Knots and BPS invariants – summary

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One of the most important goals of the contemporary theoretical physics is to find the unified description of all fundamental forces of nature. The Standard Model is a theory of electromagnetic, weak, and strong interactions, whereas gravitation is described by general relativity. Experimentally confirmed unification has not been achieved yet, but one of the most successful attempts is string theory. Among other achievements, it has provided a very sophisticated language for theoretical physics and mathematics.

Especially the intersection between string theory and topology turned out to be extremely fruitful. In 1990 Edward Witten was awarded the Fields Medal for, among others, "the only intrinsically 3-dimensional interpretation of the Jones invariants" [Ati91] in the language of Chern-Simons theory. This breakthrough can be considered as a cornerstone of the new field connecting knot theory with strings and branes.

In [OV00, LM01, LMV00] Labastida, Mariño, Ooguri and Vafa introduced topological invariants which are given in terms of HOMFLY-PT polynomials of a knot and could be interpreted as a net number of BPS states. This implies the integrality of BPS invariants – that statement, known as Labastida-Mariño-Ooguri-Vafa (LMOV) conjecture, is one of the most important unsolved problems at the intersection of knot theory with string theory. Unfortunately, the relation between HOMFLY-PT polynomials and BPS invariants is very complicated and the studies of LMOV conjecture focused on considering examples in the finite range of $\mathrm{U}(N)$ representations corresponding to the Young diagrams made of several boxes. On the other hand, mathematicians proved that HOMFLY-PT polynomials coloured by symmetric representations satisfy recursion relations encoded in A-polynomials [GLL]. Independently, BPS states were analysed in the context of quivers and Donaldson-Thomas invariants [KS, KS11, Rei12].

The main idea of this thesis is twofold. First, its aim is to find relations between A-polynomials and quivers on one side, and BPS invariants for knots on the other side. The second aim is to use these relations to find a broad spectrum of exact and explicit results on the structure of such BPS invariants, and in particular to provide a new perspective on the LMOV conjecture and prove it in a relatively general setting. The results of this thesis have been also presented in [GKS16, KS16, KRSS].

On the classical level the above aims are achieved by the introduction of a new algebraic curve called a dual A-polynomial, its extremal limit, as well as extremal BPS invariants. The analysis of solutions of dual extremal A-polynomial equations leads to exact formulas for classical extremal BPS invariants for all symmetric representations. In turn, they lead to nontrivial integrality statements in number theory, as well as to the discovery of an improved integrality conjecture, which is stronger than the known M-theory integrality predictions. All these results have been presented in [GKS16].

Furthermore, lifting the relation between dual A-polynomials and BPS invariants for knots to the quantum level reveals unexpected links with the branch of mathematics known as combinatorics on words [KS16]. In consequence quantum BPS invariants can be encoded in a combinatorial model constructed from the quantum dual A-polynomial. This model provides an explicit recursion relation satisfied by quantum BPS invariants and a proof of their integrality for a large class of knots.

This combinatorial model also leads to the relation between Donaldson-Thomas invariants corresponding to a quiver with one vertex, and BPS invariants for a framed unknot. Remarkably, this relation turns out to be more general, and in [KRSS] it is shown that generating functions of HOMFLY-PT polynomials for many other knots can be identified with motivic generating series of corresponding quivers. This leads to the formulation of a correspondence between knots and quivers, and in particular between BPS invariants for knots and integer motivic Donaldson-Thomas invariants [KRSS]. It follows that once a quiver corresponding to a given knot is identified – which has been done in [KRSS] for a large class of knots – it automatically implies the proof of the LMOV conjecture for all symmetric representations for such a knot. The proof of the LMOV conjecture along these lines is just one of many other deep consequences of the duality between knots and quivers discussed in this thesis.

To sum up, using A-polynomials and quivers to analyse BPS invariants for knots has provided a lot of new results and interesting connections, which deserve further studies.

References

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