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Report on Andrzej Chlebicki's PhD thesis:
O(N) Models and Their Anisotropic Extensions

Andrzej Chlebicki's thesis is devoted to the study of some aspects of the O(N) model using the nonperturbative renormalization group, a modern implementation of Wilson's renormalization group. The main subject is the understanding of the critical properties of this model in the vicinity of two dimensions and for N close to two.

The first chapter is an introduction to the O(N) model and the work reported in chapters 3 and 4 of the thesis. After a brief review of perturbative approaches near $d = 4$ and $d = 2$, there is a discussion of the Kosterlitz-Thouless (KT) transition driven by topological excitations (vortices). Andrzej Chlebicki then recalls the work of Cardy and Hamber (CH) on the behavior of the d-dimensional O(N) model in the vicinity of $d = N = 2$. The second part of the chapter deals with anisotropic extensions of the O(N) model. Anisotropic perturbations, when relevant at the O(N)-symmetric critical point, can make the phase transition first order or change the critical behavior if the transition remains continuous; in the latter case, the critical behavior is controlled by an anisotropic fixed point. In addition, dangerously irrelevant operators can make the critical exponents above and below the critical temperature different, as first pointed out by Nelson and more recently studied by Léonard and Delamotte. The chapter ends with a discussion of cubic perturbations to O(N) models. In addition to the Gaussian and O(N)-symmetric fixed points, there can be an Ising-like fixed point and a cubic fixed point. The latter controls the critical behavior when N is smaller than a critical value $\bar{N}_c(d)$ that depends on the dimension of the system. For the two-dimensional O(2) model, cubic transformations stabilize a low-temperature phase with long-range order, a result first shown by José et al. Apparent subsequent disagreement with Monte Carlo simulations were recently shown by Andrzej Chlebicki and Pawel Jakubczyk to be an artifact due to finite-size effects.

The second chapter is devoted to an introduction to the nonperturbative renormalization group (NPRG) and in particular the most popular approximation scheme, the derivation expansion (DE). The recent understanding of the convergence properties of the DE is emphasized, as well as the possibility to estimate the error in the evaluation of the critical exponents using the principle of minimal sensitivity (PMS) or the principle of maximum conformity (PMC). Technical issues regarding the optimal cutoff choice, possible field expansions and field parametrizations, are discussed in the case of the KT transition, the low-temperature behavior of O(N) models and anisotropic models.

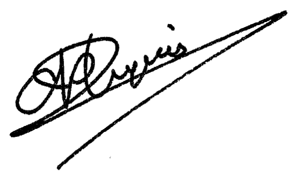
The third chapter aims at confirming (or not) the CH scenario in the framework of the NPRG. First, this scenario is discussed in detail on the basis of Eqs. (3.1-3.2), distinguishing between the cases $N > 2$ and $N < 2$. A crucial feature in the CH scenario is the non-analyticity of the critical exponents across the CH line in the (d, N) plane. Then the NPRG (and in particular its ability to detect non-analyticities in critical exponents) is benchmarked against known exact results in the two-dimensional case. Finally, the d-dimensional O(N) model is studied with the NPRG and the critical exponents are computed in the vicinity of $d = 2$ for various values of N. In the $(d > 2, N > 2)$ quadrant of the (d, N) plane, the behavior of $1/\nu$ is reminiscent of the CH scenario, but the non-analytic cusp predicted by CH as N varies is replaced by an apparently smooth crossover, which is also observed in the behavior of η. In two dimensions, the square-root non-analyticities that occur when $N \to 2^-$ are smoothed into a crossover when $d > 2$; on the large-N side of this crossover, the NPRG results precisely match the predictions of the $2 + \epsilon$ expansion showing that the long-distance physics is fully determined by spin-waves. Nevertheless, it is still possible to defined a crossover line $\tilde{N}_c(d)$ from the extrema of $\partial_N^2 \nu^{-1}$ and $\partial_N \eta$, which lies very close to the CH line. In the CH scenario, the line of non-analyticities for $d > 2$ and $N > 2$ is associated with a fixed-point collision, which must be accompanied by the vanishing of the leading correction-to-scaling exponent ω. NPRG results show that ω never approaches zero, which confirms that no fixed-point collision occurs in the $(d > 2, N > 2)$ quadrant of the (d, N) plane, thus invalidating the CH scenario. By carefully studying the fixed-point structure of the effective action, Andrzej Chlebicki shows that the crossover line can be interpreted as a rapid yet smooth transition between the perturbative regimes of the $2 + \epsilon$ and $4 - \epsilon$ expansions. As emphasized by Andrzej Chlebicki, it might be argued that the crossover line is related to the changing relevance of the vortices although a mechanism through which this might occur remains unclear. Only in the immediate vicinity of $d = 2$ could the CH scenario be valid. On the other hand, the NPRG results strongly support the fixed-point collision scenario of CH in the quadrant $(d < 2, N < 2)$. For $1 < N < 2$, the anomalous dimensions associated with the critical fixed point and the quasi-long-range-order fixed point rise slowly with decreasing dimension and collide at the lower critical dimension $d_c(N)$. The quasi-long-range-order fixed point seems to be well described by the $2+\epsilon$ expansion.

The fourth chapter is devoted to a study of the O(2) model with cubic perturbations. The leading anisotropic RG eigenvalues are computed at the O(2)-symmetric fixed point. In three dimensions, the "extended" scheme (where the effective action is expanded in powers of the anisotropic invariant τ) and the "full" scheme (where the full dependence of the effective action on the two invariants, ρ and τ, is preserved) both agree with results available in the literature, e.g. from perturbation theory or Monte Carlo simulations. The agreement between the two schemes deteriorate below three dimensions, and only the full scheme seems to be reliable, predicting that the leading anisotropic eigenvalue y_4 approaches zero for $d \to 2^+$ on an apparently non-analytic trajectory. In the second part of the chapter, two different ways of implementing the DE are compared: the "ansatz" variant, where the ansatz for the effective action is simply plugged into the flow equation, and the "truncated" variant, where the flow equation is truncated to second order in derivatives (in the case of the DE to second order). In three dimensions, both variants give similar results for ν and η (a result previously known) as well as y_4. The agreement deteriorates below three dimensions but the results remain compatible within error bars except in the immediate vicinity of $d = 2$. Only the ansatz variant, which correctly captures the KT transition in two dimensions, remains reliable for $d \to 2^+$. The failure of the truncated variant may be explained by the fact that it does not satisfy the Ward identities associated with O(2) symmetry.

Conclusion

Although the O(N) model is one of the most studied models in statistical physics, there still remains issues to be clarified. One of these concerns the behavior of the O(N) model in the vicinity of two dimensions for $N \simeq 2$. Andrzej Chlebicki has providing us with a very convincing analysis of this problem, expanding on an earlier work of Cardy and Hamper. This analysis is based on state-of-the-art functional renormalization-group techniques and requires nontrivial numerical work in order to solve the flow equations. Andrzej Chlebicki has also presented a very interesting study of the anisotropic O(2) model in dimensions $2 \leq d \leq 3$. Although this work is sometimes technically complex, the manuscript is extremely well written (and virtually free of typos).

It is therefore my pleasure to strongly recommend that Andrzej Chlebicki's thesis be defended.



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