(\psi^A)$. This completes the proof. □

This shows that catalytic transformations between pure entangled states are completely characterised by entanglement entropy. This gives an operational meaning to entanglement entropy in the single copy regime.

4.2.1 Dimension of the catalyst

Theorem 4.1 provides necessary and sufficient conditions for approximate catalytic transformations between pure entangled states, in terms of entanglement entropy. We will now show that such catalytic transformations, in general, require a catalyst with an unbounded dimension. In order to show this fact, we will make use of logarithmic negativity, given by (idZHSL98; VW02)

$$E_N(\rho) = \log_2 \|\rho^{T_A}\|_1, \tag{4.31}$$

where T_A is the partial transpose with respect to system A and $\rho = \rho^{AB}$ is a bipartite quantum state. Let us note that logarithmic negativity is additive (VW02)

$$E_N(\rho \otimes \sigma) = E_N(\rho) + E_N(\sigma). \tag{4.32}$$

We will now construct a pair bipartite pure states $|\psi\rangle$ and $|\varphi\rangle$ such that

$$E(|\psi\rangle) \geq E(|\varphi\rangle), \tag{4.33a}$$

$$E_N(|\psi\rangle) < E_N(|\varphi\rangle). \tag{4.33b}$$

In order to do this, let us define a 2-qutrit pure state $|\Psi\rangle$ as

$$|\Psi\rangle = \sin\alpha\cos\beta\,|00\rangle + \cos\alpha\cos\beta\,|11\rangle + \sin\beta\,|22\rangle\,. \tag{4.34}$$

We will define $|\psi\rangle$ and $|\varphi\rangle$ by choosing the value of the parameters as $\alpha = 1.3; \beta = 0.75$ and $\alpha = 0.7; \beta = 1$, respectively. One can easily verify that

$$E(|\psi\rangle) = 1.195, \quad E(|\varphi\rangle) = 1.157, \tag{4.35a}$$

$$E_N(|\psi\rangle) = 1.324, \quad E_N(|\varphi\rangle) = 1.361. \tag{4.35b}$$

Therefore $|\psi\rangle$ and $|\varphi\rangle$ satisfy Eqs. (4.33). Let us note that there does not exist a pair of 2-qubit pure states satisfying Eqs. (4.33). In other words, E and E_N impose the same ordering for any pair of 2-qubit pure states. Before we present the main argument, let us show that logarithmic negativity is continuous.

Lemma 4.3. *(DKMS22b) For any two bipartite quantum states ρ and σ of dimension d,*

$$|E_N(\rho) - E_N(\sigma)| \leq \frac{\sqrt{d}}{\ln 2}||\rho - \sigma||_1. \tag{4.36}$$

Proof. We will now show that for any two bipartite states ρ and σ which are arbitrarily close in trace distance, the difference of logarithmic negativities is arbitrarily small. We will make use of the fact that the trace norm and the Hilbert-Schmidt norm satisfy

$$||M||_2 \leq ||M||_1 \leq \sqrt{r}||M||_2, \tag{4.37}$$

for all matrices M. Here r is the rank of M. Also note that $||M||_2 = \left(\sum_{i,j} |M_{ij}|^2\right)^{1/2}$, where M_{ij} are the matrix elements of M. Therefore, $||M||_2$ is invariant under partial transpose i.e, $||M||_2 = ||M^{T_A}||_2$. From Eq. (4.37), we get

$$||\rho - \sigma||_2 \leq ||\rho - \sigma||_1. \tag{4.38}$$

Using the fact that Hilbert-Schmidt norm does not change under partial transpose, we have

$$||\rho^{T_A} - \sigma^{T_A}||_2 = ||\rho - \sigma||_2 \leq ||\rho - \sigma||_1. \tag{4.39}$$

Note that $|\,||A||_1 - ||B||_1| \leq ||A - B||_1$ for any two matrices A and B. Using this fact, along with Eq. (4.37) gives us,

$$|\,||\rho^{T_A}||_1 - ||\sigma^{T_A}||_1| \leq ||\rho^{T_A} - \sigma^{T_A}||_1 \leq \sqrt{r}||\rho^{T_A} - \sigma^{T_A}||_2 \leq \sqrt{r}||\rho - \sigma||_1.$$

Using the definition of logarithmic negativity, we see that

$$\begin{aligned} E_N(\rho) - E_N(\sigma) &= \log_2 ||\rho^{T_A}||_1 - \log_2 ||\sigma^{T_A}||_1 = \log_2 \frac{||\rho^{T_A}||_1}{||\sigma^{T_A}||_1} \\ &\leq \log_2 \left(1 + \frac{\sqrt{r}||\rho - \sigma||_1}{||\sigma^{T_A}||_1}\right) \leq \frac{\sqrt{r}||\rho - \sigma||_1}{\ln 2||\sigma^{T_A}||_1} \\ &\leq \frac{\sqrt{d}||\rho - \sigma||_1}{\ln 2||\sigma^{T_A}||_1} \leq \frac{\sqrt{d}}{\ln 2}||\rho - \sigma||_1. \end{aligned} \tag{4.40}$$

Here we assume $||\rho^{T_A}||_1 \geq ||\sigma^{T_A}||_1$. In the second line, we use $||\rho^{T_A} - \sigma^{T_A}||_1 \leq \sqrt{r}||\rho - \sigma||_1$, along with the relation $\ln(1 + x) \leq x$ for $x > -1$. For the third like, we considered $r \leq d$ and $||\sigma^{T_A}||_1 \geq 1$. Here d represents the dimension of the total bipartite state. By considering the opposite i.e, $||\rho^{T_A}||_1 < ||\sigma^{T_A}||_1$, one gets

$$E_N(\sigma) - E_N(\rho) \leq \frac{\sqrt{d}}{\ln 2}||\rho - \sigma||_1. \tag{4.41}$$

This completes the proof. □

We will now use the above lemma to show that an approximate catalytic transformation from $|\psi\rangle$ to $|\varphi\rangle$ would require a catalyst with unbounded dimension. We will prove this by contradiction. Let us assume $|\psi\rangle$ can be transformed to $|\varphi\rangle$ via an approximate catalytic transformation, such that the catalyst dimension is bounded. Therefore, for every $\varepsilon > 0$, there exists a LOCC operation Λ and a catalyst state τ such that

$$\left\|\Lambda\left(|\psi\rangle\langle\psi|^S \otimes \tau^C\right) - |\varphi\rangle\langle\varphi|^S \otimes \tau^C\right\|_1 \leq \varepsilon, \tag{4.42}$$

$$\mathrm{Tr}_S\left[\Lambda\left(|\psi\rangle\langle\psi|^S \otimes \tau^C\right)\right] = \tau^C, \tag{4.43}$$

Here, τ is a catalyst with a bounded dimension. Let us recall that $|\psi\rangle$ and $|\varphi\rangle$ satisfy the conditions in Eqs. (4.33). Since logarithmic negativity does not increase under LOCC operations, we have

$$E_N(|\psi\rangle\langle\psi| \otimes \tau) \geq E_N(\Lambda[|\psi\rangle\langle\psi| \otimes \tau]). \tag{4.44}$$

Note that, $\|\Lambda(|\psi\rangle\langle\psi| \otimes \tau) - |\varphi\rangle\langle\varphi| \otimes \tau\|_1$ can be made arbitrarily close to zero. If the dimension of the catalyst is finite, from lemma 4.3, it follows that the logarithmic negativity of $\Lambda(|\psi\rangle\langle\psi| \otimes \tau)$ is arbitrarily close to the logarithmic negativity of $|\varphi\rangle\langle\varphi| \otimes \tau$. Therefore,

$$E_N(|\psi\rangle\langle\psi| \otimes \tau) \geq E_N(|\varphi\rangle\langle\varphi| \otimes \tau) \Rightarrow E_N(|\psi\rangle) \geq E_N(|\varphi\rangle). \tag{4.45}$$

Here we used additivity of logarithmic negativity. Thus we arrived at a contradiction with Eq. (4.33b).

Using similar arguments, we will now show that for a catalytic transformation from $|\psi\rangle$ to $|\varphi\rangle$, non-zero correlations between the system and catalyst are required if the states fulfill Eqs. (4.33). In order to show this, consider Eq. (4.42) and assume that there exists a $\varepsilon' > 0$ such that for any ε satisfying $\varepsilon' > \varepsilon > 0$, the system and catalyst are in a product state (no correlations) i.e., $\Lambda(|\psi\rangle\langle\psi| \otimes \tau) = \rho' \otimes \tau$, where $\|\rho' - |\varphi\rangle\langle\varphi|\|_1 \leq \varepsilon$. Again using continuity and additivity properties of logarithmic negativity, we arrive at the condition: $E_N(|\psi\rangle) \geq E_N(|\varphi\rangle)$, which is contradiction with Eq. (4.33b). Therefore, for any pair of states fulfilling Eqs. (4.33) the correlations between the system and the catalyst cannot vanish. But these correlations can be made arbitrarily small.

4.3 Catalytic-asymptotic equivalence

Catalytic and asymptotic transformations may look like very distinct concepts, but lemma 4.1 provides evidence for possible connection between these two concepts. In this section, we will explore this connection and show a complete equivalence between these two kinds of transformations. In order to do this, we need to slightly change the definitions of *approximate catalysis* and *asymptotic transformations*.

We will start by introducing a variant of approximate catalysis called *correlated catalysis*. The main difference between correlated catalysis and approximate catalysis is that, correlated catalysis allows for correlations between system and catalyst, while in the case of approximate catalysis we ensure that correlations can be made arbitrarily small. Formally speaking, we say a quantum state ρ^S can be transformed into σ^S via *correlated catalysis* iff for every ε, there exists a LOCC protocol Λ and a catalyst state

τ^C such that

$$\begin{aligned} \mu^{SC} &= \Lambda(\rho^S \otimes \tau^C), \\ \|\mu^S - \sigma^S\|_1 &< \varepsilon, \;\; \mu^C = \tau^C. \end{aligned} \tag{4.46}$$

Note that $\|\mu^S - \sigma^S\|_1 < \varepsilon$, makes sure the the final state is ε-close to σ^S. It is important to note that, we allow for correlations between system and the catalyst. One can also extend this definition to general resource theories by replacing LOCC with free operations. In fact, when the target state is pure, both the notions of approximate and correlated catalysis coincide. In other words, if a quantum state ρ can be transformed into a pure state $|\psi\rangle$ via correlated catalysis, then ρ can be transformed into $|\psi\rangle$ via approximate catalysis. This can be seen from the proposition below.

Proposition 4.1. *(GKS23) For any quantum state μ^{SC},*

$$\left\|\mu^S - |\varphi\rangle\langle\varphi|^S\right\|_1 < \varepsilon \tag{4.47}$$

implies

$$\left\|\mu^{SC} - |\varphi\rangle\langle\varphi|^S \otimes \mu^C\right\|_1 < \varepsilon + 6\sqrt{\frac{\varepsilon}{2}}. \tag{4.48}$$

Proof. Fuchs-van de Graaf inequalities (see Eq. (2.17)) along with Eq. (4.47) implies the following

$$\begin{aligned} F\left(\mu^S, |\varphi\rangle\langle\varphi|^S\right) &\geq \sqrt{1 - \frac{1}{2}\left\|\mu^S - |\varphi\rangle\langle\varphi|^S\right\|_1} \\ &> \sqrt{1 - \frac{\varepsilon}{2}}. \end{aligned} \tag{4.49}$$

Note that fidelity $F(\rho, \sigma) = \mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$. Let μ^S has a purification given by

$$|\mu\rangle^{ST} = \sum_i \lambda_i |i\rangle^S |i\rangle^T. \tag{4.50}$$

Here λ_i are the Schmidt coefficients in decreasing order.Therefore Eq. (4.49) implies

$$\lambda_0 > \sqrt{1 - \frac{\varepsilon}{2}}. \tag{4.51}$$

Let now consider the purification of μ^{SC} to be $|\nu\rangle^{SCD}$. Notice that, $|\nu\rangle^{SCD}$ can be written as

$$|\nu\rangle^{SCD} = \sum_i \lambda_i |i\rangle^S |\alpha_i\rangle^{CD}, \tag{4.52}$$

with λ_i being the same Schmidt coefficients as in Eq. (4.50). $\{|\alpha_i\rangle\}$ is an orthonormal basis on CD. Let us note that

$$F\left(|\nu\rangle\langle\nu|^{SCD}, |0\rangle\langle 0|^S \otimes |\alpha_0\rangle\langle\alpha_0|^{CD}\right) = \lambda_0, \tag{4.53}$$

and the fact that the fidelity can not increase under partial trace. Using this we obtain the following

$$F\left(\mu^{SC}, |0\rangle\langle 0|^S \otimes \mathrm{Tr}_D\left[|\alpha_0\rangle\langle\alpha_0|^{CD}\right]\right) > \sqrt{1-\frac{\varepsilon}{2}}. \tag{4.54}$$

Using again Eq. (2.17), we get

$$\frac{1}{2}\left\|\mu^{SC} - |0\rangle\langle 0|^S \otimes \mathrm{Tr}_D\left[|\alpha_0\rangle\langle\alpha_0|^{CD}\right]\right\|_1 < \sqrt{\frac{\varepsilon}{2}}. \tag{4.55}$$

Let us also note that trace norm does not increase under partial trace. Therefore

$$\left\|\mu^C - \mathrm{Tr}_D\left[|\alpha_0\rangle\langle\alpha_0|^{CD}\right]\right\|_1 < 2\sqrt{\frac{\varepsilon}{2}}. \tag{4.56}$$

Let us now use the triangle inequality for trace norm, arriving at

$$\begin{aligned}\left\|\mu^{SC} - |0\rangle\langle 0|^S \otimes \mu^C\right\|_1 &\leq \left\|\mu^{SC} - |0\rangle\langle 0|^S \otimes \mathrm{Tr}_D\left[|\alpha_0\rangle\langle\alpha_0|^{CD}\right]\right\|_1 \\ &+ \left\||0\rangle\langle 0|^S \otimes \mathrm{Tr}_D\left[|\alpha_0\rangle\langle\alpha_0|^{CD}\right] - |0\rangle\langle 0|^S \otimes \mu^C\right\|_1 < 4\sqrt{\frac{\varepsilon}{2}}.\end{aligned} \tag{4.57}$$

Using again Eq. (4.55) we find

$$\left\|\mu^S - |0\rangle\langle 0|^S\right\|_1 < 2\sqrt{\frac{\varepsilon}{2}}, \tag{4.58}$$

which together with Eq. (4.47) and triangle inequality implies that

$$\left\||\varphi\rangle\langle\varphi|^S - |0\rangle\langle 0|^S\right\|_1 < \varepsilon + 2\sqrt{\frac{\varepsilon}{2}}. \tag{4.59}$$

We make use of the triangle inequality to obtain the following

$$\begin{aligned}\left\|\mu^{SC} - |\varphi\rangle\langle\varphi|^S \otimes \mu^C\right\|_1 &\leq \left\|\mu^{SC} - |0\rangle\langle 0|^S \otimes \mu^C\right\|_1 \\ &+ \left\||0\rangle\langle 0|^S \otimes \mu^C - |\varphi\rangle\langle\varphi|^S \otimes \mu^C\right\|_1 < \varepsilon + 6\sqrt{\frac{\varepsilon}{2}}.\end{aligned} \tag{4.60}$$

This completes the proof. □

The above proposition shows that achieving a pure state via correlated catalysis implies decoupling between the system and catalyst. Therefore, showing the equivalence between approximate catalysis and correlated catalysis, when the target state is pure.

Let us also introduce a variant of asymptotic transformations, called *marginal asymptotic* transformations. We say a state ρ is *marginally reducible* to σ if for every $\varepsilon, \delta > 0$, there exists a LOCC protocol Λ and natural numbers n, m such that

$$\begin{aligned}\Lambda\left(\rho^{\otimes n}\right) &= \mu_m, \\ \left\|\mu_m^{(i)} - \sigma\right\|_1 &< \varepsilon\,\forall\, i, \\ \frac{m}{n} + \delta &> 1.\end{aligned} \tag{4.61}$$

Here, μ_m is a state on m subsystems, each of them shared by Alice and Bob, and $\mu_m^{(i)}$ is the reduced state of μ_m on i-th subsystem. This notion of marginal asymptotic transformations was first studied in the context of continuous variable systems in (FLTP23). Let us now discuss the difference between asymptotic transformations and the notion of marginal reducibility. The former is requires that the final state μ_m is ε-close to m copies of σ. While the later only requires the marginals $\mu_m^{(i)}$ are ε-close to σ. Note that $\left\|\mu_m - \sigma^{\otimes m}\right\|_1 < \varepsilon$ implies $\left\|\mu_m^{(i)} - \sigma\right\|_1 < \varepsilon \, \forall \, i$, because trace distance does not increase under partial trace. Therefore, if ρ can be asymptotically converted into σ then ρ is marginally reducible to σ.

With these definitions in mind, we will now show that marginal reducibility implies correlated catalysis. We will use the same technique used in lemma 4.1.

Lemma 4.4. *(GKS23) If ρ^S is marginally reducible onto σ^S then ρ^S can be converted into σ^S via correlated catalysis.*

Proof. Let a LOCC protocol Λ converts n copies of ρ into Γ

$$\Gamma = \Lambda\left[\rho^{\otimes n}\right]. \tag{4.62}$$

Here, Γ acts on the system $S_1 \otimes S_2 \otimes \cdots \otimes S_n$, where every S_i is a copy of S. We will denote Γ_i as the reduced state of Γ on $S_1 \otimes S_2 \otimes \cdots \otimes S_i$ and $\Gamma_0 = 1$. Additionally, $\Gamma_i^{(j)}$ denotes the reduced state of Γ_i on S_j for $j \leq i$.

From the definition of marginal reducibility, we know that if ρ is marginally reducible onto σ then for every $\varepsilon > 0$ and $\delta > 0$ there exist natural numbers $m \leq n$ and an LOCC protocol Λ such that

$$\left\|\Gamma_j^{(j)} - \sigma\right\|_1 < \varepsilon, \tag{4.63a}$$

$$\frac{m}{n} + \delta > 1 \tag{4.63b}$$

for all $j \in [1, m]$.

We will now construct a catalyst τ transforming ρ into σ:

$$\tau = \frac{1}{n}\sum_{k=1}^{n} \rho^{\otimes(k-1)} \otimes \Gamma_{n-k} \otimes |k\rangle\langle k|. \tag{4.64}$$

The catalyst C acts on $S_2 \ldots S_n \otimes K$. Here K is an ancillary system held by Alice. K has a dimension of n with basis $\{|k\rangle, k \in [1, n]\}$

Let us now perform the following LOCC protocol on the system and the catalyst:

(i) Alice measures K in the basis $|k\rangle$ (rank one projective measurement) and communicates the measurement outcome to Bob. If the measurement outcome is n, Alice and Bob perform the LOCC protocol Λ (see Eq. (4.62)) on $S_1 \otimes S_2 \otimes \cdots \otimes S_n$. For all the other outcomes of Alice's measurement, both the parties do nothing.

(ii) Alice applies a unitary on K which converts $|n\rangle \to |1\rangle$ and $|i\rangle \to |i+1\rangle$.

(iii) A SWAP unitary is applied by both the parties on their parts of (S_i, S_{i+1}) and (S_1, S_n) to transform $S_i \to S_{i+1}$ and $S_n \to S_1$.

Note that the initial state of the system along with the catalyst is given by

$$\rho \otimes \tau = \frac{1}{n} \sum_{k=1}^{n} \rho^{\otimes k} \otimes \Gamma_{n-k} \otimes |k\rangle \langle k| \,. \tag{4.65}$$

After step (i), $\rho \otimes \tau$ transforms into

$$\mu^{i} = \frac{1}{n} \sum_{k=1}^{n-1} \rho^{\otimes k} \otimes \Gamma_{n-k} \otimes |k\rangle \langle k| + \frac{1}{n}\Gamma \otimes |n\rangle \langle n| \,. \tag{4.66}$$

Step (ii) transforms μ^{i} into μ^{ii}, where

$$\mu^{ii} = \frac{1}{n} \sum_{k=1}^{n} \rho^{\otimes k-1} \otimes \Gamma_{n+1-k} \otimes |k\rangle \langle k| \,. \tag{4.67}$$

If we trace out S_n from μ^{ii}, the resulting state is the initial state of the catalyst τ (see Eq. (4.64)). Therefore, step (iii) transforms μ^{ii} into μ^{SC}, satisfying $\mathrm{Tr}_S[\mu^{SC}] = \tau$. This shows that the state of the catalyst is invariant in this protocol.

Note that μ^{S} can be written as follows

$$\mu^{S} = \frac{1}{n} \sum_{k=1}^{n} \Gamma_{k}^{(k)} = \frac{1}{n} \sum_{k=1}^{m} \Gamma_{k}^{(k)} + \frac{1}{n} \sum_{k=m+1}^{n} \Gamma_{k}^{(k)}. \tag{4.68}$$

We will now show that $\left\|\mu^{S} - \sigma^{S}\right\|_1$ can be made arbitrarily small. Eqs. (4.63) along with Eq. (4.68) imply the following

$$\begin{aligned} \left\|\mu^{S} - \sigma^{S}\right\|_1 &= \frac{1}{n} \left\| \sum_{k=1}^{n} \left(\Gamma_{k}^{(k)} - \sigma^{S}\right) \right\|_1 \\ &\leq \frac{1}{n} \left\| \sum_{k=1}^{m} \left(\Gamma_{k}^{(k)} - \sigma^{S}\right) \right\|_1 \\ &+ \frac{1}{n} \left\| \sum_{k=m+1}^{n} \left(\Gamma_{k}^{(k)} - \sigma^{S}\right) \right\|_1 \\ &< \frac{m}{n}\varepsilon + 2\frac{n-m}{n} < \varepsilon + 2\delta. \end{aligned} \tag{4.69}$$

Note that $\varepsilon > 0$ and $\delta > 0$ can be chosen arbitrarily small. This completes the proof. ◻

In order to show the full equivalence, it is remaining to show that correlated catalysis implies marginal reducibility. The following lemma shows this fact, for distillable states. Let us note that distillable entanglement (E_D) of ρ^{AB}, is asymptotic transformation rate of transforming ρ^{AB} into $\left|\varphi_{+}^{AB}\right\rangle$ i.e, $E_D(\rho^{AB}) = \left(R\left(\rho^{AB} \to \left|\varphi_{+}^{AB}\right\rangle\right)\right)$. A state ρ^{AB} is said to be distillable if $E_D(\rho^{AB}) > 0$.

Lemma 4.5. *(GKS23) If ρ^{S} is distillable and can be converted into σ^{S} via correlated catalysis, then ρ^{S} is marginally reducible onto σ^{S}.*

Proof. Let τ^C be the state of the catalyst such that the following holds

$$\mu^{SC} = \Lambda\left(\rho^S \otimes \tau^C\right), \tag{4.70a}$$

$$\left\|\mu^S - \sigma^S\right\|_1 < \delta, \;\; \mu^C = \tau^C \tag{4.70b}$$

for some $\delta > 0$. Since ρ is a distillable state, it is possible to distill some singlets from a finite number of copies of ρ via LOCC. This allows us to approximate any state τ via LOCC. Precisely speaking, for any $\varepsilon > 0$, there exists is a natural number k and a LOCC map Λ' such that the following holds

$$\left\|\Lambda'\left(\rho^{\otimes k}\right) - \tau\right\|_1 < \varepsilon. \tag{4.71}$$

From now on,we call $\tau_\varepsilon = \Lambda'(\rho^{\otimes k})$ as *ε-approximation of the catalyst*.

We will now perform the following protocol on $\rho^{\otimes n}$:

1. The last k copies of the state $\rho^{\otimes n}$ are transformed into τ_ε (approximately) via LOCC, i.e.,

$$\rho^{\otimes n} \overset{\text{LOCC}}{\longrightarrow} \rho^{\otimes(n-k)} \otimes \tau_\varepsilon. \tag{4.72}$$

2. Making repeated use of the state τ_ε,we transform each of the remaining $n-k$ copies of ρ into the desired state.

Let us now analyse step 2 more carefully. Using the ε-approximation of the catalyst (τ_ε^C), Alice and Bob transform the first state ρ^{S_1} to obtain $\mu_1 \otimes \rho^{\otimes(n-k-1)}$, where μ_1 is given by

$$\mu_1^{S_1C} = \Lambda\left(\rho^{S_1} \otimes \tau_\varepsilon^C\right). \tag{4.73}$$

Note that Λ is the same LOCC protocol as in Eqs. (4.70). Since trace norm does not increase under CPTP maps, we obtain

$$\left\|\mu_1^{S_1C} - \mu^{S_1C}\right\|_1 \leq \left\|\rho^{S_1} \otimes \tau_\varepsilon^C - \rho^{S_1} \otimes \tau^C\right\|_1 < \varepsilon. \tag{4.74}$$

Recall that $\mu^C = \tau^C$. Using Eq. (4.74), we obtain the following inequalities

$$\left\|\mu_1^{S_1} - \mu^{S_1}\right\|_1 < \varepsilon, \tag{4.75}$$

$$\left\|\mu_1^C - \tau^C\right\|_1 < \varepsilon. \tag{4.76}$$

Therefore, μ_1^C is also a ε-approximation of the catalyst state. This allows Alice and Bob to carry out this procedure for other copies of ρ as well. Therefore (after performing step 2) we arrive at a state $\nu^{S_1 \ldots S_{n-k}}$ on $S_1 \ldots S_{n-k}$, such that the reduced states on S_i satisfy the following condition

$$\left\|\nu^{S_i} - \mu^{S_i}\right\|_1 < \varepsilon. \tag{4.77}$$

Using Eqs. (4.70) along with the triangle inequality of trace norm, we find that

$$\left\|\nu^{S_i} - \sigma\right\|_1 < \varepsilon + \delta. \tag{4.78}$$

From the above arguments we can see that for every $\varepsilon > 0$ and $\delta > 0$, we can transform $\rho^{\otimes n}$ into $\nu^{S_1 \dots S_{n-k}}$ fulfilling Eq. (4.78). It is important to note that k only depends on ε and δ and the integer n does not depend on neither ε nor δ. Therefore, we can choose n large enough such that making $(n-k)/n$ is arbitrarily close to 1. This proves ρ is marginally reducible onto σ. This completes the proof. □

Using lemma 4.4 along with 4.5 we arrive at the following result.

Theorem 4.2. *(GKS23) Marginal reducibility and correlated catalysis are fully equivalent for any pair of distillable states ρ and σ.*

Note that all 2-qubit entangled states are distillable (HHH97). Therefore Theorem 4.2 implies a full equivalence between correlated catalysis and marginal reducibility for all two-qubit states. When we consider states beyond two qubits, Theorem 4.2 applies even if the target σ is not a distillable state.

4.4 Catalytic distillable entanglement

We will now use the tools developed in the previous sections to show that catalysis *does not* increase the distillable entanglement of a distillable state (GKS23). At a single copy level, this result implies that approximate catalytic transformations are completely characterised by distillable entanglement, if the target state is pure and the initial state is distillable. This extends the result in theorem 4.1 to distillable initial states.

Let us write down the following definitions which will be used in this section.

- We say that a *asymptotic transformation* from ρ to σ is possible with rate r, if for any $\varepsilon > 0$ and any $\delta > 0$ there exist natural numbers m, n and a LOCC operation Λ such that

$$\Lambda\left(\rho^{\otimes n}\right) = \mu^{S_1 \dots S_m}, \tag{4.79a}$$

$$\left\|\mu^{S_1 \dots S_m} - \sigma^{\otimes m}\right\|_1 < \varepsilon, \tag{4.79b}$$

$$\frac{m}{n} + \delta > r. \tag{4.79c}$$

  The supremum of r fulfilling these properties will be called *asymptotic transformation rate $R(\rho \to \sigma)$*

- We say that a *marginal asymptotic transformation* from ρ to σ is possible with rate r, if for any $\varepsilon > 0$ and any $\delta > 0$ there exist natural numbers m, n and a LOCC operation Λ such that

$$\Lambda\left(\rho^{\otimes n}\right) = \mu^{S_1 \dots S_m}, \tag{4.80a}$$

$$\left\|\mu^{S_i} - \sigma\right\|_1 < \varepsilon \, \forall \, i \leq m, \tag{4.80b}$$

$$\frac{m}{n} + \delta > r. \tag{4.80c}$$

  The supremum of r fulfilling these properties will be called *marginal transformation rate $\tilde{R}(\rho \to \sigma)$*.

- We say that ρ can be transformed into σ via correlated catalysis at rate r, if for any $\varepsilon > 0$ and any $\delta > 0$ there exist natural numbers m, n, a catalyst τ^C and a LOCC operation Λ such that

$$\Lambda\left(\rho^{\otimes n} \otimes \tau^C\right) = \mu^{S_1 \ldots S_m C}, \tag{4.82a}$$
$$\left\|\mu^{S_1 \ldots S_m} - \sigma^{\otimes m}\right\|_1 < \varepsilon, \tag{4.82b}$$
$$\mu^C = \tau^C, \tag{4.82c}$$
$$\frac{m}{n} + \delta > r. \tag{4.82d}$$

The supremum of r fulfilling these properties will be called *catalytic transformation rate* $R_c(\rho \to \sigma)$.

- We say that a *marginal asymptotic transformation with correlated catalysis* from ρ to σ is possible with rate r, if for any $\varepsilon > 0$ and any $\delta > 0$ there exist natural numbers m, n and a LOCC operation Λ such that

$$\Lambda\left(\rho^{\otimes n} \otimes \tau^C\right) = \mu^{S_1 \ldots S_m C}, \tag{4.83a}$$
$$\left\|\mu^{S_i} - \sigma\right\|_1 < \varepsilon \ \ \forall i \leq m, \tag{4.83b}$$
$$\mu^C = \tau^C, \tag{4.83c}$$
$$\frac{m}{n} + \delta > r. \tag{4.83d}$$

The supremum of r fulfilling these properties will be called *marginal catalytic transformation rate* $\tilde{R}_c(\rho \to \sigma)$.

Eq. (4.79) and Eq. (4.80) define asymptotic transformation rate R and marginal transformation rates $\tilde{R}$ respectively. Catalytic versions of asymptotic transformation rate and marginal transformation rate are defined in Eq. (4.82) and Eq. (4.83) respectively. By definition, it follows that

$$\tilde{R}_c(\rho \to \sigma) \geq \tilde{R}(\rho \to \sigma) \geq R(\rho \to \sigma), \tag{4.84}$$
$$\tilde{R}_c(\rho \to \sigma) \geq R_c(\rho \to \sigma) \geq R(\rho \to \sigma). \tag{4.85}$$

One can also easily see that

$$\tilde{R}_c(\rho \to \sigma) = \max\{R(\rho \to \sigma), \tilde{R}(\rho \to \sigma), R_c(\rho \to \sigma), \tilde{R}_c(\rho \to \sigma)\}. \tag{4.86}$$

for any pair of states ρ and σ.

Before going to the main result, we will now give an upper bound on these rates in terms of *squashed entanglement* (see Eq. (4.21)). Since $\tilde{R}_c$ is the largest among these rates, its sufficient to provide an upper bound for $\tilde{R}_c$.

Proposition 4.2. *(GKS23) For any two bipartite states ρ and σ, $\tilde{R}_c$ is bounded as follows:*

$$\tilde{R}_c\left(\rho \to \sigma\right) \leq \frac{E_{sq}\left(\rho\right)}{E_{sq}\left(\sigma\right)}. \tag{4.87}$$

Proof. First, we will introduce *squashed transformation rate*. We say that a bipartite state ρ can be converted into σ with rate r if the following inequalities are fulfilled for all $\varepsilon, \delta > 0$:

$$\Lambda\left(\rho^{\otimes n} \otimes \tau^{C}\right) = \mu^{S_1 \ldots S_m C}, \tag{4.88a}$$

$$\left|E_{sq}(\mu^{S_i}) - E_{sq}(\sigma)\right| < \varepsilon \;\; \forall i \leq m, \tag{4.88b}$$

$$\mu^{C} = \tau^{C}, \tag{4.88c}$$

$$\frac{m}{n} + \delta > r. \tag{4.88d}$$

The squashed transformation rate is the maximal such rate, and it will be denoted by R_{sq}. By continuity of squashed entanglement (AF04), it is clear that $R_{sq}(\rho \to \sigma) \geq \tilde{R}_c(\rho \to \sigma)$.

Let us now consider a LOCC map Λ and a catalyst τ as in Eqs. (4.88). We will use the properties (additivity, monotonicity (under LOCC) and strong super-additivity) of squashed entanglement (CW04) to find

$$\begin{aligned} nE_{sq}(\rho) + E_{sq}(\tau) = E_{sq}\left(\rho^{\otimes n} \otimes \tau^{C}\right) &\geq E_{sq}\left(\mu^{S_1 \ldots S_m C}\right) \\ &\geq \sum_{i=1}^{m} E_{sq}\left(\mu^{S_i}\right) + E_{sq}(\tau). \end{aligned} \tag{4.89}$$

Therefore,

$$nE_{sq}(\rho) \geq \sum_{i=1}^{m} E_{sq}\left(\mu^{S_i}\right). \tag{4.90}$$

Let us use the above inequality along with Eqs. (4.88) to obtain

$$nE_{sq}(\rho) > m\left[E_{sq}(\sigma) - \varepsilon\right]. \tag{4.91}$$

Therefore,

$$\frac{m}{n} < \frac{E_{sq}(\rho)}{E_{sq}(\sigma) - \varepsilon}. \tag{4.92}$$

We will again use Eqs. (4.88) to arrive at

$$r < \frac{E_{sq}(\rho)}{E_{sq}(\sigma) - \varepsilon} + \delta. \tag{4.93}$$

Since $\delta, \varepsilon > 0$ can be made arbitrarily small, we can say that

$$R_{sq}(\rho \to \sigma) \leq \frac{E_{sq}(\rho)}{E_{sq}(\sigma)}. \tag{4.94}$$

Since $R_{sq}(\rho \to \sigma) \geq \tilde{R}_c(\rho \to \sigma)$, the proof is complete. □

We will now show the main result of this section, which can be mathematically phrased as the following theorem.

Theorem 4.3. *(GKS23) For any bipartite pure state $|\varphi\rangle$ and any bipartite distillable state ρ, the following holds*

$$\begin{aligned}\tilde{R}_c(\rho \to |\varphi\rangle) = \tilde{R}(\rho \to |\varphi\rangle) = R_c(\rho \to |\varphi\rangle) \\ = R(\rho \to |\varphi\rangle) = \frac{E_D(\rho)}{H(\varphi^A)}.\end{aligned} \tag{4.95}$$

As a special case, if we fix the target pure state to be a singlet $|\varphi_+\rangle$, theorem 4.3 implies that catalysis cannot increase the distillable entanglement of a distillable state. In order to prove theorem 4.3, we will first show that

$$\tilde{R}\left(\rho \to \sigma\right) = \tilde{R}_c\left(\rho \to \sigma\right) \tag{4.96}$$

holds for any bipartite distillable state ρ. We will then show that

$$R\left(\rho^{A'B'} \to \left|\varphi^{AB}\right\rangle\right) = \tilde{R}\left(\rho^{A'B'} \to \left|\varphi^{AB}\right\rangle\right). \tag{4.97}$$

holds for any pure target state $\left|\varphi^{AB}\right\rangle$. Eq. (4.85), along with Eq. (4.96) and Eq. (4.97) prove theorem 4.3.

Let us now prove the fact in Eq. (4.96).

Lemma 4.6. *(GKS23) For any bipartite distillable state ρ, the following holds for any bipartite state σ*

$$\tilde{R}\left(\rho \to \sigma\right) = \tilde{R}_c\left(\rho \to \sigma\right). \tag{4.98}$$

Proof. From the proof of lemma 4.5, we know that the catalyst τ can be approximated by performing a LOCC on a finite number (k) of copies of ρ, i.e., $\tau_{\varepsilon'} = \Lambda'(\rho^{\otimes k})$ and $\|\tau_{\varepsilon'} - \tau\|_1 < \varepsilon'$. Therefore, we call $\tau_{\varepsilon'}$ as the ε'-approximation of τ.

Similar to the proof of lemma 4.5, consider the following LOCC protocol, now acting on $n+k$ copies of ρ. In the first step, k copies of ρ are transformed into $\tau_{\varepsilon'}$ via LOCC. The total state is now given by $\rho^{\otimes n} \otimes \tau_{\varepsilon'}$. In the second step, Alice and Bob apply the LOCC protocol from Eqs. (4.83). We will denote the resulting state by μ_1, which can be written as

$$\mu_1^{S_1 \ldots S_m C} = \Lambda\left(\rho^{\otimes n} \otimes \tau^C_{\varepsilon'}\right). \tag{4.99}$$

Due to the data-processing inequality of trace norm, the following holds

$$\left\|\mu_1^{S_1 \ldots S_m C} - \mu^{S_1 \ldots S_m C}\right\|_1 \le \left\|\rho^{\otimes n} \otimes \tau_{\varepsilon'} - \rho^{\otimes n} \otimes \tau_{\varepsilon}\right\|_1 < \varepsilon'. \tag{4.100}$$

This implies the following pair of inequalities

$$\left\|\mu_1^{S_i} - \mu^{S_i}\right\|_1 < \varepsilon', \tag{4.101}$$

$$\left\|\mu_1^{C} - \tau^{C}\right\|_1 < \varepsilon'. \tag{4.102}$$

The second inequality shows that μ_1^C is also a ε'-approximation of τ. Using Eqs. (4.83) along with the triangle inequality of trace norm, we obtain

$$\left\|\mu_1^{S_i} - \sigma\right\|_1 < \varepsilon + \varepsilon' \tag{4.103}$$

for all $i \leq m$.

We will now extend this analysis to $2n + k$ copies of ρ. As earlier, we transform k copies of ρ into $\tau_{\varepsilon'}$. The results in a total state given by $\rho^{\otimes n} \otimes \rho^{\otimes n} \otimes \tau_{\varepsilon'}$. As described above in this proof, the first n copies of the state ρ together with $\tau_{\varepsilon'}$ are transformed into μ_1. This results in a total state given by $\mu_1^{S_1 \ldots S_m C} \otimes \rho^{\otimes n}$. We now transform the remaining n copies of ρ via the LOCC protocol given in Eqs. (4.83), using μ_1^C as the catalyst. Let us recall that μ_1^C is a ε'-approximation of τ^C, which is the same as for $\tau_{\varepsilon'}$. Therefore the total state of the systems $S_{m+1} \ldots S_{2m}$ will be given by $\mu_2^{S_{m+1} \ldots S_{2m} C} = \Lambda(\rho^{\otimes n} \otimes \mu_1^C)$, where (using the same arguments as above),

$$\left\| \mu_2^C - \tau^C \right\|_1 < \varepsilon', \tag{4.104}$$

$$\left\| \mu_2^{S_i} - \sigma \right\|_1 < \varepsilon + \varepsilon' \tag{4.105}$$

for all $i \in [m+1, 2m]$.

By repeating the above procedure l times, we can transform the state $\rho^{\otimes ln+k}$ into $\nu^{S_1 \ldots S_{lm}}$, satisfying $\|\nu^{S_i} - \sigma\|_1 < \varepsilon + \varepsilon'$ for all $i \in [1, lm]$. Note that this procedure works for any natural number l. By choosing a large enough l, allows us to make $\frac{lm}{ln+k}$ arbitrarily close to $\frac{m}{n}$, therefore arbitrarily close to $\tilde{R}_c(\rho \to \sigma)$. This shows that ρ can be converted into σ with rate $\tilde{R}_c(\rho \to \sigma)$ via marginal asymptotic transformations. □

We are now left to prove Eq. (4.97). In order to prove this, we will need the following proposition, which shows the strong-super additivity of asymptotic rates (when the target state is pure).

Proposition 4.3. *(GKS23) For any pure state $|\varphi\rangle^{S_3}$ and any state $\mu^{S_1 S_2}$, we have*

$$R\left(\mu^{S_1 S_2} \to |\varphi\rangle^{S_3}\right) \geq R\left(\mu^{S_1} \to |\varphi\rangle^{S_3}\right) + R\left(\mu^{S_2} \to |\varphi\rangle^{S_3}\right). \tag{4.106}$$

Proof. We will proceed as follows. For any two real numbers $\{r_1, r_2\}$, satisfying $r_1 < R\left(\mu^{S_1} \to |\varphi\rangle^{S_3}\right)$ and $r_2 < R\left(\mu^{S_2} \to |\varphi\rangle^{S_3}\right)$, we will show that $r_1 + r_2$ is a feasible rate for the transformation $\mu^{S_1 S_2} \to |\varphi\rangle^{S_3}$. Therefore, we will show that for any $\varepsilon, \delta > 0$, there exist natural numbers m, n, and a LOCC protocol Λ such that

$$\left\| \left(\Lambda \left(\mu^{S_1 S_2} \right)^{\otimes n} \right) - |\varphi\rangle \langle\varphi|^{\otimes m} \right\|_1 < \varepsilon, \tag{4.107a}$$

$$\frac{m}{n} + \delta > r_1 + r_2. \tag{4.107b}$$

Let us now fix an arbitrary $\varepsilon, \delta > 0$. We assume $\varepsilon < 1$ (without any loss of generality) and denote $S_i^{\otimes n}$ as $\mathbb{S}_i$, where $i \in \{1, 2\}$. Note that since r_i ($i \in \{1, 2\}$) are feasible, there exists natural numbers m_i, n and LOCC protocols Λ_i such that the following holds

$$\left\| \Lambda_i \left(\left(\mu^{S_i} \right)^{\otimes n} \right) - |\varphi\rangle \langle\varphi|^{\otimes m_i} \right\|_1 < \frac{\varepsilon^2}{100} < \frac{\varepsilon}{2}, \tag{4.108a}$$

$$\frac{m_i}{n} + \frac{\delta}{2} > r_i. \tag{4.108b}$$

Here, Λ_i is a LOCC map acting on $\mathbb{S}_i \equiv S_i^{\otimes n_i}$, where $i \in \{1,2\}$. From proposition 4.1 and Eq. (4.108a), it follows that

$$\left\|\Lambda_1 \otimes \mathbb{I}_2\left(\left(\mu^{S_1S_2}\right)^{\otimes n}\right) - |\varphi\rangle\langle\varphi|^{\otimes m_1} \otimes \left(\mu^{S_2}\right)^{\otimes n}\right\|_1 < \frac{\varepsilon^2}{100} + 6\frac{\varepsilon}{\sqrt{200}} < \frac{\varepsilon}{2}.$$

Since trace norm does not increase under CPTP maps, the following holds

$$\left\|\Lambda_1 \otimes \Lambda_2\left(\left(\mu^{S_1S_2}\right)^{\otimes n}\right) - |\varphi\rangle\langle\varphi|^{\otimes m_1} \otimes \Lambda_2\left(\left(\mu^{S_2}\right)^{\otimes n}\right)\right\|_1 < \varepsilon/2.$$

Now we use Eq. (4.108a) along with the triangle inequality of the trace norm, arriving at

$$\begin{aligned}\left\|\Lambda_1 \otimes \Lambda_2\left(\left(\mu^{S_1S_2}\right)^{\otimes n}\right) - |\varphi\rangle\langle\varphi|^{\otimes(m_1+m_2)}\right\|_1 &\leq \left\|\Lambda_1 \otimes \Lambda_2\left(\left(\mu^{S_1S_2}\right)^{\otimes n}\right) - |\varphi\rangle\langle\varphi|^{\otimes m_1} \otimes \Lambda_2\left(\left(\mu^{S_2}\right)^{\otimes n}\right)\right\|_1 \\ &+ \left\||\varphi\rangle\langle\varphi|^{\otimes m_1} \otimes \Lambda_2\left(\left(\mu^{S_2}\right)^{\otimes n}\right) - |\varphi\rangle\langle\varphi|^{\otimes(m_1+m_2)}\right\|_1 \\ &< \varepsilon. \end{aligned} \tag{4.109}$$

From Eq. (4.108b), it follows that

$$\frac{m_1+m_2}{n} + \delta > r_1 + r_2. \tag{4.110}$$

Eqs. (4.107) can be fulfilled by choosing $m = m_1 + m_2$, $\Lambda = \Lambda_1 \otimes \Lambda_2$. This implies, $R\left(\mu^{S_1} \to |\varphi\rangle^{S_3}\right) + R\left(\mu^{S_2} \to |\varphi\rangle^{S_3}\right)$ is a feasible rate for $\mu^{S_1S_2} \to |\varphi\rangle^{S_3}$. This completes the proof. □

With this, we are now ready to prove the fact in Eq. (4.97).

Lemma 4.7. *(GKS23) For pure target states, the standard transformation rate coincides with the marginal transformation rate :*

$$R\left(\rho^{A'B'} \to \left|\varphi^{AB}\right\rangle\right) = \tilde{R}\left(\rho^{A'B'} \to \left|\varphi^{AB}\right\rangle\right). \tag{4.111}$$

Proof. If the target state is not entangled, both $R\left(\rho^{A'B'} \to \left|\varphi^{AB}\right\rangle\right)$ and $\tilde{R}\left(\rho^{A'B'} \to \left|\varphi^{AB}\right\rangle\right)$ diverge to infinity. Therefore, we assume that the target state is entangled.

We will now introduce a new kind of asymptotic transformation. Here, we say that $\rho^{A'B'}$ is transformed into $\left|\varphi^{AB}\right\rangle$ with rate r if for any $\varepsilon, \delta > 0$ there exist natural numbers m, n and a LOCC protocol Λ such that

$$\Lambda\left(\rho^{A'B'\otimes n}\right) = \mu^{S_1\ldots S_m}, \tag{4.112a}$$

$$\left|E_D(\mu^{S_i}) - H(\varphi^A)\right| < \varepsilon \;\; \forall i \leq m, \tag{4.112b}$$

$$\frac{m}{n} + \delta > r. \tag{4.112c}$$

The supremum of such achievable rates will be denoted as $\tilde{R}_{\mathrm{m}}\left(\rho^{A'B'}, \left|\varphi^{AB}\right\rangle\right)$. Here, we are transforming ρ asymptotically into a state with marginals having distillable entanglement close to $E_D\left(\left|\varphi^{AB}\right\rangle\right) = H(\varphi^A)$. Note that the distillable entanglement can be bounded as follows (Rai01; DW05)

$$H(\rho^{A'}) - H(\rho^{A'B'}) \leq E_D(\rho^{A'B'}) \leq H(\rho^{A'}). \tag{4.113}$$

This implies that E_D is continuous near any pure state. Therefore $\tilde{R}_{\mathrm{m}}\left(\rho^{A'B'}, \left|\varphi^{AB}\right\rangle\right) \geq \tilde{R}\left(\rho^{A'B'} \to \left|\varphi^{AB}\right\rangle\right)$. Note that, for pure target states, that following holds (BBPS96b),

$$R\left(\rho^{A'B'} \to \left|\varphi^{AB}\right\rangle\right) = \frac{E_D(\rho^{A'B'})}{H(\varphi^A)}. \tag{4.114}$$

Now we will consider an LOCC protocol achieving Eqs. (4.112). Using Eq. (4.114) along with the properties of distillable entanglement (see Proposition 4.3) we find the following

$$\begin{aligned} nR\left(\rho^{A'B'} \to \left|\varphi^{AB}\right\rangle\right) = n\frac{E_D(\rho^{A'B'})}{H(\varphi^A)} = \frac{E_D\left(\rho^{A'B'\otimes n}\right)}{H(\varphi^A)} \geq \frac{E_D\left(\mu^{S_1 \ldots S_m}\right)}{H(\varphi^A)} \\ \geq \frac{1}{H(\varphi^A)} \sum_{i=1}^{m} E_D\left(\mu^{S_i}\right). \end{aligned} \tag{4.115}$$

Using the above inequality along with Eqs. (4.112), we obtain

$$nR\left(\rho \to |\varphi\rangle\right) > m\left(1 - \frac{\varepsilon}{H(\varphi^A)}\right). \tag{4.116}$$

Therefore we get

$$\frac{m}{n} < \frac{R\left(\rho \to |\varphi\rangle\right)}{1 - \frac{\varepsilon}{H(\varphi^A)}}. \tag{4.117}$$

We will use Eqs. (4.112) once again to arrive at the following

$$r < \frac{R\left(\rho \to |\varphi\rangle\right)}{1 - \frac{\varepsilon}{H(\varphi^A)}} + \delta. \tag{4.118}$$

Note that $\varepsilon, \delta > 0$ can be made arbitrarily small. Therefore, we conclude that $\tilde{R}_{\mathrm{m}}(\rho, \varphi) \leq R(\rho \to \varphi)$. Putting the above arguments together, we get

$$\tilde{R}_{\mathrm{m}}(\rho, |\varphi\rangle) \geq \tilde{R}(\rho \to |\varphi\rangle) \geq R(\rho \to |\varphi\rangle) \geq \tilde{R}_{\mathrm{m}}(\rho, |\varphi\rangle). \tag{4.119}$$

This shows that all these inequalities are equalities. This completes the proof. □

4.4.1 Single copy transformations

In this subsection, we will extend theorem 4.1 to allow for arbitrary initial states (distillable) and show that distillable entanglement completely characterises approximate catalytic transformations, when the target state is pure. This will completely characterise single copy approximate catalytic transformations when the target state is pure.

Proposition 4.4. *(GKS23) For any bipartite distillable state ρ and any bipartite pure state $|\varphi\rangle$, ρ can be converted into $|\varphi\rangle$ via approximate catalysis iff*

$$E_D(\rho) \geq E_D(|\varphi\rangle) = H(\varphi^A) \tag{4.120}$$

Proof. Theorem 4.2 shows that marginal reducibility and correlated catalysis are equivalent when ρ is a distillable state. Furthermore, since the target is a pure state, lemma 4.7 shows that marginal reducibility and asymptotic transformation are equivalent. Using these two facts, we can deduce that an asymptotic transformation exists from ρ to $|\varphi\rangle$ iff ρ can be transformed into $|\varphi\rangle$ via correlated catalysis. From (BBP$^+$96), we know that an asymptotic transformation exists from ρ to $|\varphi\rangle$ iff $E_D(\rho) \geq H(\varphi^A)$. Finally, we use proposition 4.1, which says that, when the target state is pure, correlated catalysis is equivalent to approximate catalysis. This completes the proof. □

Recently, in (LRS23), the authors showed that if a bipartite quantum state ρ is not distillable i.e, $E_D(\rho) = 0$, then it cannot be converted into any bipartite state σ with $E_D(\sigma) > 0$ via correlated catalysis. Note that, for any bipartite entangled pure state $|\psi\rangle$, $E_D(|\psi\rangle) > 0$. This result along with proposition 4.4, show that distillable entanglement completely characterises approximate catalytic transformations, if the target state is pure.

4.5 Entanglement catalysis for noisy channels

Quantum capacity of a noisy channel (Λ) corresponds to the optimal rate at which a sender can faithfully send qubits via Λ, to a receiver (Llo97; SN96; HHH00). This quantity has a close relation with the so-called coherent information of a channel Λ, given by (SN96; Llo97)

$$I(\rho, \Lambda) = H(\Lambda[\rho]) - H(\mathbb{I} \otimes \Lambda \left[\left|\psi_\rho\right\rangle\!\left\langle\psi_\rho\right| \right], \tag{4.121}$$

where $\left|\psi_\rho\right\rangle$ is a purification of ρ. Quantum capacity of Λ can be expressed in terms of coherent information as (Llo97; Sho02; Dev05):

$$Q(\Lambda) = \lim_{n\to\infty} \frac{1}{n} \max_{\rho_n} I(\rho_n, \Lambda^{\otimes n}). \tag{4.122}$$

This notion of quantum capacity has been explored for various classes of quantum channels like low noise channels (LLS18), symmetric side channels (SSW08), Pauli channels (Cer00), arbitrarily correlated noise channels (BD10) and bosonic channels (HW01; WPGG07). The central assumption in the study of quantum capacity, is that the parties have access to many copies of the quantum channel. Here we drop this assumption and assume that the parties (Alice and Bob) have access to the quantum channel only once, but have access to infinite rounds of classical communication. Additionally, we will assume that Alice and Bob have access to a quantum catalyst τ_n^{AB} [1].

In this setting, Alice and Bob also possess registers A' and B', of the same dimension $d_{A'} = d_{B'}$. Let the initial states of these registers be $|0\rangle\langle 0|^{A'}$ and $|0\rangle\langle 0|^{B'}$. The goal

[1] In this section the catalyst system will be called AB, and the primary system will be denoted by $A'B'$.

of this protocol is to entangle A' and B' using a carrier particle C which will be sent from Alice to Bob via the noisy channel Λ. Without loss of generality, we assume the carrier particle is initialized in $|0\rangle\langle 0|^C$. Therefore, the full initial state is given by

$$\sigma_n = \sigma_n^{ABA'B'C} = \tau_n^{AB} \otimes |0\rangle\langle 0|^{A'} \otimes |0\rangle\langle 0|^{B'} \otimes |0\rangle\langle 0|^C . \tag{4.123}$$

Let us now consider the following protocol, consisting of three steps:

- LOCC pre-processing: Alice and Bob perform a LOCC operation on σ_n transforming it into ν_n.
- Alice sends the carrier particle to Bob via the quantum channel Λ^C, resulting in the state $\chi_n = \Lambda^C[\nu_n]$. Note, that the particle C is with Bob now.
- Post-processing: Alice and Bob perform a LOCC protocol on χ_n, transforming it into μ_n.

We constraint the final state μ_n to satisfy the following property, for each n:

$$\mathrm{Tr}_{A'B'}\left[\mu_n^{ABA'B'}\right] = \tau_n^{AB}. \tag{4.124}$$

Eq. (4.124) makes sure that the catalyst is unchanged and can be reused. We also require that the catalyst decomposes with $A'B'$ in the limit of large n i.e,

$$\lim_{n\to\infty} \left\|\mu_n^{ABA'B'} - \tau_n^{AB} \otimes \mu_n^{A'B'}\right\|_1 = 0. \tag{4.125}$$

Eq. (4.125) restricts the correlations between two different systems, if they both are transformed using the same catalyst.

The above mentioned protocol, is the most general protocol performed by Alice and Bob, if they have access to single use of a quantum channel Λ and additionally, have access to a quantum catalyst τ_n^{AB}. As a special case, one can consider $\mu_n^{A'B'}$ to be close to maximally entangled state of m qubits i.e,

$$\lim_{n\to\infty} \left\|\mu_n - \tau_n^{AB} \otimes \left|\varphi_{2^m}^+\right\rangle\!\left\langle\varphi_{2^m}^+\right|^{A'B'}\right\|_1 = 0. \tag{4.126}$$

Note that $\left|\varphi_{2^m}^+\right\rangle$ can be used to teleport m qubits. Therefore, Alice and Bob can use this procedure to send m qubits with arbitrary accuracy, by a single use of quantum channel Λ. We will say that the channel Λ can transmit m qubits, if Eq. (4.126) holds for some natural number $m \geq 1$.

4.5.1 Catalytic quantum capacity

We will now define *catalytic quantum capacity* Q_c of a channel Λ as the maximum number of qubits the channel can faithfully transmit. Formally, let $\{\tau_n^{AB}\}$ and $\{\mu_n\}$ be a sequence of catalysts and final states respectively, such that Eqs. (4.124) and (4.126) are satisfied for some natural number $m \geq 1$. Catalytic quantum capacity of the channel Λ, is the maximum possible m.

$$Q_c(\Lambda) = \max\left\{m : \lim_{n\to\infty} \left\|\mu_n - \tau_n^{AB} \otimes \left|\varphi_{2^m}^+\right\rangle\!\left\langle\varphi_{2^m}^+\right|^{A'B'}\right\|_1 = 0\right\}. \tag{4.127}$$

If Eq. (4.127) is not satisfied for any $m \geq 1$, then catalytic quantum capacity of Λ is zero.

The key differences between the catalytic quantum capacity and standard quantum capacity are the following:

- Infinite amount of classical communication is allowed in the definition of catalytic quantum capacity, while the definition of standard quantum capacity does not allow for any classical communication.
- Catalytic quantum capacity assumes a single use of the quantum channel, whereas the standard quantum capacity is defined in the limit of infinite parallel uses of the channel.

Let us now consider a concrete example. We will here assume the system A to be a qubit and the carrier C to be a qutrit. Note that, the carrier particle is initially with Alice. Alice uses the LOCC pre-processing step to entangle the carrier C with an additional qutrit system A'', locally creating a two-qutrit maximally entangled state $\left|\varphi_3^+\right\rangle^{A''C}$. Then carrier C is sent to Bob through the noisy quantum channel Λ. Now Alice and Bob end up with a shared two-qutrit state given by

$$\xi = \mathbb{I} \otimes \Lambda\left[\left|\varphi_3^+\right\rangle\left\langle\varphi_3^+\right|\right] \tag{4.128}$$

Note that ξ is also called the Choi state of the channel Λ. Until now, Alice and Bob did not use the catalyst. They will make use of the catalyst in the post-processing step, to convert ξ into a Bell state. From proposition 4.4, we know that this is possible whenever $E_D(\xi) \geq 1$, i.e, Alice and Bob obtain a state μ_n (satisfying Eq. (4.124)), such that

$$\lim_{n\to\infty}\left\|\mu_n - \tau_n^{AB} \otimes \left|\varphi_2^+\right\rangle\left\langle\varphi_2^+\right|^{A'B'}\right\|_1 = 0. \tag{4.129}$$

This allows Alice and Bob to share a maximally entangled state (with arbitrary small error) between them.

Using the procedure mentioned above, Alice can faithfully transmit a qubit to Bob, whenever

$$E_D\left(\mathbb{I} \otimes \Lambda\left[\left|\varphi_3^+\right\rangle\left\langle\varphi_3^+\right|\right]\right) \geq 1 \tag{4.130}$$

It is important to note that, the use of maximally entangled state is not necessary in this protocol. Alice can locally create some other 2-qutrit state ρ, instead of $\left|\varphi_3^+\right\rangle\left\langle\varphi_3^+\right|$. The procedure would still work if $E_D\left(\mathbb{I} \otimes \Lambda[\rho]\right) \geq 1$. Therefore, using the same argument, Alice can transmit m-qubits to Bob via the noisy channel Λ if there exists a 2-qutrit state ρ such that

$$E_D\left(\mathbb{I} \otimes \Lambda[\rho]\right) \geq m. \tag{4.131}$$

Note that the distillable entanglement of any bipartite state ρ^{AB} can be lower bounded by the so-called hashing bound given by: (DW05)

$$E_D(\rho^{AB}) \geq H(\rho^A) - H(\rho^{AB}) \tag{4.132}$$

Eq. (4.131) along with Eq. (4.132) imply that a d-dimensional quantum channel Λ can transmit m-qubits if

$$H\left(\mathbb{I} \otimes \Lambda\left[\left|\varphi_d^+\right\rangle\left\langle\varphi_d^+\right|\right]\right) \leq \log_2 d - m. \tag{4.133}$$

In the case when $d = 3$, Eq. (4.133) says that any qutrit channel satisfying

$$H(\mathbb{I} \otimes \Lambda \left[|\varphi_3^+\rangle\langle\varphi_3^+| \right]) \leq 0.58 \tag{4.134}$$

can transmit a qubit faithfully.

Let us study the implications of Eq. (4.133) more closely. A quantum channel Λ acting on a part of bipartite state ρ can be expressed as follows

$$\mathbb{I} \otimes \Lambda(\rho) = \sum_{i=1}^{k} (\mathbb{I} \otimes K_i)\rho(\mathbb{I} \otimes K_i^\dagger), \tag{4.135}$$

where k is the minimal number of Kraus operators of the map Λ and $\sum_{i=1}^{k} K_i^\dagger K_i = \mathbb{I}$. Note that, the final state $\mathbb{I} \otimes \Lambda(\rho)$ can also be written as

$$\mathbb{I} \otimes \Lambda(\rho) = \sum_i p_i \mu_i. \tag{4.136}$$

Here, μ_i are quantum states and p_i are the probabilities given by

$$p_i = \mathrm{Tr}\left[(\mathbb{I} \otimes K_i)\rho(\mathbb{I} \otimes K_i^\dagger) \right], \tag{4.137a}$$

$$\mu_i = \frac{(\mathbb{I} \otimes K_i)\rho(\mathbb{I} \otimes K_i^\dagger)}{p_i}. \tag{4.137b}$$

Let $\rho = |\varphi_d^+\rangle\langle\varphi_d^+|$. In this case, all μ_i are pure states, given by

$$H\left(\mathbb{I} \otimes \Lambda \left[|\varphi_d^+\rangle\langle\varphi_d^+| \right] \right) \leq H(\vec{p}). \tag{4.138}$$

Here $H(\vec{p}) = -\sum_i p_i \log_2 p_i$. Eq. (4.133), along with Eq. (4.138) implies that a channel Λ can transmit m-qubits if

$$H(\vec{p}) \leq \log_2 d - m. \tag{4.139}$$

Therefore we obtain a lower bound on the catalytic quantum capacity given by

$$Q_c(\Lambda) \geq \lfloor \log_2 d - H(\vec{p}) \rfloor. \tag{4.140}$$

Eq. (4.140) implies that any quantum channel Λ of dimension $d \geq 4$ which can be decomposed into at most 2 Kraus can transmit at least one qubit.

For example consider the following channel

$$\Lambda[\rho] = (1-p)\rho + pU\rho U^\dagger. \tag{4.141}$$

Here U is an arbitrary unitary and $p \in [0, 1/2]$. Above discussion shows that, Λ can transmit m qubits if:

$$-p \log_2 p - (1-p)\log_2(1-p) \leq \log_2 d - m. \tag{4.142}$$

For the case when $d = 3$ and $m = 1$, this condition is satisfied whenever $p \in [0, 0.1403]$. From the above discussion, we also see that, when $d \geq 4$ and $m = 1$, any such channel can perfectly transmit a single qubit.

Let us now focus on the converse, providing necessary conditions a channel Λ must satisfy in order to send m-qubits perfectly. We will make use of squashed entanglement (see Eq. (4.21)) in order to give such conditions. Let us now define the amount of transmitted entanglement by a quantum channel Λ as:

$$\Delta E_{sq}(\Lambda) = \sup_{\rho^{ABC}} \left\{ E_{sq}^{A|BC}(\Lambda^C[\rho^{ABC}]) - E_{sq}^{AC|B}(\rho^{ABC}) \right\}. \tag{4.143}$$

Here the supremum is taken over all tripartite states ρ^{ABC}. We are now prove the following theorem, using these tools.

Theorem 4.4. *(DKMS22b) For a channel Λ, the catalytic quantum capacity is upper bounded as follows*

$$Q_c(\Lambda) \leq \Delta E_{sq}(\Lambda). \tag{4.144}$$

Proof. In order to show this upper bound, we will consider the most general procedure Alice and Bob could perform (see below Eq. (4.123)). In the pre-processing step, Alice and Bob perform a LOCC protocol on σ_n, transforming it into ν_n. Note that a general LOCC protocol allows Alice and Bob to attach local systems. Therefore, we denote ν_n by $\nu_n^{\tilde{A}\tilde{B}C}$. Here $\tilde{A}$ and $\tilde{B}$ include AA' and BB' respectively, additionally including the particles that Alice and Bob attached locally. Using the fact that squashed entanglement does not increase under LOCC operations, we get

$$E_{sq}^{\tilde{A}C|\tilde{B}}(\nu_n) \leq E_{sq}^{AA'C|BB'}(\sigma_n). \tag{4.145}$$

Next, Alice sends the carrier particle C via a quantum channel Λ to Bob, resulting in the state $\chi_n^{\tilde{A}\tilde{B}C} = \Lambda^C[\nu_n^{\tilde{A}\tilde{B}C}]$. It is important to note that, the squashed entanglement (between Alice and Bob) can increase in this step. The increase can be upper bounded by $\Delta E_{sq}(\Lambda)$: (see Eq. (4.143))

$$E_{sq}^{\tilde{A}|\tilde{B}C}(\chi_n) - E_{sq}^{\tilde{A}C|\tilde{B}}(\nu_n) \leq \Delta E_{sq}(\Lambda). \tag{4.146}$$

Finally, Alice and Bob perform LOCC post-processing on $\chi_n^{\tilde{A}\tilde{B}C}$, converting it into μ_n. Note that in this post-processing step, Alice and Bob trace out all the systems except $AA'BB'$. Therefore $\mu_n = \mu_n^{AA'BB'}$. Since squashed entanglement does not increase under LOCC operations, we have

$$E_{sq}^{AA'|BB'}(\mu_n) \leq E_{sq}^{\tilde{A}|\tilde{B}C}(\chi_n). \tag{4.147}$$

Therefore the total increase of squashed entanglement in this procedure is upper bounded by

$$\begin{aligned} E_{sq}^{AA'|BB'}(\mu_n) - E_{sq}^{AA'C|BB'}(\sigma_n) &\leq E_{sq}^{\tilde{A}|\tilde{B}C}(\chi_n) - E_{sq}^{\tilde{A}C|\tilde{B}}(\nu_n) \\ &\leq \Delta E_{sq}(\Lambda). \end{aligned} \tag{4.148}$$

Note that $\mu_n^{AB} = \tau_n^{AB}$ (catalyst remains unchanged in the procedure). Using strong super-additivity of squashed entangled, we get

$$E_{sq}^{AA'|BB'}(\mu_n^{AA'BB'}) \geq E_{sq}^{A|B}(\mu_n^{AB}) + E_{sq}^{A'|B'}(\mu_n^{A'B'}) \tag{4.149}$$

$$= E_{sq}^{A|B}(\tau_n^{AB}) + E_{sq}^{A'|B'}(\mu_n^{A'B'}). \tag{4.150}$$

Also, since $\sigma_n = \sigma_n^{ABA'B'C} = \tau_n^{AB} \otimes |0\rangle\langle 0|^{A'} \otimes |0\rangle\langle 0|^{B'} \otimes |0\rangle\langle 0|^{C}$, its easy to see that

$$E_{sq}^{AA'C|BB'}(\sigma_n) = E_{sq}^{A|B}(\tau_n^{AB}). \tag{4.151}$$

Using Eq. (4.148), along with Eq. (4.150) and Eq. (4.151) we obtain

$$E_{sq}^{A'|B'}(\mu_n^{A'B'}) \leq \Delta E_{sq}(\Lambda). \tag{4.152}$$

If m qubits can be faithfully transmitted via the channel Λ, then the state $\mu_n^{A'B'}$ can be made arbitrarily close to $\left|\varphi_{2^m}^+\right\rangle$. Since squashed entanglement is continuous (AF04), we get $m \leq \Delta E_{sq}(\Lambda)$. This completes the proof. □

Note that, evaluating ΔE_{sq}, is a very challenging task. However we will provide analytical upper bounds on catalytic quantum capacity, for qubit Pauli channels. Let us consider single qubit Pauli channel given by

$$\Lambda_p[\rho] = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i \tag{4.153}$$

Here $\{\sigma_i\}$ are the Pauli operators. Theorem 8 in (SADL15) shows that the best way to distribute entanglement is to send one half of a Bell state through the channel. Since ΔE_{sq} quantifies the optimal amount entanglement that can be distributed via a given quantum channel Λ, it follows that

$$\Delta E_{sq}\left(\Lambda_p\right) = E_{sq}\left(\mathbb{I} \otimes \Lambda_p\left[\left|\varphi_2^+\right\rangle\left\langle\varphi_2^+\right|\right]\right). \tag{4.154}$$

More generally, for any channel which is a tensor product of Pauli channels (possibly different), the optimal entanglement distribution procedure is to send one half of a maximally entangled state (SADL15). As a special case, for any two-qubit channel given by $\Lambda_p \otimes \Lambda_p$, the following holds

$$\Delta E_{sq}\left(\Lambda_p \otimes \Lambda_p\right) = 2E_{sq}\left(\mathbb{I} \otimes \Lambda_p\left[\left|\varphi_2^+\right\rangle\left\langle\varphi_2^+\right|\right]\right). \tag{4.155}$$

Note that the the entanglement of formation E_f is a upper bound on the squashed entanglement (CW04). Entanglement of formation for pure states is the entanglement entropy ($E_f(|\psi\rangle^{AB}) = H(\psi^A)$) and for mixed states E_f is defined as the minimal average entanglement entropy of the state (BDSW96):

$$E_f(\rho) = \min \sum_i p_i E_f(|\psi_i\rangle). \tag{4.156}$$

Here the minimisation is over all pure state decompositions of the state ρ. Therefore, from Eq. (4.155) we see that

$$\Delta E_{sq}\left(\Lambda_p \otimes \Lambda_p\right) \leq 2E_f\left(\mathbb{I} \otimes \Lambda_p\left[\left|\varphi_2^+\right\rangle\left\langle\varphi_2^+\right|\right]\right). \tag{4.157}$$

Theorem 4.4 implies that for a two-qubit channel $\Lambda_p \otimes \Lambda_p$, the catalytic quantum capacity vanishes if

$$E_f\left(\mathbb{I} \otimes \Lambda_p\left[\left|\varphi_2^+\right\rangle\left\langle\varphi_2^+\right|\right]\right) < \frac{1}{2}. \tag{4.158}$$

Note that, for 2-qubit states, the entanglement of formation has a closed expression (Woo98b). Therefore, it is easy to check whether a given Pauli channel satisfies Eq. (4.158) or not. Since $\mathbb{I} \otimes \Lambda_p \left[\left|\varphi_2^+\right\rangle\!\left\langle\varphi_2^+\right|\right]$ is diagonal in the Bell basis, the entanglement of formation is given by (BDSW96)

$$\begin{cases} E_f\left(\mathbb{I} \otimes \Lambda_p \left[\left|\varphi_2^+\right\rangle\!\left\langle\varphi_2^+\right|\right]\right) = h\left(\frac{1}{2} + \sqrt{p_{\max}(1 - p_{\max})}\right) & \text{for } p_{\max} > 1/2 \\ E_f = 0 & \text{otherwise.} \end{cases} \tag{4.159}$$

Here $p_{\max} = \max\{p_i\}$. Using Eq. (4.158) and Eq. (4.159), one can see that $Q_c(\Lambda_p \otimes \Lambda_p) = 0$, whenever $p_{\max} < 0.813$.

4.6 Conclusions

In this chapter, we first introduced various notions of catalytic transformations in the context of entanglement theory. For the case of pure states, we show that entanglement entropy characterises catalytic transformations when we allow for arbitrarily small correlations between the system and the catalyst. We then show that such catalytic transformations between pure entangled states require (in general) an unbounded catalyst. Going further, we studied the connection between catalytic transformations and asymptotic transformations. For distillable states, we show that both these settings are equivalent. Using this equivalence we show that distillable entanglement of a distillable state cannot be increased by allowing for catalysis. As a consequence, this result implies that distillable entanglement completely characterises single copy catalytic transformations when the target state is pure and the initial state is distillable. We then study the role of catalysis in transmitting quantum information via a noisy channel. We show that entangled catalysts allow us to faithfully send quantum information through a noisy quantum channel. Our results show that, as long as its not too noisy, any quantum channel can faithfully transmit qubits. Precisely speaking, we show that it is possible to faithfully transmit m qubits via a d-dimensional quantum channel whenever the von Neumann entropy of its Choi state is no more than $\log_2 d - m$. We then introduced the notion of catalytic capacity of a quantum channel, quantifying the number of qubits which can be faithfully transmitted via a noisy channel, when we allow for entangled catalysis. We then provide lower and upper bounds on the catalytic capacity.

Chapter 5

Summary and outlook

In this final chapter, we summarize the key findings and contributions of this thesis and offer suggestions for future research.

In this thesis, we studied how various resource states can be transformed into each other via free operations. We started with the notion of deterministic transformations and introduced the idea of resource monotones in Chapter 2. Resource monotones provide necessary conditions for the existence of a deterministic between two quantum states. We look at the problem of finding a finite complete set of resource monotones, which completely characterise deterministic transformations. We prove that, for a wide class of resource theories, such a set does not exist. However, we show that by allowing discontinuity, considering infinite sets and allowing for catalytic transformations one can overcome this restriction and provide a complete set of monotones.

We then go further and study theories with a single complete monotone. We show that such theories are equivalent with theories having a total order i.e, there exists a deterministic transformation transforming either ρ into σ or σ into ρ, for all ρ and σ. For such totally ordered theories, we show that all pure states are inter-convertible, via free operations. We then completely characterise all qubit resource theories having a total order. For higher dimensions ($d \geq 3$), it remains an open question to show the existence of a totally ordered resource theory. Another open question corresponds to extending the results from this chapter to resource theories of quantum channels, where quantum channels are transformed via free super-channels.

In chapter 3 we extend this notion of state transformations, allowing for a probability of failure. We study the trade-off between the success probability and the transformation error. For single copy transformations, we give upper bounds on the achievable probability for a given fidelity of transformation and achievable fidelity for a given probability of success. We then show that these single copy bounds provide non-trivial constraints on asymptotic transformations, providing upper bounds on the achievable rates.

We then focus on bipartite state transformations via SLOCC. For pure bipartite states, we provide a complete solution for single-copy transformations. For 2-qubit states, we again provide an analytical solution for stochastic-approximate state conversion when the initial state is pure. We then extend these results to the resource theory of imaginarity, where we provide a complete solution for single-copy transformations, when the initial state is pure. An important point to note is that all the analytical results about single copy transformations assume a pure initial state. It would be an interesting direction to extend these results to cases where the initial states can be potentially mixed.

Chapter 4 deals with entanglement catalysis. Firstly, we show that the entanglement entropy completely characterises catalytic LOCC transformations between bipartite pure states, when we allow for arbitrarily small correlations between the system and the catalyst. We then show an equivalence between catalytic setting and the setting of marginal asymptotic transformations. Using this equivalence we prove that catalysis does not help us distill singlets with a higher rate (asymptotically), assuming the initial state is distillable. We then show that, at a single copy level, catalytic transformations are completely characterised by the distillable entanglement, if the initial state is distillable and the target state is pure. It is an open problem to show that catalysis offers no advantage in increasing the asymptotic rate for arbitrary target states, when the initial state in pure. We leave this question for future research.

We then developed various methods to estimate the number of qubits which can be faithfully transmitted via a noisy channel. In this context we define catalytic quantum capacity of a noisy channel, as the optimal number of qubits which the channel can reliably transmit. We then provide nontrivial upper and lower bounds on the catalytic quantum capacity.

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