Topologically non-trivial black hole spacetimes

by

Maciej Ossowski

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy
Facultuy of Physics
Warsaw University

May, 2024

Abstract

Few exact solutions to the Einstein Equations have posed more difficulty in their physics interpretation than spacetimes containing the NUT parameter. Charles Misner famously said that the Taub-NUT spacetime, the simplest spacetime with the NUT parameter, is a counterexample to almost everything[1]. It is a direct and vacuum generalization of the Schwarzschild spacetime and yet is not asymptotically flat. It contains closed time-like curves close to the axis, is not geodesically complete, and contains two equivalent maximal analytic extensions[2]. All of those problems are tightly connected with the existence of a singular axis of rotation in spacetime. Usually, this singularity is associated with a conical singularity and subsequently with the existence of a cosmic (sometimes called Misner) string. Coincidently it was Misner, who proposed a non-singular interpretation, which changes the topology of the spacetime to the product of a 3-sphere and a real line. A crucial element of this interpretation is the introduction of a periodic time-like (in certain regions) coordinate. Consequently through every point of the so-called NUT region of spacetime passes a closed timelike curve, further inhibiting physical interpretation.

In this thesis, I seek to extend the Misner interpretation to a broad class of NUT solutions contained in the Plebański-Demiański solution. In particular, non-singular Misner-like spacetimes are constructed for generic spherical Type D black hole solutions, including the Kerr parameter, acceleration, the cosmological constant, and electromagnetic charges. It turns out that there always exist Killing Vector fields giving rise to a smooth space of the orbits and the $U(1)-$ PFB structure, from which the non-singular spacetime may be constructed. An immediate possible application is the construction of singularity-free non-homogeneous cosmological models.

NUT spacetimes are intrinsically connected with black hole horizons of the S^3 topology. Interestingly they split into two cases, depending on whether the $U(1)$ symmetry coincides with the horizon-generating Killing vector field. The Petrov Type D equation is a recently obtained tool in mathematical relativity allowing for obtaining the geometry of Isolated Horizon, a local generalization of the Killing horizon. In this work, a novel type of IH horizon, such that the $U(1)-$ symmetry is transversal to the null symmetry is proposed and its geometry is studied. Then the solutions to the Petrov Type D Equation on the Hopf bundle, corresponding to both of the above cases, are compared with the embedded horizon in non-singular (accelerated) Kerr-NUT-(anti-) de Sitter.

NUT spacetimes also include a generalization admitting a family of two-surface of vanishing and negative curvature. Upon compactification to a Riemann surface with a genus higher than 0, the spacetimes naturally admit a non-trivial bundle topology. A full family of Isolated Horizons of $U(1)-$ PFB structure over a compact Riemann surface will be constructed and studied via the Petrov Type D equation. Their embedding spacetimes have been found to be locally isometric to the generalized Taub-NUT-(anti-) de Sitter.

Streszczenie

Fizyczna interpretacja czasoprzestrzeni z parametrem NUTa przysparza problemów jak niewiele innych ścisłych rozwiązań Równań Einsteina. Czasoprzestrzeń Tauba-NUTa (najprostsza w tej rodzinie) została okrzyknięta przez Charlesa Misnera "kontrprzykładem do prawie wszystkiego" [1]. Chociaż są bezpośrednim i próżniowym uogólnieniem czasoprzestrzeni Schwarzwalda, to nie są asymptotycznie płaskie. Zawierają zamknięte i czasowe krzywe blisko osi symetrii orz dopuszczają dwa nierównoważne maksymalne rozszerzenia [2]. Problemy te związane są blisko związane z obecnością osobliwych osi rotacji w czasoprzestrzeni. Zwykle wiąże się je z osobliwością stożkową czasoprzestrzeni, a co za tym idzie z obecności kosmicznej struny (czasem zwanej też strun Misnera). Misner zaproponował interpretacje zmieniającą topologie czasoprzestrzeni do produktu trójwymiarowej sfery oraz prostej rzeczywistej. Kluczowym elementem jego interpretacji było wprowadzanie cyklicznej, czasowej (w zewnętrznym regionie czarnej dziury) współrzędnej. Skutkiem tego w tej interpretacji przez każdy punkt tak zwanych regionów NUTa przebiegają zamknięte krzywe czasowe, co jeszcze bardziej utrudnia fizyczną interpretacje.

W tej rozprawie rozszerzam nieosobliwą interpretacje Misnera do szerszej klasy rozwiązań z parametrem NUTa zawartych w rodzinie rozwiązań Plebańskiego-Demiańskiego, zawierających parametr Kerra, przyśpieszenie, stałą kosmologiczną oraz ładunki elektromagnetyczne. Okazuje się, że zawsze istnieje pole wektorowe Killinga posiadającące gładką przestrzeń orbit oraz indukujące w czasoprzestrzeni strukturą wiązki głównej z grupą $U(1)$. Składniki te są wykorzystane do konstrukcji nieosobliwych czasoprzestrzeni. Potencjalnym zastosowaniem powyżej konstrukcji są niejednorodne modele kosmologiczne.

Czasoprzestrzenie z parametrem NUTa są ściśle powiązane z horyzontami czarno-dziurowymi o topologii trójwymiarowej sfery. Co interesujące dzielą się one na dwa typy, w zależności od tego czy symetria $U(1)$ jest zerowa na horyzoncie. Równanie Typu D, będące od niedawna w arsenale relatywistów matematycznych, pozwala na uzyskanie geometrii Izolowanych Horyzontów, będących lokalnym uogólnieniem horyzontów Killinga. W niniejszej pracy, zostanie wprowadzony i zbadany nowy rodzaj Izolowanych Horyzontów posiadających symetrię $U(1)$ transwersalną do zerowej symetrii. Rozwiązania równania Type D Petrova zadane na horyzontach o powyższej strukturze będą porównane z horyzontami Killinga w (przyśpieszonych) czasoprzestrzeniach Kerra-NUTa-(anty-) de Sitter.

Czasoprzestrzenie z parametrem NUTa dopuszczają uogólnienie zawierającą wyróżnioną rodzinę dwu wymiarowych powierzchni o znikającej lub ujemnej krzywiźnie. Po kompak-

tyfikacji do przestrzeni Riemanna o niezerowym genusie czasoprzestrzenie te w naturalny sposób posiadają topologię nietrywialnej wiązki. Pełna rodzina Izolowanych Horyzontów o strukturze wiązki głównej z grupą $U(1)$ nad zwartymi powierzchniami Riemanna jest skonstruowana i badana dzięki rozwiązaniom równania Typu D. Ich przestrzenie zanurzające są znalezione, są one lokalnie izometryczne do uogólnionych czasoprzestrzeni Tauba-NUTa-(anty-) de Sittera.

Contents

Acknowledgments

It takes a village to write a Ph.D. dissertation;

I wish to thank my supervisor Jerzy Lewandowski. His scientific experience and expertise have been indispensable. He showed me the importance of craft in physics, and of humans in physicists.

I wish to thank my parents who are just the last link in my familial tree that led me to be here. Their support was crucial during my entire scientific journey, from my Bachelor, through the Master's and finally to the Ph.D studied.

I wish to thank Gosia, a true friend in virtue, in both life and dance. Without her, the road to finalizing this work would have been twice as long, and with double the number of dead ends.

I wish to thank my faculty friends and roommates: Andrzej, Eryk, Kacper, and Maciej. The time we have procrastinated on all the coffee and lunch breaks with them has been in fact time well spent on a discussion about our shared academic, and not only, experiences.

I wish to thank Pavel, Ivan, and David for their hospitality during my stay in Prague. This visit was mostly reinvigorating and allowed for completely new thoughts and perspectives to enter into this dissertation.

I wish to thank Karol, for his constant curiosity about the submission date which motivated me to finish, and Weronika, for the art discussions that made the last stages of writing more pleasant.

Last but not least I wish to thank Aurelia and Kobi, my four-legged friends. What they lack in understanding of theoretical physics they gain tenfold in common empathy.

ἀγεωμέτρηρος μηδεὶς εἰσἰτω μοῦ τὴν στέγην

Pseudo-Plato

1
Introduction

1.1 Notation and abbreviations

The following conventions will be assumed throughout this work unless stated otherwise.

The spacetime is a pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a four-dimensional smooth manifold and g is a Lorentzian metric tensor of the signature $(-+++)$. The Einstein summation convention is used universally whenever exactly two indices repeat, one lower and one upper.

Lowercase Greek letters will denote spacetime indices

$$\alpha, \beta, \ldots \mu, \nu, \ldots = 0, 1, 2, 3.$$

When discussing surfaces of codimension 1 and 2 such as horizons and their sections (as well as other) it useful to introduce also Latin indices

$$a, b, c \ldots = 1, 2, 3$$
$$A, B, C \ldots = 1, 2.$$

The following abbreviations will be employed throughout this work

- PFB - Principle Fibre Bundle
- KVF - Killing vector field
- IH - Isolated Horizons
- KNadS - Kerr-NUT-(anti-) de Sitter

This introductory Chapter serves as a physical and mathematical introduction to the theory of Isolated Horizons, the principal bundle structures naturally appearing in the spacetimes with NUT parameter, and finally to the Taub-NUT spacetime. On those three pillars, the results concerning the non-singular interpretation of the spacetimes with the NUT parameter and their horizons will be built.

It has to be stressed that the presence of the NUT parameter is the factor connecting the three above topics and is crucial for the constructions derived in this work. Unless stated otherwise it will be assumed that the NUT parameter is present.

1.2 Isolated horizons

The theory of Isolated Horizons (IH) is a powerful and local counterpart of the theory of Killing horizons and is crucial in a contemporary study of black hole horizons. Similarly to the Killing horizons, IHs are codimension 1 surfaces, however, their geometry may be described only in terms of local objects, i.e. given on the horizon. In particular existence of a spacetime Killing vector field is not necessary and so it is reasonable to consider them as abstract, non-embedded objects. In this approach, the geometry may still however be constrained by equations induced by the Einstein Equation. An example of such an equation is the so-called Type D equation, which will be introduced and solved in certain cases later.

A plethora of properties and theorems may be generalized from Killing horizons to Isolated ones. In particular the rigidity[3] and no-hair theorems hold[4,5]. The IHs allow for the distinction between the extremal and non-extremal horizons. Although the extremal horizons are not the main subject of the work it is worth pointing out that their investigation enjoys a great interest and specialized versions of the above properties have been discovered[6,7]. Their classification is fully known in dimension 4[8], together with various, more convenient in certain contexts, reparametrizations such as[9] and[10]. Type D extremal horizons additionally have a unique embedding into Plebański-Demiański spacetimes[11]. Einstein's Equations impose constraints on the geometry of extremal IHs in the form of the so-called Near-Horizon Equation. It has been studied in classical works such as[12,13] and[14], more recently also in[15,16]. Generalization of IHs to a higher dimension (with codimension 1) has been proposed and studied in[17]. Interestingly IHs additionally allow for a generalization of the black hole thermodynamics to the local context[18,19,20]. For a more general review of the IHs' properties, geometry, and applications, including numerical relativity, see the[12,18,21,22].

1.2.1 Unembedded isolated horizons

In this subsection, an entirely local approach to the theory of IH is presented, while the properties of IHs as hypersurfaces embedded in spacetime are presented in the next Section.

As a model for the horizons consider a manifold H endowed with:

(i) A degenerate metric tensor g of the signature $(0 + \ldots +)$,

(ii) A torsion-free covariant derivative ∇, that preserves the degenerate metric tensor

$$\nabla g = 0.$$

The triple (H, g, ∇) is called a *non-expanding horizon*. The degeneracy of g has two consequences; One of them is that ∇ can not be uniquely determined by g. The second is that ∇ may not exist at all unless it is true that for any null vector field ℓ that satisfies

$$g(\ell, \cdot) = 0,$$

generates a symmetry of g, namely that we have

$$\mathcal{L}_\ell g = 0. \tag{1.1}$$

The above property is the justification behind the name "non-expanding". Conversely, given g the latter condition is sufficient for a family of the covariant derivatives ∇ to exist. If any non-vanishing null vector field ℓ generates locally a symmetry of g, then so does $\ell' = f\ell$ for any function $f \in C^\infty(H)$. *The rotation* 1*-form potential* of a non-expanding horizon is a 1-form ω defined by a null vector field ℓ as follows,

$$\nabla_a \ell^b = \omega_a^{(\ell)} \ell^b. \tag{1.2}$$

It exists due to the fact, that the null direction has to be covariantly constant due to the metricity of ∇ [12]. The rotation 1-form is not an object intrinsic to the definition of the non-expanding horizon. If we rescale $\ell \mapsto f\ell$, then

$$\omega^{(f\ell)} = \omega^{(\ell)} + \mathrm{d} \log f. \tag{1.3}$$

The surface gravity $\kappa^{(\ell)}$ of an IH may be defined in a manner similar to a standard Killing horizon, the self-acceleration of ℓ

$$\nabla_\ell \ell = \kappa^{(\ell)} \ell \tag{1.4}$$

or equivalently

$$\kappa^{(\ell)} = \omega^{(\ell)}(\ell). \tag{1.5}$$

It depends on the choice of the representation of scaling function f

$$\kappa^{(f\ell)} = \kappa^{(\ell)} + \mathcal{L}_\ell \log f. \tag{1.6}$$

A 4-tuple: $(H, g, \nabla, [\ell])$ is called an *Isolated horizon* if (H, g, ∇) is a non-expanding horizon, and $[\ell]$ is an equivalence class of nowhere vanishing null vector fields defined by the rescaling by a constant factor

$$\ell \mapsto a_0 \ell, \quad a_0 \in \mathbb{R} \tag{1.7}$$

such that any representative of this class satisfies

$$[\mathcal{L}_\ell, \nabla] = 0. \tag{1.8}$$

If the above holds then the entire horizon connection ∇ is "time-independent", additionally if it holds for any representative of $[\ell]$ then it holds for the entire class. In the case of an isolated horizon From the above condition, it follows that the rotation potential 1-form satisfies

$$\mathcal{L}_\ell \omega^{(\ell)} = 0, \tag{1.9}$$

and is uniquely defined due to the restrictions to only constant scaling*. Consequently the superscript (ℓ) of $\omega^{(\ell)}$ may be dropped in this case.

If the isolated horizon were embedded in spacetime, then assumptions about the spacetime Riemann tensor would imply various constraints on g and ∇, which can be just introduced for the non-embedded isolated horizon as well. One of them is the $0-$th law of isolated horizon thermodynamics, namely

$$\kappa = \text{const}, \tag{1.10}$$

on every connected component of H. So far no equations restricting the geometry of an IH have been imposed, in particular, no equation being a restriction of the Einstein's Equation to the horizon holds yet. Indeed, in the embedded case it would be implied by Einstein's equations and energy inequalities, however here the constancy of the surface gravity has to be assumed. Therefore in the unembedded case, the zeroth law of IH therodynamics has to be assumed. Whenever $\kappa \neq 0$ an isolated horizon is called *non-extremal*. Similarily if $\kappa = 0$ such horizon is called *extremal*. Clearly we have $\ell(\kappa^{(\ell)}) = 0$.

The most common definition of an IH also includes an assumption concerning the topology, based on the most commonly studied spherical black holes. It is often required that it has a product topology[21]

$$H \approx S^2 \times \mathbb{R}.$$

This seems very restrictive on all three accounts: the horizon section being a 2-sphere, the null KVF having real lines as orbits and the product being trivial.

The non-spherical horizons, mostly in the toroidal case, are not considered to be of mainstream interest[23], however, they have been studied theoretically. In four-dimensional asymptotically flat spacetime with dominant energy condition satisfies, stationary black hole horizons necessarily have spherical topology[24]. This is no longer true if one, for example, includes a cosmological constant and such examples have been found[25]. Similarily the IHs of non-spherical sections have been studied[26]. The assumption that the orbits of Killing-generating KVF are lines is usually taken for granted as changing it to a circle would introduce a casual

*The above is a defining property of a structure in between non-expanding and isolated horizon, a so-called weakly isolated horizon. Weakly IH is a more stationary object than a non-expanding horizon, as it requires that ω is "time-independent" in addition to g

closed curve. Nevertheless, exactly those horizons seem to appear naturally in the presence of the NUT parameter, together with a non-trivial bundle structure.

Even if one assumes very little about the topology of IH, only that it is a 3-dimensional null surface with a preferred class of null KVFs and a connection it is possible to associate to it a *curvature scalar* K and *rotation pseudo scalar* Ω. To introduce them at a point $x \in H$, consider any 2-submanifold $S' \subset H$ transversal to ℓ at x. The curvature scalar $K(x)$ at x is the Gauss curvature of the Riemannian geometry induced on S' by the degenerate metric tensor g of H, and its value at x is independent of the choice of S' at x. The *rotation pseudo-scalar* $\Omega(x)$ is given by the pullback $d\omega^{S'}$ of $d\omega$ to S' and the 2-area form $\eta^{S'}$ of S'. It is defined as

$$d\omega^{S'} =: \Omega\, \eta^{S'}. \tag{1.11}$$

The value $\Omega(x)$ depends on the orientation of S', therefore in a neighborhood of x it is necessary to orient all the 2-surfaces consistently. To define Ω globally on H one needs a globally defined orientation of the 2-dimensional sections. Both of the functions K and Ω are constant along the null curves

$$\ell(K) = \ell(\Omega) = 0. \tag{1.12}$$

Note that even without Ω is an invariant even for non-expanding horizons when the ω 1-form is not defined uniquely, this follows directly from the transformation law 1.3.

1.2.2 Embededed horizon

The standard definition of an isolated horizon asserts that a horizon embedded in 4-dimensional spacetimes $(\mathcal{M}, g)$ is a null-submanifold H, with a future-oriented null normal vector field ℓ and that it satisfies [12]

1. H is diffeomorphic to a product $S^2 \times \mathbb{R}$ and the fibres of projection $\Pi : S^2 \times \mathbb{R} \to S^2$

2. The expansion of any vector field null at H and normal to it vanishes

3. Einstein's Equations hold on H with a stress-energy tensor $T_{\mu\nu}$ satisfying the dominant energy condition,i.e. $-T^{\mu}{}_{\nu}\ell^{\nu}$ is causal and future-directed on H.

The above set of conditions is not unique, equivalent conditions for some of the above defining properties are given in [21]. I will also relax the topological requirements similarly to the unembedded horizons. For an embedded horizon there is no need to make the horizon connection a part of a definition. A natural choice of the connection is induced by the spacetime connection. It may be introduced by

$$X^{\mu}\nabla_{\mu}Y_{\nu} := X^{\mu}\bar{\nabla}_{\mu}Y_{\nu}, \tag{1.13}$$

where $X, Y \in \mathfrak{X}(H)$ are tangent to the horizon and $\bar{\nabla}$ is the spacetime connection. It may be shown that the above preserves the tangent bundle of the horizon and is compatible with the Riemannian metric on the horizon. Therefore it is a good definition of the horizon connection. The extension to tensors of other valence follows easily[12] Connection defined in this way preserves the horizon tangent bundle and by metricity of $\bar{\nabla}$ it annihilates the horizon metric g. The remaining part of the horizon connection is then encoded in the 1-form ω given by (1.2). The surface gravity is defined in the same way as for the unembedded IHs. The difference is that now if the EEs hold on the horizon then

$$\nabla \kappa^{(\ell)} = \mathcal{L}_\ell \omega = 0, \tag{1.14}$$

and the zeroth law is given.

Next one may consider the Petrov type of the horizon, i.e. the algebraic properties of the Weyl tensor evaluated on the horizon. Due to ℓ being expansion and sheer-free (the second following from the Raychaudhuri equation) it must be one of the Weyls tensor's principal null directions. It may be shown that the double contraction of the Ricci tenor with ℓ vanishes, which implies that ℓ is a double principal null direction. Consequently, in a null tetrad

$$(\tilde{\ell}, n, \tilde{m}, \bar{\tilde{m}}), \tag{1.15}$$

where vector fields with a tilde are extensions of the frame defined on the horizon and the Weyl tensor is constant along H with respect to $\tilde{\ell}$ we have

$$\Psi_0 = \Psi_1 = 0 \text{ at the horizon}, \tag{1.16}$$

meaning that the horizon is necessarily algebraically special. The actual possible Petrov types are however limited to II, D, and O[5]. The sufficient condition for type O is vanishing of Ψ_2 which happens only if $K = \frac{1}{3}\Lambda$ and $\Omega = 0$. Otherwise, the Weyl tensor at a given point of the horizon is of Petrov type D if

$$2\Psi_3^2 - 3\Psi_2\Psi_4 = 0 \tag{1.17}$$

at this point, and type II if the equality does not hold.

1.2.3 Type D Equation

Now the Type D equation emerges from expressing Ψ_3 and Ψ_4 using the Newman-Penrose formalism in terms of Ψ_2. Further, expressing Newman-Penrose operators and coefficients in terms of m, $\bar{m}$ and the covariant derivative on S gives

$$\begin{aligned} &\Psi_2 = K + \mathrm{i}\Omega - \frac{\Lambda}{3}, \\ &\bar{m}^A \bar{m}^B \nabla_A \nabla_B \Psi_2^{-\frac{1}{3}} = 0, \end{aligned} \tag{1.18}$$

where it is assumed that $\Psi_2 \neq 0$.

Remark The Petrov D equation is a result of the spacetimes Einstein Equations imposed on the horizon geometry. However, the form (1.18) does not refer to the objects external to the horizon. Indeed, the Newman-Penrose formalism has been replaced by the co-frame and connection on the base space of the horizon, and the Weyl tensor coefficient Ψ_2 has been expressed in terms of scalar invariants of the horizon of a constant of the theory Λ. Therefore it makes sense to consider (1.18) for an unembedded horizon.

This strategy will be employed in Chapters 3 and 5. It turns out that the invariant Ψ_2 together with assumptions about the topology is enough to reconstruct the entire horizon structure. The horizon connection is further reconstructed from g^S and ω; Transport of vector tangent to S is given by the metric connection of g^S. Transport of the null vector is given by the rotation 1-form ω via its defining property (1.2).

This approach is even valid if the bundle action does not coincide with the null field ℓ. In such case to write the Equation out explicitly, consider again an arbitrary 2-dimensional submanifold $S' \subset H$ transversal to the null vector field ℓ. Introduce on it a null complex co-frame $m^{S'}_A$ $(A, B \in 1, 2)$ such that the induced metric tensor and the area 2-form read

$$g^{S'} = m^{S'}_A \bar{m}^{S'}_B + m^{S'}_B \bar{m}^{S'}_A, \quad \eta^{S'} = \mathrm{i}(\bar{m}^{S'}_A m^{S'}_B - \bar{m}^{S'}_B m^{S'}_A). \tag{1.19}$$

Using $(m^{S'}, \bar{m}^{S'})$ and $\nabla^{S'}$ one may still locally impose the Petrov type D equation. This approach is presented in Chapter 3.

Solutions of the Petrov type D equation may be used to generate further solutions. Suppose an isolated horizon H of the degenerate metric g and the rotation 1-form potential ω is a solution to the vacuum Petrov type D equation (1.18) with a cosmological constant Λ. Then so is H endowed with any

$$g' = \alpha^2 g, \quad \omega' = \omega, \quad \Lambda' = \frac{\Lambda}{\alpha^2}, \quad \alpha \in \mathbb{R} \setminus \{0\}. \tag{1.20}$$

An even wider example of a solution is given by g and ω if we require only that

$$K = \text{const}, \quad \Omega = \text{const} \tag{1.21}$$

such $\Psi_2 \neq 0$ for a proper type D. Then, an even stronger statement is true: every

$$g'' = \alpha^2 g, \quad \omega'' = \beta^2 \omega, \quad \alpha, \beta \in \mathbb{R} \setminus \{0\} \tag{1.22}$$

is another solution to (1.18) with an arbitrary cosmological constant

$$\Lambda'' = \gamma \Lambda, \quad \gamma \in \mathbb{R} \tag{1.23}$$

where again $\Psi_2 \neq 0$ is assumed.

1.3 Bundle structure of horizons and spacetime

1.3.1 Isolated horizon of the principal fiber bundle structure and null fibers

As was hinted in previous sections there is a natural way to associate a principal fiber bundle structure to any IH (and consequently for any Killing horizon). It may be seen as follows; The flow Φ_t of the KVF ℓ defines a local 1-parameter group of diffeomorphisms. By assumption, I will take ℓ to be complete, with a parameter along it taking values in the entire real line or circle, corresponding to the two connected structure groups, i.e. either $G = \mathbb{R}$ or $G = U(1)$. The flow of ℓ † generates the action on H by

$$H \times G \ni (p, t) \mapsto \Phi(p, t) := \Phi_t(p) \in H. \tag{1.24}$$

Because G is 1 dimensional and therefore abelian, the action is both left and right. The fibers H_x above any $x \in S$, are integral curves of ℓ and thus also the orbits of the group action. In summary, the action is free, transitive, and fiber-preserving and we conclude that the horizon H admits a structure of a principal fiber bundle over the space of null generators

$$G \hookrightarrow H \xrightarrow{\Pi} S \tag{1.25}$$

with a structure group G.

In this context, the 1-form ω can be associated not only with rotation of the horizon but also with a connection 1-form of the principal bundle $H \xrightarrow{\Pi} S$. This is easily seen as follows. On any principal bundle

$$G \hookrightarrow P \xrightarrow{\Pi} M \tag{1.26}$$

the vertical vectors Vect(T_pP) are distinguished and identified with the Lie algebra $\mathfrak{g}$ of the Lie group G. There is exactly one G-equivariant map from the structure group to a fiber above $p \in P$. Namely for any $p \in P$ we have the inclusion

$$\imath_p : G \ni g \mapsto p \cdot g \in G, \tag{1.27}$$

where $p \cdot g$ denote the right action of G on P. This defines the vertical vectors at p by

$$\mathrm{Vert}(T_pP) := T_e\imath_p(\mathfrak{g}).$$

The choice of the the horizontal vector (i.e. vector projecting into the base space tangent space) is non-canonical. The connection on a principal bundle is then choice of a complemented subspace Hor(T_pP) $\subset T_pP$

$$\mathrm{Vert}(T_pP) \oplus \mathrm{Hor}(T_pP) = T_pP,$$

† If horizon admits more Killing symmetries then it is possible to impose different bundle structures, with an action generated by a field transversal to ℓ. Such construction is presented in Chapter 4.

such that the choice in smooth in $p \in H$ and the subspace is G-equivariant with respect to the (right) group multiplication.

Alternatively the connection can be defined via a $\mathfrak{g}$-valued connection 1-form A[27], distinguishing the horizontal vectors as its annihilator

$$\ker A =: \mathrm{Hor}(T_pP),$$

such that the 1-form A satisfies

$$\begin{aligned} R_g^* A &= ad_{g^{-1}} \circ A \\ A(\xi) &= \xi^*. \end{aligned} \tag{1.28}$$

where $g \in G$, $\xi^* \in \mathfrak{g}$, and $\xi\big|_p := \imath_{p*}(\xi^*)$ is the corresponding fundamental vector field while the pullback of algebra valued 1-form is defined as $R_g^* A = A \circ R_{g*}$.

By the definition of the surface gravity, we have ω that $\omega(\ell) = \kappa$, which suggest a definition of a $\mathfrak{g}$-valued connection 1-form on the horizon H

$$A := \frac{\omega}{\kappa} \otimes \ell^*, \tag{1.29}$$

where $\ell^* \in \mathfrak{g}$ corresponds to the horizon developing field ℓ. The defining conditions remain to be checked. As $\mathfrak{g}$ is an abelian Lie algebra its adjoined representation is simply the identity $Ad_{g^{-1}} = Id_{\mathfrak{g}}$. The first equation is thus equivalent to A being right-invariant

$$R_g^* A = A. \tag{1.30}$$

However, the action of G on H is defined by the flow of ℓ and so this is precisely an integrated equivalent of (1.9). The second defining condition follows as

$$A(\ell) = \frac{\omega(\ell)}{\kappa} \otimes \ell^* = \ell^*. \tag{1.31}$$

The rotation 1-form may be pulled back to S by a local section $\sigma : S \to H$

$$\omega^\sigma = \sigma^* \omega. \tag{1.32}$$

This pullback is however section-dependent, and if the bundle is non-trivial, a family of different sections has to be used to cover the entire horizon. Still, the *rotation 2-form* $\mathrm{d}\omega$ provides a uniquely and globally defined 2-form on S that can be constructed using local sections

$$\tilde{\Omega}^S = \sigma^* d\omega. \tag{1.33}$$

If the horizon H is extremal, then the pullback

$$\omega^S := \sigma^* \omega \tag{1.34}$$

is independent of σ and the 1-form ω^S is globally defined on S. In that case, g^S and ω^S are insensitive to the non-trivial bundle structure.

Finally, the pseudo-scalar rotation invariant Ω is related to the curvature of the connection A. The curvature F of the principal bundle with a connection 1-form A is given by

$$F := \mathrm{d}A + [A, A], \tag{1.35}$$

where the exterior derivative acts only on the real valued 1-form and the commutation is taken in the Lie algebra. In this case the Lie algebra is 1-dimensional and so the commutator vanishes and we have

$$F = \frac{1}{\kappa}\mathrm{d}\omega \otimes \ell^* = \frac{1}{\kappa}\Omega\eta^S \otimes \ell^*, \tag{1.36}$$

where the last equality is due to (1.11). Clearly if the rotation invariant Ω vanishes then the horizon is $G-$flat.

Because the structure group is 1 dimensional the Lie algebra valued part of both connection and curvature will be omitted, bearing in mind that now the normalization condition $A(\ell) = 1$ is required.

If H admits the above bundle structure then S is simply the space of the null generators endowed with the Riemannian metric tensor g^S. The curvature scalar K becomes the Gaussian curvature of g^S. Moreover, if S is compact, as will be assumed throughout this entire work, then two more invariant may be defined. First

$$\int_S K\eta^S = 2\pi\chi_E(S), \tag{1.37}$$

where χ_E is the Euler characteristics of (S, g^S) while

$$\int_S \Omega\eta^S = 2\pi\kappa\chi_C(H), \tag{1.38}$$

where χ_C is the Chern number of the bundle H, and κ is the surface gravity introduced above.

1.3.2 Non-trivial $U(1)$-bundles over spheres

In the case of a $U(1)$-PFB over a two-dimensional sphere all non-trivial bundles may be constructed explicitly using coordinates. To do so introduce an open cover $\{V_1, V_2\}$ of S^2 together with a transition function $a_{12} : V_1 \cap V_2 \to U(1)$ defined up to gauge functions b_i

$$a_{12} \mapsto a'_{12} = b_1^{-1}a_{12}b_2, \quad b_i : V_i \to U(1), i = 1, 2. \tag{1.39}$$

A $U(1)-$ connection may be introduced by by two 1-forms $A^{S^2}_{(1)}$ and $A^{S^2}_{(2)}$ defined on V_1 and V_2, respectively, such that the following gauge transformation is satisfied

$$A^{S^2}_{(2)} = A^{S^2}_{(1)} - ia_{12}^{-1}da_{12} \;\; \text{on} \;\; V_1 \cap V_2. \tag{1.40}$$

A common choice of the cover sets are two open discs of radius $\pi - \epsilon$, centered around either of the poles

$$V_1 = S^1 \times [0, \pi - \epsilon[, \quad U_1 = S^1 \times]\epsilon, \pi]. \tag{1.41}$$

Their intersection is an open "band" around the equator

$$V_1 \cap V_2 = S^1 \times]\epsilon, \pi - \epsilon[. \tag{1.42}$$

The transition function may defined on $V_1 \cap V_2$ as

$$a_{12}(\theta, \phi) := \exp(-\mathrm{i}\chi_C \phi), \tag{1.43}$$

where $\chi_C \in \mathbb{Z}$ will be used instead of $\chi_C(H)$ for brevity's sake. To construct the local connection 1-form $A^{S^2}_{(1)}$ it is enough to consider the following 1-form

$$A^{S^2}_{(1)} = -\frac{\chi_C}{2}(\cos\theta - 1)\mathrm{d}\varphi = \chi_C \sin^2\frac{\theta}{2}\mathrm{d}\varphi, \tag{1.44}$$

while the second follows from the above-defined transition function.

$$A^{S^2}_{(2)} = -\frac{\chi_C}{2}(\cos\theta + 1)\mathrm{d}\varphi = -\chi_C \cos^2\frac{\theta}{2}\mathrm{d}\varphi. \tag{1.45}$$

The associated $U(1)-$curvature (more precisely its pullback to S reads

$$F := \mathrm{d}A^{S^2}_{(1)} = \frac{\chi_C}{2}\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi \tag{1.46}$$

Following the definition of the Chern number justifies the choice of the name of the integer parameter χ_C appearing in the transition function a_{12}

$$\int_{S^2} F = 2\pi\chi_C. \tag{1.47}$$

By an introduction of a coordinate $\psi \in [0, 2\pi[$ on a local trivialization of the PFB it is possible to express the $U(1)-$connection (as opposed to its local pullback) as

$$A = \mathrm{d}\psi - \frac{\chi_C}{2}(\cos\theta - 1)\mathrm{d}\varphi = \mathrm{d}\psi' - \frac{\chi_C}{2}(\cos\theta' + 1)\mathrm{d}\varphi', \tag{1.48}$$

where (ψ', θ', ϕ') are coordinates on another trivialization of the PFB with the transition map

$$\psi' = \psi + \chi_c\varphi, \quad \varphi' = \varphi, \quad \theta' = \theta. \tag{1.49}$$

It turns out that all $U(1)$- PFB over a two-dimensional sphere may be obtained by a suitable gluing of two solid tori into so-called "lens space". Consider a pair of 3 dimensional manifolds T^2_D and T'^2_D with coordinates $(\psi, \varphi, \theta) \in S^1 \times S^1 \times [0; \frac{\pi}{2}]$ and $(\psi', \varphi', \theta') \in S^1 \times S^1 \times [\frac{\pi}{2}, \pi]$.

T_S^2 and T'^2_S are topologically solid torii with a boundary being a two torus parametrized by either (t, φ) or (t', φ') and may be viewed as local trivializations of a given PFB over S^2. To obtain a PFB one should glue them together along their boundaries.

Consider a map from ∂T_D^2 to $\partial T'^2_D$

$$f_{p,q} : \partial T_S^2 \to \partial T'^2_S \tag{1.50}$$

such that the meridian of the first torus is mapped to the circle with the slope of p/q on the second torus, with $q = 0$ corresponding to the longitude of the torus. For $f_{p,q}$ to be a homeomorphism it is necessary that $\frac{p}{q} \in \mathbb{Q}$, otherwise the image of a meridian would wind densely $\partial T'^2_S$. Therefore it is possible to assume that $p, q \in \mathbb{Z}$.

Gluing of the boundary given by (1.50) This is equivalent to the coordinate transformation

$$\psi \mapsto q\psi + p\phi. \tag{1.51}$$

The three-manifold obtained the boundaries with the above map is called the $(p, q)-$lens space and denoted $L(p, q)$. The simplest case corresponds to the trivial circle bundle over the sphere, i.e. $L(0, 1) = S^1 \times S^2$. Comparing the above transformation with (1.49) it is clear that $U(1)-$PFB over S^2 are topologically lens spaces $L(\chi_C, 1)$

1.4 Taub-NUT solution

Taub-NUT solution is the simplest spacetime with the NUT parameter. Careful understanding of its properties, especially in the interpretation proposed by Charles Misner will be crucial for understanding the extension of the method to spacetimes containing more parameters presented in the next Chapters. In a sense understanding Taub-NUT spacetime and its PFB structure is almost all that is required and serves as the universal model of the method.

Taub-NUT spacetime is probably one of the most enigmatic exact vacuum solutions to Einstein's Equations in four dimensions. It was first discovered by Taub who solved the vacuum Einstein equations under the assumptions of spatial homogeneity i.e. existence of a 3-dimensional group of spacial symmetries, the existence of time translation, and the lack of singularities in spacetime. The Taub-NUT spacetime is obtained assuming that the algebra of generators of spacial symmetries is of Bianchi type IX, meaning the corresponding group of symmetries is universally covered by $SU(2)$ [28]. In this context, one common interpretation of Taub-NUT spacetimes is the so-called Taub universe. It is a homogeneous cosmological model that begins and ends without a curvature singularity and may be construed as a region between two non-singular horizons in the full Taub-NUT spacetime [29]. An extension of the Taub universe will be elaborated in Section (2.3.1).

Twelve years later the solution was rediscovered by Newman, Unti, and Tamburino, after whom the acronym NUT was coined, as a wider family of spacetimes containing the Schwarzschild spacetime [30] as a sub-case. They used the Newman-Penrose formalism with an assumption that the metric admits a twisting and non-shearing geodesic congruence with non-zero divergence and that the spacetime is of Petrov Type D. The resulting spacetime was an early candidate for the description of the exterior region of the rotation black hole, as the geodesic congruence has a constant but non-zero twist, with Schwarzschild spacetime being the zero twist limit. Unfortunately, Newman, Unti, and Tamburino have additionally assumed the vanishing of two spin coefficients, which are non-zero for the Kerr metric and were not able to derive the metric of a rotating black hole. Interestingly already present in the original paper is a parameter governing the sign (or vanishing) of a curvature of two-dimensional surfaces foliating the spacetime. To the spherical case of positive curvature is devoted the rest of the Introduction with the continuation in Chapters 2, 3 and 4 The Taub-NUT-(anti-) de Sitter spacetimes with two-dimensional compacted surfaces of genus higher than zero and their horizons will be revisited in Chapter 5. It was quickly realized that the two solutions are regions of one spacetime called subsequently the (spherical) Taub-NUT spacetime.

The basic idea of the research program behind this dissertation is already present in Charles Misner's seminal 1963 paper: "The Flatter Regions of Newman, Unti, and Tamburino's Generalized Schwarzschild Space". In this work, Misner proposed a construction with time coordinate periodically identified, which in turn in means that the topology of the surfaces of constant radius changes to a three-dimensional sphere, introducing the Hopf bundle structure.

The metric tensor of Taub-NUT spacetime in Schwarzschild-like coordinates reads

$$g_{TN} = -f(r)(\mathrm{d}t_0 + 2l\cos\theta\mathrm{d}\varphi)^2 + \frac{\mathrm{d}r^2}{f(r)} + (l^2 + r^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \qquad (1.52)$$

where

$$f(r) := \frac{r^2 - 2Mr - l^2}{r^2 + l^2}. \qquad (1.53)$$

In the above coordinates the metric tensor has a well-defined limit to the Schwarzschild metric by taking $l \to 0$. It shares with the Schwarzschild metric existence of a four-dimensional group of symmetries: time translation (or boost symmetry if the KVF is spacelike) generated by ∂_{t_0} and the spherical symmetries. Apart from the rotational symmetry, the full spherical symmetries may be non-obvious because of the appearance of the term $2l\cos\theta d\phi$. Then the spherical symmetry is to be understood as an existence of a 3-dimensional subalgebra of symmetries, isomorphic to $\mathfrak{so}(3)$. Contrary to Schwarzschild spacetime it may be argued that the infinitesimal symmetries generate $SU(2)$ with the orbits being 3-spheres instead of $SO(3)$ with 2-spheres orbits[31]. The algebra is explicitly exhibited later in this Chapter.

The hypersurfaces of $r = r_H$, where $f(r_H) = 0$ are Killing horizons generated by the Killing vector field $\xi = \partial_{t_0}$. One can easily calculate $r_H = M \pm \sqrt{M^2 + l^2}$. For $l \neq 0$ two distinct Killing horizons always exist and both of them are necessarily non-extremal. This changes once additional parameters such as Kerr rotation a, acceleration α, cosmological constant Λ and electric e or magnetic g charges are introduced. In the limit, $l \to 0$ one of the horizons is sent in to Schwarzschild curvature singularity at $r = 0$, while the other becomes the Schwarzschild horizon.

The coordinates in which the metric tensor (1.52) is expressed in unsurprisingly are ill-defined at the horizon $r = r_H$. This is easily solved by employing the Eddington–Finkelstein-like co-frame transformations

$$\begin{aligned} \mathrm{d}v :=& \mathrm{d}t + \frac{1}{f(r)}\mathrm{d}r, \quad \text{or} \\ \mathrm{d}u :=& \mathrm{d}t - \frac{1}{f(r)}\mathrm{d}r, \end{aligned} \tag{1.54}$$

which may be used to expresses the metric tensor in terms of the in-going coordinate v or the out-going coordinate u, covering different parts of the horizon, as

$$\begin{aligned} g_{TN} =& - f(r)(\mathrm{d}v + 2l\cos\theta \mathrm{d}\varphi)^2 + 2\mathrm{d}r(\mathrm{d}v + 2l\cos\theta\mathrm{d}\varphi) \\ &+ (l^2 + r^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \\ g_{TN} =& - f(r)(\mathrm{d}u + 2l\cos\theta \mathrm{d}\varphi)^2 - 2\mathrm{d}r(\mathrm{d}v + 2l\cos\theta\mathrm{d}\varphi) \\ &+ (l^2 + r^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \end{aligned} \tag{1.55}$$

1.4.1 Singularities of the Taub-NUT spacetime

One noticeable difference between the Taub-NUT and the Schwarzschild spacetimes is the complete lack of curvature singularities. This may be seen through an examination of the metric tensor; all of the coefficients are smooth at $r = 0$. In fact if one uses coordinates extending through the horizon such as (1.55), the metric tensor is analytic everywhere. Computation of some of the scalar invariants such as the Kretschmann scalar

$$K = \frac{48}{(l^2+r^2)^6}(l^4 + l^3M - 3l^3r + 3l^2Mr - 3r^2l^2 - 3lM\,r^2 + l\,r^3 - M\,r^3) \tag{1.56}$$

$$(l^4 - l^3M + 3l^3r + 3l^2Mr - 3r^2l^2 + 3lM\,r^2 - l\,r^3 - M\,r^3), \tag{1.57}$$

the determinant of the metric

$$\det(g_{TN})|_{r=0} = l^4\cos(2\theta), \tag{1.58}$$

or the only non-vanishing component of Weyl

$$\Psi_2 = -\frac{M + \mathrm{i}l}{(r + \mathrm{i}l)^3} \tag{1.59}$$

confirms this claim. In all examples, it may be observed that the NUT parameter "shifts the radial coordinates away from the singularity". It is therefore natural to extend the range of the radial coordinate to $r \in \mathbb{R}$ and it is not unreasonable to study also the horizon with $r_H = M - \sqrt{M^2 + l^2} < 0$ (assuming standard convention that $M > 0$).

So far the Taub-NUT metric seems to be a simple extension of the Schwarzschild metric with the benefit of regularising the curvature singularity $r = 0$. Examining the hypersurfaces $r = \mathrm{const}$ reveals many differences. In a region such that $f(r) \neq 0$ the induced metric tensor reads

$$\mathrm{d}s^2 = -f(r)(\mathrm{d}t_0 + 2l\cos\theta\mathrm{d}\phi)^2 + (l^2 + r^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{1.60}$$

If the Taub-NUT spacetime is to be interpreted as a black hole solution and an extension of the Schwarzschild spacetime it is reasonable to expect that the coordinates (θ, ϕ) are defined on a topological two-sphere. However, the metric tensor induced on the surface of $t =$const, $r =$const cannot be smoothly defined.

To see this consider the two sets of points that would correspond to the poles of the said sphere, $\{(\theta = 0, \phi)\}$ and $\{(\theta = \pi, \phi)\}$. On a two-sphere, the distance between any two points in the same set would have to be zero as in fact, there would only be one point in each of them. It is easy to see that the length of the circles at $\theta = 0$ and $\theta = \pi$ does not vanish, but is equal to $4l\pi$ instead. One possibility is that the underlying manifold is a cylinder[32] This conclusion, although it is dictated by the geometry of the two-surface of constant r and t is physically quite unsatisfactory.

One is always allowed to insist on the spherical topology, however, then the metric tensor cannot be continuous on the entire two-sphere. Consider the 1-form $\omega_0 = \mathrm{d}t_0 + 2l\cos\theta\mathrm{d}\phi$ appearing in the Taub-NUT metric tensor. In cartesian coordinates around $\theta = 0$ its angular part reads

$$\cos\theta\mathrm{d}\phi = \sqrt{1 - x^2 - y^2}\frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2},$$

and clearly diverges at the pole $\theta = 0$, corresponding to $x = 0 = y$. A similar result holds for $\theta = \pi$. Therefore this approach requires that we accept that there exists a singularity along the axis and requires an interpretation explaining its origin. This singularity is most commonly referred to as the Misner or cosmic string.

A new look at the Taub-NUT spacetimes comes as a result of considering the following family of coordinate transformations

$$t_C = t_0 + 2lC\phi, \quad \phi_C = \phi, \tag{1.61}$$

defined for any real parameter C. Two particular choices stand out, namely $C = \pm 1$, as they regularize part of the ω_0 which transform as follows

$$\begin{aligned} t_+ &= t_0 + 2l\phi, \quad \omega_+ = \mathrm{d}t_+ - 4l\sin^2\tfrac{\theta}{2}\mathrm{d}\phi_+, \\ t_- &= t_0 - 2l\phi, \quad \omega_- = \mathrm{d}t_- + 4l\cos^2\tfrac{\theta}{2}\mathrm{d}\phi_-, \end{aligned} \tag{1.62}$$

where $+$ and $-$ are used as a shorthand for $C = 1$ and $C = -1$. The new ω's appearing in the metric tensor instead of ω are now smooth at $\theta = \pi$ and $\theta = 0$, respectively. Consequently, the metric tensor induced on two-surfaces now is smooth around the respective poles. ‡ However, no choice of C regularizes both poles at the same time. This particular type of singularity, where no real curvature singularity takes place and instead may be partially regularized by the change of the coordinates is often referred to as a quasi-regular singularity[33,34]. The possibility of changing the location of the Misner string is analogous to the Dirac string for the magnetic monopole and is one of the reasons why the NUT parameter is sometimes referred to as magnetic mass, gravitomagnetic monopole moment or simply dual mass §.

It has to be said that although (1.62) is a local diffeomorphism, the global properties of the obtained spacetime are different. If the coordinate ϕ_0 was assumed to be periodic and parameterize the closed orbits of the KVF ∂_{ϕ_0} then this is no longer true about any other ϕ_C. Consider the KVF after the coordinate change

$$\partial_{\phi_C} = \partial_{\phi_0} - 2lC\partial_{t_C}. \tag{1.63}$$

Clearly if orbits of ∂_{ϕ_0} were assumed to be closed and the orbits of ∂_{t_0} were not, the orbits of ∂_{ϕ_C} do not close but instead spiral. The transformation (1.61) requires different periodic identifications of the angular coordinate for different choices of C.

The above is enough to present the first group of interpretations started by Bonnor in his 1969 paper[36]. His approach was to analyze the asymptotic behavior of the Misner string in the linearized gravity regime. Led by this analysis he proposed that the spacetime be interpreted as a superposition of Schwarzschild black hole and a massless source of the angular momentum located along half the singular axis (i.e. $C = -1$). More recent interpretations[37] prefer the symmetric location of the Misner string, i.e. $C = 0$. Manko and Ruiz[38] posit that the parts of the Misner string laying inside the horizon form a static rod of a positive mass, while outside of the horizon $r_H = M + \sqrt{M^2 + l^2}$ to be semi-infinite rotating rods of negative mass. As the outside rods rotate in opposite directions the overall angular momentum vanishes.

A phenomenon connected with the singular axis is the existence of the Closed Timelike Curves around the axis. Consider curves parametrized by ϕ_0 with the line element

$$\mathrm{d}s^2 = \big(-f(r)(2l\cos\theta)^2 + (l^2 + r^2)\sin^2\theta\big)\mathrm{d}\phi_\circ^2,$$

Its signature becomes timelike in the region where $f(r) > 0$ and sufficiently close to the axis $\theta = 0$ or $\theta = \pi$. Coordinate transformations changing the position of the Misner string also get rid of the closed timelike curves around one of the half-axis.

It is worth noting that not all authors consider the singular axis a defect of the spacetime to be regularised. Instead, the existence of the conical singularity leads to an additional thermodynamical charge associated with the black hole horizon[39,40]. Interestingly one could consider

‡Following a similar discussion as for the coordinate t_0 it would seem that the geometry would suggest a topology of on half-sphere for the said two-surface.

§The name "dual mass" hs further justification. For asymptotically flat spacetimes the mass can be computed as a charge associated with the Killing vector ∂_t. Then the NUT parameter is the dual charge, similar to electric-magnetic charge duality see e.g.[35]

the singular axis as a limit Killing horizon. So long as the appropriate Misner string is present i.e. $C \neq 1$ or $C \neq -1$, the following two Killing vector field are well defined

$$\xi_{\pm} = \partial_t + \frac{1}{2l(1 \pm C)}\partial_\phi. \tag{1.64}$$

Their norm vanishes at the Misner string's northern or southern part respectively. Then it has been shown that one may associate a thermodynamic potential associated with the surface gravity of a tube surrounding the strings, which leads to consistent thermodynamics of the Taub-NUT spacetime, accounting for the NUT parameter in the suitably modified first law of black hole thermodynamics[41,42,37].

1.4.2 Misner's interpretation

Misner's interpretation of the Taub-NUT spacetime may be summarized in the requirement that the coordinates covering smoothly both poles, i.e. with $C = 1$ and $C = -1$ coexist simultaneously in the spacetime. For this to be possible we must have the following relationship between the t coordinates

$$t_- = t_+ + 4l\phi_+, \quad t_+ = t_- + 4l\phi_- \tag{1.65}$$

If both ϕ_+ and ϕ_- are to be periodic consequently the same must be true about t_+ and t_-. This may be seen by examining the corresponding KVFs given by (1.63), i.e. if orbits of ∂_{ϕ_+} and ∂_{ϕ_-} are closed then the same must be true of orbits of ∂_{t_+} and ∂_{t_-}. The actual periodicity of any t coordinate may be deduced from (1.65) as

$$t \in \left[0, \frac{8\pi l}{m}\right), \quad m \in \mathbb{N} \setminus \{0\}, \tag{1.66}$$

with the reasoning that for every full rotation of any 2π-periodic ϕ the t coordinates has to also complete an entire number of rotations. ¶ Locally, i.e. at the level of coordinates the choice of $m \neq 0$ is irrelevant and could be gotten rid of by scaling of t's. Different choices of m lead however to different global topologies of the $U(1)$-PFB structure of the spacetime. From local conditions given by (1.65), it is yet not clear how the global topology of surfaces of r =const has changed. For $m = 1$ Misner used the fact that the patches described by coordinates $(t_-, \theta, \phi_-) \in S^1 \times [0, \frac{\pi}{2}) \times S^1$ and $(t_+, \theta, \phi_+) \in S^1 \times [\frac{\pi}{2}, \pi) \times S^1$ are cell decompositions of the three-manifold. By calculating the fundamental group and using the Poincare conjecture ‖ he concluded that the manifold is a three-sphere. The two patches in the cell decomposition are local trivializations of the $U(1)-$PFB described in the previous Section.

¶In principle the ratio of periods could be rational. This choice does not however correspond to a $U(1)$-PFB structure.

‖Interestingly it was proven almost 40 years later.

Before exhibiting explicitly the topology and bundle structure let's turn to the group structure that may be imposed on S^3. Consider the Lie group $SU(2)$ of the unitary 2×2 matrices with the determinant equal 1. It shares the lie algebra with the group of rotations of three-dimensional Euclidean space $SO(3)$, of which it is a double cover. To that $SU(2)$ is isomorphic to S^3 is immediate from the observations that

$$SU(2) = \left\{ \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} : z\bar{z} + w\bar{w} = 1 \right\} \tag{1.67}$$

defines three-sphere in $\mathbb{C}^2 \cong \mathbb{R}^4$. It allows us to endow a three-dimensional sphere with a group structure. To construct the left and right invariant fields and 1-forms on S^3 consider a group Q of quaternions with a zero quaternion removed

$$Q = \left\{ q = w + ix + jy + kz : w, x, y, z \in \mathbb{R}\,,\ w^2 + x^2 + y^2 + z^2 \neq 0 \right\}, \tag{1.68}$$

where

$$i^2 = j^2 = k^2 = ijk = -1. \tag{1.69}$$

Now Q is isomorphic to a Lie group $S^3 \times \mathbb{R}$. The left-invariant vector fields X_i and 1-forms ω_i, $i = 0, 1, 2, 3$ are defined by the left quaternion multiplication

$$L_* X_i = X_i, \quad L^* \omega_i = \omega_i, \tag{1.70}$$

the analogous definition follows for right invariant objects. Their advantage comes from the fact that they are global, well-defined objects on the Lie group. The quaternions allow for a convenient parameterization in terms of appropriately extended Euler angles (also called Hopf coordinates)

$$\mathcal{Q} \ni q = w + ix + jy + kz = \exp\frac{r}{2} \exp\frac{k\phi}{2} \exp\frac{i\phi}{2} \exp\frac{k\psi}{2}, \tag{1.71}$$

where $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$ and $\psi \in [0, 4\pi)$.

In these coordinates the left and right invariant vector fields on S^3 respectively, read[43]

$$\begin{aligned}
X_1 &= \cos\psi \partial_\theta + \frac{\sin\psi}{\sin\theta}\partial_\phi - \frac{\sin\psi}{\tan\theta}\partial_\psi, \\
X_2 &= -\sin\psi \partial_\theta + \frac{\cos\psi}{\sin\theta}\partial_\phi - \frac{\cos\psi}{\tan\theta}\partial_\psi, \\
X_3 &= \partial_\psi. \\
Y_1 &= \cos\phi \partial_\theta - \frac{\sin\phi}{\tan\theta}\partial_\phi + \frac{\sin\phi}{\sin\theta}\partial_\psi, \\
Y_2 &= -\sin\phi \partial_\theta + \frac{\cos\phi}{\tan\theta}\partial_\phi - \frac{\cos\phi}{\sin\theta}\partial_\psi, \\
Y_3 &= \partial_\phi.
\end{aligned} \tag{1.72}$$

satisfying

$$
\begin{aligned}
[X_i, X_j] &= \epsilon_{ij}{}^k X_k, \\
[Y_i, Y_j] &= -\epsilon_{ij}{}^k Y_k, \\
[X_i, Y_j] &= 0,
\end{aligned}
\tag{1.73}
$$

with dual left-invariant 1-forms

$$
\begin{aligned}
\omega^1 &= \cos\psi \mathrm{d}\theta + \sin\theta \sin\psi \mathrm{d}\phi \\
\omega^2 &= -\sin\psi \mathrm{d}\theta + \sin\theta \cos\psi \mathrm{d}\phi \\
\omega^3 &= \mathrm{d}\psi + \cos\theta \mathrm{d}\phi.
\end{aligned}
\tag{1.74}
$$

Crucially the vector fields Y_1, Y_2, Y_3 and X_3 are Killing vector fields of the Taub-NUT metric if we set $t_0 = 2l\psi$. Using the left-invariant 1-forms the metric tensor reads

$$
g = -f(r) 4l^2 (\omega^3)^2 + \frac{\mathrm{d}r^2}{f(r)} + (l^2 + r^2)\left((\omega^1)^2 + (\omega^2)^2\right). \tag{1.75}
$$

It is well defined on the manifold $S^3 \times \mathbb{R}$. The metric induced on S^3 is not a flat S^3 metric, instead, the $(\omega^3)^2$ term is perturbed. As a result, it does not admit the full six-dimensional group of the symmetries.

In the so-called *Taub region*, defined by $f(r) < 0$, we have that $g(\partial_\psi, \partial_\psi) = -4l^2 f(r) > 0$. Thus the S^3 metric is spacelike and such surfaces could represent a foliation by a constant time surfaces in a cosmological model. The so-called *NUT* region, defined by $f(r) > 0$, splits into two parts, NUT_+ for $r > r_+$ and NUT_- for $r < r_-$, where r_+ and r_- are respectively the larger and smaller roots of $f(r)$. Crucially in the NUT_+ region, usually taken to represent an exterior region of a black hole, the metric on S^3 is of mixed signature. Misner's interpretation therefore worsens the causal structure of the spacetime, with the Closed Timelike Curves parametrized by t going through any point of the NUT regions.

Consider the Taub-NUT metric in the coordinates (v_-, r, θ, ϕ_-) well defined on the horizon, where v_- is defined by Eddinton-Finkelstein transformation of t_- coordinate. The Killing horizon at r_H, where $f(r_H) = 0$ is generated by the field ∂_{v_-}, or its rescaling by any real constant. If ℓ generates the $U(1)$-group acting on the horizon, then the scaling may be fixed by requiring that the fiber coordinate ranges from 0 to 2π. In this sense, it is natural to take the horizon-generating KVF to be

$$
\ell = \frac{4l}{m} \partial_{v_-}, \tag{1.76}
$$

and introduce the fiber coordinate

$$
\psi := \frac{m}{4l} v_- \in [0, 2\pi). \tag{1.77}
$$

For this choice of scaling the horizon rotation 1-form reads

$$
\omega = \kappa(\mathrm{d}\psi + m(\cos\theta - 1)\,\mathrm{d}\phi_-), \tag{1.78}
$$

where the surface gravity reads

$$\kappa = \frac{2lf'(r_h)}{m} = \frac{1-(l^2+r_H^2)\Lambda}{r_H}\frac{2l}{m}. \tag{1.79}$$

Comparing with the $U(1)-$connection given by (1.48) it is clear that the rotation 1-form ω is proportional to the connection 1-form

$$\omega = \kappa A, \tag{1.80}$$

where A is defined by (1.48), and the integer m is equal to the Chern number χ_C. In conclusion, the horizons of Taub-NUT spacetime with the Misner identification admit the structure of a $U(1)$-PFB over the space of the null generators, diffeomorphic to a two-sphere, with the $U(1)-$ action generated by the horizon generating KVF. The total space is then a lens space $L(\chi_C, 1)$

$$U(1) \hookrightarrow L(\chi_C, 1) \xrightarrow{\pi} S^2. \tag{1.81}$$

The PFB structure may be extended to the entire spacetime, however, now the $U(1)$-connection is given by

$$A_{\text{ST}} := \frac{g(\ell,\cdot)}{g(\ell,\ell)}. \tag{1.82}$$

It reads

$$A_{\text{ST}} = \mathrm{d}\psi + m\left(\cos\theta - 1\right)\mathrm{d}\phi_- + \frac{m}{4lf(r)}\mathrm{d}r, \tag{1.83}$$

by the above definition, the connection 1-form ceases to be well defined at the horizon, as the normal of ℓ vanishes there. A transformed connection 1-form may be introduced as

$$A'_{\text{ST}} = A_{\text{ST}} + \frac{m}{4lf(r)}\mathrm{d}r. \tag{1.84}$$

Its pullback to the horizon agrees with the $U(1)-$connection defined thereon. Note that A_{ST} and A'_{ST} differ only closed 1-form on the base space $S^2 \times \mathbb{R}$ of bundle extended by a real line. Therefore they are equivalent up to a bundle automorphism, which however may not preserve the form of the metric tensor. The Taub-NUT metric tensor may be subsequently expressed in terms of the 1-form defined on the $U(1)-$PFB extended by a real line

$$U(1) \hookrightarrow L(\chi_C, 1) \times \mathbb{R} \xrightarrow{\Pi} S \times \mathbb{R}, \tag{1.85}$$

as

$$g_{\text{TN}} = -\left(\frac{2l}{\chi_C}\right)^2 f(r)A_{\text{ST}}^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2+l^2)((\omega^1)^2 + (\omega^2)^2). \tag{1.86}$$

Finally, a caveat of Misner's interpretation has to be mentioned. It has been shown that if the spacetime is extended simultaneously using the ingoing and outgoing Finkelstein-Eddingtion transformation, forming a Taub-NUT equivalent of the Kruskal-Szekeres coordinates then the resulting spacetime is not Haussdorf[2]. This is a consequence of infinitely spiraling geodesics around the singular axis. Consequently, either the extension is done using both ingoing and outgoing transformation and the resulting spacetime is non-Hausdorf, or only the ingoing or outgoing extension is applied, and the resulting spacetime is not geodesically complete. In a sense, this situation is not worse than in Taub-NUT spacetime with singular axes. Then the spacetime is still geodesically incomplete as the geodesics approaching the discontinuity at the axis of rotation must terminate[**]

1.5 Conical singularity

Consider a smooth two-dimensional manifold $\mathcal{S}$ equipped with a KVF ∂_ϕ. If its orbits are closed then such KVF is called cyclic and its integral curves are diffeomorphic to a circle. Suppose we want to find the axis of rotation symmetry generated by ∂_ϕ. It stands to reason that the axis is the set that is unchanged by the action of ∂_ϕ and we look for fixed points of the symmetry. This motivates the definition of the axis of rotation as $p \in \mathcal{M}$ such that ∂_ϕ vanishes. If additionally $\mathcal{M}$ is equipped with a Riemannian metric tensor then the vanishing of the norm $g(\partial_\phi, \partial_\phi)$ implies vanishing of ∂_ϕ. This reasoning now clearly distinguishes a sphere equipped with the axisymmetric metric tensor possessing two points, usually taken to correspond to one axis, where the norm of the rotation asymmetry vanishes, from a plane or a hyperboloid with only one such point.

If the manifold is of a higher dimension then the same may be stated. Now axis is not a point, but a higher dimensional surface[††]. If the signature of the metric is mixed then caution must be taken as the vanishing of the norm may be connected to a horizon-type phenomenon instead of an axis of rotation.

Interestingly even if the spherical topology is imposed in the spacetimes there is no guarantee that both parts of the axis correspond to the same KVF. Consider a Taub-NUT spacetime with coordinates (t^-, r, θ, ϕ_-) covering smoothly $\theta = 0$. One of the cyclic and axial KVF is simply $K_0 = \partial_{\phi_-}$. This is seen by calculating its norm

$$g(K_0, K_0) = -f(r)2l(\cos\theta - 1)^2 + (r^2 + l^2)\sin^2\theta, \tag{1.87}$$

which vanishes at $\theta = 0$, but not at $\theta = \pi$, unless $f(r)$ also vanishes which would corresponds to a horizon. Another KVF may be found $K_\pi := 4l\partial_{t_-} + \partial_{\phi_-}$ such that its norm behaves in the opposite way

$$g(K_\pi, K_\pi) = -f(r)2l(\cos\theta + 1)^2 + (r^2 + l^2)\sin^2\theta. \tag{1.88}$$

[**] Some recent results by Clément and Gal'tsov[44,45] suggest that singular axis is completely transparent to the geodesics and the spacetime is geodesically complete. There is however no consensus that their proof is correct.

[††] If the metric tensor is smooth in the region of vanishing of ∂_ϕ then the axis is a smooth submanifold defined by the locus of the function $g(\partial_\phi, \partial_\phi)$.

The KVF K_π cannot be cyclic unless the orbits of ∂_{t_-} are closed and further unless the ratio of the periods of t_- and ϕ_- are rational. If Misner's periodic time identification is applied, and so a bundle of Chern number χ_C is constructed then $K_\pi = \chi_C \partial_\psi + \partial_{\phi_-}$, where ψ and ψ_- are $2\pi-$periodic. Otherwise, K_π is still axial but not cyclic.

Consider a closed loop being an integral curve of the KVF generating the rotational symmetry and a geodesic segment from the said loop to the axis of rotation. For the purposes of this dissertation, the conical singularity is defined as an angle excess or deficit, i.e. failure of the ratio of the length of the rotational curve to the length of the geodesic segment to approach 2π in the limit to the axis. This is the standard "circumference-to-radius" test encountered in the literature[29]. For a loop generated by K_0 the situation is clear, the ratio may be calculated as 2π and no singularity is detected. The problem appears when considering orbits of K_π when ϕ has been assumed to be periodic. Griffith and Podolsky proposed to calculate the conical singularities on the surface of constant t_- and r for $\theta = 0$ and on the surface of constant t_+ and r for $\theta = \pi$. Note that if the ϕ_- is periodically identified then calculation of the conical singularity takes place of the surface of constant $t_- - 4l\phi_-$ which is a spiral. The cutoff for calculation of the circumference of the curve of the spiral is determined by the spiraling coordinate ϕ_+ going from 0 to 2π. Krtouš and Kolář proposed a similar approach in the context of the axis of rotation of higher dimensional generalization of the KNadS spacetimes. They also noticed that the definition of conical singularity depends on the choice of a timelike KVF. This timelike KVF in the standard Schwarzschild-like coordinates for Taub-NUT metric is simply ∂_t. Both of these approaches dispense with considering only closed loops and consider the length of the spiral. Both approaches are however well-defined thanks to the vanishing norm of the axial KVF ass approach the axis. This means that the curves spiral towards the axis and the length of the segments parametrized by $\phi_+ \in [0, 2\pi)$ does go to zero.

In the next Chapter, I propose treatment of the conical singularity. I propose to calculate it in the auxiliary *space of the orbits* of a chosen KVF. It formalizes the intuition that the conical singularity depends not only on the choice of ϕ coordinate but on the choice of t coordinate. It will turn out that this approach is equivalent to the one given by Krtouš and Kolář.

1.6 Plebański-Demiański spacetime

It has to be mentioned that the accelerated KNadS spacetimes, that will be the main object of study, are only some of the members of a broad Plebański-Demiański family of solutions. The family was first discovered by Debever[46], however, the most common and convenient form of the metric tensor is due to Plebański and Demiański[47]. They assumed a coordinate system adapted to two double principle null directions. Solving the Einstein-Maxwell equations possibly with the electro-magnetic field aligned with the null direction and the cosmological constant allowed them to find the most general Petrov type D spacetime in four dimensions.

Its metric tensor reads

$$\mathrm{d}s^2 = \frac{1}{(1-pr)^2}\left(-\frac{\mathcal{Q}(d\tau - p^2 d\sigma)^2}{r^2+p^2} + \frac{\mathcal{P}(d\tau + r^2 d\sigma)^2}{r^2+p^2} + \frac{r^2+p^2}{\mathcal{P}}dp^2 + \frac{r^2+p^2}{\mathcal{Q}}dr^2\right), \tag{1.89}$$

where

$$\begin{aligned} \mathcal{P}(p) &= k + 2np - \epsilon p^2 + 2mp^3 - (k + e^2 + g^2 + \tfrac{\Lambda}{3})p^4 \\ \mathcal{Q}(r) &= (k + e^2 + g^2) - 2mr + \epsilon r^2 - 2nr^3 - (k + \tfrac{\Lambda}{3})r^4, \end{aligned} \tag{1.90}$$

have to be fourth-degree polynomials in their respective variable if the metric has to satisfy the Einstein Equations. The spacetimes are parametrized by 7 real parameters $(\epsilon, k, m, n, e, g, \Lambda)$. Only three of them admit a clear physical interpretation, the cosmological constant Λ and the electric and magnetic charges e and g as they appear directly in the Eintein-Maxwell equations. The remaining four under certain conditions and identification can be shown to be related to the mass parameter, acceleration, and the Kerr and NUT parameters. As may be seen by a direct examination of the metric tensor it admits two Killing vector fields ∂_τ and ∂_σ. In physically important subcases they usually become the boost and rotation symmetry, respectively. The roots of the polynomial $\mathcal{Q}$ determine the number and position of the Killing horizons. If the polynomial $\mathcal{P}$ becomes negative then the signature is no longer Lorentzian. As the ranges of all of the coordinates as well the relation between parameters are not yet establishes this leaves many possibilities of satisfying the above condition.

A particularly important subcase appears if one demands that the principal null congruences are expanding and that the coordinate p is restricted between two roots of $\mathcal{P}$ so that $\mathcal{P} > 0$. Then a family of topological two-spheres may be distinguished in the spacetimes, with the two roots of $\mathcal{P}$ identified with the poles of the sphere. Podolsky and Griffiths found a very useful parametrization and change of coordinates in this case such that the spherical topology is exhibited explicitly. Their form of the metric tensor is studied in the next Chapter. The peculiar relations between spacetime parameters (2.3) given are precisely the result of the imposed constraints.

The Plebański-Demiański family also encompasses non-expanding solution, non-twisting solutions, and solutions with distinguished two-dimensional surfaces of vanishing or negative curvature. The survey of all possible subclasses of Petrov Type D spacetimes and their properties is beyond the scope of this work.

1.7 Results and guide to the thesis

The main achievement of this thesis is the construction of the non-singular interpretation of the Kerr-NUT-(Newman-anti-) de Sitter spacetimes and studying its properties with a particular emphasis on the Killing horizons contained therein. The construction naturally extends to the accelerated KNadS, however, then the conformal factor is a possible source of the singularity of the metric tensor. The singularity I seek to cure is connected to the axis of rotation, depending on the choice of parameters the spacetime may still have an unremovable curvature singularity. The above spacetimes are often understood to describe an external region of a rotating black hole, accelerating along the axis of the symmetry, possibly with the electromagnetic charges, on the background of a cosmological constant. However, if the NUT parameter is present a singularity of the axis of symmetry is present, most commonly detected as a conical singularity.

One possible solution is to admit that the metric tensor has a discontinuity on the singular axis. It could be subsequently interpreted as a source of angular momentum. The other possibility, proposed by Misner for Taub-NUT spacetimes, is to change the topology of the hypersurfaces r =const. In this thesis, I propose a framework to extend from Taub-NUT spacetime to the general spherical type D Plebański-Demiański black hole solution parameterized by mass parameter, Kerr parameter, electric and magnetic charge, acceleration, cosmological constant, and the NUT parameter. The last one is crucial for the proposed construction and will be assumed to be non-vanishing unless stated otherwise. As a result for every set of the above spacetime parameters a non-singular, spacetime locally isometric to accelerated KNadS may be constructed. By the non-singularity, it is meant that the metric is smooth everywhere apart from the possible curvature singularities, in particular that the symmetry axis has been regularized.

From the mathematical point of view, the construction is guided by the Misner construction for the Taub-NUT spacetime and consists of two following steps;

(i) Identify a KVF whose space of the orbits is regular, that is, it possesses no conical singularities. If the NUT parameters are non-zero it turns out that two such KVFs always exist.

(ii) Interpret the remaining part of the metric tensor with a $U(1)$-connection 1-form. Requiring that this 1-form is regular distinguishes the Misner map (Misner gluing) between the coordinates smoothly covering one part of the symmetry axis, or equivalently between the trivializations of the bundle.

The possibility of finding suitable KVF such that both above points may be realized at the same time comes as a surprise. In fact, for every set of spacetimes parameters, a pair of KVFs exists admitting smooth space of the orbit and induction of the $U(1)-$ structure.

The topology of the spacetime becomes a $U(1)$-PFB over a product of a two-sphere and a real line. All such spacetimes are constructed and presented in Chapter 2, where they are exhibited using the coordinates adopted to KVF possessing a well-defined orbit space. In those coordinates the metric tensor is manifestly smooth and the global properties of the spacetime may be studied. An important concept emerging from this construction is the definition of the spacetimes conical singularity. I discuss the possible definition of conical singularities in the presence of the NUT parameter and the fact that (for a set axial KVF) it seems to be not uniquely defined but instead is dependent on the choice of the observer, i.e. the choice of the KVF defining the space of the orbits from the point *i)* of the Misner's construction.

Any discussion of blackhole spacetimes is not complete without describing the properties of their Killing horizon, therefore the two next Chapters are devoted to studying Killing horizons appearing in the non-singular accelerated KNadS spacetimes It turns out that there are two qualitatively different types of horizons present, depending on whether the KVF generating the appropriate $U(1)-$bundle structure generates at the same time one of the Killing horizons. Based on this distinction I propose the term *projective singular horizon* when the space of the null generators of the horizon is singular. In Chapter 3 I present the properties of the non-singular horizons. If the PFB structure is generated by the horizon generating KVF then the quotient of the horizon by this field, i.e. the space of its null curves is Hausdorf. However, a further constraint on the parameters has to be satisfied is the space of the null generator has to admit a smooth metric tensor at the poles, i.e. if the conical singularity is not to be present. Perhaps surprisingly only one of the Killing horizons may be made projectively non-singular, as the generating KVF explicitly depends on the radius of the horizon. Unless the acceleration is present such horizons exist only for positive cosmological constant.

Generic horizons of the (accelerated) KNadSare projectively singular. In this case, the Killing horizon as a 3-manifold is smooth (together with the bulk spacetime), however, the space of the null curves of the horizon is defined only locally. This happens because to get to the null slice of the horizon one has to take two quotients by two different Killing vector fields, which in generic cases do not agree with each other. Their geometry in presented Chapter 4.

Part of the motivation for exploring the spacetimes with Hopf (and higher) bundle structure comes from the theory of Isolated Horizon. IHs are a structure approximating many of the Killing horizons' properties. Importantly, the Einstein Equations impose constraints on the geometry of IHs embedded as null hypersurfaces in spacetime. If the Weyl tensor is of Petrov Type D at the horizon, then this constraint takes the form of the so-called Type D equation. Together with the assumptions about the topology of both the base space of the horizon and the orbits of the horizon generating KVF the solutions to the Petrov Type D equation allow for a reconstruction of the entire IH structure. In the Introduction, I present the local approach to the IH theory. The Type D equation and the condition for the horizon to be of the Petrov Type D are possible to state even for an abstract, unembedded three-dimensional IH. The solutions to the Petrov type D equation on a Hopf bundle with the PFB structure generated by the field null on the horizon have been already found and in Chapter 3 I relate those solutions with the projectively non-singular horizon of the accelerating KNaS. I consider two main cases, with and without the acceleration parameter. Any embedded horizon of accelerated KNadS is up to discrete symmetries uniquely realized as an IH of the above structure. Although the spaces of parameters for both embedded and unembedded horizons have the same dimension it is not clear whether all IH are embeddable.

In Chapter 4 I present a new type of Isolated Horizon called, descriptively, a (spherical) horizon admitting $U(1)$-PFB structure such that the $U(1)$-action generated by a KVF transversal to the null field. I present their geometry in terms of two trivializations of the bundle structure together with a prescription for joining them into a non-trivial PFB. It turns out that for any degenerate, axis-symmetric metric tensor defined on the horizon, the PFB structure may be defined by an appropriate choice of surface gravity. The construction is modeled on the projectively singular horizons of accelerated KNadS admitting two nowhere-vanishing cyclic KVF. With the above ansatz, the Type D equation was solved. I will establish the relation between this type of IHs and the (accelerated) KNadS horizons. Similarily to the projective non-singular case, it is an open question whether all IHs with such structure are in 1-1 correspondence with spherical horizons of Plebański-Demiański spacetime.

Finally, Chapter 5 stands independently from the spherical NUT spacetimes. In the last part of the thesis, I consider IHs with $U(1)$-action generated by the horizon generating KVF, however now the base space is assumed to be any compact, orientable Riemann surface other than the sphere. With this assumption, the Petrov Type D equation is solved. The solutions have a constant Gaussian curvature and constant (non-zero if the PFB is non-trivial) rotation pseudo-scalar invariant. This implies that all horizons have a flat Riemannian metric defined on their base and $U(1)-$flat connection bundle connection. This result closes the classification of Isolated Horizons over Riemann surfaces with genus higher than 0 and bundle structure (with the group structure being $\mathbb{R}$ or $U(1)$, trivial or not) generated by the field null at the horizon. For the horizons of this type, the embedding spacetimes are constructed. They are locally isometric to the generalized Taub-NUT-(anti-) de Siter with a distinguished family of 2-surfaces of vanishing or negative curvature. In the embedding spacetimes, those 2-surfaces have been compactified from plane to a torus or from the hyperboloid to a higher genus surface and the Misner periodic time identification has to be enforced to introduce the $U(1)-$PFB structure.

To sum up, the original results consist of:

- Discovery of distinguishes KVFs such that the orbit-space does not have conical singularities.
- Generalization of the Misner non-singular construction to the (accelerated) KNadS spacetimes via imposing a $U(1)-$PFB structure.
- Analysis of the Killing Horizons container therein, together with their spaces of null geodesics.
- Generalization of the Isolated Horizon structure to the case encompassing the generic horizons of the (accelerated) KNadS spacetimes.
- Description of the horizons with the $U(1)-$structure over a compact Riemann surface and their embeddings spacetimes, also exhibiting the bundle structure.

This thesis is based on the series of papers

1. "Non-singular Kerr-NUT-de Sitter spacetimes"
   Jerzy Lewandowski, M.O., 2020, Class. Quantum Grav. 37 205007

2. ”Projectively non-singular horizons in Kerr-NUT-de Sitter spacetimes”
   Jerzy Lewandowski, M.O., 2020, Phys. Rev. D 102, 124055

3. ”Non-singular extension of the Kerr-NUT-(anti) de Sitter spacetimes”
   Jerzy Lewandowski, M.O., 2021, Phys. Rev. D 104, 024022

4. ”Isolated horizons of the Hopf bundle structure transversal to the null direction, the horizon equations and embeddability in NUT-like spacetimes”
   Denis Dobkowski-Ryłko, Jerzy Lewandowski, M.O., 2023, Phys. Rev. D 108, 104057

5. ”Vacuum Petrov type D horizons of non-trivial $U(1)$ bundle structure over Riemann surfaces with genus > 0”
   Jerzy Lewandowski, M.O., 2024, preprint.[48]

Their content approximately correspond to Chapters 2 and 3, Chapter 3, Chapter 2, Chapters 4 and 3, and Chapter 5, respectively. The results have been however reorganized and expanded for completeness and better readability.

ἔστιν δ’ ὅμως οὐδὲν ἧττον κατανοῆσαι δυνατὸν ὡς ὅ γε τέλεος ἀριθμὸς χρόνου τὸν τέλεον ἐνιαυτὸν πληροῖ τότε, ὅταν ἁπασῶν τῶν ὀκτὼ περιόδων τὰ πρὸς ἄλληλα συμπερανθέντα τάχη σχῇ κεφαλὴν τῷ τοῦ ταὐτοῦ καὶ ὁμοίως ἰόντος ἀναμετρηθέντα κύκλῳ. [...] ἵνα τόδε ὡς ὁμοιότατον ᾖ τῷ τελέῳ καὶ νοητῷ ζῴῳ πρὸς τὴν τῆς διαιωνίας μίμησιν φύσεως.

Plato, Timaeus 39d

2 Non-singular accelerated Kerr-NUT-de Sitter Spacetimes

This Chapter contains the core result of this thesis: the construction of a non-singular interpretation of the (accelerated) Kerr-NUT-(anti-) de Sitter spacetimes, in the form of a spacetime locally isometric to the (accelerated) KNadS, however with a change in topology. The construction is motivated by Misner's interpretation of Taub-NUT spacetimes presented in the Introduction. The spacetimes under consideration are a generalization of the Kerr-adS spacetimes and therefore usually they are interpreted as representing the external region of a rotating black hole, albeit with some unwanted properties due to the presence of the NUT and acceleration parameter. They are almost the most general class of spherical expanding Petrov Type D spacetime[49], with The only exception not covered is the accelerating Taub-NUT-(anti-) de Sitter which will be studied separately.

It will turn out that again, due to the presence of the NUT parameter it is possible to equip the structure with the $U(1)$-PFB structure, and more importantly with a topology such that the axes of rotation are regular. The key insight relies on the fact that two Killing vector fields admitting a regular space of the orbits always exist when the NUT parameter is present. Using such regular orbit-space as a base space a smooth spacetime with a topology of a product of a spherical 3-manifold and a real line may be constructed.

An interesting observation emerges from the fact that smooth orbit spaces always exist. The orbit space provides a geometric definition of conical singularity in the spacetime, i.e. the conicity is calculated using the orbit-space metric tensor. Therefore if conical singularity is associated with the presence and strength of a cosmic string, this fact must depend on the choice of the orbit-space and thus on the choice of the family of observers. This follows even without imposing the periodicity condition.

Compactification of the orbits of the boost symmetry (∂_t for Taub-NUT) introduces closed curves through every point of the spacetime, in some regions they are timelike. However contrary to the Taub-NUT spacetime the existence of such regions is not necessary. I will present a "cosmological interpretation" such that the section with S^3 topologies is always spacelike. This runs contrary to the typical black hole interpretation of the spacetime. On the other hand, it seems to suggest that NUT spacetimes may hold interesting connections to instantons.

Finally, a question arises about the physical nature of the NUT parameter - this work does not provide any definitive answer. In view of the results of this work, it is not the value but the presence of the NUT parameter that is the most important. The natural topology seems to discontinuously change from non-trivial to a trivial bundle if the NUT parameter vanishes. It is not true that the NUT parameter will determine the Chern number of the $U(1)$-bundle, for any NUT parameter any nontrivial bundle is admissible. However, if the topology is determined then the NUT parameter is proportional to the period of the compactified coordinate.

It has to be noted that NUT spacetimes have been of interest in different contexts for a long time. Mars and Senovilla have proposed a characterization of the KNadS in terms of a suitably defined tensor[50]. It is sometimes proposed as a spacetime description of rotating gravitational dyon[51]. Miller more than 50 years ago performed a similar analysis of Kerr-NUT (without the cosmological constant) in terms of $SU(2)$-invariant objects[43]. The proper definition of angular momentum is still not settled and instead, an object of discussion, in particular, due to the freedom of choosing the location of the Misner string[37,52,38]. Spacetimes with $U(1)$-symmetry have been also considered in the context of compact Cauchy horizons and possible cosmic censorship principle violations[53,54].

Finally, the KNadS spacetime in any dimension enjoys a benefit which, although is not necessary for the current construction, is interesting enough to warrant the mention. It is a generic spacetime admitting the so-called principal tensor, i.e. a non-degenerate closed conformal Killing-Yano 2-form[55] from which all symmetries of the spacetime and constants of the geodesic motion may be obtained (including the famous Carter constant in 4D)[56,57,58]. These results lead to many substantial calculational simplifications, e.g. separability of Hamilton-Jacobi, Klein-Gordon[59] and Dirac equations[60]. More recently they have been used to construct homogeneous symmetry operators of KNadS spacetime[61]. Interestingly the coordinates in which the Plebański-Demiański metric tensor presented in Section 1.6 are the eingenvalues of the principal tensor[62].

2.1 Accelerating KNadS spacetimes and their properties

The generic class of the accelerating Kerr-NUT-(anti-) de Sitter spacetimes in the Boyer-Linquist-like coordinate were derived by Griffiths and Podolsky as a special case of the Plebański-Demiański spacetime[49]. After a restriction of the range the p coordinate from the metric tensor (1.89) to lie between two roots of the polynomial $\mathcal{P}(p)$ may be factorized into $\mathcal{P}(\theta)\sin^2\theta$ for a suitably chosen definition of θ.

The metric tensor of the generic reads

$$\mathrm{d}s^2 = \frac{1}{F^2}\left[-\frac{\mathcal{Q}}{\Sigma}(\mathrm{d}t - A\mathrm{d}\phi)^2 + \frac{\Sigma}{\mathcal{Q}}\mathrm{d}r^2 + \frac{\Sigma}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}}{\Sigma}\sin^2\theta(a\mathrm{d}t - \rho\mathrm{d}\phi)^2\right], \tag{2.1}$$

where the functions appearing in the tensor components are defined as *

$$\begin{aligned}
F &= 1 - \frac{\alpha}{\bar{\omega}}(l + a\cos\theta)r,\\
\Sigma &= r^2 + (l + a\cos\theta)^2,\\
A &= a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta,\\
\rho &= r^2 + (l+a)^2 = \Sigma + aA,\\
\mathcal{Q} &= \left(\bar{\omega}^2k + \bar{e}^2 + \bar{g}^2\right) - 2Mr + \epsilon r^2 - 2\frac{\alpha n}{\bar{\omega}}r^3 - (\alpha k + \frac{1}{3}\Lambda)r^4,\\
\mathcal{P} &= 1 - a_3\cos\theta - a_4\cos^2\theta,
\end{aligned} \tag{2.2}$$

and the spacetime parameters enter through the following combinations

$$\begin{aligned}
a_3 &= 2\frac{\alpha a M}{\bar{\omega}} - 4\alpha^2\frac{al}{\bar{\omega}^2}k\left(\bar{\omega}^2k + \bar{e}^2 + \bar{g}^2\right) - \frac{4}{3}\Lambda al \triangleq 2\frac{\alpha a M}{\bar{\omega}} - 4\alpha^2 alk - \frac{4}{3}\Lambda al,\\
a_4 &= -\alpha^2\frac{a^2}{\bar{\omega}^2}\left(\bar{\omega}^2 + \bar{e}^2 + \bar{g}^2\right)k - \frac{1}{3}\Lambda a^2 \triangleq -\alpha^2a^2k - \frac{1}{3}\Lambda a^2,\\
\epsilon &= \frac{\bar{\omega}^2k}{(a^2 - l^2)} + 4\frac{\alpha l M}{\bar{\omega}} - (a^2 + 3l^2)\left(\frac{\alpha^2}{\bar{\omega}^2}\left(\bar{\omega}^2k + \bar{e}^2 + \bar{g}^2\right) + \frac{\Lambda}{3}\right)\\
&\triangleq \frac{\bar{\omega}^2k}{(a^2 - l^2)} + 4\frac{\alpha l M}{\bar{\omega}} - (a^2 + 3l^2)(\alpha^2k + \Lambda/3),\\
n &= \frac{\bar{\omega}^2kl}{(a^2 - l^2)} - \frac{\alpha M(a^2 - l^2)}{\bar{\omega}} + (a^2 - l^2)\left(\frac{\alpha^2}{\bar{\omega}^2}\left(\bar{\omega}^2k + \bar{e}^2 + \bar{g}^2\right) + \frac{\Lambda}{3}\right)\\
&\triangleq \frac{\bar{\omega}^2kl}{(a^2 - l^2)} - \frac{\alpha M(a^2 - l^2)}{\bar{\omega}} + (a^2 - l^2)(\alpha^2k + \Lambda/3),\\
k &= \frac{1 + 2\alpha l M\bar{\omega}^{-1} - 3\alpha^2l^2\omega^{-2}\left(\bar{e}^2 + \bar{g}^2\right) - l^2\Lambda}{3\alpha^2l^2 + \bar{\omega}^2(a^2 - l^2)^{-1}} \triangleq \frac{1 + 2\alpha l M\bar{\omega}^{-1} - l^2\Lambda}{3\alpha^2l^2 + \bar{\omega}^2(a^2 - l^2)^{-1}}
\end{aligned} \tag{2.3}$$

*Throughout this text A sometimes refers to a $U(1)$- connection 1-form. They context will differentiate them clearly.

The second equality ($\triangleq$) in the above is meant to be understood as a limit to $\bar{e} = 0 = \bar{g}$, where $\bar{e}$ and $\bar{g}$ are electric and magnetic charges. In the remaining part of this work **the uncharged case will be assumed.** This choice is motivated by the simplified form of the metric tensor in the case of a vanishing electromagnetic field. The inclusion of the charges would not lead to any changes in the construction of the non-singular spacetime for the non-accelerating case, in the accelerating case the change is not qualitative.

Seven physical parameters appear in the above metric; The electric $\bar{e}$ and magnetic charges $\bar{g}$ as well as the cosmological constant Λ appear directly in the Einstein-Maxwell equations and therefore their physical interpretation is straightforward. It has to be noted that the charges have been recalled since they appeared in the Plebańki-Demiański metric (1.89). The next parameter is M, I opt to call it the mass parameter instead of simply mass. Whether it corresponds to a conserved charge associated with ∂_t depends on the presence of other parameters, for example in the Kerr de Sitter this charge reads $\frac{M}{\left(1+a^2\frac{\Lambda}{3}\right)^2}$ [63]. The next two parameters are denoted by a and α and correspond to the famous Kerr rotation parameter[64] and the acceleration parameters respectively. The latter name originates from the Minkowski limit of the C-metric[65], i.e. black hole with only the acceleration and mass parameter when α becomes the acceleration of the test particle located at $r = 0$. The spacetime is usually interpreted as two black holes accelerating towards each other[66]. Notably, the presence of acceleration introduces an irremovable (unless the NUT parameter is present) conical singularity, often associated with the presence of a cosmic string[67]. The effect of the NUT parameter is similar to the one in Taub-NUT spacetimes, the singularities associated with it will be discussed. Finally, $\bar{\omega}$ is a gauge parameter introduced in the process of obtaining the Griffith Podolsky form from the Plebański-Demiański coordinates.

A particular advantage of the above coordinates is the possibility of taking a limit to many physically important black hole solutions, including Kerr, C-metric, possibility with electromagnetic charges, and a cosmological constant of any sign. A full diagram of possible limits may be seen in Chapter 16 of Griffiths' and Podolsky's book[29]. To obtain the zero NUT limit, i.e. the accelerating Kerr-Newman-(anti-) de Sitter spacetime and its sub-cases, one must set $\bar{\omega} = a$ before taking the limit of $l = 0$. Any other limiting case is obtained simply by putting a desired parameter to zero. Recently a convenient value of the gauge parameter $\bar{\omega}$ has been proposed[68,69] as

$$\bar{\omega} = \frac{a^2 + l^2}{a}. \tag{2.4}$$

It was shown that, after reparameterization of the spacetimes, this choice of the gauge parameters allows us to take all the limits by simply setting a desired parameter to zero. Additionally in the $\Lambda = 0$ case, it allows for a convenient factorization of the roots of $\mathcal{Q}$, clearly dividing them into the ones corresponding to the black hole and acceleration horizons. I opt to maintain the unconstrained gauge parameter $\bar{\omega} \in \mathbb{R} \setminus \{0\}$ for two reasons

1. With the choice (2.4) the limit to $\alpha = 0$ requires further redefinition of the parameters

to obtain the standard form of the metric. As $\alpha = 0$ is an important sub-case studied in this work, this would be inconvenient.

2. Any relations involving (2.2) are necessarily rather complicated due to the complicated expressions (2.3). Choosing (2.4) does not lead to substantial simplification of the formulae important for this work. In fact, the opposite is often true.

As was mentioned there is one limit that cannot be obtained from the Griffith Podolsky form of the metric tensor, namely the accelerating Taub-NUT (possibly with charges and cosmological constant). It was shown that the apparently accelerating Taub-NUT obtained as a limit $a = 0$ from the metric tensor (2.1) is simply the Taub-NUT spacetime written in different coordinates[70]. It was conjectured that the accelerating Taub-NUT spacetimes may be realized only as a Type I, with all of the horizons being of Petrov Type D[71]. As of writing this dissertation recent publications seem to have found the Type D accelerating Taub-NUT-(anti-) de Sitter, together with a universal line element allowing for taking all of the limits[72,73]. Both Petrov Type I and Type D accelerating Taub-NUT spacetimes are included in Section 2.4, where their non-singular interpretation is presented.

The two Killing symmetries inherited from the Plebański-Demiański form of the metric are now ∂_t and ∂_ϕ. If additionally, the spacetime is not rotating, i.e. $a = 0$ the rotation symmetry ∂_ϕ is enhanced to the full $\mathfrak{so}(3)$ algebra as demonstrated to the Taub-NUT spacetimes. By the usual understanding, the metric tensor describes the spacetime around a compact, spherical object. Therefore the local two-surfaces of constant t and r are assumed to be homeomorphic to a sphere. The ranges of parameters are taken to be $t \in \mathbb{R}$, $r \in \mathbb{R}$, $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$. Here $\theta = \pi$ corresponds to both the singularity of the coordinates and discontinuity of the metric tensor as is demonstrated in the next Section. Crucially the range of ϕ being 2π is a matter of convention and is not truly fixed. The rescaling freedom will be utilized in the following Sections.

Kerr-NUT-(anti-) de Sitter spacetime

The Kerr-NUT-(anti-) de Sitter spacetime is obtained by taking the limit $\alpha \to 0$. The metric tensor has a more simplified form with the conformal factor F and the gauge parameter $\bar{\omega}$ no longer appearing. It reads

$$\mathrm{d}s^2 = -\frac{\mathcal{Q}}{\Sigma}(dt - Ad\phi)^2 + \frac{\Sigma}{\mathcal{Q}}dr^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(adt - \rho d\phi)^2, \tag{2.5}$$

where

$$
\begin{aligned}
&\Sigma = r^2 + (l + a\cos\theta)^2,\\
&A = a\sin^2\theta + 4l\sin^2\tfrac{1}{2}\theta = a\sin^2\theta - 2l(\cos\theta - 1),\\
&\rho = r^2 + (l+a)^2 = \Sigma + aA,\\
&\mathcal{Q} = (a^2 - l^2) + (\bar{e}^2 + \bar{g}^2) - 2Mr + r^2 - \Lambda\big((a^2 - l^2)l^2 + (\tfrac{1}{3}a^2 + 2l^2)r^2 + \tfrac{1}{3}r^4\big),\\
&\mathcal{P} = 1 + \frac{4}{3}\Lambda al\cos\theta + \frac{\Lambda}{3}a^2\cos^2\theta.
\end{aligned}
\tag{2.6}
$$

The global analysis of the $\Lambda = 0$ limit of the above, including the case of $S^3 \times \mathbb{R}$ topology has been studied in[43]. By Miller's analysis In the Misner periodic interpretation spacetimes with the Kerr parameter still admit two equivalent extensions through the horizons, such that imposing both extensions simultaneously leads to a non-Hausdorf spacetime. Although this problem is not explicitly studied in this work, it is expected that it will persist if the cosmological constant and acceleration are included.

2.1.1 Singularities of metric tensor

The most evident source of possible singularities is the vanishing of the function defining the metric tensor. The roots of $\mathcal{Q}$ correspond to the location of the Killing horizons and thus are regular which may be seen explicitly using the Eddington-Finkelstein coordinates. They are most clearly introduced as a co-frame transformation

$$
\mathrm{d}v := \mathrm{d}t + \frac{\rho}{\mathcal{Q}}\mathrm{d}r, \quad \mathrm{d}\tilde{\phi} := \mathrm{d}\phi + \frac{a}{\mathcal{Q}}\mathrm{d}r, \tag{2.7}
$$

which is valid for all subclasses of the accelerated Kerr-NUT-(anti-) de Sitter, if the appropriate $\mathcal{Q}$ is chosen. After the transformation, the metric tensor reads

$$
\mathrm{d}s^2 = \frac{1}{F^2}\left[-\frac{\mathcal{Q}}{\Sigma}\big(\mathrm{d}v - A\mathrm{d}\tilde{\phi}\big)^2 + 2\big(\mathrm{d}v - A\mathrm{d}\tilde{\phi}\big)\mathrm{d}r + \frac{\Sigma}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}}{\Sigma}\sin^2\theta(a\mathrm{d}v - \rho\mathrm{d}\tilde{\phi})^2\right]. \tag{2.8}
$$

The above coordinates are the so-called in-going coordinates, by switching the signs of $\mathrm{d}r$ terms in (2.7) one obtains the out-going coordinates. Both choices are equivalent from the perspective of obtaining the non-singular interpretation.

The calculation of polynomial curvature invariants, for example, the Kretschmann scalar, demonstrates that vanishing Σ corresponds to a curvature singularity[29]. It takes the form of a ring-like singularity similar to the one appearing in Kerr and occurs only when $r = 0$ and $\cos\theta = -l/a$ (as opposed to $\cos\theta = 0$ in Kerr-(anti-) de Sitter). The singularity appears only if $a^2 > l^2$, which is often called the Kerr-like choice of the parameters. Otherwise, in the NUT-like choice, i.e. for $l^2 > a^2$ the curvature singularity does not appear at all. Once again introduction of the (sufficiently large) NUT parameter regularized the spacetime. The

NUT-like and Kerr-like choices lead to different global structures of the spacetime and for this reason, some authors choose to treat them as separate spacetime[29,74,75]†

To maintain the Lorentzian signature $\mathcal{P} > 0$ is required. The same condition has to be satisfied for the Plebański-Demiański metric but now thanks to a convenient factorization this condition boils down to the positivity of a quadratic polynomial in $\cos\theta$. In the non-accelerating case in may be stated as

$$\begin{aligned}
&\forall\theta \in [0,\pi]\, P > 0 \iff \\
&\left(\left(2\left|\frac{l}{a}\right| \leq 1 \wedge 0 < \Lambda < \frac{3}{4l^2}\right) \vee \left(2\frac{l}{a} > 1 \wedge 0 < \Lambda < \frac{-3}{a^2 - 4al}\right) \vee \right.\\
&\left(-2\frac{l}{a} > 1 \wedge 0 < \Lambda < \frac{-3}{a^2 + 4al}\right) \vee \left(\frac{-3}{a^2 - 4al} < \Lambda < 0 \wedge al < 0\right) \vee \\
&\left.\left(\frac{-3}{a^2 + 4al} < \Lambda < 0 \wedge al > 0\right)\right).
\end{aligned} \tag{2.9}$$

In the accelerating case, this condition becomes rather unwieldy. Whenever any choice of spacetimes parameters is chosen it should be always checked that they satisfy $\mathcal{P} > 0$. The vanishing of F corresponds to the conformal infinity at $r = \frac{\bar{\omega}}{\alpha(l + a\cos\theta)}$ for $\cos\theta > -l/a$. For $\cos\theta < -l/a$ these coordinate do now cover the conformal infinity[49].

The remaining part of the singularity is the 1-form

$$\omega = \mathrm{d}t - A\mathrm{d}\phi = \mathrm{d}t - a\sin^2\theta + 2l(\cos\theta - 1)\mathrm{d}\phi.$$

Apart from the regular "Kerr term" proportional to $\sin^2\theta$ it also has the "NUT term". For the same reason as for the Taub-NUT metric, this means that ω is not continuous at $\theta = \pi$. Note that the coordinates (t, r, θ, ϕ) are analogous to the choice of $C = -1$ in (1.61). The analog of coordinates with $C = 1$, which will be subsequently denoted by the addition of $'$, e.q. (t', r', θ', ϕ'), are introduced by

$$t = t' + 4l\phi \quad \phi = \phi'. \tag{2.10}$$

After which the metric tensor reads

$$\mathrm{d}s^2 = -\frac{1}{F^2}\left[\frac{\mathcal{Q}}{\Sigma}(\mathrm{d}t - A'\mathrm{d}\phi')^2 + \frac{\Sigma}{\mathcal{Q}}\mathrm{d}r^2 + \frac{\Sigma}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}}{\Sigma}\sin^2\theta(a\mathrm{d}t' - \rho'\mathrm{d}\phi')^2\right], \tag{2.11}$$

where the "dual functions" are introduced as

$$\begin{aligned}
A' &= a\sin^2\theta - 4l\cos^2\tfrac{1}{2}\theta = a\sin^2\theta - 2l(\cos\theta + 1),\\
\rho' &= r^2 + (l - a)^2,
\end{aligned} \tag{2.12}$$

†Interestingly the fact the both rotating black holes and NUT black holes seem to admit the same topology of surfaces of constant t and r seems to be a low dimensional coincidence. Compare the Myers-Perry spacetimes[76] where the slices have the topology of S^{2N-2}, and higher dimensional Taub-NUT spacetimes[77] with $(S^2)^{\times N-1}$ in the spacetime of dimension $2N$. The coindice only for $N = 2$.

and the following identities simplify transitioning between the coordinates

$$
\begin{aligned}
&\rho' = \Sigma + aA', \\
&A - A' = 4l, \\
&\rho - \rho' = 4al, \\
&\Sigma(\theta = 0) = \rho, \\
&\Sigma(\theta = \pi) = \rho'.
\end{aligned}
\tag{2.13}
$$

Now the 1-form $\omega' = \omega = \mathrm{d}t' - A'\mathrm{d}\phi' = \mathrm{d}t' - a\sin^2\theta' + 2l(\cos\theta' + 1)\mathrm{d}\phi'$ [‡] is continuous at $\theta = \pi$ and not at $\theta = 0$.

The inclusion of the NUT parameter means that the behavior of axial and cyclic KVFs is similar to the Taub-NUT case. Indeed, $K_0 = \partial_\phi$ is axial and cyclic for $\theta = 0$ as seen from

$$
g(K_0, K_0) = \frac{1}{F^2}\left[-\frac{\mathcal{Q}}{\Sigma}A^2 + \sin^2\theta\frac{\mathcal{P}}{\Sigma}\rho^2\right], \tag{2.14}
$$

while $K_\pi = \partial_{\phi'} = 4l\partial_t + \partial_\phi$ is axial for $\theta = \pi$ but not cyclic as seen from

$$
g(K_\pi, K_\pi) = \frac{1}{F^2}\left[-\frac{\mathcal{Q}}{\Sigma}A'^2 + \sin^2\theta\frac{\mathcal{P}}{\Sigma}\rho^2\right]. \tag{2.15}
$$

2.2 Construction of non-singular spacetimes

The main goal of this Section is the construction of a spacetime with a metric g defined on $S^3 \times \mathbb{R}$ which is locally isometric to (accelerated) the Kerr-NUT-(anti-) de Sitter metric (2.8).

The construction consists of two steps

1. Finding a regular orbit space of a combination of KVFs of (accelerating) KNadS spacetime. By regularity, it is meant that the orbit-space metric tensor is smooth, that is the conical singularities, calculated in the orbit-space do not appear.

2. Interpret the remaining part of the metric tensor as proportional to a $U(1)$-connection 1-form and impose on the spacetime $U(1)-$ PFB structure over $S^2 \times \mathbb{R}$ with the action given by the flow of the KVF admitting the regular orbit space.

This idea is conveniently expressed in the decomposition of the resulting metric tenor

$$
g = g(\xi, \xi)\omega^2 + q, \tag{2.16}
$$

[‡]If in addition to coordinate transformation (2.10) one changes which coordinate is periodically identified, then perhaps it is better to distinguish between ω and ω' explicitly. Of course, locally they are equivalent.

where q is the smooth metric on the orbit space of a KVF ξ and ω, defined by

$$\omega = \frac{g(\xi, \cdot)}{g(\xi, \xi)}, \tag{2.17}$$

is a $U(1)$-connection of the PFB whose action is generated by ξ. The first step assures that q is smooth and the second the ω is a smooth 1-form if the bundle topology is assumed. Then the entire metric tensor g has to be smooth. It has to be additionally checked that the metric is regular whenever $g(\xi, \xi) = 0$, and thus q is not well defined. Additionally, ξ distinguishes a class of adopted coordinates in which the metric tensor is manifestly regular. The desired and resulting spacetime admits the following structure of a $U(1)$-PFB

$$U(1) \hookrightarrow L(\chi_C, 1)^3 \times \mathbb{R} \to S^2 \times \mathbb{R}. \tag{2.18}$$

Crucially, the construction requires that the NUT parameter is nonvanishing.

2.2.1 Construction of the orbit-space

In Misner's construction, it was assumed that the $U(1)-$PFB is generated by the KVF which is a standard choice for the time translation or boost symmetry, namely ∂_t. Additionally, this KVF is distinguished by generating all Killing horizons and being timelike at infinity. The key to generalizing Misner's construction to the broader Plebański-Demiański family is allowing for an arbitrary KVF instead.

Consider a general linear combination of Killing vector fields

$$\xi = A\partial_v + B\partial_\phi, \quad A, B \in \mathbb{R}.$$

The space of orbits would be the same if ξ would be recalled by a real constant and so it is enough to consider the following combination

$$\xi = A(\partial_t + \tfrac{B}{A}\partial_\phi) \cong \partial_t + b\partial_\phi, \tag{2.19}$$

where $b \in \mathbb{R}$, however now the KVF ∂_ϕ is not achieved by any b.

The space of the orbits of ξ is defined as the quotient of the spacetime by the KVF ξ, i.e. spacetime where all points lying in the same orbit of ξ are identified. Equivalently any point of the orbit-space corresponds to an orbit of ξ in the spacetime. It may be equipped with the metric tensor q induced by the spacetime metric tensor g such that for $X, Y \in TM$ we have

$$q(X, Y) = g(X, Y) - \frac{g(X, \xi)g(Y, \xi)}{g(\xi, \xi)}. \tag{2.20}$$

The above metric may be seen simply as a projection orthogonal to ξ. Indeed, adding to X or Y any term proportional (by a smooth function) to ξ does not change the result, which may be seen explicitly by expanding the above formula.

The orbit-space metric is most easily exhibited in coordinates $(x^\mu) = (\tau, x^i)$ adopted to ξ in the sense that

$$\xi(\tau) = 1, \quad \xi(x^i) = 0, \tag{2.21}$$

such that we have $\partial_\tau = \xi$. Then (x^i) are well-defined coordinates in the orbit-space, which by a slight abuse of the notation they will be identified with their pullback to the spacetime.

A set of adapted coordinates satisfying the above condition is for example

$$(x^\mu) = (\tau, x^i) = (v,\ r,\ \theta,\ \hat{\phi} := -bv + \tilde{\phi}), \tag{2.22}$$

in which we have

$$\begin{aligned} q =& \frac{(1-Ab)^2\Sigma}{\mathcal{Q}(1-Ab)^2 - \mathcal{P}\sin^2\theta(a-b\rho)^2}\mathrm{d}r^2 + \frac{\mathcal{P}\sin^2\theta\Sigma(b\rho - a)}{\mathcal{Q}(1-Ab)^2 - \mathcal{P}\sin^2\theta(a-b\rho)^2} 2\mathrm{d}r\mathrm{d}\hat{\phi} + \\ &+ \frac{\Sigma}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}\mathcal{Q}\sin^2\theta\Sigma}{\mathcal{Q}(1-Ab)^2 - \mathcal{P}\sin^2\theta(a-b\rho)^2}\mathrm{d}\hat{\phi}^2. \end{aligned} \tag{2.23}$$

The common denominators are proportional to the norm of ξ given

$$g(\xi,\xi) = g_{\tau\tau} = \Sigma^{-1}\left(P\sin^2\theta(a-b\rho)^2 - \mathcal{Q}(1-Ab)^2\right). \tag{2.24}$$

By the initial assumptions about the metric functions, we have $\mathcal{P} > 0$. The coefficient of $\mathrm{d}r^2$ is an analytic function while the term proportional to $\mathrm{d}r\mathrm{d}\hat{\phi}$ is a product of $\sin^2\theta\mathrm{d}\hat{\phi}$ and a term proportional to $\mathrm{d}r$, both of which are analytical. Thus the only suspicious points are poles $\theta = 0$ and $\theta = \pi$ and the conclusion is that the orbit metric is well defined for $(r, \theta, \hat{\phi}) \in \mathbb{R}\times]0,\pi[\times[0, 2\pi\beta)$, apart from the surface where $g(\xi,\xi) = 0$.

2.2.2 Conical singularity in the orbits space

The cyclic and axial for $\theta = 0$ KVF ∂_ϕ is projected to $\partial_{\hat{\phi}}$ in the orbit space. Thus the orbit-space metric tensor inherits from the spacetime metric the axial symmetry. The desired space of the orbits has the topology of $\mathbb{R}\times S^2$, with the metric tensor on S^2 given by pullback to the surface of $r =$const. Note that if the conical singularities are present then the oribt-space is non-Hausdorf.

Clearly in the orbit-space the KVF $\partial_{\hat{\phi}}$ is axial for both $\theta = 0$ and $\theta = \pi$. Any spiraling behavior of axial KVFs is projected out in the process of going to the space of the orbits. On the surface of a constant radius the circumference L of the closed integral curves of $\partial_{\hat{\phi}}$ and the geodesic distance from the poles $\theta = 0$ or $\theta = \pi$ to the said closed curve, R_0 and R_π

respectively, may be introduced in a standard way as

$$
\begin{aligned}
L(\theta_0) &= \int_0^{2\pi\beta} \sqrt{{}^{(2)}q_{\hat{\phi}\hat{\phi}}} d\hat{\phi}, \\
R_0(\theta) &= \int_0^{\theta_0} \sqrt{{}^{(2)}q_{\theta\theta}} d\theta, \\
R_\pi(\theta) &= \int_{\theta_0}^{\pi} \sqrt{{}^{(2)}q_{\theta\theta}} d\theta,
\end{aligned}
\tag{2.25}
$$

where ${}^{(2)}q$ is the pullback of the metric to r =const

$$
{}^{(2)}q = \frac{1}{F^2}\left[\frac{\Sigma}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}\mathcal{Q}\Sigma\sin^2\theta}{\mathcal{Q}(1-Ab)^2 - \mathcal{P}\sin^2\theta(a-b\rho)^2}\mathrm{d}\hat{\phi}^2\right]. \tag{2.26}
$$

A positive, real parameter β has been introduced into the range of $\hat{\phi}$, and setting it to a proper value will be used to remove the conical singularity. Note that for $a = 0 = \alpha$ and choice of $\xi = \partial_t$, i.e. $b = 0$ the above becomes the flat metric on S^2

$$
{}^{(2)}q_{\mathrm{TN}} = (r^2 + l^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2). \tag{2.27}
$$

Therefore for Taub-NUT-(anti-) de Sitter the space of the orbits of ∂_{t_C} (for any C) is regular. Recall that the two axial KVF could be given by $K_0 = \partial_{\phi_-}$ and $K_\pi = 4l\partial_{t_-} + \partial_{\phi_-}$, where ϕ_- is assumed to be periodic. In the orbit space, i.e. up to ∂_{t_-}, both K_0 and K_π project to ∂_{ϕ_-}.

Next, consider the case of generic metric tensor but the orbits space of $\xi = \partial_t$. The orbit-space metric simplifies to

$$
{}^{(2)}q = \frac{1}{F^2}\left[\frac{\Sigma}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}\mathcal{Q}\Sigma\sin^2\theta}{\mathcal{Q} - \mathcal{P}a^2\sin^2\theta}\mathrm{d}\hat{\phi}^2\right]. \tag{2.28}
$$

The conicities are for $\theta = 0$ and $\theta = \pi$ read $\mathcal{P}(0)$ and $\mathcal{P}(\pi)$, respectively. The only remaining degree of freedom is the possibility of rescaling the rotational coordinate $\hat{\phi}$. However, unless $\mathcal{P}(0) = \mathcal{P}(\pi)$ is satisfied the rescaling remedies only the conical singularity at one of the axes. If $\alpha \neq 0$ this condition could be satisfied as

$$
2\alpha^2 l\omega^2 k - \frac{2}{3}\omega^2 l\Lambda = \alpha\omega M, \tag{2.29}
$$

which precisely reproduces the calculation by Griffith and Podolsky[49].

Finally, in the generic case, the *condition for the removal of the conical singularity* is that the limit of the circumference to the radius of loops in the orbit-space is equal to 2π when going to the poles

$$
\lim_{\theta\to 0}\frac{L(\theta)}{R(\theta)} = 2\pi = \lim_{\theta\to\pi}\frac{L(\theta)}{R_\pi(\theta)} \tag{2.30}
$$

If the above is satisfied then the conical singularity could be removed by a global rescaling of the ϕ coordinate. The condition could be solved to

$$\mathcal{P}(0) = \frac{\mathcal{P}(\pi)}{|1-4lb|}, \quad \beta = 1/\mathcal{P}(0) \tag{2.31}$$

and is valid for any $\mathcal{P}$ describing a Plebański-Demiański black hole. The parameter β defines the 2π periodic coordinate without the conical singularity

$$\varphi := \frac{\hat{\phi}}{\beta}. \tag{2.32}$$

Importantly it depends only on the spacetime parameters and the KVF parameter b and so is the same regardless of the chosen r =const surface. Note that if the NUT parameter is not present then the parameter b does not appear in the condition (2.31). Then it is either satisfied for all choices of b or cannot be satisfied at all.

Before stating its solution in full generality let me first consider a couple of special cases.

- If $\alpha = 0$ and $l = 0$ then $\mathcal{P} = 1 + \frac{\Lambda}{3}\cos^2\theta$ and the condition (2.31) is automatically satisfied with $\beta = (1+\frac{\Lambda}{3}a^2)^{-1}$. This is an example of a regularizable conical singularity happening in the Kerr-(anti-) de Sitter spacetime.

- If $l = 0$ and $\alpha \neq 0$ then (2.31) cannot be satisfied unless $\alpha m a = 0$. This reflects the fact that the C-metric and the accelerating Kerr-(anti-) de Sitter have an unremovable conical singularity.

- If $\alpha = 0$ and $a\Lambda = 0$ then $\mathcal{P} = 1$ and the proper choice is $b = 0$, i.e. $\xi = \partial_t$. This exactly recovers Misner's choice for the Taub-NUT spacetime. Additionally, it is the right choice for the Kerr-NUT spacetime. Interestingly there is another possible choice $b = \frac{1}{2b}$, i.e. $\xi = \partial_t + \frac{1}{2l}\partial_\phi$. Its orbits would be parameterized by the ϕ_0 coordinate with $C = 0$. Both ∂_t and $\partial_t + \frac{1}{2l}\partial_\phi$ after the periodic time identification will correspond to cyclic, but not axial KFVs.

For the generic solution with the NUT parameter the condition (2.31) has two branches of solutions, depending on the sign of $|1-4lb|$. For the convenience of dealing with its absolute value let's introduce a sign parameter σ as

$$\sigma := \operatorname{sgn}(1-4lb).$$

It follows that for the nonaccelerating case, $\alpha = 0$, there are two cases, either $\sigma = 1$ and

$$0 < 1 - 4lb_+ = \frac{P(\pi)}{P(0)} \quad \text{and} \quad b_+ = \frac{2a\Lambda}{3 + a^2\Lambda + 4al\Lambda} \tag{2.33}$$

or $\sigma = -1$ and

$$0 < 4lb_- - 1 = \frac{P(\pi)}{P(0)} \quad \text{and} \quad b_- = \frac{3 + a^2\Lambda}{2l(3 + a^2\Lambda + 4al\Lambda)}, \tag{2.34}$$

where a generic non-accelerating solution was assumed, i.e. $al\Lambda \neq 0$.

In the case of generic accelerating similarily there are two solutions

$$\begin{aligned} b_+ =& (a(3\alpha^3 l^2 M(l^2 - a^2) + 6\alpha^2 l\omega(l^2 - a^2) - 2\Lambda l\omega^3 + 3\alpha M\omega^2)) \\ & (l(3\alpha^2\omega(l - a)(a + l)^2(a + 3l) - \omega^3(a\Lambda(a + 4l) + 3) + \\ & 6a\alpha^3 lM(l - a)(a + l)^2 + 6a\alpha M\omega^2))^{-1}, \\ b_- =& -(\omega^3(a^2\Lambda + 3) + 3\alpha^2\omega(a^2 - l^2)(a^2 + 3l^2) + 6a^2\alpha^3 lM(a^2 - l^2)) \\ & (2l(3\alpha^2\omega(l - a)(a + l)^2(a + 3l) - \omega^3(a\Lambda(a + 4l) + 3) + \\ & 6a\alpha^3 lM(l - a)(a + l)^2 + 6a\alpha M\omega^2)). \end{aligned} \tag{2.35}$$

The above have a proper limit to (2.33) and (2.34) respectively.

I wish to emphasize that for any choice of spacetime parameters $(\alpha, l, a, M, \Lambda)$ such that the metric tensor is well defined a suitable b may be found such that the metric tensor in the orbit-space is smoothly defined. In particular, this is possible in the presence of the acceleration parameter $\alpha \neq 0$, which often is interpreted as introducing a conical singularity into the spacetime.

2.2.3 Misner construction and the bundle structure

Having obtained a smooth orbit-space metric q the remaining part of the metric tensor may be interpreted as a connection 1-form $\omega = \frac{g(\xi,\cdot)}{g(\xi,\xi)}$ and the lapse function $g(\xi, \xi)$ in the decomposition

$$g = g(\xi, \xi)\omega^2 + q. \tag{2.36}$$

To show that ω is smooth on the entire PFB it is enough to check this in its two trivializations. Using the adapted coordinates ω reads

$$\omega = \mathrm{d}\tau - \frac{(1 - Ab)\Sigma \mathrm{d}r + \left(A\mathcal{Q}(1 - Ab) - \mathcal{P}\sin^2\theta\rho(a - b\rho)\right)\mathcal{P}(0)^{-1}\mathrm{d}\varphi}{\mathcal{Q}(1 - Ab)^2 - \mathcal{P}\sin^2\theta(a - b\rho)^2}. \tag{2.37}$$

The $\mathrm{d}r$ component is analytic whenever the orbit space is well defined and so only the $\mathrm{d}\varphi$ component need to be checked. Thanks to the fact that in the coordinates (v, r, θ, ϕ) $A\mathrm{d}\phi$ is smooth at $\theta = 0$ the 1-form ω is regular at $\theta = 0$. It fails to be continuous at $\theta = \pi$ due to not vanishing there, the argument is the same as for the ω of the Taub-NUT spacetime, given in the Introduction. Explicitly we have

$$\omega_\varphi(r, \theta = 0) = 0, \quad \omega_\varphi(r, \theta = \pi) = -\frac{4l}{1 - 4lb}\frac{1}{\mathcal{P}(0)} = \sigma\frac{-4l}{\mathcal{P}(\pi)}. \tag{2.38}$$

The value of ω_ϕ at $\theta = \pi$ is precisely the obstruction to the smoothness at $\theta = \pi$, importantly its value does not depend on r.

Now a second coordinate map may be chosen, in which the other pole is covered smoothly, such that $\omega_{\varphi'}(r', \theta' = \pi) = 0$. This motivates the following definition of coordinates smoothly covering $\theta = \pi$

$$\tau = \tau' + \sigma \frac{4l}{P(\pi)}\varphi', \quad r = r', \quad \theta = \theta', \quad \varphi = \varphi' \quad \text{for}, \quad \theta, \theta' \neq 0, \pi. \tag{2.39}$$

The above transformation determines the range of τ to be

$$\tau \in \left[0, \frac{8\pi l}{\chi_C \mathcal{P}(\pi)}\right), \quad \chi_C \in \mathbb{N} \setminus \{0\},$$

where χ_C is anticipated to be the Chern number of the bundle. In this map, the 1-form ω reads

$$\omega = \mathrm{d}\tau' - \frac{(1-Ab)\Sigma \mathrm{d}r' + \left(A'\mathcal{Q}(1-Ab) - \mathcal{P}\sin^2\theta' \rho'(a-b\rho)\right)\mathcal{P}(\pi)^{-1}\mathrm{d}\varphi'}{\mathcal{Q}(1-Ab)^2 - \mathcal{P}\sin^2\theta(a-b\rho)^2}. \tag{2.40}$$

The last step is the normalization of ℓ such that its orbit are parameterized by $2\pi-$periodic coordinate

$$\psi := \tau \frac{\chi_C \mathcal{P}(\pi)}{4l} \in [0, 2\pi). \tag{2.41}$$

It is easily verified then that χ_C is the Chern number

$$\int_{S^2} \mathrm{d}\omega = 2\pi \chi_C \tag{2.42}$$

and ω is the $U(1)-$connection of the PFB characterized by χ_C.

2.2.4 Summry of the non-singular construction

The procedure so far is as follows

1. Choose the physical parameters $(\alpha, l, M, a, \Lambda)$ of the target spacetime.
2. For given parameters choose ξ according to (2.33) and (2.34) or in the accelerated case (2.35).
3. Introduce the coordinates adopted to ξ given by (2.22) and rescale the angular coordinate ϕ by $\beta = \mathcal{P}(0)^{-1}$.
4. Impose the condition the orbits of ξ are closed, with the periodic coordinate τ, and that ξ generates a $U(1)$ action on the spacetimes by its flow.

5. The remaining part of the metric is proportional to a 1-form ω. By introducing a second map determined by the obstruction $\omega_\varphi(\theta = \pi)$, this 1-form may be made to be smooth on the entire three-sphere.

6. Additionally 1-form ω, proportional to the 1-form the bullet above, may be interpreted as a $U(1)$-connection of the bundle with a Chern number χ_C, topologically being a quotient of the S^3 by a discrete group. The PFB bundle structure is explicitly exhibited using the $2\pi-$periodic coordinate ψ.

Finally, I present the non-singular metric tensor locally isometric to accelerated Kerr-NUT-(anti-) de Sitter. As was argued before the above ingredients are explicitly smooth in two coordinates patches $(\tau, r, \theta, \varphi)$ and $(\tau', r', \theta', \varphi')$. In those coordinates the metric tensor reads

$$
\begin{aligned}
g = &-\frac{\mathcal{Q}}{\Sigma}((1-bA)d\tau - \frac{A}{P(0)}d\varphi)^2 + 2dr((1-bA)d\tau - \frac{A}{P(0)}d\varphi) + \frac{\Sigma}{P}d\theta^2 + \\
&+ \frac{P}{\Sigma}\sin^2\theta((a-\rho b)d\tau - \frac{\rho}{P(0)}d\varphi)^2,
\end{aligned} \tag{2.43}
$$

and

$$
\begin{aligned}
g' = &-\frac{\mathcal{Q}}{\Sigma}((1-bA)d\tau' - \frac{A'}{P(\pi)}\sigma d\varphi')^2 + 2dr'((1-bA)d\tau' - \frac{A'}{P(\pi)}d\varphi') + \frac{\Sigma}{P}d\theta'^2 + \\
&+ \frac{P}{\Sigma}\sin^2\theta'((a-\rho b)d\tau' - \frac{\rho'}{P(\pi)}\sigma d\varphi')^2
\end{aligned} \tag{2.44}
$$

Alternatively, the rescaled coordinates ψ and ψ' may be used.

2.2.5 Caveats of the non-singular construction

Several details concerning the non-singualr construction need to be clarified.

Smoothness

The circumference-to-ratio test is determined by the limit defined in terms of the components of the metric tensor defined on S^2. As such, if the metric tensor passes this test it seems that only the continuity at the axis is guaranteed. The smoothness at other points is immediate as q and ω are composed of manifestly smooth functions unless $\theta = 0$ or $\theta = \pi$. I opt to check the smoothness at the poles explicitly in coordinates. As was argued above apart from $\theta = 0$ and $\theta = \pi$ both q and ω are built from analytic function. The smoothness at $\theta = 0$ and $\theta = \pi$ holds and was checked by direct calculation.

This may be seen as follows. Consider a two-dimensional metric tensor

$$g_E = 2(\mathrm{d}\eta^2 + \mathrm{d}\xi^2) \tag{2.45}$$

, such that coordinates (η, ξ) are regular around $(\eta, \xi) = (0, 0)$, which will be identified with the axis. After introduction of coordinates (x, ϕ)

$$\eta = \sqrt{x}\cos(a\phi), \quad \xi = \sqrt{x}\sin(a\phi), \tag{2.46}$$

i.e. such that $\eta^2 + \xi^2 = x$ it reads

$$g_E = \frac{1}{2a}\mathrm{d}x^2 + 2a^2\mathrm{d}\phi^2. \tag{2.47}$$

Next, consider 1-form

$$\gamma = a^2x^2\mathrm{d}\phi^2 = (\xi\mathrm{d}\eta - \eta\mathrm{d}\xi)^2, \tag{2.48}$$

which is smooth by the smoothness of η and ξ. Then the generic axisymmetric metric on a sphere

$$g_A = \frac{\mathrm{d}x^2}{P(x)^2} + P(x)^2\mathrm{d}\phi^2, \tag{2.49}$$

where $x = 0$ corresponds to a symmetry axis, may be decomposed using the above as

$$g_A = Q_1(x)g_E + Q_2(x)\gamma, \tag{2.50}$$

where

$$Q_1(x) = \frac{2x}{P}, \quad Q_2(x) = \frac{P - 2xa^2Q_1}{x^2a^2}. \tag{2.51}$$

By the assumption we have $P^2(x) = 0, \partial_x P^2\big|_{x=0} = 2$ and $\phi \in [0, 2\pi[$. The first one follows from $x = 0$ corresponding to the axis of the symmetry of ∂_ϕ, the second must follow if the limit of the ratio of the circumference to radius when tending to $x = 0$ equals 2π. Therefore P has a single zero at $x = 0$ and Q_1 is smooth. Function Q_2 is smooth is the appropriate choice of

$$a = \frac{4}{\partial_x P^2}\Bigg|_{x=0} \tag{2.52}$$

is made.

Whenever a metric tensor satisfies the circumference-to-radius test and its $P(x)$ is smooth except for the axis of rotation, then the metric tensor is not only continuous at the axis but also automatically smooth.

Beyond the orbit-space decomposition

Due to the factor $g(\xi,\xi)$ in the definitions of the orbit-space metric tensor and ω, they are not defined whenever ξ becomes null or vanishes. Therefore the decomposition on metric tensor g given by (2.36) is no longer valid. However, the metric tensors given by (2.43) and (2.44) may still be well defined. They could be decomposed into another set of non-singular objects, valid even if $g(\xi,\xi)=0$ as follows. First, consider the purely angular parts

$$\frac{\Sigma}{P}d\theta^2+\frac{P}{\Sigma}\sin^2\theta\left(\frac{\rho}{P(0)}\right)^2 d\varphi^2=\frac{\Sigma}{P}\left(d\theta^2+\frac{P^2}{\Sigma^2}\sin^2\theta\left(\frac{\rho}{P(0)}\right)^2 d\varphi^2\right),\quad \text{at } \theta=0, \tag{2.53}$$

$$\frac{\Sigma}{P}d\theta'^2+\frac{P}{\Sigma}\sin^2\theta'\left(\frac{\rho'}{P(\pi)}\right)^2 d\varphi'^2=\frac{\Sigma}{P}\left(d\theta'^2+\frac{P^2}{\Sigma^2}\sin^2\theta'\left(\frac{\rho'}{P(\pi)}\right)^2 d\varphi'^2\right)\quad \text{at } \theta'=\pi. \tag{2.54}$$

Employing again coordinates well defined at the corresponding poles one can see that the coefficients of $\sin^2\theta d\varphi^2$ and $\sin^2\theta' d\varphi'^2$ are smooth and tend to 1 at $\theta=0$ and, $\theta'=\pi$, respectively.

Next, consider the differential 1-forms appearing in (2.43) and (2.44)

$$\begin{aligned}\alpha_1 &= A d\varphi, \quad \alpha_2=\sin^2\theta d\varphi,\\ \alpha_1' &= A' d\varphi', \quad \alpha_2'=\sin^2\theta' d\varphi'.\end{aligned} \tag{2.55}$$

Clearly, they are smooth in their domains including the poles $\theta=0$, and $\theta'=\pi$ respectively. The remaining elements used in the definitions of the metric tensors g and g' are smooth functions in their domains, also at the respective half-axes.

In conclusion, the metric tensors g and g' given by (2.43) and (2.44) are smoothly defined on manifold constructed above, diffeomorphic to $\mathbb{R}\times S^3$ (or a quotient of S^3), and admitting the U(1) bundle structure induced by the flow of the Killing vector field ξ.

Equivalence of ξ_+ and ξ_-

In the presence of the NUT parameter the condition (2.31) has two solutions b_+ and b_-, which distinguish two KVFs ξ_+ and ξ_-, respectively. A natural question arises whether the spacetimes constructed using ξ_+ and ξ_- are equivalent and in what sense.

The equivalence is most easily seen from the transformation (2.39) where both signs of σ are equivalent up to reversing the direction of φ. This gives a direct coordinate transformation between the spacetime constructed using ξ_+ and ξ_-, preserving all topological identifications. Rewriting (2.39) using 2π-periodic fibre coordinate gives

$$\psi=\psi'+\sigma\chi_C\varphi',\quad r=r',\quad \theta=\theta',\quad \varphi=\varphi'\quad \text{for,}\quad \theta,\theta'\neq 0,\pi. \tag{2.56}$$

From the above, it is clear that depending on σ bundles with opposite χ_C are constructed. They are homeomorphic but with opposite orientations chosen.

There is another intuition behind the equivalence. Consider the auxiliary $SU(2)$ group structure that may be introduced on every hypersurface r =const. KVF ξ_- transformed expressed in the coordinates adapted to ξ_+ reads

$$\xi_- = \partial_\tau + (b_- - b_+)\partial_{\hat{\phi}} = \partial_\tau + \frac{P(\pi)}{2l}\partial_\varphi \propto \partial_\psi + \frac{1}{2}\partial_\varphi.$$

Comparing with the $SU(2)$ invariant objects introduced in the Introduction it is clear that, after changing the coordinates to those corresponding to $C = 0$, that ξ_+ and ξ_- are one of the left and right invariant vector fields of the auxiliary group structure.

Finally this fact in yet another way. My requirement for the orbit-space is that it has a removable (by a redefinition of the angle) conical singularity, i.e. that the tonicity is the same in both poles. This happens in precisely two cases, first when two axial KVFs of spacetime are projected into the same KVF in the orbits-space and second when they are projected to the same fields up to their orientation.

Starting point

Another arbitrary choice made along the way was the initial choice of coordinates. It was assumed that the metric tensor at $\theta = 0$ is regular and that the coordinate φ is periodic. Equivalently, one could start from the metric tensor (2.11) covering smoothly $\theta = \pi$. Following the analogous procedure one finds that in coordinates (v', r', θ', ϕ') the condition for removal of conical singularity on the orbits space of the Killing vector field

$$\xi' = \partial_{t'} + b'\partial_{\phi'} \tag{2.57}$$

reads

$$\frac{\mathcal{P}(0)}{1 + 4lb} = \mathcal{P}(\pi). \tag{2.58}$$

It may be check that the following relation between constants $b_\pm$ for KVF ξ and constants $b'_\pm$ for KVF ξ' takes place

$$b_\pm = \frac{b'_\pm}{1 + 4lb'_\pm}. \tag{2.59}$$

The above is precisely the condition that ξ and ξ' are the same vector field written in different coordinate systems.

2.3 The global structure of the non-singular KNadS spacetimes

In this Section, the global structure of spacetimes without the acceleration parameters is studied. For the NUT-like choice of the parameters, $l^2 > a^2$ the spacetime manifold is the entire

$\mathbb{R} \times L(\chi_C, 1)$. Otherwise, for the Kerr-like parameters $a^2 > l^2$ the vanishing of Σ produces an non-removable curvature singularity at

$$(r, \theta) = (r', \theta') = (0, \theta_c),$$

where the critical value θ_c is defined by $\cos\theta_c = -\frac{l}{a}$. The type of singularity can be characterized as ring-like, except for the case where the ring is shrunk to a point, for $l = \pm a$. Similar to the Kerr spacetime, a curve going between the $r > 0$ to $r < 0$ regions has to cross the surface

$$r = 0, \quad \theta \neq \theta_c.$$

This region has a non-vanishing 3-dimensional volume and thus it connects the two spacetime regions making it connected.

The vector field corresponding to ∂_r in $(\tau, r, \theta, \varphi)$ chart, and to $\partial_{r'}$ in the $(\tau', r', \theta', \varphi')$ chart, where both charts are well defined on the horizon, is globally defined and everywhere null, i.e. $g(\partial_r, \partial_r) = 0$. A time orientation of spacetime can be defined by either declaring either

- ∂_r and $\partial_{r'}$ to be future-directed, or
- $-\partial_r$ and $-\partial_{r'}$ to be future directed.

The coordinate transformation

$$(\tau'', r'') := (-\tau, -r) \tag{2.60}$$

maps the first case into the second (and vice versa). Due to the above, it is possible to assume that $-\partial_r$ is a future pointing vector field. However an arbitrary sign of the parameters (a, l, M, Λ) has to be admitted, as there is no symmetry in changing M to $-M$.

The orbits of all KVFs are (generically) two-dimensional surfaces endowed with the induced geometry

$$ds^2 = -\frac{\mathcal{Q}}{\Sigma}((1 - bA)d\tau - \frac{A}{P(0)}d\varphi)^2 + \frac{P}{\Sigma}\sin^2\theta((a - b\rho)d\tau - \frac{\rho}{P(0)}d\varphi)^2, \tag{2.61}$$

The signature of the above is

- $(-, +)$, if $\mathcal{Q} > 0$,
- $(+, +)$, if $\mathcal{Q} < 0$,
- $(0, +)$, i.e. null, if $\mathcal{Q} = 0$.

If an orbit is timelike (or null) at a given point, then a vector field

$$\eta = \partial_\tau + \frac{P(0)}{\rho(r_c)}(a - b\rho(r_c))\partial_\varphi, \tag{2.62}$$

where $r = r_c$ is constant, is timelike (or null). Its time orientation is encoded by the scalar product

$$g(\eta_c, -\partial_r) = -\frac{\Sigma}{\rho} < 0$$

hence it is always future pointing.

Let r_i, $i = 1, 2, 3, 4$ be the roots of the polynomial $\mathcal{Q}$. Then the surface of $r = r_i$ are Killing horizons, in fact the only possible Killing horizons as will be showed in the next Chapter. They are generated by η with r_c replaced with corresponding r_i. It follows that horizon-generating KVFs are always future-pointing. Similarly to the Kerr-(anti-) de Sitter spacetime, all of the roots cannot have the same sign, which follows directly from the Viete's formulae.

We can introduce a coordinate

$$\Omega := \frac{1}{r}$$

valid for either $r < 0$ or $r > 0$. Then the metric tensor g can be written as

$$\begin{aligned} g = \frac{1}{\Omega^2}\Bigg(&\frac{\Lambda}{3}\left((1-bA)d\tau - \frac{A}{P(0)}d\varphi\right)^2 + \frac{d\theta^2}{P} + P\sin^2\theta\left(bd\tau + \frac{d\varphi}{P(0)}\right)^2 + \\ &- 2d\Omega\left((1-bA)d\tau - \frac{A}{P(0)}d\varphi\right) + O(\Omega)\Bigg). \end{aligned}$$

The surfaces of $\Omega = 0$, corresponding to $r = \pm\infty$, define the future / past infinity equipped with an induced geometry

$$\frac{\Lambda}{3}\left((1-bA)d\tau - \frac{A}{P(0)}d\varphi\right)^2 + \frac{d\theta^2}{P} + P\sin^2\theta\left(bd\tau + \frac{d\varphi}{P(0)}\right)^2$$

of the signature depending on Λ in the known way. Applying similar procedure to the metric tensor (2.44) one arrives at

$$\begin{aligned} g' = \frac{1}{\Omega^2}\Bigg(&\frac{\Lambda}{3}\left((1-bA)d\tau' - \frac{A'}{P(\pi)}d\varphi'\right)^2 + \frac{d\theta'^2}{P} + P\sin^2\theta'\left(bd\tau' + \frac{d\varphi'}{P(0)}\right)^2 \\ &- 2d\Omega\left((1-bA)d\tau' - \frac{A'}{P(\pi)}d\varphi'\right) + O(\Omega)\Bigg), \end{aligned}$$

which on the surfaces $\Omega = 0$ gives the following geometry

$$\frac{\Lambda}{3}\left((1-bA)d\tau' - \frac{A'}{P(\pi)}d\varphi'\right)^2 + \frac{d\theta'^2}{P} + P\sin^2\theta'\left(bd\tau' + \frac{d\varphi'}{P(\pi)}\right)^2.$$

There still remains a question of extension via the outgoing Eddington-Finkelstein coordinates

$$dv := dt - \frac{\rho}{Q}dr, \quad d\tilde{\phi} := d\phi + \frac{a}{Q}dr. \tag{2.63}$$

They differ by a change of ign of r coordinate. However to preserve the metric function it is also necessary to change the sign of M, which is not a symmetry. Therefore the two extensions are not equivalent.

The schematics of global structure summarizing the above constructions and conventions are shown in Figures 2.1 and 2.2 for positive and negative values of cosmological constant, respectively.

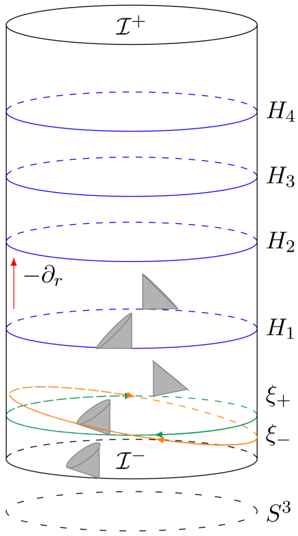

Figure 2.1: Global structure of the spacetime for $\Lambda > 0$. Each point on the cylinder corresponds to a S^2. Each circle corresponds to a S^3 although via different sections. Four uppermost, blue circles H_i correspond to surfaces of constant $r = r_i$ for $i = 1, 2, 3, 4$, i.e. Killing horizons. The green, lowermost circle is S^3 projected along the orbit of ξ_+. The orbit is closed and spacelike near both of the $\mathcal{I}$. Similarly, the orbit of ξ_- is shown in orange. Points of intersection of the orbits are circles. Future-oriented (in the direction of ∂_{-r}) parts of lightcones are shown.

2.3.1 Cosmological interpretation

So far the constructed spacetimes had a topology of $\mathbb{R} \times S^3$ (or a quotient of S^3 by a discrete group). The price that was paid is the periodicity of one of the coordinates and the introduction of the Closed Timelike Curves through any point of the spacetime whenever ξ was timelike.

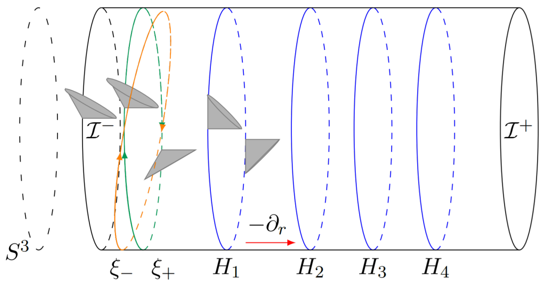

Figure 2.2: Global structure of the spacetime for $\Lambda < 0$. Each point on the cylinder corresponds to a S^2. Each circle corresponds to a S^3 although via different sections. Four rightmost, blue circles H_i correspond to surfaces of constant $r = r_i$ for $i = 1, 2, 3, 4$, i.e. Killing horizons. The green, leftmost circle is S^3 projected along the orbit of ξ. The orbit is manifestly closed and timelike near both of the $\mathcal{I}$. Similarly, the orbit of ξ_- is shown in orange. Points of intersection of the orbits are circles. Future-oriented (in the direction of ∂_{-r}) parts of lightcones are shown.

However, the presence of such curves is not necessary. The inspiration for the following construction comes from the original way in which the Taub-NUT metric was discovered by Taub. It was derived as a cosmological Bianchi IX homogeneous model[28], i.e. the algebra of symmetries action on the spacelike surfaces foliation the spacetime is $\mathfrak{su}(2)$. In the coordinates (1.52) his solution corresponds to the so called Taub region contained between two horizons located at $r_\pm = m \pm \sqrt{m^2 + l^2}$. In this region, ∂_t is spacelike, while ∂_r becomes timelike. Defining $\psi := \frac{t^\circ}{2l}$ and $\tau := r$ the metric may be written as

$$\begin{aligned} g_{TN} = &- \frac{\mathrm{d}\tau^2}{-f(\tau)} + 4l^2(-f(\tau))(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi)^2 + (l^2 + r^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) = \\ &- \frac{\mathrm{d}\tau^2}{-f(\tau)} + 4l^2(-f(\tau))(\omega^3)^2 + (l^2 + r^2)\left((\omega^1)^2 + (\omega^2)^2\right) \end{aligned} \tag{2.64}$$

where

$$f(\tau) := \frac{\tau^2 - 2m\tau - l^2}{\tau^2 + l^2} \tag{2.65}$$

and $-f(\tau)$ is positive in the Taub region. The above may be interpreted as a cosmological model without an initial or final singularity[29,28]. However, it is extensible through the initial and final surface corresponding to the horizon.

The inclusion of cosmological constant opens a possibility for finding such parameters that $-f(\tau)$ is always positive. When this happens the closed timelike curves do not appear at all and the spacetime may be viewed as an in-extendible Bianchi IX cosmological model without any singularities. Zeros of $f(\tau)$ are roots of a fourth-degree polynomial.

Given a generic fourth-degree polynomial equation with coefficients

$$ax^4 + bx^3 + c^2 + dx + e = 0 \tag{2.66}$$

one could define, aside from the discriminant Δ, the following polynomials in the coefficients

$$P = 8ac - 3b^2, \quad D = 64a^3e - 16a^2c^2 + 16a^2bd - 3b^4$$

may be used to state the sufficient condition for no real zeros to exist. If

$$\Delta > 0, \quad P > 0, \quad D > 0, \quad f(0) < 0 \tag{2.67}$$

then no roots appear, $f(r) < 0$ so and ∂_t is spacelike everywhere. Assuming $M > 0$ the inequalities (2.67) solve to

$$3M < \sqrt{6}\sqrt{-2\Lambda^2 l^6 + 3\Lambda l^4 - l^2} \wedge \left(\frac{3}{l^2} \geq 4\Lambda \vee |l|\sqrt{3\Lambda l^2\left(7 - 4\Lambda l^2\right) - 9} < 3M\right). \tag{2.68}$$

The presence of square roots additionally implicitly requires that $\Lambda > 0$. The above determines an open subset in the parameter space where the completely non-singular interpretation may exist. There is a possibility that additionally, the roots have multiplicity higher than 1, however, this does not happen unless $M = 0$.

Similarly, the KNadS spacetime in the region where ∂_r is time-like, the cosmological time $\tau := t$ may be introduced and the spacetime becomes a cosmological model with spacelike sections being topological S^3. However, the inclusion of the Kerr parameter introduces the inhomogeneity into the model. The conditions (2.67) for the function $\mathcal{Q}$ corresponding to the Kerr-NUT-(anti-) de Sitter may be solved explicitly, although the result is quite messy

$$\begin{aligned} &(\Lambda_1 < \Lambda < \Lambda_2 \quad \wedge \quad M < M_1) \vee \\ &(\Lambda_2 < \Lambda < l^{-2} \quad \wedge \quad M_1 < M < M_2) \end{aligned} \tag{2.69}$$

where

$$\begin{aligned} \Lambda_1 &= \frac{3}{a^2 + 6l^2}, \quad \Lambda_2 = \frac{3(4\sqrt{l^2 - a^2}|l| + 8l^2 - a^2)}{a^4 + 48l^4} \\ M_1 &= \frac{\sqrt{2}}{3}\sqrt{(l^2 - a^2)(1 - \Lambda l^2)(a^2\Lambda + 6\Lambda l^2 - 3))} \\ M_2 &= \frac{1}{3}\sqrt{(l^2 - a^2)(1 - \Lambda l^2)(2\Lambda(a^2 + 6l^2) - \sqrt{a^2\Lambda(a^2\Lambda + 48\Lambda l^2 - 42) + 9} - 6)}. \end{aligned} \tag{2.70}$$

By assumption the $l^2 > a^2$ it may be checked that $0 < \Lambda_1 < \Lambda_2 < l^{-2}$. This in turn implies that for $\Lambda_2 < \Lambda < l^{-2}$ the masses M_1 and M_2 are in the right order, $M_1 < M_2$, thus both branches of solution (2.69) are non-empty. Subsequently, the metric tensor reads

$$ds^2 = -\frac{\Sigma}{-\mathcal{Q}}\mathrm{d}\tau^2 + \frac{-\mathcal{Q}}{\Sigma}(\mathrm{d}t - A\mathrm{d}\phi)^2 + \frac{\Sigma}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}}{\Sigma}\sin^2\theta(a\mathrm{d}t - \rho\mathrm{d}\phi)^2, \tag{2.71}$$

where $-\mathcal{Q}(\tau)$ is always positive. Thanks to the fact that the non-singular construction is possible for any choice of parameters suitable KVF ξ may be found even if additionally the parameters have to satisfy (2.69). The non-singular construction changes the topology of surfaces of r =const (τ =const) to the S^3 with a positive definite metric tenor.

Cosmological models may be also constructed for spacetimes including the acceleration parameter α, however then one has to bear in mind that conformal factor may vanish. A similar construction has been considered in[78], where the authors have studied the impact of the Kerr parameter on non-homogeneity. However, they have assumed $\Lambda = 0$, so that the spacetime coincidently is without the conical singularity.

2.4 Accelerated Taub-NUT spacetimes

In this Section, I present two spacetimes that are not covered by the Podolsky-Griffith line metric tensor (2.1), the accelerated Taub-NUT-(anti-) de Sitter spacetimes. The goal is to show that they admit a non-singular interpretation, akin to this of the (accelerated) KNadS. At the time of writing the terminology is not yet stable and two different solutions of Petrov type I and D are referred to by this name.

Type I accelerating Taub-NUT

Chronologically first comes the Type I metric described by Podolsky and Vratny[79]

$$g = \frac{1}{F^2}\left[-\frac{\mathcal{Q}}{\mathcal{R}^2}\left(\mathrm{d}t - A\mathrm{d}\phi\right)^2 + \frac{\mathcal{R}^2}{\mathcal{Q}}\mathrm{d}r^2 + \mathcal{R}^2\left(\frac{\mathrm{d}\theta^2}{\mathcal{P}} + \mathcal{P}\sin^2\theta\mathrm{d}\phi^2\right)\right] \tag{2.72}$$

where

$$\begin{aligned}
&A(\theta) = 2l((\cos\theta - 1) - \alpha\mathcal{T}\sin^2\theta,\\
&F(r,\theta) = 1 - \alpha(r - r_-)\cos\theta,\\
&\mathcal{P}(\theta) = 1 - \alpha(r_+ - r_-)\cos\theta,\\
&\mathcal{Q} = (r - r_+)(r - r_-)(1 - \alpha(r - r_-)(1 + \alpha(r - r_-),\\
&\mathcal{T}(r,\theta) = \frac{(r - r_-)^2\mathcal{P}}{(r_+ r_-)F^2},\\
&\mathcal{R}^2(r,\theta) = \frac{1}{r_+^2 + l^2}\left(r_+^2(r - r_-)^2 + l^2(r - r_+)^2\frac{(1 - \alpha^2(r - r_-)^2)^2}{(1 - \alpha^2(r - r_-)^2\cos\theta)^4}\right),
\end{aligned} \tag{2.73}$$

where the roots $r_\pm = M + \sqrt{M^2 + l^2}$, and α, M and l are the acceleration, mass and NUT parameters. The metric solves the vacuum Einstein equations. The condition that a KVF ξ given by

$$\xi = \partial_t + b\partial_\phi, \tag{2.74}$$

admits a regular space of the orbits is formally the same as before

$$\mathcal{P}(0) = \frac{\mathcal{P}(\pi)}{|1-4lb|} \tag{2.75}$$

and solves to

$$b_+ = \frac{1}{2l(1-\alpha(r_+ - r_-))}, \quad b_- = -b_+\alpha(r_+ - r_-). \tag{2.76}$$

The regular 1-form ω reads

$$\omega = \mathrm{d}\tau + \frac{b\mathcal{Q}A^2 - \mathcal{Q}A - b\mathcal{P}\sin^2\theta\mathcal{R}^4}{\mathcal{Q}(1-bA)^2 - b^2\mathcal{P}\sin^2\theta\mathcal{R}^4}\frac{1}{\mathcal{P}(0)}\mathrm{d}\varphi, \tag{2.77}$$

which together with the smooth orbits-space metric q

$$q = \frac{\mathcal{R}^2}{F^2}\left[\frac{1}{\mathcal{Q}}\mathrm{d}r^2 + \frac{1}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{Q}\mathcal{P}\sin^2\theta}{\mathcal{Q}(1-bA)^2 - b^2\sin^2\theta\mathcal{R}^4}\mathrm{d}\varphi\right] \tag{2.78}$$

allows for the construction of a smooth metric tensor on a spacetime with the topology of a $U(1)$-PFB with a chosen χ_C. The map between the trivializations is formally the same as (2.39) and is determined by the obstruction given by

$$\omega_\varphi(\theta=\pi) = -\frac{4l}{1-4lb}\frac{1}{\mathcal{P}(0)}. \tag{2.79}$$

The construction may be extended in the expected manner to the horizon using Finkelstein-Eddington-like coordinates, similarly.

This subcase shows that the applicability of the generalized Misner procedure is broader than the Type D Plebański-Demiański solutions.

Type D accelerating Taub-NUT-(anti-) de Sitter

Recently coordinates encapsulating all Plebański-Demiański spherical black hole solutions, including the accelerating Taub-NUT-(anti-) de Sitter have been found[72,73]. Here for completeness, I present the previously elusive case of the accelerating Taub-NUT-(anti-) de Sitter. Its metric tensor is given by

$$\mathrm{d}s^2 = -f(r,x)(\mathrm{d}t - \tilde{\omega}(r,x)\mathrm{d}\phi)^2 + \frac{1}{f(r,x)}\left(G(r,x)\left(\frac{\mathrm{d}r^2}{\Delta_r(r)} + \frac{\mathrm{d}x^2}{\Delta_x(x)}\right) + \rho^2(r,x)\mathrm{d}\phi^2\right), \tag{2.80}$$

where

$$
\begin{aligned}
\Delta_r &= (1-\alpha^2 r^2)\left((r-m)^2-\sigma^2\right) \\
\Delta_x(x) &= (1-x^2)\left((1+\alpha m x)^2-\alpha^2 x^2\sigma^2\right) \\
\rho(r,x) &= \frac{\sqrt{\Delta_x(x)\Delta_r(r)}}{(1+\alpha r x)^2} \\
\tilde{\omega} &= \frac{2l}{\alpha^2 l^2+1} - \frac{\Delta_x\left(a^2-l^2-r^2\right)\left(a\alpha^2 r^2+a+2\alpha l r\right)+\Delta_r\left(a x^2+a-2 l x\right)\left(\alpha^2 x^2 (a-l)(a+l) - \right.}{\Delta_r\left(\alpha^2 x^2\left(l^2-a^2\right)+1\right)^2-\Delta_x\left(a\alpha^2 r^2+a+2\alpha l r\right)^2} \\
G &= \frac{\Delta\mathrm{r}(r)\left(\alpha^2 l^2 x^2+1\right)^2-4\alpha^2 l^2 r^2\Delta\mathrm{x}(x)}{(1+\alpha r x)^4}.
\end{aligned}
\tag{2.81}
$$

The objects necessary for the non-singular construction read

$$
q = \frac{G(r,x)}{\Delta\mathrm{r}(r) f(r,x)}\mathrm{d}r^2 + \frac{G(r,x)}{\Delta\mathrm{x}(x) f(r,x)}\mathrm{d}x^2 + \frac{f(r,x)\rho(r,x)^2}{f(r,x)^2(b\omega(r,x)-1)^2-b^2\rho(r,x)^2}\mathrm{d}\hat{\phi}^2 \tag{2.82}
$$

$$
\omega = \frac{f(r,x)^2\tilde{\omega}(r,x)(1-b\tilde{\omega}(r,x))+b\rho(r,x)^2}{b^2\rho(r,x)^2-f(r,x)^2(b\tilde{\omega}(r,x)-1)^2}\mathrm{d}\hat{\phi} \tag{2.83}
$$

Condition for removing the conical singularities in the orbit space iss

$$
\Delta_\theta(x=1) = \frac{\Delta_\theta(x=-1)}{|1-b\omega(r,x=-1)|}, \tag{2.84}
$$

where $\Delta_\theta = \frac{\Delta_x}{1-x^2}$ and tt solves to

$$
b = \frac{\alpha m\left(\alpha^2 l^2+1\right)}{-\alpha^2 l^3+2\alpha l m+l}, \quad b = \frac{1-\alpha^4 l^4}{-2\alpha^2 l^3+4\alpha l m+2l}. \tag{2.85}
$$

Obstruction at the singular pole

$$
\bar{\omega}_{\hat{\phi}}(x=-1) = -\frac{4l}{1-4bl+\alpha^2 l^2}, \tag{2.86}
$$

induces the Misner map

$$
\tau' = \tau - \frac{4l}{1-4bl+\alpha^2 l^2}\varphi, \quad r=r' \quad ,\theta=\theta', \quad \varphi=\varphi'. \tag{2.87}
$$

Contribution statement

The discussion of the conical singularity in the Introduction and its observer-dependent interpretation following in this Chapter, that the conical singularity is observer dependent are a result of discussion with Pavel Krtouš and Ivan Kolář.

The idea for the proof of smoothness of two-metric satisfying the circumference-to-radius test comes from Wojciech Kamiński.

Gelo che ti dà foco
e dal tuo foco più gelo prende!
Candida ed oscura!
Se libero ti vuol,
ti fa più servo!
Se per servo t'accetta,
ti fa Re!

From the Turandot's last riddle

3 Projectively non-singular horizons of accelerated Kerr-NUT de Sitter spacetimes

Without a doubt, the crucial property of the Plebański-Demiański family is the existence of the black hole horizons in the spacetime. If the cosmological constant is present their horizons admit various topologies, including the physically most relevant spherical black holes. This and the next Chapters are dedicated to studying the Killing horizons, modeling the black hole horizons, contained in the non-singular interpretation of (spherical) accelerating Kerr-NUT-(anti-) de Sitter spacetimes, described in the last Chapter. Particular attention is given to the simpler case of the KNadS horizons.

The defining feature of horizons discussed in this Chapter is that their spaces of the null generators are smooth. As was demonstrated in Chapter 2 for any choice of spacetime parameters there exists distinguishes KVF possessing a smooth orbit-space, the role of which is now played by the space of the null generators of the horizon. Conversely, setting the distinguished KVF to be the horizon-generating KVF ℓ imposes a constraint on the spacetime parameters. Such horizons admit a bundle structure as was described for the IH in the Introduction, with the $U(1)-$action generated by the KVF developing the Killing horizon.

It turns out that at most one of the horizons contained in the spacetimes may admit smooth space of the null generators. The properties of such horizons in the KNadS spacetimes are particularly clear; They are non-extremal, have the largest radius, and require that $\Lambda > 0$. Finally, I present the relation between the abstract Isolated Horizons obtained as solutions to the Type D equation on the Hopf bundle and the accelerated KNadS Killing horizons.

It is worth noting that the compact horizons have been of interest for a long time in the context of constructing the counter-example to the cosmic censorship conjecture. In a series of papers by Moncrief and Isenberg have shown that compact Cauchy Horizons ruled by closes null generators necessarily admit a KVF[80,81,82]. More broadly they used the horizons of this type as initial data for the construction of a generalized, non-homogeneous Taub-NUT spacetimes, not contained in the Plebańki-Demiański, admitting the $U(1)$ symmetry and PFB structure[83,84]

3.1 Singularities of the horizon

As pointed out in the Introduction, IHs distinguish two-dimensional surfaces transversal to the null KVF ℓ. Any such surface parameterizes the null orbits of the KVF and so is often called the space of the null generators. If the PFB structure with action given by the flow of ℓ is assumed then the space of the null generators is the bundle's base space.

A crucial property of the horizons considered in this Chapter is the *projectivelly (non-)singularity*. I define the horizon to be projectively (non-)singular if its space of the null generators is (non)-singular, in the sense of being a smooth two-manifold.

Three possible sources of the singularity may be distinguished

- **Curvature singularity** - caused by a curvature invariant becoming infinite at the horizon. In the Plebański-Demiański family, this type of horizon may occur if the radius of the horizon vanishes, i.e. $r_H = 0$. Conditions leading to this singularity will be discussed, however, such horizons will be excluded from further discussion due to the severe nature of the singularity.

- **Conformal singularity** - caused by the conformal factor becoming infinite at the horizon. Geometrically this corresponds to the space of the null generators intersecting the Scri. Physically they may be interpreted as an acceleration horizon bordering a region of space receding from the black hole faster than the speed of light. In the Plebański-Demiański family, this type of horizon is encountered in the presence of the acceleration parameter α.

- **Conical singularity** - caused by the singularity of the axis of rotation associated with the wrong choice of the range of the parameter. The angle deficit (or surplus) may be different at both poles of the two-sphere and thus rescalling to the correct range of 2π may be impossible at both parts of the axis simultaneously. Regularising one of the poles, unless some other constraints are implied, should be always possible. Importantly there is no other singularity associated with the conical singularity.

3.2 Geometry of the non-singular horizons

To study the geometry of the horizon it is necessary to use a coordinate system covering at least part of it. In the following, I will choose the in-going Finkelstein-Eddington coordinates defined by (2.7), such that the metric tensor reads (2.8). Analogous analysis could be done in the out-going coordinates system.

The Killing horizon is then a set such that the norm of ξ vanishes, without ξ being a zero vector, and that the surface is null[85]. Those conditions are expressed as

$$\begin{aligned} \xi^\mu \xi_\mu &= 0, \\ (\xi^\alpha \xi_\alpha)^{,\mu} (\xi^\beta \xi_\beta)_{,\mu} &= 0. \end{aligned} \tag{3.1}$$

The KVF ∂_t does not vanish at any point of the spacetime, thus as long as ξ contains a part that is proportional to ∂_t it does not vanish. Conditions (3.1) are annihilated by both ∂_t and ∂_ϕ and so we can conclude that any constant linear combination of those fields is tangent to the horizon. As was argued in the previous Chapter to capture all KVFs up to scale (except for ∂_ϕ, which however clearly does not generate a Killing horizon) it is enough to consider

$$\xi = \partial_v + b\partial_\phi.$$

Consider then the two-dimensional orbit of the symmetries generated by ∂_v and ∂_ϕ. The metric tensor q_{orbit} defined thereon is simply a restriction of g to a distribution spanned by those vector fields. It reads

$$q_{\mathrm{orbit}} = \frac{1}{F^2}\left(-\frac{\mathcal{Q}}{\Sigma}(\mathrm{d}v - A\mathrm{d}\phi)^2 + \frac{\mathcal{P}}{\Sigma}\sin^2\theta(a\mathrm{d}v - \rho\mathrm{d}\phi)^2\right), \tag{3.2}$$

in the in-going Eddington-Finkelstein coordinates. If a particular combination of the KVFs would be Killing horizon generating then its orbit would have to be null (at some hypersurface) and so the above metric tensor would have to degenerate there. Therefore it is enough to study the degeneracy conditions. The 1-forms

$$\mathrm{d}v - A\mathrm{d}\phi, \quad a\mathrm{d}v - \rho\mathrm{d}\phi$$

are linearly dependent if and only if $\rho - aA = 0$. However, this would correspond precisely to the vanishing of Σ and the curvature singularity, due to the relation $\Sigma = \rho - aA$. And so the degeneracy condition reads

$$\det(q_{\mathrm{orbit}}) = \frac{\mathcal{P}\mathcal{Q}\sin^2\theta}{F^4} = 0.$$

Vanishing of $\sin^2\theta$ is the spherical coordinates singularity and by assumption, we have $\mathcal{P} > 0$. The only remaining possibility is that we must have $\mathcal{Q} = 0$. This implies that the Killing horizon is located on the hypersurface $r = r_H$, where r_H is the root of $\mathcal{Q}$. To determine the coefficient b of the linear combination consider the norm of ξ

$$g(\xi, \xi) = \frac{\mathcal{P}\sin^2\theta(a - b\rho)^2 - \mathcal{Q}(1 - Ab)^2}{F^2\Sigma}. \tag{3.3}$$

Restricting it to $r = r_H$ and demanding that it vanishes yields $b = \frac{a}{\rho_H}$. The horizon-generating KVF reads

$$\ell = \partial_v + \frac{a}{\rho_H}\partial_\phi, \tag{3.4}$$

where $\rho_H := \rho\big|_{r=r_H}$. In the generic case, there are up to 4 distinct roots of $\mathcal{Q}$, with repeating roots corresponding to extremal horizons, and so there are up to 4 Killing horizons present in the spacetime. They correspond to the hypersurfaces $r = r_H$ and importantly they are developed by different KVF, each defined by the corresponding root.

Consider the metric tenor q_H on the horizon, i.e. spacetimes metric tensor pulled back to $r = r_H$.

$$q_H = \frac{1}{F_H^2}\left(\frac{\Sigma_H}{P}\mathrm{d}\theta^2 + \frac{P}{\Sigma_H}\sin^2\theta(a\mathrm{d}v - \rho_H\mathrm{d}\phi)^2\right). \tag{3.5}$$

The fact that the horizon is a null surface is equivalently expressed as the degeneracy of q_H, which is easily confirmed by the introduction of adapted coordinates (τ, x_H^i), $i = 2, 3$, such that

$$\xi(\tau) = 1, \quad \xi_H(x_H^i) = 0. \tag{3.6}$$

Take for example the coordinates

$$\tau = v, \quad x_H^2 = \theta, \quad x_H^3 = \hat{\phi} := -\frac{a}{\rho_H}v + \phi, \tag{3.7}$$

such that the metric tensor reads

$$q_H = \frac{1}{F_H^2}\left(\frac{\Sigma_H}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}}{\Sigma_H}\sin^2\theta\rho_H^2\mathrm{d}\hat{\phi}^2\right). \tag{3.8}$$

and its signature is explicitly degenerate $(0, +, +)$

With a slight abuse of the notation, the coordinates $(v, \theta, \hat{\phi})$ may be used as spacetime coordinates, horizon coordinates, and also the coordinates on the space of the null generators of the horizon, equipped with the Riemannian metric tensor ${}^{(2)}q_H$.

For the metric tensor ${}^{(2)}q_H$ the natural choice of the axial Killing vector field is $\partial_{\hat{\phi}}$. Equivalently for q_H, the natural choices are $\partial_{\hat{\phi}}$ or ∂_ϕ. The axis is defined by the condition

$${}^{(2)}q_H(\partial_{\tilde{\phi}}, \partial_{\tilde{\phi}}) = \frac{1}{F_H^2}\frac{\mathcal{P}}{\Sigma_H}\rho_H^2\sin^2\theta = 0, \tag{3.9}$$

which identifies it as $\theta = 0$ and $\theta = \pi$. As was the case for the orbit-space of a generic ξ, the same vector field is axial for both parts of the axis.

The condition for removal of the conical singularity can be stated again through the limit of circumference-to-radius ratio and it reads

$$\frac{\mathcal{P}(0)}{r_H^2 + (l+a)^2} = \frac{\mathcal{P}(\pi)}{r_H^2 + (l-a)^2}, \quad \beta = 1/\mathcal{P}(0). \tag{3.10}$$

The particular solutions will depend on chosen $\mathcal{P}$, however, the corresponding $2\pi-$periodic coordinates always is defined as

$$\varphi := \frac{\hat{\phi}}{\beta} \in [0; 2\pi[. \tag{3.11}$$

Coordinates $(\tau = v, \theta, \varphi)$ are used in the remaining part of this Chapter.

Finally the well-defined metric tensor on the space of the null generators with the topology of the two dimensional sphere reads

$${}^{(2)}q_H = \frac{1}{F_H^2}\left(\frac{\Sigma_H}{\mathcal{P}}\mathrm{d}\theta^2 + \frac{\mathcal{P}}{\Sigma_H}\sin^2\theta\rho_H^2\mathcal{P}^{-1}(0)\mathrm{d}\varphi^2\right), \tag{3.12}$$

and its associated area 2-form is

$${}^{(2)}\epsilon_H = \frac{\rho_H}{F_H^2\mathcal{P}(0)}\sin\theta\mathrm{d}\theta\wedge d\varphi. \tag{3.13}$$

The horizon rotation 1-form defined by (1.2) reads

$$\omega = \kappa\left(\mathrm{d}\psi - \chi_C\frac{A}{4l}\frac{\rho_H'}{\Sigma_H}\mathrm{d}\varphi\right) + \frac{F_H\partial_\theta\Sigma_H - \Sigma_H\partial_\theta F_H}{2\Sigma_H F_H}\mathrm{d}\theta + \frac{\mathcal{P}\rho_H r_H a\sin^2\theta}{\Sigma\mathcal{P}(0)}\mathrm{d}\varphi. \tag{3.14}$$

A straightforward calculation gives its exterior derivative as

$$\begin{aligned}\mathrm{d}\omega = &\left(\frac{ar_H\rho_H}{\Sigma^2}\partial_\theta\frac{\mathcal{P}\sin^2\theta}{\Sigma} - \frac{\mathcal{Q}'(r_H)}{2}\partial_\theta\frac{A}{\Sigma_H}\right)\frac{1}{\mathcal{P}(0)}\mathrm{d}\theta\wedge\mathrm{d}\varphi = \\ &\Bigg(ar_H\left(\mathcal{P}'\Sigma_H + \mathcal{P}\left(2\cos\theta\rho_H^\circ + al(3+\cos 2\theta)\right)\right) \\ &+ 2\frac{\mathcal{Q}'(r_H)}{\rho_H}(l + a\cos\theta)\Bigg)\frac{\rho_H}{\Sigma_H^2}\frac{\sin\theta}{\mathcal{P}(0)}\mathrm{d}\theta\wedge\mathrm{d}\varphi.\end{aligned} \tag{3.15}$$

Comparison with the area 2-form gives the pseudo-scalar rotational invariant Ω

$$\Omega = \left(ar_H\left(\mathcal{P}'\Sigma_H + \mathcal{P}\left(2\cos\theta\rho_H^\circ + al(3+\cos 2\theta)\right)\right) + 2\frac{\mathcal{Q}'(r_H)}{\rho_H}(l + a\cos\theta)\right)\frac{F_H^2}{\Sigma_H^2} \tag{3.16}$$

3.2.1 U(1)-bundle structure of the horizon

In the case of rotating spherical horizons, similarily to the case of Taub-NUT's horizons, one can define a U(1)-principle bundle explicitly. Consider a 2π-periodic fibre coordinate ψ defined on the horizon

$$\psi := \tau\frac{\chi_C\mathcal{P}(\pi)}{4l} \in [0, 2\pi), \tag{3.17}$$

where $\chi_C \in \mathbb{N}$ is expected to be the Chern number of the horizon bundle. The ℓ rescaled such that $\ell(\psi) = 1$ is

$$\ell = \frac{4l}{\chi_C \mathcal{P}(\pi)} \partial_\tau = \frac{4l}{\chi_C \mathcal{P}(\pi)} \left(\partial_v + \frac{a}{\rho_H} \partial_\phi \right) \tag{3.18}$$

The surface gravity may be computed as the self-acceleration of the recalled ℓ and it reads

$$\kappa = \frac{4l}{m} \frac{1}{\mathcal{P}(\pi)} \frac{\partial_r \mathcal{Q}}{2\rho} \bigg|_{r=r_H} = \frac{4l}{m} \frac{1}{\mathcal{P}(0)} \frac{\partial_r \mathcal{Q}}{2\rho'} \bigg|_{r=r_H}, \tag{3.19}$$

where $\rho' = r^2 + (l-a)^2$.

The candidate for the U(1)-connection 1-form is the rotation 1-form of the horizon given by (3.14). As before, it is required to be well defined everywhere on the horizon, including $\theta = \pi$. Examining ω it is clear that θ and the second term proportional to $\mathrm{d}\varphi$ are smooth. Indeed, their coefficients are smooth functions, proportional to $\sin\theta$ and $\sin^2\theta$ which vanish sufficiently fast when tending to the poles. The value of the first term of ω_φ at $\theta = \pi$, if it is non-zero, is an obstacle to smoothness at the second poles and it reads

$$\omega_\varphi(\theta = \pi) = -\frac{4l}{\mathcal{P}(0)} \frac{\rho_H}{\rho'_H} = -\frac{4l}{\mathcal{P}(\pi)}, \tag{3.20}$$

where in the last equality the constraint on the parameters has been used. It motivates the definition of the following transition map between local trivialization of the horizon:

$$\tau = \tau' + \frac{4l}{\mathcal{P}(\pi)} \varphi', \quad \theta = \theta', \quad \varphi = \varphi' \text{ for } \theta, \theta' \neq 0, \pi, \tag{3.21}$$

or using the bundle-adapted coordinates

$$\psi = \psi' + \chi_C \varphi', \quad \theta = \theta', \quad \varphi = \varphi' \text{ for } \theta, \theta' \neq 0, \pi, \tag{3.22}$$

In the second chart, we have

$$\omega = \kappa \left(\mathrm{d}\psi' - \frac{A'}{4l} \frac{\rho_H}{\Sigma_H} \mathrm{d}\varphi' \right) + \frac{F_H \partial_{\theta'} \Sigma_H - \Sigma_H \partial_{\theta'} F_H}{2 \Sigma_H F_H} \mathrm{d}\theta' + \frac{\mathcal{P} \rho_H r_H a \sin^2\theta'}{\Sigma \mathcal{P}(0)} \mathrm{d}\varphi'. \tag{3.23}$$

To define the connection 1-form it is enough to rescale ω by the factor κ^{-1}. However even simpler choice is possible, namely the part of (3.14) is proportional to κ, subsequently recalled by κ^{-1}. The $U(1)-$connection A reads

$$A := \mathrm{d}\psi - \chi_C \frac{A}{4l} \frac{\rho'_H}{\Sigma_H} \mathrm{d}\varphi = \mathrm{d}\psi' - \frac{A'}{4l} \frac{\rho_H}{\Sigma_H} \mathrm{d}\varphi'. \tag{3.24}$$

A straightforward integral of the $U(1)$-curvature's pullback

$$\int_{S^2} F := \int_{S^2} \mathrm{d}A = 2\pi \chi_C \tag{3.25}$$

confirms that χ_C is the Chern number characterizing the horizon.

3.3 Projectively non-singular Kerr-NUT-(anti-)de Sitter

In this Section, a simpler case of an unaccelerated black hole, i.e. with $\alpha = 0$ is considered. This drastically simplifies the form of the functions appearing in the metric tensor, in particular, we have $F = 1$. First let's note that there are even simpler sub-cases which either solve (3.10) automatically or do not do it at all. They are as follows:

- For $a = 0$ we have $\mathcal{P} = 1$ satisfying the conditions, together with $\beta = 1$. This class corresponds to Taub-NUT-(anti-) de Sitter.
- For $l = 0$ we have $\mathcal{P} = 1 + a^2\frac{\Lambda}{3}\cos^2\theta$ and $\beta = (1 + a^2\frac{\Lambda}{3})^{-1}$, which corresponds to Kerr-(anti-) de Sitter. Further limits to $a = 0$ and Λ also always satisfy the condition. Of course if the NUT parameter vanishes there is no need to impose a non-trivial bundle structure in the first place.
- For $\Lambda = 0$, and therefore $\mathcal{P} = 1$, the condition (3.26) reads

  $$r_H^2 + (l + a)^2 = r_H^2 + (l - a)^2.$$

  Its only solutions are $a = 0$ or $l = 0$ and so this case reduces to one of the previous.

These special sub-cases where at least one of the a, l, or Λ vanishes have been known in the literature. However, their distinguished position among the non-singular black hole solutions could not have been pointed out without considering the full Kerr-NUT-(anti-) de Sitter spacetime.

The new case is obtained only after assuming that the parameters are generic, i.e. $al\Lambda \neq 0$. Then the condition (3.10) solves to

$$\Lambda = \frac{3}{a^2 + 2l^2 + 2r_H^2}. \tag{3.26}$$

In the rest of this Chapter, the parameters will be assumed to take generic, non-zero values unless a sub-case is explicitly studied.

3.3.1 Properties on non-singular horizons

Due to the fact that M enter $\mathcal{Q}$ linearly it is convenient to solve for it in terms of the horizon radius r_H as

$$M = \frac{a^4 - 2a^2l^2 + l^4 + 2a^2r_H^2 - 6l^2r_H^2 + r_H^4}{2a^2r_H + 4l^2r_H + 4r_H^3}, \tag{3.27}$$

and instead use the parameters $(r_H, l, a, \alpha, \Lambda)$.

Using the associated area 2-form the area of a non-singular horizon may be calculated as

$$A = 4\pi\frac{\rho_0}{\mathcal{P}(0)} = 4\pi\frac{3}{2}\frac{1}{\Lambda}.$$

Then the area-radius is simply $R^2 = \frac{3}{2}\frac{1}{\Lambda}$.

An immediate question of extremality comes to mind. Suppose that r_H is the radius of a non-singular horizon. Then its extremality is equivalent to r_H being a repeated root of $\mathcal{Q}$ with Λ constrained by (3.26). For r_H to be a root of multiplicity 2 the condition

$$a^4 - 2a^2\left(l^2 - r_H^2\right) + \left(l^2 + r_H^2\right)^2 = 0, \tag{3.28}$$

would have to be satisfied, yet the above has no real solution. This agrees with [11], where the authors have found that the extremal Kerr-NUT-(anti-) de Sitter horizons necessarily have angle deficits.

Location of the non-singular horizon

The next question is the location of the non-singular horizon with respect to the other horizons. This problem amounts to studying the polynomial $\mathcal{Q}$ with Λ and M given by (3.26) and (3.27). The first constraint assures that the discuses horizon is projectively non-singular, while the second exchanges the mass parameter M for r_H of a given horizon. To clarify the notation let's denote it as $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}(y)$, where y is a more convenient variable defined by

$$y = \frac{r}{r_H} - 1.$$

The simplification comes from the fact that now $r = r_H$ corresponds to $y = 0$ and for $y > 0$ we have $|r| > |r_H|$, irrespective of the sign of r_H. The polynomial reads

$$\tilde{\mathcal{Q}}(y) = -\frac{y\left(a^4 - 2a^2\left(l^2 - r_H^2\right) + l^4 + 2l^2 r_H^2(2y+1) + r_H^4\left(y^3 + 4y^2 + 4y + 1\right)\right)}{a^2 + 2\left(l^2 + r_H^2\right)}. \tag{3.29}$$

The tool that will be used to analyze the location of the root r_H is the Descartes rule of sign[86]. It may be stated as follows:

- Compute all of the coefficients of the polynomial.
- Introduce an index tracking how many times the sign of the consecutive coefficients has changed. If a coefficient vanishes then the sign does not change.
- The index is an upper bound on the number of positive roots of a given polynomial. The actual number may be smaller due to the existence of pairs of complex roots.

Up to a positive factor of $a^2 + 2(l^2 + r_H^2)$ the coefficients of $\tilde{\mathcal{Q}}$ may be listed as

$$\begin{aligned}
\tilde{\mathcal{Q}}_0 &= 0, \\
\tilde{\mathcal{Q}}_1 &= -a^4 + 2a^2(l^2 - r_H^2) - (l^2 + r_H^2)^2, \\
\tilde{\mathcal{Q}}_2 &= -4r_H^2(a^2 + l^2), \\
\tilde{\mathcal{Q}}_3 &= -4r_H^4, \\
\tilde{\mathcal{Q}}_4 &= -r_H^4.
\end{aligned} \tag{3.30}$$

The coefficient $\tilde{\mathcal{Q}}_1$ can be easily checked (e.g. treating it as a quadratic function of a^2) to be always negative. This implies that the number of changes of signs is zero and so there are no positive roots $y > 0$ of $\tilde{\mathcal{Q}}$. Remembering that $y = 0$ corresponds to $r = r_H$ it follows if other horizons exist, the radius of the non-singular one is of the largest absolute value, i.e. that $|r_H| > |r_i|$ where r_i is a radius of any other horizon.

Horizons at $r_H = 0$

The constraints (3.26) and (3.27) implicitly assume that the radius does not vanish, $r_H \neq 0$. It turns out this constraint does not lead to a loss of generality as it is not possible that a protectively non-singular horizon of the Kerr-NUT-(anti-) de Sitter spacetime is located at the surface of $r_H = 0$.

Suppose on the contrary that $\mathcal{Q}$ admits a root of $r_H = 0$. This is equivalent to its constant term being equal to zero and so we must have

$$(a^2 - l^2)(1 - l^2\Lambda) = 0.$$

This in turn implies that

$$a^2 = l^2, \quad \text{or} \quad 1 - l^2\Lambda = 0.$$

The first case, $a = l$, together with $r_H = 0$ implies that the function $\Sigma_H = l^2(\cos\theta + 1)^2$ vanishes whenever $\theta = \pi$. However, it may be shown that the scalar curvature of the metric tensor ${}^{(2)}q_H$ on the space of the null generators of the horizon is proportional to Σ_H^{-1} and therefore $\theta = \pi$ actually corresponds to the curvature singularity. A similar result follows for $a = -l$, with curvature singularity at $\theta = 0$ instead. The last possibility is that $\Lambda = 1/l^2$. Solving (3.26) with this value again requires that $a^2 = l^2$. In conclusion, the projectively non-singular horizon of the Kerr-NUT-de Sitter spacetime located at $r_H = 0$ possesses a curvature singularity and therefore without loss of generality may be excluded from further investigation.

3.4 Projectively non-singular accelerated Kerr-NUT-(anti-) de Sitter horizons

In this Section the properties of horizons with non-vanishing acceleration parameter will be investigated, i.e. $\alpha \neq 0$ will be assumed. The condition for removal of the angular deficit is formally the same as (3.10) but with a more complicated definition of $\mathcal{P}$. It solves to

$$\Lambda = \frac{1}{\xi}\frac{3\alpha\bar{\omega}\left(a^2 - l^2 + r_H^2\right)^2 + 6\alpha^2 l r_H^3(a^2 - l^2) + 6lr_H\bar{\omega}^2}{\alpha\bar{\omega}\left(l^2 + r_H^2\right)d^- d^+ + 2lr_H\bar{\omega}^2 d + 2\alpha^2 l r_H^3 \bar{d}}, \tag{3.31}$$

where some particular combinations of parameters have been introduced

$$\begin{aligned}
d^{-} &= a^2 - l^2 - 2lr_H + r_H^2, \\
d^{+} &= a^2 - l^2 + 2lr_H + r_H^2, \\
d &= a^2 + 2\left(l^2 + r_H^2\right), \\
\bar{d} &= 2l^2\left(l^2 + r_H^2\right) - a^2\left(2l^2 + r_H^2\right).
\end{aligned} \tag{3.32}$$

The relation between the radius of the horizon r_H and the mass parameter is given by

$$\begin{aligned}
M =& l\bar{\omega}\Big(d^{-}d^{+} + 4\alpha l r_H^3\left(-a^2 + l^2 - r_H^2\right)\Big)\Big(\alpha\bar{\omega}\left(l^2 + r_H^2\right)d^{+}d^{-} + \\
& 2lr_H\bar{\omega}^2 d + 2\alpha^2 l r_H^3\left(2l^2\left(l^2 + r_H^2\right) - a^2\left(2l^2 + r_H^2\right)\right)\Big)^{-1}.
\end{aligned} \tag{3.33}$$

In the case of $\alpha = 0$ the above have an appropriate limit to the un-accelerated condition (3.26) and (3.27), irregardless of the value of the gauge parameter $\bar{\omega}$.

Here it must be stated that precisely one horizon in a given spacetime can be made protectively non-singular. This is implied by the nature of any solution to (3.26) where r_H appears explicitly. If it is satisfied for some r_H then it cannot be satisfied for any other horizon.

3.4.1 Extremal non-singular horizons

Finally, the extremal horizons are present in the accelerated case. The condition is obtained identically to the non-accelerated case and reads

$$\begin{aligned}
&\left(a^4 - 2a^2\left(l^2 - r_H^2\right) + \left(l^2 + r_H^2\right)^2\right) \\
&\left(\alpha r_H\bar{\omega}\left(a^2 + r_H^2\right) + \alpha l^2(-r_H)\bar{\omega} + l\left(\bar{\omega}^2 - \alpha^2 r_H^4\right)\right) = 0.
\end{aligned} \tag{3.34}$$

The first term is the same the condition for the non-accelerated horizon to be extremal and thus never vanishes. Vanishing of the second term gives

$$a^2 = \frac{\alpha l^2 r_H\bar{\omega} + \alpha^2 l r_H^4 - l\bar{\omega}^2 - \alpha r_H^3\bar{\omega}}{\alpha r_H\bar{\omega}}. \tag{3.35}$$

It has to be noted that if the horizon is extremal then the space of the null generators is a globally defined section and the horizons are Kerr-like[87]. Thus the $U(1)-$ bundle trivializes and is usually considered to be simply a $\mathbb{R}-$ bundle. This is in agreement with the fact that for $\kappa = 0$ it is no longer possible to defined the $U(1)-$connection by rescaling ω by κ^{-1}.

3.4.2 Horizons located at $r_H = 0$

With the inclusion of the acceleration, a qualitatively distinguished type of horizon appears if one sets $r_H = 0$. The necessary condition for its existence is a choice of the parameters such that the constant coefficient of $\mathcal{Q}$ vanishes. This condition reads

$$\bar{\omega}\left(a^2 - l^2\right)\left(\Lambda l^2\bar{\omega} - 2\alpha l m - \bar{\omega}\right) = 0. \tag{3.36}$$

The gauge parameter $\bar{\omega}$ does not vanish by assumption and the possibility of $a^2 = l^2$ leads to the same problem of curvature singularity as for the not accelerating case. Thus it follows that

$$1 - \Lambda l^2 + \frac{2\alpha l M}{\omega} = 0.$$

If additionally, the horizon is to be protectively non-singular the condition for the vanishing of the conical singularity (3.10) must be imposed. Solving them together implies that

$$\Lambda = \frac{3}{l^2} > 0, \quad \bar{\omega} = lM\alpha$$

The above significantly simplifies the metric tensor, now depends only on mass M parameter, the cosmological constant $\Lambda = \frac{3}{l^2}$ and a ratio of Kerr and NUT parameters

$$\chi := \frac{a}{l}. \tag{3.37}$$

In the simplified form the functions appearing in the definition of g read

$$\begin{aligned}
F &= 1 - \frac{r}{lM}(l + a\cos\theta) = 1 - \frac{r}{M}(1 + \chi\cos\theta),\\
\mathcal{P} &= \frac{1}{l^2}(l + a\cos\theta)^2 = (1 + \chi\cos\theta)^2,\\
\Sigma &= r^2 + \frac{\Lambda}{3}(1 + \chi\cos\theta)^2,\\
\mathcal{Q} &= -\frac{r}{l^2}\left(r^3 + (a^2 - l^2)r + 2ml^2\right) = r\left(-\frac{\Lambda}{3}r^3 + (1 - \chi^2)r - 2M\right),
\end{aligned} \tag{3.38}$$

with an even simpler form at the horizon $r_H = 0$

$$F\bigg|_{r=0} = 1, \quad \frac{\Sigma}{\mathcal{P}}\bigg|_{r=0} = \frac{3}{\Lambda}. \tag{3.39}$$

Surprisingly, after the necessary rescaling of the axial coordinate by the factor of $\beta = \mathcal{P}(0)^{-1} = (1+\chi)^{-2}$ the metric on the space of the null generators becomes the flat metric on the two-dimensional sphere

$${}^{(2)}q = \frac{3}{\Lambda}\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right), \tag{3.40}$$

with the radius $R = \sqrt{\frac{3}{\Lambda}}$ and $(\theta, \varphi) \in [0;\pi] \times [0; 2\pi[$.

Additionally, the horizon invariants also benefit from the simplified form, in particular, r_H explicitly appears in ω and Ω. They read now

$$\kappa = -\frac{4l}{\chi C}\frac{2\Lambda M}{3(1-\chi^2)^2}, \tag{3.41}$$

$$\Omega = \left(\frac{3}{\Lambda}\right)^{3/2}\frac{2M}{(1 + \chi\cos\theta)^3}. \tag{3.42}$$

3.5 Connection to the Type D equation

My initial motivation for the study of the Plebański-Demiański solutions with the U(1)-bundle structure has its origin in the local theory of the Isolated Horizons[88]. In this Section, the problem of embeddability of the axisymmetric solutions to the Petrov Type D equation on the Hopf bundle will be tackled. Solutions of such equation provide the geometry of the Isolated Horizon in the form of the Gaussian curvature of the section of the horizon and the rotation pseudo-scalar invariant Ω. Using the assumptions on the structure of the horizon one can deduce the degenerate metric q on the horizon, together with the rotation 1-form ω. Given the assumptions on the bundle structure, reconstruction of the bundle connection and curvature follows easily.

To solve the Petrov Type D equation on a surface with a bundle structure over a sphere, it is most convenient to introduce an axially adopted frame

$$m^A\partial_A = \frac{1}{R\sqrt{2}}\left(\frac{1}{P(x)^2}\mathrm{d}x^2 + P(x)^2\mathrm{d}\varphi^2\right) \tag{3.43}$$

such that the metric tensor on the space of the null generators reads

$${}^{(2)}q_D = R^2\left(\frac{1}{P^2}\mathrm{d}x^2 + P^2\mathrm{d}\varphi^2\right), \tag{3.44}$$

where $x \in [-1;1]$ and $\varphi \in [0, 2\pi[$. Constant R will be referred to as the area radius and P^2 as the frame coefficient[89]. Note that the φ coordinate has already been fixed to be 2π-periodic. Together with the assumed axial symmetry, this makes ${}^{(2)}q_D$ unique up to choosing the origin of φ. The smoothness of ${}^{(2)}q_D$ is guaranteed by the assumption that

$$P^2(x)\bigg|_{\pm1} = 0, \quad \partial_x P^2(x)\bigg|_{\pm1} = \mp 2. \tag{3.45}$$

Detailed geometric descriptions of Isolated Horizons with Hopf bundle structure have been obtained in[90]. The authors have distinguished three classes of solutions to the Petrov Type D equation on the Hopf bundle. In each class, one of the parameters is a "continuous Chern number" $n := \kappa\, \chi_C$, where κ is the surface gravity and $\chi_C \in \mathbb{Z}$ is the Chern number*. The classes are as follows

Class I. Parameterised by 2 real parameters n and R. This class is characterised by

$$\begin{aligned} &R^2 > 0,\\ &P^2 = 1 - x^2\\ &\Omega = \frac{n}{2R^2}. \end{aligned} \tag{3.46}$$

*In the original work Chern number is denoted as m.

Class II. Parameterised by 2 real parameters $\bar{\alpha}$ and n. This class is characterised by

$$
\begin{aligned}
R^2 &= \frac{3}{\Lambda} > 0, \\
P^2 &= 1 - x^2 \\
\Omega &= -\frac{2\bar{\alpha}\left(1 - \left(\frac{n\Lambda}{12\bar{\alpha}}\right)^2\right)^2}{\left(x - \frac{n\Lambda}{12\bar{\alpha}}\right)^3}.
\end{aligned}
\tag{3.47}
$$

Class III. Parameterised by 3 real parameters η, γ and n. This class is characterised by

$$
\begin{aligned}
R^2 &\neq \frac{3}{\Lambda}, \\
P^2 &= \frac{(1-x^2)\Big(\big(x - \frac{1}{2}\eta n(1 - \frac{1}{6}\Lambda\gamma)\big)^2 + \eta^2 + \frac{1-x^2}{1-\frac{1}{6}\Lambda\gamma}\Big)}{\big(x - \frac{1}{2}\eta n(1 - \frac{1}{6}\Lambda\gamma)\big)^2 + \eta^2}, \\
\Omega &= \mathrm{Im}\left[\frac{2\mathrm{i}\left(1 - \eta^2\left(\frac{1}{2}n\left(\frac{\Lambda}{6}\gamma - 1\right) + \mathrm{i}\right)^2\right)}{\eta\gamma\left(x + \frac{1}{2}\eta n\left(\frac{\Lambda}{6}\gamma - 1\right) + \mathrm{i}\eta\right)^3}\right].
\end{aligned}
\tag{3.48}
$$

By construction both Isolated Horizons and the accelerated Kerr-NUT-(anti-) de Sitter horizons have the structure of the principle U(1)-bundle and the same Chern number χ_C may be chosen. Thus to establish the embeddability only the relation between the local parameters ($n, \bar{\alpha}, R, \eta$ and γ) and the black hole parameters (M, a, l, α) is required. The cosmological constant Λ is treated as a constant of the theory. As mentioned before only non-extremal horizons with the Hopf bundle structure are considered. This relation is obtained by comparing either the metric tensors ${}^{(2)}q_D$ and ${}^{(2)}q_H$ or comparing the pseudo-scalar invariants Ω obtained for the embedded and unembedded horizons.

Class I has been shown to be embeddable in Taub-NUT-(anti-) de Sitter, unless $n = 0$ in which case the bundle stricture trivializes and the horizon is embeddable in Schwarzschild-(anti-)de Sitter[90,91]. The frame function $P(x)^2 = 1 - x^2$ can be shown to read simply $P^2(\theta) = \sin^2\theta$ after the coordinate change $x = -\cos\theta$.

3.5.1 Embedding into Kerr-NUT-de Sitter

Embeddings of Classes II and III constitute new results and are presented in the two following Sections. Before embedding the generic Class II solution into the accelerated spacetime it is instructive to consider embeddings into the Kerr-NUT-de Sitter spacetime. Obtained relations will be much simpler and clearer than for the generic accelerated class.

For the abstract horizon the relation between the radius R and one of the parameters γ is given by[90]

$$
R^2 = \frac{\gamma}{\frac{\Lambda}{3}\gamma - 2}
\tag{3.49}
$$

and the horizon area 2-form is simply

$$^{(2)}\epsilon_D = R^2 \mathrm{d}x \wedge \mathrm{d}\varphi. \tag{3.50}$$

Comparing the integrated areas gives the relation between R and the black hole parameters

$$R^2 = \frac{\rho_H}{\mathcal{P}(0)}. \tag{3.51}$$

Given that on both sides of the comparison the φ coordinate parameterises integral curves of the axial Killing vector and is 2π-periodic, it suffices to compare φ-components of the metric. It gives

$$P^2 = R^2 \frac{\mathcal{P}}{\Sigma_H} \sin^2\theta. \tag{3.52}$$

The relation between the coordinates θ and x has to be obtained from the comparison of the remaining part of the metric

$$\frac{R^2}{P^2}\mathrm{d}x^2 = \frac{\Sigma_H}{\mathcal{P}}\mathrm{d}\theta^2 \implies \mathrm{d}x = \sin\theta \mathrm{d}\theta. \tag{3.53}$$

With a boundary condition $x(0) = -1$ the above uniquely solves to

$$x = -\cos\theta. \tag{3.54}$$

Using the above the right-hand side of (3.51) may be expressed using the black hole parameters and x, while the left-hand side is taken from the solution to the Petrov Type D equation and is expressed in terms of γ, η and n. Now comparing the coefficients of x at both sides gives

$$\begin{aligned} \gamma &= -\frac{6}{\Lambda} = -2(a^2 + 2l^2 + 2r_H^2), \\ n &= \pm\frac{l}{r_H}, \\ \eta &= \pm\frac{r_H}{a}, \end{aligned} \tag{3.55}$$

where the same sign should be chosen for both n and η. It can be noted, however, that both branches correspond to the same Kerr-NUT-de Sitter spacetime. KNadS metric tensor admits two discrete symmetries

$$\begin{aligned} (t, a, l) &\mapsto (-t, -a, -l), \\ (\varphi, a, l) &\mapsto (-\varphi, -a, -l). \end{aligned}$$

Importantly they fix the sign of r_H (equivalently of M) and thus both branches of solutions for n, η and γ are equivalent. From the above and (3.49) it also follows that

$$R^2 = \frac{3}{2\Lambda}. \tag{3.56}$$

3.5.2 Embedding into accelerated Kerr-NUT-(anti-) de Sitter

The accelerated case follows closely the previous Section however the formulae are more lengthy. Although recently a value for the gauge parameter

$$\bar{\omega} = \frac{a^2 + l^2}{a},$$

has been proposed I opt to keep $\bar{\omega}$ free. Comparing the areas resulting from the area forms (3.50) and (3.13) gives

$$R^2 = \frac{\rho_H}{\mathcal{P}(0)}\zeta, \tag{3.57}$$

where

$$\zeta := \frac{\bar{\omega}^2}{(\bar{\omega} + r_H\alpha(a-l))(\bar{\omega} - r_H\alpha(a+l))}, \tag{3.58}$$

is a constant originating from integration of the conformal factor F^{-2}. Consistently, ζ reduces to 1 in the limit $\alpha \to 0$. In the presence of acceleration, the frame coefficient takes the form

$$P^2 = \frac{\mathcal{P}}{F^2\Sigma_H\zeta}\rho_H \sin^2\theta. \tag{3.59}$$

The comparison of $\mathrm{d}x^2$ and $\mathrm{d}\theta^2$ components gives

$$\begin{aligned} R^2\frac{1}{P^2}\mathrm{d}x^2 &= \frac{1}{F^2}\frac{\Sigma_H}{\mathcal{P}}\mathrm{d}\theta^2 \\ \mathrm{d}x &= \frac{1}{F^2\zeta}\sin\theta\mathrm{d}\theta, \end{aligned} \tag{3.60}$$

which after imposing the boundary condition $x(\theta = 0) = -1$ solves to

$$x = \frac{\cos\theta(\bar{\omega} - \alpha l r_H) - a\alpha r_H}{a\alpha r_H\cos\theta + \alpha l r_H - \bar{\omega}}, \tag{3.61}$$

or equivalently

$$\cos\theta = \frac{a\alpha r_H + (\alpha l r - \bar{\omega})x}{\bar{\omega} - \alpha l r_H - a\alpha r_H x}, \tag{3.62}$$

both of which have a proper limit to un-accelerated case. The proper behavior of various quantities when one sets α to 0 is an expected quality, one of the advantages of the coordinate introduced by Podolsky and Griffith[92] is that all of the sub-classes of the Plebański-Demiański black holes are captures in one coordinate system (possibly one the to choose a proper value of the gauge parameter $\bar{\omega}$). Outside of the metric coefficients, this is however not necessarily given.

The comparison of the coefficients of powers of x in the frame function P^2 expressed in two different ways results in

$$\begin{aligned}
\gamma &= \xi^2 \Bigg(\frac{2a^2\left(\alpha^2 r_H^2\left(2l^2+r_H^2\right)-\bar{\omega}^2\right)-4\left(l^2+r_H^2\right)\left(\bar{\omega}-\alpha l r_H\right)^2}{\bar{\omega}^2} - \\
&\qquad \frac{\alpha\rho_H\rho'_H\left(l^2+r_H^2\right)}{l r_H \bar{\omega}} \Bigg), \\
\eta &= \pm\frac{r_H}{a}\frac{\left(\bar{\omega}-\alpha l r_H\right)^2-a^2\alpha^2 r_H^2}{\alpha^2 r_H^4+\bar{\omega}^2}, \\
n &= \pm 4l \frac{\alpha r_H \bar{\omega}\left(a^2+r_H^2\right)-\alpha l^2 r_H \bar{\omega}+l\left(\bar{\omega}^2-\alpha^2 r_H^4\right)}{4 l r_H\left(\left(\bar{\omega}-\alpha l r_H\right)^2-a^2\alpha^2 r_H^2\right)+\alpha\bar{\omega}\rho_H\rho'_H},
\end{aligned} \tag{3.63}$$

where $\rho'_H = r_H^2 + (a-l)^2$ is the "dual function" to ρ_H.

It should be stressed that the above establishes the connection between two triples of free parameters. On the side of the Type D solution, they are (n, η, γ), and on the spacetime side they are (M, a, l, α) which are however subject to the constraining equation (3.31), giving in total three degrees of freedom. Their relation is however clear only in one direction; Any embedded horizon of the accelerated KNadS may be realized as a solution to the Petrov Type D equation. It is not obvious if and when the above is actually a bijection. One could certainly expect that the solutions to the Petrov Type D equation contain more horizons. In particular, the Podolsky-Griffiths form of the metric does not encompass the accelerated Taub-NUT (both of Petrov Type I and D), which contained Type D horizons.

The necessary condition, in the form of the local invertibility, is more available to study and amounts to the Jacobian of (3.63) being well defined. For the purpose of the calculation on the parameters, α in this case, has to be expressed in terms of the other three via the constraint

(3.31). The Jacobian reads

$$J = -\Bigg(4\bar{\omega}^3(a^4+2a^2(r_H^2-l^2)+(l^2+r_H^2)^2)(\alpha\bar{\omega}(l^2+r_H^2)d^+d^-+$$
$$+2lr_H\bar{\omega}^2(a^2+2(l^2+r_H^2))+2\alpha^2 l r_H^3(2l^2(l^2+r_H^2)-a^2(2l^2+r_H^2)))^2$$
$$(\alpha r_H\bar{\omega}(a^2+r_H^2)+\alpha l^2(-r_H)\bar{\omega}+l(\bar{\omega}^2-\alpha^2 r_H^4))\Bigg)$$
$$\Bigg(a^2 l r_H^2(\alpha^2 r_H^4+\bar{\omega}^2)(\alpha\bar{\omega}d^+d^-+4\alpha^2 l r_H^3(l^2-a^2))+4lr_H\bar{\omega}^2)$$
$$(-l\bar{\omega}^5 d^+d^-(a^2+l^2+r_H^2)+4\alpha^5 l^2 r_H^7(a-l)^2(a+l)^2(a^2(2l^2+r_H^2)+$$
$$-2l^2(l^2+r_H^2))-\alpha^4 l r_H^4\bar{\omega}(l^2-a^2)d^+d^-$$
$$(5l^2(l^2+r_H^2)-a^2(5l^2+4r_H^2))+\alpha^2 l\bar{\omega}^3 d^+d^-$$
$$((l^3-a^2l)^2+r_H^4(9a^2-7l^2)+r_H^2(4a^4+3a^2l^2-7l^4)+r_H^6)+$$
$$+4\alpha l^2 r_H\bar{\omega}^4(a^6+6a^4r_H^2+a^2(-3l^4-2l^2r_H^2+8r_H^4)+2(l^6-2l^4r_H^2-2l^2r_H^4+r_H^6))+$$
$$+\alpha^3 r_H\bar{\omega}^2(a^2-l^2)(r_H^8(4a^2-11l^2)+l^2(a^2-l^2)^4+2r_H^6(3a^4-20a^2l^2+21l^4)+$$
$$+r_H^2(a^8-14a^4l^4+24a^2l^6-11l^8)+2r_H^4(2a^6-11a^4l^2-4a^2l^4+21l^6)+r_H^{10}))\Bigg)^{-1}.$$

Unsurprisingly the denominator of the Jacobian vanishes whenever one of the parameters, with the exception of α, is set to zero. All of the cases with non-generic parameters are already known in the literature or studied separately in this work. Further vanishing of the denominator corresponds to the vanishing of one of the parameters (η, n, γ).

Now there are two more conditions for the vanishing of the Jacobian. The first one corresponds exactly to the condition (3.35) and the case of the extremal horizon. This is not particularly surprising as the parameter $n = \kappa m$ is defined by the surface gravity κ. Similarly, there is another possibility, namely that

$$\omega = \frac{\alpha(l^2+r_H^2)d^+d^- \pm \alpha\Big((l^2+r_H^2)^2(2r_H^2(a^2-3l^2)+(a^2-l^2)^2+r_H^4)^2-16l^2r_H^4 d\bar{d}\Big)^{1/2}}{4lr_H d} \tag{3.64}$$

corresponding to $\gamma = 0$. This case is excluded by the assumption $\gamma \neq 0$ stated in [90].

Embedding of Class II solutions

This Class of solutions of the Petrov Type D equation may be viewed as an intermediary between the other two. Similarly to the Class I horizons the space of the null generator is simply a smooth two-sphere, however similarly to the Class III horizons, the rotation invariant Ω is not constant and depends on x instead.

It turns out that these horizons are embeddable as the horizons of accelerated Kerr-NUT-(anti-) de Sitter located at $r_H = 0$. Indeed the horizons at $r_H = 0$ have spherical geometry and following the same steps as for the previous embeddings one arrives at the relation

$$x = -\cos\theta.$$

Expressing the surface gravity using $n = \kappa m$ and using above gives the rotation pseudo scalar Ω (3.42) as

$$\Omega = \frac{\Lambda}{6} n\chi \frac{(1-\chi^{-2})^2}{(x-\chi^{-1})^3}. \tag{3.65}$$

Comparison with the Ω calculated for Class II of solution of the Type D equation yields the full set of relations between the parameters parameters

$$\begin{aligned} R^2 &= \frac{3}{\Lambda} = l^2, \\ n &= \kappa\chi_C = \frac{8M}{\sqrt{\frac{3}{\Lambda}}(1-\chi^2)^2} \\ \bar{\alpha} &= \chi n \frac{\Lambda}{12} = \frac{2M\chi}{\left(\sqrt{\frac{3}{\Lambda}}\right)^3 (1-\chi^2)^2} \end{aligned} \tag{3.66}$$

which is invertible to

$$\begin{aligned} M &= \frac{R\left(n^2 - 16\bar{\alpha}^2 R^4\right)^2}{8n^3} \\ \chi &= \frac{4\bar{\alpha}R^2}{n}. \end{aligned} \tag{3.67}$$

[...]
Natura twórcza, wieczna, bezcelowa,
Nie da się zmieścić w naszych pojęć ramie,
Więc wiecznym prawom, co w swem łonie chowa,

I konieczności, co nas wszystkich łamie,
Ludzkiej dążności narzucamy znamię,
Na ludzki język tłómacząc jej słowa.
Adam Asnyk, Nad Głębiami, XXV

4
Projectively singular horizons of accelerated Kerr-NUT-(anti-) de Sitter

In this Chapter, the horizons appearing in the generic non-singular interpretation of the accelerated KNadS spacetimes and their IH counterparts are studied. A novel class of IH admitting the topology of a Hopf bundle allowing for cases when the null direction is not tangent to the bundle fibers is required to describe the structure of generic KNadS horizons. In order to characterize them in a coordinate-independent way, I will construct the three-dimensional manifold, on which the horizon structure is considered. The construction is modeled on a manifold admitting an auxiliary structure of the Lie group $SU(2)$, such that the right action of the subgroup $U(1)$ provides the Hopf bundle structure, while the left action of $U(1)$ gives a cyclic KVF. The null vector field defining the canonical null symmetry of the horizon is assumed to be an arbitrary linear combination of the generators of the left and right actions of $U(1)$. Generically its orbits are not closed. The horizon geometry is assumed to be invariant with respect to the two-dimensional group generated by the right and left actions of $U(1)$ mentioned above. The surface gravity is assumed to be constant, the same as for IHs described in the Introduction.

The first result of this Chapter is the characterization of the geometry of such horizons, i.e. the degenerate metric tensor and the rotation 1-form potential, by spherically symmetric data defined on a (topological) two-dimensional sphere, i.e. a metric tensor defined up to a constant rescaling with conical singularities at the poles, and a function regular at the poles - the rotation pseudo-scalar invariant. The surface gravity is determined by that data. If it is zero, then one more function is needed to reconstruct the rotation 1-form potential. Given any data characterized above, it is possible to construct a degenerate metric tensor and rotation potential 1-form defined regularly on the Hopf bundle. This kind of data was considered in the literature as singular solutions to horizon equations, either extremal vacuum horizon [13] or the Petrov Type D vacuum horizon [93].

The second part of the result is concerned with imposing the Type D Equation and deriving all the solutions in this class. They are characterized by conically singular data defined on S^2 and may be used to construct the corresponding horizons of the Hopf bundle structure and the null direction transversal to the fibers. It is a nonobvious conclusion is that every solution gives rise to a nonsingular horizon, provided the two-dimensional metric tensor has at most conical singularities at the poles - the corresponding rotation scalar emerges as a regular function. Finally, the explicit embedding into the non-singular accelerated KNadS spacetimes is presented.

While the solutions to the Petrov Type D equation with conical singularities can be used to construct singularity-free horizons with the Hopf bundle structure, this is not necessary. Without performing the topological identification one could instead construct a horizon with a product topology $S \times \mathbb{R}$, where S is only diffeomorphic to the sphere due to the conical singularities. Such horizons could be embedded as a Cauchy horizon in a spacetime possessing a cosmic string and have been of interest for a long time in the context of a cosmic string snapping and a subsequent generation of gravitational waves [94,92]. Recently there has also been a resurgence of mathematical interest in spacetimes with conical singularities in the field of the black hole thermodynamics. It has been shown that a consistent formulation of the thermodynamics of spacetimes with a conical singularity is possible, with contributions from the cosmic string [39,95].

4.1 Isolated horizons with $U(1)-$ PFB structure admitting conical singularities

The structure of horizons studied in this Chapter differs from those presented in Chapters 3 and 5 in the following way. The projectively non-singular horizons naturally admitted a bundle structure such that the space of the null generators is simultaneously the base space of the bundle and the $G-$action was given by the flow of the KVF ℓ null at the horizon. Here the $U(1)$- action is generated by a generic linear combination of KVFs ξ different from ℓ. As a consequence, the space of the null generators is not transversal to ξ and in fact, is not globally well defined.

The non-trivial PFB structure may be introduced as follows. As a starting point, consider two isolated horizons:

$$\mathcal{H} = S^2 \backslash \{p_\pi\} \times S^1 \qquad \text{and} \qquad \mathcal{H}' = S^2 \backslash \{p_0\} \times S^1. \tag{4.1}$$

$\mathcal{H}$ equipped with coordinates (θ, φ, v) and $\mathcal{H}'$ with (θ', φ', v') which in turn provide the following metrics on the two horizons:

$$g_{ab}dx^a dx^b = U^2(\theta)\left[d\theta^2 + f^2(\theta)\sin^2\theta\Big(\frac{1}{f(0)}d\varphi + \Big(\frac{1}{f(\pi)} - \frac{1}{f(0)}\Big)dv\Big)^2\right] \tag{4.2}$$

and

$$g'_{ab}dx'^a dx'^b = U^2(\theta')\left[d\theta'^2 + f^2(\theta')\sin^2\theta'\Big(\frac{1}{f(\pi)}d\varphi' + \Big(\frac{1}{f(0)} - \frac{1}{f(\pi)}\Big)dv'\Big)^2\right]. \tag{4.3}$$

The coordinates (θ, φ) and (θ', φ') are standard spherical coordinates with typical ranges, $\theta \in [0, \pi)$, $\theta' \in (0, \pi]$, $\phi, \phi' \in [0, 2\pi)$, whereas v and v' are 2π-periodic U(1)-coordinates, $v, v' \in [0, 2\pi)$. The conformal factor U^2 is assumed to be smooth everywhere on the horizon.

Notice that both metrics g and g' admit conical singularities at $\theta = \pi$ and $\theta' = 0$, respectively, unless $f(0) = f(\pi)$. Here a general case when such an equality does not necessarily occur is considered. Next, a gluing of the two charts is defined by the following transformation law:

$$\theta = \theta', \qquad \varphi = -\varphi', \qquad v = v' - \phi', \qquad \text{for } \theta, \theta' \neq 0, \pi. \tag{4.4}$$

Therefore, the two charts (v, θ, φ) and $(v', \theta', \phi'$ together with the transition map (4.4) provide an atlas on H. Metrics (4.2) and (4.3) uniquely define a smooth, symmetric, rank 2 tensor on a three-dimensional null surface H. The transition map (4.4) generates a Hopf bundle structure, where the $U(1)$ action is determined by the vector field ∂_v and $\partial_{v'}$, respectively. Notice, however, that it does not coincide with the null vector fields ℓ. We may choose ℓ to be of the form:

$$\begin{aligned}\ell &= f(\pi)\partial_v + \Big(f(\pi) - f(0)\Big)\partial_\varphi \\ &= f(0)\partial_{v'} + \Big(f(0) - f(\pi)\Big)\partial_{\varphi'},\end{aligned} \tag{4.5}$$

which is equivalent to fixing its normalization. Next, on H a rotation 1-form potential ω may be defined such that:

$$\omega(\ell) = \kappa = \text{const.} \qquad \text{and} \qquad \mathcal{L}_\ell \omega = 0. \tag{4.6}$$

It follows that in the coordinates (θ, ϕ, v) the rotation 1-form potential is of the form:

$$\omega = \frac{2\kappa}{f(\pi) + f(0)}\Big(dv - \frac{1}{2}d\varphi\Big) + h(\theta)\Big(\frac{1}{f(0)}d\varphi + \Big(\frac{1}{f(\pi)} - \frac{1}{f(0)}\Big)dv\Big) + dw(\theta), \tag{4.7}$$

where $h(\theta)$ is *a priori* an arbitrary function. Note that because ∂_v and not ℓ generates the $U(1)-$ action and $\omega(\partial_v) \neq \kappa$, the 1-form $\frac{\omega}{\kappa}$ no longer has the interpretation of the $U(1)$-connection form. Requiring ω to be well-defined at $\theta = 0$ results in the following condition

$$\omega_\phi(0) = \frac{h(0)}{f(0)} - \frac{\kappa}{f(\pi) + f(0)} = 0. \tag{4.8}$$

Similarly, for the primed coordinates (v', θ', φ') we have

$$\omega = \frac{2\kappa}{f(\pi)+f(0)}\Big(dv' - \frac{1}{2}d\varphi'\Big) - h(\theta)\Big(\frac{1}{f(\pi)}d\varphi' + \Big(\frac{1}{f(0)} - \frac{1}{f(\pi)}\Big)dv'\Big) + dw(\theta') \tag{4.9}$$

and, to be well-defined at $\theta = \pi$, it needs to satisfy

$$\omega_{\varphi'}(\pi) = -\frac{h(\pi)}{f(\pi)} - \frac{\kappa}{f(\pi)+f(0)} = 0. \tag{4.10}$$

Instead of the rotation 1-form potential ω, one may operate with the rotation pseudoscalar Ω, defined by (1.11), where:

$$\begin{aligned}\eta &= U^2(\theta)f(\theta)\sin\theta d\theta \wedge \Big(\frac{1}{f(0)}d\varphi + \Big(\frac{1}{f(\pi)} - \frac{1}{f(0)}\Big)dv\Big) \\ &= -U^2(\theta')f(\theta')\sin\theta' d\theta' \wedge \Big(\frac{1}{f(\pi)}d\varphi' + \Big(\frac{1}{f(0)} - \frac{1}{f(\pi)}\Big)dv'\Big).\end{aligned} \tag{4.11}$$

Furthermore, the rotation pseudo scalar Ω determines the function $h(\theta)$ up to an additive constant c:

$$h(\theta) = \int \Omega(\theta)U^2(\theta)f(\theta)\sin\theta d\theta =: h_0(\theta) + c. \tag{4.12}$$

The integration constant c can be recovered from the first constraint (4.8):

$$c = \frac{f(0)\kappa}{f(\pi)+f(0)} - h_0(0). \tag{4.13}$$

Then, the second condition (4.10) becomes a constraint on surface gravity κ, which is of the form

$$\kappa = h_0(0) - h_0(\pi). \tag{4.14}$$

Therefore, it is the surface gravity κ which defines an appropriate way of gluing two horizons, $\mathcal{H}$ and $\mathcal{H}'$. Consequently, the horizon H defined by the two charts, (v, θ, φ) and (v', θ', φ'), is diffeomorphic to S^3.

If the resulting $\kappa \neq 0$, then the term $\mathrm{d}w(x)$ in ω can always be set to be zero, due to transformations $v = v' + \tilde{w}(x)$. In the case $\kappa = 0$, we additionally need

$$w = w(x) \tag{4.15}$$

to be regular on the entire sphere.

Several general remarks about the overall construction are in order.
Remark 1: the cyclic symmetry. The construction is modeled on the Hopf bundle where S^3 also admits a part of the $SU(2)$ group structure. In the auxiliary $SU(2)$ structure ∂_v would correspond to a right-invariant vector field and

$$\partial_\phi + \frac{1}{2}\partial_v = -\left(\partial_{\phi'} + \frac{1}{2}\partial_{v'}\right), \tag{4.16}$$

to a left-invariant vector field. Furthermore, it is assumed that both of them are symmetries of the degenerate horizon metric and that the vector field ℓ spanning the null direction is their linear combination. In the generic case, the orbits of ℓ are dense in the orbits spanned by ℓ and ∂_v. Similar constructions have been recently studied in [82].
Remark 2: nongeneric sub-cases. Given the Hopf bundle $H \to S^2$:

- Every pair (g_{AB}, ω_A) defined on the entire S^2 without conical singularities corresponds to an extremal horizon structure ($\kappa = 0$) such that ℓ is tangent to the fibers of the Hopf bundle $H \to S^2$. Notice, however, that such structure in general does not satisfy the vacuum extremal equation introduced in Remark 3 below.
- if g_{AB} is regular on the entire S^2, then ℓ is tangent to the fibers of the bundle.

Remark 3: extremal vacuum horizons. Given a 2-dimensional manifold endowed with a metric tensor g_{AB} and a differential 1-form ω_{AB}, the vacuum extremal horizon equation with a cosmological constant Λ reads

$$D_{(A}\omega_{B)} + \omega_A\omega_B - \frac{1}{2}R_{AB} + \frac{1}{2}\Lambda g_{AB} = 0, \tag{4.17}$$

where D_A is the torsion-free, metric connection corresponding to g_{AB} and R_{AB} is its Ricci tensor. Extremal isolated horizon H is vacuum-extremal if there is a constant Λ such that the vacuum-extremal horizon equation is satisfied for every local section transversal to the null vector ℓ by the pullback of the degenerate metric tensor and the rotation 1-form potential ω. A new conclusion coming from this section is that a regular vacuum extremal horizon structure on the Hopf bundle H is obtained by implementing the above construction to any solution of the vacuum extremal horizon equation defined on S^2 such that

1. (g_{AB}, ω_A) is axisymmetric,
2. g_{AB} is regular except the poles where it can have conical singularities,
3. ω_A is regular everywhere.

Since all the axisymmetric solutions of (4.17) on S^2 with conical singularities are known [13,96] it should be possible to check by inspection which of them satisfy the conditions of the above construction.

Next, I will compare the resulting family of solutions with those defined by horizons in the generalized Plebański-Demiański spherical black hole spacetimes known from the literature[29].

To solve the Petrov Type D equation on a spare with conical singularities it is again convenient to introduce an axially adopted frame. Only the axially symmetric solution will be considered, the existence of solutions without the axial symmetry is still an open question. The metric tensor on S reads

$$g_{AB}dx^A dx^B = R^2\left(\frac{1}{P^2(x)}dx^2 + P^2(x)d\psi^2\right), \tag{4.18}$$

where $x \in [-1;1]$. For these metrics, the topologically non-trivial horizon geometry construction method presented in the previous chapter will be applied. Consequently, the range of dependence of the ψ-coordinate is irrelevant at this point, ψ may take values in an a priori undetermined interval. In particular, it is not assumed that the conical singularity is removable, i.e. that there exists a rescaling of ψ such that both poles are regular. The complex frame vector and its dual may be written in the following form:

$$m^A\partial_A = \frac{1}{R\sqrt{2}}\left(P(x)\partial_x + i\frac{1}{P(x)}\partial_\psi\right), \quad \bar{m}_A dx^A = \frac{R}{\sqrt{2}}\left(\frac{1}{P(x)}dx - iP(x)d\psi\right). \tag{4.19}$$

In contrast to the previous solutions to the Type D equation[89,90] here a more general structure is considered and the condition on the frame coefficient P^2 is relaxed, by requiring it to be vanishing only on the poles. That is

$$P^2(1) = 0 = P^2(-1). \tag{4.20}$$

In particular the condition for lack of conical singularity, which in terms of the frame coefficient P^2 reads

$$\partial_x P^2\Big|_{x=\pm 1} = \mp 2, \tag{4.21}$$

is not assumed. If the conical singularities are present at the poles, then the range of ψ is not chosen in agreement with a given pole. Consequently, (4.21) may be rescaled by positive constants different in both poles and is relaxed to

$$\partial_x P^2\Big|_{x=-1} > 0, \quad \partial_x P^2\Big|_{x=1} < 0. \tag{4.22}$$

By changing the range of the coordinate ψ or equivalently by its rescaling, one may get rid of the conical singularity at any one of the poles and retrieve the corresponding condition in (4.21). Hence, the condition (4.22) will be assumed throughout this Section. Consequently, the 2-metric (4.18) itself generically will not be differentiable nor continuous simultaneously on both of the poles. It will lead to a new four-parameter family of solutions to the Type D equation.

Finally, the regularity condition that has to be satisfied by the function Ω on the poles reads

$$g^{AB} D_A \Omega D_B \Omega = 0. \tag{4.23}$$

In (x, ψ) coordinates, it is of the form

$$\left(\left(\frac{d\Omega}{dx} \right)^2 P^2 \right) \bigg|_{x=\pm 1} = 0. \tag{4.24}$$

Every solution (g, Ω) that satisfies the conditions (4.20), (4.22) and (4.24) gives rise to an isolated horizon structure on the Hopf bundle H, according to the construction presented in the previous Section, that is at least continuous at the points of the fibers corresponding to the poles of the sphere. The actual horizons determined below by those conditions and by the Petrov Type D equation will be everywhere smooth and even analytic.

Similar to the case without the conical singularities, the axial symmetry Type D equation reads

$$\partial_x^2 \Psi_2^{-\frac{1}{3}} = 0, \tag{4.25}$$

with a solution given by

$$\Psi_2 = \frac{1}{(c_1 x + c_2)^3}, \tag{4.26}$$

where c_1 and c_2 are complex constants. Expressing the Gaussian curvature K in terms of the frame coefficient P^2 and its derivatives yields:

$$\frac{1}{(c_1 x + c_2)^3} = \frac{1}{4R^2} \partial_x^2 P^2 - \frac{1}{2} i\Omega + \frac{\Lambda}{6}. \tag{4.27}$$

The above complex equation may be split into its real and imaginary parts:

$$\mathrm{Re}\left[\frac{1}{(c_1 x + c_2)^3} \right] = \frac{1}{4R^2} \partial_x^2 P^2 + \frac{\Lambda}{6}, \tag{4.28}$$

$$\mathrm{Im}\left[\frac{1}{(c_1 x + c_2)^3} \right] = -\frac{1}{2} \Omega. \tag{4.29}$$

The remaining part of this Section, dealing with the solutions to the above equations with an assumption that the metric coefficient P^2 vanishes at the poles (4.20).

4.2.1 Special solution for vanishing c_1

First, consider the case when the complex parameter c_1 vanishes. Consequently, Eq. (4.28) becomes

$$A = \partial_x^2 P^2 \tag{4.30}$$

for some real parameter A. Its solution is of the form

$$P^2(x) = \tfrac{1}{2}Ax^2 + Bx + C, \tag{4.31}$$

where B and C are real constants. Requiring the continuity of the frame coefficient P^2 (4.20) yields

$$P^2(x) = C(1 - x^2). \tag{4.32}$$

Using the above expression one may obtain the following metric tensor g_{AB}:

$$\begin{aligned} g_{AB}dx^A dx^B &= R^2\left(\frac{1}{P^2(x)}dx^2 + P^2(x)d\psi^2\right) \\ &= R'^2\left(\frac{1}{P'^2(x)}dx^2 + P'^2(x)d\varphi^2\right), \end{aligned} \tag{4.33}$$

where the rescaled area radius and angular coordinate are used, namely $R'^2 = R^2/C$ and $d\varphi = Cd\psi$. Therefore, the metric remains of the form (4.18) with the frame coefficient

$$P'^2(x) = 1 - x^2 \tag{4.34}$$

which on the poles not only satisfies the continuity (4.20) but also the differentiability condition

$$\partial_x P'^2\Big|_{x=\pm 1} = \mp 2. \tag{4.35}$$

Consequently, the 2-metric g_{AB} lacks conical singularities and is differentiable at the poles $x = \pm 1$. Furthermore, from the imaginary part of the Type D equation (4.29) and the vanishing of the complex constant c_1 follows that the rotation pseudoscalar Ω is constant:

$$\Omega = \text{const.} \tag{4.36}$$

Since g_{AB} is regular on the entire sphere, the solution falls in the nongeneric case of Remark 2, either the first built if $\Omega = 0$ or the second built is $\Omega \neq 0$. In the latter case the null flow is tangent to the fibers of the Hopf bundle and such solutions were considered in[90].

4.2.2 Generic solution for nonvanishing c_2

A general axisymmetric solution to the real part of the Petrov Type D equation (4.28) is of the form:

$$P^2 = 2R^2 \mathrm{Re}\left[\tfrac{1}{c_1^2(c_1 x + c_2)}\right] - \tfrac{1}{3}R^2\Lambda x^2 + a_0 x + b_0, \tag{4.37}$$

where a_0 and b_0 are real integration constants. Introducing a new parametrization

$$\begin{aligned}
A &:= \frac{c_1\bar{c}_2 + \bar{c}_1 c_2}{c_1\bar{c}_1}, & D &:= \frac{1}{R^2}\Big(Aa_0 + b_0\Big) - \frac{\Lambda}{3}B,\\
B &:= \frac{c_2\bar{c}_2}{c_1\bar{c}_1}, & E &:= \frac{c_1^3 + \bar{c}_1^3}{c_1^3\bar{c}_1^3} + \frac{1}{R^2}\Big(Ba_0 + Ab_0\Big),\\
C &:= \frac{a_0}{R^2} - \frac{\Lambda}{3}A, & F &:= \frac{c_1^2 c_2 + \bar{c}_1^2\bar{c}_2}{c_1^3\bar{c}_1^3} + \frac{b_0}{R^2}B,
\end{aligned} \tag{4.38}$$

yields a more compact expression for the frame coefficient

$$P^2 = R^2\frac{-\frac{\Lambda}{3}x^4 + Cx^3 + Dx^2 + Ex + F}{x^2 + Ax + B}. \tag{4.39}$$

If additionally Eq. (4.20) is satisfied on the poles then two constraints for parameters C, D, E and F extist, that is

$$P^2(1) = -\frac{\Lambda}{3} + C + D + E + F = 0, \tag{4.40}$$

$$P^2(-1) = -\frac{\Lambda}{3} - C + D - E + F = 0, \tag{4.41}$$

from which it follows that

$$F = \frac{\Lambda}{3} - D \qquad \text{and} \qquad E = -C. \tag{4.42}$$

Consequently, the frame coefficient P^2 can be written as an expression manifestly satisfying constraints (4.20), namely:

$$P^2 = R^2\frac{-\frac{\Lambda}{3}x^2 + Cx + D - \frac{\Lambda}{3}}{x^2 + Ax + B}(x^2 - 1). \tag{4.43}$$

Next, let's change parameter D to $D' + \frac{\Lambda}{3}$ so that

$$P^2 = R^2\frac{-\frac{\Lambda}{3}x^2 + Cx + D'}{x^2 + Ax + B}(x^2 - 1), \tag{4.44}$$

and drop the prime to make the expression more elegant:

$$P^2 = R^2 \frac{-\frac{\Lambda}{3}x^2 + Cx + D}{x^2 + Ax + B}(x^2 - 1). \tag{4.45}$$

Solving the imaginary part of the Type D equation (4.29) allows to find the rotation pseudoscalar Ω:

$$\begin{aligned}\Omega = \Bigg(& x^3\Big(- 3AC + 3A^3C - 9ABC + 6(1+B)D + A^4\Lambda - 4A^2B\Lambda + 2B(1+B)\Lambda - A^2(3D+\Lambda)\Big) \\ & + x^2\Big(9A^2BC - 18B(1+B)C + 9AD - 9ABD + 3A^3B\Lambda - 3AB(1+3B)\Lambda\Big) \\ & + x\Big(- 9ABC + 9AB^2C + 9A^2D - 18B(1+B)D + 3A^2B^2\Lambda - 6B^2(1+B)\Lambda\Big) \\ & - 3A^2BC + 6B^2(1+B)C + 3A^3D - 9ABD - 3AB^2D + A(-1+B)B^2\Lambda\Bigg) \\ & \Bigg/ \Big(3\sqrt{4B - A^2}\big(B + x(A+x)\big)^3\Big).\end{aligned} \tag{4.46}$$

Notice that for the frame coefficient P^2 (4.45) and rotation pseudoscalar Ω (4.46) to be well-defined the parameters A, B, C and D must satisfy the following conditions:

$$\text{for } \Lambda < 0: \quad A^2 - 4B < 0 \quad \wedge \quad \tfrac{3}{2\Lambda}\left(C \pm \sqrt{C^2 + \tfrac{4}{3}\Lambda D}\right) \in (-\infty, -1] \cup [1, \infty), \tag{4.47}$$
$$\text{for } \Lambda > 0: \quad A^2 - 4B < 0 \quad \wedge \quad C^2 + \tfrac{4}{3}\Lambda D < 0, \tag{4.48}$$

where the second column corresponds to the condition that $P^2 > 0$.

The case when the denominator of Ω (4.46) vanishes, namely

$$A^2 = 4B, \tag{4.49}$$

will be investigated separately. It follows from the definition of parametrization (A, B, C, D) in (4.38) that this specific case corresponds to

$$\mathrm{Im}\left[\frac{c_2}{c_1}\right] = 0. \tag{4.50}$$

Moreover, the parametrization degenerates from four to only two independent parameters and is not sufficient to cover the solution to the imaginary part of the Type D equation (4.29) - the rotation pseudoscalar Ω. Consequently, one has to introduce a third parameter. Notice, that whenever (4.50) holds, the invariant Ψ_2 (4.26) is of the form:

$$\Psi_2 = \frac{1}{c_1^3(x + A')^3}, \tag{4.51}$$

for some parameter $A' \in \mathbb{R}$. It is, then, convenient to introduce a pair of real parameters, B' and C', such that

$$B' = \frac{1}{c_1^3} + \frac{1}{\bar{c}_1^3}, \qquad iC' = \frac{1}{c_1^3} - \frac{1}{\bar{c}_1^3}. \tag{4.52}$$

Consequently, the real (4.28) and imaginary (4.29) parts of the Type D equation are of the form

$$\frac{B'}{2(x+A')^3} = \frac{1}{4R^2}\partial_x^2 P^2 + \frac{\Lambda}{6}, \tag{4.53}$$

$$\frac{-C'}{2(x+A')^3} = -\frac{1}{2}\Omega, \tag{4.54}$$

and $A' \in (-\infty, -1) \cup (1, \infty)$. Their solutions read

$$P^2 = R^2 \frac{B' - \frac{\Lambda}{3}(A'^2 - 1)(A' + x)}{(A'^2 - 1)(A' + x)}(x^2 - 1), \tag{4.55}$$

$$\Omega = \frac{C'}{(x + A')^3}. \tag{4.56}$$

Additionally, parameters A' and B' need to be restricted, so that the frame coefficient is positive everywhere on the sphere (apart from the poles where it has to vanish), meaning the following must hold:

$$\frac{B'}{A' + x} < \frac{\Lambda}{3}\left(A'^2 - 1\right), \tag{4.57}$$

for any $x \in (-1, 1)$.

The condition (4.20) has been ensured from the beginning of the derivation. Remarkably, the conical singularity condition (4.22) is satisfied now automatically for both solutions parametrized by (A, B, C, D) and (A', B', C'). So is the regularity condition (4.24), due to the condition (4.20) and the regularity of Ω as a function of x.

Finally, one can calculate the rescaling constants β, β'used to rescale ψ and ψ' to 2π- periodic coordinates namely

$$\begin{aligned} \beta &= \left.\frac{-2}{\partial_x P^2}\right|_{x=1} = -\frac{1 + A + B}{R^2\left(C + D - \frac{\Lambda}{3}\right)}, \\ \beta' &= \left.\frac{2}{\partial_x P^2}\right|_{x=-1} = \frac{1 - A + B}{R^2(C - D + \frac{\Lambda}{3})}. \end{aligned} \tag{4.58}$$

Hence, the solutions (g_{AB}, Ω), that is the pairs (4.45), (4.46) and (4.55), (4.56), define uniquely a vacuum isolated horizon structure of the Petrov Type D on the Hopf bundle H, that continues at the points of the fibers over the poles of the sphere. The case when also $\beta = \beta'$ is satisfied has already been studied in Chapter 3.

The novel horizons defined in this Chapter owe their structure to the horizons appearing in the non-singular interpretation of the accelerated KNadS spacetimes, such that the closed KVF ξ in not null at the horizon, i.e. $\xi \neq \ell$. Both the horizons in accelerated KNadS and the abstract solutions given by (4.45) have 4 parameters, (α, a, M, l) and (A, B, C, D), respectively, where Λ is treated as a constant of a theory. It is therefore natural to expect that solutions to the Petrov Type D equation are embeddable in the accelerated KNadS spacetimes.

The metric tensors (2.43) and (2.44) are well-defined for vanishing $\mathcal{Q}$, and one may consider null surfaces characterized by $r = r_H$. Their metrics are degenerate and read

$$ds^2_{\mathcal{H}} = \frac{1}{F^2}\left[\frac{\Sigma_H}{\mathcal{P}}d\theta^2 + \frac{\mathcal{P}}{\Sigma_H}\sin^2\theta\left(\Big(a - \rho_H b\Big)d\tau - \frac{\rho_H}{\mathcal{P}(0)}d\hat{\phi}\right)^2\right], \tag{4.59}$$

and

$$ds'^2_{\mathcal{H}'} = \frac{1}{F^2}\left[\frac{\Sigma_H}{\mathcal{P}}d\theta'^2 + \frac{\mathcal{P}}{\Sigma_H}\sin^2\theta\left(\Big(a - \rho_H b\Big)d\tau' - \frac{\rho'_H}{\mathcal{P}(\pi)}\sigma d\hat{\phi}'\right)^2\right]. \tag{4.60}$$

Now to descend to the space of null generators introduce new coordinates φ and φ',

$$\varphi = \hat{\phi} - \frac{\mathcal{P}(0)}{\rho_H}(a - \rho_H b)\tau, \qquad \varphi' = \hat{\phi}' - \frac{\mathcal{P}(\pi)}{\rho'_H}(a - \rho_H b)\sigma\tau', \tag{4.61}$$

which yields

$$ds^2_{\mathcal{H}} = \frac{1}{F^2}\left[\frac{\Sigma_H}{\mathcal{P}}d\theta^2 + \frac{\mathcal{P}\rho_H^2}{\mathcal{P}^2(0)\Sigma_H}\sin^2\theta d\varphi^2\right] \tag{4.62}$$

and

$$ds'^2_{\mathcal{H}'} = \frac{1}{F^2}\left[\frac{\Sigma_H}{\mathcal{P}}d\theta'^2 + \frac{\mathcal{P}\rho'^2_H}{\mathcal{P}^2(\pi)\Sigma_H}\sin^2\theta\sigma d\varphi'^2\right]. \tag{4.63}$$

Notice that the above procedure is an analog of the horizon gluing presented in Sec 4.1. Moreover, the rotation 1-form potential ω is well-defined as it satisfies conditions (4.8) and (4.10) on the poles. On the other hand, the conditions for a well-defined 1-form ω stated at the beginning of this Chapter provide the correct spacetime gluing introduced in (2.39).

This can be seen in two ways. First the 1-form ω can be calculated and expressed in coordinates (τ, θ, φ) and $(\tau', \theta', \varphi')$ adopted to the space of null generators. Since both ω and κ depend on the normalization of ℓ let's at first make the obvious choice of (expressed in the advanced coordinates (2.8))

$$\ell = \partial_v + \frac{a}{\rho_H}\partial_{\tilde{\phi}}. \tag{4.64}$$

Then, κ reads simply

$$\kappa = \frac{1}{2}\frac{\mathcal{Q}'(r)\big|_{r=r_H}}{\rho_H}, \tag{4.65}$$

and ω is of the form

$$\omega = \kappa dv + \omega_\theta d\theta + \frac{\omega_{\phi_H}}{\mathcal{P}(0)} d\varphi = \kappa\frac{\rho'_H}{\rho_H}(1-4lb)dv' + \omega_\theta d\theta' + \sigma\left(4l\kappa + \omega_{\phi_H}\frac{\rho'_H}{\rho_H}\right)\frac{1}{\mathcal{P}(\pi)}d\varphi', \tag{4.66}$$

where:

$$\omega_\theta = -\frac{2\Sigma\partial_\theta F + aF\partial_\theta A}{2F\Sigma}\Bigg|_{r=r_H} = a\sin\theta\frac{\frac{r_H^3\alpha}{\bar{\omega}} + l + a\cos\theta}{F_H\Sigma_H} \tag{4.67}$$

$$\omega_{\phi_H} = -\frac{a\rho r\mathcal{P}\sin^2\theta + Q'A\Sigma}{\Sigma^2}\Bigg|_{r=r_H} = -\frac{a\rho_H r_H\mathcal{P}\sin^2\theta}{\Sigma_H^2} - \kappa A\frac{\rho_H}{\Sigma_H}, \tag{4.68}$$

making ω explicitly regular at $\theta = 0$ and $\theta = \pi$ in unprimed and primed coordinates, respectively.

In the above the fact that orbits of Killing vector field $\xi = \partial_t + b\partial_\phi$ with the constant b satisfying condition (2.31) have been compactified and the axis of rotation is regular in spacetime, has been used. However, condition (2.31) can be recovered purely from the horizon data by the procedure described in the Sec. 4.1. The starting point is the degenerate metric tensor on the horizon (4.59) adopted to the Killing vector field ξ generating the action transversal to the horizon. The first step is putting (4.59) into the form of (4.2). This is achieved by taking

$$U^2(\theta) = \frac{\Sigma_H}{\mathcal{P}F^2}, \quad f(\theta) = \frac{\mathcal{P}\rho_H}{\Sigma_H}, \quad f(0) = \mathcal{P}(0), \quad f(\pi) = \mathcal{P}\frac{\rho_H}{\rho'_H}, \tag{4.69}$$

and defining $v = \gamma\tau$, where

$$\gamma = \frac{a - \rho_H b}{\rho_H}\left(\frac{1}{f(0)} - \frac{1}{f(\pi)}\right)^{-1}.$$

The second step is to normalize ℓ so that having chosen (4.69) the vector fields are of the same form as (4.5). To do so let's rescale

$$\ell = \frac{f(\pi)}{\gamma}\left(\partial_v + \frac{a}{\rho_H}\partial_{\tilde{\phi}}\right). \tag{4.70}$$

With the above normalization the rotation 1-form ω can be calculated and put in the form (4.7) with

$$h(\theta) = \frac{\omega_{\phi_H}\gamma}{f(\pi)} + \frac{\kappa f(0)}{f(0) + f(\pi)}. \tag{4.71}$$

Then, solving condition (4.14) precisely reproduces relation (2.31) between the Killing vector field parameter b and the spacetime parameters.

For the sake of simplicity, it is possible to express both of the above horizon metrics by one expression

$$g_{ab}dx^a dx^b = \frac{1}{F^2}\left(\frac{\Sigma_H}{\mathcal{P}}d\theta^2 + \frac{\mathcal{P}}{\Sigma_H}\rho_H^2 \sin^2\theta \beta^2 d\varphi^2\right), \tag{4.72}$$

where β is of a different form depending on the chart, that is

$$\beta_{\mathcal{H}} = \frac{1}{\mathcal{P}(0)} \qquad \text{and} \qquad \beta_{\mathcal{H}'} = \frac{\rho'_H}{\mathcal{P}(\pi)\rho_H}. \tag{4.73}$$

4.3.1 Comparison of the 2D geometries

In this Section the embeddability via the comparison of the two metrics of two-dimensional surfaces

$$R^2\left(\frac{1}{P^2}dx^2 + P^2\beta^2 d\varphi^2\right) = \frac{1}{F^2}\left(\frac{\Sigma_H}{\mathcal{P}}d\theta^2 + \frac{\mathcal{P}}{\Sigma_H}\rho_H^2\sin^2\theta\beta^2 d\varphi^2\right), \tag{4.74}$$

will be studied. Notice, that the metric on the left-hand side above, satisfying the Type D equation, is of the general axisymmetric form, with the coordinate $\varphi := \frac{\psi}{\beta} \in [0, 2\pi[$, which allows for a construction of the isolated horizon H diffeomorphic to S^3, just as the accelerated Kerr-NUT-(anti-)de Sitter horizon.

The comparison follows similar lines as in the projectively non-singular case with for the area radius R written in terms of the accelerated Kerr-NUT-(anti-)de Sitter parameters reading

$$R^2 = \frac{\rho_H\bar{\omega}^2}{(\bar{\omega} + r_H\alpha(a-l))(\bar{\omega} - r_H\alpha(a+l))} =: \rho_H\xi. \tag{4.75}$$

Therefore, the frame coefficient written explicitly in terms of θ reads

$$\begin{aligned} P^2 &= \frac{\mathcal{P}}{F^2\Sigma_H\xi}\rho_H\sin^2\theta \\ &= \frac{1 - a_3\cos\theta - a_4\cos^2\theta}{(1 - \frac{\alpha}{\bar{\omega}}(l + a\cos\theta)r_H)^2(r_H^2 + (l + a\cos\theta)^2)\xi}\rho_H\sin^2\theta. \end{aligned} \tag{4.76}$$

On the other hand, the general solution (4.45) to the real part of the Type D equation (4.28), which together with (3.60) yields

$$P^2 = \frac{\beta\bar{\omega}^2\rho_H(-\frac{\Lambda}{3}x^2 + Cx + D)}{(x^2 + Ax + B)\big((a-l)r\alpha + \bar{\omega}\big)\big(-(a+l)r\alpha + \bar{\omega}\big)}(x^2 - 1). \tag{4.77}$$

Comparing the two expressions for P^2 gives

$$\frac{(-\frac{\Lambda}{3}x^2+Cx+D)(x^2-1)}{(x^2+Ax+B)}=\frac{(1-a_3\cos\theta-a_4\cos^2\theta)(\bar{\omega}+r_H\alpha(a-l))^2(\bar{\omega}-r_H\alpha(a+l))^2}{(1-\frac{\alpha}{\bar{\omega}}(l+a\cos\theta)r_H)^2(r_H^2+(l+a\cos\theta)^2)\bar{\omega}^4}\sin^2\theta. \tag{4.78}$$

The above expression together with (3.61) [or (3.62)] provides the relation between parameters (A,B,C,D) and (r_H,α,l,a):

$$A=\frac{2lr_H^4\alpha^2-2r_H\alpha\bar{\omega}(a^2-l^2+r_H^2)-2l\bar{\omega}^2}{a(r_H^4\alpha^2+\bar{\omega}^2)}, \tag{4.79}$$

$$B=\frac{a^4r_H^2\alpha^2-2a^2lr_H\alpha(lr_H\alpha-\bar{\omega})+(l^2+r_H^2)(lr_H\alpha-\bar{\omega})^2}{a^2(r_H^4\alpha^2+\bar{\omega}^2)}, \tag{4.80}$$

$$\begin{aligned}C=&\frac{1}{a\bar{\omega}(lr_H\alpha-\bar{\omega})((a^2-l^2)l\alpha-r_H\bar{\omega})(r_H^4\alpha^2+\bar{\omega}^2)}\Big[-r_H^4\alpha^5(a^2-l^2)^2(a^2-l^2-r_H^2)+\tfrac{4}{3}lr_H\Lambda\bar{\omega}^5\\&+r_H^2\alpha^3\bar{\omega}^2\Big(2(a^2-3l^2)(a^2-l^2-r_H^2)+\big((l^3-a^2l)^2-(a^4-11a^2l^2+14l^4)r_H^2+(a^2+l^2)r_H^4\big)\tfrac{\Lambda}{3}\Big)\\&+\alpha\Big((l^2+r_H^2)+(l^4-14l^2r_H^2+r_H^4)\tfrac{\Lambda}{3}-a^2(1+l^2\tfrac{\Lambda}{3}-r_H^2\Lambda)\Big)\bar{\omega}^4-4lr_H^3\alpha^4\bar{\omega}(a^2-l^2)\\&\Big(a^2-r_H^2+l^2(r_H^2\tfrac{\Lambda}{3}-1)\Big)\\&-2lr_H\alpha^2\Big(2(-a^2+l^2+r_H^2)+\big((a^2-l^2)^2+2(2a^2-5l^2)r_H^2+r_H^4\big)\tfrac{\Lambda}{3}\Big)\bar{\omega}^3\Big],\end{aligned} \tag{4.81}$$

$$\begin{aligned}D=&\frac{-1}{a^2\bar{\omega}^2(lr_H\alpha-\bar{\omega})((a^2-l^2)l\alpha-r_H\bar{\omega})(r_H^4\alpha^2+\bar{\omega}^2)}\Big[(a^2-l^2)^4r_H^5\alpha^6\\&+lr_H^4\alpha^5(a^2-l^2)^2(5a^2-5l^2+r_H^2)\bar{\omega}\\&-r_H^3\alpha^4(a^2-l^2)\Big((a^2-5l^2)(2a^2-2l^2+r_H^2)-a^2(a^2-4l^2)r_H^2\tfrac{\Lambda}{3}\Big)\bar{\omega}^2\\&-lr_H^2\alpha^3\Big(2(3a^2-5l^2)(a^2-l^2+r_H^2)-a^2\big((a^2-l^2)^2+2(3a^2-5l^2)r_H^2+r_H^4\big)\tfrac{\Lambda}{3}\Big)\bar{\omega}^3\\&+r_H\alpha^2\Big((a^2-5l^2)(a^2-l^2+2r_H^2)+a^2\big(-l^4+10l^2r_H^2-r_H^4+a^2(l^2-2r_H^2)\big)\tfrac{\Lambda}{3}\Big)\bar{\omega}^4\\&-l\alpha\Big((l^2-5r_H^2)-a^2(1-\tfrac{4}{3}r_H^2\Lambda)\Big)\bar{\omega}^5-r_H\bar{\omega}^6\Big].\end{aligned} \tag{4.82}$$

Using the above relations it is possible to retrieve the expression for rotation pseudoscalar Ω for the accelerated Kerr-NUT-(anti)-de Sitter spacetime:

$$\Omega=\mathrm{Im}\Bigg[\frac{(-iM+l)\Big(a^2\alpha^2r_H^2-(l\alpha r_H-\omega)^2\Big)^3}{\omega^3(-a^2\alpha r_H+(l-ir_H)(l\alpha r_H-\omega)-ax(i\alpha r_H^2-\omega))^3}\Bigg]. \tag{4.83}$$

Furthermore, the Jacobian of the map between the parameterizations reads

$$J = 8r_H^2\Big(\big(\bar\omega - lr_H\alpha\big)^2 - a^2r_H^2\alpha^2\Big)^6\Big(\big(a^2-l^2\big)^2\big(a^2-l^2-r_H^2\big)\alpha^4r_H^4 \tag{4.84}$$

$$- 4\big(a^2-l^2\big)\big(-a^2+l^2+r_H^2\big)l\alpha^3\bar\omega r_H^3 + r_H^2\alpha^2\Big(a^4\big(-2+l^2\Lambda+\tfrac{1}{3}r_H^2\Lambda\big)$$

$$+ l^2\big(l^2+r_H^2\big)\big(-6+(l^2+r_H^2)\Lambda\big) \tag{4.85}$$

$$+ a^2\big(2(4l^2+r_H^2) - (2l^4+l^2r_H^2+\tfrac{1}{3}r_H^4)\Lambda\big)\Big)\bar\omega^2$$

$$- 2lr_H\alpha\Big(\tfrac{1}{3}a^4\Lambda + a^2(2-\tfrac{2}{3}l^2\Lambda) + \big(l^2+r_H^2\big)\big(-2+(l^2+r_H^2)\Lambda\big)\Big)\bar\omega^3$$

$$+ \Big(a^2(1-l^2\Lambda+\tfrac{1}{3}r_H^2\Lambda) + (l^2+r_H^2)(-1+(l^2+r_H^2)\Lambda)\Big)\bar\omega^4\Big)$$

$$\Big/ \Big(a^7\bar\omega^3(-lr_H\alpha+\bar\omega)^2((l^2-a^2)l\alpha+r_H\bar\omega)^2(r_H^4\alpha^2+\bar\omega^2)^5\Big). \tag{4.86}$$

It approaches infinity in the following cases:

$$J \to \infty \ \text{ for } a = 0 \ \vee \ \bar\omega = 0 \ \vee \ \bar\omega = lr_H\alpha \ \vee \ \bar\omega = \tfrac{(a^2-l^2)l\alpha}{r_H} \tag{4.87}$$

which coincide with the values for which parameters A, B, C, D are ill-defined. On the other hand, the Jacobi matrix J vanishes if

$$r_H = 0 \text{ or } r_H = \tfrac{\bar\omega}{(l\pm a)\alpha}, \tag{4.88}$$

or whenever

$$\big(a^2-l^2\big)^2\big(a^2-l^2-r_H^2\big)\alpha^4r_H^4 - 4\big(a^2-l^2\big)\big(-a^2+l^2+r_H^2\big)l\alpha^3r_H^3\bar\omega$$

$$+ r_H^2\alpha^2\Big(a^4\big(-2+l^2\Lambda+\tfrac{1}{3}r_H^2\Lambda\big) + l^2\big(l^2+r_H^2\big)\big(-6+(l^2+r_H^2)\Lambda\big) \tag{4.89}$$

$$+ a^2\big(2(4l^2+r_H^2) - (2l^4+l^2r_H^2+\tfrac{1}{3}r_H^4)\Lambda\big)\Big)\bar\omega^2$$

$$- 2lr_H\alpha\Big(\tfrac{1}{3}a^4\Lambda + a^2(2-\tfrac{2}{3}l^2\Lambda) + \big(l^2+r_H^2\big)\big(-2+(l^2+r_H^2)\Lambda\big)\Big)\bar\omega^3$$

$$+ \Big(a^2(1-l^2\Lambda+\tfrac{1}{3}r_H^2\Lambda) + (l^2+r_H^2)(-1+(l^2+r_H^2)\Lambda)\Big)\bar\omega^4 = 0. \tag{4.90}$$

The first part of the alternative in (4.88) requiring $r_H = 0$ further constrains the parameters since the zeroth coefficient of the polynomial $\mathcal{Q}$ has to vanish. Solving this constraint gives

$$\alpha = \frac{\omega(l^2\Lambda-1)}{2lM}. \tag{4.91}$$

Then, the map between parameter spaces degenerates to

$$
\begin{aligned}
A &= -\frac{2l}{a}, \\
B &= \frac{l^2}{a^2}, \\
C &= \frac{\Lambda l^2 + 3}{3al}, \\
D &= -\frac{1}{a^2},
\end{aligned}
\tag{4.92}
$$

However, the above implies that $A^2 - 4B = 0$, which corresponds to the case when rotation pseudoscalar Ω (4.46) is not well-defined and the alternative parameterization, (A', B', C'), is applicable. Comparing the 2-metrics on the slices of the isolated horizon and accelerated Kerr-NUT-(anti-)de Sitter horizon with $r_H = 0$, provides the following relation between our parameters A', B' and a, l

$$A' = -\frac{l}{a}, \tag{4.93}$$

$$B' = \frac{l^2 - a^2}{la^3}\left(1 - \frac{\Lambda}{3}l^2\right). \tag{4.94}$$

The rotation scalar Ω for the vanishing r_H reads

$$\Omega = \frac{M}{(l - ax)^3}. \tag{4.95}$$

It is, then, straightforward to find the correspondence between parameters M and C' using the expressions (4.95) and (4.56). The result yields

$$C' = -\frac{M}{a^3}. \tag{4.96}$$

It is noteworthy that horizons at $r_H = 0$ enjoy the benefit of a simplified form. The conformal factor reads simply $F = 1$ and consequently $x = -\cos\theta$. The horizon radius can be expressed as $R^2 = \beta(a^2 + l^2)$ and the frame coefficient as

$$P^2 = (a^2 + l^2)\frac{(x^2 - 1)\,(a\Lambda lx - 3)}{3l(l - ax)}. \tag{4.97}$$

The smoothness condition (3.31) together with the value of the proper rescaling parameter β read

$$\beta = \frac{l^2}{a^2 + l^2}, \quad \Lambda = \frac{3}{l^2}, \tag{4.98}$$

which inserted in the above expression for P^2 and R^2 gives

$$P^2 = 1 - x^2, \quad R^2 = \frac{3}{\Lambda}. \tag{4.99}$$

Therefore this class of horizons is a conical singularity generalization of Class II of horizons presented in the previous Chapter.

The second part of (4.88), i.e. the condition that $r_H = \frac{\bar{\omega}}{(l\pm a)\alpha}$ corresponds to an acceleration horizon. It can be seen by examining the conformal factor F, which now reads

$$F = 1 - \frac{l + a\cos\theta}{l + a}, \tag{4.100}$$

and vanishes for $\theta = 0$. This implies that the horizon intersects Scri at the surface of $\theta = 0$. Moreover, the parametrization (A, B, C, D) explicitly excludes this case as R^2 defined by (4.75) becomes infinite.

The last equation (4.89) is equivalent to the following condition for the cosmological constant

$$\Lambda = 3\big(a^2 - l^2 - r_H^2\big)\Big(\big(\bar{\omega} - \alpha l r_H\big)^2 - a^2\alpha^2 r_H^2\Big)^2 \Big/ \tag{4.101}$$

$$\Bigg(\bar{\omega}^2\Big(- 3\big(l^2 + r_H^2\big)^2\big(\bar{\omega} - \alpha l r_H\big)^2 - a^4\alpha r_H(3\alpha l^2 r_H + \alpha r_H^3 - 2l\bar{\omega})$$

$$+ a^2\Big(6\alpha^2 l^4 r_H^2 + \alpha^2 r_H^6 - 8\alpha l^3\bar{\omega} r_H - \bar{\omega}^2 r_H^2 + 3l^2(\bar{\omega}^2 + \alpha^2 r_H^4)\Big)\Big)\Bigg). \tag{4.102}$$

Interestingly, the above expression for Λ is precisely the condition for the horizon to be extremal, therefore, the relation is locally invertible if and only if the horizon is nonextremal.

In conclusion, the relation between the metric on a slice of isolated horizon and horizons of accelerated Kerr-NUT-(anti-)de Sitter is locally invertible for parameters corresponding to the generic case when four distinct horizons exist.

The explicit map between the spacetime parameters and the parameters A, B, C, D (or A', B', C') is determined only in one direction. However, in the neighborhood of a bifurcated Type D horizon, a unique Type D spacetime may be constructed[97]. All four-dimensional $\Lambda-$vacuum Type D solutions to Einstein equations are contained in the Plebański-Demiański family which suggests that the spacetimes constructed in the neighborhood of the spherical horizon would also be a spherical accelerated Kerr-NUT-(anti-)de Sitter. Then for such a spacetime one may construct the spacetime admitting the bundle structure, containing the nonbifurcated horizon of the Hopf bundle structure. If that were the case the relation between the parameters would be a bijection, up to parameters defining the same spacetime.

Contribution statement

The results in this Chapter are a product of collaboration with Denis Dobkowski-Ryłko and Jerzy Lewandowski. In particular, the presented solution to the Type D equation on a sphere with conical singularities is due to Denis Dobkowski-Ryłko.

What am I? I am zero—nothing. What shall I be tomorrow? I may be risen from the dead, and have begun life anew. For still, I may discover the man in myself, if only my manhood has not become utterly shattered.

Fyodor Dostoevsky, The Gambler

5 Spacetime and horizons of non-spherical topology

This Chapter deals with non-extremal horizons and spacetimes with the topology of the $U(1)-$PFB over a Riemann surface with a genus higher than 0. The $U(1)-$ action is generated by the flow of the KVF ℓ null at the horizon and its structure follow closely the cases described in the Introduction and Chapter 3.

Type D horizons with global sections being a compact, orientable Riemann surface have already been studied in [26], as solutions to the Type D Equation. The equation takes the form of the vanishing second holomorphic derivative of a complex function formed from Gaussian curvature K and the rotation pseudo-invariant Ω. If the Euler characteristic of the surface is non-positive, the only solutions are geometries of constant curvature and the rotation two-form vanishing at every point. A rotating solution with a trivial bundle structure necessarily has a spherical section of the horizon. The non-trivial bundle structure relaxes the above conclusions; The Gaussian curvature still must be constant, i.e. that the space of the null generator is equipped with a conformally flat metric. However, the rotation invariant Ω may be constant but non-vanishing. Crucially either the non-vanishing NUT parameter or the cosmological constant must be assumed due to Hawking's famous topology theorem [24,98].

I will solve the Type D equation with the non-trivial bundle assumption and using this data I will construct all IHs over Riemann surfaces with genus higher than zero and $U(1)-$PFB structure. This closes the classification of the IHs with with bundle structure over compact Riemann surfaces, generated by the flow of ℓ. The trivial spherical horizons were studied in [89], non-trivial spherical in [90] and trivial higher-genus in [26]. A possible remaining case in the theory of IHs with PFB structure are horizons generated by a KVF transversal to ℓ, similar to the horizons in Chapter 4. It is not clear whether this construction could be naturally extended to the higher genus case.

In the last part of this Chapter, I construct embedding spacetimes for those horizons. They are locally isometric to the generalized Taub-NUT-(anti-) de Sitter spacetimes, such that the surfaces of constant t and r have constant vanishing or negative curvature. The embedding spacetimes for toroidal horizons are locally isometric to the quotient of planar Taub-NUT-(anti-) de Sitter spacetimes. They may be expressed using standard coordinates on tours, parametrizing a pair of circles. For horizons with genus higher than 1, the embedding spacetime is locally isometric to the negative curvature, hyperbolic spacetimes. In this case, an abstract approach is necessary as coordinates on higher genus surfaces, respecting the gluing of the hyperboloid to the compact surface are not readily available. Then the embeddings are found by assuming the $U(1) - PFB$ ansatz the form of the metric, such that the spacetime is a product of the horizon and a real line. Subsequently, the Einstein Equations are solved and in this sense, the result may be seen as a partial generalization of the Birkhoff theorem for the NUT-like spacetimes. A similar result, where only the topology of the group orbits in spherical Taub-NUT was assumed was used to show a stronger version of the spherical, NUT-like Birkhoff theorem[99].

The constructed horizons are characterized by: the genus p of the base space S of the horizon, the Chern number χ_C accounting for all possible $U(1)-$ bundles one could construct over S, a constant curvature metric tensor on S accounting for all possible ways of compactifing a torus or a hyperboloid and a $U(1)$-connection with constant curvature. Due to the non-zero cohomology group of the higher genus Riemann surfaces the $U(1)$-connections satisfying the same curvature condition may still differ by a closed but not exact 1-form. This leads to locally diffeomorphic horizons and thus spacetimes which are however different. In the non-trivial case, the surface gravity κ is not free but is instead determined by the above. Upon embedding K and Ω are determined by the radius of the horizon r_H and the NUT parameter l. The NUT parameter determines only whether the bundle structure is trivial or not, by itself it does not determine the topological invariant χ_C.

The main focus of this Chapter is on the non-trivial horizons and their embeddings. However trivial $U(1)$-horizons and their embeddings into the generalized Schwrschild-(anti-) de Sitter spacetimes are also discussed. Similarly, the solutions to the Einstein Equations with $U(1)-$PFB ansatz are immediately extendible to the case of the spherical base space, corresponding to the standard Taub-NUT-(anti-) de Sitter spacetimes.

In the literature, higher-genus black holes are usually referred to as topological black holes[100,101], or if the genus is equal to one simply torus-like black holes[25]. Even though these black holes seem to be the generalization of Kerr spacetimes to the higher genus case, their horizons are still flat and of constant rotation scalar. Kerr-NUT topological black holes have been discussed in[102].

5.1 Vacuum Petrov type D isolated horizons whose null flow has PFB structure over a Riemann surface of genus $g \geq 1$

The results concerning horizons of trivial bundle topology presented in this Section are based on the solution to Type D equation obtained by the authors of[26].

The solutions are split into the toroidal and genus > 1 cases. For the torus they introduce a complex coordinates $(z, \bar{z})$

$$z = \frac{a\phi + b\psi + \mathrm{i}\psi}{\sqrt{2}},$$

where ψ and ϕ are 2π-periodic coordinates on torus such that the flat metric read as in (5.5). In $(z, \bar{z})$ coordinates the metric tensor and its area two-form take form

$$g^S = \frac{2}{P^2}\mathrm{d}z\mathrm{d}\bar{z}, \quad \eta = \mathrm{i}\frac{1}{P^2}\mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{5.1}$$

with the null frame

$$m^A\partial_A = P\partial_z. \tag{5.2}$$

The Type D equation reads

$$\partial_{\bar{z}}\left(P^2\partial_{\bar{z}}\Psi_2^{-1/3}\right), \tag{5.3}$$

where $\Psi_2 = K + \mathrm{i}\Omega + \frac{\Lambda}{3}$ is the horizon invariant. The author showed that $P^2\partial_{\bar{z}}\Psi_2^{-1/3}$ is a holomorphic, and thus constant, function on the torus. Due to the compactness of torus, it then follows that Ψ_2, together with the Gaussian curvature K and rotation invariant Ω are constant. For torus, the Gauss-Bonnet theorem gives $K = 0$, independently of the chosen global bundle structure. The two-form $\Omega\eta$ by the definition of Ω is subject to (1.38) defining the Chern number invariant. Therefore in the trivial case $\Omega = 0$, as was shown in[26].

For the higher genus case, S was covered by charts such that the metric tensor on S reads the same a (5.1). It was shown that $P^2\partial_{\bar{z}}\Psi_2^{-1/3}$ is a component of a globally defined, holomorphic vector field. From the fact that the space of holomorphic vector fields for a surface of genus > 1 has dimension zero, it follows that Ψ_2 must be constant. Again the curvature of S and Ω are subject to Gauss-Bonnet theorem and the Chern invariant definition.

It will revisit also the case of the trivial horizons with the purpose of describing the PFB structure defined thereon.

5.1.1 The horizons over 2-torus

If the space of the null directions S of the vacuum type D isolated horizon H is a 2-dimensional torus

$$S = S_1 \times S_1, \tag{5.4}$$

then it was shown that $K = 0$, implying that the metric g^S on S has to be flat. Therefore, there exist cyclic coordinates $(\phi, \psi) \in [0, 2\pi) \times [0, 2\pi)$ that set a coordinate system on $S^1 \times S^1$, such that

$$g^S = \frac{1}{P_0^2}(a^2 \mathrm{d}\phi^2 + 2ab\, \mathrm{d}\psi \mathrm{d}\phi + (1 + b^2)\mathrm{d}\psi^2), \tag{5.5}$$

with real constants $a > 0,\ b$ and $P_0 > 0$. The above admits the following discrete symmetries

1. If $b \neq 0$, then we can assume $b > 0$ without loss of generality, due to either $\psi \mapsto -\psi$ or $\phi \mapsto -\phi$ and the remaining coordinate freedom is a simultaneous change $(\psi, \phi) \mapsto (-\psi, -\phi)$.

2. If $b = 0$, then the remaining coordinate freedom is independently $\psi \mapsto -\psi$ and / or $\phi \mapsto -\phi$.

The rotation 1-form potential depends on the bundle structure. Therefore it is natural to split considerations into the trivial and non-trivial cases.

The trivial bundle case

In the case of a horizon of the structure of a trivial PFB, Eq (1.38) and the fact the Ω is constant imply

$$\Omega = 0. \tag{5.6}$$

Therefore, a corresponding G-connection A (1.29) is flat, and it can be represented by a single closed 1-form on S,

$$A^S = \alpha \mathrm{d}\phi + \beta \mathrm{d}\psi + \mathrm{d}f, \quad \alpha, \beta \in \mathbb{R},\ f \in C^\infty(S_1 \times S_2), \tag{5.7}$$

defined modulo global gauge transformations described below.

The horizon over S has the topology

$$H = G \times S \tag{5.8}$$

where $G = \mathbb{R},\ U(1)$. Let τ be a coordinate along G with (τ, ψ, ϕ) being the corresponding coordinate system on H such that the null vector field ℓ, defined up to a multiplication by a real constant, is

$$\ell = \partial_\tau. \tag{5.9}$$

Then, the degenerate metric tensor of H is given by just the same formula as above, namely

$$g = \frac{1}{P_0^2}(a^2 \mathrm{d}\phi^2 + 2ab\, \mathrm{d}\psi \mathrm{d}\phi + (1 + b^2)\mathrm{d}\psi^2). \tag{5.10}$$

The corresponding G-connection on H viewed as the PFB is

$$A = (\mathrm{d}\tau + \alpha \mathrm{d}\phi + \beta \mathrm{d}\psi + \mathrm{d}f) \otimes \partial_\tau, \tag{5.11}$$

while the formula for the rotation 1-form potential such that (5.6) reads

$$\omega = \kappa \left(\mathrm{d}\tau + \alpha \mathrm{d}\phi + \beta \mathrm{d}\psi + \mathrm{d}f \right). \tag{5.12}$$

The term $\mathrm{d}f$ may be eliminated by a change of the coordinates

$$\tau = \tau' + \pi^* f, \quad f \in C^\infty(S). \tag{5.13}$$

so that ω reads

$$\omega = \kappa \left(\mathrm{d}\tau' + \alpha \mathrm{d}\phi + \beta \mathrm{d}\psi \right). \tag{5.14}$$

If the gauge group $G = \mathbb{R}$ then, as explained above, rescaling freedom of the vector field ℓ (equivalently the coordinate τ) allows us to set $\kappa = 1$. The coefficients α and β may be made non-negative $\alpha, \beta \geq 0$, however, their range can not be further reduced. They are true degrees of freedom.

If the structure group of the PFB of H is $G = U(1)$, then the scaling ambiguity of the vector field ℓ can be fixed, by choosing τ to range $[0, 2\pi)$. In that case, the value of the surface gravity κ is an intrinsically meaningful characteristic of the horizon. On the other hand, there are more gauge transformations, meaning bundle automorphisms, namely

$$\tau = \tau' + n\psi + m\phi, \quad m, n \in \mathbb{Z}.$$

Hence, in the $U(1)$ case, without loss of generality, it is sufficient to consider $\alpha, \beta \in [0, 1)$.

In conclusion, the meaningful degrees of freedom of g and ω split into the following cases

1. If $b = 0$ and $G = \mathbb{R}$, then $\kappa \in \mathbb{R} \setminus \{0\}$, and $\alpha, \beta \geq 0$.

2. If $b > 0$ and $G = \mathbb{R}$, then $\kappa \in \mathbb{R} \setminus \{0\}$, and $\alpha, \beta \in \mathbb{R}$, which are determined up to $(\alpha, \beta) \mapsto (-\alpha, -\beta)$

3. If $G = U(1)$, then $\kappa \in \mathbb{R}^+$, $\alpha, \beta \in [0, 1)$, modulo $\alpha \mapsto 1 - \alpha$ and / or $\beta \mapsto 1 - \beta$, in the $b = 0$ case, and modulo $(\alpha, \beta) \mapsto (1 - \alpha, 1 - \beta)$, in the $b > 0$ case.

The corresponding covariant derivative ∇ is determined on H by the metric connection of g^S and the rotation 1-form ω[5].

Non-trivial bundle case

Now let's a nontrivial $U(1)$-bundle over $S := S^1 \times S^1$ (there are no non-trivial $\mathbb{R}-$bundles, due to $\mathbb{R}$ being contractible). It may be characterized using an open cover $\{V_1, V_2\}$ of $S^1 \times S^1$ and a transition function $a_{12} : V_1 \cap V_2 \to U(1)$ defined modulo

$$a_{12} \mapsto a'_{12} = b_1^{-1} a_{12} b_2, \quad b_i : V_i \to U(1), i = 1, 2. \tag{5.15}$$

It is sufficient to find just one pair $(\tilde{A}^S_{(1)},\ \tilde{A}^S_{(2)})$, and every other $(A^S_{(1)},\ A^S_{(2)})$ will be given by a globally defined 1-form a on $S^1 \times S^1$, namely

$$A^S_{(i)} = \tilde{A}^S_{(i)} + a. \tag{5.16}$$

The connection $A^S_{(i)}$ will be used to solve the condition that Ω=const provided its Hodge dual is a constant function,

$$*\mathrm{d}A^S_{(i)} = \mathrm{const}, \tag{5.17}$$

where the area 2-form η of the metric tensor (5.5) is

$${}^{(2)}\eta = \frac{a}{P_0^2}\mathrm{d}\psi \wedge \mathrm{d}\phi. \tag{5.18}$$

The existence of A^S satisfying (5.17) is demonstrated explicitly for S being a torus using coordinates. The existence is also obvious for a U(1) bundle automorphic to the bundle of the orthonormal frames. For higher genus surfaces the existence will be taken as an assumption.

A possible choice of the open cover is (S^1 is below identified with the interval $[0, 2\pi)$ endowed with the compact topology)

$$\begin{aligned} &V_1 = \big(\epsilon\,;\, 2\pi-\epsilon\big] \times S^1, \quad V_2 = \big[0\,;\, \pi-\epsilon\big) \cup \big(\pi+\epsilon\,;\, 2\pi\big] \times S^1, \\ &V_1 \cap V_2 = \big(\epsilon\,;\, \pi-\epsilon\big) \times S^1 \cup \big(\pi+\epsilon\,;\, 2\pi-\epsilon\big) \times S^1, \end{aligned}$$

for $0 < \epsilon < \frac{\pi}{2}$.

Using the gauge freedom (1.39) it is possible to set the transition function to

$$a_{12}(\psi,\phi) := \begin{cases} 1 & \text{at } \big(\epsilon\,;\, \pi-\epsilon\big) \times S^1 \\ \exp(-\mathrm{i}\chi_C\phi) & \text{at } \big(\pi+\epsilon\,;\, 2\pi-\epsilon\big) \times S^1,\ \ \chi_C \in \mathbb{Z}. \end{cases} \tag{5.19}$$

Next step is to construct the local connection 1-forms $\tilde{A}^S_{(i)}$, each defined in V_i. Consider a 1-form

$$\tilde{A}^S_{(1)} = \chi_C \frac{\psi}{2\pi}\mathrm{d}\phi. \tag{5.20}$$

It is well defined in V_1, as ψ does not have a discontinuity there. Next, define on $V_1 \cap V_2$

$$\tilde{A}^S_{(2)} = \tilde{A}^S_{(1)} - \mathrm{i}a_{12}^{-1}\mathrm{d}a_{12}, \tag{5.21}$$

and extend it by the analyticity to V_2. It follows that

$$\tilde{A}^S_{(2)} = \begin{cases} \chi_C \frac{\psi}{2\pi}\mathrm{d}\phi & \text{at } \big[0\,;\, \pi-\epsilon\big) \times S^1, \\ \chi_C \frac{\psi-2\pi}{2\pi}\mathrm{d}\phi & \text{at } \big(\pi+\epsilon\,;\, 2\pi\big] \times S^1. \end{cases} \tag{5.22}$$

To restore symmetry between the two elements of the covering, consider a function ψ' continuous on V_2 but with a step of 2π at $\psi = \pi$. It can be defined piece-wise as

$$\psi' = \begin{cases} \psi & \text{at } \big[0\,;\ \pi - \epsilon\big) \times S^1, \\ \psi - 2\pi & \text{at } \big(\pi + \epsilon\,;\ 2\pi\big] \times S^1. \end{cases} \tag{5.23}$$

Using ψ' one may rewrite $\tilde{A}^S_{(2)}$ to

$$\tilde{A}^S_{(2)} = \chi_C \frac{\psi'}{2\pi} \mathrm{d}\phi. \tag{5.24}$$

which continuously covers the points such that $\psi = 0$, i.e. $\psi' = \pi$. The G-curvature

$$\tilde{F} := \mathrm{d}\tilde{A}^S_{(2)} = \chi_C \frac{1}{2\pi} \mathrm{d}\psi \wedge \mathrm{d}\phi = \chi_C \frac{1}{2\pi} \mathrm{d}\psi' \wedge \mathrm{d}\phi, \tag{5.25}$$

together with the integral (justifying the choice of the name for χ_C)

$$\int_{S^1 \times S^1} \tilde{F} = 2\pi \chi_C \tag{5.26}$$

exhausts all the possible values as χ_C ranges $\mathbb{Z}$. Combined with the result of the general theory of principal bundles about the equivalence for the classes of isomorphism of principal $U(1)-$bundles over a manifold M [103]

$$\mathrm{Prin}_{U(1)}(M) \cong H^2(M; \mathbb{Z}), \tag{5.27}$$

where $H^2(M;\mathbb{Z})$ is the second cohomology group of M with coefficients in group $\mathbb{Z}$. For any compact and orientable Riemann surface, including sphere, torus and all surfaces considered in the subsequent Sections, this group is isomorphic to $\mathbb{Z}$ which reassures us that we have not missed above any $U(1)$-PFB.

A general connection $A^S_{(i)}, i = 1, 2$ such that

$$*\mathrm{d}A^S_{(i)} = \mathrm{const}$$

is given by (5.16) and an arbitrary closed 1-form a, namely

$$\tilde{A}^S_{(1)} = \chi_C \frac{\psi}{2\pi} \mathrm{d}\phi + \alpha \mathrm{d}\psi + \beta \mathrm{d}\phi, \quad \tilde{A}^S_{(2)} = \chi_C \frac{\psi'}{2\pi} \mathrm{d}\phi + \alpha \mathrm{d}\psi' + \beta \mathrm{d}\phi', \tag{5.28}$$

where due to the global gauge transformations the constants α, β can be restricted to the interval $[0, 1)$.

The horizon-bundle H is obtained by gluing $S^1 \times V_1$ endowed with coordinates (τ, ψ, ϕ) with $S^1 \times V_2$ equipped with coordinates (τ', ψ', ϕ'), using the relation (5.23) and the following relation between the remaining unprimed and primed coordinates, valid on $S^1 \times (V_1 \cap V_2)$,

$$\phi = \phi', \;\; \tau = \tau' - \begin{cases} 0 & \text{at } \big(\epsilon\,;\, \pi - \epsilon\big) \times S^1 \\ \chi_C \phi' & \text{at } \big(\pi + \epsilon\,;\, 2\pi - \epsilon\big) \times S^1, \;\; m \in \mathbb{Z}. \end{cases} \tag{5.29}$$

The null vector field ℓ on H is

$$\ell = \begin{cases} \partial_\tau & \text{at } S^1 \times V_1 \\ \partial_{\tau'} & \text{at } S^1 \times V_2, \end{cases} \tag{5.30}$$

while the degenerate metric tensor on H reads

$$g = \begin{cases} \frac{1}{P_0^2}(a^2 \mathrm{d}\phi^2 + 2ab\, \mathrm{d}\psi \mathrm{d}\phi + (1 + b^2)\mathrm{d}\psi^2) & \text{at } S^1 \times V_1, \\ \frac{1}{P_0^2}(a^2 \mathrm{d}\phi'^2 + 2ab\, \mathrm{d}\psi' \mathrm{d}\phi' + (1 + b^2)\mathrm{d}\psi'^2). & \text{at } S^1 \times V_2. \end{cases} \tag{5.31}$$

Finally, the rotation 1-form potential on H is

$$\omega = \begin{cases} \kappa \left(\mathrm{d}\tau + \frac{\chi_C}{2\pi}\psi \mathrm{d}\phi + \alpha \mathrm{d}\psi + \beta \mathrm{d}\phi\right), & \text{at } S^1 \times V_1; \\ \kappa \left(\mathrm{d}\tau' + \frac{\chi_C}{2\pi}\psi' \mathrm{d}\phi' + \alpha \mathrm{d}\psi' + \beta \mathrm{d}\phi'\right), & \text{at } S^1 \times V_2. \end{cases} \tag{5.32}$$

with the corresponding rotation pseudo scalar

$$\Omega = \kappa P_0^2 \frac{\chi_C}{2\pi a}. \tag{5.33}$$

To sum up a general vacuum Petrov type D isolated horizon (H, g, ℓ, ∇) whose null flow coincide with the $U(1)$ symmetry has the structure of a principal $U(1)$ fiber bundle over 2-torus $S_1 \times S_1$ is defined as follows:

- The manifold structure of H is given by considering $U(1) \times V_1$ endowed with the angle coordinate system (τ, ψ, ϕ) and $U(1) \times V_2$ endowed with the angle coordinate system (τ', ψ', ϕ'), and glueing them with (5.23, 5.29),
- The generator ℓ of the null flow is given by (5.30),
- The rotation 1-form potential ω is given by (5.32),
- The degenerate metric tensor g is given by (5.31),
- A degenerate metric tensor g and a rotation 1-form potential on H determine the covariant derivative ∇.

The constants $a > 0$, b, α, β, $P_0 \neq 0$, $\kappa \in \mathbb{R} \setminus \{0\}$, $\chi_C \in \mathbb{Z}$ are arbitrary. Their range may be reduced by the residual discrete coordinate transformation discussed above. If $\chi_C \neq 0$, then the structure group of the bundle generated by the flow of ℓ is non-trivial, and the horizon does not admit global, spacelike sections. If $\chi_C = 0$, then the bundle is trivial.

The formulae (5.30, 5.31, 5.32) with $\chi_C = 0$ define also a general vacuum Petrov type D isolated horizon $(H, [\ell], g, \omega)$ whose null flow has the structure of a principal $\mathbb{R}$ fiber bundle over 2-torus $S_1 \times S_1$ if we assume that τ is coordinate on $\mathbb{R}$. In that case, rescalling τ we can set the surface gravity κ to take arbitrarily fixed non-zero value.

The definition of IH for higher genus space of null generators follow similarity.

5.1.2 Solutions of Type D equation on a bundle over Riemann surface with genus ≥ 2

Consider an isolated horizon $(H, g, \nabla, [\ell])$ such that the null flow has the structure of a principal fiber bundle

$$\pi : H \to S, \quad G = \mathbb{R},\ U(1), \tag{5.34}$$

over a Riemann surface S of genus $g \geq 2$. The metric tensor g^S defined on S has the Ricci tensor

$$R^S = K g^S, \tag{5.35}$$

where due to the type D equation $K = \text{const}$[26], and as such it can be determined by the total area of S and its genus, namely

$$K = \frac{4\pi(1 - \text{genus})}{\text{Area}(g^S)}. \tag{5.36}$$

The degenerate metric tensor on H is the pullback of g^S

$$g = \pi^* g^S. \tag{5.37}$$

The trivial bundle case

As before we have $\Omega = 0$, hence the corresponding G-connections (1.29) are flat. In a consequence, to ensure that the Type D condition $\Psi_2 \neq 0$ holds it is also required that $K \neq \frac{\Lambda}{3}$. The horizon over S has the product topology $H = G \times S$ consistent with $\pi : H \to S$, with possible groups $G = \mathbb{R},\ U(1)$. Let τ be a coordinate along G extended naturally to H using the product structure and such that

$$\ell(\tau) = 1. \tag{5.38}$$

Then a flat connection on H can be written in terms of a closed 1-form A^S on S,

$$A = \left(\mathrm{d}\tau + \pi^* A^S\right) \otimes \partial_\tau, \tag{5.39}$$

while the formula for the corresponding rotation 1-form potential such that (5.6) reads

$$\omega = \kappa \left(d\tau + \pi^* A^S \right). \tag{5.40}$$

Now if the gauge group $G = \mathbb{R}$ then, as explained above, using the scaling freedom of the vector field ℓ that defines the isolated horizon structure we can fix $\kappa = 1$. The automorphisms of the bundle (choice of the Cartesian product) are given by transformations

$$\tau = \tau' + \pi^* f, \quad f \in C^\infty(S). \tag{5.41}$$

On the other hand, if the gauge group $G = U(1)$, then it is possible to fix the scaling of ℓ such that the corresponding coordinate on $U(1)$ ranges from 0 to 2π. Hence a value of κ in (5.40) becomes meaningful. However now, the automorphisms of the bundle are

$$e^{i\tau} = e^{i\tau' + i\pi^* f}, \quad \ell(f) = 0, \tag{5.42}$$

where the smoothness (or suitable differentiability class) is required of the exponentiated function.

The non-trivial bundle case

In this case

$$G = U(1), \tag{5.43}$$

and the non-equivalent bundles over Riemann surfaces are numbered by the Chern invariant $\chi_C \in \mathbb{Z}$. The rotation 1-form potential is given by

$$\omega = \kappa A \tag{5.44}$$

where κ is the surface gravity, and A is a 1-form on H such that $A \otimes \ell^*$ is a connection 1-form of the PFB considered thereon, and

$$\mathrm{d}A = \tilde{\Omega} \pi^* \eta^S, \quad \tilde{\Omega} = \mathrm{const}, \tag{5.45}$$

where η^S stands for the area 2-form of the metric tensor g^S. If there is a 1-form A satisfying the above condition for any $\tilde{\Omega} = \mathrm{const}$ then, as was explained above, it necessarily follows that

$$\tilde{\Omega} = \frac{2\pi \chi_C}{\mathrm{Area}(g^S)} = \frac{K \chi_C}{2(1-g)}, \tag{5.46}$$

where $\mathrm{Area}(g^S)$ is the area S according to the metric tensor g^S, and K is the constant Gauss curvature of g^S. Given a bundle with a specific A, every other connection 1-form A' can be written as

$$A' = A + \pi^* a, \quad \mathrm{d}a = 0. \tag{5.47}$$

The gauge equivalence classes $[a]$ are defined by the gauge transformations

$$a \mapsto a - ig^{-1}\mathrm{d}g, \quad g \in C^\infty(S, U(1))$$

as explained in detail in the case of the torus in Sec. 2.3.1. Finally, the rotation pseudo-scalar Ω is

$$\Omega = \frac{2\pi\chi_C\kappa}{\text{Area}(g^S)} = \frac{K\kappa\chi_C}{2(1-g)}. \tag{5.48}$$

If $\chi_C \neq 0$ and since $\kappa \neq 0$, also we have that $\Omega \neq 0$. Hence the Gauss curvature K has no forbidden values.

5.2 Embeddability of the Isolated Horizons with genus ≥ 1

In this Section, I seek to find the embedding spacetimes for the vacuum Petrov type D Isolated Horizons $(H, g, \nabla, [\ell])$ constructed in the first part of this Chapter. The approach is to assume that the embedding spacetime admits a $U(1)$-PFB structure, specifically that it has the topology $\mathbb{R} \times H$, and then solve the Einstein equations in terms of elements manifestly compatible with the PFB structure, that is globally defined on H. I solve the Einstein Equations for a general, generically non-trivial case, while the trivial limit is possible by assuming that a certain topological charge is zero. The obtained solution contains as a limit the embedding spacetimes for the trivial bundle horizons. In the last part of this Section, the construction of toroidal embedding spacetimes is exibited, where an explicit use of the coordinates on a torus is possible. The horizons of spherical topology with a non-trivial U(1)-bundle structure have been already found to be embeddable in the accelerated Kerr-NUT-(anti-) de Sitter spacetimes in the generic case, and Taub-NUT-(anti-) de Sitter spacetimes for constant gaussian curvature K and rotation invariant Ω as described in the INtroduction and Chapter 3. However, because the calculations presented in this ection naturally generalize to the case of spherical horizon the spherical spacetimes are also inclued.

In the global construction of the vacuum spacetimes containing Petrov type D horizons the following data will be used:

- A 2 dimensional compact and orientable Riemann surface S without boundary of genus p.
- A positive definite metric tensor g^ϵ on S, of constant Gaussian curvature normalized to $\epsilon = 0, \pm 1$ *.
- A $U(1)-$principal fiber bundle $\pi : H \to S$.

*g^S will be reserved for the metric on S induced by the horizon metric. Wi will turn out that g^ϵ and g^S differ only by a constant due to different, constant Gaussian curvatures.

- A connection 1-form A on H satisfying the constant curvature condition,compare with (1.11),
$$\mathrm{d}A = \tilde{\Omega}\pi^*\eta^\epsilon, \quad \tilde{\Omega} = \mathrm{const} \tag{5.49}$$
where η^ϵ is the 2-area tensor induced by g^ϵ on S.

Actually, the possible values of $\tilde{\Omega}$ are constrained by the topological invariants if the genus of S is higher than one, namely

$$\tilde{\Omega} := \frac{2\pi\chi_C}{\mathrm{Area}(g^\epsilon)} = \frac{\epsilon\chi_C}{2(1-\mathrm{genus})}.$$

From the previous sections it is clear that a family of the vacuum Petrov type D isolated horizons is associated with these data. They are given by the following family of pairs (g, ω), degenerate metric tensor and rotation 1-form potential, namely

$$g = \frac{1}{k^2}\pi^* g^\epsilon, \quad \omega = \kappa A, \quad k, \kappa = \mathrm{const} \neq 0, \tag{5.50}$$

If the genus is not equal to zero, then the scaling is already determined by the Gauss-Bonnect theorem. In this case, k is simply the Gaussian curvature of the horizon. For torus, the area is not determined and k remains arbitrary.

Given the above ingredients one may build the following spacetime

$$M = \mathbb{R} \times H, \tag{5.51}$$

and endow it with a metric tensor

$$g = -h(r)A^2 + \frac{\mathrm{d}r^2}{f(r)} + R(r)g^\epsilon, \tag{5.52}$$

where r is a coordinate on $\mathbb{R}$, and the functions $f(r)$, $h(r)$ and $R(r)$ are arbitrary and to be determined by solving Einstein's equations. With a typical, slight abuse of the notation, objects defined on the bundle H and their extension to the spacetime will be identified. Now the goals are as follows

- Impose the vacuum Einstein Equations with cosmological constant to solve for the functions $f(r)$, $h(r)$ and $R(r)$.
- Construct an Eddington-Finkelstein extension of the metric (5.52) covering possible Killing horizons.
- Show that for every Killing horizon derived in the previous Section) the degenerate metric tensor g and the rotation 1-form potential ω, both defined on H, have the form (5.50) and that the constants k, κ take all possible non-zero values.

It is useful to introduce the orthonormal co-frame (e^A_ϵ), $A \in 2,3$ on S. The structure equations for g^ϵ depend only on the connection component $\Gamma^2_{\epsilon\,3}$. They read

$$\begin{aligned} \mathrm{d}e^2_\epsilon + \Gamma^2_{\epsilon\,3} \wedge e^3_\epsilon = 0, \\ \mathrm{d}e^3_\epsilon - \Gamma^2_{\epsilon\,3} \wedge e^2_\epsilon = 0. \end{aligned} \tag{5.53}$$

and are enough to write the only, up to symmetries, component of the two-dimensional curvature 2-form $R^A_{\epsilon\;B}$:

$$R^2_{\epsilon\,3} = \mathrm{d}\Gamma^2_{\epsilon\,3} = R^2_{\epsilon\,323}\, e^2_\epsilon \wedge e^3_\epsilon = \epsilon\, e^2_\epsilon \wedge e^3_\epsilon \tag{5.54}$$

To solve the spacetime Einstein Equations I employ the orthonormal co-frame corresponding to g

$$\begin{aligned} e^0 &= \sqrt{h}A, \\ e^1 &= \frac{\mathrm{d}r}{\sqrt{f}}, \\ e^2 &= \sqrt{R}\, e^2_\epsilon, \\ e^3 &= \sqrt{R}\, e^3_\epsilon. \end{aligned} \tag{5.55}$$

Then the definitions of the objects on the bundle over S allow us to write the structure equations

$$\begin{aligned} \mathrm{d}e^0 &= -\frac{h'\sqrt{f}}{2h} e^0 \wedge e^1 + \sqrt{h}\frac{N\tilde{\Omega}}{R} e^2 \wedge e^3, \\ \mathrm{d}e^1 &= 0, \\ \mathrm{d}e^2 &= \frac{R'\sqrt{f}}{2R} e^1 \wedge e^2 - \Gamma^2_{\epsilon\,3} \wedge e^3, \\ \mathrm{d}e^3 &= \frac{R'\sqrt{f}}{2R} e^1 \wedge e^3 + \Gamma^2_{\epsilon\,3} \wedge e^2. \end{aligned} \tag{5.56}$$

where by $'$ the derivative with respect to r is denoted. The detailed computations of spacetimes connection and curvature may be found in the Appendix.

Adding G_{00} and G_{11} allows to solve for f and g

$$G_{00} + G_{11} = \Lambda - \Lambda = 0 = \frac{\tilde{\Omega}^2 h(r)}{2r^4} - \frac{f'(r)}{r} + \frac{f(r)h'(r)}{rh(r)}. \tag{5.57}$$

Integrating the above yields the relation between $f(r)$ and $h(r)$

$$\frac{h(r)}{f(r)} = \frac{4r^2 c_1}{r^2 - \tilde{\Omega}^2 c_1}, \tag{5.58}$$

where c_1 is an integration constant. The above motivates a following coordinate change.

$$\bar{r} = \sqrt{r^2 + l^2}, \tag{5.59}$$

where the integration constant has been reparametrized $l^2 := -\tilde{\Omega}^2 c_1$ for $\tilde{\Omega} c_1 < 0$.

The metric tensor using the new coordinate reads

$$g = -hA^2 + \frac{\mathrm{d}\bar{r}^2}{\bar{f}} + (\bar{r}^2 + l^2) g^\epsilon, \tag{5.60}$$

where $\bar{f} := f\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}r}\right)^2 = f\frac{\bar{r}^2}{r^2}$. After the transformation we have

$$\partial_{\bar{r}}\left(\frac{\bar{f}}{h}\right) = \frac{\mathrm{d}r}{\mathrm{d}\bar{r}}\partial_r\left(\frac{f}{h}\frac{\bar{r}^2}{r^2}\right) = 0. \tag{5.61}$$

It follows that h and $\bar{f}$ are proportional $h =: N\bar{f}$ where N is a real constant. Note that the apparent singularity at $r = \tilde{\Omega}^2 c_1$ of (5.58) appears only for the non-trivial bundles, if $\chi_C = 0$ there is no need to introduce an additional parameter l^2 and the role of proportionality constant between h and f is played by the integration constant c_1. The form of the metric tensor given by (5.60) is therefore as general as the one given by (5.52). In the subsequent, p the bar above r and h is dropped. The Einstein Equations will be solved for the metric tenor

$$g = -f(r)NA^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 + l^2) g^\epsilon, \tag{5.62}$$

where l is an arbitrary parameter. Because the above metric is formally the same as previously, only with f replaced by Nh and specified R, it is possible to reuse the calculations of connection and curvature. Again it follows that

$$G_{00} + G_{11} = 0 = \frac{f(r)\left(N\tilde{\Omega}^2 - 4l^2\right)}{2\left(l^2 + r^2\right)^2}, \tag{5.63}$$

which establishes

$$N = \frac{4l^2}{\tilde{\Omega}^2}. \tag{5.64}$$

Inserting the above value into G_{00} and solving $G_{00} = -\Lambda$ gets

$$f(r) = \frac{\epsilon\left(r^2 - l^2\right) + c_1 r - \Lambda\left(\frac{1}{3}r^4 + 2l^2r^2 - l^4\right)}{(r^2 + l^2)}. \tag{5.65}$$

with the resulting metric reading

$$g = -\left(\frac{l\mathrm{Area}(g^\epsilon)}{\pi\chi_C}\right)^2 f(r)A^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 + l^2)g^\epsilon. \tag{5.66}$$

Note that for the toroidal case, the function $f(r)$ is already the same as of the toroidal Taub-NUT-(anti-) de Sitter case if one renames $c_1 = -2M$. The remaining freedom is the choice of metric on the torus S.

The extension through the horizon requires the introduction of an (in-going) Eddington-Finkelstein co-frame transformation

$$\left(\frac{l\mathrm{Area}(g^S)}{\pi\chi_C}\right) A' := \left(\frac{l\mathrm{Area}(g^S)}{\pi\chi_C}\right) A + \frac{\mathrm{d}r}{f(r)}. \tag{5.67}$$

or equivalently

$$e'^0 := e^0 + e^1. \tag{5.68}$$

Similarly to the torus case, introducing A preserves the bundle structure, equivalently one could have started with an ansatz in the Eddington-Finkelstein form

$$g = -\left(\frac{l\mathrm{Area}(g^S)}{\pi\chi_C}\right)^2 f(r)A'^2 + 2\left(\frac{l\mathrm{Area}(g^S)}{\pi\chi_C}\right) A'\mathrm{d}r + (r^2+l^2)g^S. \tag{5.69}$$

If the Gaussian curvature of S is non-vanishing the area is already determined by the normalization of $|\epsilon| = 1$. Then in all expressions that will follow one could replace

$$\frac{l\mathrm{Area}(g^S)}{\pi\chi_C} = \frac{2l|1-p|}{\chi_C}. \tag{5.70}$$

Note that ℓ is a Killing vector field of $\tilde{\Pi}^* g^S$ and therefore may be naturally extended to a Killing vector field of g generating a horizon at the hypersurface $r = r_H$, where $f(r_H) = 0$.

Now K and Ω of the horizon with a metric tensor on section S have to be calculated

$${}^{(2)}g_H = (r^2+l^2)g^\epsilon. \tag{5.71}$$

By assumption (S, g^ϵ) has the Gaussian curvature ϵ, giving the Gaussian curvature K of the horizon as

$$K = \frac{\epsilon}{r_H^2 + l^2} \tag{5.72}$$

The horizon rotation 1-form ω and the surface gravity κ read

$$\begin{aligned}
\omega \otimes \ell &:= \nabla\ell\Big|_H = \left(\frac{l\mathrm{Area}(g^S)}{\pi\chi_C}\right)\frac{f'}{2}\Big|_H e'^0 \otimes e_0 = \left(\frac{l\mathrm{Area}(g^S)}{\pi\chi_C}\right)\frac{f'}{2}\Big|_H A \otimes \ell,\\
\kappa &:= \nabla_\ell \ell\Big|_H = \left(\frac{l\mathrm{Area}(g^S)}{\pi\chi_C}\right)\frac{f'}{2}\Big|_H,\\
\omega &= \kappa A.
\end{aligned} \tag{5.73}$$

The rotation invariant Ω associated with the horizon in generalized Taub-NUT-(anti-) de Sitter can be calculated using

$$\mathrm{d}\omega = \Omega\eta_H, \tag{5.74}$$

where η_H is the volume form associated with g_H, as

$$\Omega = \frac{\kappa}{r_H^2 + l^2} \frac{2\pi\chi_C}{\text{Area}(g^S)} = \frac{2\pi\chi_C\kappa}{\text{Area}({}^{(2)}g_H)} = \frac{\epsilon - (r_H^2 + l^2)\Lambda}{r_H(r_H^2 + l^2)} \tag{5.75}$$

which exactly reproduces the abstract solution (5.48).

Having constructed the toroidal Taub-NUT-(anti-) de Sitter horizon with a structure of a non-trivial $U(1)-$principal bundle over torus is it natural to ask if the bundle structure extends to the bulk spacetimes.

Take the group action generated by the flow of the same Killing vector field ℓ, now acting on the whole spacetime instead of only the horizon, then the connection 1-form may be defined as

$$A_{\text{ST}} := \frac{g(\ell, \cdot)}{g(\ell, \ell)}. \tag{5.76}$$

To see that the above indeed is a real-valued part of the connection 1-form one has to check the conditions

$$A_{\text{ST}}(\ell) = 1, \quad \mathcal{L}_\ell A_{\text{ST}} = 0, \tag{5.77}$$

which follow immediately from ℓ being a Killing vector of the spacetime.

The PFB structure of π used in the construction of the spacetime extends naturally to the PFB structure Π of the entire spacetime by extension of the real line associated with the coordinate r.

$$U(1) \hookrightarrow P \times \mathbb{R} \xrightarrow{\Pi} S \times \mathbb{R}. \tag{5.78}$$

The connection 1-form defined as (5.76) is simply

$$A_{\text{ST}} = A' - \left(\frac{\pi\chi_C}{l\text{Area}(g^S)}\right)\frac{\mathrm{d}r}{f(r)} = A \tag{5.79}$$

Finally note that A (and consequently A', ω and A_{ST}) for a given Ω^S is defined uniquely only up to 1-forms $\alpha_1, \alpha_2, \ldots, \alpha_{2p}$ generating first de Rham cohomology group $H^1(S, \mathbb{R}) \cong \mathbb{R}^{2g}$, where p is the genus of S. In this sense also the constructed spacetime (5.52) is also unique only up to the α's.

The metric tensors derived above are locally isometric to the generalized Taub-NUT-(anti-) de Sitter (or in the trivial case to the generalized Schwarzschild-(anti-) de Sitter) given by[29]

$$g = -f(r)\left(\mathrm{d}t + l\frac{\mathrm{i}(\zeta\mathrm{d}\bar{\zeta} - \bar{\zeta}\mathrm{d}\zeta)}{1 + \frac{1}{2}\epsilon\zeta\bar{\zeta}}\right)^2 + f(r)^{-1}\mathrm{d}r^2 + \left(r^2 + l^2\right)\frac{2\mathrm{d}\zeta\mathrm{d}\bar{\zeta}}{(1 + \frac{1}{2}\epsilon\zeta\bar{\zeta})^2}, \tag{5.80}$$

where

$$f(r) = \frac{\epsilon(r^2 - l^2) - 2Mr - \Lambda(\frac{1}{3}r^4 + 2l^2r^2 - l^4)}{r^2 + l^2}, \tag{5.81}$$

and the NUT and mass parameters are denoted by l and M, respectively, and $\epsilon = 0, \pm 1$ so that the last term in the above is a flat metric tensor defined on a surface of constant Gaussian curvature equal to ϵ.

The difference between (5.80) and (5.62) is only in the compactification of the surfaces parameterized by $(\zeta, \bar{\zeta})$. Indeed, all tori may be constructed by a suitable identification of sides of a parallelogram, i.e. a quotient $\mathbb{R}^2/\mathbb{Z}^2$. Similarly, higher genus Riemann surfaces may be constructed as the quotient of a hyperbolic plane by a discrete subgroup $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ generated by $2p$ elements $a_1, b_1, \ldots, a_p, b_p$ satisfying single defining relation

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_p a_p^{-1} b_p^{-1} = 1. \tag{5.82}$$

Then this surface is a hyperbolic $4p$-gon with a pair of edges identified.

5.2.1 Toroidal Taub-NUT-(anti-) de Sitter and non-trivial bundles

Although the toroidal case is covered by the construction in the previous Section, there is an advantage to consider it separately. Namely on a torus global coordinates may be easily introduced which allows for an explicit construction of both the connection A and the metric on the base space g^S.

Consider the metric tensor (5.62) with $f(r)$ and N given by (5.65) and (5.64) and $\epsilon = 0$. The metric g^ϵ may be given for torus as

$$g^\epsilon = (a^2 \mathrm{d}\phi^2 + 2ab\, \mathrm{d}\phi \mathrm{d}\psi + (1 + b^2)\mathrm{d}\psi^2)$$

which implies that

$$A = \mathrm{d}\tau + \frac{\chi_C}{2\pi} \psi \mathrm{d}\phi, \tag{5.83}$$

where $\tau \in [0; 2\pi)$ and ∂_τ generates the Killing horizon.

After introducing Finkelstein-Eddington coordinates it is easy to see that the non-degenerate metric tensor on a toroidal section of the horizon is

$${}^{(2)}g_H = (r_H^2 + l^2) g^\epsilon, \tag{5.84}$$

which compared with the Isolated Horizon given by the solution to the Petrov Type D equation yields $r_H^2 + l^2 = P_0^{-2}$. Comparing the rotation 1-form $\omega = \kappa A$ for the above horizon with the solution to the Petrov Type D equation given by (5.32) it seems that only the horizons with $\alpha = 0 = \beta$ are embeddable. To embed the general class it is sufficient to introduce a local coordinate transformation

$$\tau \mapsto \tau + \alpha\psi + \beta\phi. \tag{5.85}$$

Importantly the above is only a local diffeomorphism. It does not extend to global diffeomorphism as the transformation is not well defined whenever the coordinates (ψ, ϕ) are not well defined, namely if $\psi = 0$ or $\phi = 0$. Knowing the first de Rham cohomology group of the torus $H^1_{\mathrm{dR}} \cong \mathbb{R}^2$ concludes that the above transformation accounts for all solutions to (5.49)). Finally, the rotation invariant of the horizon reads

$$\Omega = \frac{\chi_C \kappa}{2\pi(r_H^2 + l^2)} = \frac{l\Lambda}{r_H}. \tag{5.86}$$

It should be noted that the formula for Ω (5.75) does not have limit to $\Lambda = 0$ in the toroidal case. Then we have

$$f(r) = \frac{-2Mr}{r^2 + l^2}, \quad r_H = 0, \tag{5.87}$$

which results

$$\Omega = -\frac{2M}{l^3}. \tag{5.88}$$

5.2.2 Generalized Schwarzschild-(anti-) de Sitter and trivial horizons

If the horizon bundle is trivial i.e. $\chi_C = 0 = \Omega^S$ then the solutions (5.69) obtained in the previous Section do not have a well defined limit. Instead, one has to go back to the derivation earlier in that Section. Due to the triviality the structure equation containing $\mathrm{d}A = 0$ changes. Now the choice of $R(r)$ giving the proportionality between $f(r)$ and $h(r)$ is standard in the Schwarzschild case, i.e. $R(r) = r^2$. From the fact that $G_{00} + G_{11} = 0$ follows

$$h(r) = c_1 f(r). \tag{5.89}$$

Solving $G_{00} = -\Lambda$ gives

$$f(r) = \epsilon + \frac{c_2}{r} - \frac{\Lambda r^2}{3}, \tag{5.90}$$

recovering the generalized Schwarzschild-(anti-) de Sitter for $c_2 = -2M$ with the metric tensor reading

$$g = -c_1 f(r) A^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 g^\epsilon. \tag{5.91}$$

If the fiber is $\mathbb{R}$ then c_1 is arbitrary and may be set to 1, similarly to the arbitrary choice rescaling of the time coordinate in Schwarzschild spacetime. Otherwise is the fiber is $U(1)$ and c_1 is determined by the range of the fibre coordinate.

5.3 Detailed Einstein Equations for the spacetimes with $U(1)-$PFB structure

The (Levi-Civita) connection 1-forms $\Gamma^{\mu}{}_{\nu}$ may be written as

$$\begin{aligned}
&\mathrm{d}e^{\mu} + \Gamma^{\nu}{}_{\alpha} \wedge e^{\alpha} = 0, \\
&\Gamma^{0}{}_{1} = \Gamma^{1}{}_{0} = \frac{h'\sqrt{f}}{2h} e^{0}, \\
&\Gamma^{0}{}_{2} = \Gamma^{2}{}_{0} = \frac{N\tilde{\Omega}}{2R}\sqrt{h} e^{3}, \\
&\Gamma^{0}{}_{2} = \Gamma^{3}{}_{0} = -\frac{N\tilde{\Omega}}{2R}\sqrt{h} e^{2}, \\
&\Gamma^{1}{}_{2} = -\Gamma^{2}{}_{1} = -\frac{R'\sqrt{f}}{2R} e^{2}, \\
&\Gamma^{1}{}_{3} = -\Gamma^{3}{}_{1} = -\frac{R'\sqrt{f}}{2R} e^{3}, \\
&\Gamma^{2}{}_{3} = -\Gamma^{3}{}_{2} = \Gamma^{2}_{\epsilon 3} + \frac{N\tilde{\Omega}}{2R}\sqrt{h} e^{0}.
\end{aligned} \tag{5.92}$$

Similarly for the curvature 2-forms $\Omega^{\mu}{}_{\nu}$

$$\begin{aligned}
&\Omega^{\mu}{}_{\nu} = \mathrm{d}\Gamma^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\alpha} \wedge \Gamma^{\alpha}{}_{\nu}, \\
&\Omega^{0}{}_{1} = -Ae^{0} \wedge e^{1} + Be^{2} \wedge e^{3}, \\
&\Omega^{0}{}_{2} = -Ce^{0} \wedge e^{2} + \frac{1}{2}Be^{1} \wedge e^{3}, \\
&\Omega^{0}{}_{3} = -Ce^{0} \wedge e^{3} - \frac{1}{2}Be^{1} \wedge e^{2}, \\
&\Omega^{1}{}_{2} = \frac{1}{2}Be^{0} \wedge e^{3} - De^{1} \wedge e^{2}, \\
&\Omega^{1}{}_{3} = -\frac{1}{2}Be^{0} \wedge e^{2} - De^{1} \wedge e^{3}, \\
&\Omega^{2}{}_{3} = -Be^{0} \wedge e^{1} + Ee^{2} \wedge e^{3},
\end{aligned} \tag{5.93}$$

where

$$
\begin{aligned}
A &= \frac{(2fhh'' - f\,h'^2 + f'hh')}{4h^2}, \\
B &= \frac{\tilde{\Omega}\sqrt{\mathrm{f}}\ (h'R - hR')}{2R^2\sqrt{\mathrm{h}}}, \\
C &= \frac{(\tilde{\Omega}^2 h^2 + RR'fh')}{4R^2 h}, \\
D &= \frac{(RR'f' + 2RR''f - f\,R'^2)}{4R^2}, \\
E &= \frac{\left(3\tilde{\Omega}^2 h - f\,R'^2 + 4\epsilon R\right)}{4R^2}.
\end{aligned}
\tag{5.94}
$$

By standard conventions we have

$$
\Omega^\mu{}_\nu = \tfrac{1}{2} R^\mu{}_{\nu\alpha\beta} e^\alpha \wedge e^\beta, \quad R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}, \quad \mathrm{Ric} = R^\alpha{}_\alpha, \quad E_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\mathrm{Ric}\, g_{\mu\nu}
$$

and so

$$
R_{00} = A + 2C, \quad R_{11} = -A - 2D, \quad R_{22} = R_{33} = -C - D + E, \tag{5.95}
$$

$$
\mathrm{Ric} = -2A - 4C - 4D + 2E, \tag{5.96}
$$

$$
G_{00} = -2D + E, \quad G_{11} = 2C - E, \quad E_{22} = E_{33} = A + C + D. \tag{5.97}
$$

Il faut imaginer Sisyphe heureux.
Albert Camus, Le Mythe de Sisyphe

Summary and Conclusions

This work sought to establish the non-singular interpretation of spacetime with the NUT parameter with a particular emphasis on the spherical solutions. The main obstacle is a plethora of pathological behaviors associated with the axis of rotation of the spacetime if the surfaces of constant t have the standard topology of $S^2 \times \mathbb{R}$ First, both parts of the axis are associated with different axial KVFs, only one of which may be chosen to be cyclic. This poses a problem for a consistent definition of conical singularity. Second, is that even if the conical singularity may be defined for a non-cyclic axial KVF then NUT spacetimes generically possess an irremovable conical singularity. Often this type of singularity is associated with the presence of cosmic (sometimes called Misner) string. The metric tensor is clearly not defined at the singular axis.

For the simplest NUT spacetime, i.e. the Taub-NUT spacetime a different interpretation has existed for the last 60 years. It was proposed by Misner and is based on the periodic identification of time coordinates. It also cures both of the above problems. Periodicity of t implies that the orbits of both axial KVFs are closed. It also allows for separate removal of conical singularity at both parts of the axis. The topology of the spacetime changes to a $U(1)$ bundle over $S^2 \times \mathbf{R}$. Therefore in no single coordinate patch, a two-dimensional sphere without conical singularity may be exhibited. However, they exist as a part of S^3, in the form of local trivializations of the PFB.

I have proposed a method of generalizing Misner's method to generic spacetime with NUT parameter and used it to construct non-singular spacetimes describing rotating, accelerating, electromagnetic charges black holes on the background of cosmological constant of any sign. The new construction is based on the realization that any linear combination of KVFs may be used for the introduction of the $U(1)$-PFB structure. The base space of the bundle is then the orbit space of a suitably chosen KVF. Quite surprisingly it turns out that if the NUT parameter is present there always exist two KVFs such that their orbit-space is smooth and therefore a suitable candidate for the base space. Subsequently, the orbits of the KVF may be compactified to a circle. Another naturally emerging object is the $U(1)-$ connection 1-form. Together with the orbit-space metric they may be used to construct a smooth metric tensor with a possible exception of curvature singularity, or the singularity of the conformal factor in the accelerated case. The definition of conical singularity I opt to use is the conical singularity calculated in the orbit space of some KVF. It turns out that it is independent of the chosen

KVF if and only if the NUT parameter vanishes. This fact may be interpreted as the conical singularity being an observer-dependent quantity in the presence of the NUT parameter. It points to a need for a better understanding of the conical singularity when considered in four-dimensional spacetime.

Next, I discuss the Killing horizons of the non-singular (accelerated) KNadS spacetimes. In a natural way, two different two-different classes of horizons arise, depending on if the space of the null generators coincides with the orbits with the base space of the $U(1)$- bundle pullbacked to the horizon. Both types have the structure of Hopf (or higher Chern number) bundle. For the first type the $U(1)$-action is given by the flow of the horizon-generating KVF, while for the second it is a generic combination. It turns out that in the second case, the space of the null geodesic is defined only locally and that IHs with such a structure was previously not known. I propose the structure of the local counterpart of this horizon and describe it as the horizons of the Hopf bundle structure with the action generated by a KVF transversal to the null direction. In both cases, the Type D equation is solved and the local solutions are compared with the embedded Killing horizons. In both cases the bijection between is not established, it follows only that any embedded spherical horizon is obtained from an IH, not that every IH is embeddable in the accelerated KNadS. The solutions to the Petrov Type D equation in the transversal case may be alternatively used to construct horizons with a product topology, however with base space being a sphere with conical singularities.

The last part of this work is concerned with the horizons of $U(1)-$PFB structure over Riemann surfaces with genus higher than 0. Based on the results obtained for the horizons with a trivial bundle structure I solve the Type D Equation and construct all Type D horizons of this type. This result was the penultimate missing part of the classification of Type D IH in dimension 4, the last is the existence of non-axially symmetric spherical horizons, which remains an open question. Intuitively horizons of higher genus should be embeddable in flat and negative curvature generalizations of Taub-NUT-(anti-) de Sitter. This fact was however more difficult to establish directly than expected, mainly due to the lack of suitable coordinates covering Riemann surfaces of genus higher than 1. By solving the Einstein Equations with an ansatz constructed from the objects corresponding to a $U(1)-$ bundle of the constant $U(1)-$ curvature I have found the embedding spacetimes. They are in fact the expected generalizations of the NUT spacetimes, after the compactification of a family of two-dimensional surfaces.

In this work, I have described the geometry and topology of spacetimes with the NUT parameter which has the advantage of admitting a smooth metric tensor. Several open problems still remain. After the periodic identification, through every point of the region, we would like to associate with the exterior of a black hole a closed timeline curve passes. Whether the black hole interpretation is possible to maintain is not clear. Similarly, the physical interpretation of the NUT parameter remains an open problem. One clear answer contained in this work is that it is connected with a non-zero Chern number of the spacetime. The physical interpretation of other parameters in the presence of NUT parameter also requires revision.

In particular, even without imposing periodicity the conical singularity seems to be observer-dependent. If one would like to associate the conical singularity with the tension of a cosmic string then this (or even the presence of said string) then it seems that it must be observer-dependent.

References

[1] C. W. Misner. Taub-Nut Space as a Counterexample to almost anything. In Jürgen Ehlers, editor, *Relativity Theory and Astrophysics. Vol.1: Relativity and Cosmology*, volume 8, page 160. 1967.

[2] Petr Hajicek. Extensions of the taub and nut spaces and extensions of their tangent bundles. *Communications in Mathematical Physics*, 17(2):109–126, Jun 1970.

[3] Jerzy Lewandowski and Adam Szereszewski. Axial symmetry of kerr spacetime without the rigidity theorem. *Phys. Rev. D*, 97:124067, Jun 2018.

[4] Jerzy Lewandowski and Tomasz Pawłowski. Geometric Characterizations of the Kerr Isolated Horizon. *International Journal of Modern Physics D*, 11, 12 2001.

[5] Denis Dobkowski-Ryłko, Jerzy Lewandowski, and Tomasz Pawłowski. The Petrov type D isolated null surfaces. *Classical and Quantum Gravity*, 35(17):175016, aug 2018.

[6] Jerzy Lewandowski and Tomasz Pawlowski. Extremal isolated horizons: a local uniqueness theorem. *Classical and Quantum Gravity*, 20(4):587–606, jan 2003.

[7] Maciej Dunajski and James Lucietti. Intrinsic rigidity of extremal horizons, 2023.

[8] Hari K. Kunduri and James Lucietti. A classification of near-horizon geometries of extremal vacuum black holes. *Journal of Mathematical Physics*, 50(8):082502, 08 2009.

[9] Eryk Buk and Jerzy Lewandowski. Axisymmetric, extremal horizons in the presence of a cosmological constant. *Phys. Rev. D*, 103:104004, May 2021.

[10] David Matejov and Jiří Podolský. Uniqueness of extremal isolated horizons and their identification with horizons of all type d black holes*. *Classical and Quantum Gravity*, 38(13):135032, jun 2021.

[11] David Matejov and Jiri Podolský. Extremal isolated horizons with a cosmological constant and the related unique type d black holes. *Phys. Rev. D*, 105:064016, Mar 2022.

[12] Abhay Ashtekar, Christopher Beetle, and Jerzy Lewandowski. Geometry of generic isolated horizons. *Classical and Quantum Gravity*, 19(6):1195–1225, mar 2002.

[13] Hari Kunduri and James Lucietti. Classification of near-horizon geometries of extremal black holes. *Living Reviews in Relativity*, 16, 06 2013.

[14] Hari K Kunduri, James Lucietti, and Harvey S Reall. Near-horizon symmetries of extremal black holes. *Classical and Quantum Gravity*, 24(16):4169, jul 2007.

[15] Maciej Kolanowski, Jerzy Lewandowski, and Adam Szereszewski. Extremal horizons stationary to the second order: New constraints. *Phys. Rev. D*, 100:104057, Nov 2019.

[16] E. Buk, J. Lewandowski, and A. Szereszewski. Lie point symmetries of near-horizon geometry equation. *Phys. Rev. D*, 102:124064, Dec 2020.

[17] Jerzy Lewandowski and Tomasz Pawlowski. Quasi-local rotating black holes in higher dimension: geometry. *Classical and Quantum Gravity*, 22(9):1573, apr 2005.

[18] Abhay Ashtekar, Christopher Beetle, and Jerzy Lewandowski. Mechanics of rotating isolated horizons. *Phys. Rev. D*, 64:044016, Jul 2001.

[19] Abhay Ashtekar, Christopher Beetle, and Stephen Fairhurst. Isolated horizons: a generalization of black hole mechanics. *Classical and Quantum Gravity*, 16(2):L1, feb 1999.

[20] Abhay Ashtekar, Stephen Fairhurst, and Badri Krishnan. Isolated horizons: Hamiltonian evolution and the first law. *Physical Review D - Particles, Fields, Gravitation and Cosmology*, 62(10):1–29, November 2000.

[21] Abhay Ashtekar and Badri Krishnan. Isolated and Dynamical Horizons and Their Applications. *Living Reviews in Relativity*, 7(1):10, Dec 2004.

[22] Abhay Ashtekar and Badri Krishnan. Dynamical horizons and their properties. *Phys. Rev. D*, 68:104030, Nov 2003.

[23] D McManus. A toroidal source for the kerr black hole geometry. *Classical and Quantum Gravity*, 8(5):863, may 1991.

[24] S. W. Hawking. Black holes in general relativity. *Communications in Mathematical Physics*, 25(2):152–166, Jun 1972.

[25] Chao guang Huang and Can bin Liang. A torus-like black hole. *Physics Letters A*, 201(1):27–32, 1995.

[26] Denis Dobkowski-Ryłko, Wojciech Kamiński, Jerzy Lewandowski, and Adam Szereszewski. The Petrov type D equation on genus >0 sections of isolated horizons. *Physics Letters B*, 783:415 – 420, 2018.

[27] Mikio Nakahara. *Geometry, topology and physics*. Graduate student series in physics. Hilger, Bristol, 1990.

[28] A. H. Taub. Empty space-times admitting a three parameter group of motions. *Annals of Mathematics*, 53(3):472–490, 1951.

[29] Jerry B. Griffiths and Jiří Podolský. *Exact Space-Times in Einstein's General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2009.

[30] E. Newman, L. Tamburino, and T. Unti. Empty-space generalization of the schwarzschild metric. *Journal of Mathematical Physics*, 4(7):915–923, 1963.

[31] Charles W. Misner. The Flatter Regions of Newman, Unti, and Tamburino's Generalized Schwarzschild Space. *Journal of Mathematical Physics*, 4(7):924–937, 1963.

[32] David A. Feinblum. Global Singularities and the Taub-NUT Metric. *Journal of Mathematical Physics*, 11(9):2713–2721, 10 2003.

[33] G. F. R. Ellis and B. G. Schmidt. Singular space-times. *Gen. Rel. Grav.*, 8:915–953, 1977.

[34] Robert Geroch. Local Characterization of Singularities in General Relativity. *Journal of Mathematical Physics*, 9(3):450–465, 10 2003.

[35] Uri Kol. Dual komar mass, torsion and riemann-cartan manifolds, 2020.

[36] W. B. Bonnor. A new interpretation of the NUT metric in general relativity. *Mathematical Proceedings of the Cambridge Philosophical Society*, 66(1):145–151, 1969.

[37] Remigiusz Durka. The first law of black hole thermodynamics for taub–nut spacetime. *International Journal of Modern Physics D*, 31(04):2250021, 2022.

[38] V S Manko and E Ruiz. Physical interpretation of the nut family of solutions. *Classical and Quantum Gravity*, 22(17):3555, aug 2005.

[39] Michael Appels, Ruth Gregory, and David Kubizňák. Black hole thermodynamics with conical defects. *Journal of High Energy Physics*, 2017(5):116, May 2017.

[40] Ruth Gregory and Andrew Scoins. Accelerating black hole chemistry. *Physics Letters B*, 796:191–195, 2019.

[41] Robie A. Hennigar, David Kubizňák, and Robert B. Mann. Thermodynamics of lorentzian taub-nut spacetimes. *Phys. Rev. D*, 100:064055, Sep 2019.

[42] Alvaro Ballon Bordo, Finnian Gray, Robie A Hennigar, and David Kubizňák. Misner gravitational charges and variable string strengths. *Classical and Quantum Gravity*, 36(19):194001, sep 2019.

[43] J. G. Miller. Global analysis of the Kerr-Taub-NUT metric. *Journal of Mathematical Physics*, 14(4):486–494, 1973.

[44] Gérard Clément, Dmitri Gal'tsov, and Mourad Guenouche. Rehabilitating spacetimes with nuts. *Physics Letters B*, 750:591–594, 2015.

[45] Gérard Clément, Dmitri Gal'tsov, and Mourad Guenouche. Nut wormholes. *Phys. Rev. D*, 93:024048, Jan 2016.

[46] R Debever. On type d expanding solutions of einstein–maxwell equations. *Bull. Soc. Math. Belg.*, 23:360–376, 1971.

[47] J.F Plebański and M Demiański. Rotating, charged, and uniformly accelerating mass in general relativity. *Annals of Physics*, 98(1):98 – 127, 1976.

[48] Jerzy Lewandowski and Maciej Ossowski. Type d equation on non-trivial bundles over riemann surfaces with genus >0 - in preparation, 2023.

[49] J. B. Griffiths and J. Podolský. A new look at the Plebański–Demiański family of solutions. *International Journal of Modern Physics D*, 15(03):335–369, 2006.

[50] Marc Mars and José Senovilla. A Spacetime Characterization of the Kerr-NUT-(A)de Sitter and Related Metrics. *Annales Henri Poincaré*, 16, 07 2013.

[51] N. Dadhich and Z. YA. Turakulov. Gravitational field of a rotating gravitational dyon. *Modern Physics Letters A*, 17(15n17):1091–1096, 2002.

[52] Remigiusz Durka. The first law of black hole thermodynamics for taub-nut spacetime, 2019.

[53] James Isenberg and Vincent Moncrief. Asymptotic behaviour in polarized and half-polarized u(1) symmetric vacuum spacetimes. *Classical and Quantum Gravity*, 19(21):5361, oct 2002.

[54] Y. Choquet-Bruhat and V. Moncrief. Future global in time einsteinian spacetimes with u(1) isometry group. *Annales Henri Poincaré*, 2(6):1007–1064, Dec 2001.

[55] David Kubiznák and Valeri P Frolov. The hidden symmetry of higher dimensional kerr–nut–ads spacetimes. *Classical and Quantum Gravity*, 24(3):F1, jan 2007.

[56] Don N. Page, David Kubizňák, Muraari Vasudevan, and Pavel Krtouš. Complete integrability of geodesic motion in general higher-dimensional rotating black-hole spacetimes. *Phys. Rev. Lett.*, 98:061102, Feb 2007.

[57] Pavel Krtouš, David Kubizňák, Don N. Page, and Muraari Vasudevan. Constants of geodesic motion in higher-dimensional black-hole spacetimes. *Phys. Rev. D*, 76:084034, Oct 2007.

[58] Pavel Krtouš, David Kubizná́k, Don N. Page, and Valeri P. Frolov. Killing-yano tensors, rank-2 killing tensors, and conserved quantities in higher dimensions. *Journal of High Energy Physics*, 2007(02):004, feb 2007.

[59] Valeri P. Frolov, Pavel Krtouš, and David Kubiznák. Separability of hamilton-jacobi and klein-gordon equations in general kerr-nut-ads spacetimes. *Journal of High Energy Physics*, 2007(02):005, feb 2007.

[60] Marco Cariglia, Pavel Krtouš, and David Kubizňák. Dirac equation in kerr-nut-(a)ds spacetimes: Intrinsic characterization of separability in all dimensions. *Phys. Rev. D*, 84:024008, Jul 2011.

[61] Finnian Gray and David Kubiznak. Homogeneous symmetry operators in kerr–nut–ads spacetimes, 2024.

[62] David Kubizňák and Pavel Krtouš. Conformal killing-yano tensors for the plebański-demiański family of solutions. *Phys. Rev. D*, 76:084036, Oct 2007.

[63] Abhay Ashtekar, Béatrice Bonga, and Aruna Kesavan. Asymptotics with a positive cosmological constant: I. basic framework. *Classical and Quantum Gravity*, 32(2):025004, dec 2014.

[64] Roy P. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, 11:237–238, Sep 1963.

[65] Kenneth Hong and Edward Teo. A new form of the c-metric. *Classical and Quantum Gravity*, 20(14):3269, jul 2003.

[66] J B Griffiths, P Krtouš, and J Podolský. Interpreting the c-metric. *Classical and Quantum Gravity*, 23(23):6745, oct 2006.

[67] David Kofroň. On the nature of cosmic strings in black hole spacetimes. *General Relativity and Gravitation*, 52(9):91, Sep 2020.

[68] Ji ří Podolský and Adam Vrátný. New improved form of black holes of type d. *Phys. Rev. D*, 104:084078, Oct 2021.

[69] Ji ří Podolský and Adam Vrátný. New form of all black holes of type d with a cosmological constant. *Phys. Rev. D*, 107:084034, Apr 2023.

[70] J B Griffiths and J Podolský. Accelerating and rotating black holes. *Classical and Quantum Gravity*, 22(17):3467, aug 2005.

[71] Ji ří Podolský and Adam Vrátný. Accelerating nut black holes. *Phys. Rev. D*, 102:084024, Oct 2020.

[72] Marco Astorino. Equivalence principle and generalized accelerating black holes from binary systems. *Phys. Rev. D*, 109:084038, Apr 2024.

[73] Marco Astorino. Most general type-d black hole and accelerating reissner-nordstrom-nut-(a)ds, 2024.

[74] Ivan Kolář and Pavel Krtouš. Symmetry axes of kerr-nut-(a)ds spacetimes. *Phys. Rev. D*, 100:064014, Sep 2019.

[75] Ivan Kolář and Pavel Krtouš. Nut-like and near-horizon limits of kerr-nut-(a)ds spacetimes. *Phys. Rev. D*, 95:124044, Jun 2017.

[76] R.C Myers and M.J Perry. Black holes in higher dimensional space-times. *Annals of Physics*, 172(2):304–347, 1986.

[77] Robert Mann and Cristian Stelea. New multiply nutty spacetimes. *Physics Letters B*, 634(5):448–455, 2006.

[78] Andrés Anabalón, Sebastian F. Bramberger, and Jean-Luc Lehners. Kerr-nut-de sitter as an inhomogeneous non-singular bouncing cosmology. *Journal of High Energy Physics*, 2019(9):96, 2019.

[79] Jiří Podolský and Adam Vrátný. Accelerating nut black holes. *Phys. Rev. D*, 102:084024, Oct 2020.

[80] Vincent Moncrief and James Isenberg. Symmetries of cosmological cauchy horizons. *Communications in Mathematical Physics*, 89(3):387–413, Sep 1983.

[81] James Isenberg and Vincent Moncrief. Symmetries of cosmological Cauchy horizons with exceptional orbits. *Journal of Mathematical Physics*, 26(5):1024–1027, 05 1985.

[82] Vincent Moncrief and James Isenberg. Symmetries of cosmological cauchy horizons with non-closed orbits. *Communications in Mathematical Physics*, 374(1):145–186, Feb 2020.

[83] Vincent Moncrief. Neighborhoods of cauchy horizons in cosmological spacetimes with one killing field. *Annals of Physics*, 141(1):83–103, 1982.

[84] Vincent Moncrief. The space of (generalized) taub-nut spacetimes. *Journal of Geometry and Physics*, 1(1):107–130, 1984.

[85] Piotr T. Chruściel. Geometry of Black Holes. Oxford University Press, 08 2020.

[86] R. Descartes and P.J. Olscamp. *Discourse on Method, Optics, Geometry, and Meteorology*. Hackett Classics Series. Hackett Pub., 2001.

[87] Jerzy Lewandowski and Tomasz Pawlowski. Extremal isolated horizons: a local uniqueness theorem. *Classical and Quantum Gravity*, 20(4):587, jan 2003.

[88] Jerzy Lewandowski and Maciej Ossowski. Non-singular Kerr-NUT-de Sitter spacetimes. *Classical and Quantum Gravity*, 37(20):205007, sep 2020.

[89] Denis Dobkowski-Ryłko, Jerzy Lewandowski, and Tomasz Pawłowski. Local version of the no-hair theorem. *Phys. Rev. D*, 98:024008, Jul 2018.

[90] Denis Dobkowski-Ryłko, Jerzy Lewandowski, and István Rácz. Petrov type D equation on horizons of nontrivial bundle topology. *Phys. Rev. D*, 100:084058, Oct 2019.

[91] Denis Dobkowski-Ryłko, Jerzy Lewandowski, and Maciej Ossowski. Isolated horizons of the hopf bundle structure transversal to the null direction, the horizon equations and embeddability in nut-like spacetimes, 2023.

[92] J Podolský and J B Griffiths. The collision and snapping of cosmic strings generating spherical impulsive gravitational waves. *Classical and Quantum Gravity*, 17(6):1401, mar 2000.

[93] István Rácz. Stationary black holes as holographs. *Classical and Quantum Gravity*, 24(22):5541–5571, oct 2007.

[94] S A Hayward. Snapping of cosmic strings. *Classical and Quantum Gravity*, 6(9):L179, sep 1989.

[95] Alvaro Ballon Bordo, Finnian Gray, and David Kubizňák. Thermodynamics of rotating nutty dyons. *Journal of High Energy Physics*, 2020(5):84, May 2020.

[96] Carmen Li and James Lucietti. Transverse deformations of extreme horizons. *Classical and Quantum Gravity*, 33(7):075015, mar 2016.

[97] M J Cole, I Rácz, and J A Valiente Kroon. Killing spinor data on distorted black hole horizons and the uniqueness of stationary vacuum black holes. *Classical and Quantum Gravity*, 35(20):205001, sep 2018.

[98] Stephen W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time (Cambridge Monographs on Mathematical Physics)*. Cambridge University Press, 1975.

[99] Schiden Yohannes and Domenico Giulini. A group-theoretic characterisation of taub-nut spacetime, 2021.

[100] D. Klemm, V. Moretti, and L. Vanzo. Rotating topological black holes. *Phys. Rev. D*, 57:6127–6137, May 1998.

[101] R. B. Mann. Topological black holes – outside looking in, 1997.

[102] Natxo Alonso-Alberca, Patrick Meessen, and Tomás Ortín. Supersymmetry of topological kerr-newman-taub- nut-ads spacetimes. *Classical and Quantum Gravity*, 17(14):2783, jul 2000.

[103] Ralph L.Cohen. Lecture notes in bundles, homotopy, and manifolds, February 2023.