To: Academic Council of Physical Sciences, University of Warsaw

From: Taro Kimura, IMB/UBFC

Subject: Doctoral dissertation report for Mr. Helders Larraguível

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Summary This is a report of the doctoral dissertation entitled "A-polynomials, symmetries and permutohedra for quivers and knots" authored by Mr. Helders Larraguível. The dissertation demonstrates that various (apparently different) mathematical concepts are elegantly unified in the frame work of quantum field theory (QFT) and string/M theory. Firstly, the author explains a new approach to knot theory, called the knot-quiver correspondence, and clarifies an interesting combinatorial structure in representation theory of quiver, called the permutohedra, which potentially yields a novel topological knot invariant. Secondly, the author explains the notion of quiver A-polynomial, which provides a new insight on non-perturbative aspects of three dimensional supersymmetric (SUSY) gauge theory. These results have been published in internationally recognized academic journals. The dissertation clearly summarizes these new results together with a comprehensive review of the background materials.

Presentation In Chapter 1, the author provides a general introduction to the subject of the dissertation together with the outline. In particular, the author explains how three dimensional SUSY gauge theory and Chern–Simons theory interplay in the context of knot theory and related concepts in mathematics.

Chapter 2 is devoted to a review of the materials used in the later parts of the dissertation. Since the subjects discussed in the dissertation are quite broad, gauge theory, knot theory, quiver representation theory, algebraic and enumerative geometry, this review part plays an important role for the presentation of the dissertation. The author starts with three dimensional SUSY gauge theory, and discusses the partition function on the geometry $\mathbb{R}^2 \times S^1$, called the (K-theoretic) vortex partition function. This partition function plays a central role in the whole dissertation as the quiver generating series. The author then explains several aspects of knot theory with emphasis on its relation to Chern–Simons gauge theory. In particular, the author introduces various know invariants, Jones polynomial, HOMFLY-PT polynomial, Khovanov homology, superpolynomial, and A-polynomial. In fact, identifying the superpolynomial with the quiver generating function, one may arrive at the know-quiver correspondence, which is the main subject of this dissertation. The author also provides a physical argument for this correspondence form string/M theory perspective.

Chapter 3 is devoted to detailed study of the knot-quiver correspondence. Since, under this correspondence, one may obtain several different quivers for a single knot, it is quite important to elucidate the equivalence class for the quivers. Although its equivalence structure is not so simple, the author shows that it is systematically described by the combinatorial object that they call the permutohedra. The author discusses various examples, including torus knots, twist knots, and also the stand-alone examples, to verify the combinatorial structure among equivalent quivers together with the permutohedra.

In Chapter 4, extending the framework of the know-quiver correspondence, the author introduces the notion of quiver A-polynomial, which may provide a new algebraic/geometric characterization of three dimensional SUSY gauge theory. Starting with any quiver, the author establishes how to construct the corresponding A-polynomial based on the quiver generating series, where one may discuss the quantum/classical Nahm equation and its asymptotic behavior based on the WKB approach. The author discusses mainly the rank two examples to see how this formalism efficiently works.

In Chapter 5, the author provides conclusion and perspectives.

Novelty and contribution The author with the collaborators have established new concepts, permutohedra and quiver A-polynomial, on which the novelty of this dissertation relies. I expect that these concepts would further promote interplay between physics and mathematics, and thus they would be important contributions to the community. In addition, the dissertation is clearly written and self-contained. In particular, the review part of the dissertation would be an efficient introduction to intersecting area of various concepts in physics and mathematics. Also from this point of view, this dissertation makes a valuable contribution to the scientific community.

Suggestions I have a few suggestions/questions for the dissertation as listed below:

1. The author discusses the equivalence between quivers in the context of knot-quiver correspondence. Meanwhile, there is another type of equivalence class for quivers in the context of cluster algebra: quivers are "equivalent" if they are converted to each other under the operation called the mutation. In fact, the cluster algebra also appears in the study of knot theory. Hence, I am wondering if there is a potential link between them. I would suggest seeing similarity and difference between the combinatorial structure of quiver discussed in the dissertation and the cluster mutation.

2. I suspect that the process around eq. (4.1.7) can be interpreted as an inverse operation of surface defect insertion. Usually, in the context of four-dimensional $\mathcal{N} = 2$ gauge theory, we start with the algebraic curve (Seiberg–Witten curve) and then quantize it to obtain the quantum integrable system (quantum curve). The curve is parametrized as $\Sigma = \{(x, y) \in \mathbb{C} \times \mathbb{C}^\times \mid P(x, y) = 0\}$, so that there are only two variables. The defect insertion corresponds to considering the orbifold. For the four-dimensional U(N) gauge theory defined on $\mathbb{C} \times \mathbb{C}$, we consider the partial quotient space $\mathbb{C} \times (\mathbb{C}/\mathbb{Z}_N)$ for the full defect. In this case, the original (complexified) coupling constant splits into N fractional ones, $\mathfrak{q} = e^{2\pi i \tau} \to \mathfrak{q}_1 \cdots \mathfrak{q}_N$. Then, these fractional couplings are to be identified with the coordinates of the corresponding N-particle integrable system. See, e.g., Section 2.2.2 of https://arxiv.org/abs/1908.04928 for more details. Hence, the process to obtain the A-polynomial ((quantum) algebraic curve) seems like an inverse of the surface defect insertion explained above.

3. I suggest correcting typos in the manuscript. I would send a list separately.

Conclusion From all these points of view, I believe that the dissertation meets the following criteria:

1. The doctoral dissertation presents the candidate's general theoretical knowledge in a discipline or disciplines as well as the ability to independently conduct scientific work.

2. The subject of the doctoral dissertation is an original solution to a scientific problem or an original solution in the field of applying the results of own scientific research in the economic of social sphere.

Thus, I would like to provide the conclusion "positive" for the dissertation.

Declaration I do not have any conflict of interest in regard of this dissertation. I never worked with the author scientifically, nor I never worked in the same organization.

With best regards,

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