Summary of PhD thesis

“Recursive Structures for Nahm Sums”

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In the past half-century there has been a tendency towards unification in theoretical physics. Famously known string theory provides a hypothetical framework for quantum theory of gravity, filling the gap between Einstein's general relativity and the Standard Model of particle physics. Vast developments on the crossroads of string theory, low-dimensional topology and quantum field theory led to many results which provided an idea for this thesis.

Nahm sums are the q-hypergeometric series of a special form which have many incarnations in theoretical physics and mathematics. Originally appeared in the context of rational two-dimensional conformal field theories [NRT93], Nahm sums can be also thought of as characters of moduli spaces of quiver representations [KS08] (quiver is a directed graph, and a representation assigns vector spaces and linear maps to its vertices and edges, respectively). Such series encode certain numerical invariants (motivic Donaldson-Thomas invariants), which in turn can be associated to Bogomol'nyi-Prasad-Sommerfield (BPS) states of supersymmetric gauge theories related to such quivers [KS08; GMN10]. On the other hand, Nahm sums generate quantum knot invariants, such as colored HOMFLY-PT polynomials (named due to Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki and Traczyk), for a particular selection of quivers. These polynomials arise from Chern-Simons theory on a three-sphere as expectation values of Wilson loops along a knot, as shown by Edward Witten in his 1989 groundbreaking paper [Wit89]. The knots-quivers correspondence [Kuc+17; Kuc+19] states that for every knot there exists a symmetric quiver which encodes quantum polynomials of this knot. From string theory perspective, Nahm sum for a knot (commonly referred to as Ooguri-Vafa partition function [OV00]) is conjectured to count M2-branes inside the resolved conifold — a special type of Calabi-Yau manifold. Therefore, Nahm sums provide a bridge between Chern-Simons theory and string theory, via quantum knot invariants.

The primary purpose of this thesis is to study the recursive structures – topological recursion, Wentzel–Kramers–Brillouin (WKB) expansions, recursions from quantum curves – for Nahm sums for generic quivers, not necessarily related to knots. The secondary purpose is to shed more light on the knots-quivers correspondence. The results of this thesis are presented in [Bor+19; Lar+20; Nos21; Jan+21]. Besides, during my master studies I was working on topics beyond the scope of this thesis, and these results are presented in [INP15] and [NP16]. The thesis can be divided into three main parts.

In the first part, whose results are also presented in [Nos21], we show that quiver A-polynomials, defined as the resultants for the semi-classical limit of the associated Nahm sum for a quiver, pass the K-theoretic quantization criterion from [GS12], and are therefore quantizable. Alongside we reveal an interesting combinatorial structure of quiver A-polynomials and the explicit formulas for their "extremal" versions (involving their Newton polytopes). It follows that classical quiver A-polynomials can be lifted to a well-defined quantum objects.

The second part, whose results are also presented in [Lar+20], deals mainly with the topological recursion for Nahm sums. It is an algorithm to compute correlation functions of an arbitrary order, for a given spectral curve [Eyn14] (the term "spectral curve" is taken from matrix model theory, where the topological recursion first appeared). It is also called "topological", because the recursion runs over the Euler characteristics of correlation functions, pictorially presented as punctured surfaces of a non-negative genus. In our study we take spectral curves to be quiver A-polynomials and apply topological recursion to a selection of quivers in order to confirm that it reconstructs the associated Nahm sum. Topological recursion therefore provides a way to quantization, and allows to find an explicit expression for quantum A-polynomials which take form of recursions for the coefficients of the corresponding Nahm sum. This problem is interesting because the remodelling theorem [EO15] states that topological recursion invariants from a B-model spectral curve can be reinterpreted as topological string amplitudes in the A-model. By using spectral curves from quivers we establish the correspondence between knot invariants and topological string counting in explicit examples, and also identify the A-model when there is no explicit relation to a knot. Our results mainly confirm the consistency of the topological recursion results with the corresponding Nahm sum. We also study quantum Airy structures and give a general definition of an r-Airy structure, also presented in [Bor+19]. Although not being directly related to Nahm sums, quantum Airy structures generalize the topological recursion and make a direct connection to algebraic structures coming from conformal field theories.

Having shown that quiver A-polynomials are quantizable and then having applied to them topological recursion, in the last part we consider properties of quantum objects, i.e. Nahm sums associated to various knots. We study the local equivalence relation for quivers and the identities between various Nahm sums, which allow for a better understanding of the knots-quivers correspondence. It has been known that there may be more than one quiver corresponding to the same knot (we declare such quivers as equivalent), and in this thesis we explore the conditions for two quivers to be equivalent. In principle it allows to find all equivalent quivers for a given knot recursively, starting from some initial quiver. The results of this part are also presented in [Jan+21].

Summing up, we obtain a few interesting results in relation to Nahm sums, in particular we show that quiver A-polynomials are quantizable and topological recursion reconstructs the associated Nahm sums for quivers. We also determine various equivalent quivers for a given knot and relate them to identities between their Nahm sums. These results also leave the scope for future research.

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