Protecting quantum resources from noise

UNIVERSITY OF WARSAW, FACULTY OF PHYSICS

DOCTORAL THESIS

Author: Manfredi Scalici
Supervisor: Dr. hab. Alexander Streltsov

February 28, 2024

Abstract

Quantum technologies, a diverse and rapidly expanding field, are at the forefront of a modern scientific revolution. Harnessing the intrinsic properties of quantum mechanics, these technologies have the potential to revolutionize many fields apart from physics, such as cryptography and computer science.

Among these new features, the capability of quantum matter to be in a superposition of states is certainly the furthest from a classical physical intuition. Turns out that a computer whose logical bits are made from quantum particles, i.e. a quantum computer, can exploit this property to hasten certain algorithms by a significant amount.

Another property fundamentally connected with the quantum world is entanglement, a phenomenon where the quantum states of two or more particles become correlated in such a way that the state of one particle is dependent on the state of another, even when they are physically separated. This feature too is a key part of many quantum algorithms. Those are the reasons why these properties are regarded as "resources" to perform computations or other technological tasks.

Unfortunately, these very valuable features for technological applications are very fragile if the systems exhibiting them are not isolated, and there is no such thing in the physical reality as an isolated system. When a quantum computer is exposed to its environment, for example, interactions with external particles or radiation can disturb the delicate superposition of states, turning a quantum bit into a classical one, only able to assume values 0 or 1 in a deterministic fashion. This phenomenon is called decoherence and it constitutes the main cause of errors and performance degradation of quantum algorithms. The entangled particles must remain isolated from external influences too, to preserve their unique entangled states. Unlike classical computers, which are relatively robust and resilient to external influences, quantum computers are then highly sensitive to their surroundings due to the delicate quantum states they rely on. This sensitivity has significant implications for the design and operation of quantum computing systems.

In order to extend the lifespan of quantum states and enable longer computation times, quantum computers must be then shielded from decoherence-inducing factors, such as temperature fluctuations, electromagnetic radiation, and vibrations, or must operate in a highly controlled and isolated environment.

The first attempts to protect a quantum machine from the noise induced by

the environment are quantum error correction codes, where, using extra qubits, it is possible to recover part of the information lost due to the noise. In this thesis, we will provide other protocols or scenarios in which it is possible to preserve or enhance quantum resources stored in noisy quantum systems. The work will be structured as follows:

In Chapter 1, we will describe a general framework to study various quantum resources in a unified way, the so-called Quantum Resource Theories.

In Chapter 2, we will describe the way a quantum system interacts with its environment and how this interaction leads to the loss of coherence, putting the accent on a particular property of the reduced dynamic of the system, the so-called Markovianity.

In Chapter 3, we will first show that, for longer times, Markovian dynamics always destroy correlations between a system in evolution and an isolated qubit ancilla, a very valuable resource in informational tasks, and that the best way to preserve those correlations is to consider a dynamics which is non-Markovian at any time.

In Chapter 4, we will study a kind of map that helps preserve resources if applied as pre-processing before the noise acts, i.e. the dilution map.

In Chapter 5 we will answer a slightly different question, i.e what the optimal protocol to generate resource is if we equip a resource theory with a resource-generating map.

Streszczenie

Technologie kwantowe stanowią różnorodną i dynamicznie rozwijającą się dziedzinę, będąc kluczowym elementem współczesnej rewolucji naukowej. Wykorzystując szczególne właściwości mechaniki kwantowej, technologie te mają potencjał zrewolucjonizowania wielu dziedzin poza fizyką, takich jak kryptografia czy informatyka.

Spośród tych szczególnych właściwości zdolność materii kwantowej do znajdowania się w superpozycji stanów jest z pewnością najbardziej odległa od klasycznej intuicji fizycznej. Komputer, którego bity logiczne są zbudowane z cząstek kwantowych, a więc komputer kwantowy, może wykorzystać tę właściwość do przyspieszenia działania pewnych algorytmów w znacznym stopniu.

Inną właściwością fundamentalnie związaną z kwantowym światem jest splątanie – zjawisko, w którym kwantowe stany dwóch lub więcej cząstek stają się skorelowane w taki sposób, że stan jednej cząstki zależy od stanu innej, nawet gdy są one fizycznie oddzielone. Ta cecha również stanowi kluczowy element wielu algorytmów kwantowych. Ze wskazanych powodów właściwości te są uważane za „zasoby" konieczne do przeprowadzania obliczeń lub innych zadań technologicznych.

Niestety te bardzo wartościowe dla zastosowań technologicznych właściwości są bardzo delikatne, jeśli systemy je wykazujące nie są izolowane – w rzeczywistości fizycznej zaś nie istnieje coś takiego jak izolowany system. Gdy komputer kwantowy jest wystawiony na oddziaływanie z otoczeniem, na przykład interakcje z zewnętrznymi cząstkami lub promieniowaniem, może to zakłócić delikatną superpozycję stanów, zamieniając kwantowy bit w klasyczny, zdolny jedynie przyjmować wartości 0 lub 1 w sposób deterministyczny. To zjawisko nazywane jest dekoherencją i stanowi główną przyczynę błędów i ograniczenia wydajności algorytmów kwantowych. Splątane cząstki również muszą pozostać izolowane od zewnętrznych oddziaływań, aby zachować swoje szczególne splątane stany. W przeciwieństwie do komputerów klasycznych, które są stosunkowo odporne na oddziaływania zewnętrzne, komputery kwantowe są bardzo wrażliwe na swoje otoczenie ze względu na trudne do utrzymania stany kwantowe, na których są opatrte. Ta wrażliwość ma istotne implikacje dla projektowania i działania systemów obliczeń kwantowych.

Aby przedłużyć żywotność stanów kwantowych i umożliwić dłuższe czasy obliczeń, komputery kwantowe muszą być chronione przed czynnikami wywołującymi dekoherencję, takimi jak fluktuacje temperatury, promieniowanie elektro-

magnetyczne i drgania, lub muszą działać w wysoce kontrolowanym i izolowanym środowisku.

Pierwsze próby ochrony urządzeń kwantowych przed szumem wywołanym przez otoczenie stanowią algorytmy korekcji błędów kwantowych, w których za pomocą dodatkowych kubitów można odzyskać część informacji utraconej z powodu zakłóceń. W tej pracy przedstawimy inne protokoły i metody, umożliwiające zachowanie lub zwiększenie zasobów kwantowych przechowywanych w systemach kwantowych w obecności zakłóceń. Struktura pracy jest następująca:

W Rozdziale 1 opisujemy ogólną metodykę opisywania różnych zasobów kwantowych w jednolity sposób, czyli tzw. teorie zasobów kwantowych.

W Rozdziale 2 opisujemy sposób, w jaki system kwantowy oddziałuje ze swoim otoczeniem i jak ta interakcja prowadzi do utraty koherencji, kładąc nacisk na szczególną właściwość zredukowanej dynamiki systemu, tzw. własność Markowa.

W Rozdziale 3 najpierw wykazujemy, że dla dłuższych czasów dynamika markowowska zawsze niszczy korelacje między ewoluującym układem a izolowanym kubitem pomocniczym, bardzo przydatnym zasobem w procedurach kwantowych, oraz że najlepszym sposobem na zachowanie tych korelacji jest branie pod uwagę procesów niemarkowowskich w każdej chwili czasu.

W Rozdziale 4 badamy pewien rodzaj przekształcenia pomagającego zachować zasoby, jeśli zostanie zastosowane do przygotowania układu przed działaniem szumu, czyli tzw. „rozcieńczanie zasobu”.

W Rozdziale 5 odpowiemy na trochę inne pytanie, tj. jaka jest optymalna procedura generowania zasobów, jeśli wyposażymy teorię zasobu w przekształcenie generujące dany zasób.

Publications

Most of the results and ideas exposed in this thesis have been extracted from the following publications:

- [MWS$^+$22]: Optimally preserving quantum correlations and coherence with eternally non-Markovian dynamics.

  Marek Miller, Kang-Da Wu, Manfredi Scalici, Jan Kołodyński, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo and Alexander Streltsov

  New J. Phys. **24** 053022 — Published May 2022.

- [MSFAS24]: Power of noisy quantum states and the advantage of resource dilution.

  Marek Miller, Manfredi Scalici, Marco Fellous-Asiani and Alexander Streltsov

  Phys. Rev. A 109, 022404 – Published 2 February 2024.

- [SNS24]: Coherence generation with Hamiltonians.

  Manfredi Scalici, Moein Naseri, and Alexander Streltsov

  Uploaded on arxiv on 27 February 2024.

Acknowledgements

I am thankful for the support and guidance provided by the "Fist Team" project, conducted under the International Research Agendas program of the Foundation for Polish Science, co-financed by the European Union through the European Regional Development Fund.

I extend my deepest gratitude to my Ph.D. supervisor, Dr. hab. Alexander Streltsov, for his unwavering support and invaluable mentorship. Your guidance has played a pivotal role in shaping my understanding.

I would like to thank Dr Marek Miller too for the hours and the patience he spent in discussing profound physical and mathematical concepts with me. Those sessions really helped me expand my knowledge and develop a method of research.

I am also profoundly grateful to Ray and Varun for their meticulous proofreading of my thesis and insightful comments. Your feedback has been immensely valuable. Additionally, I would like to express my appreciation to Marcin for his assistance in writing the Polish abstract.

Contents

Chapter 1

Quantum Resource Theories

The theoretical research in quantum physics has recently moved from questions about the fundamental laws of nature to ones related to useful practical applications in various contexts such as information and communication science. More precisely, in a real laboratory there are often restrictions to the operations we can perform. These restrictions can arise, for example, from technological limitations or from locality in a distributed scenario and divide the space of the physical quantum states (the Hilbert space) into free states that can be generated easily and non-free states that represent a resource.

The value of resourceful states lies not only in their rarity but most importantly in the fact that one can consume them to overcome these limitations, justifying us to call them resources.

Resource theories aim to provide a general underlying mathematical framework describing the majority of these real-life scenarios. Before presenting this structure, we will introduce the physical scenario regarding the most famous of these: the protocol of quantum teleportation and how this leads to the resource theory of entanglement. This example will be the term of comparison to build the general structure of a resource theory. Then, after going into the mathematical details of a resource theory, we will discuss the problems of state convertibility and how to quantify a resource, then proceed to list a few relevant examples.

1.1 Physical scenario of entanglement theory: LOCC operations

Consider two distant laboratories A and B, the first operated by Alice, the second by Bob. Both of them can perform any type of quantum operation in their respective laboratories, acting on the systems that are contained inside those laboratories.

We will now briefly review the basics of the mathematical formalism which describes quantum states and operations. The state of a quantum system is described by a positive semidefinite matrix of trace 1 that we will denote by ρ, i.e. $\rho \geq 0$ and $\mathrm{Tr}[\rho] = 1$. Such a matrix will be called a **density matrix**. A quantum operation acting on a quantum system ρ is a completely positive map, meaning it has to preserve the positivity of ρ, the density matrix of system A, even in the presence of another system B. In other words, a positive map Λ is such that

$$\rho \geq 0 \implies \Lambda(\rho) \geq 0 \tag{1.1}$$

and a completely positive map is defined by the property

$$\rho^{AB} \geq 0 \implies \Lambda_A \otimes \mathbb{I}_B(\rho^{AB}) \geq 0 \tag{1.2}$$

An operation that is completely positive and trace preserving (CPTP map) possesses an operator-sum representation, i.e.

$$\Lambda(\rho) = \sum_j K_j \rho K_j^\dagger. \tag{1.3}$$

The operators K_j are called Kraus operators and they satisfy $\sum_j K_j K_j^\dagger = \mathbb{I}$ [NC11].

A quantum operation does not need to preserve the trace of a quantum state. This condition can indeed be relaxed to trace non-increasing, and one can consider the action of each single Kraus operator $K_j \rho K_j^\dagger$, as a trace non-increasing map.

This is the structure of the operations that both Alice and Bob can perform locally in their respective laboratories, like this $\Lambda_A \otimes \Lambda_B(\rho^{AB})$. It is forbidden for them to transfer any quantum system or subsystem between the laboratories, but they can send and receive information through classical signals before and after any round of quantum operations. The protocols which follow the above described rules are called LOCC (Local Operations and Classical Communication). See [HHHH09] for more details about this setting.

Let us now focus on the states that Alice and Bob can manipulate with those operations. A bipartite quantum state ρ^{AB} is said to be **separable** if it can be written as

$$\rho^{AB} = \sum_k p_k \rho_k^A \otimes \rho_k^B \tag{1.4}$$

with ρ_k^A local systems of A and ρ_k^B local systems of B, and p_k probabilities summing up to 1. Any other state that is not possible to write in that form is said to be **entangled**. It is clear that local quantum operations cannot turn a separable state into an entangled one. If we consider entanglement an intrinsic property of certain bipartite quantum states, we say that LOCC cannot create entanglement. The initial amount of entanglement given to Alice and Bob can only be manipulated, sometimes in order to achieve tasks otherwise impossible.

We will now describe a protocol showing such a practical application of an entangled state, called **quantum teleportation** [BBC$^+$93].

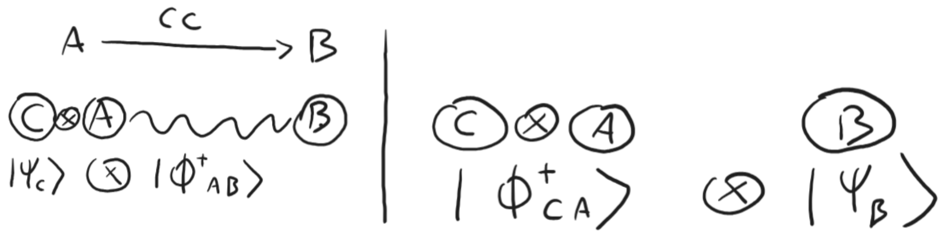

Figure 1.1: Scheme of Alice and Bob's laboratories and, on the left, the initial configuration of their systems in the teleportation protocol. In Alice's lab is present a particle Ψ_C, whose state she wishes to transfer to Bob's laboratory, plus another particle that belongs to a maximally entangled state Φ^+_{AB} together with a particle owned by Bob. The total initial state is then $|ABC\rangle = |\Psi_C\rangle \otimes |\Phi^+_{AB}\rangle$. Alice and Bob can perform local operations (LO) and classical communications (CC) to share information about the outcomes of those operations. On the right, after the teleportation protocol, entanglement between the two laboratories is destroyed and the state of Ψ_c is successfully transferred to Bob's particle, meaning the total final state is $|ABC\rangle = |\Phi^+_{AC}\rangle \otimes |\Psi_B\rangle$, with $\Psi_B = \Psi_C$.

1.1.1 Quantum teleportation

Alice and Bob share a singlet or **maximally entangled state**, i.e. a pure two-qubit bipartite state of the form

$$|\Phi^+_{AB}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{1.5}$$

Moreover, Alice possesses a generic pure qubit state

$$|\Psi_C\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,. \tag{1.6}$$

The goal of the task would be to transfer this state into the qubit in Bob's lab, by the mean of some LOCC protocol.

The overall state can be written as

$$\begin{aligned}|ABC\rangle = |\Psi_C\rangle \otimes |\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}(\alpha\,|0_C\rangle + \beta\,|1_C\rangle) \otimes (|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle) = \\ = \frac{1}{\sqrt{2}}(\alpha\,|00_{CA}\rangle \otimes |0_B\rangle + \alpha\,|01_{CA}\rangle \otimes |1_B\rangle + \beta\,|10_{CA}\rangle \otimes |0_B\rangle + \beta\,|11_{CA}\rangle \otimes |1_B\rangle)\end{aligned} \tag{1.7}$$

At this point, we will change the basis in CA from computational to Bell.

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle) \tag{1.8}$$

$$|11\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Phi^-\rangle) \tag{1.9}$$

$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle) \tag{1.10}$$

$$|10\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle - |\Psi^-\rangle) \tag{1.11}$$

Obtaining

$$\begin{aligned} |ABC\rangle = \frac{1}{2}(&|\Phi^+_{AC}\rangle \otimes (\alpha\,|0_B\rangle + \beta\,|1_B\rangle) + |\Phi^-_{AC}\rangle \otimes (\alpha\,|0_B\rangle - \beta\,|1_B\rangle) + \\ &+ |\Psi^+_{AC}\rangle \otimes (\alpha\,|1_B\rangle + \beta\,|0_B\rangle) + |\Psi^-_{AC}\rangle \otimes (\alpha\,|1_B\rangle - \beta\,|0_B\rangle)) = \\ &= \frac{1}{2}\sum_{i=0}^{3} |\Phi_i\rangle_{AC} \otimes \sigma_i\,|\Psi_B\rangle \end{aligned} \tag{1.12}$$

Where $|\Phi_i\rangle$ are the different Bell states in Eq. 1.8-1.11. This means that the following protocol transfers the state $|\Psi\rangle$ to Bob's lab:

1. Alice performs a local PVM (Projective Valued Measurement) in the Bell basis on AC projecting the state in one of the above four terms.
2. She sends the outcome with a classical signal to Bob.
3. Bob performs a conditional local unitary depending on the outcome he receives, in order to undo the σ_i's, being then able to perfectly "recreate" the state $|\Psi_C\rangle$ (which originally belonged to Alice) in his particle $|\Psi_B\rangle$, assisted by Alice.

Notice that:

1. Only LOCC operations were used in the protocol.
2. The entire procedure was possible by consuming an entangled state shared between A and B, which makes entanglement a resource for teleportation.

1.1.2 Imperfect quantum teleportation and entanglement distillation

Let us now consider a shared state that is not perfectly entangled, but it is a noisy maximally entangled state mixed with a maximally mixed state.

$$\rho = p\,|\Phi^+\rangle\langle\Phi^+| + (1-p)\frac{\mathbb{I}_4}{4} \tag{1.13}$$

with $0 \leq p \leq 1$.

It was shown in [HHH99], that the performance of a general teleportation protocol $\Lambda_T(|\Psi\rangle\langle\Psi|)$, where $|\Psi\rangle$ is the input state we wish to teleport, measured through the average fidelity over all possible input states:

$$\bar{F}(\rho) = \int \langle\Psi| \Lambda_T(|\Psi\rangle\langle\Psi|) |\Psi\rangle \, d\Psi \tag{1.14}$$

can attain a maximum proportional to the singlet fraction:

$$F(\rho) = \max_i \langle\Phi^i| \rho |\Phi^i\rangle \tag{1.15}$$

maximized over all possible Bell states. Between $\bar{F}(\rho)$ and $F(\rho)$ there is the linear relation $\vec{F} = \frac{2F(\rho)+1}{3}$. Then this simple example shows that the performance is given by the linear relation:

$$\bar{F}(\rho) = p + \frac{1-p}{2} \tag{1.16}$$

which is maximum where the shared state is maximally entangled and monotonically decreasing otherwise. Then the Bell basis measurement protocol performed by Alice can never perfectly recreate the original qubit $|\Psi\rangle$. So the performances of the teleportation protocol are a function of the mixing coefficient of the shared state between Alice and Bob, suggesting there is a property we can quantify that plays the role of a resource in this operation. This function must reach its maximum in correspondence of $|\Phi^+\rangle$ and decrease for any other bipartite state.

Before performing a quantum teleportation Alice and Bob would then desire to share a maximally entangled state. Now if they possess only one copy of a pure bipartite state $|\Psi_{AB}\rangle$ it is impossible to convert it to $|\Phi^+\rangle$ by means of any LOCC operation as stated by Nielsen theorem [Nie99], which we will briefly review in the following.

Consider two normalized vectors of dimension d, $\vec{v}$ and $\vec{w}$ whose components are ordered in a decreasing manner. It is said that $\vec{v} \succ \vec{w}$ if

$$\sum_{i=1}^{k} v_i \geq \sum_{i=1}^{k} w_i \tag{1.17}$$

$\forall k \in 1, ..., d-1$.

Nielsen theorem states that $|\Psi\rangle$ can be converted to $|\Phi\rangle$ through LOCC iff the vector of the Schmidt coefficients of $|\Phi\rangle$ majorizes the vector of the coefficients of $|\Psi\rangle$.

The Schmidt coefficients [NC11] are a decomposition of a pure bipartite quantum state. For a general dimension d of the Hilbert spaces of both local systems A and B

$$|\Psi\rangle^{AB} = \sum_{i=1}^{d} \sqrt{\lambda_i} \, |i\rangle^{A} \otimes |i\rangle^{B} \tag{1.18}$$

where $\{|i\rangle^A\}$ and $\{|i\rangle^B\}$ are two orthogonal basis of system A and B respectively, and λ_i are called Schmidt coefficients.

It is easy to see that both reduced states $\rho^{A(B)}$ will have the form

$$\rho^{A(B)} = \mathrm{Tr}_{B(A)}[|\Psi\rangle\langle\Psi|^{AB}] = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|^{A(B)} \tag{1.19}$$

On the other hand, any other pure bipartite state $|\Psi_{AB}\rangle$ can be created starting from $|\Phi^+\rangle$.

Suppose otherwise that they have multiple copies of a less entangled state. For example, they share two copies of the bipartite state $|\Psi_{AB}\rangle = a|00\rangle + b|11\rangle$, meaning they share the state

$$|\Psi_{AB}\rangle^{\otimes 2} = a^2 |00_{AB}\rangle^{\otimes 2} + b^2 |11_{AB}\rangle^{\otimes 2} + ab(|00_{AB}\rangle \otimes |11_{AB}\rangle + |11_{AB}\rangle \otimes |00_{AB}\rangle) \tag{1.20}$$

Alice can perform a Bell measurement in her lab to project this state into a state proportional to a singlet $|\Psi^-\rangle = |01\rangle + |10\rangle$ with probability $(ab)^2$. Notice that this state differs from $|\Phi^+\rangle$ just by a rotation of one qubit. In the limit of many shared copies, there exists a procedure called **entanglement distillation** or **concentration** (see Appendix and [BBPS96]) which allows converting n copies of Ψ into m copies of Φ^+, provided n is very large. Moreover, the probability of successful conversion goes to 1 as $n \to \infty$. In this asymptotic setting, it is then possible to concentrate the resource called entanglement and store it for application in future tasks.

There are many other features of quantum systems that have or might have a useful technological application in communication cryptography or computing. Then there is the need to formalize all these situations in a unique mathematical framework called quantum resource theories, which we will present in the following section.

1.2 Resource theories

The main reference for the next three sections is the review by E. Chitambar and G. Gour [CG19].

As described above, the first approach in a resource theory is the identification of a set of free operations F_{op} in the space of CP (completely positive) maps acting on quantum states.

The most basic properties that every resource theory should satisfy are

1. The do nothing operation $\mathbb{I}$ is included in the set of free operations, i.e. $\mathbb{I} \in F_{op}$

2. The set of free operations is closed under composition

$$\forall \Lambda_1, \Lambda_2 \in F_{op}, \Lambda_1 \circ \Lambda_2 \in F_{op} \tag{1.21}$$

Analogously we define a set of free states inside the Hilbert space. A free operation must be resource-non-generating (RNG), even if applied repeatedly to a free state.

$$\rho \in F_s, \Lambda \in F_{op} \implies \Lambda(\rho) \in F_s \tag{1.22}$$

This is usually referred to as the golden rule of quantum resource theories.

Moreover, resource theories are usually equipped with a tensor product structure, in the following sense:

1. Free operations can act on a subsystem

   $$\Phi \in F_{op}(H_A \to H_B) \implies \mathbb{I}_C \otimes \Phi \in F_{op}(H_C \otimes H_A \to H_C \otimes H_B) \tag{1.23}$$

   where in parenthesis we denote the input and output Hilbert space of the operation.

2. Appending free states is a free operation.

3. Discarding systems, i.e. the trace, is a free operation.

In particular, the first property, combined with composition, implies that one can perform local operations independently in different laboratories, in the following way: $\Phi' \otimes \Phi = (\Phi' \otimes \mathbb{I}) \circ (\mathbb{I} \otimes \Phi)$. In other words if $\Phi, \Phi' \in F_{op}$, then $\Phi \otimes \Phi' \in F_{op}$. The properties second and third imply that preparing a free state is a free operation

$$\Phi(X) = Tr[X]\sigma \tag{1.24}$$

when $\sigma \in F_s$. Consequently, any state can be converted into a free one. Moreover, other direct implications of properties 2 and 3 are that the set of free states is closed under tensor product and partial trace.

$$\rho \in F_s(H_A), \sigma \in F_s(H_B) \implies \rho \otimes \sigma \in F_s(H_A \otimes H_B) \tag{1.25}$$

$$\rho^{AB} \in F_s(H_A \otimes H_B) \implies \rho^A \in F_S(H_A) \text{ and } \rho^B \in F_s(H_B) \tag{1.26}$$

In some cases, a resourceful state can be used to execute an operation that normally is not allowed. For example, having access to the state $|\Phi^+_{AB}\rangle$ allows us to implement a teleportation channel with LOCC. We will now write this statement in a more formal way: A state $\sigma \notin F_s$ can be used to implement $\Lambda \notin F_{op}$ if there exists $\Phi \in F_{op}$ such that $\forall \rho \; \Phi(\rho \otimes \sigma) = \Lambda(\rho)$.

Before moving on we want to mention one last desirable property for a resource theory, which is convexity of the set of free states, meaning that, if $\rho, \sigma \in F_s$, then any convex combination

$$p\rho + (1-p)\sigma \in F_s \tag{1.27}$$

is still a free state for all $p \in [0, 1]$.

1.2.1 Different classes of free operations

Now let us review typical ways to construct the set of free operations from the free states. The main ones are

- Taking the maximal set of operations that maps free states to free states, we obtain the set of **resource non-generating maps**. In this case, the request is that overall this map does not generate any resource.
- Adding the freedom of acting on a system correlated with an ancilla, these kinds of maps will be called **completely resource non-generating**.
- The resource non-generating condition can be strengthened, imposing that each of the trace non-increasing maps $K_j \rho K_j^\dagger$ do not take a free state out of its set, i.e. $\forall \rho \in F_s, j = 1, ..., r$

$$\frac{K_j \rho K_j^\dagger}{Tr[K_j \rho K_j^\dagger]} \in F_s \tag{1.28}$$

  where r is the Kraus rank of the operation. We will call this kind of maps **stochastically resource non-generating**.
- Finally, we can consider a smaller set of free operations which are the maps that destroy any form of resource from any state. We will denote this sort of map by the symbol Δ, call it **resource destroying map**, and we will assign the two properties
  - it maps every free state to itself, i.e. $\forall \rho \in F_s$, $\Delta(\rho) = \rho$
  - it maps every state to a free state.

In the next section, we will discuss the problem of state convertibility.

1.3 Convertibility

One important question studied in quantum resource theories is whether it is possible to convert two arbitrary states into each other through free operations. In particular, we would like to manipulate the resource we possess in order to maximize the performance of informational tasks like the above-mentioned teleportation.

Mathematically, free operations induce a pre-order on the Hilbert space, i.e. a relation between two states which is reflexive and transitive. We will write

$$\rho \xrightarrow{\Phi} \sigma \text{ if } \exists \Phi \in F_{op} : \sigma = \Phi(\rho) \tag{1.29}$$

Physically this means that, if we can transform ρ to σ, then σ does not contain more resource than ρ. Because free operations are closed under composition we have transitivity, i.e.

$$\rho \xrightarrow{\Phi} \sigma, \sigma \xrightarrow{\Phi} \gamma \implies \rho \xrightarrow{\Phi} \gamma. \tag{1.30}$$

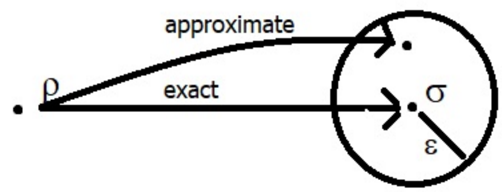

Figure 1.2: Schematic picture of the concepts of exact and approximate state conversion. In the exact process, a quantum state ρ is exactly converted into σ, without any error. In the approximate process, we are content to send ρ inside a ball of radius ε around σ.

If both $\rho \overset{\Phi}{\to} \sigma$ and $\sigma \overset{\Phi}{\to} \rho$ hold, we say that $\rho \overset{\Phi}{\approx} \sigma$, meaning ρ is equivalent (in terms of resources) to σ.

There are many variants of this question, depending on the specific restrictions on the tasks one is allowed to perform, the most notable being:

- Single-shot: the task is to convert one copy of the initial state ρ to one copy of the final state σ. This scenario has many subcategories:
  - Exact 1.2: we require that the conversion is achieved with zero error, i.e.
  $$\Phi(\rho) = \sigma. \tag{1.31}$$
  We will denote this by
  $$\rho \overset{\Phi}{\to} \sigma \tag{1.32}$$
  - Stochastic 1.3: the task is achieved probabilistically.
  $$\Phi(\rho) = \sum_j \Phi_j(\rho) \otimes |j\rangle \langle j| \tag{1.33}$$
  A probabilistic map is applied to ρ and an outcome is selected by measuring an observable on some ancillary system, whose eigenvectors are $\{|j\rangle\}$. We say that the conversion is successful if there is a non-zero probability $p_j = Tr[\Phi_j(\rho)] > 0$ that $\frac{\Phi_j(\rho)}{p_j} = \sigma$ is the desired

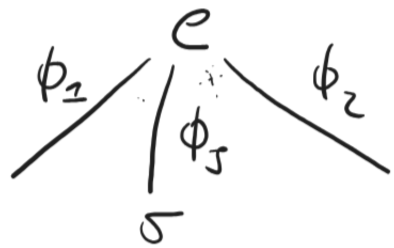

Figure 1.3: Stochastic state conversion: a map Φ_j is applied on ρ depending on the outcome of a random variable and the target state σ is reached only one of the possible outcomes, occurring with probability $\mathrm{Tr}[\Phi_j(\rho)]$.

state we want to generate. We can denote it by

$$\rho \xrightarrow{\Phi_j} \sigma \tag{1.34}$$

– Approximate 1.2: We do not require to obtain the state σ exactly after the conversion but we are satisfied with a state inside a ball of fixed radius ε in fidelity, centered around σ. In other words

$$\Phi(\rho) = \sigma' \tag{1.35}$$

with $F(\sigma, \sigma') \geq 1 - \varepsilon$. We will denote this process by

$$\rho \xrightarrow{\Phi}_{\varepsilon} \sigma \tag{1.36}$$

- Asymptotic 1.4: We have access to multiple copies of the initial state $\rho^{\otimes n}$ and we would like to convert them to multiple copies of the final state $\sigma^{\otimes m}$. Typically, we are interested in rates of conversion, meaning the ratio m/n that is achievable in such a conversion. We will say that a ratio R is achievable if $\forall R' < R$ and $\varepsilon \in [0, 1]$, $\exists n$:

$$\rho^{\otimes n} \xrightarrow{\Phi}_{\varepsilon} \sigma^{\otimes nR'} \tag{1.37}$$

  Finally, the optimal rate of conversion $R(\rho \to \sigma)$ is the supremum over all the achievable rates.

  We say that an asymptotic process is reversible if $R(\rho \to \sigma)R(\sigma \to \rho) = 1$, i.e. there are no copies of ρ lost performing a conversion to σ and going back, in the limit $n \to \infty$.

Reversible processes establish equivalence classes between quantum states, after the identification of a special state all the states in a certain class can be

Figure 1.4: Asymptotic state conversion: from n copies of the initial state to m of the final state. The interesting quantity is the rate m/n in the limit $n \to \infty$.

converted into and which can generate back all the states in this class. A natural choice for this state is the one that contains the maximal amount of resource (according to one of the quantifiers we will list ahead). For example, in the resource theory of entanglement restricted to the class of pure bipartite states every conversion rate can be expressed as $R(\Psi \to \Phi) = R(\Psi \to \Phi^+)R(\Phi^+ \to \Phi)$, with $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle^A \otimes |i\rangle^B$ maximally entangled state. The two quantities $R(\Psi \to \Phi^+)$ and $R^{-1}(\Phi^+ \to \Phi)$ are called respectively distillable entanglement and entanglement cost of ρ. See [BBPS96] and section 1 of Appendix.

We will now introduce a new class of operations, which becomes free only when acting in the asymptotic limit of many copies, that is, the asymptotically resource non-generating operations. A conversion under this set of operations is denoted by

$$\rho^{\otimes n} \xrightarrow{\Phi_n}_{\varepsilon} \sigma^{\otimes nR'} \tag{1.38}$$

where Φ_n can generate ε_n amount of resource, but this must vanish in the limit of a great number of copies of ρ, i.e. $\lim_{n\to\infty} \varepsilon_n = 0$. We will see that the optimal rate R under this class of operations is connected to entropic quantities.

1.4 Resource quantifiers

As it is evident by the quantum teleportation example, another desirable tool a resource theory should have are functions of quantum states which faithfully return a value proportional to the content of resource that the state contains, so one can classify the usefulness of states in performing certain tasks.

A resource quantifier of states should be a function from the Hilbert space to the non-negative real axis $f : H \to \mathcal{R}_{\geq 0}$. Where we used the convention that the minimum of the quantifier achieved on the free states is 0. Using an axiomatic approach we would first like to list a series of desired properties of these functions.

1. Faithfulness: $\forall \rho \in F_s$, $f(\rho) = 0$.

   It is not obvious that $f(\rho) = 0 \implies \rho \in F_s$ because there might be a task for which some resource state is still useless, making the corresponding quantifier zero. For example, we cannot extract singlets from any bound entangled state [HHH98].

2. Monotonicity: $\forall \Phi \in F_{op}$ $f(\rho) \geq f(\Phi(\rho))$.

3. Strong Monotonicity: $f(\rho) \geq \sum_i p_i f(\sigma_i)$ with $\rho \xrightarrow{\Phi_i} \sigma_i$, $\sigma_i = \frac{\Phi_i(\rho)}{p_i}$, $\Phi_i(\rho) = K_i \rho K_i^\dagger$ some stochastic map and $p_i = \mathrm{Tr}[K_i \rho K_i^\dagger]$ [Vid00].

   We can show that any function that is convex linear on quantum-classical states $\sum_i p_i \sigma_i \otimes |i\rangle\langle i|$, i.e. $f(\sum_i p_i \sigma_i \otimes |i\rangle\langle i|) = \sum_i p_i f(\sigma_i \otimes |i\rangle\langle i|)$, monotonic, and is unchanged by attaching and discarding a classical flag is also strongly monotonic [Vid00].

   $$f(\rho) \geq f(\sum_i p_i \sigma_i \otimes |i\rangle\langle i|) = \sum_i p_i f(\sigma_i \otimes |i\rangle\langle i|) = \sum_i p_i f(\sigma_i) \quad (1.39)$$

   Strongly monotonic functions can be used to obtain a bound on the maximal probability of obtaining outcome σ from stochastic transformations [Vid00]:

   $$P_\rho^{max}(\sigma) \leq \frac{f(\rho)}{f(\sigma)} \quad (1.40)$$

4. Convexity: $f(\sum_i p_i \rho_i) \leq \sum_i p_i f(\rho_i)$

5. Subadditivity: $f(\rho \otimes \sigma) \leq f(\rho) + f(\sigma)$. When equality holds we say the function is additive. A technique called regularization allows the creation of additive quantifiers on many copies of the same state.

   $$f^\infty(\rho) = \lim_{n\to\infty} \frac{1}{n} f(\rho^{\otimes n}) \quad (1.41)$$

6. Asymptotic continuity:

   $$|f(\rho) - f(\sigma)| \leq k\varepsilon \log d + c(\varepsilon) \quad (1.42)$$

   with $\varepsilon = \frac{1}{2}||\rho - \sigma||_1$ trace distance between ρ and σ, c a function such that $c(\varepsilon) \overset{\varepsilon\to 0}{\to} 0$, and k a constant. This property assures that, in the asymptotic limit, two evaluations of the functions are close when the corresponding states are similar in trace distance.

Now we will show that asymptotically continuous functions bound the achievable rates in asymptotic convertibility [HOH02]. As mentioned before R is an achievable asymptotic conversion rate if $\forall \varepsilon > 0$ $\exists n$ and $\Phi \in F_{op}$: $\rho^{\otimes n} \xrightarrow{\Phi} \sigma_n$, with $F(\sigma_n, \sigma^{\otimes nR}) > 1 - \varepsilon$.

Thanks to the Fuchs-Van der Graaf inequality

$$1 - F(\rho,\sigma) \leq D(\rho,\sigma) \leq \sqrt{1 - F(\rho,\sigma)^2} \tag{1.43}$$

with $D(\rho,\sigma) = \frac{1}{2}||\rho - \sigma||_1$, we can say this corresponds to $\frac{1}{2}||\sigma_n - \sigma^{\otimes nR}||_1 < \sqrt{2\varepsilon}$ in terms of trace distance. Then, if f is asymptotically continuous and a monotone

$$f(\rho^{\otimes n}) \geq f(\sigma_n) \geq f(\sigma^{\otimes nR}) - k'\sqrt{\varepsilon} nR \log d - c(\varepsilon) \tag{1.44}$$

Now let's divide both sides of this inequality by n and take the limit $n \to \infty$

$$f^{\infty}(\rho) \geq R f^{\infty}(\sigma) - O(\sqrt{\varepsilon}) \tag{1.45}$$

This bound is saturated for a reversible process.

1.4.1 Contractive divergences

Now we will review some common constructions of resource quantifiers. An important construction of quantifiers is through the use of a contractive divergence, i.e. functions that satisfy

$$d(\rho,\sigma) \geq d(\Phi(\rho), \Phi(\sigma)) \tag{1.46}$$

for any Φ, CPTP map. We define the quantifier by taking the minimum distance with the set of free states, i.e.

$$R_d(\rho) = \min_{\sigma \in F_s} d(\rho,\sigma) \tag{1.47}$$

This quantifier will be automatically monotonic since

$$R_d(\Phi(\rho)) = \inf_{\tau \in F_s} d(\Phi(\rho), \tau) \leq \inf_{\sigma \in F_s} d(\Phi(\rho), \Phi(\sigma)) \leq \inf_{\sigma \in F_s} d(\rho,\sigma) = R_d(\rho) \tag{1.48}$$

In addition, the following property, known as joint convexity, guarantees that the resulting quantifier $R_d(\rho)$ will be convex:

$$d\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) \leq \sum_i p_i d(\rho_i, \sigma_i) \tag{1.49}$$

An important class of such contractive functions is given by entropies.

1.4.2 Entropic measures

In this thesis, we will mainly refer to two families of functions after which many relevant entropies are derived. These are:

- Quantum relative Renyi Entropies

$$D_\alpha(\rho||\sigma) = \begin{cases} +\infty & \text{if } \alpha \notin (0,1) \wedge \text{supp}(\rho) \not\subset \text{supp}(\sigma) \\ \frac{1}{\alpha - 1} \log(Tr[\rho^\alpha \sigma^{1-\alpha}]) & \text{otherwise} \end{cases} \tag{1.50}$$

- Sandwiched Renyi entropies or divergencies

$$\tilde{D}_\alpha(\rho||\sigma) = \begin{cases} +\infty & \text{if } \alpha \notin (0,1) \wedge \text{supp}(\rho) \not\subset \text{supp}(\sigma) \\ \frac{1}{\alpha-1}\log(Tr[\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}]) & \text{otherwise} \end{cases} \tag{1.51}$$

$\forall \alpha \in (0,+\infty) - \{1\}$. [Pet86].

Then, if $\text{supp}(\rho) \subset \text{supp}(\sigma)$

-
$$D_0(\rho||\sigma) = \lim_{\alpha\to 0} D_\alpha(\rho||\sigma) = -\log(Tr[\Pi_\rho\sigma]) \tag{1.52}$$
with Π_ρ projector into the support of ρ.

- Quantum relative entropy
$$\lim_{\alpha\to 1} D_\alpha(\rho||\sigma) = \lim_{\alpha\to 1} \tilde{D}_\alpha(\rho||\sigma) = S(\rho||\sigma) = -Tr[\rho(\log\sigma - \log\rho)] \tag{1.53}$$

- Max relative entropy
$$\lim_{\alpha\to +\infty} \tilde{D}_\alpha(\rho||\sigma) = D_{max}(\rho||\sigma) = \inf\{\lambda : \rho \leq 2^\lambda\sigma\} \tag{1.54}$$

One can also set $\sigma = \mathbb{I}$ to obtain

- Reiny entropy
$$S_\alpha(\rho) = -D_\alpha(\rho||\mathbb{I}) = -\frac{1}{\alpha-1}\log(Tr[\rho^\alpha]) \tag{1.55}$$

-
$$H_{min}(\rho) = -D_{max}(\rho||\sigma) = -\log||\rho||_\infty \tag{1.56}$$
where $||\rho||_\infty$ is the infinity norm, i.e. the largest eigenvalue of ρ.

- Von Neumann entropy
$$S(\rho) = -S(\rho||\mathbb{I}) + \log d = -Tr[\rho\log\rho] \tag{1.57}$$

-
$$H_{max}(\rho) = -D_0(\rho||\mathbb{I}) = \log\text{rank}(\rho). \tag{1.58}$$

See [MLDS$^+$13] for more details. Since Reiny entropies are contractive (we speak about Data Processing inequality) in the range $\alpha \in [0,2]$ and the divergencies for $\alpha \in [1/2,+\infty)$, we can use them to define the resource measures in these ranges.

$$R_\alpha(\rho) = \inf_{\sigma\in F_s} D_\alpha(\rho||\sigma) \tag{1.59}$$

$$\tilde{R}_\alpha(\rho) = \inf_{\sigma\in F_s} \tilde{D}_\alpha(\rho||\sigma) \tag{1.60}$$

Setting $\alpha = 1$ we obtain the relative entropy of the resource

$$R_{rel}(\rho) = \inf_{\sigma \in F_s} S(\rho||\sigma) \tag{1.61}$$

This is asymptotically continuous whenever F_s is convex and contains the maximally mixed state [DH99].

Entropic quantities are also connected to the hypothesis testing problem. We recall that in this scenario we want to distinguish between two states, ρ and σ, using a two-outcome POVM $\{T, \mathbb{I} - T\}$, with probabilities of making a mistake:

$$\alpha(T) = Tr[(\mathbb{I} - T)\rho] \tag{1.62}$$

$$\beta(T) = Tr[T\sigma] \tag{1.63}$$

The task of minimizing one error probability under the constrain that the other is smaller than ε can be connected to the hypothesis testing relative entropy [WR12]:

$$D_H^{\varepsilon}(\rho||\sigma) = \sup_{0 \leq T \leq \mathbb{I}, \alpha(T) \leq \varepsilon} - \log \beta(T) \tag{1.64}$$

Its regularization gives the usual relative entropy

$$\lim_{n \to \infty} \frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n}||\sigma^{\otimes n}) = S(\rho||\sigma) \tag{1.65}$$

It is known that [BBG$^+$], if

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \min_{\sigma_n \in F_s^{\otimes n}} D_H^{\varepsilon}(\rho^{\otimes n}||\sigma_n) = R_{rel}^{\infty}(\rho) \tag{1.66}$$

where $F_s^{\otimes n}$ is the set of free states in the Hilbert space of dimension $\dim(\rho)^n$, the corresponding resource theory is reversible under asymptotically resource non-generating operations and the asymptotic rate of conversion is given by:

$$R(\rho \to \sigma) = \frac{R_{rel}^{\infty}(\rho)}{R_{rel}^{\infty}(\sigma)} \tag{1.67}$$

See also [BaG15]. This applies in particular to the resource theory of coherence, as shown in [BBG$^+$], the resource theory of thermodynamics ([BHO$^+$13]), and the resource theory of entanglement restricted to pure bipartite states, see Appendix 7.1. Unfortunately, even if they have this nice contractive property, the entropies cannot be used as a proper metric because triangle inequality doesn't hold for them.

Special cases of these rates are the ones towards the maximally resourceful state $R_D(\rho) = R(\rho \to \Phi^+)$, namely the distillable resource of ρ, or how many Φ^+ we can extract per unit of copies of ρ in an asymptotic conversion, and from it $R_C^{-1}(\rho) = R(\Phi^+ \to \rho)$, the resource cost, or how many copies of Φ^+ we need to spend per unit of copy of ρ we generate. These quantities are computed using the convention $R_{rel}^{\infty}(\Phi^+) = 1$.

Now follow a few other examples of noteworthy resource theories.

1.5 Other examples of resource theories

1.5.1 Thermodynamics

This is a theory about energy exchange between systems [NW18].

- The free operations consist in putting a physical system in contact with a thermal bath at a fixed temperature $\beta = 1/(k_B T)$ and letting them evolve (exchange heat) through an energy-preserving unitary $[U, H_S + H_B] = 0$, in formulas
$$\Phi(\rho) = Tr_B[U(\rho \otimes \gamma_B)U^\dagger] \tag{1.68}$$
Such operations are also called thermal.

- The Gibbs state $\gamma = \frac{e^{-\beta H}}{Tr[e^{-\beta H}]}$ is the free state of this theory, defined to be at thermal equilibrium with the bath. The Gibbs state is the only Completely passive state [AF13]. A passive state is such that no work can be extracted from it, meaning
$$Tr[U\rho U^\dagger H_S] \geq Tr[\rho H_S] \tag{1.69}$$
$\forall U$ unitaries. ρ is completely passive if $\rho^{\otimes n}$ is passive $\forall n$.

Any other state is then specified by the couple (ρ, H), including its Hamiltonian H.

An important subclass of thermal operations is given by the noisy operations, when the Hamiltonians involved are fully degenerate, i.e. proportional to the identity [GMN$^+$15]. In this case, the free state is the maximally mixed state. Operations that preserve such state are called unital and they have the same power as unitaries mixtures, i.e. maps whose Kraus operators are given by unitaries. The resource theory of thermodynamics under noisy operations is usually called purity.

Now we will list the main quantifiers used in this theory, connected to single-shot or asymptotic state transition.

- The criterion for single-shot state transition when the target state is diagonal is called thermo-majorization [HO13] and it consists in plotting $\{\sum_{i=0}^k e^{-\beta E_i}, \sum_{i=0}^k p_i\}$, where p_i are the diagonal elements of ρ in the energy eigenbasis and E_i are the eigenenergies of the Hamiltonian. These coordinates are such that the product $p_i e^{-\beta E_i}$ is in increasing order. If the plot of ρ is above the plot of σ we say that ρ thermomajorises σ and is possible to convert ρ into σ via thermal operations. The classical version of this fact is that there exists a stochastic matrix M such that $diag(\rho) = M diag(\sigma)$.

  In the case of noisy operations, this criterion reduces to the usual majorization for the eigenvalues of ρ and σ.

- Since there is only one free state the asymptotic quantifier of this theory is given by
$$R(\rho) = S(\rho||\gamma). \tag{1.70}$$
Notice that is already additive, so it doesn't need regularization. States that maximize this distance would be the maximally resourceful ones. For example, in the resource theory of noisy operations, pure states play this role.

1.5.2 Coherence

The possibility to create superpositions of states is one of the unique features of quantum mechanics. The resource theory of coherence aims to define and quantify this feature [SAP17]. This theory is special because it is basis-dependent. There are different approaches to characterize the free states

- Usually, one takes the eigenbasis of some observable T, $\{|i\rangle\}$ and defines free states, also called incoherent, as the ones diagonal in this basis or, alternatively, if T is nondegenerate, we define free states as the ones who commute with T.
$$\rho_f = \sum_{i=0}^{d-1} \rho_{ii} |i\rangle \langle i| \tag{1.71}$$
$$[\rho_f, T] = 0 \tag{1.72}$$

- Another characterization of the free states is that they do not change after the application of the Resource Destroying Map of this theory, the completely dephasing map in the eigenbasis of T, i.e.

$$\Delta(\rho) = \sum_{i=0}^{d-1} \langle i| \rho |i\rangle |i\rangle \langle i| \tag{1.73}$$
$$\rho_f = \Delta(\rho_f) \tag{1.74}$$

In the resource theory of coherence, there are several ways to define free operations, depending on the physical scenario.

- The first class of incoherent operations we will define are the Translationally covariant Incoherent Operations (TIO), i.e. all maps that commute with a unitary evolution whose generator is given by T, i.e., for any s we have:
$$[\Phi, U] = \Phi \circ U - U \circ \Phi = 0 \tag{1.75}$$
$$\text{with } U = e^{-iTs}. \tag{1.76}$$
Note that the free states of this class are also the ones that commute with U.

- The second class, representing the maximal set of Resource Non-Generating operations is made by the Maximally Incoherent Operations (MIO).

  An example of an operation which is MIO, because it doesn't change the total amount of coherence, i.e. the amplitude of off-diagonal elements, but is not TIO is the following

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |2\rangle) \tag{1.77}$$

  In fact the unitary U in the definition of TIO might apply a different phase to the ket $|1\rangle$ with respect to the ket $|2\rangle$, meaning this operation won't commute with it.

- After that we can list the Stochastically Resource Non-Generating Operations, called Incoherent Operations (IO) as the ones that commute with the dephasing map for each of their Kraus operator

$$K_j \Delta(\rho) K_j^\dagger = \Delta(K_j \rho K_j^\dagger) \tag{1.78}$$

  $\forall j = 1, ...,$ Kraus rank. This property is satisfied by the following form of Kraus operators

$$K_j = \sum_{k=0}^{d-1} c_{jk} |f(k)\rangle \langle k| \tag{1.79}$$

  with $f(k)$ any function acting on the eigenvectors labels. If the Kraus operators are unitaries

$$U = \sum_{k=0}^{d-1} e^{i\varphi_k} |\pi(k)\rangle \langle k| \tag{1.80}$$

  with π any permutation of the eigenvectors.

- If a map arises from Incoherent unitaries in a Stinespring dilation (see next chapter), it is called Strictly Incoherent Operation (SIO).

- If a map overall commute with the dephasing map

$$\Phi(\Delta(\rho)) = \Delta(\Phi(\rho)) \tag{1.81}$$

  it is said Dephasing Covariant Incoherent Operation (DIO).

There is an entire class of Maximally Coherent states, generated as the orbit of

$$|+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \tag{1.82}$$

under the action of incoherent unitaries $U|+\rangle$.

Coherence gives metrological advantage in phase estimation tasks, i.e. if we want to estimate the phase φ of the unitary $U = e^{-iH\varphi}$ it is better to use a probe state which has some coherence in the eigenbasis of H, since such a state, after the application of U will have φ encoded as a relative phase, which is physically observable opposite to the global phase we will end up if we use an incoherent state. More precisely the ultimate bound on the variance of φ after repeating this experiment N times is given by

$$\Delta^2\varphi \geq \frac{1}{NI(\rho, H)} \tag{1.83}$$

with

$$I = 2\sum_{k,l} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k - \lambda_l} |\langle k| H |l\rangle|^2 \tag{1.84}$$

Quantum Fisher Information, λ_k and $|k\rangle$ eigenvalues and eigenvectors of the density matrix ρ. This function is non-zero if the Hamiltonian has some non-zero off-diagonal elements or, equivalently, if the probe state ρ contains some coherence. Moreover, if the state ρ is higher dimensional and contains some entanglement, the scaling of the bound can be improved to N^{-2} [MLS04].

- The most simple quantifier of coherence one can provide is the so-called l_1-norm of coherence [BCP14], consisting in the sum of the magnitude of the off-diagonal elements of a density matrix $\rho = \sum_{i,j=1}^{d} \rho_{ij} |i\rangle \langle j|$, written in the incoherent basis. That is

$$C_{l_1} = \sum_{i \neq j} |\rho_{ij}| \tag{1.85}$$

- $\Delta(\rho)$ also provides the minimizer for the asymptotic quantifier of distillable coherence [WY16], even under MIO, meaning

$$C_D(\rho) = S(\rho||\Delta(\rho)) = S(\Delta(\rho)) - S(\rho) \tag{1.86}$$

  The theory became reversible for pure states under Incoherent Operations, making this quantity equal to coherence cost.

A connection between coherence and entanglement has been explored in [SSD$^+$15]. For example, the CNOT unitary gate which flips a qubit if the control is $|1\rangle$ and does nothing if the control is $|0\rangle$ cannot create coherence from an incoherent target qubit and ancilla. Nevertheless, it can convert the coherence of one of the two qubits into entanglement, i.e.

$$U_{CNOT}(|+\rangle \otimes |0\rangle) = |\Phi^+\rangle \tag{1.87}$$

Having presented the theory of quantum resources, in the next chapter, we will describe the formalism of quantum open system and show an example of how the interaction with an environment can lead to the loss of coherence. Moreover, we will characterize an important class of dynamics called Markovian, having the property that the evolution at every time step is independent of the past times.

Chapter 2

Quantum Non-Markovian processes

This chapter is based on the review papers by Ángel Rivas, Susana F. Huelga and Martin B. Plenio [RHP14] and [RH12].

Closed systems are an idealization and all the quantum systems in nature are open, meaning they are interacting with some environment. Usually, such interactions destroy the resources described in the previous chapter. In fact, this field is connected to the problem of why we don't observe the features of quantum mechanics in macroscopic objects even if these are made by many quantum particles. In other words, how does the transition from quantum to classical happen? [Sch19] Connected to this issue in the next chapter we will prove that a Markovian type of noise, which will be defined in this chapter, always destroys correlations between two qubits when applied locally to one of them. No matter how advanced our laboratories are, the presence of an environment is unavoidable. It follows that the formalism we are going to introduce in the next section is the proper way of describing them.

After that, in section 2.2 we will review the classical theory of stochastic processes with an emphasis on Markovian processes.

In section 2.3 we will extend the definition of Markovianity to the quantum realm and in sections 2.4 and 2.5 we will characterize this property and provide some famous measures of it.

2.1 Open systems dynamics

It is known that an isolated quantum system evolves following a unitary dynamic, i.e. $\Lambda_{t_1,t_0}(\rho_{t_0}) = U(t_1,t_0)\rho(t_0)U^\dagger(t_1,t_0)$. Now we will review the usual way to model the evolution of an open system. The physical picture is that the system is interacting and becoming entangled with some big environment, i.e. constituted by a large number of degrees of freedom. We will denote the system by S and the environment by E. The first step is then to define the total SE

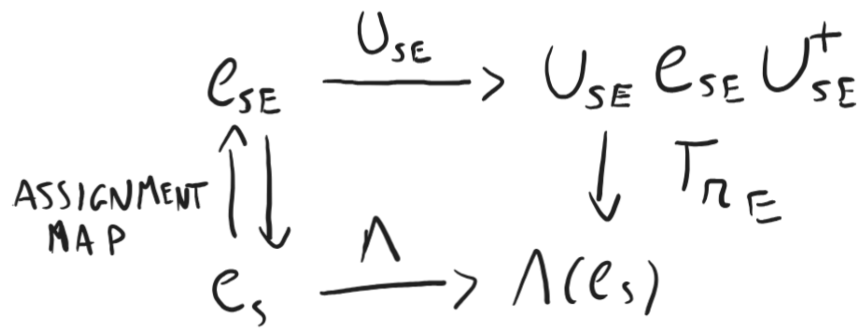

Figure 2.1: Schematic description of an open quantum system evolution through three steps: 1. An assignment map transforms the state of the system ρ_S into one of the composite system plus environment ρ_{SE}. This map must be reversible by tracing out the environment and uniquely defined. The standard choice is then a product map $\rho_S \rightarrow \rho_S \otimes \rho_E$, with a fixed $\rho_E = \frac{e^{-\beta H_E}}{Tr[e^{-\beta H_E}]}$, usually the Gibbs state at thermal equilibrium of the environment, with $\beta = \frac{1}{kT}$. 2. The system and environment interact and evolve unitarily as a closed system. 3. The environment is traced out after the interaction, leading to the effective map $\Lambda(\rho_S)$.

system starting from a state of the system S at time t_0, $\rho_S(t_0)$. This is done through the so-called **assignment map** $\rho_S(t_0) \rightarrow \rho_{SE}(t_0)$. The assignment map must be invertible, in the sense that $\mathrm{Tr}_E[\rho_{SE}(t_0)] = \rho_S(t_0)$. In the second step ρ_{SE} will evolve driven by the coupling between the two subsystems. In particular, since SE is a closed system it will evolve according to some unitary $U(t_0, t_1)$ from the initial time t_0 to a final time t_1. Remember that we have only access to the dynamic of one of the subsystems, S. For this reason, after letting SE evolve for some time we will discard the environment E by the operation of partial trace. Formally

$$\Lambda_{t_1,t_0}(\rho_S(t_0)) = \rho_S(t_1) = Tr_B[U(t_1, t_0)\rho_{SE}(t_0)U^\dagger(t_1, t_0)] \tag{2.1}$$

See fig. 2.1. Up to this point $\rho_{SE}(t_0)$ can contain correlations between the system and the environment. We will give an example to show why correlations are undesirable to model the evolution of the system. Let us start with an initial maximally mixed state $\rho_S(t_0) = \mathbb{I}_S/d_S$. This can be obtained in two ways through the assignment map, by tracing out E from a product state $\rho_{SE}(t_0) = \frac{\mathbb{I}_S}{d_S} \otimes \frac{\mathbb{I}_E}{d_E}$ or from a maximally correlated state $\rho_{SE}(t_0) = |\Phi^+_{SE}\rangle \langle\Phi^+_{SE}|$. In general, these two initial states of SE give different dynamics for the reduced state $\rho_S(t_1)$, when put into eq. 2.1, even though the initial state of S is still $\mathbb{I}_S/d_S$. This means that if we allow for correlations between S and E at time 0, the map 2.1

might be not uniquely defined. Moreover, if the assignment map has to preserve mixtures, in the sense that $p\rho_S^1 + (1-p)\rho_S^2 \to p\rho_{SE}^1 + (1-p)\rho_{SE}^2$ it can be shown that system and environment must be initially uncorrelated

$$\rho_{SE}(t_0) = \rho_S(t_0) \otimes \rho_E(t_0) \tag{2.2}$$

where $\rho_S(t_0)$ and $\rho_E(t_0)$ are marginals of $\rho_{SE}(t_0)$. ρ_E is fixed and it's usually given by the Gibbs state of the environment at thermal equilibrium, $\rho_E(t_0) = \frac{e^{-\beta H_E}}{Tr[e^{-\beta H_E}]}$, with $\beta = \frac{1}{kT}$. If one allows for correlations in the initial state the map 2.1 is not even positivity preserving for all $\rho_S(t_0)$. See [Pec94] for more details. With this assumption the map 2.1 is usually referred to as **Stinespring dilation** [NC11].

Let us now consider an eigendecomposition of $\rho_E(t_0) = \sum_i \lambda_i |\Psi_i\rangle \langle\Psi_i|$. Then we can write the map 2.1 as

$$\begin{aligned} \rho_S(t_1) &= Tr_B[U(t_1,t_0)\rho_S(t_0) \otimes \rho_E(t_0)U^\dagger(t_1,t_0)] \\ = \sum_i &\lambda_i Tr_B[U(t_1,t_0)\rho_S(t_0) \otimes |\Psi_i\rangle \langle\Psi_i| U^\dagger(t_1,t_0)] \\ &= \sum_\alpha K_\alpha(t_1,t_0)\rho_S(t_0)K_\alpha^\dagger(t_1,t_0) \end{aligned} \tag{2.3}$$

with $\alpha = \{i,j\}$ and $K_\alpha = \sqrt{\lambda_i}\langle\Psi_j| U(t_1,t_0) |\Psi_i\rangle$ Kraus operators, which obey

$$\sum_\alpha K_\alpha(t_1,t_0)K^\dagger(t_1,t_0) = \mathbb{I} \tag{2.4}$$

i.e. they preserve the trace of $\rho_S(t_0)$. The existence of these operators guarantees that the Stinespring dilation generates a completely positive map.

The most simple example of a map based on such a dilation procedure is given by

$$\Lambda_{(t_1,t_0)}(\rho_S(t_0)) = Tr_2[U_{SWAP}\rho_S(t_0) \otimes \rho_S(t_1)U_{SWAP}^\dagger] = \rho_S(t_1), \tag{2.5}$$

where U_{SWAP} exchange systems 1 and 2. This means that there always exists a completely positive map connecting two arbitrary time points t_0 and t_1 with $t_0 \leq t_1$.

Before to proceed with the theory of Markovianity, we will show a simple model which can explain the loss of coherence when a quantum system is put in contact with an environment.

2.1.1 A simple decoherence model

A simple model can show that the interaction with the environment destroys coherence in quantum states, a phenomenon called **decoherence**.

Consider a qubit coupled with an environment with an interaction tailored such that the environment assumes the state $|E_1\rangle$ when the qubit is in the state

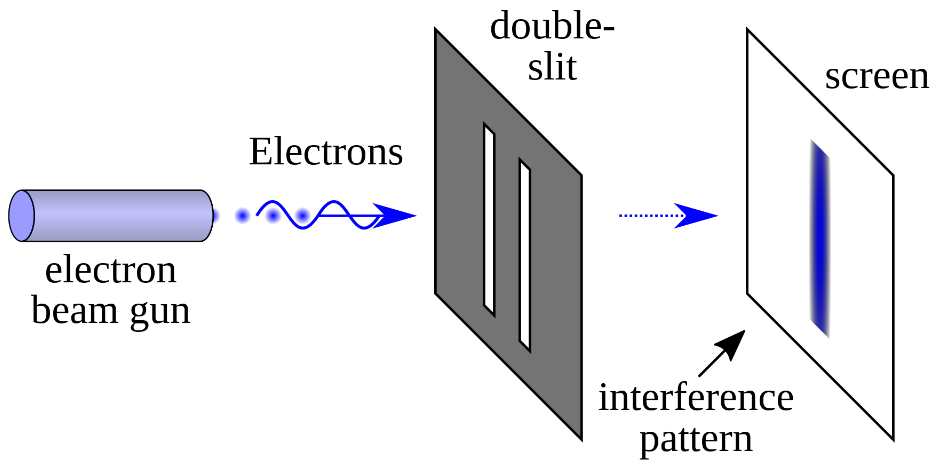

Figure 2.2: Double slit experiment. From `https://en.wikipedia.org/wiki/Double-slit_experiment`: a beam of electrons is sent into a barrier with two slits. Interference patterns typical of waves behavior appear on the screen after the barrier if the path of the electrons is not measured.

$|0\rangle$ and $|E_2\rangle$ when the qubit is in the state $|1\rangle$. For the moment, we assume that $|E_1\rangle$ and $|E_2\rangle$ are not necessarily orthogonal. The qubit is also initially prepared in a superposition of $|0\rangle$ and $|1\rangle$ in the following way:

$$|\Psi\rangle_S = \alpha\,|0\rangle + \beta\,|1\rangle \tag{2.6}$$

with α and β complex numbers such that $|\alpha|^2 + |\beta|^2 = 1$. Its density matrix will be

$$\rho_S = |\Psi\rangle\,\langle\Psi|_S = |\alpha|^2\,|0\rangle\,\langle 0| + |\beta|^2\,|1\rangle\,\langle 1| + \alpha\beta^*\,|0\rangle\,\langle 1| + \alpha^*\beta\,|1\rangle\,\langle 0| \tag{2.7}$$

with non-zero off-diagonal elements. Notice that those are the ones that produce the interference pattern in experiments like the double slit one (see Fig. 2.2), where $|0\rangle$ is associated with for example the left slit path of a particle and $|1\rangle$ with the right path.

In fact, writing the two associated wave functions as $\Psi_1(x) = \langle x|0\rangle$ and $\Psi_1(x) = \langle x|1\rangle$, the probability of finding the particle in position x is

$$p(x) = \mathrm{Tr}[\rho_S\,|x\rangle\,\langle x|] = \langle x|\,\rho_S\,|x\rangle = |\alpha|^2|\Psi_1(x)|^2 + |\beta|^2|\Psi_2(x)|^2 + 2Re[\alpha\beta^*\Psi_1^*(x)\Psi_2(x)^*] \tag{2.8}$$

where the last term is the interference one and it is proportional to the off-diaonal elements of ρ_S.

Let us now write the total system-environment state after the interaction.

$$|\Psi\rangle_{SE} = \alpha\,|0\rangle \otimes |E_1\rangle + \beta\,|1\rangle \otimes |E_2\rangle \tag{2.9}$$

Notice that this state is entangled between system and environment. The reduced density matrix of the system will be

$$\rho_S = \mathrm{Tr}_E[|\Psi\rangle\,\langle\Psi|_{SE}] = |\alpha|^2\,|0\rangle\,\langle 0| + |\beta|^2\,|1\rangle\,\langle 1| + \alpha\beta^*\,|0\rangle\,\langle 1|\,\langle E_2|E_1\rangle + \alpha^*\beta\,|1\rangle\,\langle 0|\,\langle E_1|E_2\rangle \tag{2.10}$$

Notice that if the states of the environment would be orthogonal, the off diagonal elements of the reduced density matrix would disappear, meaning that would not be no longer possible to observe interference fringes in a double slit experiment. In fact, computing the probability $p(x)$ in this case leads to

$$p(x) = |\alpha|^2|\Psi_1(x)|^2 + |\beta|^2|\Psi_2(x)|^2 + 2Re[\alpha\beta^*\Psi_1^*(x)\Psi_2(x)^*\,\langle E_2|E_1\rangle] \tag{2.11}$$

This is an example in which an interaction with the environment kills a very unique feature of quantum mechanics such us superposition.

We will now make an excursus on the classical Markovian theory to have a better understanding of the matter.

2.2 Classical stochastic processes

A classical random variable X is a function from the triple (Ω, Σ, p) to the triple $(\mathbb{R}, B, p)$, where:

- Ω is the set of all the possible outcomes. For the sake of simplicity, we will restrict to the case in which $card(\Omega) = N$, i.e. the set is finite and countable.
- Σ is the subset of all the possible partitions of Ω including the empty set $\emptyset$ and the entire Ω itself. This is usually referred to as σ-algebra.
- p: function $\Sigma \to [0, 1]$, with the property, called σ-additivity, that, if two sets are disjoint, i.e. $\Sigma_1 \bigcap \Sigma_2 = 0$, then $p(\Sigma_1 \bigcup \Sigma_2) = p(\Sigma_1) + p(\Sigma_2)$. Moreover $p(\Omega) = 1$ (see also Kolmogorov laws). This function will represent the probability of the realizations corresponding to Σ.
- The conditional probability of event Σ_1 conditioned to event Σ_2 is defined as $p(\Sigma_1|\Sigma_2) = \frac{p(\Sigma_1 \bigcap \Sigma_2)}{p(\Sigma_2)}$.
- B is the Borel set, i.e. the union of all the open sets in $\mathbb{R}$.

See also [BP07], Chapter 1. Now we can define a classical stochastic process as a family of random variables, parametrized by a single real value parameter, i.e. $\{X(t), t \in I \subset \mathbb{R}\}$. We will sample discrete time points t_n from the interval I and denote with x_n a realization of the random variable $X(t_n)$. Then we will define $p(x_n, t_n|x_{n-1}, t_{n-1}; ...; x_0, t_0)$ as the probability that event x_n occur, conditioned on all the previous events from t_0 to t_{n-1}.

Definition 2.1. A stochastic process is said to be Markovian if:

$$p(x_n, t_n|x_{n-1}, t_{n-1}; ...; x_0, t_0) = p(x_n, t_n|x_{n-1}, t_{n-1}) \tag{2.12}$$

$\forall t \subset I$ with $t_n \geq t_{n-1} \geq ... \geq t_0$.

This means that the conditional probability of the stochastic event at time t_n, providing all the previous events have occurred, depends only on the immediately previous event at time t_{n-1}. That is the reason a Markovian process is also called memoryless. Notice that for a non-Markov process then two points conditional probabilities $p(x_n, t_n|x_{n-1}, t_{n-1})$ are not well defined.

An important property of such processes is the following:

$$\begin{aligned} &p(x_3, t_3; x_2, t_2; x_1, t_1) = p(x_3, t_3|x_2, t_2; x_1, t_1)p(x_2, t_2; x_1, t_1) = \\ &p(x_3, t_3|x_2, t_2; x_1, t_1)p(x_2, t_2|x_1, t_1)p(x_1, t_1) = \\ &p(x_3, t_3|x_2, t_2)p(x_2, t_2|x_1, t_1)p(x_1, t_1) \end{aligned} \tag{2.13}$$

where $p(A; B) = p(A|B)p(B)$ is the joint probability of events A and B, which by definition is equal to the probability of the event A conditioned to the event B times the absolute probability of the event B.
Now we sum over x_2 and we divide by $p(x_1, t_1)$:

$$\begin{aligned} &\frac{1}{p(x_1, t_1)} \sum_{x_2} p(x_3, t_3; x_2, t_2; x_1, t_1) = \\ &\frac{1}{p(x_1, t_1)} p(x_3, t_3; x_1, t_1) = \frac{1}{p(x_1, t_1)} p(x_3, t_3|x_1, t_1)p(x_1, t_1) = \\ &\frac{1}{p(x_1, t_1)} \sum_{x_2} p(x_3, t_3|x_2, t_2)p(x_2, t_2|x_1, t_1)p(x_1, t_1) \end{aligned} \tag{2.14}$$

$$p(x_3, t_3|x_1, t_1) = \sum_{x_2} p(x_3, t_3|x_2, t_2)p(x_2, t_2|x_1, t_1) \tag{2.15}$$

which is known as **Chapman-Kolmogorov equation**, or **divisibility property**.

Now we will introduce the concept of transition matrix. A transition matrix $T(x_1, t_1|x_0, t_0)$ is such that connects two one-point probabilities of a stochastic process in the following way

$$p(x_1, t_1) = \sum_{x_0} T(x_1, t_1|x_0, t_0)p(x_0, t_0) \tag{2.16}$$

since all the outcomes at time t_1 are complete, i.e. $\sum_{x_1} p(x_1, t_1) = 1$, a desired property of the transition matrix is that $\sum_{x_1} T(x_1, t_1|x_0, t_0) = 1$ and since probabilities are positive quantities, i.e. $p(x_1, t_1) \geq 0$ it follows that $T(x_1, t_1|x_0, t_0) \geq 0$. These two properties define what is called a **stochastic matrix**.

Now from the initial time t_0 the quantity $p(x_1, t_1|x_0, t_0)$ is well defined and we can do the association $T(x_1, t_1|x_0, t_0) = p(x_1, t_1|x_0, t_0)$. This follows from the definition of conditional probability $p(x_1, t_1; x_0, t_0) = p(x_1, t_1|x_0, t_0)p(x_0, t_0)$, which implies $p(x_1, t_1) = \sum_{x_0} p(x_1, t_1|x_0, t_0)p(x_0, t_0)$. Starting from any other initial time t_1 of a general stochastic process unfortunately will not yield the same equality, i.e. $T(x_2, t_2|x_1, t_1) \neq p(x_2, t_2|x_1, t_1)$ simply because $p(x_2, t_2|x_1, t_1)$ is not well defined. Remember in fact that in general events happening at time t_2 depend on both t_0 and t_1 times. In a Markovian process, though $p(x_2, t_2|x_1, t_1; x_0, t_0) = p(x_2, t_2|x_1, t_1)$ and we can write $T(x_2, t_2|x_1, t_1) = p(x_2, t_2|x_1, t_1)$ for every $t_2 \geq t_1$.

This means that for a Markovian process, the transition matrices obey the Chapman-Kolmogorov equation derived above. In summary, they have to obey the three properties

$$\sum_{x_1} T(x_1, t_1|x_0, t_0) = 1 \tag{2.17}$$

$$T(x_1, t_1|x_0, t_0) \geq 0 \tag{2.18}$$

$$T(x_3, t_3|x_1, t_1) = \sum_{x_2} T(x_3, t_3|x_2, t_2)T(x_2, t_2|x_1, t_1) \tag{2.19}$$

In particular, the last equation means that the evolution from time t_1 to time t_3 can always be expressed as a composition of two single-step processes, one from time t_1 to time t_2 and the other from time t_2 to time t_3. Such a property is also called **divisiblity** and the entire process will be called **divisible**.

Notice that it is always possible to satisfy Eq. 2.17 and 2.19 in the following way:

$$\begin{aligned} T(x_2, t_2|x_1, t_1) = &\sum_{x_0} T(x_2, t_2|x_0, t_0)T(x_0, t_0|x_1, t_1) = \\ &\sum_{x_0} p(x_2, t_2|x_0, t_0)[p(x_1, t_1|x_0, t_0)]^{-1} \end{aligned} \tag{2.20}$$

but the problem is to ensure Eq. 2.18.

An important characterization of divisible processes is given by a property called contractivity under L_1-norm, which we will present in the following section.

2.2.1 Contractive property

Let us consider a random variable X that can have two possible associated probability distributions, $p_1(x)$ and $p_2(x)$, respectively drawn out with probability q and $1-q$. Then the minimal probability of failing to guess the correct distribution is given by:

$$P_{min}(fail) = \frac{1 - ||w||_1}{2} \tag{2.21}$$

where $||v||_1 = \sum_x |v(x)|$ is the L_1-norm of a vector, and $w(x) = qp_1(x) - (1 - q)p_2(x)$.

We will provide a proof of the quantum case of this theorem later on, which includes the classical one as a particular case.

Now the L_1-norm has a very important property called contractivity under divisible processes, which is formulated in the following way:

Theorem 2.1. *A transition matrix belongs to a divisible process if and only if it contracts the trace norm of every vector, meaning* $\forall v$ *and* t_1, t_2 *such that* $t_1 \leq t_2$*:*

$$||v(x_2)||_1 = ||\sum_{x_1} T(x_2,t_2|x_1,t_1)v(x_1)||_1 \leq ||v(x_1)||_1 \tag{2.22}$$

Proof. The first implication to prove is that if $T(x_2,t_2|x_1,t_1)$ is a stochastic matrix, then it is also a contraction of $v(x_1)$

$$\begin{aligned} ||\sum_{x_1} T(x_2,t_2|x_1,t_1)v(x_1)||_1 &= \sum_{x_2} |\sum_{x_1} T(x_2,t_2|x_1,t_1)v(x_1)| \leq \\ &\leq \sum_{x_1,x_2} T(x_2,t_2|x_1,t_1)|v(x_1)| = ||v(x_1)||_1. \end{aligned} \tag{2.23}$$

Where we used the triangle inequality for the modulus and the properties of a stochastic matrix.

The opposite implication is, assuming any vector $v(x_1)$ is contracted by the action of $T(x_2,t_2|x_1,t_1)$, then $T(x_2,t_2|x_1,t_1) \geq 0$. We just need this property since we already shown Eq. 2.17 and 2.18 are easily satisfied. We will consider a probability vector, i.e. $v(x) = p(x) \geq 0$ and $\sum_x p(x) = 1$. We will also assume that $\sum_{x_2} T(x_2,t_2|x_1,t_1) = 1$ is true.

$$\begin{aligned} ||p(x_1)||_1 = \sum_{x_1} p(x_1) = \sum_{x_1,x_2} T(x_2,t_2|x_1,t_1)p(x_1) \leq \sum_{x_2} |\sum_{x_1} T(x_2,t_2|x_1,t_1)p(x_1)| = \\ = \sum_{x_2} |p(x_2)| \leq ||p(x_1)||_1 \end{aligned} \tag{2.24}$$

Meaning

$$\sum_{x_1,x_2} T(x_2,t_2|x_1,t_1)p(x_1) = \sum_{x_2} |\sum_{x_1} T(x_2,t_2|x_1,t_1)p(x_1)| \tag{2.25}$$

which is only possible if the transition matrix is positive. □

It follows that, for Non-Markovian stochastic processes the probability of distinguishing two distributions can grow again at a certain time $t_1 \geq t_0$. This is what is called **backflow of information**. Consider the task of distinguishing the two probability distributions $p_1(x)$ and $p_2(x)$. Then if we are in the presence of a Markov process it is better to sample them as soon as possible. On the contrary, for a non-Markov process, it might be better to wait more time before performing the sampling.

After presenting the main points of the classical Markovian theory we are ready to discuss its generalization to the quantum case.

2.3 Quantum scenario

In the quantum case, it is impossible to sample a system at different times without perturbing it, so we have to restrict the theory to one-point probabilities, for which markovianity and divisibility coincide.
Consider an initial density matrix decomposed in some eigenbasis:

$$\rho(t_0) = \sum_x p_i(x, t_0) |x\rangle \langle x| \tag{2.26}$$

This state evolves in such a way that only the eigenvalues change, not the eigenvectors.

$$\rho(t) = \sum_i p_i(x, t) |x\rangle \langle x| \tag{2.27}$$

This is equivalent to a classical stochastic process.

Then we can think about these eigenvalues being transformed by stochastic matrices

$$p(x_1, t_1) = \sum_{x_0} T(x_1, t_1|x_0, t_0)p(x_0, t_0) \tag{2.28}$$

and if the matrices satisfy the divisibility property we speak about a Markov process.

This process can be written in terms of a linear quantum map bringing the state from t_0 to t_1.

$$\begin{aligned}\rho(t_1) = \Lambda_{t_1,t_0}(\rho(t_0)) = \sum_{x_0} p(x_0, t_0)\Lambda_{t_1,t_0}(|x_0\rangle \langle x_0|) = \\ \sum_{x_1,x_0} T(x_1, t_1|x_0, t_0)p(x_0, t_0) |x_1\rangle \langle x_1|\end{aligned} \tag{2.29}$$

since $T(x_1, t_1|x_0, t_0)$ is a stochastic matrix this map preserves positivity and trace of $\rho(t_0)$. Now we want to bring $\rho(t_1)$ from time t_1 to time t_3.

$$\begin{aligned}\rho(t_3) = \Lambda_{t_3,t_1}(\rho(t_1)) = \sum_{x_1} p(x_1, t_1)\Lambda_{t_3,t_1}(|x_1\rangle \langle x_1|) = \\ \sum_{x_3,x_1} T(x_3, t_3|x_1, t_1)p(x_1, t_1) |x_3\rangle \langle x_3| = \\ \sum_{x_3,x_1,x_2} T(x_3, t_3|x_2, t_2)T(x_2, t_2|x_1, t_1)p(x_1, t_1) |x_3\rangle \langle x_3| = \\ \sum_{x_1} p(x_1, t_1)\Lambda_{t_3,t_2}(\Lambda_{t_2,t_1}(|x_1\rangle \langle x_1|))\end{aligned} \tag{2.30}$$

Where the last steps are valid if the divisibility property of the process holds. Then we can say the dynamical map obeys the following composition law:

$$\Lambda_{t_3,t_1} = \Lambda_{t_3,t_2} \circ \Lambda_{t_2,t_1} \tag{2.31}$$

This is usually also referred to as **divisibility property**. Notice that each of the composed maps is physical, i.e. positivity and trace-preserving $\forall t_1 \leq t_2 \leq t_3$. A two-point map $\Lambda_{t_3,t_2} = \Lambda^{-1}_{t_3,t_2} \circ \Lambda_{t_3,t_1}$ from any intermediate time to the final is called **propagator**. If the propagator is positive we call the dynamic P-divisible. If it's completely positive we speak of CP-divisibility. A much stronger property is the **semigroup** one, i.e. $\Lambda_{t_1+t_2,0} = \Lambda_{t_1,0} \circ \Lambda_{t_2,0}$. Finally, in the general quantum setting, we require these properties to hold even when the eigenbasis of the density matrix is not preserved during the dynamics. We will now provide a simple physical model for a Markovian quantum dynamic.

2.3.1 Collisional model

In this model, we consider an environment consisting of many identical particles $\rho_E^{\otimes N}$, with N very large. The interaction between the system and environment is discretized in time steps $t_1, ..., t_n$, and it happens with only one particle of the environment per time. The n-th interaction is of the form:

$$\rho_S(t_{n+1}) = Tr_E[U(t_{n+1}, t_n)\rho_S(t_n) \otimes \rho_E U^\dagger(t_{n+1}, t_n)] = \Lambda_{(t_{n+1},t_n)}[\rho_S(t_n)] \tag{2.32}$$

Notice that, after having interacted with the system, the interacting particle of the environment is thrown away. In the next step, the system interacts with another identical particle ρ_E through the unitary $U(t_{n+1}, t_{n+2})$, and so on. See Fig. 2.3 and [CLGP22].

It is easy to see that the total dynamics from $t_1, ..., t_n$ is a concatenation of all these completely positive dynamics, making it Markovian then.

$$\rho_S(t_{n+2}) = \Lambda_{(t_{n+2},t_{n+1})}\Lambda_{(t_{n+1},t_n)}[\rho_S(t_n)] \tag{2.33}$$

In the limit $t_{n+1} - t_n \to 0$, for every n, it is possible to recover a continuous dynamics obeying a Lindblad master equation (see next section). We will now review the main approaches to characterize or quantify quantum non-Markovianity.

2.4 Characterization of quantum Markovianity

The characterization of quantum Markovian dynamics we will now present is based on the property of being **differentiable**.

Definition 2.2. A quantum dynamics is differentiable if the generator

$$\mathcal{L}_t := \lim_{\varepsilon \to 0} \frac{\Lambda_{(t+\varepsilon,t)} - \mathbb{I}}{\varepsilon} \tag{2.34}$$

is well-defined. A sufficient condition for the existence of the generator is the semigroup one.

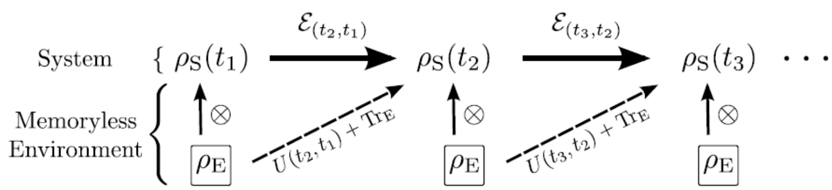

Figure 2.3: [RHP14] Collisional model: at each step, the system is put in contact in an uncorrelated way with a big environment ρ_E, it interacts through the unitary $U(t_n, t_{n+1})$, the environment is thrown away and it is reset to the initial state ρ_E for the next step.

A closed expression for the equation of motion of open quantum systems is known only in the case of Markovian dynamics.

Theorem 2.2. *A quantum dynamics is Markovian iff the* ***master equation*** *[Pea12]*

$$\frac{d\rho(t)}{dt} = \mathcal{L}_t(\rho(t)) = -i[H(t),\rho(t)] + \sum_k \gamma_k(t)(V_k(t)\rho(t)V_k^\dagger(t) - \frac{1}{2}\{V^\dagger(t)V(t),\rho(t)\}) \tag{2.35}$$

can be written with positive decoherence rates $\gamma_k(t) \geq 0$, $\forall k$.

Proof. We will prove the implication: Markovian dynamics $\Longrightarrow$ positive Lindbladian.

Let's write the evolution between two generic times t_1 and t_2 in Kraus decomposition

$$\Lambda_{(t_2,t_1)}(\rho) = \sum_\alpha K_\alpha(t_2,t_1)\rho K_\alpha^\dagger(t_2,t_1) \tag{2.36}$$

Let $\{F_j, j = 1, ..., N^2\}$ be a complete orthonormal basis of operators with respect to the Hilbert-Schmidt norm: $(F_i, F_j)_{HS} = Tr(F_j^\dagger F_k) = \delta_{jk}$, such that $F_{N^2} = \frac{\mathbb{I}}{\sqrt{N}}$ and the rest are traceless, i.e. $Tr(F_j) = 0\ \forall j = 1, ..., N^2 - 1$.

Clearly, the Kraus operators can be expanded in such a basis

$$K_\alpha(t_2,t_1) = \sum_{i=1}^{N^2} c_i^\alpha(t_2,t_1)F_i \tag{2.37}$$

$$c_i^\alpha(t_2,t_1) = (F_i, K_\alpha(t_2,t_1))_{HS} \tag{2.38}$$

$$\Lambda_{(t_2,t_1)}(\rho) = \sum_{i,j=1}^{N^2}\sum_\alpha c_i^\alpha(t_2,t_1)(c_j^\alpha)^*(t_2,t_1)F_i\rho F_j^\dagger \tag{2.39}$$

with $c_i^\alpha(t_2,t_1)$ continuous and differentiable. The matrix

$$c_{ij} = \sum_{\alpha} c_i^{\alpha}(t_2,t_1)(c_j^{\alpha})^*(t_2,t_1) \tag{2.40}$$

is positive semidefinite.
Let's choose $t_1 = t$ and $t_2 = t+\varepsilon$. Then we can write the generator as

$$\mathcal{L}_t(\rho) = \lim_{\varepsilon\to 0} \sum_{i,j=1}^{N^2} \frac{c_{ij}(t+\varepsilon,t)F_i\rho F_j^{\dagger} - \mathbb{I}}{\varepsilon} = \lim_{\varepsilon\to 0}\left[\frac{1}{N}\frac{c_{N^2N^2}(t+\varepsilon,t)-N}{\varepsilon}\rho \right.$$
$$\left. +\frac{1}{\sqrt{N}}\sum_{i=1}^{N^2-1}\left(\frac{c_{iN^2}(t+\varepsilon,t)}{\varepsilon}F_i\rho + \frac{c_{N^2i}(t+\varepsilon,t)}{\varepsilon}\rho F_i^{\dagger}\right) + \sum_{i,j=1}^{N^2-1}\frac{c_{ij}(t+\varepsilon,t)}{\varepsilon}F_i\rho F_j^{\dagger}\right] \tag{2.41}$$

and defining

$$a_{N^2N^2}(t) = \lim_{\varepsilon\to 0}\frac{c_{N^2N^2}(t+\varepsilon,t)-N}{\varepsilon} \tag{2.42}$$

$$a_{iN^2}(t) = \lim_{\varepsilon\to 0}\frac{c_{iN^2}(t+\varepsilon,t)}{\varepsilon}\ ,\ j=1,...,N^2-1 \tag{2.43}$$

$$a_{ij}(t) = \lim_{\varepsilon\to 0}\frac{c_{ij}(t+\varepsilon,t)}{\varepsilon}\ ,\ i,j=1,...,N^2-1 \tag{2.44}$$

$$F(t) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N^2-1} a_{iN^2}(t)F_i \tag{2.45}$$

$$G(t) = \frac{a_{N^2N^2}}{2N}\mathbb{I} + \frac{1}{2}(F^{\dagger} + F(t)) \tag{2.46}$$

$$H(t) = \frac{1}{2i}(F^{\dagger} - F(t)) \tag{2.47}$$

$$\mathcal{L}_t(\rho) = -i[H(t),\rho] + \{G(t),\rho\} + \sum_{i,j=1}^{N^2-1} a_{ij}(t)F_i\rho F_j^{\dagger} \tag{2.48}$$

The map preserves the trace so

$$0 = Tr(\mathcal{L}_t(\rho)) = Tr\left(\left[2G(t) + \sum_{i,j=1}^{N^2-1} a_{ij}(t)F_i^{\dagger}F_j\right]\rho\right) \tag{2.49}$$

$\forall\rho$ which implies that

$$G(t) = -\frac{1}{2}\sum_{i,j=1}^{N^2-1} a_{ij}(t)F_i^{\dagger}F_j \tag{2.50}$$

Then

$$\mathcal{L}_t(\rho) = -i[H(t),\rho] + \sum_{i,j=1}^{N^2-1} a_{ij}(t)[F_i\rho F_j^\dagger - \frac{1}{2}\{F_i^\dagger F_j,\rho\}] \tag{2.51}$$

Being $a_{ij}(t)$ another positive semidefinite matrix because of the positivity of c_{ij}, it can be diagonalized by a unitary transformation

$$\sum_{i,j} u_{mi}(t)a_{ij}(t)u_{nj}^*(t) = \gamma_m\delta_{mn} \tag{2.52}$$

with $\gamma_m \geq 0$. Then with the new set of operators

$$V_k(t) = \sum_{j=1}^{N^2-1} u_{kj}^* F_j \tag{2.53}$$

$$\mathcal{L}_t(\rho) = -i[H(t),\rho] + \sum_{k=1}^{N^2-1} \gamma_k(t)[V_k\rho F_k^\dagger - \frac{1}{2}\{V_k^\dagger V_k,\rho\}] \tag{2.54}$$

□

For a first quantitative measure of non-Markovianity one can then define the functions

$$f_j(t) := max\{-\gamma_j(t),0\} \tag{2.55}$$

which monitors the signs of the single decay rates. The total non-Markovianity measure inside the time interval I will be

$$N = \int_I f(t)dt \tag{2.56}$$

with $f(t) = \sum_{j=1}^{d^2-1} f_j(t)$. We will provide other two equivalent measures in the next section.

2.5 Measures of quantum Non-Markovianity

The first measure we will propose uses an important characterization of completely positive maps called **Choi-Jamiolkowski isomorphism** [Cho75], stating that a quantum map Λ is Completely Positive iff the matrix

$$\Lambda \otimes \mathbb{I}(|\Phi^+\rangle\langle\Phi^+|) \geq 0 \tag{2.57}$$

is positive semidefinite, where $|\Phi^+\rangle = \frac{1}{\sqrt{d}}\sum_i |ii\rangle$ is the maximally entangled state. Using this characterization we can say that

$$||\Lambda_{(t,t_1)} \otimes \mathbb{I}(|\Phi^+\rangle\langle\Phi^+|)||_1 = \begin{cases} 1 & \text{iff } \Lambda_{(t,t_1)} \text{ is CP} \\ >1 & \text{otherwise.} \end{cases} \tag{2.58}$$

Then we define the function

$$g(t) := \lim_{\varepsilon \to 0^+} \frac{||\Lambda_{(t+\varepsilon,t)} \otimes \mathbb{I}(|\Phi^+\rangle \langle\Phi^+|)||_1 - 1}{\varepsilon} \tag{2.59}$$

and study the sign of it over a certain time interval to determine if the map is CP divisible or not.

If the process is differentiable we can write

$$g(t) = \lim_{\varepsilon \to 0^+} \frac{||[\mathbb{I} + \varepsilon(\mathcal{L}_t \otimes \mathbb{I})](|\Phi^+\rangle \langle\Phi^+|)||_1 - 1}{\varepsilon} \tag{2.60}$$

The second measure we will propose makes use of the contractivity property of a positive map, a property that holds in the quantum case too.

2.5.1 Contractivity

Theorem 2.3. *A trace-preserving linear map is positive iff it is a contraction [PGWPR06] for the trace norm* $||A||_1 = Tr(\sqrt{A^\dagger A})$.

Proof. First, we will prove the only if implication.

Let us split our operator into two orthogonal positive parts

$$\sigma = \sigma^+ - \sigma^- \tag{2.61}$$

$$||\sigma||_1 = ||\sigma^+||_1 + ||\sigma^-||_- \tag{2.62}$$

with σ^+ containing the positive eigenvalues of σ, and σ^- the negatives, but with changed sign. Trace preservation implies $||\Lambda(\rho)||_1 = ||\rho||_1$ on positive operators ρ. Then, since the map can mix the orthogonal subspaces of the supports of σ^+ and σ^- we will use triangular inequality of the trace norm

$$||\Lambda(\sigma)||_1 \leq ||\Lambda(\sigma^+)||_1 + ||\Lambda(\sigma^-)||_1 = ||\sigma^+||_1 + ||\sigma^-||_1 = ||\sigma||_1 \tag{2.63}$$

For the other implication

$$||\rho||_1 = Tr(\rho) = Tr(\Lambda(\rho)) \leq ||\Lambda(\rho)||_1 \leq ||\rho||_1 \tag{2.64}$$

Then

$$Tr(\Lambda(\rho)) = ||\Lambda(\rho)||_1 \tag{2.65}$$

for every ρ, making the map positive. □

This implies that the inverse of a map, even if it might exist mathematically, is never physical unless we are talking of a unitary, since an invertible physical map must basically leave the trace norm invariant. We refer once again to the review [RH12] for the complete proof of this fact. An invertible physical map is called **reversible**.

The trace norm has a very important physical application, connected to the single shot distinguishability of two quantum states in a similar fashion that the

L1 norm was connected to the distinguishability of two probability distributions [Hel69]. We will now give the proof of the quantum version of this fact.

The task will be to distinguish two quantum states by means of a binary quantum measurement (POVM) $\{T, \mathbb{I} - T\}$, the outcome of T meaning we are observing ρ_1 and the outcome of $\mathbb{I} - T$ signaling the presence of ρ_2. The state ρ_1 is prepared with probability q and the state ρ_2 with $1 - q$.

Theorem 2.4. *The minimum error probability in the one-shot state discrimination task between ρ_1 and ρ_2 is given by*

$$P_{min}(fail) = \frac{1 - ||\Delta||_1}{2} \tag{2.66}$$

where $\Delta = q\rho_1 - (1 - q)\rho_2$.

Proof. The average probability of failing the discrimination task is given by:

$$P(fail) = qTr(\rho_1(\mathbb{I} - T)) + (1 - q)Tr(\rho_2 T) \tag{2.67}$$

at this point we need to minimize among all possible binary POVMs, then $P_{min}(fail)$ can be rewritten as:

$$P_{min}(fail) = q - max_{0 \leq T \leq \mathbb{I}} Tr(\Delta T) = q - \mathrm{Tr}(\Delta^+) \tag{2.68}$$

where $\Delta = q\rho_1 - (1 - q)\rho_2$.
Notice that we used the fat that $Tr(\Delta T)$ is upper bounded by $Tr(\Delta^+)$, with Δ^+ the positive part of the matrix Δ, meaning the matrix with only the positive eigenvalues of Δ. In fact, this can be decomposed in $\Delta = \Delta^+ - \Delta^-$, with $\Delta^+, \Delta^- \geq 0$, Δ^- containing the negative eigenvalues of Δ with opposite sign. Moreover combining the 2 relations:

$$Tr(\Delta) = Tr(\Delta^+) - Tr(\Delta^-) = 2q - 1 \tag{2.69}$$

$$||\Delta||_1 = Tr(\Delta^+) + Tr(\Delta^-) \tag{2.70}$$

and solving for

$$Tr(\Delta^+) = \frac{||\Delta||_1 + 2q - 1}{2} \tag{2.71}$$

Substituting in 2.68 we obtain 2.21. □

The classical case is retrieved when the two states are diagonal in the same basis and the binary measurements commute.

Then we can define a second measure of non-Markovianity based on the contractivity of the map in the presence of an ancilla:

$$\lim_{\varepsilon \to 0^+} \frac{||\Delta(t + \varepsilon)||_1 - ||\Delta(t)||_1}{\varepsilon} \tag{2.72}$$

where $\Delta(t_0) = q\rho_{1A} + (1 - q)\rho_{2A}$ is the enlarged Helstrom matrix and $\Delta(t) = \Lambda_{(t,t_0)} \otimes \mathbb{I}(\Delta)$, which takes into account the presence of an inert ancillary system.

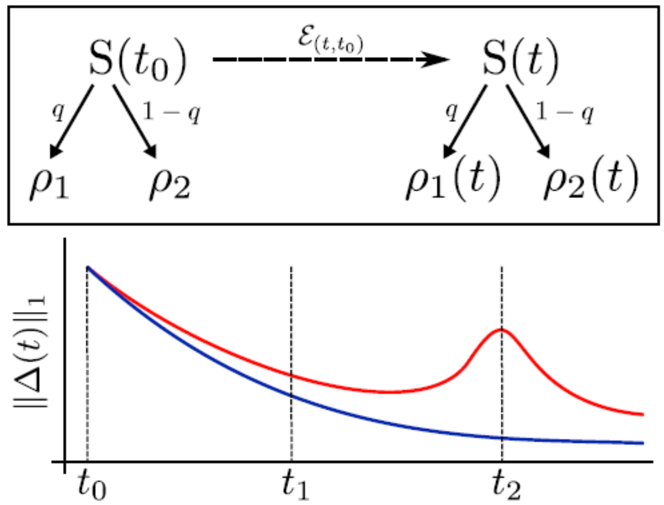

Figure 2.4: [RHP14] Evolution of the distinguishability between two states ρ_1 and ρ_2 undergoing two different dynamics. Red curve: Overall the dynamic is positive so the map will be contractive, but there is an intermediate time such that the derivative of $||\Delta(t)||_1$ can be positive. That is a signature of a negative propagator, i.e. non-Markovianity. Blue curve: The dynamic is positive and positive-divisible, hence it is always contractive.

In other words, we are imagining a state discrimination task as the one defined in Theorem 2.4, but with time-dependent states $\rho_1(t)$ and $\rho_2(t)$. Depending on the type of dynamic (Markovian or not) they are subject to, the probability of distinguishing them can only decrease or at a particular time t^* can increase. See figure 2.4. In the non-Markovian case is then convenient to wait for this time t^*, but in the Markovian one, the experimenter should perform the measurement as fast as possible.

Having equipped ourselves with the theories of quantum resources and open systems we are now ready to present the original results of this thesis in the next two chapters.

Chapter 3

Preserving quantum resources with non-Markovian dynamics

In this chapter, based on the original results published in the work [MWS$^+$22], we show that there is an advantage of non-Markovian types of noises over Markovian ones in preserving resources such as quantum correlations or coherence. The family of noises we will study does not have to reduce to a trivial unitary evolution at any time, hence we will consider a decoherence matrix always separated from 0.

In the next section, we will review the basics of qubit dynamics and we will prove a statement about correlations between two qubits when one is subject to a Markovian noise and the other plays the role of an ancillary system, i.e. their correlations inevitably decay exponentially in time. In particular, our setting is made explicit in Fig. 3.1: a qubit is prepared in a correlated state with another two-level particle, eventually in a maximally correlated state. Moreover, the same qubit is put in contact with an environment that generates the noise. We wish to study the behavior of the correlations between the two qubits over time when the environment is such that induces Markovian dynamics.

In section 3.2 we will describe the covariant maps, a particular class of maps that arise naturally when a specific basis is fixed.

In section 3.3 we allow those maps to be non-Markovian and show an example in which a map that is always non-Markovian is the best choice for preserving the initial correlations. Then we proceed, in section 3.4, to study the evolution of different correlations and resource measures under this dynamic and give a practical scenario of an application in quantum metrology.

Finally, in section 3.5 we will present the setup and results of an experiment that simulates this optimal example using single photons interacting with a quartz crystal.

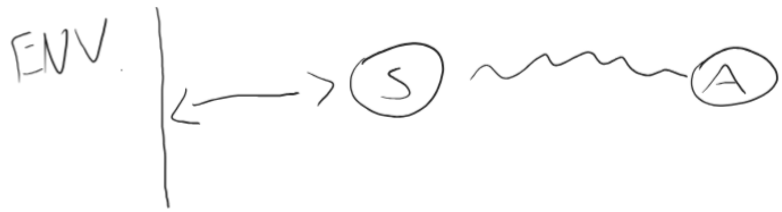

Figure 3.1: Setting studied in this chapter: a qubit is prepared in a correlated state with another two-level particle, eventually in a maximally correlated state. Moreover, the same qubit is put in contact with an environment that generates the noise.

3.1 Qubit dynamics

For $d = 2$, the Lindblad jump operators are given by the Pauli matrices σ_i $i = 1, 2, 3$. They are given by, expressed in the computational basis $\{|0\rangle, |1\rangle\}$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.1}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{3.2}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.3}$$

and they obey the commutation and anti-commutation relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \tag{3.4}$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{I} \tag{3.5}$$

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{I} + i\varepsilon_{ijk}\sigma_k \tag{3.6}$$

The Lindbladian reads

$$\mathcal{L}_t(\rho) = \frac{1}{2}\sum_{i,j=1}^{3} \gamma_{ij}(t)\left(\sigma_i\rho\sigma_j - \frac{1}{2}\{\sigma_i\sigma_j, \rho\}\right) \tag{3.7}$$

We make use of the Bloch vector representation of a qubit density matrix

$$\rho(t) = \frac{1}{2}\left(\mathbb{I} + \sum_{i=1}^{3} r_i(t)\sigma_i\right) \tag{3.8}$$

with $||\vec{r}(t)|| \leq 1$. Putting this form into eq. 3.7 and using the commutation relations

$$\mathcal{L}_t(\rho) = \frac{1}{2}\sum_{i,j=1}^{3}\left\{\left[\frac{1}{2}(\gamma_{ij}(t)+\gamma_{ji}(t)) - \gamma_{ii}(t)\right] r_j(t) + \xi_j(t)\right\}\sigma_j \tag{3.9}$$

where $\xi_k(t) = i\sum_{i,j=1}^{3}\varepsilon_{ijk}\gamma_{ij}(t)$ is a vector given by the antisymmetric part of the decoherence matrix and representing a non-unital shift of the Bloch vector.

This means that the corresponding differential equations for the Bloch vector components are

$$\dot{\vec{r}}(t) = (\gamma^S(t) - Tr[\gamma(t)]\mathbb{I})\vec{r}(t) + \vec{\xi}(t) \tag{3.10}$$

with $\gamma^S(t) = \frac{1}{2}(\gamma_{ij}(t)+\gamma_{ji}(t))$ symmetric part of the decoherence matrix.

We remember that the solution of an inhomogeneous differential equation of the first order [HC80]

$$y' + p(t)y = f(t) \tag{3.11}$$

is given by a solution of the homogeneous equation $y' + p(t)y = 0$, i.e. $y(t) = Ae^{P(t)}$, with $P(t) = -\int_0^t p(t')dt'$, plus a particular one of the 3.11, i.e. $y(t) = v(t)e^{P(t)}$, with $v'(t) = e^{-P(t)}f(t)$.

In other words, all solutions have the form

$$y(t) = v(t)e^{P(t)} + Ae^{P(t)} \tag{3.12}$$

Analogously, if X_t is the solution to the homogeneous differential equation $\dot{\vec{r}}(t) = A_t\vec{r}(t)$, with $A_t = \gamma^S(t) - Tr\gamma(t)\mathbb{I}$, meaning $\frac{d}{dt}X_t = A_tX_t$, then the solution to the inhomogeneous equation is given by

$$\vec{r}(t) = X_t\vec{r}_0(t) + X_t\int_0^t X_s^{-1}\vec{\xi}(s)ds \tag{3.13}$$

Note that the second term of the solution is independent of the initial condition.

Let us now consider a full rank decoherence matrix $\gamma(t) \geq c\mathbb{I}$, $\forall t \geq T$, with $c > 0$. This assumption guarantees that the evolution never became a trivial unitary, even at large time scales. Moreover, the positivity of $\gamma(t)$ implies that $\mathcal{L}_t(\rho)$ is the generator of a Markovian evolution $\Lambda_t(\rho)$. Then we can prove the following property:

Proposition 1

$$\min_{\sigma^A\otimes\sigma^B} ||\Lambda_t \otimes \mathbb{I}(\rho^{AB}) - \sigma^A\otimes\sigma^B||_1 \leq 2e^{-2ct} \tag{3.14}$$

Proof. Since $\gamma \geq c\mathbb{I}$, it follows that $\mathrm{Tr}[\gamma] \geq c$, and we can rewrite the differential equation as

$$\dot{\vec{r}}(t) = (A'_t - 2c\mathbb{I})\vec{r}(t) + \vec{\xi}(t) \tag{3.15}$$

with $A'_t = \gamma_t^S - Tr(\gamma)\mathbb{I} + 2c\mathbb{I} \leq \gamma_t^S + c\mathbb{I} \leq 0$. The solution to this equation is

$$\vec{r}(t) = e^{-2ct} X'_t \vec{r}_0 + e^{-2ct} X'_t \int_0^t e^{2cs} (X'_s)^{-1} \vec{\xi}_s ds \tag{3.16}$$

with $\frac{d}{dt} X'_t = A'_t X'_t$. Defining

$$\vec{\eta}(t) = e^{-2ct} X'_t \int_0^t e^{2cs} (X'_s)^{-1} \vec{\xi}_s ds \tag{3.17}$$

it is true that $|\vec{r}(t) - \vec{\eta}(t)| \leq 2^{-ct} |\vec{r}_0|$, which means that

$$||\Lambda_t(\rho_0) - \tilde{\rho}(t)||_1 \leq e^{-2ct} \tag{3.18}$$

with $\tilde{\rho}(t) = \frac{1}{2}(\mathbb{I} + \vec{\eta}(t) \cdot \vec{\sigma})$, for all initial states ρ_0.

This implies that the distance between the two maps Λ_t and $\Phi_t(\rho) = Tr(\rho)\tilde{\rho}(t)$, $||\Lambda_t - \Phi_t|| = \min_\rho ||\Lambda_t(\rho) - \Phi_t(\rho)||_1$ is also exponentially bounded

$$||\Lambda_t - \Phi_t|| \leq e^{-2ct} \tag{3.19}$$

Finally, using the relation

$$||\Lambda_1 \otimes \mathbb{I}_d - \Lambda_2 \otimes \mathbb{I}_d|| \leq d||\Lambda_1 - \Lambda_2|| \tag{3.20}$$

with the substitution $\Lambda_1 = \Lambda_t$ and $\Lambda_2 = \Phi_t$, and noticing that

$$\Phi_t \otimes \mathbb{I}(\rho^{AB}) = \Phi(\rho^A) \otimes \rho^B \tag{3.21}$$

we end up with

$$||\Lambda_t \otimes \mathbb{I}(\rho^{AB}) - \Phi_t(\rho^A) \otimes \rho^B|| \leq 2e^{-2ct} \tag{3.22}$$

□

Every two-qubit state is then sent exponentially close to a product one by the action of a Markovian, never ending evolution, if we wait for a sufficiently long time. This means that any form of correlation is destroyed by a Markovian noise acting locally on one qubit.

Before moving on to the problem of preserving quantum resources we will now describe the class of maps we will focus on in this chapter.

3.2 Covariant maps

One of the quantum resources we wish to preserve is given by quantum coherence. Since, as we saw in Chapter 1, this is a basis-dependent quantity, we focus our attention on covariant dynamical maps [FGL20],[SKanHDDan16], i.e. maps with the property

$$\Lambda_t(U\rho U^\dagger) = U\Lambda_t(\rho)U^\dagger \tag{3.23}$$

of commuting with a unitary, that, without loss of generality, we will fix to be a rotation around the z-axis of the Bloch sphere $U = e^{-i\sigma_z\varphi}$. This definition is basically equivalent to the one of TIO.

The most general form of a Lindbladian for such maps is given by the following decoherence matrix, written in the Pauli basis

$$\gamma(t) = \begin{pmatrix} a(t) & ix(t) & 0 \\ -ix(t) & a(t) & 0 \\ 0 & 0 & f(t) \end{pmatrix} \tag{3.24}$$

with eigenvalues $\gamma_\pm(t) = a(t) \pm x(t)$ in the $\sigma_\pm = \frac{1}{2}(\sigma_x \pm \sigma_y)$ basis and $f(t)$ in σ_z basis.

The diagonal form of the Lindbladian is then given by

$$\mathcal{L}_t(\rho) = \gamma_+(t)\left(\sigma_+\rho\sigma_- - \frac{1}{2}\{\rho, \sigma_-\sigma_+\}\right) + \gamma_-(t)\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\rho, \sigma_+\sigma_-\}\right) + \gamma_z(t)(\sigma_z\rho\sigma_z - \rho) \tag{3.25}$$

with $\gamma_+(t)$ being a gain rate, $\gamma_-(t)$ a dissipation and $\gamma_z(t)$ a dephasing one.

Let's write 3.10 in this case.

$$\gamma^S - Tr(\gamma)\mathbb{I} = \begin{pmatrix} -(a(t) + f(t)) & ix(t) & 0 \\ -ix(t) & -(a(t) + f(t)) & 0 \\ 0 & 0 & -2a(t) \end{pmatrix} \tag{3.26}$$

$$\vec{\xi}(t) = \begin{pmatrix} 0 \\ 0 \\ x(t) \end{pmatrix} \tag{3.27}$$

$$\dot{r}_1(t) = -(a(t) + f(t))r_1(t) \tag{3.28}$$

$$\dot{r}_2(t) = -(a(t) + f(t))r_2(t) \tag{3.29}$$

$$\dot{r}_3(t) = -2a(t)r_3(t) - 2x(t) \tag{3.30}$$

with solutions

$$r_1(t) = e^{-A(t)-F(t)}r_1(0) \tag{3.31}$$

$$r_2(t) = e^{-A(t)-F(t)}r_2(0) \tag{3.32}$$

$$r_3(t) = e^{2A(t)}r_3(0) - l_z(t) \tag{3.33}$$

with $A(t) = \int_0^t a(\tau)d\tau$, $F(t) = \int_0^t f(\tau)d\tau$ and $l_z(t) = 2e^{-A(t)}\int_0^t x(\tau)e^{4A(\tau)}d\tau$.

Calling $\lambda(t) = e^{-A(t)-F(t)}$ and $\lambda_z(t) = e^{-2A(t)}$ the effective evolution of a generic initial density matrix will be

$$\Lambda_t(\rho) = \frac{1}{2}\left(\mathbb{I} + \lambda(t)(r_1(0)\sigma_1 + r_2(0)\sigma_2) + (\lambda_z(t) - l_z(t))r_3(0)\sigma_z\right) \tag{3.34}$$

Decoherence rates and lastly defined factors are connected through the relations

$$\gamma_{\pm}(t) = \frac{\lambda_z(t)}{2}\frac{d}{dt}\left(\frac{1 \pm l_z(t)}{\lambda_z(t)}\right) \tag{3.35}$$

$$\gamma_z(t) = \frac{1}{4}\frac{d}{dt}\ln\frac{\lambda_z(t)}{\lambda^2(t)} \tag{3.36}$$

Those have to be all positive for a CP-divisible dynamic.

Let's write the Choi matrix of this dynamic. This will correspond to our setting of Fig. 3.1 in which we prepare an initial maximally entangled state of the system and the ancilla and the noise acts locally on the qubit system.

$$\Omega_{\Lambda_t} = \Lambda_t \otimes \mathbb{I}(|\Phi^+\rangle\langle\Phi^+|) = \frac{1}{4}\sum_{i,j=0}^{1}\Lambda_t(|i\rangle\langle j|)\otimes|i\rangle\langle j| =$$
$$\frac{1}{4}\begin{pmatrix} 1+\lambda_z(t)+l_z(t) & 0 & 0 & 2\lambda(t) \\ 0 & 1-\lambda_z(t)-l_z(t) & 0 & 0 \\ 0 & 0 & 1-\lambda_z(t)+l_z(t) & 0 \\ 2\lambda(t) & 0 & 0 & 1+\lambda_z(t)-l_z(t) \end{pmatrix} \tag{3.37}$$

The conditions for the positivity of this matrix, then the complete positivity of the map are

$$\lambda_z(t) + |l_z(t)| \leq 1 \tag{3.38}$$

$$4\lambda^2(t) + l_z^2(t) \leq (1+\lambda_z(t))^2 \tag{3.39}$$

The Bloch ball is contracted into an ellipsoid with axes λ, λ, λ_z and shifted by l_z along the z-axis 3.3. As long as these conditions are satisfied the matrix $\gamma(t)$ doesn't have to be positive semidefinite if we don't want to impose CP-divisibility. In the next section, we will relax the CP-divisibility condition, allowing the $\gamma(t)$ matrix to be negative and search among this class of evolutions for the one that best preserves quantum correlations.

3.3 Preserving quantum correlations

The first condition for complete positivity 3.38 means that the image space of the Bloch vector will never be outside the Bloch sphere. This implies, from the differential equation 3.30 for the coordinate $r_3(t)$, that $\frac{x(t)}{a(t)} \leq 1$, and by definition of $\gamma_{\pm}(t) = a(t) \pm x(t)$, the only possible negative eigenvalue of the decoherence matrix $\gamma(t)$ can be $f(t)$.

The condition for the separability of the Choi state is given by the Peres criterion [Per96]., which for a two-qubit state is necessary and sufficient. It states that a bipartite state $\rho = \sum_{ijkl} \rho_{ijkl} |i\rangle \langle j|^A \otimes |k\rangle \langle l|^B$ is separable iff

$$\rho^{T_B} \geq 0 \tag{3.40}$$

where

$$\rho^{T_B} = \sum_{ijkl} \rho_{ijkl} |i\rangle \langle j|^A \otimes |l\rangle \langle k|^B \tag{3.41}$$

is the partial transposition of ρ with respect to the system B. This condition applied to the Choi state reads:

$$\Omega_{\Lambda_t}^{T_B} = \frac{1}{4} \sum_{i,j=0}^{1} \Lambda_t(|i\rangle \langle j|) \otimes |j\rangle \langle i| =$$

$$\frac{1}{4} \begin{pmatrix} 1+\lambda_z(t)+l_z(t) & 0 & 0 & 0 \\ 0 & 1-\lambda_z(t)-l_z(t) & 2\lambda(t) & 0 \\ 0 & 2\lambda(t) & 1-\lambda_z(t)+l_z(t) & 0 \\ 0 & 0 & 0 & 1+\lambda_z(t)-l_z(t) \end{pmatrix} \geq 0 \tag{3.42}$$

which leads to

$$-\lambda_z(t) + |l_z(t)| \leq 1 \tag{3.43}$$

$$4\lambda^2(t) + l_z^2(2) \leq (1-\lambda_z(t))^2 \tag{3.44}$$

The first condition is automatically true if the first condition for complete positivity is true. Violating the second we guarantee that the map is not entanglement breaking at that time. Together, the violation of condition 3.44 and the condition 3.39, read (see Fig. 3.2)

$$(1-\lambda_z(t))^2 < 4\lambda^2(t) + l_z^2(t) \leq (1+\lambda_z(t))^2 \tag{3.45}$$

Then by saturating the CP condition, we obtain the dynamic that best preserves quantum correlations among this family.

We now give an explicit solution for the simple, but didactical case of $a(t) = a$ and $x(t) = x$ constants.

The evolution factors became

$$\lambda(t) = e^{-at-\int_0^t f(t')dt'} \tag{3.46}$$

$$\lambda_z(t) = e^{-2at} \tag{3.47}$$

$$l_z(t) = e^{-2at} \int_0^t e^{2at'} x dt' = \frac{x}{a}(1-e^{-2at}) \tag{3.48}$$

The saturated CP condition reads

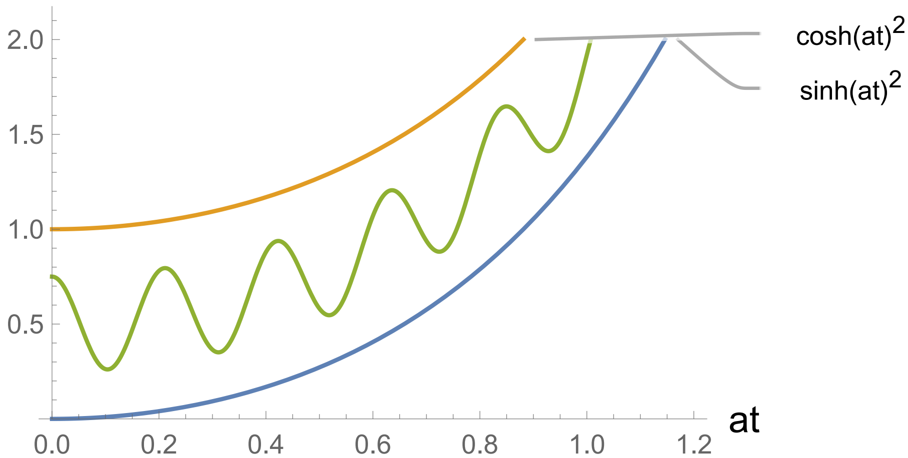

Figure 3.2: In green a generic $f(t)$ fulfilling the condition $(1-\lambda_z(t))^2 < 4\lambda(t)^2 + l_z(t)^2 \leq (1+\lambda_z(t))^2$, which converge to one function as $t \to \infty$. In blue and orange lower and upper bounds of the condition.

$$4e^{-2at}e^{-2\int_0^t f(t')dt'} + \frac{x^2}{a^2}(1-e^{-2at})^2 = (1+e^{-2at})^2 \tag{3.49}$$

$$e^{-2\int_0^t f(t')dt'} = \cosh^2 at - \frac{x^2}{a^2}\sinh^2 at = h(t) \tag{3.50}$$

$$f(t) = -\frac{\dot{h(t)}}{2h(t)} = -\frac{1}{2}a\left(1-\frac{x^2}{a^2}\right)\frac{\sinh 2at}{\cosh^2 at - \frac{x^2}{a^2}\sinh^2 at} \tag{3.51}$$

This optimal eigenvalue of the decoherence matrix is always negative, then it gives rise to an **eternally non-Markovian dynamics**. Notice that $\gamma(t)$ once again never crosses 0, meaning the evolution never stops. Particular cases of this are the unital one, for $x = 0$, which is known in the literature as Hall dynamic [HCLA14], and $x = a$ which is the amplitude damping, i.e. the entire Bloch sphere being mapped on one pole.

In the next section, we will proceed to analyze the behavior of different quantum resources over time, under the action of this simple example.

3.4 Resources behaviour

With the optimal choice of $f(t)$ 3.51 the factor $\lambda(t)$ becomes:

$$\lambda(t) = \frac{1}{2}\sqrt{(1+e^{-2at})^2 - \frac{x^2}{a^2}(1-e^{-2at})^2} \tag{3.52}$$

This is connected with the l_1-norm of coherence since for a qubit this is given by

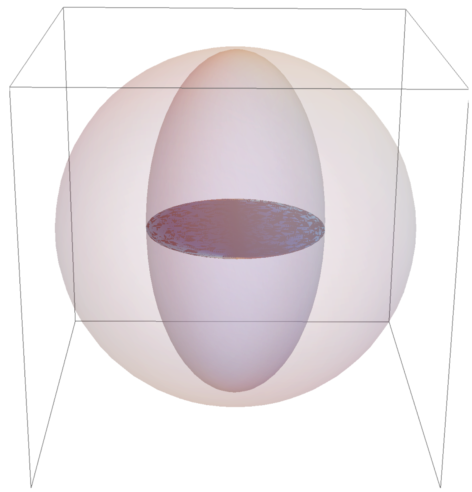

Figure 3.3: Bloch sphere evolution for $x = 0$, the final image is a disk inside the ellipsoid described by the complete positivity condition $4\lambda(t)^2 + l_z(t)^2 \leq (1 + \lambda_z(t))^2$.

$$C_{l_1}(\rho) = \sqrt{r_1^2 + r_2^2} = \lambda(t)\sqrt{r_1^2(0) + r_2^2(0)} \tag{3.53}$$

i.e. the distance to the z-axis. The evolution of this quantifier is then given by

$$C_{l_1}(t) = C_{l_1}(0)\frac{1}{2}\sqrt{(1 + e^{-2at})^2 - \frac{x^2}{a^2}(1 - e^{-2at})^2} \tag{3.54}$$

In the limit $t \to \infty$ the Bloch sphere approaches a flat disk of radius $\frac{1}{2}\sqrt{1 - \frac{x^2}{a^2}}$ with center in $\frac{x}{a}$ around the z-axis (see Fig. 3.3). This means this family of non-Markovian evolutions preserves coherence of a qubit even at infinite times, the maximum of coherence being preserved for the unital version of it, i.e. $x = 0$.

This preserved coherence can be exploited in a quantum metrological task as follows.

Let us suppose we want to estimate the frequency of precession of a spin $1/2$ around the z-axis, evolving according to our non-Markovian dynamic and the unitary which commutes with it, i.e. $U = e^{-i\sigma_z \frac{\omega}{2} t}$. Quantum metrology tells us that the ultimate bound on the precision of this estimation is given by the Cramer-Rao Bound

$$\Delta^2\omega \geq \frac{1}{F(U\Lambda_t(\rho)U^\dagger)} \tag{3.55}$$

where $F(\rho)$ is the quantum Fisher information of the state in which the parameter we want to estimate is encoded. There is a closed formula for it for qubits [ZSM+13]

$$F(\rho) = |\dot{\vec{r}}|^2 + \frac{(\vec{r}\cdot\dot{\vec{r}})^2}{1-r^2} \tag{3.56}$$

with $\dot{\vec{r}} = \partial\vec{r}/\partial\omega$. The scalar product $\vec{r}\cdot\dot{\vec{r}} = 0$ is vanishing for a phase covariant dynamics such as the one in consideration and $\dot{\vec{r}} = \omega C_{l_1}(t)(\cos\omega t, -\sin\omega t, 0)$, implying that $|\dot{\vec{r}}|^2 = \omega^2 C_{l_1}^2$. Then the quantum Fisher information is proportional to the l_1 norm of coherence and preserving the latter implies preserving the first. This means that the noise we are considering is the best to perform a phase estimation of a unitary along the z axis.

Let us now look at the negativity [VW02] of the Choi state to study the behavior of the entanglement between the qubit noisy system and the ancilla

$$E(\Omega_{\Lambda_t}) = \frac{||\Omega_{\Lambda_t}^{T_B}||_1 - 1}{2} = \frac{1}{2}e^{-2at} \tag{3.57}$$

The entanglement of this state is finite for every finite time and dies for $t\to\infty$.

The mutual information $I(\Omega_{\Lambda_t}) = S(Tr_B[\Omega_{\Lambda_t}]) + S(Tr_A[\Omega_{\Lambda_t}]) - S(\Omega_{\Lambda_t})$, instead is non vanishing at infinity, being

$$\lim_{t\to\infty} I(\Omega_{\Lambda_t}) = \frac{h(p)}{2} \tag{3.58}$$

with $h(p) = -p\log_2 p - (1-p)\log_2(1-p)$ the binary entropy of $p = \frac{1+\frac{x}{a}}{2}$.

Part of these correlations are also quantum, in the form of quantum discord [ACB16], which is defined as the difference between mutual information and purely classical information.

$$Q(\rho^{AB}) = I(\rho^{AB}) - C(\rho^{AB}) \tag{3.59}$$

So we need to quantify $C(\rho^{AB})$ in order to determine $Q(\rho^{AB})$, based on [ARA10].

Sending the reader to the appendix for further details, we obtain that the quantum discord at infinite time is given by

$$\lim_{t\to\infty} Q(\Omega_{\Lambda_t}) = \frac{h\left(\frac{1+\frac{x}{a}}{2}\right)}{2} + h\left(\frac{1+\frac{\sqrt{1-\frac{x^2}{a^2}}}{2}}{2}\right) - 1 \tag{3.60}$$

In fig. 3.4 one can see all the different correlations of the Choi state at infinity.

In the next section, we will describe a possible experimental implementation of the evolution so far studied, showing that is possible to practically realize it in a laboratory.

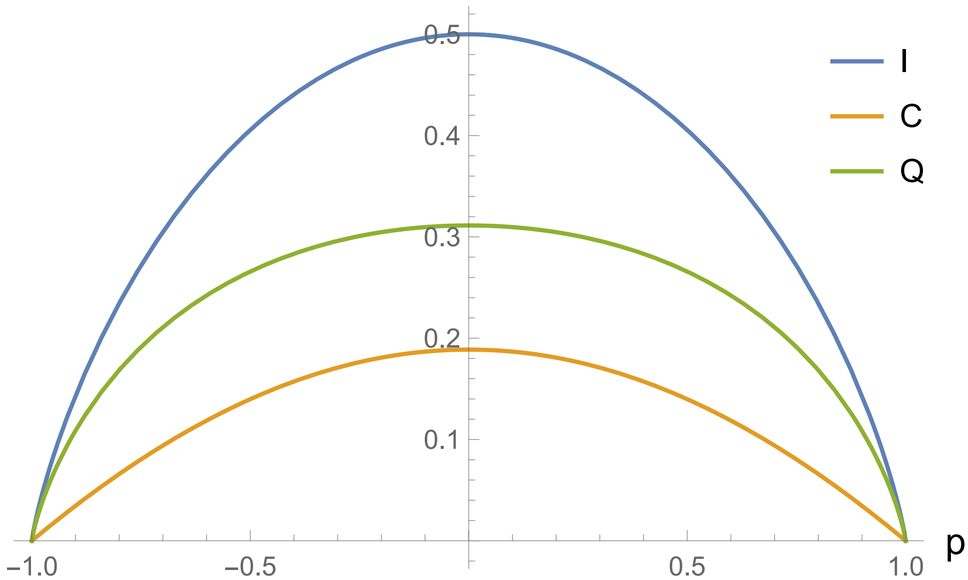

Figure 3.4: Plots of the different correlations included in the Choi state at infinite time as a function of $p = \frac{1+\frac{x}{a}}{2}$. In blue, mutual information, in orange classical correlations and in green quantum correlations. Notice that $I = C + Q$.

3.5 Experimental implementation of the non-Markovian dynamic

The unital version of the eternally non-Markovian dynamic, obtained for $x = 0$, was implemented experimentally using a photon whose vertical and horizontal polarization states $\{|V\rangle, |H\rangle\}$ plays the role of the computational basis $\{|0\rangle, |1\rangle\}$ and the environment is simulated through the many degrees of freedom of its frequency distribution:

$$|\Phi_E\rangle = \int g(\omega) |\omega\rangle \, d\omega \tag{3.61}$$

the distribution $g(\omega)$ is chosen to be a Gaussian with standard deviation δ and centered in ω_0

$$|g(\omega)|^2 = \frac{1}{\sqrt{2\pi}\delta} e^{-\frac{(\omega-\omega_0)^2}{2\delta^2}} \tag{3.62}$$

the two "systems" interact via a quartz crystal which couples them while the photon travels inside, through the effective unitary

$$U(t) = \int d\omega \, |\omega\rangle \langle\omega| \otimes \left(e^{-in_H\omega t} |H\rangle \langle H| + e^{-in_V\omega t} |V\rangle \langle V|\right) \tag{3.63}$$

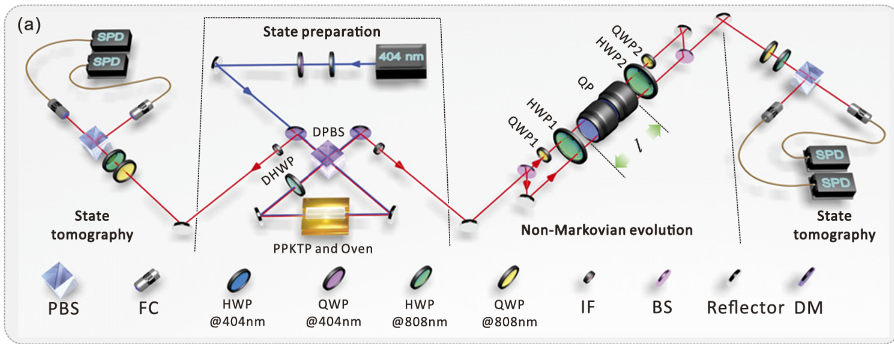

Figure 3.5: Scheme of the experiment proposed in this section: an entangled photon pair is generated in the state preparation section. One of them is sent to a state tomography apparatus to the left, and the other is sent to a simulation of our theoretical eternally non-Markovian dynamics, performed, among the others through a quartz crystal which couples the polarizations degrees of freedom of the photon with its frequency spectrum. Finally, after the evolution, the second photon is tomographied too.

where n_H and n_V are the refraction indexes of light inside the crystal corresponding to the two different polarizations.

This Stinespring-like interaction leads to an effective decoherence factor $|k(t)|$ with

$$k(t) = \int d\omega |g(\omega)|^2 e^{-i\Delta n \omega t} = e^{-\frac{\Delta n^2 \delta^2 t^2}{2} - i\Delta n \omega_0 t} \tag{3.64}$$

with $\Delta n = n_H - n_V$. One can always convert time t into distance l traveled through the crystal via the relation $t = \frac{l}{c}$, with c speed of light inside the crystal.

In Fig. 3.5 we can observe a detailed scheme of the experimental setup: first an entangled photon pair is prepared, then one of the photons is processed through optical plates and the quartz crystals in a way that leads to the density matrix:

$$\Lambda_t(\rho) = \frac{1}{2}(\mathbb{I} + |k(t)|z_0\sigma_z) + \frac{1}{2}(1 + |k(t)|)(x_0\sigma_x + y_0\sigma_y) \tag{3.65}$$

Finally, the decoherence factor is measured by evaluating the spectrum of the process matrix $F_{ij} = Tr[\sigma_i \Lambda_t(\sigma_j)]$, which is

$$|\lambda_i| = \left(1, \frac{1}{2}(1 + |k(t)|), \frac{1}{2}(1 + |k(t)|), |k(t)|\right) \tag{3.66}$$

The first component of this vector corresponds to the center of the Bloch sphere and the last three components to the ones of the Bloch vector of a density matrix undergoing this evolution. As expected by our theoretical results the x

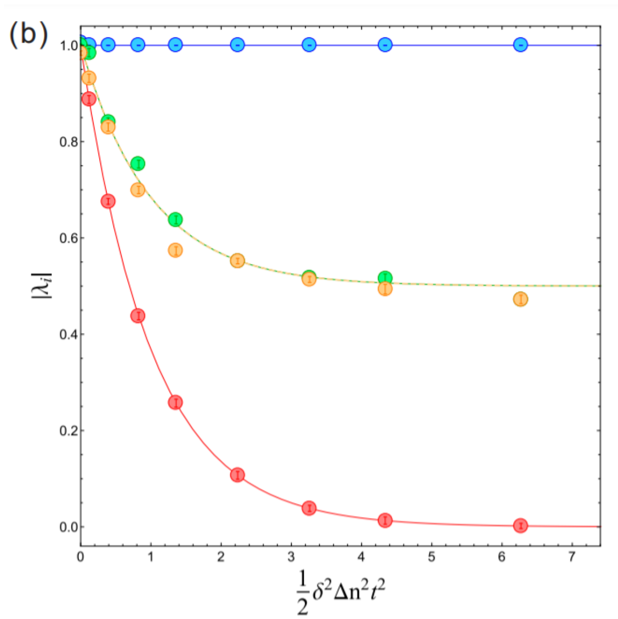

Figure 3.6: The dots represent the experimental values of the components of the vector 3.65 or the Bloch sphere components versus time. x and y component in green and yellow, z component in red and in blue the center of the Bloch sphere. In continuous lines are plotted the theoretical values of $\lambda(t)$, $\lambda_z(t)$ and $1 - l_z(t)$, for $x = 0$. The match is inside the error bars.

and y components of the density matrix converge to 1/2, the z component to 0, and the center stays constant. See in Fig. 3.6 the plot of these factors over time or optical path of the photon inside the crystal. Thus we can conclude that the setup described, successfully simulates the unital version of the eternally non-Markovian dynamics.

3.6 Conclusions

In conclusion, in this chapter, we showed that Markovian dynamics inexorably depletes any form of correlation between a noisy qubit and its ancillary system. We then proceed to study a particular but relevant class of dynamics called covariant maps, important in scenarios where one basis is fixed, like the resource theory of coherence. Allowing these sorts of maps to have a negative decoherence matrix we showed that the best possible noise one can choose to preserve those correlations is an eternally non-Markovian one, meaning the decoherence matrix keeps at least one negative eigenvalue forever. We also provided an example in which part of the preserved correlations are quantum discord and coherence is kept different from 0 at infinite time too, having an important application in the field of quantum metrology. Finally, an experimental implementation of this example was described in the last section, showing a practical realization of the eternally non-Markovian dynamics on an optical table.

Chapter 4

Advantage of dilution

The original results presented in this chapter are extracted from the paper [MSFAS24].

As seen in previous chapters distillation is a procedure to extract maximally resourceful states from many copies of less resourceful ones. The inverse procedure, in which imperfect states are created starting from an ensemble of golden ones, is called dilution. Distillation of maximally resourceful states is often useful since they are the states that most tasks need. However, it is unclear how dilution can be useful. In the following, we give an example in which dilution provides an advantage in terms of protecting some resource from some noise.

Let us try to formulate a general framework before going to the particular examples.

We start with n copies of a maximally resourceful state, which we know will be subject to noise, i.e. a resource-degrading map Λ. For our purposes the definition of resource-degrading map will be: for any resource measure R and any input state of the map Ψ it holds that

$$R(\Lambda(\Psi)) < R(\Psi) \tag{4.1}$$

with a strict inequality. We are allowed though to pre-process and post-process these copies before and after the noise with free maps Φ_1 and Φ_2 and then try to distill back as many golden states as we can. In formulas

$$F(\Lambda, n, k) = \max_{\Phi_1, \Phi_2} F(\Phi_2(\Lambda^{\otimes m}(\Phi_1(|\Phi^+\rangle\langle\Phi^+|^{\otimes n}))), |\Phi^+\rangle\langle\Phi^+|^{\otimes k}) \tag{4.2}$$

The biggest rate $r = \frac{k}{n}$ such that this fidelity goes to 1 in the limit of large n will be the resource protection rate, i.e

$$R_p(\Lambda) = \sup\{r : \lim_{n\to\infty} F(\Lambda, n, \lfloor rn \rfloor) = 1\} \tag{4.3}$$

We will propose a procedure in which the pre-processing Φ_1 is given by the dilution protocol (see Fig. 4.1) into the state Ψ, which lower bounds $R_p(\Lambda)$.

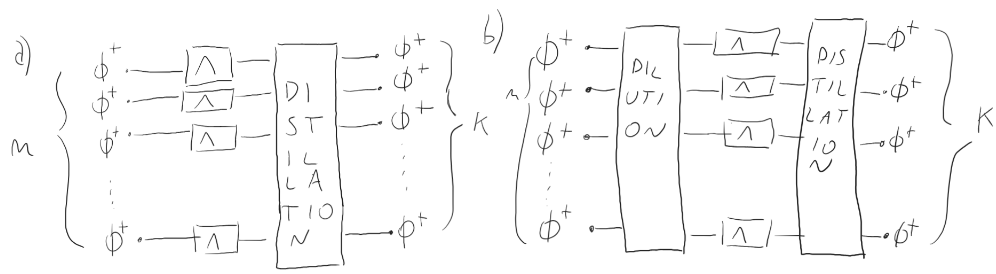

Figure 4.1: Two different scenarios of a collection of maximally resourceful states Φ^+ subject to local noise Λ. In picture a) no pre-processing is applied to the ensemble and only distillation as post-processing. In Figure b) dilution is executed as a form of pre-processing and still distillation as post-processing. We will show that the scenario b) is advantageous in preserving more Φ^+ states per unit of initial ones in the limit $n \to \infty$.

For reversible resource theories, the resource protection rate of this protocol is given by

$$R(\Phi^+ \to \Psi) R(\Lambda(\Psi) \to \Phi^+) = \frac{R_D(\Lambda(\Psi))}{R_C(\Psi)} = \frac{R^\infty_{rel}(\Lambda(\Psi))}{R^\infty_{rel}(\Psi)} \tag{4.4}$$

We remember in fact, that these rates are given by the regularized relative entropy of the resource.

$$R(\rho \to \sigma) = \frac{R^\infty_{rel}(\rho)}{R^\infty_{rel}(\sigma)} \tag{4.5}$$

with $R^\infty_{rel}(\rho) = \lim_{n\to\infty} \frac{R_{rel}(\rho^{\otimes n})}{n}$ and $R_{rel}(\rho) = \inf_{\sigma \in F} S(\rho||\sigma)$. Finally $R^\infty_{rel}(\Phi^+) = 1$. Even in some cases where the theory is not reversible, it is possible to apply a similar formula for the protection rate, while the measure will not be given anymore by a relative entropy. We will see an example of this situation in entanglement theory (section 5.2), where the noise will bring our initial pure state into mixed.

We will say there is an advantage of using dilution in protecting resource from noise if for any other target state of the dilution Ψ it holds

$$\frac{R^\infty_{rel}(\Lambda(\Psi))}{R^\infty_{rel}(\Psi)} > \frac{R^\infty_{rel}(\Lambda(\Phi^+))}{R^\infty_{rel}(\Phi^+)} \tag{4.6}$$

We will now provide a series of examples in which this condition is verified.

4.1 Resource theory of thermodynamics and purity

As mentioned in Chapter 1, in resource theory of thermodynamics the system is defined by a density matrix ρ and a Hamiltonian H_S. The free state is the one in thermal equilibrium with a bath at temperature T, i.e. the Gibbs state $\gamma_S = \frac{e^{-\beta H_S}}{Tr[e^{-\beta H_S}]}$. Free operations are thermal and are implemented by letting the system interact with the bath through an energy-preserving unitary, i.e.

$$\Lambda(\rho) = Tr_B(U(\rho \otimes \gamma_B)U^\dagger) \tag{4.7}$$

with $[U, H_S + H_B] = 0$. Being the set of free state of dimension 0, the regularized relative entropy is given by $R^\infty_{rel}(\rho) = S(\rho||\gamma) = Tr(\rho \log_2 \rho) - Tr(\rho \log_2 \gamma)$. This quantity also gives the asymptotic rate of conversion

$$R(\rho \to \sigma) = \frac{S(\rho||\gamma)}{S(\sigma||\gamma)} \tag{4.8}$$

For simplicity, we will now focus on the qubit case. Let us consider a qubit Hamiltonian $H = E_0 |E_0\rangle\langle E_0| + E_1 |E_1\rangle\langle E_1|$. We are in possession of n qubits initialized in the excited eigenstate of the Hamiltonian $|E_1\rangle$. The thermal noise we choose is of the form

$$\Lambda(\rho) = p\gamma_S + (1-p)\Delta(\rho) \tag{4.9}$$

with $\Delta(\rho) = \sum_{i=0}^{1} \langle E_i| \rho |E_i\rangle |E_i\rangle\langle E_i|$ completely dephasing map in the energy eigenbasis and $0 \leq p \leq 1$.

The resource protection rate is given by $\frac{S(\Lambda(\rho)||\gamma)}{S(\rho||\gamma)}$ and dilution into ρ provides an advantage with respect to do nothing before the noise whenever

$$\frac{S(\Lambda(\rho)||\gamma)}{S(\rho||\gamma)} > \frac{S(\Lambda(|E_1\rangle\langle E_1|)||\gamma)}{S(|E_1\rangle\langle E_1|\,||\gamma)} \tag{4.10}$$

Since $\Delta(\rho)$ destroys any coherence we might as well dilute into a diagonal state in the energy eigenbasis

$$\rho = (1-q)|E_0\rangle\langle E_0| + q|E_1\rangle\langle E_1| \tag{4.11}$$

so that $\Delta(\rho) = \rho$. In Fig. 4.2 it is shown, for the case of noise parameter $p = 0.9$ and temperature of the bath $T = 0.3$, that the diluted state which maximizes the resource protection rate corresponds to $q \approx 0.82$. Notice that at this value of q the rate is higher than at $q = 1$, the value corresponding to the no dilution scenario, meaning there is an advantage for a certain range of q since the function is continuous. In further examples, it will occur that the optimal state to dilute in order to maximize the resource protection rate should be as close as possible to the free one. This is not true for the resource theory or thermodynamic, since the Gibbs state in this scenario corresponds to a parameter $q \approx 0.03$.

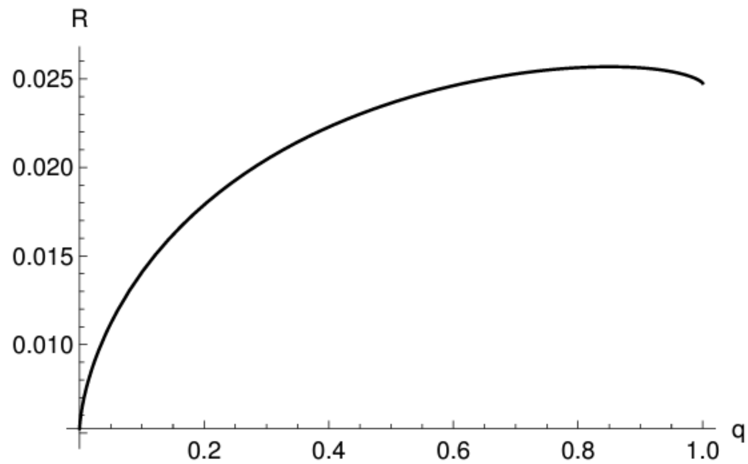

Figure 4.2: Resource protection rate as a function of the dilution parameter q of the target state $\rho = (1-q)\,|E_0\rangle\,\langle E_0| + q\,|E_1\rangle\,\langle E_1|$. The noise parameter is set to $p = 0.9$ and the temperature of the bath is $T = 0.3$. The best performance is achieved for $q \approx 0.82$, where the rate is higher than the no dilution case for $q = 1$, confirming an advantage in performing it. Moreover notice that the parameter for the Gibbs state is $q \approx 0.03$, meaning the optimal state is quite far from it.

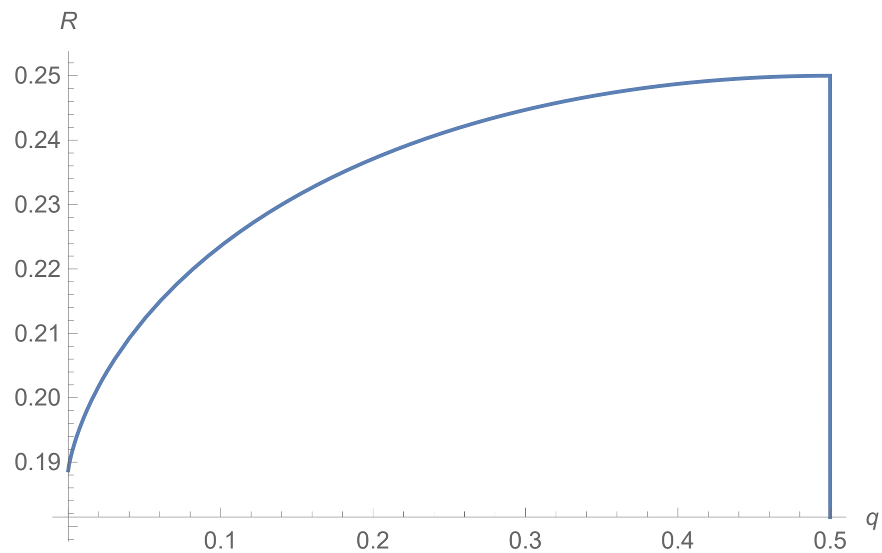

Figure 4.3: Purity protection rate as a function of the parameter q of the diluted state in Eq. 4.14. The noise parameter is set to $p = 0.5$. It is visible that the maximum is attained for $q \to 1/2$, i.e. for diluting the least pure state possible.

A particular but relevant case of the resource theory of quantum thermodynamics is the one of purity, obtained when the Hamiltonians are degenerate, i.e. proportional to the identity. The free Gibbs state becomes then the maximally mixed one $\frac{\mathbb{I}_d}{d}$, the free operations are unital ones and the resource protection rate is

$$\frac{S(\Lambda(\rho)||\mathbb{I}_d/d)}{S(\rho||\mathbb{I}_d/d)} \tag{4.12}$$

with $S(\rho||\mathbb{I}_d/d) = -S(\rho) + \log_2 d$.

Let us first, once again for the sake of simplicity, consider the qubit case ($d = 2$). The noise in consideration will be depolarizing

$$\Lambda(\rho) = p\frac{\mathbb{I}}{2} + (1-p)\rho \tag{4.13}$$

the states we dilute into will be a mixture of the pure states of the computational basis

$$\rho = (1-q)\left|0\right\rangle\left\langle 0\right| + q\left|1\right\rangle\left\langle 1\right| \tag{4.14}$$

We see from the plot 4.3, where we set the noise parameter to be $p = 0.5$, that diluting always brings an advantage in protecting purity from depolarizing noise, since the no dilution scenario is obtained for $q = 0$. Moreover, the maximum protection rate is attained in the limit $q \to 1/2$, i.e. a maximally mixed state, meaning the more we dilute purity the better. We give the analytical formula of the protection rate 4.12 for completeness

$$\frac{1+\frac{1}{2}\left[(\frac{3}{2}-q)\log_2\frac{1}{2}(\frac{3}{2}-q)+(\frac{1}{2}+q)\log_2\frac{1}{2}(\frac{1}{2}+q)\right]}{1+q\log_2 q+(1-q)\log_2(1-q)} \tag{4.15}$$

We can also prove this result for general dimension d, i.e.

$$\sup_\rho \frac{S(\Lambda(\rho)||\mathbb{I}/d)}{S(\rho||\mathbb{I}/d)} = \lim_{\rho\to\mathbb{I}/d}\frac{S(\Lambda(\rho)||\mathbb{I}/d)}{S(\rho||\mathbb{I}/d)} \tag{4.16}$$

Let us think of the noise as a semigroup

$$\rho_t = \Lambda_t(\rho_0) = p_t\rho_0 + (1-p_t)\frac{\mathbb{I}}{d} \tag{4.17}$$

with, for example, $p_t = e^{-\gamma t}$. Then $S(\Lambda_t(\rho)||\mathbb{I}/d) = \log_2 d - S(\Lambda_t(\rho))$ is a monotonically decreasing and convex function of t. For $t\to\infty$ it goes to the maximally mixed state, i.e. $\frac{\mathbb{I}}{d}$. We give the expression when $\rho_0 = |0\rangle\langle 0|$, so it can be verified

$$\rho_t = \begin{pmatrix} \frac{1+p_t(d-1)}{d} & 0 & ... \\ 0 & \frac{1-p_t}{d} & ... \\ ... & ... & \frac{1-p_t}{d} \end{pmatrix} \tag{4.18}$$

$$\begin{aligned} f(t) = S(\rho_t||\mathbb{I}/d) = \log_2 d + \sum_{i=1}^{d}\lambda_i(t)\log_2\lambda_i(t) = \log_2 d \\ +\frac{1}{d}\left((p_t(d-1)+1)\log_2\frac{p_t(d-1)+1}{d} + (1-p_t)(d-1)\log_2\frac{1-p_t}{d}\right) \end{aligned} \tag{4.19}$$

where $\lambda_i(t)$ are the eigenvalues of ρ_t, i.e. $\frac{1+p_t(d-1)}{d}$ with multiplicity 1 and $\frac{1-p_t}{d}$ with multiplicity $d-1$.

Because of monotonicity it holds that $f(t) > f(t+\delta)$ for any $\delta > 0$ and because of convexity $\frac{d}{dt}f(t)|_t < \frac{d}{dt}f(t)|_{t+\delta}$. Then the function

$$\frac{f(t+\delta)}{f(t)} = \frac{S(\rho_{t+\delta}||\mathbb{I}/d)}{S(\rho_t||\mathbb{I}/d)} = \frac{S(\Lambda_\delta(\rho_t))||\mathbb{I}/d)}{S(\rho_t||\mathbb{I}/d)} \tag{4.20}$$

which is nothing that the purity protection rate, is monotonically increasing in the maximally mixed state direction $\forall\ \delta$ and it attains its maximum there. In the last passage, we used the semigroup property of the depolarizing noise $\rho_{t+\delta} = \Lambda_\delta(\rho_t)$. That is because its derivative is strictly positive

$$\frac{d}{dt}\frac{f(t+\delta)}{f(t)} = \frac{\frac{df}{dt}|_{t+\delta}f(t) - \frac{df}{dt}|_t f(t+\delta)}{f^2(t)} > 0 \tag{4.21}$$

4.1.1 Correlations do not help

Is it better to dilute into correlated pure states $|\Psi_k\rangle$ of dimension 2^k instead of a collection of k product states $|\Psi\rangle^{\otimes k}$ of dimension 2? To answer this question we write the purity protection rate

$$\frac{S(\Lambda^{\otimes k}(\Psi_k)||\mathbb{I}_{2^k}/2^k)}{S(\Psi_k||\mathbb{I}_{2^k}/2^k)} = 1 - \frac{S(\Lambda^{\otimes k}(\Psi_k))}{k} \tag{4.22}$$

where we used the fact that $S(\Psi_k||\mathbb{I}_{2^k}/2^k) = k$. We want to maximize this rate, meaning minimizing $S(\Lambda^{\otimes k}(\Psi_k))$. According to [Kin02] the minimal output entropy of two local unital channels $\Lambda_1 \otimes \Lambda_2$ is additive. This means that there exist two pure qubit states $|\Phi_1\rangle$ and $|\Phi_{k-1}\rangle$ such that

$$S(\Lambda^{\otimes k}(\Psi_k)) \geq S(\Lambda(\Phi_1)) + S(\Lambda^{\otimes k-1}(\Phi_{k-1})) = S(\Lambda^{\otimes k}(\Phi_1 \otimes \Phi_{k-1})) \tag{4.23}$$

We iterate the procedure until

$$S(\Lambda^{\otimes k}(\Psi_k)) \geq kS(\Lambda(\Phi)) \tag{4.24}$$

so the maximal purity protection rate under local unital noise is achieved for product states and no correlations in the diluted state are needed to improve the protocol.

4.1.2 Finite copies

In this section and the next, we present some original results which are not yet published.

So far all the results we obtained work in the asymptotic limit. One can ask if there is some finite number of diluted copies N such that diluting still would provide an advantage in protecting resources from noise. For the advantage to appear in the asymptotic limit, such a finite number must exist. It turns out this number is not $N = 2$. Before starting with the proof we provide a useful theorem which we are going to use later.

Let's call $K_k(\rho)$ the function which returns the sum of the k largest eigenvalues of the density matrix ρ, i.e.

$$K_k(\rho) = \max_{\tau} Tr[\rho\tau] \tag{4.25}$$

where τ are all the possible rank k projectors, with the properties $Tr[\tau] = k, \tau^2 = \mathbb{I}, \tau \geq 0$.

It can be proven that this is a convex function,i.e.

$$K_k(A + B) \leq K_k(A) + K_k(B) \tag{4.26}$$

To do this let us assume $\rho = A + B$, τ' is the optimal projector for $A + B$, and σ and γ are respectively the ones for A and B. Then it follows:

$$K_k(A+B) = Tr[(A+B)\tau'] = Tr[A\tau']+Tr[B\tau'] \leq Tr[A\sigma]+Tr[B\gamma] = K_k(A)+K_k(B) \tag{4.27}$$

this proves convexity of $K_k(\rho)$.

The following theorem also holds.

Theorem 4.1. *Let ρ^{AB} be a bipartite state living in the Hibert space $H_{d_A} \otimes H_{d_B}$. Then $\forall k_1, k_2 \in \mathbb{N}$, with $k_1 \leq d_1$ and $k_2 \leq d_2$*

$$\rho^A_{k_1} + \rho^B_{k_2} - Tr[\rho^{AB}] \leq \rho^{AB}_{k_1 k_2} \tag{4.28}$$

with ρ_k sum of the k largest eigenvalues of the matrix ρ.

Proof. We will write $\rho^A_{k_1} = Tr[\rho^A \tau]$ and $\rho^B_{k_2} = Tr[\rho^B \sigma]$, with $Tr[\tau] = k_1, \tau^2 = I, \tau \geq 0$ and $Tr[\sigma] = k_2, \sigma^2 = I, \sigma \geq 0$. Moreover $\mathbb{I} - \tau = P_\tau$ and $\mathbb{I} - \sigma = P_\sigma$

$$\begin{aligned} \rho^A_{k_1} + \rho^B_{k_2} - Tr[\rho^{AB}] &= Tr[\tau \otimes \mathbb{I}\rho^{AB}] + Tr[\mathbb{I} \otimes \sigma\rho^{AB}] - Tr[\mathbb{I} \otimes \mathbb{I}\rho^{AB}] = \\ = Tr[\mathbb{I}\otimes\sigma\rho^{AB}] - Tr[P_\tau\otimes\mathbb{I}\rho^{AB}] &= Tr[\tau\otimes\sigma\rho^{AB}] + Tr[P_\tau\otimes\sigma\rho^{AB}] - Tr[P_\tau\otimes\mathbb{I}\rho^{AB}] = \\ &= Tr[\tau \otimes \sigma\rho^{AB}] - Tr[P_\tau \otimes P_\sigma \rho^{AB}] \leq Tr[\tau \otimes \sigma\rho^{AB}] \leq \rho^{AB}_{k_1 k_2} \end{aligned} \tag{4.29}$$

□

We will also need to slightly modify the general protocol of purity protection to adapt it to the finite copies regime.

Our protocol starts with n pure states $|\Psi^{\otimes n}\rangle$ (in Hilbert space S), to which we attach $m - n$ ancillas $\frac{\mathbb{I}}{d}^{\otimes(m-n)}$ (in Hilbert space E). We then (pre-)process this with a unital operation Λ_1, apply the noise Λ_N, which is also unital but resource degrading, and post-processing with a third unital Λ_2, such that the following quantity, i.e. the fidelity between the post-processed state and the original pure state, is optimized

$$P(\psi, n) = \max_{\Lambda_1, \Lambda_2} \langle\Psi^{\otimes n}| \operatorname{Tr}_E \left[\Lambda_2 \circ \Lambda_N \circ \Lambda_1 \left(|\Psi\rangle\langle\Psi|^{\otimes n} \otimes \left(\frac{\mathbb{I}}{d}\right)^{\otimes(m-n)}\right)\right] |\Psi^{\otimes n}\rangle \tag{4.30}$$

Let $\Lambda_N \circ \Lambda_1 \left(|\Psi\rangle\langle\Psi|^{\otimes n} \otimes \left(\frac{\mathbb{I}}{d}\right)^{\otimes(m-n)}\right) = \rho^{SE}$ and $d^{m-n} = d_E$. We now show that

$$\max_{\Lambda_2} \langle\Psi^{\otimes n}| Tr_E \Lambda_2(\rho^{SE}) |\Psi^{\otimes n}\rangle = \rho^{SE}_{d_E} \tag{4.31}$$

where ρ_k denotes the sum of the k-largest eigenvalues of the density matrix ρ.

Firstly, note that

$$\max_{\Lambda_2} \langle\Psi^{\otimes n}| Tr_E \Lambda_2(\rho^{SE}) |\Psi^{\otimes n}\rangle \leq (Tr_E \Lambda_2(\rho^{SE}))_1. \tag{4.32}$$

From 4.1, by setting $k_1 = 1$ and $k_2 = d_E$ it follows that:

$$(Tr_E \Lambda_2(\rho^{SE}))_1 \leq (\Lambda_2(\rho^{SE}))_{d_E} \leq \rho^{SE}_{d_E} \tag{4.33}$$

The second inequality follows from majorization. Putting together 4.32 and 4.33,

$$\max_{\Lambda_2} \langle\Psi^{\otimes n}| Tr_E \Lambda_2(\rho^{SE}) |\Psi^{\otimes n}\rangle \leq (Tr_E \Lambda_2(\rho^{SE}))_1 \leq (\Lambda_2(\rho^{SE}))_{d_E} \leq \rho^{SE}_{d_E} \tag{4.34}$$

All the above inequalities can be saturated by choosing post-processing unital (Λ_2) to be a unitary, such that the eigenvectors corresponding to first "d_E largest" eigenvalues (of $\Lambda_2(\rho^{SE})$) are as follows:

$$|\Psi^{\otimes n}\rangle \otimes |\varphi_i\rangle\,,\; i : 1 \to d_E. \tag{4.35}$$

Here, $|\varphi_i\rangle$ are arbitrary orthogonal basis of the Hilbert space E. Therefore, the optimisation in 4.30, reduces to the following optimisation

$$P(\psi,n) = \max_{\Lambda_1}\left(\Lambda_N \circ \Lambda_1\left(|\Psi\rangle\langle\Psi|^{\otimes n} \otimes \left(\frac{\mathbb{I}}{d}\right)^{\otimes(m-n)}\right)\right)_{d_E} \tag{4.36}$$

Moreover, using the fact that we can express the unital map Λ_1 as a convex combination of unitaries $\Lambda_1(\rho) = \sum_i p_i U_i \rho U_i^\dagger$, because they have the same conversion power:

$$P(\psi,n) = \max_{U_i,p_i}\left(\sum_i p_i\Lambda_N\left(U_i\left(|\Psi\rangle\langle\Psi|^{\otimes n} \otimes \left(\frac{\mathbb{I}}{d}\right)^{\otimes(m-n)}\right)U_i^\dagger\right)\right)_{d_E} \tag{4.37}$$

$$\leq \max_{U_1}\left(\Lambda_N\left(U_1\left(|\Psi\rangle\langle\Psi|^{\otimes n} \otimes \left(\frac{\mathbb{I}}{d}\right)^{\otimes(m-n)}\right)U_1^\dagger\right)\right)_{d_E} \tag{4.38}$$

where we used convexity of the function $K_{d_E}(\rho)$ and majorized everything with U_1, the unitary which makes the initial state the most resilient to the noise, in the sense that the eigenvalues will be preserved. Then the whole maximization can be done considering only unitaries.

$$P(\psi,n) = \max_{U_1}\left(\Lambda_N \circ U_1\left(|\Psi\rangle\langle\Psi|^{\otimes n} \otimes \left(\frac{\mathbb{I}}{d}\right)^{\otimes(m-n)}\right)\right)_{d_E} \tag{4.39}$$

Now we will use these results to check if diluting into two qudits, i.e. $m = 2$ and $n = 1$, with the new setting designed for finite copies, provides any advantage.

4.1.3 Dilution into 2 qudits

Let's consider a depolarizing noise:

$$\Lambda_N(\rho) = p\rho + (1-p)\frac{\mathbb{I}_d}{d} \tag{4.40}$$

we will need to play a bit with its definition to obtain a more useful form for our purposes:

$$\Lambda(\rho) = p\sum_{i,j}\langle i|\rho|j\rangle|i\rangle\langle j| + \frac{1-p}{d}Tr[\rho]\sum_i|i\rangle\langle i| =$$

$$= p\sum_{i\neq j}\langle i|\rho|j\rangle|i\rangle\langle j| + \sum_i\left(p\langle i|\rho|i\rangle + \frac{1-p}{d}\sum_j\langle j|\rho|j\rangle\right)|i\rangle\langle i| \tag{4.41}$$

So for the matrix elements $|i\rangle\langle j|$, if $i \neq j$,$\Lambda(|i\rangle\langle j|) = p|i\rangle\langle j|$, and if $i = j$, $\Lambda(|i\rangle\langle i|) = p|i\rangle\langle i| + \frac{1-p}{d}\mathbb{I}_d$.

which means

$$\Lambda(|i\rangle\langle j|) = p|i\rangle\langle j| + \frac{1-p}{d}\delta_{ij}\mathbb{I}_d \tag{4.42}$$

Now we can study its action on a general 2-qdit state:

$$\rho^{AB} = \sum_{i,j,k,l=0}^{d-1} \rho_{ijkl}|i\rangle\langle j| \otimes |k\rangle\langle l| \tag{4.43}$$

which is

$$\begin{aligned}\Lambda\otimes\Lambda(\rho^{AB}) = p^2\rho^{AB} + p(1-p)\left(\sum_{i,j,k}\rho_{ijkk}|i\rangle\langle j|\otimes\frac{\mathbb{I}}{d} + \frac{\mathbb{I}}{d}\otimes\sum_{i,j,k}\rho_{kkij}|i\rangle\langle j|\right) + \\ +(1-p)^2\frac{\mathbb{I}}{d^2} = p^2\rho^{AB} + p(1-p)\left(\rho^A\otimes\frac{\mathbb{I}}{d} + \frac{\mathbb{I}}{d}\otimes\rho^B\right) + (1-p)^2\frac{\mathbb{I}}{d^2}\end{aligned} \tag{4.44}$$

Now we are interested in the sum of the d largest eigenvalues, i.e. the function $K_d(\rho)$. Using convexity and theorem 1 ($||\rho^A||_\infty + ||\rho^B||_\infty \leq ||\rho^{AB}||_\infty + 1$):

$$\begin{aligned}K_d(\Lambda\otimes\Lambda(\rho^{AB})) \leq p^2 + p(1-p)(||\rho^A||_\infty + ||\rho^B||_\infty) + \frac{(1-p)^2}{d} \leq \\ p((1-p)||\rho^{AB}||_\infty + 1) + \frac{(1-p)^2}{d}\end{aligned} \tag{4.45}$$

If $\rho^{AB} = U(|\Psi\rangle\langle\Psi| \otimes \frac{\mathbb{I}_d}{d})U^\dagger$, with $|\Psi\rangle\langle\Psi| \otimes \frac{\mathbb{I}_d}{d} = \frac{1}{d}\sum_{i=0}^{\frac{d-1}{2}}|i\rangle\langle i|$, and U the unitary pre-processing, on the right-hand side we have that $||U(\rho^{AB})||_\infty \leq || \, |\Psi\rangle\langle\Psi| \otimes \frac{\mathbb{I}_d}{d}||_\infty = \frac{1}{d}$, because unitaries cannot increase the highest eigenvalue of a density matrix, leading to the bound $\frac{1+p(d-1)}{d}$. But this bound is saturated exactly when the pre-processing is an identity operation. In fact on the left-hand side:

$$\begin{aligned}&\Lambda(|\Psi\rangle\langle\Psi|)\otimes\Lambda\left(\frac{\mathbb{I}_d}{d}\right) = \\ &\left(p|\Psi\rangle\langle\Psi| + (1-p)\frac{\mathbb{I}_d}{d}\right)\otimes\frac{\mathbb{I}_d}{d} = \left(\frac{1+p(d-1)}{d}|\Psi\rangle\langle\Psi| + \frac{1-p}{d}|\Psi^\perp\rangle\langle\Psi^\perp|\right)\otimes\frac{\mathbb{I}_d}{d}\end{aligned} \tag{4.46}$$

which implies the sum of the d largest eigenvalues of the initial, not pre-processed, noisy state is also $\frac{1+p(d-1)}{d}$. This means that is not possible to improve the fidelity with the initial pure state by diluting in just two qudits.

4.2 Entanglement

In this setting, two spatially separated parties, Alice and Bob share n singlets $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ which they want to preserve from Bob's local noise in his lab. As before they first dilute them into $n/S(\Psi^A)$ copies of a less entangled state $|\Psi\rangle = \cos\alpha\,|00\rangle + \sin\alpha\,|11\rangle$ with

$$\Psi^A = Tr_B[|\Psi\rangle\langle\Psi|] = \begin{pmatrix} \cos^2\alpha & 0 \\ 0 & \sin^2\alpha \end{pmatrix} \tag{4.47}$$

then this state becomes noisy, i.e. $\sigma = \mathbb{I}\otimes\Lambda(|\Psi\rangle\langle\Psi|)$, and finally they can distill back $nE_D(\sigma)/S(\Psi^A)$ singlets from it. All of this in the limit of large n. The noise protection by dilution is effective if

$$\frac{E_D(\sigma)}{S(\Psi^A)} > E_D(\mathbb{I}\otimes\Lambda(|\Psi^-\rangle\langle\Psi^-|)) \tag{4.48}$$

We can choose $\Lambda(\rho) = K_0\rho K_0^\dagger + K_1\rho K_1^\dagger$ as qubit phase damping with Kraus operators

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix} \tag{4.49}$$

$$K_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} \tag{4.50}$$

with $0 < \lambda < 1$, or, an equivalent description of this noise is

$$\Lambda(\rho) = (1-p)\rho + p\sigma_z\rho\sigma_z \tag{4.51}$$

with $p = \frac{1}{2}(1-\sqrt{1-\lambda})$. With this choice of the noise, σ will be a maximally correlated state of the form $\sigma = \sum_{i,j=0}^{1}\sigma_{ij}\,|ii\rangle\langle jj|$. In particular

$$\sigma = \begin{pmatrix} \cos^2\alpha & 0 & 0 & \sqrt{1-\lambda}\cos\alpha\sin\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\lambda}\cos\alpha\sin\alpha & 0 & 0 & \sin^2\alpha \end{pmatrix} \tag{4.52}$$

with eigenvalues $(0, 0, \frac{1}{2} - \frac{\sqrt{2\lambda\cos(4\alpha)-2\lambda+4}}{4}, \frac{1}{2} + \frac{\sqrt{2\lambda\cos(4\alpha)-2\lambda+4}}{4})$.
For this kind of states $E_D(\sigma^{AB})$ has a closed expression [LDS18] which is

$$\begin{aligned} E_D(\sigma^{AB}) &= S(Tr_B[\sigma^{AB}]) - S(\sigma^{AB}) = \\ &= h(\cos^2\alpha) - h\left(\frac{1}{2} + \frac{\sqrt{2\lambda\cos(4\alpha) - 2\lambda + 4}}{4}\right) \end{aligned} \tag{4.53}$$

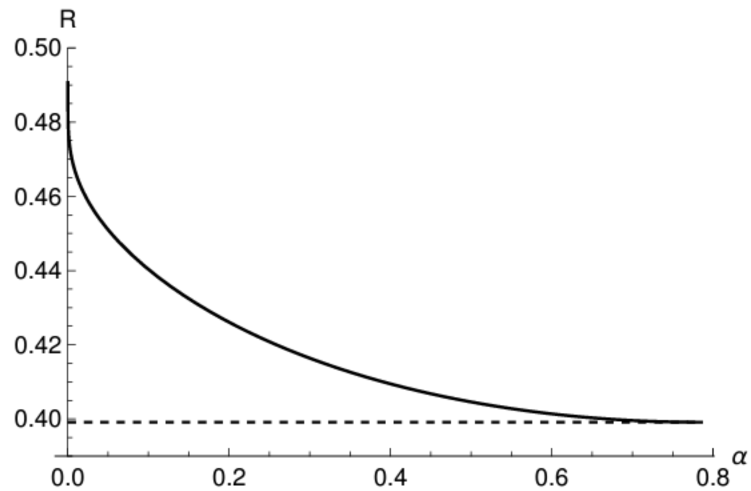

Figure 4.4: The solid line shows the entanglement protection rate $R = E_D(\sigma)/S(\Psi^A)$ as a function of the dilution parameter α of the target state $|\Psi\rangle = \cos\alpha\,|00\rangle + \sin\alpha\,|11\rangle$ and for the noise parameter $\lambda = 0.5$. The dotted line corresponds to the no-dilution scenario. Any value of the dilution parameter brings an advantage to the entanglement protection rate, with maximum protection achieved in the limit of a separable target state.

and $S(\Psi^A) = h(\cos^2\alpha)$. In fig. 4.4 is shown the plot of the entanglement protection rate $R = E_D(\sigma)/S(\Psi^A)$ as a function of the dilution parameter α and for the choice of the noise parameter $\lambda = 0.5$. Any choice of α is advantageous in preserving entanglement with respect to the no pre-processing scenario. The less entanglement the diluted state contains, the more protection rate is achieved. In the next section, we compare our protocol of pre-processing through dilution with a more famous one, the 3-qubit error correction code.

4.2.1 Comparison with error correction

We will review the standard 3-qubit error correction protocol [Rof19]. This is designed to be effective against a bit-flip noise. We will then initialize our singlet in the $\{|+\rangle, |-\rangle\}$ basis, with $|\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}$, because $\sigma_z\,|\pm\rangle = |\mp\rangle$. In this way $|\Phi^+\rangle = \frac{|++\rangle + |--\rangle}{\sqrt{2}}$.

First, we will attach two additional qubits, initialized in the same way, in Bob's lab

$$|\Psi_{enc}\rangle = \frac{1}{\sqrt{2}}(|+_A(+++)_B\rangle + |-_A(---)_B\rangle) \tag{4.54}$$

Then the phase flip noise will act locally on this state, producing

$$\mathbb{I} \otimes \Lambda^{\otimes 3}(|\Psi_{enc}\rangle\langle\Psi_{enc}|) = (1-p)^3 |\Psi_{enc}\rangle\langle\Psi_{enc}| + p(1-p)^2 \sum_{i=2}^{4} \sigma_z^i |\Psi_{enc}\rangle\langle\Psi_{enc}| \sigma_z^i$$
$$+p^2(1-p) \sum_{2\leq i<j\leq 4} \sigma_z^i\sigma_z^j |\Psi_{enc}\rangle\langle\Psi_{enc}| \sigma_z^i\sigma_z^j + p^3 \sigma_z^2\sigma_z^3\sigma_z^4 |\Psi_{enc}\rangle\langle\Psi_{enc}| \sigma_z^2\sigma_z^3\sigma_z^4 \quad (4.55)$$

The next step is to measure the observables $\sigma_x^2\sigma_x^3$ and $\sigma_x^3\sigma_x^4$ in Bob's lab, and apply conditional recovery unitaries R depending on the outcome of these measurements, according to the table 4.2.1.

After this procedure, the density matrix will have the form

$$\begin{aligned}\rho_{rec} = ((1-p)^3 + 3p(1-p)^2) |\Psi_{enc}\rangle\langle\Psi_{enc}| + \\ (3p^2(1-p) + p^3)\sigma_z^2\sigma_z^3\sigma_z^4 |\Psi_{enc}\rangle\langle\Psi_{enc}| \sigma_z^2\sigma_z^3\sigma_z^4\end{aligned} \quad (4.56)$$

This procedure is able to correct only a single phase flip per time, not a simultaneous flip of more than one qubit, or, equivalently, the encoded state is recovered up to order p. In fact, after discarding the ancillary qubits, we obtain

$$\begin{aligned}\rho_{dec} = ((1-p)^3 + 3p(1-p)^2) |\Phi^+\rangle\langle\Phi^+| + \\ (3p^2(1-p) + p^3)\sigma_z^2 |\Phi^+\rangle\langle\Phi^+| \sigma_z^2\end{aligned} \quad (4.57)$$

This state is maximally correlated, so the number of singlets we can distill back is

$$E_D(\rho_{dec}) = S(Tr_B[\rho_{dec}]) - S(\rho_{dec}) = 1 - h((1-p)^3 + 3p(1-p)^2) \quad (4.58)$$

In order to compare this procedure with the dilution we choose to dilute into the same number of extra qubits, i.e. our dilution rate must be 3 $1/S(\Psi^A) = 3$, attained setting $\alpha = 0.25$. We see from Fig. 4.5 that the error correction procedure achieves a better distillable entanglement than the entanglement protection rate studied in the previous section.

$(\sigma_x^2\sigma_x^3, \sigma_x^3\sigma_x^4)$	R
$(+1,+1)$	I
$(+1,-1)$	σ_z^4
$(-1,+1)$	σ_z^2
$(-1,-1)$	σ_z^3

Table 4.1: Table of the conditional recovery unitaries, conditioned on the outcomes of $\sigma_x^2\sigma_x^3$ and $\sigma_x^3\sigma_x^4$, that Bob has to perform in order to obtain 4.56 from 4.55.

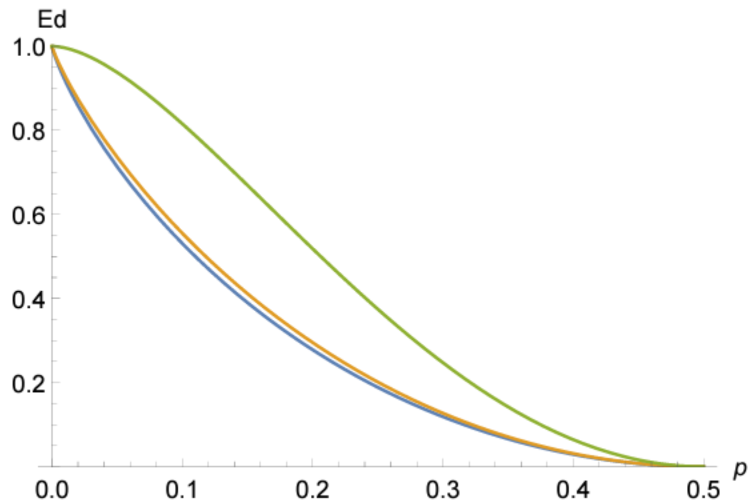

Figure 4.5: Plot of different distillation rates against the noise parameter p. Green: after error correction protocol, i.e. 4.58. Orange: after dilution, i.e. the entanglement protection rate for $\alpha = 0.25$. Blue: no pre-processing.

4.3 Coherence

We start with a collection of n maximally coherent states $|+\rangle^{\otimes n}$, in the fixed incoherent computational basis, i.e. $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. The noise will be given by the ampliutude damping channel $\Lambda(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger$, with

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \tag{4.59}$$

$$K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{4.60}$$

The coherence protection rate will be

$$\frac{C_D(\Lambda(\mu))}{C_D(\mu)} \tag{4.61}$$

with advantage when

$$\frac{C(\Lambda(\mu))}{C(\mu)} > C(\Lambda(|+\rangle\langle+|)) \tag{4.62}$$

with $C(\rho) = S(\rho||\Delta(\rho)) = S(\Delta(\rho)) - S(\rho)$ asymptotic distillable coherence rate and $\Delta(\rho) = \sum_{i=1}^d \langle i|\rho|i\rangle |i\rangle\langle i|$ completely dephasing map in the incoherent basis.

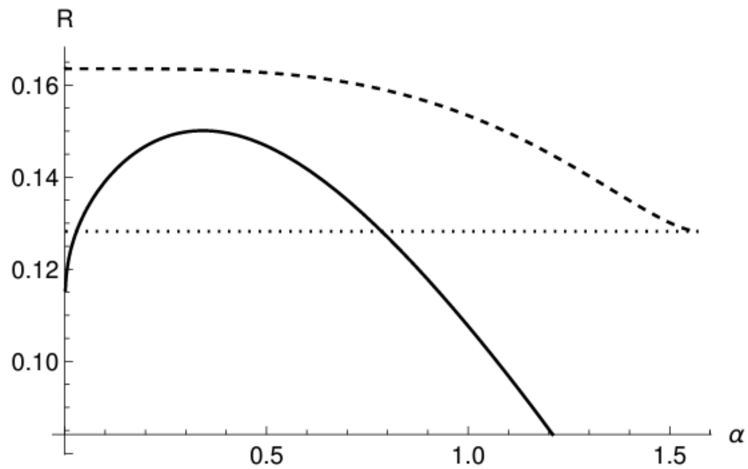

Figure 4.6: Coherence protection rate when the diluted target state is a pure state $|\Psi\rangle = \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle$ (solid line) or a mixed state $\mu = \sin^2\alpha\,|+\rangle\langle+| + \cos^2\alpha\frac{\mathbb{I}}{2}$ (dashed line) plotted against the dilution parameter α. No dilution is plotted in a straight dotted line. There is an advantage in both cases for certain ranges of α, but mixed states perform better.

We will dilute into two different types of target states, pure $\mu = |\Psi\rangle\langle\Psi|$ with $|\Psi\rangle = \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle$ and mixed $\mu = \sin^2\alpha\,|+\rangle\langle+| + \cos^2\alpha\frac{\mathbb{I}}{2}$. In Fig. 4.6 are shown the performances of pure states, in the solid curve, and mixed state, dashed curve, against the dilution parameter α. The straight dotted line represents the no-dilution scenario. The plot shows that dilution in both pure and mixed states provides an advantage in protecting coherence from amplitude-damping noise, but perhaps surprisingly mixed states perform even better in this task.

4.4 Conclusions

To summarize, in this chapter we proposed a general protocol to protect quantum resources from noise. First, we defined such a protocol. Then we showed that a kind of operation whose usefulness in a quantum technological task was never pointed out, dilution into less resourceful states, turns out to be decisive in preserving those resources from some particular choices of noise models if applied as pre-processing to many copies of a maximally resourceful state. Moreover, for the resource theory of purity and entanglement, it seems that the less resource is contained in the diluted state, the better the protection performances, and in the resource theory of coherence, mixed states perform better than pure ones. We also provided a comparison with the most famous protocol to fight noise in a quantum computer, i.e. error correction. The latter still seems to show better performances.

Chapter 5

Resource Generation

The original results presented in this chapter are mainly uploaded to the arxiv at the link: [SNS24]. Here, we will study the possibility of generating resources for a finite amount of time t, through a resource generating map Λ_t and we will study how to optimize the resource production, i.e. which kind of maps and input states are best suitable for this task. We will still restrict ourselves to quantum dynamical semigroups, i.e. CPTP maps with the property:

$$\Lambda_{t+s}(\rho) = \Lambda_s \circ \Lambda_t(\rho), \ \forall \rho, t, s \geq 0. \tag{5.1}$$

A resource-generating map is such that, for any amount of time $t \geq 0$ and any resource measure R, it exists at least one input state ρ such that:

$$R(\Lambda_t(\rho)) \geq R(\rho). \tag{5.2}$$

In particular, the question we will attempt to answer is the following: what is the best way to generate resources if we have access to such a map for a finite amount of time and to an infinite collection of any input state, including resourceful states? We underline we work in the limit of many identical and independent copies of states. In order to preserve this setting, the resource generation map will be applied locally to each of the states of the collection. The relevant resource quantifier will be the distillable resource, i.e. the rate $R(\rho) = R_D(\rho) = R(\rho \to \Phi^+)$. Moreover, we will restrict to reversible resource theories, for which the following holds:

$$R(\rho \to \Phi^+)R(\Phi^+ \to \rho) = \frac{R_D(\rho)}{R_C(\rho)} = 1. \tag{5.3}$$

We remind that, under such an assumption, the rate of resource distillation is given by the following minimization of the quantum relative entropy

$$R_D(\rho) = \min_{\sigma \in F_s} S(\rho||\sigma) \tag{5.4}$$

with $S(\rho||\sigma) = \mathrm{Tr}[\rho \log \rho] - \mathrm{Tr}[\rho \log \sigma]$ and the minimization performed over the set of free states.

In the next section, we will introduce the resource-generating power and show that it is equivalent to a more practical quantity to compute, the derivative of the resource quantifier evaluated a t time 0 and maximized over all possible initial states. The consideration that the amount of resource of a state is equivalent to the time needed to generate will follow.

In Section 5.2, we will present an operational interpretation of the resource-generating power. In the following sections, we will consider the particular examples of resource theories of coherence, entanglement, and purity.

5.1 Resource-generating power

We formalize the resource-generating power as follows:

$$P(\Lambda_t) = \max_{\rho_i, t} \frac{R(\Lambda_t(\rho_i)) - R(\rho_i)}{t}. \tag{5.5}$$

Here, we maximize over all the initial states ρ_i, even resourceful ones, and time $t \geq 0$. This quantity takes inspiration from the coherence-generating power introduced in [TRS21]. Essentially, it quantifies the maximum resource gain per unit of time after applying the map Λ_t. We assert that the maximization over time can be subsumed within the maximization over the initial state and that this quantity is equivalent to the derivative of the resource quantifier computed at time $t = 0$ and maximized over all possible initial states. In fact, according to the Lagrange or mean value theorem, if $R(\Lambda_t(\rho))$ is continuous and differentiable in t, there exists a $t_0 \in [0, t]$ such that:

$$\frac{R(\Lambda_t(\rho_i)) - R(\rho_i)}{t} = \frac{d}{dt} R(\Lambda_t(\rho_i))\Big|_{t=t_0}. \tag{5.6}$$

Furthermore, by leveraging the semigroup property of the map:

$$\begin{aligned} \frac{d}{dt} R(\Lambda_t(\rho_i))\Big|_{t=t_0} &= \lim_{\Delta t \to 0} \frac{R(\Lambda_{t_0+\Delta t}(\rho_i)) - R(\Lambda_{t_0}(\rho_i))}{\Delta t} = \\ = \lim_{\Delta t \to 0} \frac{R(\Lambda_{\Delta t}(\Lambda_{t_0}(\rho_i)) - R(\Lambda_{t_0}(\rho))}{\Delta t} &= \frac{d}{dt} R(\Lambda_t(\Lambda_{t_0}(\rho_i)))\Big|_{t=0}. \end{aligned} \tag{5.7}$$

Thus, the maximization over time t is encapsulated in t_0, allowing us to regard $\Lambda_{t_0}(\rho_i)$ as a new initial state μ. Notice that, since ρ_i spans the entire set of density matrices, also does μ, simply by choosing $\Lambda_{t_0} = \mathbb{I}$ as the identity map. Consequently, we can express $P(\Lambda_t)$ as:

$$P(\Lambda_t) = \sup_{\mu} \frac{d}{dt} R(\Lambda_t(\mu))\Big|_{t=0}. \tag{5.8}$$

We have just shown that the resource-generating power maximized over all possible times t is equivalent to the resource-generation rate computed at time

$t = 0$ and maximized over all possible initial states μ. This means that the resource generated after an application of the map for time T is upper bounded by:

$$R(\Lambda_T(\rho)) - R(\rho) \leq \frac{d}{dt} R(\Lambda_t(\mu^*))\bigg|_{t=0} T \tag{5.9}$$

where μ^* is the state that maximizes the right-hand side of Eq. 5.8. From now on, this will be the primary quantity of interest in this chapter. In the next section, we suggest a protocol for extracting resources from the optimal state μ^*—the one maximizing 5.8—whose performance is characterized by the same quantity.

5.2 Operational Interpretation

Let us assume we start with an ensemble of the optimal state μ^* maximizing 5.8. In general, this state will cost some amount of resource, but we will show that this is negligible compared to the resource we can generate. Then, we can:

1. Apply the resource-generating map locally to each of these replicas for a duration δt on the initial state, yielding $\Lambda_{\delta t}(\mu^*)$.
2. In the distillation stage, employing free operations, expend a portion of the resource within the ensemble of states $\Lambda_{\delta t}(\mu^*)$ to reset the replicas to their initial state μ^*. This involves utilizing $R_C(\mu^*) = R_D(\mu^*) = R(\mu^*)$ amount of resource. The remaining amount, $R(\Lambda_{\delta t}(\mu^*)) - R(\mu^*)$, is stored in a resource battery. The time used to implement this stage is irrelevant since only free operations are involved.
3. We restart the cycle from point 2.

The assembly of states μ^* may potentially contain some amount of resource. Nevertheless, we can demonstrate that repeating the cycle numerous times essentially renders the resource cost of μ^* negligible. The total resource gain per unit of time after point 2. will in fact be:

$$\frac{R(\Lambda_{\delta t}(\mu^*)) - 2R(\mu^*)}{\delta t}. \tag{5.10}$$

where the second $R(\mu^*)$ is due to the resource we spend to create μ^* at the beginning of the cycle. We repeat this procedure N times, with $N \to \infty$, obtaining

$$\lim_{N\to\infty} \frac{R(\Lambda_{\delta t}(\mu^*)) - R(\mu^*) - \frac{R(\mu^*)}{N}}{\delta t} = \frac{R(\Lambda_{\delta t}(\mu^*)) - R(\mu^*)}{\delta t}, \tag{5.11}$$

making the initial time of preparation T negligible. Finally, we also take the limit of short time $\delta t \to 0$, obtaining the derivative of the resource quantifier computed at time $t = 0$

$$\frac{dR(\Lambda_t(\mu^*))}{dt}\Big|_{t=0}. \tag{5.12}$$

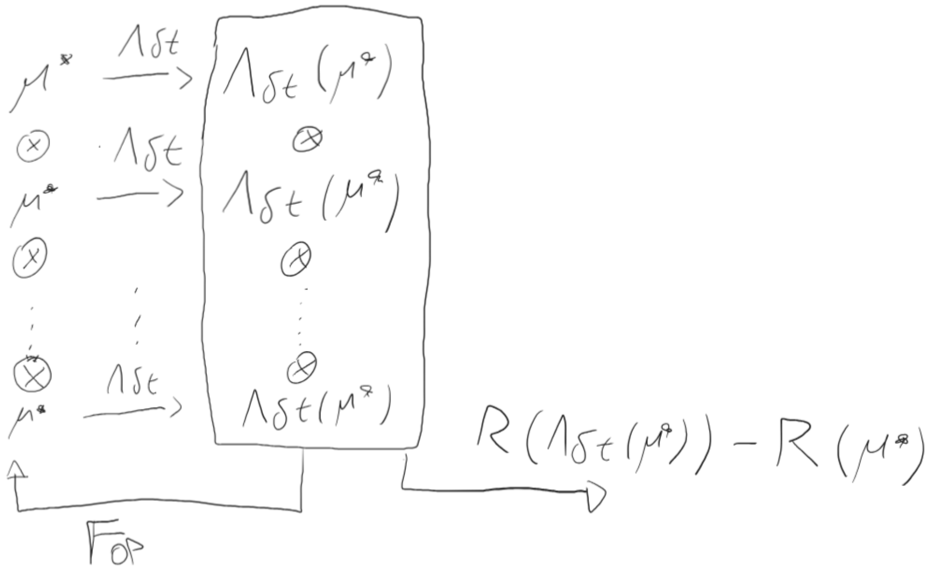

Figure 5.1: Schematic depicture of the resource generation protocol described in 5.2. 1. We start with a collection of μ^* states which maximize the rate 5.8. 2. We apply the resource-generating map locally, to each of the copies μ^*, for a time δt and obtain $\Lambda_{\delta t}(\mu^*)$. 3. We spend part of the resource of this last state to generate back the state μ^*, with a free operation, spending $R(\mu^*)$ amount of resource. We then save the difference $R(\Lambda_{\delta t}(\mu^*)) - R(\mu^*)$ in a resource battery and restart the cycle from point.

Remember that μ^* is the state which maximizes this rate. Thus, we can interpret the resource-generating power $P(\Lambda_t)$ as the optimal charging rate of a resource battery in this task. In Fig. 5.1 we present a scheme of the protocol we just described.

Let us now find the optimal initial state and resource-generating map in three particular cases: the resource theory of coherence, entanglement, and purity.

5.3 Resource theory of coherence

We recall that the quantifier of coherence in the asymptotic limit, i.e. when we have access to many copies of the same state is the regularized relative entropy of coherence

$$R(\rho) = S(\rho||\Delta(\rho)) = S(\Delta(\rho)) - S(\rho) \tag{5.13}$$

with

$$\Delta(\rho) = \sum_{i=1}^{d} \langle i| \rho |i\rangle |i\rangle \langle i| \tag{5.14}$$

completely dephasing channel in the fixed incoherent basis $\{|i\rangle\}$ of the Hilbert space of dimension d, with $i = 1, ..., d$. Moreover $S(\rho) = -\lambda_i \log \lambda_i$ is the Von Neumann entropy of ρ, wtih λ_i its eigenvalues. This quantifier operationally represents the optimal rate of asymptotic conversion from an ensemble of states ρ to another ensemble of maximally coherent states $|+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle$.

A seamless method for establishing coherence involves aligning the state with a coherent basis, whose eigenvectors are of the form

$$|+_i\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} e^{i\varphi_j} |j\rangle \tag{5.15}$$

where each vector $|+_i\rangle$, with $i = 1, ..., d$ has different set of phases $\{\varphi_j\}$, such that the $|+_i\rangle$ are mutually orthogonal. Consequently, we opt for a resource-generating map in the form of a unitary evolution, denoted as $U_t = e^{-iHt}$, where H represents a Hamiltonian and our evolved state will be $\rho_t = U_t \rho U_t^\dagger$. This unitary will keep the eigenvalues of ρ constant, therefore

$$\left.\frac{dS(\rho_t)}{dt}\right|_{t=0} = 0 \tag{5.16}$$

and

$$\left.\frac{dR(\rho_t)}{dt}\right|_{t=0} = \left.\frac{dS(\Delta(\rho_t))}{dt}\right|_{t=0} \tag{5.17}$$

Before moving on we prove an important identity for the derivative of the von Neumann entropy, i.e.

$$\frac{d}{dt} S(\rho) = -\operatorname{Tr}[\dot{\rho} \ln \rho] \tag{5.18}$$

Let us use the decomposition of the density operator in its eigenbasis:

$$\rho = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| \tag{5.19}$$

$$\dot{\rho} = \sum_i \dot{\lambda}_i |\lambda_i\rangle \langle \lambda_i| + \sum_i \lambda_i (|\dot{\lambda}_i\rangle \langle \lambda_i| + |\lambda_i\rangle \langle \dot{\lambda}_i|) \tag{5.20}$$

$$\ln \rho = \sum_i \ln \lambda_i |\lambda_i\rangle \langle \lambda_i| \tag{5.21}$$

$$\rho \ln \rho = \sum_i \lambda_i \ln \lambda_i |\lambda_i\rangle \langle \lambda_i| \tag{5.22}$$

$$\frac{d}{dt} Tr[\rho \ln \rho] = Tr[\frac{d}{dt} \rho \ln \rho] \tag{5.23}$$

$$\begin{aligned}\frac{d}{dt}\rho\ln\rho = \sum_i \big(&\dot{\lambda}_i \ln\lambda_i \left|\lambda_i\right\rangle\left\langle\lambda_i\right| + \\ +\dot{\lambda}_i \left|\lambda_i\right\rangle\left\langle\lambda_i\right| + \lambda_i\ln\lambda_i(&|\dot{\lambda_i}\rangle\left\langle\lambda_i\right| + \left|\lambda_i\right\rangle\langle\dot{\lambda_i}|)\big)\end{aligned} \tag{5.24}$$

Please observe that the second term traced out is essentially $\mathrm{Tr}[\dot{\rho}] = 0$. Additionally, the third term is also eliminated through tracing out because:

$$\langle\lambda_i|\dot{\lambda_i}\rangle + \langle\dot{\lambda_i}|\lambda_i\rangle = \frac{d}{dt}\left\langle\lambda_i|\lambda_i\right\rangle = 0 \tag{5.25}$$

So, finally:

$$\frac{d}{dt}S(\rho) = -\sum_i \dot{\lambda}_i \ln\lambda_i = -\,\mathrm{Tr}[\dot{\rho}\ln\rho] \tag{5.26}$$

where $\dot{\rho} = \frac{d\rho}{dt}$. This equation was also known in Lemma 1 of [BN88]. In the following, we give more details about the last equality:

$$\begin{aligned}\dot{\rho}\ln\rho = \sum_i \dot{\lambda}_i \ln\lambda_i \left|\lambda_i\right\rangle\left\langle\lambda_i\right| + \sum_i \lambda_i\ln\lambda_i |\dot{\lambda_i}\rangle\left\langle\lambda_i\right| + \\ \sum_{i,j}\lambda_i\ln\lambda_j\left|\lambda_i\right\rangle\langle\dot{\lambda_i}|\lambda_j\rangle\left\langle\lambda_j\right|\end{aligned} \tag{5.27}$$

$$\begin{aligned}\mathrm{Tr}[\dot{\rho}\ln\rho] = \sum_i \dot{\lambda}_i\ln\lambda_i + \sum_i \lambda_i\ln\lambda_i(\langle\lambda_i|\dot{\lambda_i}\rangle + \langle\dot{\lambda_i}|\lambda_i\rangle) = \\ \sum_i \dot{\lambda}_i\ln\lambda_i\end{aligned} \tag{5.28}$$

At this point, we have:

$$\begin{aligned}\frac{dR(\rho)}{dt} = -\sum_{i=1}^{d}\dot{\rho_{ii}}\ln\rho_{ii} = -\,\mathrm{Tr}\left[\left(\frac{d}{dt}\Delta(\rho)\right)\ln\Delta(\rho)\right] = \\ = -\,\mathrm{Tr}[\Delta(\dot{\rho})\ln\Delta(\rho)].\end{aligned} \tag{5.29}$$

where, in the last step, we have used the fact that we can move the derivative directly to the density matrix:

$$\frac{d}{dt}\Delta(\rho) = \frac{d}{dt}\sum_{i=1}^{d}\left\langle i\right|\rho\left|i\right\rangle\left|i\right\rangle\left\langle i\right| = \sum_{i=1}^{d}\left\langle i\right|\frac{d\rho}{dt}\left|i\right\rangle\left|i\right\rangle\left\langle i\right| = \Delta\left(\frac{d\rho}{dt}\right) \tag{5.30}$$

This is because the incoherent basis is fixed in our dynamics. In other words, we have shown that the derivative and the dephasing operation commute. Now we apply the von Neumann equation $\dot{\rho} = -i[H,\rho]$, evaluated at time $t=0$:

$$\left.\frac{d}{dt}S(\Delta(\rho))\right|_{t=0} = i\,\mathrm{Tr}[\Delta([H,\rho])\ln\Delta(\rho)] \tag{5.31}$$

Notice that we do not want to write $\frac{d}{dt}\Delta(\rho) = -i[H, \Delta(\rho)]$, since we are first rotating the density matrix and then dephasing it, not the opposite. To compute the maximum coherence generating power, we then maximize the following quantity:

$$\max_{\rho,H} i\,\mathrm{Tr}[\Delta([H,\rho])\ln\Delta(\rho)] \tag{5.32}$$

over the input density matrix ρ and a bounded Hamiltonian with respect to the Hilbert-Schmidt norm. The Hamiltonian must be bounded to guarantee a finite time of the application of the resource generating unitary $U = e^{iHt}$. We remember the definition of the Hilbert-Schmidt norm [GGK90] for an operator $A = \sum_{i,j} a_{ij} |i\rangle\langle j|$ being

$$||A||^2_{HS} \equiv \mathrm{Tr}[AA^\dagger] = \sum_{i,j} |a_{ij}|^2 \tag{5.33}$$

Let us now show, before proceeding, that we can drop the dephasing operation applied to $[H,\rho]$ in the expression 5.32. This fact will simplify the calculations

$$[H,\rho] = \sum_{i,j,k}(E_{ik}\rho_{kj} - \rho_{ik}E_{kj})|i\rangle\langle j| \tag{5.34}$$

$$\Delta([H,\rho]) = \sum_{i,k}(E_{ik}\rho_{ki} - \rho_{ik}E_{ki})|i\rangle\langle i| \tag{5.35}$$

$$\ln\Delta(\rho) = \sum_i \ln\rho_{ii}|i\rangle\langle i| \tag{5.36}$$

$$[H,\rho]\ln\Delta(\rho) = \sum_{i,j,k}(E_{ik}\rho_{kj} - \rho_{ik}E_{kj})\ln\rho_{jj}|i\rangle\langle j| \tag{5.37}$$

$$\Delta([H,\rho])\ln\Delta(\rho) = \sum_{i,k}(E_{ik}\rho_{ki} - \rho_{ik}E_{ki})\ln\rho_{ii}|i\rangle\langle i| \tag{5.38}$$

$$\mathrm{Tr}[[H,\rho]\ln\Delta(\rho)] = \sum_{i,k}(E_{ik}\rho_{ki} - \rho_{ik}E_{ki})\ln\rho_{ii} = \mathrm{Tr}[\Delta([H,\rho])\ln\Delta(\rho)] \tag{5.39}$$

where $H = \sum_{ik} E_{ik}|i\rangle\langle k|$ and $\rho = \sum_{ik}\rho_{ik}|i\rangle\langle k|$ are written in the incoherent basis. So the expression to maximize is equivalent to

$$\max_{\rho,||H||_{HS}\leq 1} iTr[[H,\rho]\ln\Delta(\rho)] \tag{5.40}$$

Moreover, using the cyclic property of the trace, we can further modify the above expression by moving the commutator between H and ρ to ρ and $\ln\Delta(\rho)$:

$$\begin{aligned} Tr[[H,\rho]\ln\Delta(\rho)] &= Tr[H\rho\ln\Delta(\rho)] - Tr[\rho H\ln\Delta(\rho)] \\ &= Tr[H\rho\ln\Delta(\rho)] - Tr[H\ln\Delta(\rho)\rho] = Tr[H[\rho,\ln\Delta(\rho)]] \end{aligned} \tag{5.41}$$

At this stage, we can employ the Cauchy-Schwarz inequality formulated for the Hilbert-Schmidt inner product [Yan95] between $A = \sum_{i,j} a_{ij}|i\rangle\langle j|$ and $B = \sum_{i,j} b_{ij}|i\rangle\langle j|$:

$$|\operatorname{Tr}[A^\dagger B]| = |\sum_{i,j} a^*_{ji} b_{ji}| \leq ||A||_{HS}||B||_{HS} = \sqrt{\sum_{i,j} |a_{ij}|^2}\sqrt{\sum_{i,j} |b_{ij}|^2} \tag{5.42}$$

Applying this to our scenario, where A is the Hamiltonian H and B is the commutator $[\rho, \ln \Delta(\rho)]$, we obtain:

$$\operatorname{Tr}[H[\rho, \ln \Delta(\rho)]] \leq ||H||_{HS}||[\rho, \ln \Delta(\rho)]||_{HS} \leq ||[\rho, \ln \Delta(\rho)]||_{HS} \tag{5.43}$$

where we considered the constraint $||H||_{HS} \leq 1$. Then, we seek the density matrix that maximizes $||[\rho, \ln \Delta(\rho)]||_{HS}$. The maximum of 5.43 is achieved when H and $[\rho, \ln \Delta(\rho)]$ are proportional, implying $H = \alpha[\rho, \ln \Delta(\rho)]$, or

$$E_{ij} = \alpha \rho_{ij} \ln \frac{\rho_{ii}}{\rho_{jj}} \tag{5.44}$$

for all their matrix elements, with α being a constant of proportionality. We will now show that this constant must be purely imaginary. First, we remember that the commutator between two Hermitian matrices $C = [A, B] = AB - BA$ is anti-Hermitian:

$$C^\dagger = [A, B]^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = BA - AB = [B, A] = -[A, B] = -C \tag{5.45}$$

This means that it can be written as a Hermitian operator time the imaginary unit, i.e. $C = iA_{Herm}$. Now, if an anti-Hermitian operator C is proportional to a Hermitian operator B_{Herm}, i.e. $B_{Herm} = \alpha C$, it follows that

$$\alpha C = B_{Herm} = B^\dagger_{Herm} = \alpha^* C^\dagger = -\alpha^* C \tag{5.46}$$

indicating that $\alpha = -\alpha^*$, i.e. α is purely imaginary.

The specific commutator in our case is $C = [\rho, \ln \Delta(\rho)]$ and the Hermitian operator is $B_{Herm} = H$. Considering $\rho_{ij} = |\rho_{ij}|e^{i\varphi_{ij}}$ and $E_{ij} = |E_{ij}|e^{i\gamma_{ij}}$:

$$e^{i(\gamma_{ij} - \varphi_{ij})}|E_{ij}| = \alpha|\rho_{ij}| \ln \frac{\rho_{ii}}{\rho_{jj}} \tag{5.47}$$

Since the right-hand side is purely imaginary, it implies $e^{i(\varphi_{ij} - \gamma_{ij})} = i \implies \gamma_{ij} - \varphi_{ij} = \frac{\pi}{2}$. In essence, the optimal evolution occurs in a plane or subspace orthogonal to the Hamiltonian direction.

Explicitly writing the commutator $[\rho, \ln \Delta(\rho)]$:

$$[\rho, \ln \Delta(\rho)] = \sum_{i,j} \rho_{ij} (\ln \rho_{ii} - \ln \rho_{jj}) |i\rangle \langle j| \tag{5.48}$$

The Hilbert-Schmidt norm of this commutator is given by:

$$||[\rho, \ln \Delta(\rho)]||^2_{HS} = \sum_{i,j} |\rho_{ij}|^2 \ln^2 \frac{\rho_{jj}}{\rho_{ii}} \tag{5.49}$$

This quantity is bounded using the positivity condition of the density matrix. According to Sylvester's criterion [HJ85], the density matrix ρ is positive semidefinite if and only if all the minors satisfy:

$$\begin{vmatrix} \rho_{ii} & \rho_{ij} \\ \rho_{ji} & \rho_{jj} \end{vmatrix} = \rho_{ii}\rho_{jj} - |\rho_{ij}|^2 \geq 0 \tag{5.50}$$

This bound is achieved for pure states. Specifically, considering a pure state $|\Psi\rangle = \sum_{i=1}^{d} c_i |i\rangle$, the corresponding density matrix $|\Psi\rangle\langle\Psi|$ has diagonal and off-diagonal elements given by $\rho_{ii} = |c_i|^2$ and $\rho_{ij} = c_i c_j^*$. Consequently, the quantity in Equation 5.49 is maximized when the state is pure. By renaming the squared moduli of the coefficients $|c_i|^2$ as probabilities p_i, our objective is to maximize the function

$$F(p_1, ..., p_d) \equiv F(\vec{p}) = 2\sum_{i>j} p_i p_j \ln^2 \frac{p_i}{p_j} \tag{5.51}$$

over all possible probability distributions $\vec{p} = (p_1, ..., p_d)$ such that $\sum_{i=1}^{d} p_i = 1$. Given that the function $F(\vec{p})$ exhibits symmetry when i and j are interchanged and evaluates to 0 when $i = j$, it can be reformulated as follows:

$$F(\vec{p}) = \sum_{i,j} p_i p_j \ln^2 \frac{p_i}{p_j} \tag{5.52}$$

We now state and prove the following lemma, connecting $F(\vec{p})$ with the variance of the surprisal function $I(\vec{p}) = -\ln \vec{p}$, which is equal to the surprisal operator $S = -\ln \Delta(\rho)$:

Lemma 5.1.

$$\frac{F(\vec{p})}{2} = \sum_{i=1}^{d} p_i \ln^2 p_i - \left(\sum_{i=1}^{d} p_i \ln p_i\right)^2 = \Delta^2 S. \tag{5.53}$$

with $\Delta^2 S$ variance of the surprisal function $I(\vec{p}) = -\ln \vec{p}$.

Proof. After some algebra we have:

$$
\begin{aligned}
\Delta^2 S = \sum_{i=1}^{d} p_i \ln^2 p_i - \sum_{i=1}^{d} p_i^2 \ln^2 p_i - 2 \sum_{i>j=1}^{d} p_i p_j \ln p_i \ln p_j = \\
= \sum_{i=1}^{d} p_i (1-p_i) \ln^2 p_i - 2 \sum_{i>j=1}^{d} p_i p_j \ln p_i \ln p_j \\
= \sum_{i=1}^{d} p_i \left(\sum_{j\neq i}^{d} p_j \right) \ln^2 p_i - 2 \sum_{i>j=1}^{d} p_i p_j \ln p_i \ln p_j = \\
= \sum_{i>j=1}^{d} p_i p_j (\ln^2 p_i + \ln^2 p_j - 2 \ln p_i \ln p_j) = \\
\sum_{i>j=1}^{d} p_i p_j (\ln p_i - \ln p_j)^2
\end{aligned}
\tag{5.54}
$$

□

The maximum of this function is known from [RW15] to be attained for a binary spectrum of probabilities with multiplicities 1 and $d-1$.

In fact, employing the Lagrange multiplier method, we introduce the Lagrangian function and its gradient:

$$
L = \frac{1}{2} \sum_{i,j} p_i p_j \ln^2 \frac{p_i}{p_j} - \lambda \left(\sum_i p_i - 1 \right) \tag{5.55}
$$

$$
\frac{\partial L}{\partial p_k} = \sum_j p_j \ln \frac{p_k}{p_j} \left(2 + \ln \frac{p_k}{p_j} \right) - \lambda = 0 \tag{5.56}
$$

Let f_k be defined as:

$$
f_k = \sum_j p_j \ln \frac{p_k}{p_j} \left(2 + \ln \frac{p_k}{p_j} \right) \tag{5.57}
$$

Equation 5.56 implies that all f_k must equal a constant λ for every $i = 1, ..., d$. This equivalence implies that a single set of probabilities satisfies the system of equations:

$$
\begin{aligned}
f_i - f_j = p_i \ln \frac{p_i}{p_j} \left(2 + \ln \frac{p_j}{p_i} \right) + p_j \ln \frac{p_i}{p_j} \left(2 + \ln \frac{p_i}{p_j} \right) \\
+ \sum_{k\neq i,j}^{d} p_k \left[\ln \frac{p_i}{p_k} \left(2 + \ln \frac{p_i}{p_k} \right) - \ln \frac{p_j}{p_k} \left(2 + \ln \frac{p_j}{p_k} \right) \right] = 0
\end{aligned}
\tag{5.58}
$$

for all $i = 1, ..., d$. Notice that these equations are antisymmetric by swapping i with j, but they are neither symmetric nor antisymmetric by swapping i with k or j with k. Possible solutions include:

- A trivial solution $p_i = p_j = 1/d$, for all $i, j = 1, ..., d$, resulting in all equations being identities $(0 = 0)$. This corresponds to the minimum of the target function in 5.32, achieved with the maximally coherent state $|+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle$.

- A set of probabilities with only two distinct values: $p_1 = \gamma/n_1$, occurring n_1 times and $p_2 = (1-\gamma)/n_2$, occurring n_2 times, forming a binary spectrum for $0 \leq \gamma \leq 1$. In this case, equations 5.58 reduce to a single equation:

$$\gamma \ln \frac{n_2}{n_1} \frac{\gamma}{1-\gamma} \left(2 + \ln \frac{n_1}{n_2} \frac{1-\gamma}{\gamma} \right) + (1-\gamma) \ln \frac{n_2}{n_1} \frac{\gamma}{1-\gamma} \left(2 + \ln \frac{n_2}{n_1} \frac{\gamma}{1-\gamma} \right) = 0 \tag{5.59}$$

  We can solve this equation graphically for all values of the multiplicities n_1 and $n_2 = d - n_1$. In particular, in Fig. 5.2 is shown the special case of $n_1 = n_2$. Substituting this kind of spectrum in Eq. 5.51 we obtain

$$F(\vec{p}) = \gamma(1-\gamma) \ln^2 \frac{n_2}{n_1} \frac{\gamma}{1-\gamma} \tag{5.60}$$

  Then all these solutions are actually local maxima and the ratio of the multiplicities n_2/n_1 must be maximized in order to find the biggest, i.e. $n_1 = 1$ and $n_2 = d - 1$.

- No solution exists for Equation 5.58 with more than two distinct probabilities. The lack of symmetry under the interchange of j with any of the k implies that if a solution for $f_i - f_j = 0$ is $(p_1^*, ..., p_i^*, ..., p_j^*, ..., p_k^*, ..., p_d^*)$, the solution for $f_i - f_k = 0$ must be $(p_1^*, ..., p_i^*, ..., p_k^*, ..., p_j^*, ..., p_d^*)$. This makes it impossible for a single set of probabilities to solve the entire system of equations. For instance, in the case of $d = 3$, consider two equations:

$$\begin{aligned} f_1 - f_2 = {} & p_1 \ln \frac{p_1}{p_2} \left(2 + \ln \frac{p_2}{p_1} \right) + p_2 \ln \frac{p_1}{p_2} \left(2 + \ln \frac{p_1}{p_2} \right) \\ & + p_3 \left[\ln \frac{p_1}{p_3} \left(2 + \ln \frac{p_1}{p_3} \right) - \ln \frac{p_2}{p_3} \left(2 + \ln \frac{p_2}{p_3} \right) \right] = 0 \end{aligned} \tag{5.61}$$

$$\begin{aligned} f_1 - f_3 = {} & p_1 \ln \frac{p_1}{p_3} \left(2 + \ln \frac{p_3}{p_1} \right) + p_3 \ln \frac{p_1}{p_3} \left(2 + \ln \frac{p_1}{p_3} \right) \\ & + p_2 \left[\ln \frac{p_1}{p_2} \left(2 + \ln \frac{p_1}{p_2} \right) - \ln \frac{p_3}{p_2} \left(2 + \ln \frac{p_3}{p_2} \right) \right] = 0 \end{aligned} \tag{5.62}$$

  If a solution for $f_1 - f_2 = 0$ exists, denoted as (p_1^*, p_2^*, p_3^*), with $p_1^* \neq p_2^* \neq p_3^*$, the solution for $f_1 - f_3 = 0$ must be (p_1^*, p_3^*, p_2^*), as $f_1 - f_3 = 0$ results from $f_1 - f_2 = 0$ by swapping p_2 with p_3. Note that this solution does not need to be unique. However, $(p_1^*, p_2^*, p_3^*) \neq (p_1^*, p_3^*, p_2^*)$, making it impossible for one set of different probabilities to solve the entire system of equations. Therefore, the only way to solve the system is to choose $p_1 = p_2$, $p_2 = p_3$, or $p_3 = p_1$, leading to equations similar to 5.59 for

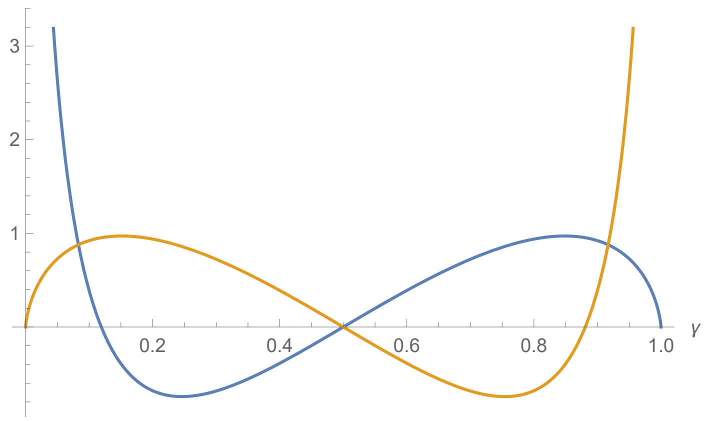

Figure 5.2: Plot of the two terms of eq. 5.59 when $n_1 = n_2$. Intersections of them give solutions of eq. 5.59. There are three of them, in this case, symmetric around $\gamma = 0.5$, in general, even changing the ratio n_1/n_2 three solutions always exist but are not symmetric around the center.

specific values, such as $p_1 = \gamma$ and $p_2 = \frac{1-\gamma}{2}$. This argument extends to cases with $d > 3$, as the additional terms in equations 5.3 retain the symmetry property.

The optimal state for maximizing the coherence production rate is given by:

$$|\Psi\rangle = \sqrt{\gamma^*}\,|0\rangle + \sqrt{1-\gamma^*}\,|+'\rangle \tag{5.63}$$

where $|+'\rangle = \frac{1}{\sqrt{d-1}}\sum_{i=1}^{d}|i\rangle$, and γ^* is the graphical solution of 5.59 for $n_1 = 1$ and $n_2 = d-1$. Importantly, this solution is independent of the Hamiltonian, which only determines the optimal subspace of the evolution, as mentioned. The result emphasizes that a maximally coherent state is not advantageous in this context for resource generation; instead, the optimal state always possesses a finite amount of coherence.

In Fig. 5.2 and Fig. 5.3 are shown, respectively the two sides of Eq. 5.59, i.e. the gradient condition, and the function 5.60 we wish to maximize for the case $n_1 = n_2$. The intersections of the two graphs in 5.2 gives γ^* and $1-\gamma^*$, while in plot 5.3 we can see that those correspond to global maxima and the maximally coherent state ($\gamma = 0.5$) correspond to a global minimum. In the case $n_1 \neq n_2$ the minimum shifts at $\frac{n_1}{n_2}\frac{\gamma}{1-\gamma} = 1$ and only one maxima becomes global.

In the accompanying Figure 5.4, the plot illustrates the variation of the solution γ^*, of 5.59, as a function of the dimension d of ρ, considering the case $n_1 = 1$ and $n_2 = d-1$. The graph reveals that the optimal coherence increases with the dimension d but never reaches the value of $\gamma^* = 1/2$, corresponding to the maximally coherent state.

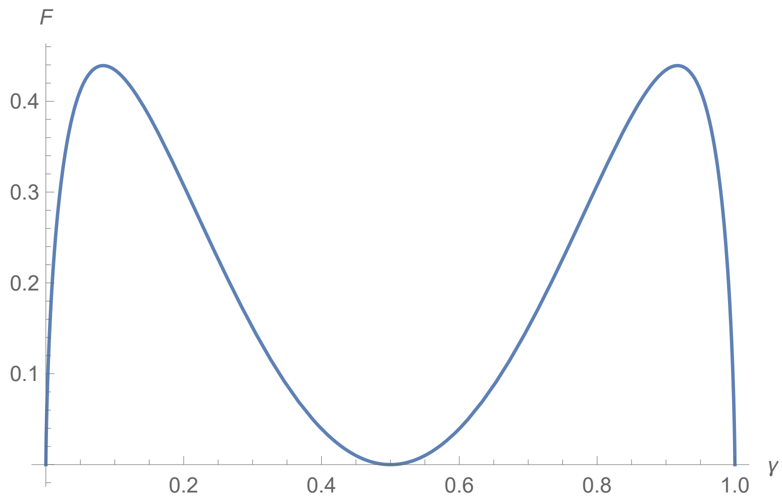

Figure 5.3: Plot of the function 5.60 for $n_1 = n_2$. it is evident that γ^* and $1-\gamma^*$ correspond to global maxima and the maximally coherent state ($\gamma = 0.5$) corresponds to a global minimum. In the case $n_1 \neq n_2$ the minimum shifts at $\frac{n_1}{n_2}\frac{\gamma}{1-\gamma} = 1$ and only one maxima becomes global.

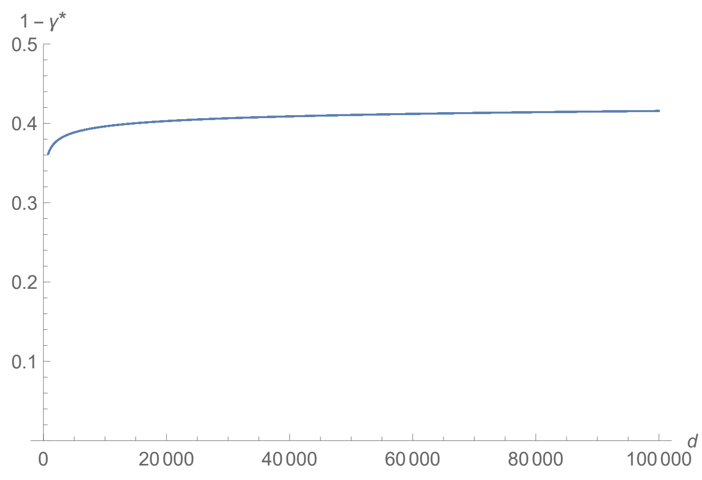

Figure 5.4: Plot of the solution γ^* of 5.59 for the case $n_1 = 1$ and $n_2 = d - 1$ over the dimension d of ρ. There is an asymptote for large dimensions but it is not given by the maximally coherent state, corresponding to $\gamma^* = 1/2$.

The optimal Hamiltonian, as per 5.47, is the generator of rotations in the two-dimensional space formed by $|0\rangle$ and $|+'\rangle$:

$$H = -\frac{i}{\sqrt{2}}(|+'\rangle \langle 0| - |0\rangle \langle +'|) \tag{5.64}$$

The coefficients can be derived from Equation 5.47 by substituting the optimal state provided by Equation 5.63. This yields:

$$|E_{0+'}| = |\alpha|\sqrt{\gamma^*(1-\gamma^*)} \ln \frac{\gamma^*}{1-\gamma^*} = |E_{+'0}|. \tag{5.65}$$

Computing its norm:

$$||H||_{HS}^2 = 2|\alpha|^2 \gamma^*(1-\gamma^*) \ln^2 \frac{1-\gamma^*}{\gamma^*}. \tag{5.66}$$

We can then select $|\alpha|^2$ to equate the norm to a constant, for example 1.

In the special case of a qubit ($d = 2$), $|+'\rangle = |1\rangle$. In this particular dimension, the Bloch sphere depiction becomes quite handy. Passing to spherical coordinates $\gamma = \cos^2(\theta/2)$ and the function to maximize becomes:

$$F(\theta) = \frac{\sin^2\theta}{4} \ln \frac{\cos^2(\theta/2)}{\sin^2(\theta/2)} \tag{5.67}$$

We can envision the optimal state residing within a plane perpendicular to the Hamiltonian direction, along the y-axis, with an optimal angle $\theta^* = \arccos(2\gamma^* - 1)$ relative to the z-axis, irrespective of the Hamiltonian. See Fig. 5.5.

Additionally, as discussed in 5.1, the resource generation rate in 5.8 reduces to the coherence-generating power studied in [TRS21] when choosing the appropriate coherence quantifier. The earlier study established that correlations with an additional system not evolving during the protocol, i.e., an ancilla, do not enhance the coherence-generating power of a channel. Therefore, this conclusion is applicable in the present case as well.

In the next section, we tackle the same problem for the resource theory of purity.

5.4 Resource Theory of purity

In the context of the resource theory of purity, the distillable purity, denoted as $P(\rho)$ is expressed by the equation:

$$P(\rho) = R(\rho \to |0\rangle \langle 0|) = \log_2 d - S(\rho) \tag{5.68}$$

Here, d represents the dimension, and $S(\rho)$ is the von Neumann entropy of the state ρ. The set of free states is composed only of the maximally mixed state $\frac{\mathbb{I}_d}{d}$. Notably, unitaries, being free operations, do not alter or influence the

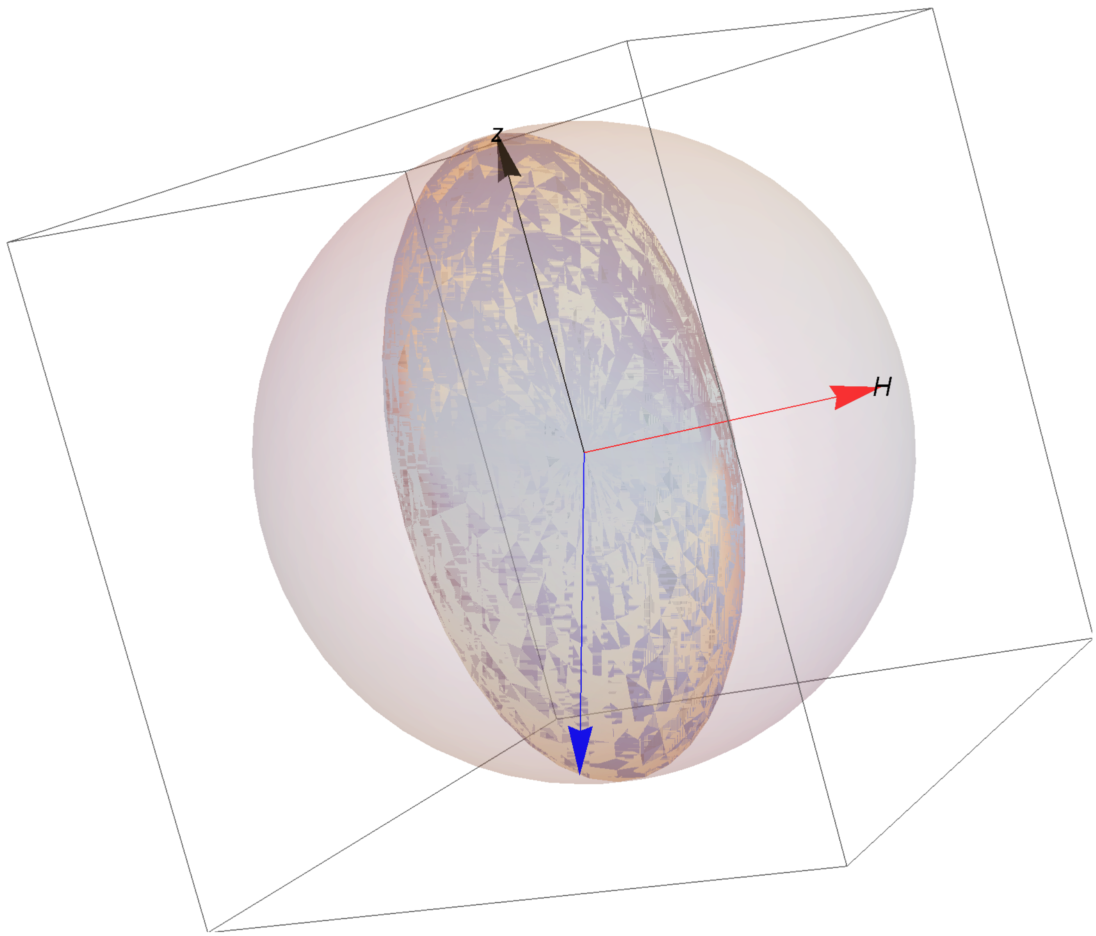

Figure 5.5: Schematic depiction of the optimal plane, orthogonal to the Hamiltonian σ_y, in which the optimal state 5.63 lives in the qubit case $d = 2$. This, in blue, forms an optimal angle $\theta^* = \arccos(2\gamma^* - 1)$ with the incoherent axis z.

purity of the system. Therefore, a logical choice for a purity-generating map is a non-unitary Lindbladian.

In the following, we will analyze the qubit case ($d = 2$). In this dimension, the time derivative of the asymptotic quantifier is given by

$$\frac{d}{dt}P(\rho)\bigg|_{t=0} = \frac{\dot{r}}{2}\ln\frac{1+r}{1-r} \tag{5.69}$$

where r is the length of the Bloch vector $\vec{r}$, such that the density matrix can be expressed as follows $\rho = \frac{\mathbb{I}_2+\vec{r}\cdot\vec{\sigma}}{2}$, with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, vector of the Pauli matrices. In particular, we will make use of the vector identity

$$\dot{r} = \frac{\vec{r}\cdot\dot{\vec{r}}}{r} = \hat{e}\cdot\dot{\vec{r}} \tag{5.70}$$

with $\hat{e} = \frac{\vec{r}}{r}$, unit vector defining the direction of $\vec{r}$. Now, considering a general Lindbladian, as we mentioned in Chapter 3 Eq. 3.10, the differential equation governing the evolution of the Bloch vector is given by:

$$\dot{\vec{r}} = A\vec{r} + \vec{b} \tag{5.71}$$

where $A = \gamma^S - \text{Tr}[\gamma]\mathbb{I}$ and $b_k = \sum_{ij}\varepsilon_{ijk}\gamma_{ij}$, represent the symmetric and antisymmetric part of the decoherence matrix γ of the Lindbladian, with $i, j = 1, 2, 3$. Since A is symmetric, it is diagonalizable with real eigenvalues denoted as $\{a_1, a_2, a_3\}$. The matrix represents a contraction of the Bloch sphere, therefore its eigenvalues must be negative. Moreover, $\vec{b}$ gives the non-unital shift of the center of the Bloch sphere.

Substituting 5.70 into 5.71 yields:

$$\dot{r} = r\hat{e}\cdot A\hat{e} + \hat{e}\cdot\vec{b} \tag{5.72}$$

Substituting this result into Eq. 5.69, we obtain:

$$\frac{d}{dt}P(\rho)\bigg|_{t=0} = \frac{(r\hat{e}\cdot A\hat{e} + \hat{e}\cdot\vec{b})}{2}\ln\frac{1+r}{1-r} \tag{5.73}$$

We can now express $\hat{e}$ and $\vec{b}$ in terms of the normalized eigenvectors of the symmetric matrix A, denoted as $\{\hat{a_1}, \hat{a_2}, \hat{a_3}\}$. That is, $\hat{e} = e_1\hat{a_1} + e_2\hat{a_2} + e_3\hat{a_3}$. Consequently $A\hat{e} = e_1a_1\hat{a_1} + e_2a_2\hat{a_2} + e_3a_3\hat{a_3}$. This leads to $\hat{e}\cdot A\hat{e} = e_1^2a_1 + e_2^2a_2 + e_3^2a_3$ and $\hat{e}\cdot\vec{b} = e_1b_1 + e_2b_2 + e_3b_3$.

To determine the optimal direction $\hat{e}$ on the Bloch sphere, we seek the maxima of the Lagrangian:

$$L = r\hat{e}\cdot A\hat{e} + \hat{e}\cdot\vec{b} + \lambda(|\hat{e}| - 1) \tag{5.74}$$

subject to the constraint $|\hat{e}| = 1$, where λ is the Lagrange multiplier. Using the decomposition specified above:

$$L = r\sum_{i=1}^{3} e_i^2 a_i + \sum_{i=1}^{3} e_i b_i + \lambda(\sum_{i=1}^{3} e_i^2 - 1) \tag{5.75}$$

and the null-gradient equation to solve:

$$\frac{\partial L}{\partial e_i} = 2re_i a_i + b_i + 2\lambda e_i = 0 \tag{5.76}$$

with solution

$$e_i = -\frac{b_i}{2(\lambda + ra_i)} \tag{5.77}$$

which can be rewritten in a compact form as

$$\hat{e} = -\frac{1}{2}(\lambda \mathbb{I}_3 + rA)^{-1}\vec{b} \tag{5.78}$$

Putting this solution back into the constrain $|\hat{e}| = 1$, we obtain a quadratic equation in λ:

$$\frac{b_1^2}{(ra_1+\lambda)^2} + \frac{b_2^2}{(ra_2+\lambda)^2} + \frac{b_3^2}{(ra_3+\lambda)^2} = 4, \tag{5.79}$$

and back into the function we wish to maximize:

$$\max_{r,A,\vec{b}} \ln\left(\frac{1+r}{1-r}\right) \sum_{i=1}^{3} \left(\frac{r}{4}\frac{a_i b_i^2}{(ra_i+\lambda)^2} - \frac{1}{2}\frac{b_i^2}{(ra_i+\lambda)}\right) \tag{5.80}$$

In the special case where $\vec{b} = (0,0,b_3)$ is an eigenvector of A:

$$\lambda = \frac{b_3 - 2ra_3}{2} \tag{5.81}$$

$$\hat{e} = (0,0,-1) \tag{5.82}$$

$$\max_{r,b_3,a_3} \frac{1}{2}\ln\frac{1+r}{1-r}(ra_3 - b_3) \tag{5.83}$$

Note that this instance provides an upper limit for the right-hand side of Eq. 5.72. This can be demonstrated using the Cauchy-Schwarz inequality once more, but this time applying the version suited for vectors to Eq. 5.72:

$$r\hat{e}\cdot A\hat{e} + \hat{e}\cdot\vec{b} \leq r||A\hat{e}||\cdot||\hat{e}|| + ||\vec{b}||\cdot||\hat{e}|| = r||A\hat{e}|| + ||\vec{b}|| \tag{5.84}$$

Here, we've utilized the fact that $||\hat{e}|| = 1$.

Recalling that the conditions for reaching the limit set by Eq. 5.84 are:

$$A\hat{e} = \alpha\hat{e} \tag{5.85}$$

$$\vec{b} = \beta\hat{e} \tag{5.86}$$

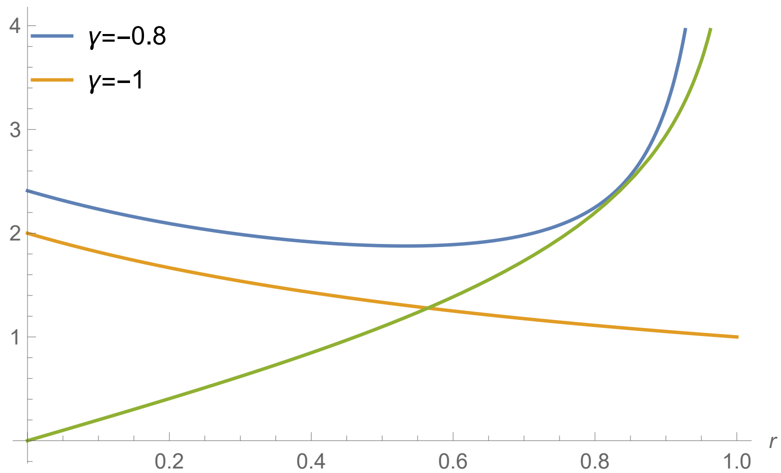

Figure 5.6: Left side of Eq. 5.87, in green, and right sides for $\gamma = -0.8$, in blue, and $\gamma = -1$, in orange. We can see that solutions exists only in this range of γ.

This implies that the optimal direction $\vec{e}$ must align either along the non-unitary shift one or one of the eigenvectors of matrix A.

Moreover, $b_3 > |a_3|$, for the map to be positive, i.e. the shift of the center of the Bloch sphere must not overcome the speed of contraction. Renaming the ratio of these two parameters as $\gamma = \frac{a_3}{b_3}$, and computing the derivative respect to r, we obtain the following transcendental equation:

$$\ln \frac{1+r}{1-r} = \left(\frac{1}{\gamma} - r \right) \frac{2}{1-r^2} \tag{5.87}$$

In Fig. 5.6 the two sides of this equation are plotted against r and for a range of $-1 \leq \gamma \leq -0.8$, for which solutions exist. The optimal input state for purity generation in this special case will then be:

$$\rho^* = \frac{\mathbb{I}_2 + r^*(\gamma)\sigma_3}{2} \tag{5.88}$$

where $r^*(\gamma)$ is the graphical solution of Eq. 5.87, function of γ. Notice that $\gamma = 1$ corresponds to the amplitude damping map.

5.4.1 Addition of ancillas

Furthermore, we establish that the addition of ancillas does not enhance the purity production scheme. Considering the evolution of one side of a bipartite state ρ^{AB}, by applying the data processing inequality on mutual information, we obtain:

$$I(A:B) := S(\rho^A) + S(\rho^B) - S(\rho^{AB}) \tag{5.89}$$

$$I(\Lambda_t(A):B) \leq I(A:B) \tag{5.90}$$

This inequality implies:

$$S(\Lambda_t(\rho^A)) + S(\rho^B) - S(\Lambda_t \otimes 1(\rho^{AB})) \leq S(\rho^A) + S(\rho^B) - S(\rho^{AB}) \tag{5.91}$$

$$S(\Lambda_t(\rho^A)) - S(\rho^A) \leq S(\Lambda_t \otimes 1(\rho^{AB})) - S(\rho^{AB}) \tag{5.92}$$

indicating that adding ancillas to the system does not improve the purity production rate, as $P(\rho) = \log d - S(\rho)$.

5.5 Resource Theory of entanglement

A parallel analysis can be conducted for the resource theory of entanglement in bipartite settings, where Alice and Bob share a pure state and can only implement local operations in their respective laboratories.

In this context, the free states are products of the form $|\Psi_A\rangle \otimes |\Phi_B\rangle$, and the maximally resourceful state is the maximally entangled one $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$, where d is the local dimension of the two laboratories.

The rate for asymptotic conversion of an iid ensemble of generic pure states $|\Psi\rangle$ into maximally entangled states is given by the distillable entanglement of Ψ:

$$E(\Psi) = R(\Psi \to \Phi^+) = S(Tr_B[|\Psi\rangle\langle\Psi|]) = -Tr[\rho^A \ln \rho^A] \tag{5.93}$$

Here, $\rho^A = \mathrm{Tr}_B[|\Psi\rangle\langle\Psi|]$ is the density matrix of Alice obtained by tracing out Bob's system [PV07]. This quantity remains invariant under the application of local unitary operations (on Alice's or Bob's side). However, Alice and Bob can enhance this rate by applying a global unitary operation, acting jointly on the two laboratories. This type of operation will serve as our resource-generating map in this setting.

The generator of this dynamics can be represented by an interaction Hamiltonian between the two laboratories:

$$\frac{d}{dt} |\Psi\rangle\langle\Psi| \Big|_{t=0} = -i[H, |\Psi\rangle\langle\Psi|]. \tag{5.94}$$

Our objective is to maximize the following quantity:

$$\frac{d}{dt} E(\Psi) \Big|_{t=0} = -Tr[\dot{\rho}^A \ln \rho^A] \tag{5.95}$$

over the initial pure bipartite state $|\Psi\rangle$ and the non-local bounded Hamiltonian H, with $||H||_{HS} = 1$. Since tracing out Bob's system and evolving the state are commuting operations, we can write:

$$\dot{\rho}^A = -iTr_B([H, |\Psi\rangle\langle\Psi|]). \tag{5.96}$$

We choose to perform the computation in the Schmidt basis of $|\Psi\rangle$, $\{|e_i\rangle\}$ for Alice and $\{|f_i\rangle\}$ for Bob:

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |e_i\rangle \otimes |f_i\rangle \tag{5.97}$$

$$|\Psi\rangle \langle\Psi| = \sum_{i,j=1}^{d} \sqrt{\lambda_i \lambda_j}\, |e_i\rangle \langle e_j| \otimes |f_i\rangle \langle f_j| \tag{5.98}$$

$$\rho_A = Tr_B[|\Psi\rangle \langle\Psi|] = \sum_{i=1}^{d} \lambda_i\, |e_i\rangle \langle e_i| \tag{5.99}$$

$$H = \sum_{i,j,k,l=1}^{d} E_{ij}^{kl}\, |e_i\rangle \langle e_j| \otimes |f_k\rangle \langle f_l| \tag{5.100}$$

Here the square of the Schmidt coefficients $\sqrt{\lambda_i}$ sum up to one: $\sum_i \lambda_i = 1$. The lower indices of E_{ij}^{kl} correspond to the basis of system A, and the upper indices correspond to the basis of system B. In this form, Hermiticity corresponds to swapping both upper and lower indices together $E_{ij}^{kl} = (E_{ji}^{lk})^*$.

We can rewrite Eq. 5.95 as

$$\begin{aligned} \frac{d}{dt} E(\Psi)\Big|_{t=0} &= -\operatorname{Tr}[\dot{\rho}^A \ln \rho^A] = \\ = i \operatorname{Tr}_A[\operatorname{Tr}_B[[H, \Psi]] \log \rho^A] &= i \operatorname{Tr}[H[\Psi, \log \rho^A \otimes \mathbb{I}_B]] \end{aligned} \tag{5.101}$$

Using the Cauchy-Schwarz inequality for the Hilbert-Schmidt norm, we can bound the quantity 5.95 with $||[\Psi, \log \rho^A \otimes \mathbb{I}_B]||_{HS}$. Let us compute this commutator:

$$[\Psi, \log \rho^A \otimes \mathbb{I}_B] = \sum_{i,j} \sqrt{\lambda_i \lambda_j} (\log \lambda_j - \log \lambda_i)\, |e_i\rangle \langle e_j| \otimes |f_i\rangle \langle f_j| \tag{5.102}$$

Then:

$$||[\Psi, \log \rho^A \otimes \mathbb{I}_B]||_{HS}^2 = \sum_{i,j} \lambda_i \lambda_j \log^2 \frac{\lambda_i}{\lambda_j} \tag{5.103}$$

This expression is identical to the one found in the coherence case, with the Schmidt coefficients replacing the diagonal elements of the density matrix. Moreover, the Cauchy-Schwarz bound is saturated when the two matrices H and $[\rho, \log \rho^A \otimes \mathbb{I}_B]$ are proportional:

$$E_{ij}^{ij} = \alpha \sqrt{\lambda_i \lambda_j} \log \frac{\lambda_i}{\lambda_j} \tag{5.104}$$

These results align with previous analyses by [DVC$^+$01] and [Bra07], where the optimal state and Hamiltonian for entanglement production rate 5.95 were:

$$|\Psi\rangle = \sqrt{\lambda^*}\,|00\rangle + \sqrt{1-\lambda^*}\,|\Phi^+\rangle \tag{5.105}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{d-1}}\sum_{i=1}^{d-1}|ii\rangle \tag{5.106}$$

$$H = -i(|\Phi^+\rangle\langle 00| - |00\rangle\langle\Phi^+|) \tag{5.107}$$

In the two-qubit case, we can write the Schmidt coefficients as $\lambda_1 = \cos^2(\theta/2)$ and $\lambda_2 = \sin^2(\theta/2)$ with $\theta \in [0, \pi]$. Eq. 5.103 then becomes:

$$||[\Psi, \log\rho^A \otimes \mathbb{I}_B]||^2_{HS} = \frac{\sin^2\theta}{2}\log^2\frac{\cos^2(\theta/2)}{\sin^2(\theta/2)} \tag{5.108}$$

which is the same function we wished to maximize in the single qubit coherence case, i.e. Eq. 5.67.

5.6 Conclusions

In this chapter, we explored a possible extension of resource theories, in which the application of a resource-generating map is allowed for a finite time t. In this enhanced setting we formulated a protocol to optimally extract resource from an initial state ρ. In the specific contexts of the resource theory of coherence, purity, and entanglement we answer the question: what is the optimal initial state and resource-generating dynamic to utilize in the protocol? For coherence and entanglement, where the dynamic taken into consideration is a unitary this optimal state turns out to be a two-level system and the optimal Hamiltonian, generator of the dynamic, is a rotation in the subspace where the system lives. Moreover, the coefficient of the linear combination which gives the optimal state is independent of the Hamiltonian, which only determines the best plane of rotation. For the resource theory of coherence, the coherence-generating power is not affected by the addition of an inert ancillary system. Finally, we analyzed the same problem for the resource theory of purity in the case of a qubit ($d = 2$). In this case, the optimal dynamic is a general Lindbladian and the optimal state depends on its parameters, which in a special case we showed to be the speed of contraction of the Bloch sphere and its non-unital shift. Moreover, we were also able to prove the non-utility of an ancilla.

Chapter 6

Summary and outlook

Here, we recap once again the main results of this thesis and we offer directions for future investigations.

In this work, various ways to protect valuable resources for quantum technological tasks from environmental noise were analyzed.

In Chapter 3, we focused on which type of noise produced by the environment could be optimal to preserve coherence in a qubit and various correlations with an isolated ancilla. Our analysis outcome was that such an environment must possess an infinite amount of non-Markovianity, i.e. always retain memory about the information that the system exchanges with it. We also demonstrated that is possible to simulate such a dynamic on an optical table. Apart from the trivial extension to higher dimensions and more general types of noises more research in this direction could be spent on developing a physical model for such an eternal non-Markovian dynamic, leading then to practical applications in the field of quantum computing.

In Chapter 4, we presented a general protocol that consists of applying free maps on the system containing resource before and after the noise acts to destroy it. We thoroughly investigated one type of such maps in the limit of many identical copies of the system, i.e. the map that dilutes these copies in more of a less resourceful state, called the dilution map. We provided examples that show that in many resource theories and many different choices of noise such a map can be successful in extracting more resource from the noisy state than the situation when no pre-processing is done. Moreover in many of those situations, the less resourceful the target diluted state the better the performance of this task turns out to be. In one example, the one involving the resource theory of coherence, we also showed that diluting into mixed states provides a better coherence protection rate than diluting into pure states. Finally, we compared our method with one of the most famous pre-processing maps in literature, the error correction code, the latter performing better. The work of this chapter can be extended in many directions, from generalizing noise and the dimension of the states involved to finding the real optimal pre-processing map. Since all these results are valid in the asymptotic limit a natural question that we just scratched in this work is whether is it possible to obtain similar results in the finite copies regime. We investigated the case $N = 2$ but this analysis can be generalized for $N > 2$ but still finite.

Finally, in Chapter 5 we analyzed a possible extension of a resource theory in which the experimenter is allowed to generate resource for a finite amount of time t, through the application of a map Λ_t on a previously prepared quantum state ρ. In this scenario, we formulated a protocol to optimally extract quantum resource from the prepared quantum state in the infinite copies regime, where the resource quantifier is given by the distillable resource. Then, for the specific resource theories of coherence purity and entanglement, we found the optimal initial state and dynamic to utilize in this protocol, maximizing the resource production rate. In the resource theory of coherence and entanglement, when the resource-generating map is given by a unitary, the optimal state belongs to a two-dimensional space and the coefficients of the linear combination between the two basis vectors are independent of the Hamiltonian which generates the dynamic. The latter only determines the optimal plane in which the evolution happens. Moreover, for the resource theory of coherence adding ancillas doesn't improve the resource generation rate. In the context of the resource theory of purity we analyze a qubit case when the resource-generating map is given by a general Lindbladian. There we found that the optimal state depends form the parameters of the Lindbladian, in particular the ratio between the speed of contraction of the Bloch sphere along one direction, and the non-unital shift of the center of the sphere along the same direction. In this case, as well additional ancillary systems do not improve the protocol. A further direction where to expand this line of research would be to find the optimal state as a function of any given resource-generating dynamic. Moreover, it is still unclear if adding ancillary systems enhances the entanglement generation rate.

Chapter 7

Appendix

7.1 Entanglement cost and distillation

In this section, we give an overview on the two important asymptotic conversion rates to and from the maximally resourceful state in the context of the resource theory of entanglement, the **distillable entanglement** and the **entanglement cost** respectively. We will only consider the problem for pure state and for theat class we will prove reversibility, i.e. the two rates coincide. We redirect the interested reader to Alexander Streltsov's lecture notes http://qot.cent.uw.edu.pl/teaching/.

7.1.1 Typical sequences

Consider a sequence of Independent and Identically Distributed random variables (IID) of length m $\{x_1, ..., x_m\}$. For example m launches of a coin with head probability p. The probability of such a sequence is gonna factorize in the following way $p(x_1, ...x_m) = p(x_1)...p(x_m)$. Such a sequence is called ε-typical if

$$2^{-m(H(p(x))+\varepsilon)} \leq p(x_1, ..., x_m) \leq 2^{-m(H(p(x))-\varepsilon)} \tag{7.1}$$

with $H(p(x)) = -\sum_{x_i} p(x_i) \log_2 p(x_i)$ Shannon entropy of the probability distribution.

Two important theorems hold in the case of ε-typical sequences.

Theorem 7.1. *$\forall \varepsilon > 0$ and $\delta > 0$ $\exists m' : \forall m > m'$*

$$\sum_{\varepsilon-typical} p(x_1)...p(x_m) > 1 - \delta \tag{7.2}$$

Theorem 7.2. *$\forall \varepsilon > 0$ and $\delta > 0$ $\exists m' : \forall m > m'$ the number N of ε-typical sequences satisfies*

$$(1 - \delta)2^{m(H(p(x))-\varepsilon)} \leq N \leq 2^{m(H(p(x))+\varepsilon)} \tag{7.3}$$

7.1.2 Entanglement Dilution

Asymptotic conversion from n copies of a maximally entangled state to m copies of a generic pure state $|\Psi\rangle$, with an asymptotically vanishing error ε_n, i.e. $|\Phi^+\rangle^{\otimes n} \overset{LOCC}{\rightarrow}_{\varepsilon_n} |\Psi\rangle^{\otimes m}$ with $\lim_{n\to\infty} \varepsilon_n = 0$.

The minimum ratio n/m in the asymptotic limit is called entanglement cost of $|\Psi\rangle$, i.e. $\min_{\lim_{n\to\infty}} \frac{n}{m} = E_C(|\Psi\rangle)$.

Theorem 7.3.

$$E_C(|\Psi\rangle) \leq S(\rho_\Psi) \tag{7.4}$$

with $\rho_\Psi = Tr_B[|\Psi\rangle \langle\Psi|]$.

Proof. The target state $|\Psi\rangle$ can be written in Schmidt decomposition

$$|\Psi\rangle = \sum_x \sqrt{p(x)}\, |x\rangle^A \otimes |x\rangle^B \tag{7.5}$$

$$|\Psi\rangle^{\otimes m} = \sum_{x_1,\ldots,x_m} \sqrt{p(x_1)...p(x_m)}\, |x_1...x_m\rangle^A \otimes |x_1...x_m\rangle^B \tag{7.6}$$

Let's define the state

$$|\Phi_m\rangle = \sum_{\varepsilon-\text{typical}} \sqrt{p(x_1)...p(x_m)}\, |x_1...x_m\rangle^A \otimes |x_1...x_m\rangle^B \tag{7.7}$$

this state is undernormalized. To make it so we divide by the overlap between the two states

$$\langle\Psi_m|\Phi_m\rangle = \sum_{\varepsilon-\text{typical}} p(x_1)...p(x_m) \tag{7.8}$$

in this way

$$|\Phi'_m\rangle = \frac{1}{\sqrt{\langle\Phi_m|\Psi_m\rangle}} |\Phi_m\rangle \tag{7.9}$$

Because of theorem 1 of typical sequences, this overlap goes to 1 in the limit of large sequences

$$\lim_{m\to\infty} \left(\sum_{\varepsilon-\text{typical}} p(x_1)...p(x_m) \right) = 1 \tag{7.10}$$

meaning for large m the two states are basically equivalent.

To create the desired shared state Alice can prepare it locally and teleport half of it to Bob, consuming $\lceil \log_2 k \rceil$ singlets, with k Schmidt rank of $|\Phi_m\rangle$, which classically correspond to the number of typical sequences. Because of theorem 2 $k \leq 2^{m(S(\rho_\Psi)+\varepsilon)}$. □

7.1.3 Entanglement Distillation

The inverse process of entanglement dilution, i.e. $|\Psi\rangle^{\otimes m} \overset{LOCC}{\rightarrow}_{\varepsilon_n} |\Phi^+\rangle^{\otimes n}$, $\max_{m\to\infty} \frac{n}{m} = E_D(|\Psi\rangle)$

Theorem 7.4.

$$E_D(|\Psi\rangle) \geq S(\rho_\Psi) \tag{7.11}$$

Proof. Alice performs the projective measurement

$$\Pi_0 = \sum_{\varepsilon-\text{typical}} |x_1...x_m\rangle \langle x_1...x_m| \tag{7.12}$$

the probability of outcome being

$$p_0 = Tr[(\Pi_0 \otimes \mathbb{I}) |\Psi_m\rangle \langle\Psi_m|] = \sum_{\varepsilon-\text{typical}} p(x_1)...p(x_m) \tag{7.13}$$

and the post measurement state

$$\frac{1}{\sqrt{p_0}}(\Pi_0 \otimes \mathbb{I}) |\Psi_m\rangle = |\Phi'_m\rangle \tag{7.14}$$

The largest Schmidt coefficient of $|\Phi_m\rangle$, by definition of typical sequences, is at most

$$p(x_1)...p(x_m) \leq 2^{-m(S(\rho_\Psi)-\varepsilon)} \tag{7.15}$$

and the one of the normalized state $|\Phi'_m\rangle$, because of theorem 1 is at most

$$\frac{2^{-m(S(\rho_\Psi)-\varepsilon)}}{1-\delta} \leq 2^{-n} \tag{7.16}$$

where in the last equation we choose n to fulfill it. Then our state is majorized by a singlet of rank 2^n and by Nielsen theorem it can be converted into it, at a rate bounded by Eq. 7.16. □

7.1.4 Reversibility

Let's suppose ∃LOCC : $|\Psi\rangle^{\otimes m} \rightarrow |\Phi^+\rangle^{\otimes n}$, such that

$$\frac{n}{m} \approx S \gtrsim S(\rho_\Psi) \tag{7.17}$$

This means that if Alice and Bob starts with k singlets, they can convert them in m copies of a state $|\Psi\rangle$ at a rate $\frac{k}{m} \approx S(\rho_\Psi)$. If the first statement is true they can convert back these states into $n \approx mS = k\frac{S}{S(\rho_\Psi)} > k$, which is impossible by LOCC. Then cost and distillation must be equal to $S(\rho_\Psi)$ making the theory reversible.

7.2 X states

In this section we will present some calculations and formulas useful for the understanding of chapter 3 results. We will start from properties of the X-shaped bipartite states, to which, in particular, the class of Choi state of covariant maps belongs:

$$\rho_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix} \tag{7.18}$$

positivity conditions

$$\rho_{22}\rho_{33} \geq |\rho_{23}|^2 \tag{7.19}$$
$$\rho_{11}\rho_{44} \geq |\rho_{14}|^2 \tag{7.20}$$

separability conditions

$$\rho_{22}\rho_{33} \geq |\rho_{14}|^2 \tag{7.21}$$
$$\rho_{11}\rho_{44} \geq |\rho_{23}|^2 \tag{7.22}$$

eigenvalues

$$\lambda_0 = \frac{1}{2}\left[(\rho_{11} + \rho_{44}) + \sqrt{(\rho_{11} - \rho_{44})^2 + 4|\rho_{14}|^2}\right] \tag{7.23}$$
$$\lambda_1 = \frac{1}{2}\left[(\rho_{11} + \rho_{44}) - \sqrt{(\rho_{11} - \rho_{44})^2 + 4|\rho_{14}|^2}\right] \tag{7.24}$$
$$\lambda_2 = \frac{1}{2}\left[(\rho_{22} + \rho_{33}) + \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{23}|^2}\right] \tag{7.25}$$
$$\lambda_3 = \frac{1}{2}\left[(\rho_{22} + \rho_{33}) - \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{23}|^2}\right] \tag{7.26}$$

The reduced states and corresponding entropies are

$$\rho^A = \begin{pmatrix} \rho_{11} + \rho_{22} & 0 \\ 0 & \rho_{33} + \rho_{44} \end{pmatrix} \tag{7.27}$$
$$\rho^B = \begin{pmatrix} \rho_{11} + \rho_{33} & 0 \\ 0 & \rho_{22} + \rho_{44} \end{pmatrix} \tag{7.28}$$
$$S(\rho^A) = -\left[(\rho_{11} + \rho_{22})\log_2(\rho_{11} + \rho_{22}) + (\rho_{44} + \rho_{33})\log_2(\rho_{33} + \rho_{44})\right] \tag{7.29}$$
$$S(\rho^B) = -\left[(\rho_{11} + \rho_{33})\log_2(\rho_{11} + \rho_{33}) + (\rho_{44} + \rho_{22})\log_2(\rho_{22} + \rho_{44})\right] \tag{7.30}$$

7.2.1 Choi state of covariant maps

$$\rho_{11} = \frac{1}{4}(1 + \lambda_z(t) + l_z(t)) \tag{7.31}$$
$$\rho_{22} = \frac{1}{4}(1 - \lambda_z(t) - l_z(t)) \tag{7.32}$$
$$\rho_{33} = \frac{1}{4}(1 - \lambda_z(t) + l_z(t)) \tag{7.33}$$
$$\rho_{44} = \frac{1}{4}(1 + \lambda_z(t) - l_z(t)) \tag{7.34}$$
$$\rho_{23} = \rho_{32} = 0 \tag{7.35}$$
$$\rho_{14} = \rho_{41} = \frac{\lambda(t)}{2} \tag{7.36}$$
$$\lambda_0(t) = \frac{1}{4}\left[(1 + \lambda_z(t)) + \sqrt{l_z^2(t) + 4\lambda^2(t))}\right] \tag{7.37}$$
$$\lambda_1(t) = \frac{1}{4}\left[(1 + \lambda_z(t)) - \sqrt{l_z^2(t) + 4\lambda^2(t))}\right] \tag{7.38}$$
$$\lambda_2(t) = \rho_{22} \tag{7.39}$$
$$\lambda_3(t) = \rho_{33} \tag{7.40}$$
$$S(Tr_B[\Omega_{\Lambda_t}]) = 1 \tag{7.41}$$
$$S(Tr_A[\Omega_{\Lambda_t}]) = -\left[\frac{1 + l_z(t)}{2}\log_2\frac{1 + l_z(t)}{2} + \frac{1 - l_z(t)}{2}\log_2\frac{1 - l_z(t)}{2}\right] \tag{7.42}$$
$$\lim_{t\to\infty}\lambda_0(t) = \frac{1}{2} \tag{7.43}$$
$$\lim_{t\to\infty}\lambda_1(t) = 0 \tag{7.44}$$
$$\lim_{t\to\infty}\lambda_2(t) = \frac{1 + \frac{x}{a}}{4} \tag{7.45}$$
$$\lim_{t\to\infty}\lambda_2(t) = \frac{1 - \frac{x}{a}}{4} \tag{7.46}$$
$$\lim_{t\to\infty}S(\Omega_{\Lambda_t}) = 1 + \frac{1}{2}h\left(\frac{1 + \frac{x}{a}}{2}\right) \tag{7.47}$$

7.2.2 Computation of the classical correlations of the Choi state

Here we will provide details on the computation of the classical correlations and subsequently of the quantum discord for the Choi state of covariant maps and of the eternally non-Markovian dynamics. This section is based on the paper [ARA10].

Let us introduce rank one projectors $\{B_k\}$ which describe a local measurement on Bob's side. They have the properties of being complete $\sum_k B_k = \mathbb{I}$, Hermitian $B_k = B_k^\dagger$ and idempotent $B_k^2 = B_k$.

They can be written as

$$B_k = V\Pi_i V^\dagger \tag{7.48}$$

with $\Pi = |i\rangle\langle i|$ projector along the computational basis element $|i\rangle$ and V unitary transformation in $SU(2)$ when local dimension is 2

$$V = t\mathbb{I} + i\vec{y}\cdot\vec{\sigma} \tag{7.49}$$

with $t^2 + y_1^2 + y_2^2 + y_3^2 = 1$.
The post-measurement bipartite state is

$$\rho_k = \frac{1}{p_k}(\mathbb{I}\otimes B_k)\rho(\mathbb{I}\otimes B_k) \tag{7.50}$$

with $p_k = Tr[(\mathbb{I}\otimes B_k)\rho(\mathbb{I}\otimes B_k)]$.
The classical correlations contained in state ρ are then given by

$$C(\rho) = \sup_{\{B_k\}} I(\rho|\{B_K\}) \tag{7.51}$$

with $I(\rho|\{B_K\})$ conditional mutual information.

$$I(\rho|\{B_K\}) = S(\rho^A) - S(\rho|\{B_k\}) \tag{7.52}$$

and $S(\rho|\{B_k\})$ conditional entropy

$$S(\rho|\{B_k\}) = \sum_k p_k S(\rho_k) \tag{7.53}$$

When ρ is an X state, the post measurement ensamble $\{\rho_i, p_i\}$ with

$$p_0 = (\rho_{11} + \rho_{33})k + (\rho_{22} + \rho_{44})l \tag{7.54}$$
$$p_1 = (\rho_{11} + \rho_{33})l + (\rho_{22} + \rho_{44})k \tag{7.55}$$

can be carachterized by the eigenvalues of ρ_0 and ρ_1

$$v_\pm(\rho_0) = \frac{1\pm\theta}{2} \tag{7.56}$$
$$v_\pm(\rho_1) = \frac{1\pm\theta'}{2} \tag{7.57}$$

with

$$\theta = \sqrt{\frac{[(\rho_{11}-\rho_{33})k + (\rho_{22}-\rho_{44})l]^2 + \Theta}{[(\rho_{11}+\rho_{33})k + (\rho_{22}+\rho_{44})l]^2}} \tag{7.58}$$

$$\theta' = \sqrt{\frac{[(\rho_{11}-\rho_{33})l + (\rho_{22}-\rho_{44})k]^2 + \Theta}{[(\rho_{11}+\rho_{33})l + (\rho_{22}+\rho_{44})k]^2}} \tag{7.59}$$

$$\Theta = 4kl[|\rho_{14}|^2 + |\rho_{23}|^2 + 2Re(\rho_{14}\rho_{23})] + 16mRe(\rho_{14}\rho_{23}) + 16nIm(\rho_{14}\rho_{23}) \tag{7.60}$$

$$m = (ty_1 + y_2y_3)^2 \tag{7.61}$$

$$n = (ty_2 - y_1y_3)(ty_1 + y_2y_3) \tag{7.62}$$

$$k = t^2 + y_3^2 \tag{7.63}$$

$$l = y_1^2 + y_2^2 \tag{7.64}$$

$$k + l = 1 \tag{7.65}$$

The entropies of the post measurement states are then

$$S(\rho_0) = -\left[\frac{1+\theta}{2}\log_2\frac{1+\theta}{2} + \frac{1-\theta}{2}\log_2\frac{1-\theta}{2}\right] \tag{7.66}$$

$$S(\rho_1) = -\left[\frac{1+\theta'}{2}\log_2\frac{1+\theta'}{2} + \frac{1-\theta'}{2}\log_2\frac{1-\theta'}{2}\right] \tag{7.67}$$

and the conditional entropy we want to minimize over the unitary parameters k, l, m, n

$$S(\rho_X|\{B_k\}) = p_0S(\rho_0) + p_1S(\rho_1) \tag{7.68}$$

In the case of our Choi state

$$\theta = \sqrt{\frac{\lambda_z^2(t)(k-l)^2 + 4kl\lambda^2(t)}{[(1+l_z(t))k + (1-l_z(t))l]^2}} \tag{7.69}$$

$$\theta' = \sqrt{\frac{\lambda_z^2(t)(k-l)^2 + 4kl\lambda^2(t)}{[(1+l_z(t))l + (1-l_z(t))k]^2}} \tag{7.70}$$

In the limit $t \to \infty$

$$\theta = \sqrt{\frac{kl(1-\frac{x^2}{a^2})}{[(1+\frac{x}{a})k + (1-\frac{x}{a})l]^2}} \tag{7.71}$$

$$\theta' = \sqrt{\frac{kl(1-\frac{x^2}{a^2})}{[(1+\frac{x}{a})l + (1-\frac{x}{a})k]^2}} \tag{7.72}$$

The minimum of the conditional entropy is attained in one of these cases

- $k = l = 1/2$
- $k = 0$ and $l = 1$
- $k = 1$ and $l = 0$

In the last two cases $\theta = \theta' = 0$ and $S(\rho_0) = S(\rho_1) = 1$ which is the maximum value, so not the one we search.

In the first case $\theta = \theta' = \frac{\sqrt{1-\frac{x^2}{a^2}}}{2} = \theta_{min}$. So

$$C(\Omega_{\Lambda_\infty}) = 1 - S(\rho_0)|_{\theta_{min}} \tag{7.73}$$

Then the quantum discord is equal to

$$\lim_{t\to\infty} Q(\Omega_{\Lambda_t}) = \frac{h\left(\frac{1+\frac{x}{a}}{2}\right)}{2} + h\left(\frac{1+\frac{\sqrt{1-\frac{x^2}{a^2}}}{2}}{2}\right) - 1 \tag{7.74}$$

Bibliography

[ACB16] Gerardo Adesso, Marco Cianciaruso, and Thomas R. Bromley. An introduction to quantum discord and non-classical correlations beyond entanglement, 2016. arXiv:1611.01959.

[AF13] Robert Alicki and Mark Fannes. Entanglement boost for extractable work from ensembles of quantum batteries. *Phys. Rev. E*, 87:042123, Apr 2013. URL: https://link.aps.org/doi/10.1103/PhysRevE.87.042123, doi:10.1103/PhysRevE.87.042123.

[ARA10] Mazhar Ali, A. R. P. Rau, and G. Alber. Quantum discord for two-qubit x states. *Phys. Rev. A*, 81:042105, Apr 2010. URL: https://link.aps.org/doi/10.1103/PhysRevA.81.042105, doi:10.1103/PhysRevA.81.042105.

[BaG15] Fernando G. S. L. Brandão and Gilad Gour. Reversible framework for quantum resource theories. *Phys. Rev. Lett.*, 115:070503, Aug 2015. URL: https://link.aps.org/doi/10.1103/PhysRevLett.115.070503, doi:10.1103/PhysRevLett.115.070503.

[BBC+93] Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters. Teleporting an unknown quantum state via dual classical and einstein-podolsky-rosen channels. *Phys. Rev. Lett.*, 70:1895–1899, Mar 1993. URL: https://link.aps.org/doi/10.1103/PhysRevLett.70.1895, doi:10.1103/PhysRevLett.70.1895.

[BBG+] Mario Berta, Fernando G. S. L. Brandão, Gilad Gour, Ludovico Lami, Martin B. Plenio, Bartosz Regula, and Marco Tomamichel. On a gap in the proof of the generalised quantum steinamp;apos;s lemma and its consequences for the reversibility of quantum resources. *Quantum*, 7. URL: http://dx.doi.org/10.22331/q-2023-09-07-1103, doi:10.22331/q-2023-09-07-1103.

[BBPS96] Charles H. Bennett, Herbert J. Bernstein, Sandu Popescu, and Benjamin Schumacher. Concentrating partial entanglement by local operations. *Phys. Rev. A*, 53:2046–2052, Apr 1996. URL: https://link.aps.org/doi/10.1103/PhysRevA.53.2046, doi:10.1103/PhysRevA.53.2046.

[BCP14] T. Baumgratz, M. Cramer, and M. B. Plenio. Quantifying coherence. *Phys. Rev. Lett.*, 113:140401, Sep 2014. URL: https://link.aps.org/doi/10.1103/PhysRevLett.113.140401, doi:10.1103/PhysRevLett.113.140401.

[BHO+13] Fernando G. S. L. Brandão, Michał Horodecki, Jonathan Oppenheim, Joseph M. Renes, and Robert W. Spekkens. Resource theory of quantum states out of thermal equilibrium. *Physical Review Letters*, 111(25), December 2013. URL: http://dx.doi.org/10.1103/PhysRevLett.111.250404, doi:10.1103/physrevlett.111.250404.

[BN88] Fabio Benatti and Heide Narnhofer. Entropy behaviour under completely positive maps. *Letters in Mathematical Physics*, 15:325–334, 01 1988. doi:10.1007/BF00419590.

[BP07] Heinz-Peter Breuer and Francesco Petruccione. *The Theory of Open Quantum Systems*. Oxford University Press, 01 2007. doi:10.1093/acprof:oso/9780199213900.001.0001.

[Bra07] Sergey Bravyi. Upper bounds on entangling rates of bipartite hamiltonians. *Physical Review A*, 76(5), nov 2007. URL: https://doi.org/10.1103%2Fphysreva.76.052319, doi:10.1103/physreva.76.052319.

[CG19] Eric Chitambar and Gilad Gour. Quantum resource theories. *Rev. Mod. Phys.*, 91:025001, Apr 2019. URL: https://link.aps.org/doi/10.1103/RevModPhys.91.025001, doi:10.1103/RevModPhys.91.025001.

[Cho75] Man-Duen Choi. Completely positive linear maps on complex matrices. *Linear Algebra and its Applications*, 10(3):285–290, 1975. URL: https://www.sciencedirect.com/science/article/pii/0024379575900750, doi:https://doi.org/10.1016/0024-3795(75)90075-0.

[CLGP22] Francesco Ciccarello, Salvatore Lorenzo, Vittorio Giovannetti, and G. Massimo Palma. Quantum collision models: Open system dynamics from repeated interactions. *Physics Reports*, 954:1–70, April 2022. URL: http://dx.doi.org/10.1016/j.physrep.2022.01.001, doi:10.1016/j.physrep.2022.01.001.

[DH99] Matthew J. Donald and Michal Horodecki. Continuity of relative entropy of entanglement, 1999. arXiv:quant-ph/9910002.

[DVC+01] W. Dür, G. Vidal, J. I. Cirac, N. Linden, and S. Popescu. Entanglement capabilities of nonlocal hamiltonians. *Phys. Rev. Lett.*, 87:137901, Sep 2001. URL: https://link.aps.org/doi/10.1103/PhysRevLett.87.137901, doi:10.1103/PhysRevLett.87.137901.

[FGL20] S. N. Filippov, A. N. Glinov, and L. Leppäjärvi. Phase covariant qubit dynamics and divisibility. *Lobachevskii Journal of Mathematics*, 41(4):617–630, apr 2020. URL: https://doi.org/10.1134%2Fs1995080220040095, doi:10.1134/s1995080220040095.

[GGK90] Israel Gohberg, Seymour Goldberg, and Marinus A. Kaashoek. *Hilbert-Schmidt Operators*, pages 138–147. Birkhäuser Basel, Basel, 1990. doi:10.1007/978-3-0348-7509-7_9.

[GMN+15] Gilad Gour, Markus P. Müller, Varun Narasimhachar, Robert W. Spekkens, and Nicole Yunger Halpern. The resource theory of informational nonequilibrium in thermodynamics. *Physics Reports*, 583:1–58, 2015. The resource theory of informational nonequilibrium in thermodynamics. URL: https://www.sciencedirect.com/science/article/pii/S037015731500229X, doi:https://doi.org/10.1016/j.physrep.2015.04.003.

[HC80] J.K. Hale and Karreman Mathematics Research Collection. *Ordinary Differential Equations*. Pure and applied mathematics. R. E. Krieger, 1980. URL: https://books.google.pl/books?id=NW1BxwEACAAJ.

[HCLA14] Michael J. W. Hall, James D. Cresser, Li Li, and Erika Andersson. Canonical form of master equations and characterization of non-markovianity. *Phys. Rev. A*, 89:042120, Apr 2014. URL: https://link.aps.org/doi/10.1103/PhysRevA.89.042120, doi:10.1103/PhysRevA.89.042120.

[Hel69] Carl W. Helstrom. Quantum detection and estimation theory. *Journal of Statistical Physics*, 1:231–252, 1969. URL: https://api.semanticscholar.org/CorpusID:12758217.

[HHH98] Michał Horodecki, Paweł Horodecki, and Ryszard Horodecki. Mixed-state entanglement and distillation: Is there a "bound" entanglement in nature? *Phys. Rev. Lett.*, 80:5239–5242, Jun 1998. URL: https:

//link.aps.org/doi/10.1103/PhysRevLett.80.5239, doi:10.1103/PhysRevLett.80.5239.

[HHH99] Pawel Horodecki, Michal Horodecki, and Ryszard Horodecki. General teleportation channel, singlet fraction and quasi-distillation, 1999. arXiv:quant-ph/9807091.

[HHHH09] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki. Quantum entanglement. *Reviews of Modern Physics*, 81(2):865–942, jun 2009. URL: https://doi.org/10.1103%2Frevmodphys.81.865, doi:10.1103/revmodphys.81.865.

[HJ85] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.

[HO13] Michał Horodecki and Jonathan Oppenheim. Fundamental limitations for quantum and nanoscale thermodynamics. *Nature Communications*, 4(1), June 2013. URL: http://dx.doi.org/10.1038/ncomms3059, doi:10.1038/ncomms3059.

[HOH02] Michał Horodecki, Jonathan Oppenheim, and Ryszard Horodecki. Are the laws of entanglement theory thermodynamical? *Phys. Rev. Lett.*, 89:240403, Nov 2002. URL: https://link.aps.org/doi/10.1103/PhysRevLett.89.240403, doi:10.1103/PhysRevLett.89.240403.

[Kin02] Christopher King. Additivity for unital qubit channels. *Journal of Mathematical Physics*, 43(10):4641–4653, sep 2002. URL: https://doi.org/10.1063%2F1.1500791, doi:10.1063/1.1500791.

[LDS18] Felix Leditzky, Nilanjana Datta, and Graeme Smith. Useful states and entanglement distillation. *IEEE Transactions on Information Theory*, 64(7):4689–4708, July 2018. URL: http://dx.doi.org/10.1109/TIT.2017.2776907, doi:10.1109/tit.2017.2776907.

[MLDS+13] Martin Müller-Lennert, Frédéric Dupuis, Oleg Szehr, Serge Fehr, and Marco Tomamichel. On quantum rényi entropies: A new generalization and some properties. *Journal of Mathematical Physics*, 54(12), December 2013. URL: http://dx.doi.org/10.1063/1.4838856, doi:10.1063/1.4838856.

[MLS04] M. W. Mitchell, J. S. Lundeen, and A. M. Steinberg. Super-resolving phase measurements with a multiphoton entangled state. *Nature*, 429(6988):161–164, May 2004. URL: http://dx.doi.org/10.1038/nature02493, doi:10.1038/nature02493.

[MSFAS24] Marek Miller, Manfredi Scalici, Marco Fellous-Asiani, and Alexander Streltsov. Power of noisy quantum states and the advantage of resource dilution. *Phys. Rev. A*, 109:022404, Feb 2024. URL: https://link.aps.org/doi/10.1103/PhysRevA.109.022404, doi:10.1103/PhysRevA.109.022404.

[MWS+22] Marek Miller, Kang-Da Wu, Manfredi Scalici, Jan Kołodyński, Guo-Yong Xiang, Chuan-Feng Li, Guang-Can Guo, and Alexander Streltsov. Optimally preserving quantum correlations and coherence with eternally non-markovian dynamics. *New Journal of Physics*, 24(5):053022, may 2022. URL: https://dx.doi.org/10.1088/1367-2630/ac6820, doi:10.1088/1367-2630/ac6820.

[NC11] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. Cambridge University Press, 2011. URL: https://www.amazon.com/Quantum-Computation-Information-10th-Anniversary/dp/1107002176?SubscriptionId=AKIAIOBINVZYXZQZ2U3A&tag=chimbori05-20&linkCode=xm2&camp=2025&creative=165953&creativeASIN=1107002176.

[Nie99] M. A. Nielsen. Conditions for a class of entanglement transformations. *Phys. Rev. Lett.*, 83:436–439, Jul 1999. URL: https://link.aps.org/doi/10.1103/PhysRevLett.83.436, doi:10.1103/PhysRevLett.83.436.

[NW18] Nelly Huei Ying Ng and Mischa Prebin Woods. *Resource Theory of Quantum Thermodynamics: Thermal Operations and Second Laws*, page 625–650. Springer International Publishing, 2018. URL: http://dx.doi.org/10.1007/978-3-319-99046-0_26, doi:10.1007/978-3-319-99046-0_26.

[Pea12] Philip Pearle. Simple derivation of the lindblad equation. *European Journal of Physics*, 33(4):805–822, April 2012. URL: http://dx.doi.org/10.1088/0143-0807/33/4/805, doi:10.1088/0143-0807/33/4/805.

[Pec94] Philip Pechukas. Reduced dynamics need not be completely positive. *Phys. Rev. Lett.*, 73:1060–1062, Aug 1994. URL: https://link.aps.org/doi/10.1103/PhysRevLett.73.1060, doi:10.1103/PhysRevLett.73.1060.

[Per96] Asher Peres. Separability criterion for density matrices. *Physical Review Letters*, 77(8):1413–1415, August 1996. URL:

http://dx.doi.org/10.1103/PhysRevLett.77.1413, doi:10.1103/physrevlett.77.1413.

[Pet86] Dénes Petz. Quasi-entropies for finite quantum systems. *Reports on Mathematical Physics*, 23(1):57–65, 1986. URL: https://www.sciencedirect.com/science/article/pii/0034487786900674, doi:https://doi.org/10.1016/0034-4877(86)90067-4.

[PGWPR06] David Pérez-García, Michael M. Wolf, Denes Petz, and Mary Beth Ruskai. Contractivity of positive and trace-preserving maps under lp norms. *Journal of Mathematical Physics*, 47(8), August 2006. URL: http://dx.doi.org/10.1063/1.2218675, doi:10.1063/1.2218675.

[PV07] Martin B. Plenio and S. Virmani. An introduction to entanglement measures. *Quant. Inf. Comput.*, 7:1–51, 2007. arXiv:arXiv:quant-ph/0504163.

[RH12] ´Angel Rivas and Susana F. Huelga. *Open Quantum Systems*. Springer Berlin Heidelberg, 2012. URL: https://doi.org/10.1007%2F978-3-642-23354-8, doi:10.1007/978-3-642-23354-8.

[RHP14] Ángel Rivas, Susana F Huelga, and Martin B Plenio. Quantum non-markovianity: characterization, quantification and detection. *Reports on Progress in Physics*, 77(9):094001, Aug 2014. URL: http://dx.doi.org/10.1088/0034-4885/77/9/094001, doi:10.1088/0034-4885/77/9/094001.

[Rof19] Joschka Roffe. Quantum error correction: an introductory guide. *Contemporary Physics*, 60(3):226–245, July 2019. URL: http://dx.doi.org/10.1080/00107514.2019.1667078, doi:10.1080/00107514.2019.1667078.

[RW15] David Reeb and Michael M. Wolf. Tight bound on relative entropy by entropy difference. *IEEE Transactions on Information Theory*, 61(3):1458–1473, March 2015. URL: http://dx.doi.org/10.1109/TIT.2014.2387822, doi:10.1109/tit.2014.2387822.

[SAP17] Alexander Streltsov, Gerardo Adesso, and Martin B. Plenio. Colloquium: Quantum coherence as a resource. *Reviews of Modern Physics*, 89(4), October 2017. URL: http://dx.doi.org/10.1103/RevModPhys.89.041003, doi:10.1103/revmodphys.89.041003.

[Sch19] Maximilian Schlosshauer. The quantum-to-classical transition and decoherence, 2019. arXiv:1404.2635.

[SKanHDDan16] Andrea Smirne, Jan Kołodyński, Susana F. Huelga, and Rafał Demkowicz-Dobrzański. Ultimate precision limits for noisy frequency estimation. *Phys. Rev. Lett.*, 116:120801, Mar 2016. URL: https://link.aps.org/doi/10.1103/PhysRevLett.116.120801, doi:10.1103/PhysRevLett.116.120801.

[SNS24] Manfredi Scalici, Moein Naseri, and Alexander Streltsov. Coherence generation with hamiltonians, 2024. arXiv:2402.17567.

[SSD+15] Alexander Streltsov, Uttam Singh, Himadri Shekhar Dhar, Manabendra Nath Bera, and Gerardo Adesso. Measuring quantum coherence with entanglement. *Physical Review Letters*, 115(2), July 2015. URL: http://dx.doi.org/10.1103/PhysRevLett.115.020403, doi:10.1103/physrevlett.115.020403.

[TRS21] Masaya Takahashi, Swapan Rana, and Alexander Streltsov. Creating and destroying coherence with quantum channels. *arXiv:2105.12060*, 2021. URL: https://arxiv.org/abs/2105.12060.

[Vid00] Guifré Vidal. Entanglement monotones. *Journal of Modern Optics*, 47(2-3):355–376, feb 2000. URL: https://doi.org/10.1080%2F09500340008244048, doi:10.1080/09500340008244048.

[VW02] G. Vidal and R. F. Werner. Computable measure of entanglement. *Physical Review A*, 65(3), February 2002. URL: http://dx.doi.org/10.1103/PhysRevA.65.032314, doi:10.1103/physreva.65.032314.

[WR12] Ligong Wang and Renato Renner. One-shot classical-quantum capacity and hypothesis testing. *Phys. Rev. Lett.*, 108:200501, May 2012. URL: https://link.aps.org/doi/10.1103/PhysRevLett.108.200501, doi:10.1103/PhysRevLett.108.200501.

[WY16] Andreas Winter and Dong Yang. Operational resource theory of coherence. *Physical Review Letters*, 116(12), March 2016. URL: http://dx.doi.org/10.1103/PhysRevLett.116.120404, doi:10.1103/physrevlett.116.120404.

[Yan95] Xinmin Yang. Some trace inequalities for operators. *Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics*, 58(2):281–286, 1995. doi:10.1017/S1446788700038295.

[ZSM+13] Wei Zhong, Zhe Sun, Jian Ma, Xiaoguang Wang, and Franco Nori. Fisher information under decoherence in bloch representation. *Phys. Rev. A*, 87:022337, Feb 2013. URL: https://link.aps.org/doi/10.1103/PhysRevA.87.022337, doi:10.1103/PhysRevA.87.022337.