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Classical and quantum Kneser graphs

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Abstract

We present a construction of quantum Kneser graphs, which yields classical Kneser graphs in the commutative case. We also generalize this construction to obtain quantum versions of generalized Kneser graphs, such as Johnson graphs. We discuss several basic properties of the constructed quantum graphs, such as their reflexivity and regularity. Then, we investigate their quantum chromatic numbers and establish upper bounds on them by constructing explicit colorings. We also investigate spectral lower bounds on quantum chromatic numbers for several examples and formulate a conjecture about their asymptotic behavior. Lastly, we present an alternate approach to quantization of Kneser graphs and a few results obtained for the quantum Mycielski construction.

Keywords

Quantum graphs, Kneser graphs, coloring, chromatic number

Title of the thesis in Polish language

Klasyczne i kwantowe grafy Knesera

Contents

1. Introduction and basic definitions

1.1. **Non-commutative geometry.** Non-commutative geometry is a branch of mathematics that studies non-commutative, "quantum" generalizations of certain objects known from "classical" mathematics, such as sets, topological spaces, groups and graphs. It is formulated primarily in the language of non-commutative algebras (hence the name), for example C*-algebras.

Definition 1.1. An algebra A is a C*-algebra if it is a Banach $*$-algebra (that is, an algebra over $\mathbb{C}$ which is also a Banach space, equipped with an antilinear, antimultiplicative involution, denoted by $*$) in which $\|x^*x\| = \|x\|^2$ for every $x \in \mathsf{A}$.

Examples of C*-algebras which are particularly important in the study of finite quantum graphs are matrix algebras (equipped with the usual Hermitian adjoint as the involution), as well as their direct sums. Indeed, all finite-dimensional C*-algebras are isomorphic to (finite) direct sums of matrix algebras.

Non-commutative geometry originated in the papers of Woronowicz [22, 23, 24], who introduced the notion of compact quantum groups. Later, Wang [18] and Banica [1, 2] interpreted finite-dimensional C*-algebras as quantum sets (also known as finite quantum spaces) and introduced their quantum automorphism groups. The basic idea behind this field of research is inspired by the Gelfand–Naimark theorem.

Theorem 1.2 (Gelfand–Naimark)**.** *The category of commutative, unital* C*-*algebras is anti-equivalent to the category of compact Hausdorff topological spaces.*

Phrased differently, this theorem states that every commutative and unital C*-algebra is isomorphic to the C*-algebra of complex-valued, continuous functions on a certain compact topological space. Additionally, every $*$-homomorphism between commutative C*-algebras comes from a continuous function between their corresponding topological spaces (more precisely, it is given by the pullback with respect to such a function). It allows us to translate topological properties of a given space X into algebraic properties of its function algebra $\mathrm{C}(X)$.

Non-commutative geometry expands this idea to non-commutative C*-algebras, which are treated as algebras of complex-valued, continuous functions on some quantum spaces. Because of this, we often denote C*-algebras as $\mathrm{C}(\mathbb{X})$, where $\mathbb{X}$ is a quantum space. These quantum spaces are formally defined as objects dual (in the category-theoretic sense) to their corresponding C*-algebras. Such non-classical objects often have interesting topological and geometric properties. Here, we focus primarily on quantum graphs, but it may be illustrative to first introduce, as an example, the notion of compact quantum groups as defined by Woronowicz.

Definition 1.3. A compact quantum group $\mathbb{G}$ is described by a unital C*-algebra $\mathrm{C}(\mathbb{G})$ equipped with a $*$-homomorphism $\Delta : \mathrm{C}(\mathbb{G}) \to \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G})$ (knows as a comultiplication) satisfying the following 2 properties:

$$(\mathrm{id} \otimes \Delta)\,\Delta = (\Delta \otimes \mathrm{id})\,\Delta, \tag{1.1}$$

$$\overline{\Delta(\mathrm{C}(\mathbb{G}))\,(\mathbb{1} \otimes \mathrm{C}(\mathbb{G}))} = \mathrm{C}(\mathbb{G}) \otimes \mathrm{C}(\mathbb{G}) = \overline{\Delta(\mathrm{C}(\mathbb{G}))\,(\mathrm{C}(\mathbb{G}) \otimes \mathbb{1})}. \tag{1.2}$$

Objects defined in this way are noncommutative generalizations of classical groups in the spirit of the Gelfand-Naimark theorem. To better conceptualize them as groups, we note that if $\mathbb{G} = G$ is a classical group and $\mathrm{C}(G)$ is the C*-algebra of complex-valued functions on G, then the comultiplication is given by

$$\Delta(f)(x, y) = f(xy), \tag{1.3}$$

where $x, y \in G$ and $f \in \mathrm{C}(G)$. It is then easy to verify that the first condition in the definition corresponds to the associativity of the group operation ($\forall x, y, z \in G : (xy)z = x(yz)$). The second condition, on the other hand, is equivalent to the following 2 cancellation conditions:

$$xy_1 = xy_2 \Rightarrow y_1 = y_2, \tag{1.4}$$

$$x_1 y = x_2 y \Rightarrow x_1 = x_2. \tag{1.5}$$

It can be shown [21, 15] that any compact space equipped with an associative binary operation satisfying the cancellation properties is a group, which justifies the name "compact quantum group."

1.2. **GNS construction.** Before we introduce quantum graphs, it is useful to first describe the Gelfand–Naimark–Segal construction, originally introduced by Gelfand and Naimark [12]. The construction makes it possible to represent any C*-algebra equipped with a state (or, slightly more generally, a positive linear functional) as an operator algebra acting on some Hilbert space. This is done in the following way.

Let A be a C*-algebra with a positive functional $\psi : \mathsf{A} \to \mathbb{C}$ (which means that for any $x \in \mathsf{A}$ we have $\psi(x^* x) \geq 0$). For simplicity, we also assume that A is unital (with a unit element $\mathbb{1}$), but it is not strictly necessary for the construction. We define a sesquilinear form on A given by

$$\langle x | y \rangle = \psi(x^* y). \tag{1.6}$$

Consider the set $I = \{x \in \mathsf{A} | \psi(x^* x) = 0\}$. It can be shown that I is a vector subspace of A. In fact, by applying the Cauchy–Schwarz inequality, we can show that I is a left ideal in A. This allows us to define an inner product on A/I by the formula

$$\langle x + I | y + I \rangle = \psi(x^* y) \tag{1.7}$$

and define the Hilbert space $\mathscr{H}$ as the Cauchy completion of A/I with respect to the norm induced by this inner product.

We now define a representation π of A on A/I by the formula

$$\pi(x)(y + I) = xy + I. \tag{1.8}$$

It can be shown that $\pi(x)$ is a bounded operator for every $x \in \mathsf{A}$, therefore it can be uniquely extended to $\mathscr{H}$. Additionally, we can show that $\pi(x^*) = \pi(x)^\dagger$, so π is a $*$-representation.

Lastly, it can be shown that $\xi = \mathbb{1} + I$ is a cyclic vector for π (or, in other words, $\pi(\mathsf{A})\xi$ is dense in $\mathscr{H}$) and that $\psi(x) = \langle \xi | \pi(x) \xi \rangle$, so, in a sense, the representation π contains information about the positive functional ψ.

It is also worth noting that, if ψ is faithful (that is, $\psi(x^* x) = 0$ implies $x = 0$), then I is trivial and $\mathscr{H}$ is simply the completion of A with respect to the norm induced by ψ.

1.3. **Quantum graphs.** Intuitively, a classical graph is a collection of vertices, which are connected by edges. This informal definition can be formalized in the following way.

Definition 1.4. A classical graph G is an ordered pair (V, E), where V is a set and $E \subset V \times V$.

In this definition, elements of V are identified with vertices, while E is a relation on V which determines whether any given pair of vertices is connected by an edge. We will only consider finite graphs, so we may, without loss of generality, assume that $V = \overline{1, n} = \{1, 2, \ldots, n\}$. We will also write $i \sim j$ if $(i, j) \in E$.

For a given graph G, we define the adjacency matrix as

$$A_{i,j} = \begin{cases} 1, & \text{if } i \sim j \\ 0, & \text{otherwise} \end{cases}. \tag{1.9}$$

Such a matrix contains all information necessary to specify a graph. We can treat this matrix as an operator on the C*-algebra of complex-valued functions on vertices of G, which we shall denote by $\mathrm{C}(G)$. This operator satisfies the following properties:

- all components of A are equal to 0 or 1, so it is idempotent with respect to the Schur (component-wise) product: $A \odot A = A$,
- all components of A are real.

Quantum graphs, first introduced by Weaver [19, 20], generalize this concept to a quantum setting by replacing the classical set of vertices with a quantum set described by some noncommutative algebra and defining quantum graphs in the following way.

Definition 1.5. A quantum graph on a von Neumann algebra $\mathcal{M} \subset \mathrm{B}(\mathscr{H})$ is a weak*-closed subspace $\mathcal{V} \subset \mathrm{B}(\mathscr{H})$ which is a bimodule over the commutant $\mathcal{M}'$ (that is, $\mathcal{M}'\mathcal{V}\mathcal{M}' \subset \mathcal{V}$) and satisfies the following 2 properties:

- $\mathcal{M}' \subset \mathcal{V}$,
- $\mathcal{V}^* = \mathcal{V}$.

This definition relies on Weaver's notion of quantum relations. In particular, the 2 conditions listed in the definition correspond to the relation being reflexive and symmetric, just like the adjacency relation defined on the set of a graph's vertices (as Weaver focused on reflexive graphs, which have loops at each vertex). The reflexivity condition, which can be expressed in simpler terms as $\mathbb{1} \in \mathcal{V}$, is often omitted to study more general quantum graphs and their properties. We shall also make that choice, since we want to study quantum colorings, which, just like classical colorings, do not exist for reflexive graphs.

In this thesis, we mostly utilize a different definition of quantum graphs, which instead relies on generalizing the adjacency matrix. Though these 2 approaches seem very different at first, they are actually equivalent to each other, as discussed by Gromada [13] and Daws [8].

Definition 1.6. A quantum graph is a triple $\mathcal{G} = (\mathrm{C}(\mathcal{G}), \psi, A)$, where $\mathrm{C}(\mathcal{G})$ is a finite-dimensional, unital C*-algebra, while $\psi : \mathrm{C}(\mathcal{G}) \to \mathbb{C}$ and $A : \mathrm{C}(\mathcal{G}) \to \mathrm{C}(\mathcal{G})$ are linear maps satisfying the following properties (where by $L^2(\mathcal{G})$ we denote the GNS Hilbert space resulting by the GNS construction applied to the state ψ):

- ψ is a δ-form – a real, positive and faithful functional, for which there exists a real number δ such that $mm^\dagger = \delta^2 \mathrm{id}$, where $m : L^2(\mathcal{G}) \otimes L^2(\mathcal{G}) \ni x \otimes y \mapsto xy \in L^2(\mathcal{G})$ and id is the identity map on $L^2(\mathcal{G})$,
- $A(x^*) = (A(x))^*$ for every $x \in \mathrm{C}(\mathcal{G})$,
- $m(A \otimes A) m^\dagger = \delta^2 A$.

To understand why quantum graphs are defined in such a way, it is useful to examine what happens when $\mathrm{C}(\mathcal{G})$ is commutative. In such a case, the first of those properties — ψ being a δ-form — boils down to ψ being given by integration with respect to a uniform measure. Similarly, the second and third properties correspond to the adjacency matrix having real entries and being an idempotent of the Schur product, which implies that its entries must all be equal to either 0 or 1.

In the quantum adjacency matrix formulation, a quantum graph is said to be reflexive if $m(A \otimes \mathrm{id}) m^\dagger = \delta^2 \mathrm{id}$ and irreflexive if $m(A \otimes \mathrm{id}) m^\dagger = 0$. In this thesis, we primarily focus on irreflexive graphs.

To obtain a less abstract characterization of δ-forms on finite-dimensional C*-algebras, we note that any such algebra A can be decomposed as a direct sum of matrix algebras:

$$\mathsf{A} = \bigoplus_{a=1}^{n} \mathsf{Mat}_{d_a}(\mathbb{C}). \tag{1.10}$$

This means that any element $x \in \mathsf{A}$ can be uniquely written as a formal sum $x = \sum_{a=1}^{n} x_a$, where $x_a \in \mathsf{Mat}_{d_a}(\mathbb{C})$. Any functional $\psi : \mathsf{A} \to \mathbb{C}$ can therefore be expressed in the form

$$\psi(x) = \sum_{a=1}^{n} \mathrm{Tr}(\rho_a x_a), \tag{1.11}$$

where $\rho_a \in \mathsf{Mat}_{d_a}(\mathbb{C})$. In such a case, it can be shown that ψ is a δ-form if and only if every ρ_a is positive and invertible, and satisfies $\mathrm{Tr}\left(\rho_a^{-1}\right) = \delta^2$. This also means that there is a unique (up to scaling) tracial δ-form on a given C*-algebra A, given by

$$\psi\left(x\right)=\frac{1}{\delta^{2}}\sum_{a=1}^{n}d_{a}\operatorname{Tr}\left(x_{a}\right). \tag{1.12}$$

Another way to see δ-forms arise naturally is to analyze the GNS representation of the C*-algebra A — in other words, represent each element of A as the operator of left multiplication by that element:

$$\pi\left(x\right):\mathsf{A}\ni y\mapsto xy\in\mathsf{A}. \tag{1.13}$$

Then each $\pi\left(x\right)$ is a linear operator on A (treated as a vector space), so it makes sense to calculate its trace. To do that, we introduce a basis on A consisting of matrix units on each block, denoted by $e^{a}_{i,j}\in\mathsf{Mat}_{d_a}\left(\mathbb{C}\right)$, as well as its dual basis, denoted by $\phi^{a}_{i,j}$, such that $\phi^{a}_{i,j}\left(e^{b}_{k,l}\right)=\delta_{a,b}\delta_{i,k}\delta_{j,l}$ (for $a,b\in\overline{1,n}$, $i,j\in\overline{1,d_a}$ and $k,l\in\overline{1,d_b}$). We can then write

$$\pi\left(e^{a}_{i,j}\right)y=\sum_{b=1}^{n}\sum_{k,l=1}^{d_a}\phi^{b}_{k,l}\left(y\right)e^{a}_{i,j}e^{b}_{k,l}=\sum_{b=1}^{n}\sum_{k,l=1}^{d_a}\phi^{b}_{k,l}\left(y\right)\delta_{a,b}\delta_{j,k}e^{a}_{i,l}=\sum_{l=1}^{d_a}\phi^{a}_{j,l}\left(y\right)e^{a}_{i,l}, \tag{1.14}$$

which means that

$$\operatorname{Tr}\left(\pi\left(e^{a}_{i,j}\right)\right)=\sum_{l=1}^{d_a}\phi^{a}_{j,l}\left(e^{a}_{i,l}\right)=\sum_{l=1}^{d_a}\delta_{i,j}=d_a\delta_{i,j}. \tag{1.15}$$

As a result, we obtain

$$\operatorname{Tr}\left(\pi\left(x\right)\right)=\sum_{a=1}^{n}d_{a}\operatorname{Tr}\left(x_{a}\right), \tag{1.16}$$

where the traces on the right-hand side of the equation are the usual matrix traces on $\mathsf{Mat}_{d_a}\left(\mathbb{C}\right)$. This means that the trace on A represented as a subspace of $\mathrm{B}\left(\mathsf{A}\right)$ (via left multiplication) is a δ-form for $\delta=1$. As a result, we may say that a tracial δ-form on A is the most natural, canonical trace.

It is worth noting that there are multiple equivalent conventions related to normalization of ψ, used by different authors. Sometimes, it is assumed that $\psi\left(\mathbb{1}\right)=1$ — that is, ψ is a state. Another popular convention involves normalizing ψ in such a way that $\delta=1$. We choose to follow neither of those conventions (unless specified otherwise), instead allowing the reader to choose the one preferable to them.

Additionally, we note that it is possible to consider quantum graphs with a functional that is not a δ-form, as mentioned by Daws [8]. However, as pointed out by Banica [1, 2] (who also introduced the notion of δ-forms), tracial δ-forms are natural traces on C*-algebras and massively simplify many issues related to quantum groups' actions on quantum spaces, particularly those groups' representations. For this reason, it is common to assume that ψ is a δ-form, even though it is not, strictly speaking, a necessary condition for quantum graphs.

1.3.1. *Connection between different definitions.* We now examine in more detail the connection between Definitions 1.5 and 1.6, and provide a method of translating between them. For simplicity of calculations, we assume that ψ is a tracial δ-form on $\mathrm{C}\left(\mathcal{G}\right)$ with $\delta=1$.

Firstly, we notice that the subspace $\mathcal{V}\subset\mathrm{B}\left(\mathscr{H}\right)$ can be uniquely characterized by the orthogonal projection onto it, called the edge projection and denoted by $P\in\mathrm{B}\left(\mathrm{B}\left(\mathscr{H}\right)\right)$. Since $\mathcal{V}$ is a bimodule over $\mathcal{M}'$, this projection is a bimodule map and, by a theorem given by Smith [16] (though originally proved by Haagerup in an unpublished paper), can be identified with an element $\sum_{i=1}^{r}p_i\otimes q_i\in\mathcal{M}\otimes\mathcal{M}^{\mathrm{op}}$ (where $\mathcal{M}^{\mathrm{op}}$ is the opposite algebra of $\mathcal{M}$, which is equal to $\mathcal{M}$ as a vector space, but multiplication is defined in the opposite way) in the following way:

$$P\left(x\right)=\sum_{i=1}^{r}p_{i}xq_{i}. \tag{1.17}$$

Then, we can identify $\mathcal{M}$ with $\mathrm{C}\left(\mathcal{G}\right)$, define the adjacency operator $A\in\mathrm{B}\left(\mathrm{C}\left(\mathcal{G}\right)\right)$ by the formula

$$A\left(x\right)=\sum_{i=1}^{r}\psi\left(xq_{i}\right)p_{i} \tag{1.18}$$

and check that it satisfies all the required conditions. To do that, we consider (for simplicity) a quantum graph on $\mathcal{M} = \mathrm{B}(\mathscr{H})$, where $\mathscr{H}$ is d-dimensional, defined by a subspace $\mathcal{V}$ and introduce an orthogonal basis $\{f_i\}_{i=1}^{p}$ of $\mathcal{V}$ (with respect to the standard Hilbert–Schmidt inner product on $\mathrm{B}(\mathscr{H})$). Then, we have

$$P(x) = \sum_{i=1}^{p} \langle f_i|x\rangle f_i = \sum_{i=1}^{p} \operatorname{Tr}(f_i^* x) f_i = \sum_{i=1}^{p} \sum_{j,k=1}^{d} e_{j,k} f_i^* x e_{k,j} f_i, \tag{1.19}$$

where $e_{j,k}$ are matrix units in $\mathrm{B}(\mathscr{H})$. This means that

$$\begin{aligned} A(x) = \sum_{i=1}^{p} \sum_{j,k=1}^{d} \psi(x e_{k,j} f_i) e_{j,k} f_i^* &= d \sum_{i=1}^{p} \sum_{j,k=1}^{d} \operatorname{Tr}(x e_{k,j} f_i) e_{j,k} f_i^* \\ &= d \sum_{i=1}^{p} \sum_{j,k=1}^{d} \operatorname{Tr}\left(e_{j,k}^* f_i x\right) e_{j,k} f_i^* = d \sum_{i=1}^{p} f_i x f_i^*. \end{aligned} \tag{1.20}$$

As a result, we see that

$$A(x)^* = d \sum_{i=1}^{p} (f_i x f_i^*)^* = d \sum_{i=1}^{p} f_i x^* f_i^* = A(x^*) \tag{1.21}$$

and

$$\begin{aligned} m(A \otimes A) m^\dagger(e_{i,j}) &= \frac{1}{d} m(A \otimes A) \left(\sum_{k=1}^{d} e_{i,k} \otimes e_{k,j} \right) = \frac{1}{d} \sum_{k=1}^{d} A(e_{i,k}) A(e_{k,j}) \\ &= d \sum_{k=1}^{d} \sum_{l,l'=1}^{p} f_l e_{i,k} f_l^* f_{l'} e_{k,j} f_{l'}^* = d \sum_{l,l'=1}^{p} f_l \operatorname{Tr}(f_l^* f_{l'}) e_{i,j} f_{l'}^* = d \sum_{l,l'=1}^{p} f_l \delta_{l,l'} e_{i,j} f_{l'}^* = d \sum_{l=1}^{p} f_l e_{i,j} f_l^*, \end{aligned} \tag{1.22}$$

so A satisfies the definition of a quantum adjacency operator.

Conversely, if we start with a quantum adjacency operator A, we can obtain from it the edge projection (identified with an element of $\mathcal{M} \otimes \mathcal{M}^{\mathrm{op}}$) by utilizing the following equation:

$$P = (A \otimes \mathrm{id}) m^\dagger(\mathbb{1}), \tag{1.23}$$

where id represents an identity map from $\mathcal{M}$ to $\mathcal{M}^{\mathrm{op}}$ (which makes sense due to the fact that these two algebras have the same underlying vector space) and m is the multiplication map (like in Definition 1.6). Indeed, using the same conventions as earlier, we obtain

$$P(x) = \frac{1}{d} \sum_{i,j=1}^{d} A(e_{i,j}) x e_{j,i} = \sum_{i,j=1}^{d} \sum_{k=1}^{p} f_k e_{i,j} f_k^* x e_{j,i} = \sum_{k=1}^{p} f_k \operatorname{Tr}(f_k^* x) = \sum_{k=1}^{p} \langle f_k|x\rangle f_k, \tag{1.24}$$

so P is an orthogonal projection onto a subspace $\mathcal{V}$ spanned by the vectors f_k.

In the end, we may see that the various definitions of quantum graphs are equivalent.

1.3.2. *Physical motivation.* To give a more physical motivation for studying quantum graphs, aside from their interesting mathematical properties, we shall now present the notion of confusability graphs of classical communication channels and their generalization to quantum channels.

Definition 1.7. A classical channel F from an input set X to an output set Y is characterized by a collection of real numbers $(p(y|x))_{x \in X, y \in Y}$, satisfying the following properties:

- $\forall x \in X, y \in Y : 0 \leq p(y|x) \leq 1$,
- $\forall x \in X : \sum_{y \in Y} p(y|x) = 1$.

Physically, $p(y|x)$ describes the probability of receiving y as the output after sending x through the channel as the input.

We say that 2 inputs, x and x', are confusable if the channel has a non-zero probability of sending them both to the same output, or, in other words, if $\exists y \in Y : p(y|x) p(y|x') > 0$. This can be encoded using the notion of a confusability graph.

Definition 1.8. The confusability graph of a classical channel F is the graph whose vertices are all possible inputs of F, such that $x \sim x'$ whenever $\exists y \in Y : p(y|x)\, p(y|x') > 0$.

The notion of confusability graphs can be extended to quantum channels, understood as completely positive, trace-preserving, linear maps between matrix algebras. Any such quantum channel can be expressed as

$$\Phi : \mathsf{Mat}_n(\mathbb{C}) \ni \rho \mapsto \sum_{i=1}^{r} K_i \rho K_i^\dagger \in \mathsf{Mat}_m(\mathbb{C}), \tag{1.25}$$

where K_i are Kraus operators, given by matrices with m rows and n columns, satisfying the condition $\sum_{i=1}^{r} K_i^\dagger K_i = \mathbb{1}$.

It is sensible to say that quantum states given by density matrices ρ and σ are distinguishable if $\operatorname{Tr}(\rho\sigma) = 0$. Otherwise, they are confusable.

Let us investigate when $\Phi(\rho)$ and $\Phi(\sigma)$ are distinguishable. Since density matrices are always diagonalizable, we may express them as $\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$ and $\sigma = \sum_{i=1}^{n} q_i |\phi_i\rangle\langle\phi_i|$, where $\forall i \in \overline{1,n} : 0 \le p_i, q_i \le 1$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$. Then we can write

$$\begin{aligned}\operatorname{Tr}(\Phi(\rho)\,\Phi(\sigma)) &= \sum_{i,j=1}^{r} \sum_{k,l=1}^{n} p_k q_l \operatorname{Tr}\left(K_i|\psi_k\rangle\langle\psi_k|K_i^\dagger K_j|\phi_l\rangle\langle\phi_l|K_j^\dagger\right) \\ &= \sum_{i,j=1}^{r} \sum_{k,l=1}^{n} p_k q_l \operatorname{Tr}\left(\langle\phi_l|K_j^\dagger K_i|\psi_k\rangle\langle\psi_k|K_i^\dagger K_j|\phi_l\rangle\right) = \sum_{i,j=1}^{r} \sum_{k,l=1}^{n} p_k q_l \left|\left\langle \psi_k \middle| K_i^\dagger K_j \phi_l \right\rangle\right|^2. \end{aligned} \tag{1.26}$$

Denoting $\mathcal{V} = \operatorname{span}\left\{K_i^\dagger K_j \middle| i,j \in \overline{1,r}\right\}$, we can now see that the condition $\operatorname{Tr}(\Phi(\rho)\,\Phi(\sigma)) = 0$ is equivalent to the claim that

$$\forall \psi \in \operatorname{ran}\rho, \phi \in \operatorname{ran}\sigma : \langle\psi|\mathcal{V}\phi\rangle = \{0\}. \tag{1.27}$$

In addition to being a subspace of $\mathsf{Mat}_n(\mathbb{C})$, $\mathcal{V}$ is also an operator system — it contains $\mathbb{1} = \sum_{i=1}^{r} K_i^\dagger K_i$ and is closed with respect to adjunction. This means that $\mathcal{V}$ is a quantum graph according to the definition used by Weaver. Therefore, quantum graphs can encode the notions of distinguishability and confusability of states by quantum communication channels.

1.3.3. *Quantum chromatic numbers.* For classical graphs, a chromatic number is defined as the minimal number of colors needed to color the vertices of a given graph in such a way that no two adjacent vertices have the same color. This notion can be generalized to quantum graphs and colorings in the following way.

Definition 1.9. A quantum c-coloring of a quantum graph $\mathcal{G} = (\mathrm{C}(\mathcal{G}), \psi, A)$ is a collection $\{P_i\}_{i=1}^{c}$ of self-adjoint projections in $\mathrm{C}(\mathcal{G}) \otimes \mathrm{B}(\mathcal{K})$, where $\mathcal{K}$ is some Hilbert space, which satisfies the following conditions:

- $\sum_{i=1}^{c} P_i = \mathbb{1} \otimes \mathbb{1}$,
- $\forall i \in \overline{1,c} : P_i (A \otimes \mathrm{id})(P_i) = 0$.

The quantum chromatic number of $\mathcal{G}$ is the minimal c for which a quantum c-coloring exists.

This definition of a quantum coloring differs from the original one, formulated in Weaver's operator system formalism. This original definition, obtained first in [7], is formulated as follows.

Definition 1.10. A quantum c-coloring of a quantum graph $\mathcal{G}$ on $\mathcal{M} \subset \mathrm{B}(\mathscr{H})$ defined by an operator system $\mathcal{V} \subset \mathrm{B}(\mathscr{H})$ is a collection $\{P_i\}_{i=1}^{c}$ of self-adjoint projections in $\mathcal{M} \otimes \mathrm{B}(\mathcal{K})$, where $\mathcal{K}$ is some Hilbert space, which satisfies the following conditions:

- $\sum_{i=1}^{c} P_i = \mathbb{1} \otimes \mathbb{1}$,
- $\forall i \in \overline{1,c} : P_i (\mathcal{V} \otimes \mathbb{1}) P_i = \{0\}$.

These two definitions, however, are equivalent, as proved by the following lemma.

Lemma 1.11. *Let $\mathcal{G} = (\mathrm{C}(\mathcal{G}), \psi, A)$, where $\mathrm{C}(\mathcal{G}) = \bigoplus_{a=1}^{n} \mathsf{Mat}_{d_a}(\mathbb{C})$ and ψ is tracial, be a quantum graph, and let $\mathcal{V} \subset \mathsf{Mat}_{d_1+d_2+\ldots+d_n}(\mathbb{C})$ be its corresponding operator system. Let $\mathcal{K}$ be a Hilbert space and let $P \in \mathrm{C}(\mathcal{G}) \otimes \mathrm{B}(\mathcal{K})$ be a self-adjoint projection. Then the following conditions are equivalent:*

(1) $P\left(\mathcal{V}\otimes\mathbb{1}\right)P=\{0\}$,
(2) $P\left(A\otimes\mathrm{id}\right)(P)=0$.

Proof. Thanks to Equation (1.20), we know that there exists a basis of $\mathcal{V}$, $\{v_i\}_{i=1}^p$, for which $A(x)=\sum_{i=1}^p v_i x v_i^*$.

The implication (1) $\implies$ (2) is easy to prove. If $P\left(\mathcal{V}\otimes\mathbb{1}\right)P=\{0\}$, then in particular $\forall i: P\left(v_i\otimes\mathbb{1}\right)P=0$, so

$$P\left(A\otimes\mathrm{id}\right)(P)=\sum_{i=1}^p P\left(v_i\otimes\mathbb{1}\right)P\left(v_i^*\otimes\mathbb{1}\right)=0. \tag{1.28}$$

The implication (2) $\implies$ (1) is slightly more difficult to prove. By multiplying $P\left(A\otimes\mathrm{id}\right)(P)$ by P from the right side, we obtain

$$P\left(A\otimes\mathrm{id}\right)(P)P=\sum_{i=1}^p P\left(v_i\otimes\mathbb{1}\right)P\left(v_i^*\otimes\mathbb{1}\right)P=\sum_{i=1}^p \left(P\left(v_i\otimes\mathbb{1}\right)P\right)\left(P\left(v_i\otimes\mathbb{1}\right)P\right)^*, \tag{1.29}$$

which is a sum of non-negative matrices. Therefore, if $P\left(A\otimes\mathrm{id}\right)(P)=0$, then $P\left(v_i\otimes\mathbb{1}\right)P=0$ for every i. Since $\{v_i\}_{i=1}^n$ is a basis of $\mathcal{V}\otimes\mathbb{1}$, this implies that $P\left(\mathcal{V}\otimes\mathbb{1}\right)P=0$. ☐

Since finding quantum colorings and calculating chromatic numbers of graphs is often difficult, it is useful to find bounds on the value of chromatic numbers. Many such bounds, based on the spectral properties of the quantum adjacency operator, are presented by Ganesan [11]. We shall present here a proof of one of those bounds — Hoffman's bound — which we utilize later. The proof begins with the following lemma.

Lemma 1.12. *Let A be a self-adjoint matrix, partitioned into blocks as*

$$A=\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n}\\ A_{2,1} & A_{2,2} & \cdots & A_{2,n}\\ \vdots & \vdots & \ddots & \vdots\\ A_{n,1} & A_{n,2} & \cdots & A_{n,n}\end{bmatrix}, \tag{1.30}$$

and let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ be functions that map matrices to their smallest and largest eigenvalue, respectively. Then

$$(n-1)\,\lambda_{\min}(A)+\lambda_{\max}(A)\leq\sum_{i=1}^n\lambda_{\max}\left(A_{i,i}\right). \tag{1.31}$$

Proof. Firstly, we consider the case of $n=2$. Let $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ be a normalized eigenvetor of A (so $\|x_1\|^2+\|x_2\|^2=1$) corresponding to its biggest eigenvalue and define $y=\begin{bmatrix}\frac{\|x_2\|}{\|x_1\|}x_1\\-\frac{\|x_1\|}{\|x_2\|}x_2\end{bmatrix}$. Then a direct calculation shows that

$$\lambda_{\max}(A)+\lambda_{\min}(A)\leq\langle x|Ax\rangle+\langle y|Ay\rangle=\frac{\langle x_1|A_{1,1}x_1\rangle}{\|x_1\|^2}+\frac{\langle x_2|A_{2,2}x_2\rangle}{\|x_2\|^2}\leq\lambda_{\max}\left(A_{1,1}\right)+\lambda_{\max}\left(A_{2,2}\right), \tag{1.32}$$

which is the desired inequality for $n=2$. To obtain the general case, we utilize induction over n. ☐

Using this lemma, we can now prove Hoffman's bound for quantum graphs.

Theorem 1.13. *Let $\mathcal{G}=\left(\mathrm{C}(\mathcal{G}),\psi,A\right)$ be an irreflexive quantum graph. Then*

$$\chi_q(\mathcal{G})\geq 1+\frac{\lambda_{\max}(A)}{\left|\lambda_{\min}(A)\right|}, \tag{1.33}$$

where $\chi_q(\mathcal{G})$ denotes the quantum chromatic number.

Proof. Let $\{P_i\}_{i=1}^c\in\mathrm{C}(\mathcal{G})\otimes\mathrm{B}(\mathcal{K})$ be a coloring of $\mathcal{G}$. We denote $\tilde{A}=A\otimes\mathrm{id}$ and, by associating projections P_i with operators of left multiplication by P_i (or, in other words, $P_i: x\mapsto P_i x$), define

$$\tilde{A}_{i,j}=P_i\tilde{A}P_j:\operatorname{ran}P_j\to\operatorname{ran}P_i. \tag{1.34}$$

Applying the previous lemma to the partition of $\tilde{A}$ given by blocks $\tilde{A}_{i,j}$ leads to

$$(c-1)\,\lambda_{\min}\left(\tilde{A}\right)+\lambda_{\max}\left(\tilde{A}\right)\leq\sum_{i=1}^{c}\lambda_{\max}\left(\tilde{A}_{i,i}\right). \tag{1.35}$$

Since $\tilde{A}$ has the same eigenvalues as A (though with different multiplicities), $\tilde{A}_{i,i}=0$ (because projections P_i constitute a coloring) and $\lambda_{\min}(A)\leq 0$ (because A is traceless due to irreflexivity), we obtain

$$\chi_q(\mathcal{G})\geq c\geq 1-\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}=1+\frac{\lambda_{\max}(A)}{|\lambda_{\min}(A)|}. \tag{1.36}$$

□

1.4. **Kneser graphs.** Our primary goal is defining quantum versions of Kneser graphs and investigating their properties. Before we do that, we first need to define these graphs in a classical setting.

Definition 1.14. Let X be a finite set with cardinality n. For $0\leq k\leq n$, denote by $\binom{X}{k}$ the set of k-element subsets of X, that is

$$\binom{X}{k}=\{A\subset X|\,|A|=k\}. \tag{1.37}$$

The Kneser graph, denoted by $K(n,k)$, is a graph (V,E) whose vertices are k-element subsets of X, which are adjacent if and only if they are disjoint; in other words

$$V=\binom{X}{k}, \tag{1.38}$$

$$E=\{(A,B)\in V\times V|A\cap B=\varnothing\}. \tag{1.39}$$

Kneser graphs have several interesting properties. Firstly, $K(n,1)$ is isomorphic to the complete graph with n vertices. Those graphs are regular, vertex-transitive, edge-transitive and arc-transitive.

Definition 1.15. A classical graph $G=(V,E)$ is:

- regular if all its vertices have the same degree (number of edges connected to it),
- vertex-transitive if for any 2 vertices $v_1,v_2\in V$ there exists an automorphism $f\in\mathrm{Aut}(G)$ for which $f(v_1)=v_2$,
- edge-transitive if for any 2 edges $e_1,e_2\in E$ there exists an automorphism $f\in\mathrm{Aut}(G)$ for which $f(e_1)=e_2$,
- arc-transitive if for any 2 edges $(v_1,w_1),(v_2,w_2)\in E$ there exists an automorphism $f\in\mathrm{Aut}(G)$ for which $f(v_1)=v_2$ and $f(w_1)=w_2$.

To see that $K(n,k)$ has the above properties, we note that the degree of the vertex corresponding to $A\subset X$ is equal to the number of k-element sets $B\subset X\backslash A$, which is given by $\binom{n-k}{k}$, where $n=|X|$. As a result, all vertices have the same degree. Additionally, it is easy to see that the automorphisms from the definitions of vertex-transitivity, edge-transitivity and arc-transitivity can be induced by permutations of the underlying set X — given sets $A_1,B_1,A_2,B_2\subset X$ satisfying $A_1\cap B_1=A_2\cap B_2=\varnothing$, $|A_1|=|A_2|$ and $|B_1|=|B_2|$, we can always find a permutation σ of X for which $\sigma(A_1)=A_2$ and $\sigma(B_1)=B_2$.

The chromatic number of $K(n,k)$ is equal to $\max(n-2k+2,1)$, which was first proved by László Lovász [14] by using the Borsuk–Ulam theorem [5], which gave rise to the field of topological combinatorics. Later, Imre Bárány found a simpler proof, utilizing Gale's theorem in addition to Borsuk–Ulam theorem.

Theorem 1.16 (Borsuk–Ulam)**.** *Let S_l be an l-dimensional sphere and let $\{A_i\}_{i=1}^{l+1}$ be a partition of S_l into open sets. Then A_i contains a pair of antipodal points for some $i\in\overline{1,l+1}$.*

It is important to mention that, in its original form, the Borsuk–Ulam theorem involves a partition of S_l into $l+1$ closed sets. The version used here, however, is a consequence of the original one.

Theorem 1.17 (Gale). *Let l and k be non-negative integers. For $a \in S_l \subset \mathbb{R}^{l+1}$ let $H(a) = \{x \in S_l \,|\, \langle a|x\rangle > 0\}$. Then there exists a $(2k+l)$-element set $V \subset S_l$ such that $|H(a) \cap V| \geq k$ for every $a \in S_l$.*

We now reproduce here Bárány's proof.

Theorem 1.18. *The chromatic number of $K(n,k)$ is equal to* $\max(n-2k+2, 1)$.

Proof. If $n < 2k$, then $K(n,k)$ has no edges and therefore its chromatic number is equal to 1 (since no vertices are adjacent).

If $n \geq 2k$, then the map $\binom{\overline{1,n}}{k} \ni A \mapsto \min(A \cup \{n-2k+2\})$ is a coloring of $K(n,k)$ using $n-2k+2$ colors. Therefore, it suffices to prove that a coloring using $n-2k+1$ colors does not exist.

Take $l = n-2k$. Then the set $V \subset S_l$ from Gale's theorem has n elements. We identify V with the set X from the definition of Kneser graphs. Suppose now that $K(n,k)$ has an $(n-2k+1)$-coloring, which means that each k-element subset of V is assigned one of $n-2k+1$ colors, which we label with the set $\overline{1, n-2k+1}$. We now define a family of subsets $\{A_i\}_{i=1}^{n-2k+1}$ of S_l in the following way: A_i consists of all points $a \in S_l$ for which $H(a) \cap V$ contains a subset colored with the i-th color. It is easy to see that each A_i is open and, by Gale's theorem, each point of S_l is an element of at least one of the sets A_i. Therefore, $\{A_i\}_{i=1}^{n-2k+1}$ is a partition of S_l into $n-2k+1 = l+1$ open subsets.

By applying the Borsuk–Ulam theorem, we now see that there exists $j \in \overline{1, n-2k+1}$ for which A_j contains a pair of antipodal points, a and $-a$. This means that there exist k-element sets $A, B \subset V$ that are both colored with the j-th color, such that $A \subset H(a) \cap V$ and $B \subset H(-a) \cap V$. However, this means that $A \cap B \subset H(a) \cap H(-a) \cap V = \varnothing$. Therefore, vertices of $K(n,k)$ corresponding to A and B are adjacent and colored with the same color, which is a contradiction. As a result, $K(n,k)$ does not have any coloring using $n-2k+1$ (or less) colors. ☐

Additionally, Kneser graphs can be used to define an interesting graph invariant known as fractional chromatic number. The fractional chromatic number of G, denoted by $\chi_f(G)$ is defined as the infimum of $\frac{n}{k}$ taken over all pairs (n,k) for which there exists a graph homomorphism from G to $K(n,k)$. Since the ordinary chromatic number can be defined in a similar way using homomorphisms from G to complete graphs instead, we can regard Kneser graphs as fractional generalizations of complete graphs.

Lastly, it should be noted that the definition of Kneser graphs can be slightly generalized by replacing the adjacency condition with

$$E = \{(A,B) \in V \times V \,|\, |A \cap B| \in S\}, \tag{1.40}$$

where S is a fixed subset of $\overline{0,k}$. Ordinary Kneser graphs correspond to the choice of $S = \{0\}$, but other possibilities also exist. For example, picking $S = \{k-1\}$ leads to Johnson graphs, while picking $S = \overline{0,s}$ leads to generalized Kneser graphs as defined by Tristan Denley [10]. Quantum versions of those graphs shall also be constructed.

2. Definition of quantum Kneser graphs

Let $\mathrm{C}(\mathbb{X})$ be a finite-dimensional C*-algebra equipped with a δ-form $\psi : \mathrm{C}(\mathbb{X}) \to \mathbb{C}$. Then $\mathrm{C}(\mathbb{X})$ has the form

$$\mathrm{C}(\mathbb{X}) = \bigoplus_{a=1}^{n} \mathsf{B}_a, \tag{2.1}$$

where $\forall a : \mathsf{B}_a = \mathsf{Mat}_{d_a}(\mathbb{C})$. Every δ-form on $\mathrm{C}(\mathbb{X})$ can be written as

$$\psi = \bigoplus_{a=1}^{n} \psi_a, \tag{2.2}$$

where $\psi_a(x) = \mathrm{Tr}(\rho^a x)$. Here $\rho^a \in \mathsf{B}_a$ are all positive, invertible matrices satisfying the condition $\mathrm{Tr}\left((\rho^a)^{-1}\right) = \delta^2$. In other words, for $x = \sum_{a=1}^{n} x^a$, where $x^a \in \mathsf{B}_a$, $\psi(x) = \sum_{a=1}^{n} \mathrm{Tr}(\rho^a x^a)$ (Tr represents the standard, unnormalized trace — the sum of diagonal entries of a matrix).

Since all matrices ρ^a in the definition of ψ are positive, we may without loss of generality assume that they are all diagonal and denote their eigenvalues as ρ^a_i, where $i \in \overline{1, d_a}$. Then for the matrix units $e^a_{i,j} \in \mathsf{B}_a = \mathsf{Mat}_{d_a}(\mathbb{C})$ of the a-th matrix component we have $\psi\left(e^a_{i,j}\right) = \rho^a_i \delta_{i,j}$.

For $k \in \overline{1, n}$ we define

$$\mathrm{C}\left(\binom{\mathbb{X}}{k}\right) = \bigoplus_{1 \le a_1 < a_2 < \ldots < a_k \le n} \left(\bigoplus_{\sigma \in S_k} \mathsf{B}_{a_{\sigma(1)}} \otimes \mathsf{B}_{a_{\sigma(2)}} \otimes \ldots \otimes \mathsf{B}_{a_{\sigma(k)}}\right)_{\mathrm{sym}}, \tag{2.3}$$

where $(\cdot)_{\mathrm{sym}}$ denotes the symmetric (invariant with respect to permuting the legs of the tensor product) part of the algebra contained in parenthesis — in this case, it is the algebra spanned by elements of the form

$$\bigotimes_{a \in A}^{\mathrm{sym}} x^a = \sum_f x^{f(1)} \otimes x^{f(2)} \otimes \ldots \otimes x^{f(k)}, \tag{2.4}$$

where A is a k-element subset of $\overline{1, n}$, $\forall a \in A : x^a \in \mathsf{B}_a$ and the sum is taken over all bijections $f : \overline{1, k} \to A$. Multiplication in this algebra is given by

$$\left(\bigotimes_{a \in A}^{\mathrm{sym}} x^a\right)\left(\bigotimes_{\tilde{a} \in \tilde{A}}^{\mathrm{sym}} y^{\tilde{a}}\right) = \sum_{f, \tilde{f}} x^{f(1)} y^{\tilde{f}(1)} \otimes x^{f(2)} y^{\tilde{f}(2)} \otimes \ldots \otimes x^{f(k)} y^{\tilde{f}(k)}. \tag{2.5}$$

The terms in which $f(i) \neq \tilde{f}(i)$ for some $i \in \overline{1, k}$ all evaluate to 0. Non-zero contributions, therefore, only come from terms in the sum for which $f = \tilde{f}$, which may only happen when $A = \tilde{A}$. In such a case, we have

$$\left(\bigotimes_{a \in A}^{\mathrm{sym}} x^a\right)\left(\bigotimes_{\tilde{a} \in \tilde{A}}^{\mathrm{sym}} y^{\tilde{a}}\right) = \delta_{A, \tilde{A}} \sum_f x^{f(1)} y^{f(1)} \otimes x^{f(2)} y^{f(2)} \otimes \ldots \otimes x^{f(k)} y^{f(k)} = \delta_{A, \tilde{A}} \bigotimes_{a \in A}^{\mathrm{sym}} x^a y^a. \tag{2.6}$$

Since permutations of legs of the tensor product $\mathrm{C}(\mathbb{X})^{\otimes k}$ are $*$-homomorphisms, $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ is a C^*-subalgebra of $\mathrm{C}(\mathbb{X})^{\otimes k}$. We define $\iota_k : \mathrm{C}\left(\binom{\mathbb{X}}{k}\right) \hookrightarrow \mathrm{C}(\mathbb{X})^{\otimes k}$ as the natural embedding of $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ in $\mathrm{C}(\mathbb{X})^{\otimes k}$. Additionally, we define $\psi_k = \psi^{\otimes k} \iota_k$ and denote the multiplication map on $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ by m_k.

Lemma 2.1. *ψ_k is a δ_k-form on $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$, where $\delta_k = \frac{\delta^k}{\sqrt{k!}}$.*

Proof. We claim that the action of $m_k^\dagger$ on basis elements of $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ can be expressed as

$$m_k^\dagger\left(\bigotimes_{a \in A}^{\mathrm{sym}} e^a_{i_a, j_a}\right) = \frac{1}{k!} \sum_{\alpha \in \times_{a \in A} \overline{1, d_a}} \left(\prod_{a \in A} \rho^a_{\alpha_a}\right)^{-1} \bigotimes_{a \in A}^{\mathrm{sym}} e^a_{i_a, \alpha_a} \otimes \bigotimes_{a \in A}^{\mathrm{sym}} e^a_{\alpha_a, j_a}. \tag{2.7}$$

To show that this is the case, we must verify the following condition:

$$\psi_k(x^* y z) = (\psi_k \otimes \psi_k)\left(\left(m_k^\dagger(x)\right)^* (y \otimes z)\right). \tag{2.8}$$

Setting

$$x = \bigotimes_{a \in A}^{\mathrm{sym}} e^a_{i_a, j_a}, \tag{2.9}$$

$$y = \bigotimes_{a \in A'}^{\mathrm{sym}} e^a_{i'_a, j'_a}, \tag{2.10}$$

$$z = \bigotimes_{a \in A''}^{\mathrm{sym}} e^a_{i''_a, j''_a}, \tag{2.11}$$

we obtain

$$\psi_k\left(x^*yz\right)=\psi_k\left(\delta_{A,A'}\delta_{A',A''}\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{j_a,i_a}e^a_{i'_a,j'_a}e^a_{i''_a,j''_a}\right)=\delta_{A,A'}\delta_{A',A''}\psi_k\left(\bigotimes_{a\in A}^{\mathrm{sym}}\delta_{i_a,i'_a}\delta_{j'_a,i''_a}e^a_{j_a,j''_a}\right)$$
$$=k!\delta_{A,A'}\delta_{A',A''}\prod_{a\in A}\delta_{i_a,i'_a}\delta_{j'_a,i''_a}\delta_{j_a,j''_a}\rho^a_{j_a}. \quad (2.12)$$

On the other hand, the above expression for $m_k^\dagger$ yields

$$\left(\psi_k\otimes\psi_k\right)\left(\left(m_k^\dagger\left(x\right)\right)^*\left(y\otimes z\right)\right)$$
$$=\left(\psi_k\otimes\psi_k\right)\left(\left(\frac{1}{k!}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{\alpha_a,i_a}\otimes\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{j_a,\alpha_a}\right)\left(\bigotimes_{a\in A'}^{\mathrm{sym}}e^a_{i'_a,j'_a}\otimes\bigotimes_{a\in A''}^{\mathrm{sym}}e^a_{i''_a,j''_a}\right)\right)$$
$$=\left(\psi_k\otimes\psi_k\right)\left(\frac{1}{k!}\delta_{A,A'}\delta_{A,A''}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}\bigotimes_{a\in A}^{\mathrm{sym}}\delta_{i_a,i'_a}e^a_{\alpha_a,j'_a}\otimes\bigotimes_{a\in A}^{\mathrm{sym}}\delta_{\alpha_a,i''_a}e^a_{j_a,j''_a}\right)$$
$$=\frac{1}{k!}\delta_{A,A'}\delta_{A,A''}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}\psi_k\left(\bigotimes_{a\in A}^{\mathrm{sym}}\delta_{i_a,i'_a}e^a_{\alpha_a,j'_a}\right)\psi_k\left(\bigotimes_{a\in A}^{\mathrm{sym}}\delta_{\alpha_a,i''_a}e^a_{j_a,j''_a}\right)$$
$$=k!\delta_{A,A'}\delta_{A,A''}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}\left(\prod_{a\in A}\delta_{i_a,i'_a}\delta_{\alpha_a,j'_a}\rho^a_{\alpha_a}\right)\left(\prod_{a\in A}\delta_{\alpha_a,i''_a}\delta_{j_a,j''_a}\rho^a_{j_a}\right)$$
$$=k!\delta_{A,A'}\delta_{A,A''}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\prod_{a\in A}\delta_{i_a,i'_a}\delta_{\alpha_a,j'_a}\delta_{\alpha_a,i''_a}\delta_{j_a,j''_a}\rho^a_{j_a}=k!\delta_{A,A'}\delta_{A',A''}\prod_{a\in A}\delta_{i_a,i'_a}\delta_{j'_a,i''_a}\delta_{j_a,j''_a}\rho^a_{j_a}. \quad (2.13)$$

Having checked that the given formula for $m_k^\dagger$ is indeed correct, we can write

$$m_km_k^\dagger\left(\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{i_a,j_a}\right)=\frac{1}{k!}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}m_k\left(\left(\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{i_a,\alpha_a}\right)\otimes\left(\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{\alpha_a,j_a}\right)\right)$$
$$=\frac{1}{k!}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}\left(\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{i_a,\alpha_a}\right)\left(\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{\alpha_a,j_a}\right)=\frac{1}{k!}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{i_a,j_a}$$
$$=\frac{1}{k!}\prod_{a\in A}\left(\sum_{\alpha_a=1}^{d_a}\left(\rho^a_{\alpha_a}\right)^{-1}\right)\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{i_a,j_a}=\frac{1}{k!}\prod_{a\in A}\left(\delta^2\right)\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{i_a,j_a}=\frac{\delta^{2k}}{k!}\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{i_a,j_a}, \quad (2.14)$$

which implies that $m_km_k^\dagger=\delta_k^2\mathrm{id}$, where $\delta_k=\frac{\delta^k}{\sqrt{k!}}$. □

We define recursively a sequence of operators $\tilde{A}_k\in\mathrm{B}\left(\mathrm{C}\left(\mathbb{X}\right)^{\otimes k}\right)$ by the following formula:

$$\tilde{A}_k=\frac{\delta^{2k}}{k!}\left(\psi^\dagger\psi\right)^{\otimes k}-\sum_{l=0}^{k-1}\left(\sum_{1\le i_1<i_2<\ldots<i_l\le k}\left(\tilde{A}_l\right)_{i_1,i_2,\ldots,i_l}\right), \quad (2.15)$$

where subscripts in $\left(\tilde{A}_l\right)_{i_1,i_2,\ldots,i_l}$ signify the legs of the tensor product on which the operators act as $\tilde{A}_l$. We also define $A_k=\iota_k^\dagger\tilde{A}_k\iota_k$, where † denotes the Hermitian adjoint (to distinguish it from the C*-algebra involution).

Theorem 2.2. *The operators $\tilde{A}_k$, defined as above, are given by the following formula:*

$$\tilde{A}_k=\sum_{l=0}^{k}\frac{\left(-1\right)^{k-l}\delta^{2l}}{l!}\sum_{1\le i_1<i_2<\ldots<i_l\le k}\left(\left(\psi^\dagger\psi\right)^{\otimes l}\right)_{i_1,i_2,\ldots,i_l}. \quad (2.16)$$

Proof. We start by rewriting the formula (2.15) using the summation over sets of the form $B = \{i_1, i_2, \ldots, i_l\}$, obtaining

$$\sum_{B \subset \overline{1,k}} \left(\tilde{A}_{|B|}\right)_B = \frac{\delta^{2k}}{k!} \left(\psi^\dagger \psi\right)^{\otimes k}. \tag{2.17}$$

Similarly, the equation (2.16) can be rewritten as

$$\tilde{A}_k = \sum_{B \subset \overline{1,k}} \frac{(-1)^{k-|B|} \, \delta^{2|B|}}{|B|!} \left(\left(\psi^\dagger \psi\right)^{\otimes |B|}\right)_B. \tag{2.18}$$

We now have to check that the operators given by the formula (2.18) satisfy the condition (2.17). To do this, we write

$$\sum_{B \subset \overline{1,k}} \left(\tilde{A}_{|B|}\right)_B = \sum_{B \subset \overline{1,k}} \sum_{C \subset B} \frac{(-1)^{|B|-|C|} \, \delta^{2|C|}}{|C|!} \left(\left(\psi^\dagger \psi\right)^{\otimes |C|}\right)_C, \tag{2.19}$$

which, after exchanging the order of summations, becomes

$$\sum_{C \subset \overline{1,k}} \sum_{\substack{B \subset \overline{1,k} \\ B \supset C}} \frac{(-1)^{|B|-|C|} \, \delta^{2|C|}}{|C|!} \left(\left(\psi^\dagger \psi\right)^{\otimes |C|}\right)_C. \tag{2.20}$$

Since B must be a superset of C, it can be expressed as $C \cup B'$, where $B' \subset \overline{1,k} \backslash C$. Therefore, the expression we just obtained can be further rewritten as

$$\sum_{C \subset \overline{1,k}} \left(\sum_{B' \subset \overline{1,k} \backslash C} (-1)^{|B'|} \right) \frac{\delta^{2|C|}}{|C|!} \left(\left(\psi^\dagger \psi\right)^{\otimes |C|}\right)_C. \tag{2.21}$$

Since the summand of $\sum_{B' \subset \overline{1,k} \backslash C} (-1)^{|B'|}$ only depends on the cardinality of B', we can express this sum as

$$\sum_{l=0}^{k-|C|} \binom{k-|C|}{l} (-1)^l = (1-1)^{k-|C|} = \delta_{k,|C|}, \tag{2.22}$$

where we used the binomial formula. Therefore,

$$\sum_{B \subset \overline{1,k}} \left(\tilde{A}_{|B|}\right)_B = \sum_{C \subset \overline{1,k}} \delta_{k,|C|} \frac{\delta^{2|C|}}{|C|!} \left(\left(\psi^\dagger \psi\right)^{\otimes |C|}\right)_C = \frac{\delta^{2k}}{k!} \left(\psi^\dagger \psi\right)^{\otimes k}. \tag{2.23}$$

□

We also introduce operators $\tilde{A}_{k,l}$, defined as

$$\tilde{A}_{k,l} = \sum_{1 \le i_1 < i_2 < \ldots < i_l \le k} \left(\tilde{A}_l\right)_{i_1, i_2, \ldots, i_l}. \tag{2.24}$$

By convention, $\tilde{A}_{k,0} = \mathrm{id}^{\otimes k}$. Additionally, $\tilde{A}_{k,k} = \tilde{A}_k$. The following lemma gives us a more convenient formula for those operators.

Lemma 2.3. *The operators $\tilde{A}_{k,l}$, defined as above, are given by*

$$\tilde{A}_{k,l} = \sum_{r=0}^{l} \binom{k-r}{k-l} \frac{(-1)^{l-r} \, \delta^{2r}}{r!} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le k} \left(\left(\psi^\dagger \psi\right)^{\otimes r}\right)_{i_1, i_2, \ldots, i_r}. \tag{2.25}$$

Proof. We start by rewriting the definition of $\tilde{A}_{k,l}$ using summation over subsets, similarly to the method used in the proof of the previous lemma. In this way, we obtain

$$\tilde{A}_{k,l} = \sum_{\substack{B \subset \overline{1,k} \\ |B| = l}} \left(\tilde{A}_l\right)_B = \sum_{\substack{B \subset \overline{1,k} \\ |B| = l}} \sum_{C \subset B} \frac{(-1)^{l-|C|} \, \delta^{2|C|}}{|C|!} \left(\left(\psi^\dagger \psi\right)^{\otimes |C|}\right)_C. \tag{2.26}$$

Exchanging the order of summations yields

$$\sum_{C\subset\overline{1,k}}\sum_{\substack{B\supset C\\ B\subset\overline{1,k}\\ |B|=l}}\frac{(-1)^{l-|C|}\,\delta^{2|C|}}{|C|!}\left(\left(\psi^\dagger\psi\right)^{\otimes|C|}\right)_C=\sum_{C\subset\overline{1,k}}\left(\sum_{\substack{B\supset C\\ B\subset\overline{1,k}\\ |B|=l}}1\right)\frac{(-1)^{l-|C|}\,\delta^{2|C|}}{|C|!}\left(\left(\psi^\dagger\psi\right)^{\otimes|C|}\right)_C. \tag{2.27}$$

The quantity $\sum_{\substack{B\supset C\\ B\subset\overline{1,k}\\ |B|=l}}1$ is equal to the number of l-element supersets of C contained in $\overline{1,k}$ — in other words, to $\binom{k-|C|}{l-|C|}$. Therefore

$$\tilde{A}_{k,l}=\sum_{C\subset\overline{1,k}}\binom{k-|C|}{l-|C|}\frac{(-1)^{l-|C|}\,\delta^{2|C|}}{|C|!}\left(\left(\psi^\dagger\psi\right)^{\otimes|C|}\right)_C, \tag{2.28}$$

which, after returning to the original notation, becomes

$$\tilde{A}_{k,l}=\sum_{r=0}^{l}\binom{k-r}{k-l}\frac{(-1)^{l-r}\,\delta^{2r}}{r!}\sum_{1\le i_1<i_2<\ldots<i_r\le k}\left(\left(\psi^\dagger\psi\right)^{\otimes r}\right)_{i_1,i_2,\ldots,i_r}. \tag{2.29}$$

□

Lastly, we define $A_{k,l}=\iota_k^\dagger\tilde{A}_{k,l}\iota_k$. To evaluate those operators on elements of $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$, we utilize the following lemma.

Lemma 2.4. *Let us define*

$$\Psi_{k,l}=\frac{1}{l!}\iota_k^\dagger\left(\sum_{1\le i_1<i_2<\ldots<i_l\le k}\left(\left(\psi^\dagger\psi\right)^{\otimes l}\right)_{i_1,i_2,\ldots,i_l}\right)\iota_k. \tag{2.30}$$

Then, if $A\subset\overline{1,n}$ and $\forall a\in A: x^a\in\mathsf{B}_a$, we have

$$\Psi_{k,l}\left(\bigotimes_{a\in A}^{\text{sym}}x^a\right)=\sum_{\substack{B\subset A\\ |B|=k-l}}\left(\prod_{a\in A\setminus B}\psi\left(x^a\right)\right)\sum_{\substack{C\subset\overline{1,n}\setminus B\\ |C|=l}}\bigotimes_{a\in C}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in B}^{\text{sym}}x^a. \tag{2.31}$$

Proof. Let $A=\{a_1,a_2,\ldots,a_k\}$. Then we can write

$$\iota_k\left(\bigotimes_{a\in A}^{\text{sym}}x^a\right)=\sum_{\sigma\in S_k}\bigotimes_{i=1}^{k}x^{a_{\sigma(i)}}. \tag{2.32}$$

Let us also rewrite the formula for $\Psi_{k,l}$ as

$$\Psi_{k,l}=\frac{1}{l!}\iota_k^\dagger\left(\sum_{\substack{\tilde{B}'\subset\overline{1,k}\\ |\tilde{B}'|=l}}\left(\left(\psi^\dagger\psi\right)^{\otimes l}\right)_{\tilde{B}'}\right)\iota_k. \tag{2.33}$$

Therefore, we calculate:

$$\Psi_{k,l}\left(\bigotimes_{a\in A}^{\text{sym}}x^a\right)=\frac{1}{l!}\iota_k^\dagger\left(\sum_{\substack{\tilde{B}'\subset\overline{1,k}\\ |\tilde{B}'|=l}}\sum_{\sigma\in S_k}\left(\left(\psi^\dagger\psi\right)^{\otimes l}\right)_{\tilde{B}'}\left(\bigotimes_{i=1}^{k}x^{a_{\sigma(i)}}\right)\right). \tag{2.34}$$

The expression $\left(\left(\psi^\dagger\psi\right)^{\otimes l}\right)_{\tilde{B}'}\left(\bigotimes_{i=1}^{k}x^{a_{\sigma(i)}}\right)$ evaluates to

$$\bigotimes_{i=1}^{k}\left(\chi\left(i\in\tilde{B}'\right)\psi\left(x^{a_{\sigma(i)}}\right)\mathbb{1}+\chi\left(i\notin\tilde{B}'\right)x^{a_{\sigma(i)}}\right), \tag{2.35}$$

where $\mathbb{1}$ is the unit of $\mathrm{C}(\mathbb{X})$ and χ is a logical statement evaluation function (which assigns 1 to true statements and 0 to false ones). Denoting $\chi\left(i \in \tilde{B}'\right)\mathbb{1} + \chi\left(i \notin \tilde{B}'\right)x^{a_{\sigma(i)}}$ by $y_{\tilde{B}'}^{a_{\sigma(i)}}$, we express it as

$$\left(\prod_{i\in\tilde{B}'}\psi\left(x^{a_{\sigma(i)}}\right)\right)\bigotimes_{i=1}^{k} y_{\tilde{B}'}^{a_{\sigma(i)}}. \tag{2.36}$$

Returning to the original calculation, we now have

$$\Psi_{k,l}\left(\bigotimes_{a\in A}^{\mathrm{sym}} x^a\right) = \frac{1}{l!}\iota_k^\dagger\left(\sum_{\substack{\tilde{B}'\subset\overline{1,k}\\ \left|\tilde{B}'\right|=l}}\sum_{\sigma\in S_k}\left(\prod_{i\in\tilde{B}'}\psi\left(x^{a_{\sigma(i)}}\right)\right)\bigotimes_{i=1}^{k} y_{\tilde{B}'}^{a_{\sigma(i)}}\right). \tag{2.37}$$

We now replace summation over $\tilde{B}'$ with summation over $\sigma\left(\tilde{B}'\right)$, obtaining

$$\Psi_{k,l}\left(\bigotimes_{a\in A}^{\mathrm{sym}} x^a\right) = \frac{1}{l!}\iota_k^\dagger\left(\sum_{\substack{\tilde{B}'\subset\overline{1,k}\\ \left|\tilde{B}'\right|=l}}\sum_{\sigma\in S_k}\left(\prod_{i\in\sigma^{-1}\left(\tilde{B}'\right)}\psi\left(x^{a_{\sigma(i)}}\right)\right)\bigotimes_{i=1}^{k} y_{\sigma^{-1}\left(\tilde{B}'\right)}^{a_{\sigma(i)}}\right). \tag{2.38}$$

Since $\prod_{i\in\sigma^{-1}\left(\tilde{B}'\right)}\psi\left(x^{a_{\sigma(i)}}\right) = \prod_{i\in\tilde{B}'}\psi\left(x^{a_i}\right)$, we can write

$$\Psi_{k,l}\left(\bigotimes_{a\in A}^{\mathrm{sym}} x^a\right) = \frac{1}{l!}\sum_{\substack{\tilde{B}'\subset\overline{1,k}\\ \left|\tilde{B}'\right|=l}}\left(\prod_{i\in\tilde{B}'}\psi\left(x^{a_i}\right)\right)\sum_{\sigma\in S_k}\iota_k^\dagger\left(\bigotimes_{i=1}^{k} y_{\sigma^{-1}\left(\tilde{B}'\right)}^{a_{\sigma(i)}}\right). \tag{2.39}$$

Now we define $B' = \left\{a_i \middle| i\in\tilde{B}'\right\}\subset A$ and note that

$$\iota_k^\dagger\left(\bigotimes_{i=1}^{k} y_{\sigma^{-1}\left(\tilde{B}'\right)}^{a_{\sigma(i)}}\right) = \frac{l!}{k!}\sum_{\substack{C\subset\overline{1,n}\setminus\left(A\setminus B'\right)\\ |C|=l}}\bigotimes_{a\in C}^{\mathrm{sym}}\mathbb{1}^a\overset{\mathrm{sym}}{\otimes}\bigotimes_{a\in A\setminus B'}^{\mathrm{sym}} x^a, \tag{2.40}$$

obtaining

$$\Psi_{k,l}\left(\bigotimes_{a\in A}^{\mathrm{sym}} x^a\right) = \frac{1}{l!}\sum_{\substack{B'\subset A\\ \left|\tilde{B}'\right|=l}}\left(\prod_{a\in B'}\psi\left(x^a\right)\right)\sum_{\sigma\in S_k}\frac{l!}{k!}\sum_{\substack{C\subset\overline{1,n}\setminus\left(A\setminus B'\right)\\ |C|=l}}\bigotimes_{a\in C}^{\mathrm{sym}}\mathbb{1}^a\overset{\mathrm{sym}}{\otimes}\bigotimes_{a\in A\setminus B'}^{\mathrm{sym}} x^a. \tag{2.41}$$

Lastly, by replacing B' in the summation with $B = A\setminus B'$ and performing the summation over permutations, we obtain

$$\Psi_{k,l}\left(\bigotimes_{a\in A}^{\mathrm{sym}} x^a\right) = \sum_{\substack{B\subset A\\ |B|=k-l}}\left(\prod_{a\in A\setminus B}\psi\left(x^a\right)\right)\sum_{\substack{C\subset\overline{1,n}\setminus B\\ |C|=l}}\bigotimes_{a\in C}^{\mathrm{sym}}\mathbb{1}^a\overset{\mathrm{sym}}{\otimes}\bigotimes_{a\in B}^{\mathrm{sym}} x^a. \tag{2.42}$$

□

These operators satisfy a useful property, given by the following lemma.

Lemma 2.5. *Operators $A_{k,l}$ on $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ satisfy the following condition:*

$$m_k\left(A_{k,l}\otimes A_{k,\tilde{l}}\right)m_k^\dagger = \delta_{l,\tilde{l}}\delta_k^2 A_{k,l}. \tag{2.43}$$

Proof. We can express the results of acting with A_k and $m_k^\dagger$ on basis elements of $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ as

$$A_{k,l}\left(\bigotimes_{a\in A}^{\text{sym}} e^a_{i_a,j_a}\right) = \sum_{r=0}^{k}\binom{k-r}{k-l}(-1)^{l-r}\,\delta^{2r}\sum_{\substack{B\subset A\\ |B|=k-r\\ C\subset\overline{1,n}\setminus B\\ |C|=r}}\left(\prod_{a\in A\setminus B}\rho^a_{i_a}\,\delta_{i_a,j_a}\right)\bigotimes_{a\in C}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in B}^{\text{sym}} e^a_{i_a,j_a} \tag{2.44}$$

and

$$m_k^\dagger\left(\bigotimes_{a\in A}^{\text{sym}} e^a_{i_a,j_a}\right) = \frac{1}{k!}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}\left(\bigotimes_{a\in A}^{\text{sym}} e^a_{i_a,\alpha_a}\right)\otimes\left(\bigotimes_{a\in A}^{\text{sym}} e^a_{\alpha_a,j_a}\right). \tag{2.45}$$

The right-hand side of equation (2.44) can be rewritten as

$$\sum_{\substack{B\subset A\\ C_1\subset\overline{1,n}\setminus A\\ C_2\subset A\setminus B\\ |C_1|+|C_2|=k-|B|}}\binom{|B|}{k-l}(-1)^{l-k+|B|}\,\delta^{2(k-|B|)}\left(\prod_{a\in A\setminus B}\rho^a_{i_a}\,\delta_{i_a,j_a}\right)\bigotimes_{a\in C_1\cup C_2}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in B}^{\text{sym}} e^a_{i_a,j_a} \tag{2.46}$$

by partitioning the set C into $C_1\subset\overline{1,n}\setminus A$ and $C_2\subset A\setminus B$. Next, we introduce $C_2' = (A\setminus B)\setminus C_2$, obtaining

$$(-1)^{k-l}\sum_{\substack{B\subset A\\ C_1\subset\overline{1,n}\setminus A\\ C_2'\subset A\setminus B\\ |C_1|=|C_2'|}}\binom{|B|}{k-l}(-1)^{|B|}\,\delta^{2(k-|B|)}\left(\prod_{a\in A\setminus B}\rho^a_{i_a}\,\delta_{i_a,j_a}\right)\bigotimes_{a\in C_1\cup\left((A\setminus B)\setminus C_2'\right)}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in B}^{\text{sym}} e^a_{i_a,j_a}. \tag{2.47}$$

Lastly, we notice that the condition $B\subset A\wedge C_2'\subset A\setminus B$ (where $\wedge$ denotes logical conjunction) is equivalent to $C_2'\subset A\wedge B\subset A\setminus C_2'$, leading to

$$(-1)^{k-l}\sum_{\substack{C_1\subset\overline{1,n}\setminus A\\ C_2'\subset A\\ |C_1|=|C_2'|\\ B\subset A\setminus C_2'}}\binom{|B|}{k-l}(-1)^{|B|}\,\delta^{2(k-|B|)}\left(\prod_{a\in A\setminus B}\rho^a_{i_a}\,\delta_{i_a,j_a}\right)\bigotimes_{a\in C_1\cup\left((A\setminus B)\setminus C_2'\right)}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in B}^{\text{sym}} e^a_{i_a,j_a}. \tag{2.48}$$

Therefore, we can write:

$$m_k\left(A_{k,l}\otimes A_{k,\tilde{l}}\right)m_k^\dagger\left(\bigotimes_{a\in A}^{\text{sym}}e^a_{i_a,j_a}\right)$$
$$=\frac{1}{k!}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}A_k\left(\bigotimes_{a\in A}^{\text{sym}}e^a_{i_a,\alpha_a}\right)A_k\left(\bigotimes_{a\in A}^{\text{sym}}e^a_{\alpha_a,j_a}\right)$$
$$=\frac{(-1)^{2k-l-\tilde{l}}}{k!}\sum_{\alpha\in\times_{a\in A}\overline{1,d_a}}\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}\sum_{\substack{C_1\subset\overline{1,n}\setminus A\\ C_2'\subset A\\ |C_1|=|C_2'|\\ B\subset A\setminus C_2'}}\sum_{\substack{\widetilde{C}_1\subset\overline{1,n}\setminus A\\ \widetilde{C}_2'\subset A\\ |\widetilde{C}_1|=|\widetilde{C}_2'|\\ \widetilde{B}\subset A\setminus\widetilde{C}_2'}}\binom{|B|}{k-l}\binom{\left|\widetilde{B}\right|}{k-\tilde{l}}$$
$$(-1)^{|B|+\left|\widetilde{B}\right|}\delta^{4k-2|B|-2\left|\widetilde{B}\right|}\left(\prod_{a\in A\setminus B}\rho^a_{\alpha_a}\delta_{i_a,\alpha_a}\right)\left(\prod_{a\in A\setminus\widetilde{B}}\rho^a_{\alpha_a}\delta_{\alpha_a,j_a}\right)$$
$$\left(\bigotimes_{a\in C_1\cup\left((A\setminus B)\setminus C_2'\right)}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in B}^{\text{sym}}e^a_{i_a,\alpha_a}\right)\left(\bigotimes_{a\in\widetilde{C}_1\cup\left((A\setminus\widetilde{B})\setminus\widetilde{C}_2'\right)}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in\widetilde{B}}^{\text{sym}}e^a_{\alpha_a,j_a}\right).\quad(2.49)$$

We notice that

$$\left(\prod_{a\in A}\rho^a_{\alpha_a}\right)^{-1}\left(\prod_{a\in A\setminus B}\rho^a_{\alpha_a}\delta_{i_a,\alpha_a}\right)\left(\prod_{a\in A\setminus\widetilde{B}}\rho^a_{\alpha_a}\delta_{\alpha_a,j_a}\right)$$
$$=\left(\prod_{a\in B\cap\widetilde{B}}\rho^a_{\alpha_a}\right)^{-1}\left(\prod_{a\in(A\setminus B)\cap\widetilde{B}}\delta_{i_a,\alpha_a}\right)\left(\prod_{a\in B\cap\left(A\setminus\widetilde{B}\right)}\delta_{\alpha_a,j_a}\right)\left(\prod_{a\in(A\setminus B)\cap\left(A\setminus\widetilde{B}\right)}\rho^a_{\alpha_a}\delta_{i_a,\alpha_a}\delta_{\alpha_a,j_a}\right)\quad(2.50)$$

and

$$\left(\bigotimes_{a\in C_1\cup\left((A\setminus B)\setminus C_2'\right)}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in B}^{\text{sym}}e^a_{i_a,\alpha_a}\right)\left(\bigotimes_{a\in\widetilde{C}_1\cup\left((A\setminus\widetilde{B})\setminus\widetilde{C}_2'\right)}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in\widetilde{B}}^{\text{sym}}e^a_{\alpha_a,j_a}\right)=0\quad(2.51)$$

whenever $C_1\neq\widetilde{C}_1$ or $C_2'\neq\widetilde{C}_2'$. If $C_1=\widetilde{C}_1$ and $C_2'=\widetilde{C}_2'$, then

$$\left(\bigotimes_{a\in C_1\cup\left((A\setminus B)\setminus C_2'\right)}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in B}^{\text{sym}}e^a_{i_a,\alpha_a}\right)\left(\bigotimes_{a\in\widetilde{C}_1\cup\left((A\setminus\widetilde{B})\setminus\widetilde{C}_2'\right)}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in\widetilde{B}}^{\text{sym}}e^a_{\alpha_a,j_a}\right)$$
$$=\left(\bigotimes_{a\in C_1}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in(A\setminus B)\setminus C_2'}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in B}^{\text{sym}}e^a_{i_a,\alpha_a}\right)\left(\bigotimes_{a\in C_1}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in\left(A\setminus\widetilde{B}\right)\setminus C_2'}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in\widetilde{B}}^{\text{sym}}e^a_{\alpha_a,j_a}\right)$$
$$=\bigotimes_{a\in C_1\cup\left(\left(A\setminus\left(B\cup\widetilde{B}\right)\right)\setminus C_2'\right)}^{\text{sym}}\mathbb{1}^a\overset{\text{sym}}{\otimes}\bigotimes_{a\in\left((A\setminus B)\setminus C_2'\right)\cap\widetilde{B}}^{\text{sym}}e^a_{\alpha_a,j_a}\overset{\text{sym}}{\otimes}\bigotimes_{a\in B\cap\left(\left(A\setminus\widetilde{B}\right)\setminus C_2'\right)}^{\text{sym}}e^a_{i_a,\alpha_a}\overset{\text{sym}}{\otimes}\bigotimes_{a\in B\cap\widetilde{B}}^{\text{sym}}e^a_{i_a,j_a}.\quad(2.52)$$

Therefore

$$
\begin{aligned}
& m_k \left(A_{k,l} \otimes A_{k,\tilde{l}} \right) m_k^\dagger \left(\bigotimes_{a\in A}^{\text{sym}} e^a_{i_a,j_a} \right) \\
&\quad = \frac{(-1)^{l+\tilde{l}}}{k!} \sum_{\substack{C_1\subset\overline{1,n}\setminus A \\ C_2'\subset A \\ |C_1|=|C_2'| \\ B,\widetilde{B}\subset A\setminus C_2'}} \sum_{\alpha\in\times_{a\in A}\overline{1,d_a}} \binom{|B|}{k-l}\binom{\left|\widetilde{B}\right|}{k-\tilde{l}} (-1)^{|B|+\left|\widetilde{B}\right|} \delta^{4k-2|B|-2\left|\widetilde{B}\right|} \\
&\quad \left(\prod_{a\in B\cap\widetilde{B}} \rho^a_{\alpha_a}\right)^{-1} \left(\prod_{a\in(A\setminus B)\cap\widetilde{B}} \delta_{i_a,\alpha_a}\right) \left(\prod_{a\in B\cap\left(A\setminus\widetilde{B}\right)} \delta_{\alpha_a,j_a}\right) \left(\prod_{a\in(A\setminus B)\cap\left(A\setminus\widetilde{B}\right)} \rho^a_{\alpha_a}\delta_{i_a,\alpha_a}\delta_{\alpha_a,j_a}\right) \\
&\quad \bigotimes_{a\in C_1\cup\left(\left(A\setminus\left(B\cup\widetilde{B}\right)\right)\setminus C_2'\right)}^{\text{sym}} \mathbb{1}^a \overset{\text{sym}}{\otimes} \bigotimes_{a\in\left((A\setminus B)\setminus C_2'\right)\cap\widetilde{B}}^{\text{sym}} e^a_{\alpha_a,j_a} \overset{\text{sym}}{\otimes} \bigotimes_{a\in B\cap\left(\left(A\setminus\widetilde{B}\right)\setminus C_2'\right)}^{\text{sym}} e^a_{i_a,\alpha_a} \overset{\text{sym}}{\otimes} \bigotimes_{a\in B\cap\widetilde{B}}^{\text{sym}} e^a_{i_a,j_a} \\
&\quad = \frac{(-1)^{l+\tilde{l}}}{k!} \sum_{\substack{C_1\subset\overline{1,n}\setminus A \\ C_2'\subset A \\ |C_1|=|C_2'| \\ B,\widetilde{B}\subset A\setminus C_2'}} \sum_{\alpha\in\times_{a\in B\cap\widetilde{B}}\overline{1,d_a}} \binom{|B|}{k-l}\binom{\left|\widetilde{B}\right|}{k-\tilde{l}} (-1)^{|B|+\left|\widetilde{B}\right|} \delta^{4k-2|B|-2\left|\widetilde{B}\right|} \\
&\quad \left(\prod_{a\in B\cap\widetilde{B}} \rho^a_{\alpha_a}\right)^{-1} \left(\prod_{a\in A\setminus\left(B\cup\widetilde{B}\right)} \rho^a_{i_a}\delta_{i_a,j_a}\right) \bigotimes_{a\in C_1\cup\left(\left(A\setminus\left(B\cup\widetilde{B}\right)\right)\setminus C_2'\right)}^{\text{sym}} \mathbb{1}^a \overset{\text{sym}}{\otimes} \bigotimes_{a\in B\cup\widetilde{B}}^{\text{sym}} e^a_{i_a,j_a}. \qquad (2.53)
\end{aligned}
$$

We can further simplify this expression by noticing that

$$
\sum_{\alpha\in\times_{a\in B\cap\widetilde{B}}\overline{1,d_a}} \left(\prod_{a\in B\cap\widetilde{B}} \rho^a_{\alpha_a}\right)^{-1} = \prod_{a\in B\cap\widetilde{B}} \left(\sum_{\alpha_a=1}^{d_a} \left(\rho^a_{\alpha_a}\right)^{-1}\right) = \delta^{2\left|B\cap\widetilde{B}\right|}, \qquad (2.54)
$$

which follows from the fact that ψ is a δ-form. As a result,

$$
\begin{aligned}
& m_k \left(A_{k,l} \otimes A_{k,\tilde{l}} \right) m_k^{\dagger} \left(\bigotimes_{a \in A}^{\mathrm{sym}} e^a_{i_a, j_a} \right) \\
&= \frac{(-1)^{l+\tilde{l}}}{k!} \sum_{\substack{C_1 \subset \overline{1,n} \setminus A \\ C_2' \subset A \\ |C_1| = |C_2'| \\ B, \widetilde{B} \subset A \setminus C_2'}} \binom{|B|}{k-l} \binom{\left|\widetilde{B}\right|}{k-\tilde{l}} (-1)^{|B| + \left|\widetilde{B}\right|} \delta^{4k - 2|B| - 2\left|\widetilde{B}\right| + 2\left|B \cap \widetilde{B}\right|} \left(\prod_{a \in A \setminus \left(B \cup \widetilde{B}\right)} \rho^a_{i_a} \delta_{i_a, j_a} \right) \\
&\qquad \bigotimes_{a \in C_1 \cup \left(\left(A \setminus \left(B \cup \widetilde{B} \right) \right) \setminus C_2' \right)}^{\mathrm{sym}} \mathbb{1}^a \overset{\mathrm{sym}}{\otimes} \bigotimes_{a \in B \cup \widetilde{B}}^{\mathrm{sym}} e^a_{i_a, j_a} \\
&= \frac{(-1)^{l+\tilde{l}} \delta^{2k}}{k!} \sum_{\substack{C_1 \subset \overline{1,n} \setminus A \\ C_2' \subset A \\ |C_1| = |C_2'| \\ B, \widetilde{B} \subset A \setminus C_2'}} \binom{|B|}{k-l} \binom{\left|\widetilde{B}\right|}{k-\tilde{l}} (-1)^{\left|B \cup \widetilde{B}\right| + \left|B \cap \widetilde{B}\right|} \delta^{2\left(k - \left|B \cup \widetilde{B}\right|\right)} \left(\prod_{a \in A \setminus \left(B \cup \widetilde{B}\right)} \rho^a_{i_a} \delta_{i_a, j_a} \right) \\
&\qquad \bigotimes_{a \in C_1 \cup \left(\left(A \setminus \left(B \cup \widetilde{B} \right) \right) \setminus C_2' \right)}^{\mathrm{sym}} \mathbb{1}^a \overset{\mathrm{sym}}{\otimes} \bigotimes_{a \in B \cup \widetilde{B}}^{\mathrm{sym}} e^a_{i_a, j_a} \\
&= \frac{(-1)^{l+\tilde{l}} \delta^{2k}}{k!} \sum_{\substack{C_1 \subset \overline{1,n} \setminus A \\ C_2' \subset A \\ |C_1| = |C_2'| \\ D \subset A \setminus C_2'}} \left(\sum_{\substack{B, \widetilde{B} \subset D \\ B \cup \widetilde{B} = D}} \binom{|B|}{k-l} \binom{\left|\widetilde{B}\right|}{k-\tilde{l}} (-1)^{\left|B \cap \widetilde{B}\right|} \right) \\
&\qquad (-1)^{|D|} \delta^{2(k - |D|)} \left(\prod_{a \in A \setminus D} \rho^a_{i_a} \delta_{i_a, j_a} \right) \bigotimes_{a \in C_1 \cup \left((A \setminus D) \setminus C_2' \right)}^{\mathrm{sym}} \mathbb{1}^a \overset{\mathrm{sym}}{\otimes} \bigotimes_{a \in D}^{\mathrm{sym}} e^a_{i_a, j_a}.
\end{aligned} \tag{2.55}
$$

Finally, denoting $|B|$ and $\left|B \cap \widetilde{B}\right|$ by r and $\tilde{r}$, respectively, we can write

$$
\begin{aligned}
& \sum_{\substack{B, \widetilde{B} \subset D \\ B \cup \widetilde{B} = D}} \binom{|B|}{k-l} \binom{\left|\widetilde{B}\right|}{k-\tilde{l}} (-1)^{\left|B \cap \widetilde{B}\right|} = \sum_{r=0}^{|D|} \sum_{\tilde{r}=0}^{r} \binom{|D|}{r} \binom{r}{\tilde{r}} \binom{r}{k-l} \binom{|D| - r + \tilde{r}}{k - \tilde{l}} (-1)^{\tilde{r}} \\
&= \sum_{r=0}^{|D|} \binom{|D|}{r} \binom{r}{k-l} \left(\sum_{\tilde{r}=0}^{r} \binom{r}{\tilde{r}} \binom{|D| - r + \tilde{r}}{k - \tilde{l}} (-1)^{\tilde{r}} \right) = \sum_{r=0}^{|D|} \binom{|D|}{r} \binom{r}{k-l} \binom{|D| - r}{k - \tilde{l} - r} (-1)^r \\
&= \sum_{r=0}^{|D|} \binom{|D|}{k-\tilde{l}} \binom{r}{k-l} \binom{k-\tilde{l}}{r} (-1)^r = \binom{|D|}{k-\tilde{l}} \sum_{r=0}^{|D|} \binom{r}{k-l} \binom{k-\tilde{l}}{r} (-1)^r \\
&= \binom{|D|}{k-\tilde{l}} \sum_{r=0}^{|D|} \binom{k-\tilde{l}}{k-l} \binom{l-\tilde{l}}{r-k+l} (-1)^r = \binom{|D|}{k-\tilde{l}} \binom{k-\tilde{l}}{k-l} \sum_{r=0}^{|D|} \binom{l-\tilde{l}}{r-k+l} (-1)^r \\
&= \binom{|D|}{k-\tilde{l}} \binom{k-\tilde{l}}{k-l} (-1)^{k-l} \sum_{r=k-l}^{k-\tilde{l}} \binom{l-\tilde{l}}{r-k+l} (-1)^{r-k+l} 1^{k-r-\tilde{l}} \\
&= \binom{|D|}{k-\tilde{l}} \binom{k-\tilde{l}}{k-l} (-1)^{k-l} (1-1)^{l-\tilde{l}} = \binom{|D|}{k-l} (-1)^{k-l} \delta_{l,\tilde{l}}.
\end{aligned} \tag{2.56}
$$

Therefore

$$
\begin{aligned}
&m_k\left(A_{k,l}\otimes A_{k,\tilde{l}}\right)m_k^{\dagger}\left(\bigotimes_{a\in A}^{\mathrm{sym}} e^a_{i_a,j_a}\right)\\
&\quad=\frac{(-1)^{l+\tilde{l}}\,\delta^{2k}}{k!}\sum_{\substack{C_1\subset\overline{1,n}\setminus A\\ C_2'\subset A\\ |C_1|=|C_2'|\\ D\subset A\setminus C_2'}}\binom{|D|}{k-l}(-1)^{k-l}\,\delta_{l,\tilde{l}}\,(-1)^{|D|}\,\delta^{2(k-|D|)}\left(\prod_{a\in A\setminus D}\rho^a_{i_a}\delta_{i_a,j_a}\right)\\
&\qquad\qquad\bigotimes_{a\in C_1\cup\left((A\setminus D)\setminus C_2'\right)}^{\mathrm{sym}}\mathbb{1}^a\overset{\mathrm{sym}}{\otimes}\bigotimes_{a\in D}^{\mathrm{sym}}e^a_{i_a,j_a}\\
&\quad=\delta_{l,\tilde{l}}\frac{\delta^{2k}}{k!}(-1)^{k-l}\sum_{\substack{C_1\subset\overline{1,n}\setminus A\\ C_2'\subset A\\ |C_1|=|C_2'|\\ D\subset A\setminus C_2'}}\binom{|D|}{k-l}(-1)^{|D|}\,\delta^{2(k-|D|)}\left(\prod_{a\in A\setminus D}\rho^a_{i_a}\delta_{i_a,j_a}\right)\\
&\qquad\qquad\bigotimes_{a\in C_1\cup\left((A\setminus D)\setminus C_2'\right)}^{\mathrm{sym}}\mathbb{1}^a\overset{\mathrm{sym}}{\otimes}\bigotimes_{a\in D}^{\mathrm{sym}}e^a_{i_a,j_a}=\delta_{l,\tilde{l}}\delta_k^2A_{k,l}\left(\bigotimes_{a\in A}^{\mathrm{sym}}e^a_{i_a,j_a}\right).
\end{aligned}
\tag{2.57}
$$

Therefore, we can conclude that $m_k\left(A_{k,l}\otimes A_{k,\tilde{l}}\right)m_k^{\dagger}=\delta_{l,\tilde{l}}\delta_k^2A_{k,l}$. ☐

We can now easily prove that the operators A_k are valid adjacency operators of quantum graphs.

Proposition 2.6. *The triple $\mathcal{K}(\mathbb{X},k)=\left(\mathrm{C}\left(\binom{\mathbb{X}}{k}\right),\psi_k,A_k\right)$ is a quantum graph.*

Proof. To show that $\left(\mathrm{C}\left(\binom{\mathbb{X}}{k}\right),\psi_k,A_k\right)$ is a quantum graph, we need to verify that A_k is a quantum adjacency matrix — in other words, it satisfies the following 2 conditions:

$$\forall x\in\mathrm{C}\left(\binom{\mathbb{X}}{k}\right):A_k\left(x^*\right)=A_k\left(x\right)^*,\tag{2.58}$$

$$m_k\left(A_k\otimes A_k\right)m_k^{\dagger}=\delta_k^2A_k,\tag{2.59}$$

where m_k represents the multiplication on $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ (understood as a linear map from $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)\otimes\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ to $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$). The first of these conditions is obviously satisfied, while the second one follows immediately from Lemma 2.5. ☐

We may now finally define quantum Kneser graphs.

Definition 2.7. A quantum Kneser graph is any quantum graph of the form $\mathcal{K}(\mathbb{X},k)=\left(\mathrm{C}\left(\binom{\mathbb{X}}{k}\right),\psi_k,A_k\right)$, defined as above.

The justification of that name is provided by Proposition 2.12.

As a digression, we note that it is possible to introduce a slightly more general class of quantum graphs. For $S\subset\overline{0,k}$, we define operators $A_{k,S}$ as

$$A_{k,S}=\sum_{l\in k-S}A_{k,l},\tag{2.60}$$

where $k-S=\{k-s|s\in S\}$. It is an immediate consequence of Lemma 2.5 that those operators satisfy $m_k\left(A_{k,S}\otimes A_{k,S}\right)m_k^{\dagger}=\delta_k^2A_{k,S}$, therefore they too define quantum graphs on the quantum set $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$. Those can be seen as generalized quantum Kneser graphs, which we also justify in Proposition 2.12. For now, we provide a simple example, obtained by taking $S=\{k-1\}$, which, in the classical case, leads to Johnson graphs.

Definition 2.8. A quantum Johnson graph is any quantum graph of the form $\mathcal{J}(\mathbb{X},k) = \left(\mathrm{C}\left(\binom{\mathbb{X}}{k}\right), \psi_k, A\right)$, where

$$A = A_{k,\{k-1\}} = \delta^2 \iota_k^\dagger \left(\sum_{l=1}^{k} \left(\psi^\dagger \psi\right)_l\right) \iota_k - k\,\mathrm{id}. \tag{2.61}$$

Having defined quantum Kneser graphs, we can now proceed to an investigation of their properties. The following 2 propositions show that these graphs are irreflexive, undirected and, under certain additional conditions, regular.

Proposition 2.9. *Quantum Kneser graphs are undirected and irreflexive.*

Proof. It is obvious from definition that A_k is a self-adjoint operator, so $\mathcal{K}(\mathbb{X},k)$ is undirected. Their irreflexivity, on the other hand, follows immediately from Lemma 2.5 due to the fact that $A_{k,0} = \mathrm{id}$. $\square$

Proposition 2.10. *Let* $\mathrm{C}(\mathbb{X}) = \bigotimes_{a=1}^{n} \mathsf{Mat}_d(\mathbb{C})$ *and let* ψ *be tracial. Then* $\mathcal{K}(\mathbb{X},k)$ *is regular of degree*

$$\sum_{l=0}^{k} \binom{k}{l} \binom{n-l}{k-l} (-1)^l d^{2(k-l)} \tag{2.62}$$

for every $k \in \overline{1,n}$.

Proof. To prove that $\mathcal{K}(\mathbb{X},k)$ is regular, we have to show that the unit of $\binom{\mathbb{X}}{k}$ is an eigenvector of A_k. We calculate

$$\begin{aligned} A_k(\mathbb{1}) &= \sum_{\substack{A\subset\overline{1,n}\\|A|=k}} A_k\left(\bigotimes_{a\in A}^{\mathrm{sym}} \mathbb{1}^a\right) \\ &= \sum_{\substack{A\subset\overline{1,n}\\|A|=k}} \sum_{B\subset A} (-1)^{|B|} \delta^{2(k-|B|)} \left(\prod_{a\in A\setminus B} \psi(\mathbb{1}^a)\right) \sum_{\substack{C\subset\overline{1,n}\setminus B\\|C|=k-|B|}} \bigotimes_{a\in C}^{\mathrm{sym}} \mathbb{1}^a \overset{\mathrm{sym}}{\otimes} \bigotimes_{a\in B}^{\mathrm{sym}} \mathbb{1}^a. \end{aligned} \tag{2.63}$$

Since $\psi(\mathbb{1}^a) = \frac{d^2}{\delta^2}$ and $|A\setminus B| = k - |B|$, we can rewrite this as

$$\sum_{\substack{A\subset\overline{1,n}\\|A|=k}} \sum_{B\subset A} (-1)^{|B|} d^{2(k-|B|)} \sum_{\substack{C\subset\overline{1,n}\setminus B\\|C|=k-|B|}} \bigotimes_{a\in B\cup C}^{\mathrm{sym}} \mathbb{1}^a. \tag{2.64}$$

Denoting $B\cup C$ as D lets us rewrite it further as

$$\sum_{\substack{A\subset\overline{1,n}\\|A|=k}} \sum_{B\subset A} (-1)^{|B|} d^{2(k-|B|)} \sum_{\substack{D\subset\overline{1,n}\\D\supset B\\|D|=k}} \bigotimes_{a\in D}^{\mathrm{sym}} \mathbb{1}^a. \tag{2.65}$$

Exchanging the order of summations leads us to

$$\sum_{\substack{D\subset\overline{1,n}\\|D|=k}} \sum_{B\subset D} \sum_{\substack{A\subset\overline{1,n}\\A\supset B\\|A|=k}} (-1)^{|B|} d^{2(k-|B|)} \bigotimes_{a\in D}^{\mathrm{sym}} \mathbb{1}^a = \sum_{\substack{D\subset\overline{1,n}\\|D|=k}} \sum_{B\subset D} \left(\sum_{\substack{A\subset\overline{1,n}\\A\supset B\\|A|=k}} 1\right) (-1)^{|B|} d^{2(k-|B|)} \bigotimes_{a\in D}^{\mathrm{sym}} \mathbb{1}^a. \tag{2.66}$$

The sum $\sum_{\substack{A\subset\overline{1,n}\\ A\supset B\\ |A|=k}} 1$ is equal to the number of k-element supersets of B contained in $\overline{1,n}$ — in other words, to $\binom{n-|B|}{k-|B|}$. Taking that into account, we obtain

$$\sum_{\substack{D\subset\overline{1,n}\\ |D|=k}} \sum_{B\subset D} \binom{n-|B|}{k-|B|} (-1)^{|B|} d^{2(k-|B|)} \bigotimes_{a\in D}^{\mathrm{sym}} \mathbb{1}^a. \tag{2.67}$$

Replacing summation over sets B with summation over $l=|B|$ yields

$$\sum_{\substack{D\subset\overline{1,n}\\ |D|=k}} \sum_{l=0}^{k} \binom{k}{l}\binom{n-l}{k-l} (-1)^l d^{2(k-l)} \bigotimes_{a\in D}^{\mathrm{sym}} \mathbb{1}^a = \sum_{l=0}^{k} \binom{k}{l}\binom{n-l}{k-l} (-1)^l d^{2(k-l)} \mathbb{1}. \tag{2.68}$$

Therefore, we obtain

$$A_k(\mathbb{1}) = \sum_{l=0}^{k} \binom{k}{l}\binom{n-l}{k-l} (-1)^l d^{2(k-l)} \mathbb{1}, \tag{2.69}$$

which means that $\mathbb{1}$ is an eigenvector of A_k, so the graph is regular. The corresponding eigenvalue is the degree of regularity. ☐

The condition that all algebras B_a are the same is sufficient, but not necessary for regularity. However, graphs that do not satisfy it may fail to be regular, as the following example demonstrates.

Example 2.11. Let $\mathrm{C}(\mathbb{X}) = \mathsf{Mat}_{d_1}(\mathbb{C}) \oplus \mathsf{Mat}_{d_2}(\mathbb{C}) \oplus \mathsf{Mat}_{d_3}(\mathbb{C})$ and let ψ be tracial. Then $\mathcal{K}(\mathbb{X},2)$ is not guaranteed to be regular. Indeed,

$$\begin{aligned} A_2(\mathbb{1}) = A_2\left(\mathbb{1}^1 \overset{\mathrm{sym}}{\otimes} \mathbb{1}^2 + \mathbb{1}^1 \overset{\mathrm{sym}}{\otimes} \mathbb{1}^3 + \mathbb{1}^2 \overset{\mathrm{sym}}{\otimes} \mathbb{1}^3\right) \\ &= \left(d_1^2 d_2^2 + d_1^2 d_3^2 + d_2^2 d_3^2 - d_1^2 - d_2^2 - 2d_3^2 + 1\right) \mathbb{1}^1 \overset{\mathrm{sym}}{\otimes} \mathbb{1}^2 \\ &+ \left(d_1^2 d_2^2 + d_1^2 d_3^2 + d_2^2 d_3^2 - d_1^2 - d_3^2 - 2d_2^2 + 1\right) \mathbb{1}^1 \overset{\mathrm{sym}}{\otimes} \mathbb{1}^3 \\ &+ \left(d_1^2 d_2^2 + d_1^2 d_3^2 + d_2^2 d_3^2 - d_2^2 - d_3^2 - 2d_1^2 + 1\right) \mathbb{1}^2 \overset{\mathrm{sym}}{\otimes} \mathbb{1}^3, \end{aligned} \tag{2.70}$$

which is a only a multiple of $\mathbb{1}^1 \otimes^{\mathrm{sym}} \mathbb{1}^2 + \mathbb{1}^1 \otimes^{\mathrm{sym}} \mathbb{1}^3 + \mathbb{1}^2 \otimes^{\mathrm{sym}} \mathbb{1}^3$ if $d_1 = d_2 = d_3$.

The following proposition justifies the name of "quantum Kneser graphs" by showing that they boil down to classical Kneser graphs in the commutative case.

Proposition 2.12. *If* $\mathrm{C}(\mathbb{X})$ *is commutative, generalized quantum Kneser graphs correspond to generalized classical Kneser graphs.*

Proof. Due to the way $A_{k,S}$ is defined, it is enough to consider cases in which S only has one element, corresponding to the adjacency operator given by $A_{k,l}$.

If $\mathrm{C}(\mathbb{X})$ is commutative, then $\forall a : \mathsf{B}_a = \mathsf{Mat}_1(\mathbb{C}) \simeq \mathbb{C}$. Additionally, $\forall a : \rho^a = \frac{1}{\delta^2}$. Let e^a denote the unit element of B_a and let

$$e^A = \bigotimes_{a\in A}^{\mathrm{sym}} e^a \in \mathrm{C}\left(\binom{\mathbb{X}}{|A|}\right) \tag{2.71}$$

for any $A \subset \overline{1,n}$.

Then we can write

$$A_{k,l}\left(e^{A}\right)=(-1)^{k-l}\sum_{\substack{C_1\subset\overline{1,n}\backslash A\\ C_2'\subset A\\ |C_1|=\left|C_2'\right|\\ B\subset A\backslash C_2'}}\binom{|B|}{k-l}(-1)^{|B|}e^{C_1\cup\left(A\backslash C_2'\right)}$$

$$=(-1)^{k-l}\sum_{\substack{C_1\subset\overline{1,n}\backslash A\\ C_2'\subset A\\ |C_1|=\left|C_2'\right|}}\left(\sum_{B\subset A\backslash C_2'}\binom{|B|}{k-l}(-1)^{|B|}\right)e^{C_1\cup\left(A\backslash C_2'\right)}. \quad (2.72)$$

By replacing summation over the sets B with summation over cardinalities r of B, we can rewrite

$$\sum_{B\subset A\backslash C_2'}\binom{|B|}{k-l}(-1)^{|B|}=\sum_{r=0}^{\left|A\backslash C_2'\right|}\binom{\left|A\backslash C_2'\right|}{r}\binom{r}{k-l}(-1)^{r}$$

$$=\sum_{r=k-l}^{\left|A\backslash C_2'\right|}\binom{\left|A\backslash C_2'\right|}{k-l}\binom{\left|A\backslash C_2'\right|-k+l}{r-k+l}(-1)^{r}$$

$$=\binom{\left|A\backslash C_2'\right|}{k-l}(-1)^{k-l}\sum_{r=0}^{\left|A\backslash C_2'\right|-k+l}\binom{\left|A\backslash C_2'\right|-k+l}{r}(-1)^{r}$$

$$=\binom{\left|A\backslash C_2'\right|}{k-l}(-1)^{k-l}\,\delta_{\left|A\backslash C_2'\right|,k-l}=(-1)^{k-l}\,\delta_{\left|C_2'\right|,l}. \quad (2.73)$$

Therefore

$$A_{k,l}\left(e^{A}\right)=\sum_{\substack{C_1\subset\overline{1,n}\backslash A\\ C_2'\subset A\\ |C_1|=\left|C_2'\right|=l}}e^{C_1\cup\left(A\backslash C_2'\right)}. \quad (2.74)$$

We see that $C_1\cup\left(A\backslash C_2'\right)$ always has $k-l$ common elements with A, which implies that $A_{k,l}$ is the adjacency matrix of a generalized classical Kneser graph with $S=\{k-l\}$. $\square$

3. Colorings of quantum Kneser graphs

Having defined quantum Kneser graphs and investigated their basic properties, we may now focus on their chromatic numbers. For simplicity, we are going to focus only on the tracial case. First, we establish an upper bound on the quantum chromatic numbers by providing an explicit construction of a coloring.

Proposition 3.1. *Let* $\mathrm{C}(\mathbb{X})=\bigoplus_{a=1}^{n}\mathsf{B}_a$, *where* $\mathsf{B}_a=\mathsf{Mat}_{d_a}(\mathbb{C})$ *and* $d_1\leq d_2\leq\ldots\leq d_n$, *and let* ψ *be tracial. Then*

$$\chi_q\left(\mathcal{K}\left(\mathbb{X},k\right)\right)\leq\sum_{a=1}^{n-k+1}d_a^2. \quad (3.1)$$

Proof. For $a\in\overline{1,n-k+1}$ and $p,q\in\overline{1,d_a}$ we define projections $P^a_{p,q}\in\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)\otimes\bigotimes_{a'=1}^{n-k+1}\mathsf{B}_{a'}$ as

$$P^a_{p,q}=\frac{1}{d_a}\sum_{\substack{B\subset\overline{a+1,n}\\ |B|=k-1\\ i,j\in\overline{1,d_a}}}\exp\left(\frac{2\pi\mathrm{i}p\,(i-j)}{d_a}\right)$$

$$\left(e^a_{i,j}\overset{\mathrm{sym}}{\otimes}\overset{\mathrm{sym}}{\bigotimes_{b\in B}}\mathbb{1}^b\right)\otimes\mathbb{1}^1\otimes\ldots\otimes\mathbb{1}^{a-1}\otimes e^a_{i+q,j+q}\otimes\mathbb{1}^{a+1}\otimes\ldots\otimes\mathbb{1}^{n-k+1}. \quad (3.2)$$

These projections are inspired by those given by Brannan, Ganesan and Harris [7] for complete quantum graphs. It is easy to verify through direct computation that they are self-adjoint and $\sum_{a=1}^{n-k+1}\sum_{p,q=1}^{d_a} P_{p,q}^a = \mathbb{1}_{\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)} \otimes \bigotimes_{a'=1}^{n-k+1}\mathbb{1}^{a'}$, so they form a partition of unity. Additionally,

$$
\begin{aligned}
&(A_k \otimes \mathrm{id})\left(P_{p,q}^a\right)\\
&= \frac{1}{d_a}\sum_{\substack{B\subset\overline{a+1,n}\\ |B|=k-1\\ i,j\in\overline{1,d_a}}}\left(\sum_{\substack{C\subset B\\ D\subset\left(\overline{1,n}\setminus(B\cup\{a\})\right)\cup C\\ |D|=|C|}}(-1)^{k-|C|}\exp\left(\frac{2\pi \mathrm{i}p\,(i-j)}{d_a}\right)\left(\prod_{c\in C}d_c^2\right)e_{i,j}^a \overset{\mathrm{sym}}{\otimes}\bigotimes_{b\in(B\setminus C)\cup D}^{\mathrm{sym}}\mathbb{1}^b\right.\\
&\qquad\left.+\sum_{\substack{C\subset B\\ D\subset\left(\overline{1,n}\setminus B\right)\cup C\\ |D|=|C|+1}}(-1)^{k-|C|-1}\exp\left(\frac{2\pi \mathrm{i}p\,(i-j)}{d_a}\right)\left(\prod_{c\in C}d_c^2\right)d_a\delta_{i,j}\bigotimes_{b\in(B\setminus C)\cup D}^{\mathrm{sym}}\mathbb{1}^b\right)\\
&\qquad\otimes\mathbb{1}^1\otimes\ldots\otimes\mathbb{1}^{a-1}\otimes e_{i+q,j+q}^a\otimes\mathbb{1}^{a+1}\otimes\ldots\otimes\mathbb{1}^{n-k+1}.
\end{aligned}\tag{3.3}
$$

To calculate $(A_k \otimes \mathrm{id})\left(P_{p,q}^a\right)P_{p,q}^a$, we notice that

$$
\left(e_{i,j}^a \overset{\mathrm{sym}}{\otimes}\bigotimes_{b\in(B\setminus C)\cup D}^{\mathrm{sym}}\mathbb{1}^b\right)\left(e_{i',j'}^a \overset{\mathrm{sym}}{\otimes}\bigotimes_{b\in B'}^{\mathrm{sym}}\mathbb{1}^b\right)=0 \tag{3.4}
$$

if $(B\setminus C)\cup D \neq B'$, so this product can only be non-zero if $D\subset\overline{a+1,n}$ and $(B\setminus C)\cup D = B'$. At the same time,

$$
\left(\bigotimes_{b\in(B\setminus C)\cup D}^{\mathrm{sym}}\mathbb{1}^b\right)\left(e_{i',j'}^a \overset{\mathrm{sym}}{\otimes}\bigotimes_{b\in B'}^{\mathrm{sym}}\mathbb{1}^b\right)=0 \tag{3.5}
$$

if $a\notin(B\setminus C)\cup D$, so this product can only be non-zero if $a\in D$, which means that $D=\{a\}\cup D'$, where $D'\subset\overline{a+1,n}$ and $(B\setminus C)\cup D' = B'$.

Therefore, we can conclude that

$$
\begin{aligned}
&(A_k \otimes \mathrm{id})\left(P_{p,q}^a\right)P_{p,q}^a\\
&\quad= \frac{1}{d_a^2}\sum_{\substack{B\subset\overline{a+1,n}\\ |B|=k-1\\ i,j\in\overline{1,d_a}}}\sum_{\substack{B'\subset\overline{a+1,n}\\ |B'|=k-1\\ i',j'\in\overline{1,d_a}}}\left(\sum_{\substack{C\subset B\\ D\subset\left(\overline{a+1,n}\setminus B\right)\cup C\\ |D|=|C|}}(-1)^{k-|C|}\left(\prod_{c\in C}d_c^2\right)e_{i,j}^a \overset{\mathrm{sym}}{\otimes}\bigotimes_{b\in(B\setminus C)\cup D}^{\mathrm{sym}}\mathbb{1}^b\right.\\
&+\left.\sum_{\substack{C\subset B\\ D'\subset\left(\overline{a+1,n}\setminus B\right)\cup C\\ |D'|=|C|}}(-1)^{k-|C|-1}\left(\prod_{c\in C}d_c^2\right)d_a\delta_{i,j}\mathbb{1}^a \overset{\mathrm{sym}}{\otimes}\bigotimes_{b\in(B\setminus C)\cup D'}^{\mathrm{sym}}\mathbb{1}^b\right)\exp\left(\frac{2\pi \mathrm{i}p\,(i-j+i'-j')}{d_a}\right)\\
&\qquad\left(e_{i',j'}^a \overset{\mathrm{sym}}{\otimes}\bigotimes_{b\in B'}^{\mathrm{sym}}\mathbb{1}^b\right)\otimes\mathbb{1}^1\otimes\ldots\otimes\mathbb{1}^{a-1}\otimes\left(e_{i+q,j+q}^a e_{i'+q,j'+q}^a\right)\otimes\mathbb{1}^{a+1}\otimes\ldots\otimes\mathbb{1}^{n-k+1}.
\end{aligned}\tag{3.6}
$$

We note that

$$e^a_{i+q,j+q}e^a_{i'+q,j'+q} = \delta_{j,i'}e^a_{i+q,j'+q}, \tag{3.7}$$

$$\left(e^a_{i,j} \overset{\text{sym}}{\otimes} \bigotimes_{b\in(B\setminus C)\cup D}^{\text{sym}} \mathbb{1}^b\right)\left(e^a_{i',j'} \overset{\text{sym}}{\otimes} \bigotimes_{b\in B'}^{\text{sym}} \mathbb{1}^b\right) = \delta_{(B\setminus C)\cup D,B'}\delta_{j,i'}e^a_{i,j'} \overset{\text{sym}}{\otimes} \bigotimes_{b\in(B\setminus C)\cup D}^{\text{sym}} \mathbb{1}^b, \tag{3.8}$$

$$\left(\mathbb{1}^a \overset{\text{sym}}{\otimes} \bigotimes_{b\in(B\setminus C)\cup D'}^{\text{sym}} \mathbb{1}^b\right)\left(e^a_{i',j'} \overset{\text{sym}}{\otimes} \bigotimes_{b\in B'}^{\text{sym}} \mathbb{1}^b\right) = \delta_{(B\setminus C)\cup D',B'}e^a_{i',j'} \overset{\text{sym}}{\otimes} \bigotimes_{b\in(B\setminus C)\cup D'}^{\text{sym}} \mathbb{1}^b, \tag{3.9}$$

which allows us to conclude that

$$\begin{aligned}
&(A_k \otimes \mathrm{id})\left(P^a_{p,q}\right)P^a_{p,q} \\
&\quad= \frac{1}{d_a^2}\sum_{\substack{B\subset\overline{a+1,n}\\|B|=k-1\\i,j,i',j'\in\overline{1,d_a}}}\ \sum_{\substack{C\subset B\\D\subset\left(\overline{a+1,n}\setminus B\right)\cup C\\|D|=|C|}}\left((-1)^{k-|C|}\left(\prod_{c\in C}d_c^2\right)\delta_{j,i'}e^a_{i,j'} \overset{\text{sym}}{\otimes} \bigotimes_{b\in(B\setminus C)\cup D}^{\text{sym}} \mathbb{1}^b\right. \\
&\qquad\qquad\left. - (-1)^{k-|C|}\left(\prod_{c\in C}d_c^2\right)d_a\delta_{i,j}e^a_{i',j'} \overset{\text{sym}}{\otimes} \bigotimes_{b\in(B\setminus C)\cup D}^{\text{sym}} \mathbb{1}^b\right) \\
&\quad\delta_{j,i'}\exp\left(\frac{2\pi\mathrm{i}p\left(i-j+i'-j'\right)}{d_a}\right)\otimes\mathbb{1}^1\otimes\ldots\otimes\mathbb{1}^{a-1}\otimes e^a_{i+q,j'+q}\otimes\mathbb{1}^{a+1}\otimes\ldots\otimes\mathbb{1}^{n-k+1} \\
&\quad= \frac{1}{d_a^2}\sum_{\substack{B\subset\overline{a+1,n}\\|B|=k-1\\i,j,j'\in\overline{1,d_a}}}\ \sum_{\substack{C\subset B\\D\subset\left(\overline{a+1,n}\setminus B\right)\cup C\\|D|=|C|}}(-1)^{k-|C|}\left(\prod_{c\in C}d_c^2\right)\exp\left(\frac{2\pi\mathrm{i}p\left(i-j'\right)}{d_a}\right) \\
&\qquad\left(e^a_{i,j'} \overset{\text{sym}}{\otimes} \bigotimes_{b\in(B\setminus C)\cup D}^{\text{sym}} \mathbb{1}^b - d_a\delta_{i,j}e^a_{j,j'} \overset{\text{sym}}{\otimes} \bigotimes_{b\in(B\setminus C)\cup D}^{\text{sym}} \mathbb{1}^b\right) \\
&\qquad\otimes\mathbb{1}^1\otimes\ldots\otimes\mathbb{1}^{a-1}\otimes e^a_{i+q,j'+q}\otimes\mathbb{1}^{a+1}\otimes\ldots\otimes\mathbb{1}^{n-k+1} \\
&\quad= \frac{1}{d_a^2}\sum_{\substack{B\subset\overline{a+1,n}\\|B|=k-1\\i,j'\in\overline{1,d_a}}}\ \sum_{\substack{C\subset B\\D\subset\left(\overline{a+1,n}\setminus B\right)\cup C\\|D|=|C|}}(-1)^{k-|C|}\left(\prod_{c\in C}d_c^2\right)\exp\left(\frac{2\pi\mathrm{i}p\left(i-j'\right)}{d_a}\right) \\
&\qquad\left(d_ae^a_{i,j'} \overset{\text{sym}}{\otimes} \bigotimes_{b\in(B\setminus C)\cup D}^{\text{sym}} \mathbb{1}^b - d_ae^a_{i,j'} \overset{\text{sym}}{\otimes} \bigotimes_{b\in(B\setminus C)\cup D}^{\text{sym}} \mathbb{1}^b\right) \\
&\qquad\otimes\mathbb{1}^1\otimes\ldots\otimes\mathbb{1}^{a-1}\otimes e^a_{i+q,j'+q}\otimes\mathbb{1}^{a+1}\otimes\ldots\otimes\mathbb{1}^{n-k+1} = 0.
\end{aligned} \tag{3.10}$$

Therefore $(A_k \otimes \mathrm{id})\left(P^a_{p,q}\right)P^a_{p,q} = 0$, so the family of projections $P^a_{p,q}$ is a valid quantum coloring consisting of $\sum_{a=1}^{n-k+1} d_a^2$ colors, which means that $\chi_q\left(\mathcal{K}\left(\mathbb{X},k\right)\right) \leq \sum_{a=1}^{n-k+1} d_a^2$. $\square$

Proposition 3.2. *The upper bound for $\chi_q\left(\mathcal{K}\left(\mathbb{X},k\right)\right)$ given by Proposition 3.1 is reached for $k = 1$ and for $k = n$.*

Proof. If $k = 1$, then $\mathrm{C}\left(\binom{\mathbb{X}}{1}\right) \simeq \mathrm{C}\left(\mathbb{X}\right)$ and $A_1 = \delta^2\psi^\dagger\psi - \mathrm{id}$, so $\mathcal{K}\left(\mathbb{X},1\right)$ is simply the complete graph on the quantum set $\mathbb{X}$. The chromatic number of such a graph is well known, thanks to explicit colorings found by Brannan, Ganesan and Harris [7], as well as upper bounds given by Ganesan [11], to be $\dim\left(\mathrm{C}\left(\mathbb{X}\right)\right) = \sum_{a=1}^{n} d_a^2$, which coincides with our upper bound.

If, on the other hand, $k = n$, then

$$\mathrm{C}\left(\binom{\mathbb{X}}{n}\right) = \bigotimes_{a\in\overline{1,n}}^{\text{sym}} \mathsf{B}_a \simeq \bigotimes_{a=1}^{n} \mathsf{B}_a, \tag{3.11}$$

while the action of adjacency operator on basis elements is given by

$$A_n\left(\bigotimes_{a\in\overline{1,n}}^{\mathrm{sym}} e^a_{i_a,j_a}\right)=\sum_{B\in\overline{1,n}}(-1)^{n-|B|}\,\delta^{2|B|}\left(\prod_{a\in B}d_a\delta_{i_a,j_a}\right)\bigotimes_{a\in B}^{\mathrm{sym}}\mathbb{1}^a\overset{\mathrm{sym}}{\otimes}\bigotimes_{a\in\overline{1,n}\setminus B}^{\mathrm{sym}} e^a_{i_a,j_a}$$
$$=\bigotimes_{a\in\overline{1,n}}^{\mathrm{sym}}\left(\delta^2 d_a\delta_{i_a,j_a}\mathbb{1}^a-e^a_{i_a,j_a}\right)=\bigotimes_{a\in\overline{1,n}}^{\mathrm{sym}}\left(\delta^2\psi_a^\dagger\psi_a-\mathrm{id}\right)\left(e^a_{i_a,j_a}\right). \quad (3.12)$$

We may therefore identify $\mathcal{K}(\mathbb{X},n)$ as a tensor product of complete graphs on B_a. If $d_1\le d_2\le\ldots\le d_n$, then the biggest and smallest eigenvalues of A_n are, respectively, equal to $\prod_{a=1}^n\left(d_a^2-1\right)$ and $-\prod_{a=2}^n\left(d_a^2-1\right)$, so the spectral lower bound for the quantum chromatic number, given by Theorem 1.13, is equal to d_1^2 and, once again, coincides with our upper bound. □

4. Spectra of quantum Kneser graphs

Proposition 3.1 gives us upper bounds on quantum chromatic numbers of quantum Kneser graphs. Lower bounds on them can be calculated using spectral properties of the quantum adjacency operators A_k. To do so, the following lemma is useful.

Lemma 4.1. *For every $a\in\overline{1,n}$ let $\{x^a_i\}_{i=1}^{d_a^2-1}$ be a basis of $\ker\psi_a\subset\mathsf{B}_a$. Let $A\subset\overline{1,n}$ satisfy $|A|\in\overline{0,k}$ and let $i_a\in\overline{1,d_a^2-1}$ for each $a\in A$. Then*

$$\operatorname{span}\left\{\bigotimes_{a\in A}^{\mathrm{sym}}x^a_{i_a}\overset{\mathrm{sym}}{\otimes}\bigotimes_{b\in B}^{\mathrm{sym}}\mathbb{1}^b\,\middle|\,B\subset\overline{1,n}\setminus A,|B|=k-|A|\right\} \quad (4.1)$$

is an invariant subspace for A_k. Furthermore, $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ can be decomposed as a direct sum of subspaces of that form.

Proof. For a given $A\subset\overline{1,n}$, let $B\subset\overline{1,n}\setminus A$ and $|B|=k-|A|$. Then

$$A_k\left(\bigotimes_{a\in A}^{\mathrm{sym}}x^a_{i_a}\overset{\mathrm{sym}}{\otimes}\bigotimes_{b\in B}^{\mathrm{sym}}\mathbb{1}^b\right)=\sum_{\substack{C_1\subset A\\ C_2\subset B}}(-1)^{k-|C_1|-|C_2|}\,\delta^{2(|C_1|+|C_2|)}\left(\prod_{c\in C_1}\psi\left(x^c_{i_c}\right)\right)\left(\prod_{c\in C_2}\psi\left(\mathbb{1}^c\right)\right)$$
$$\sum_{D\subset\left(\overline{1,n}\setminus(A\cup B)\right)\cup C_1\cup C_2}\ \bigotimes_{c\in A\setminus C_1}^{\mathrm{sym}}x^c_{i_c}\overset{\mathrm{sym}}{\otimes}\bigotimes_{c\in(B\setminus C_2)\cup D}^{\mathrm{sym}}\mathbb{1}^c. \quad (4.2)$$

Since $x^c_{i_c}\in\ker\psi$ for every $c\in C_1\subset A$, the product $\prod_{c\in C_1}\psi\left(x^c_{i_c}\right)$ is equal to 0 if C_1 is non-empty and 1 otherwise. Therefore we may write

$$A_k\left(\bigotimes_{a\in A}^{\mathrm{sym}}x^a_{i_a}\overset{\mathrm{sym}}{\otimes}\bigotimes_{b\in B}^{\mathrm{sym}}\mathbb{1}^b\right)=\sum_{\substack{C\subset B\\ D\subset\left(\overline{1,n}\setminus(A\cup B)\right)\cup C}}(-1)^{k-|C|}\,\delta^{2|C|}\left(\prod_{c\in C}\psi\left(\mathbb{1}^c\right)\right)\bigotimes_{c\in A}^{\mathrm{sym}}x^c_{i_c}\overset{\mathrm{sym}}{\otimes}\bigotimes_{c\in(B\setminus C)\cup D}^{\mathrm{sym}}\mathbb{1}^c. \quad (4.3)$$

Since $C\subset B\subset\overline{1,n}\setminus A$ and $D\subset\left(\overline{1,n}\setminus\left(A\cup B\right)\right)\cup C\subset\overline{1,n}\setminus A$, the obtained expression is a linear combination of elements of the form ${\bigotimes^{\mathrm{sym}}}_{a\in A}\,x^a_{i_a}\otimes^{\mathrm{sym}}{\bigotimes^{\mathrm{sym}}}_{b\in B'}\,\mathbb{1}^b$ for some B'. Therefore, the subspace spanned by elements of that form is invariant.

Since these subspaces obviously have trivial intersections, to prove that $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ decomposes (as a vector space) as their direct sum, it suffices to show that the sum of their dimensions is equal to $\dim\left(\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)\right)$. This sum is given by

$$\sum_{\substack{A\subset\overline{1,n}\\|A|\leq k}}\left(\prod_{a\in A}\left(d_a^2-1\right)\right)\binom{n-|A|}{k-|A|}=\sum_{\substack{A\subset\overline{1,n}\\|A|\leq k}}\sum_{B\subset A}(-1)^{|A|-|B|}\binom{n-|A|}{k-|A|}\prod_{a\in B}d_a^2$$

$$=\sum_{\substack{B\subset\overline{1,n}\\|B|\leq k}}\left(\sum_{\substack{A\subset\overline{1,n}\\A\supset B\\|A|\leq k}}\binom{n-|A|}{k-|A|}(-1)^{|A|-|B|}\right)\prod_{a\in B}d_a^2. \tag{4.4}$$

We now calculate the middle sum by changing summation over sets A into summation over $l=|A|$, obtaining

$$\sum_{\substack{A\subset\overline{1,n}\\A\supset B\\|A|\leq k}}\binom{n-|A|}{k-|A|}(-1)^{|A|-|B|}=\sum_{l=|B|}^{k}\binom{n-|B|}{l-|B|}\binom{n-l}{k-l}(-1)^{l-|B|}$$

$$=\sum_{l=|B|}^{k}\binom{n-|B|}{n-l}\binom{n-l}{n-k}(-1)^{l-|B|}=\sum_{l=|B|}^{k}\binom{n-|B|}{n-k}\binom{k-|B|}{k-l}(-1)^{l-|B|}$$

$$=\binom{n-|B|}{n-k}\sum_{l=|B|}^{k}\binom{k-|B|}{l-|B|}(-1)^{l-|B|}=\binom{n-|B|}{n-k}(1-1)^{k-|B|}=\delta_{|B|,k}, \tag{4.5}$$

where we utilize properties of the binomial coefficients and, at the end, the binomial formula. As a result, the sum of the invariant subspaces' dimensions is

$$\sum_{\substack{B\subset\overline{1,n}\\|B|\leq k}}\delta_{|B|,k}\prod_{a\in B}d_a^2=\sum_{\substack{B\subset\overline{1,n}\\|B|=k}}\prod_{a\in B}d_a^2=\dim\left(\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)\right), \tag{4.6}$$

which ends the proof. □

This lemma greatly simplifies the search for eigenvalues of A_k, as it allows us to split $\mathrm{C}\left(\binom{\mathbb{X}}{k}\right)$ into A_k-invariant subspaces and calculate the eigenvalues of restrictions of A_k to each of them separately. This method was used to find the eigenvalues in the following, simple examples. In all of them $\mathrm{C}(\mathbb{X})=\bigoplus_{a=1}^{n}\mathsf{Mat}_d(\mathbb{C})$ for some n and d, while ψ is tracial. In these examples, we utilize results obtained by Ganesan [11] to calculate lower bounds for quantum chromatic numbers.

Example 4.2. Let $n=3$ and $k=2$. The eigenvalues of the corresponding quantum Kneser graphs are showcased in Table 1. This spectrum, together with Proposition 3.1, imposes the

Eigenvalue	Multiplicity
$1-2d^2$	$3d^2-3$
$1-d^2$	2
1	$3d^4-3d^2$
$3d^4-4d^2+1$	1

Table 1. Spectrum of a regular, tracial quantum Kneser graph with $n=3$ and $k=2$

following bounds on the quantum chromatic number in this case:

$$\frac{3d^2-2}{2d^2-1}d^2=1+\frac{3d^4-4d^2+1}{2d^2-1}\leq\chi_q\left(\mathcal{K}\left(\mathbb{X},2\right)\right)\leq 2d^2. \tag{4.7}$$

Eigenvalue	Multiplicity
$1-3d^2$	$4d^2-4$
$1-2d^2$	3
1	$6d^4-4d^2$
$6d^4-6d^2+1$	1

TABLE 2. Spectrum of a regular, tracial quantum Kneser graph with $n=4$ and $k=2$

Example 4.3. Let $n=4$ and $k=2$. The eigenvalues of the corresponding quantum Kneser graphs are showcased in Table 2. This spectrum, together with Proposition 3.1, imposes the following bounds on the quantum chromatic number in this case:

$$\frac{6d^2-3}{3d^2-1}d^2 = 1+\frac{6d^4-6d^2+1}{3d^2-1} \leq \chi_q\left(\mathcal{K}\left(\mathbb{X},2\right)\right) \leq 3d^2. \tag{4.8}$$

Example 4.4. Let $n=4$ and $k=3$. The eigenvalues of the corresponding quantum Kneser graphs are showcased in Table 3. This spectrum, together with Proposition 3.1, imposes the

Eigenvalue	Multiplicity
$-3d^4+4d^2-1$	$4d^2-4$
$-d^4+2d^2-1$	3
-1	$4d^6-6d^4+2$
d^2-1	$8d^2-8$
$2d^2-1$	$6d^4-12d^2+6$
$4d^6-9d^4+6d^2-1$	1

TABLE 3. Spectrum of a regular, tracial quantum Kneser graph with $n=4$ and $k=3$

following bounds on the quantum chromatic number in this case:

$$\frac{4d^2-2}{3d^2-1}d^2 = \frac{4d^4-6d^2+2}{3d^4-4d^2+1}d^2 = 1+\frac{4d^6-9d^4+6d^2-1}{3d^4-4d^2+1} \leq \chi_q\left(\mathcal{K}\left(\mathbb{X},3\right)\right) \leq 2d^2. \tag{4.9}$$

It is worth noting that, in Examples 4.2 and 4.4, all eigenvalues or their multiplicities are equal to 0 if $d=1$, as those Kneser graphs have no edges in the classical case. Another pattern we may notice concerns the lower bounds on quantum chromatic numbers in those cases, or specifically their asymptotic behavior for large values of d. It is formulated as the following conjecture.

Conjecture 4.5. Let $\mathrm{C}\left(\mathbb{X}_d\right) = \bigoplus_{a=1}^{n} \mathsf{Mat}_d\left(\mathbb{C}\right)$ with a tracial δ-form ψ. Then

$$\frac{n}{k} \leq \lim_{d\to\infty} \frac{\chi_q\left(\mathcal{K}\left(\mathbb{X}_d,k\right)\right)}{d^2} \leq n-k+1. \tag{4.10}$$

5. Conclusions and results

We have successfully constructed a family of quantum graphs that generalize Kneser graphs beyond the classical setting. We have investigated their properties, focusing primarily on their quantum chromatic numbers, and obtained lower and upper bounds on those numbers. These bounds, in certain cases, coincide with the chromatic numbers themselves. The formulated definition of quantum Kneser graphs also allows for certain generalizations, such as quantum Johnson graphs.

We have also found another, alternative way to construct quantum graphs that, in a slightly different sense, generalize the properties of Kneser graphs to a quantum setting, as outlined in Appendix A. These graphs are defined in a less explicit manner than the construction shown and analyzed in the main part of this thesis, which makes them more difficult to study. Nonetheless, they appear interesting and should warrant further investigation in the future.

Additionally, as part of joint work with Bochniak, Kasprzak and Sołtan outlined in Appendix B, we have defined quantum twin vertices, quantum distinguishing labelings and quantum distinguishing numbers. These notions were then employed in the study of the quantum Mycielski construction [4] and its properties.

There are still certain remaining open questions concerning the topic of our research. The exact values of quantum chromatic numbers of Kneser graphs (both classical and quantum) are still unknown in most cases. Furthermore, quantum automorphism groups of quantum Kneser graphs still need to be investigated. Lastly, the alternative method of quantizing Kneser graphs, outlined in Appendix A, requires further investigation, as their properties (including their chromatic numbers and automorphism groups) are still largely unknown.

Appendix A. An alternative quantization of Kneser graphs

Here, we present an alternative method of obtaining quantum graphs that, in some sense, generalize classical Kneser graphs to a quantum setting. We start with the following observation.

Remark A.1. Let F be a classical communication channel with input set $X = \binom{Y}{k}$ and output set Y, such that $|Y| = n$, given by

$$p\left(y|x\right) = \frac{1}{k}\chi\left(y \in x\right) \tag{A.1}$$

(in other words, it sends k-element subsets of Y to their elements randomly). Then the Kneser graph $K\left(n,k\right)$ is the complement of F's confusability graph.

This fact allows us to define Kneser graphs as complements of confusability graphs of certain classical channels. We shall now show how this can be extended to a quantum setting.

Let $\mathscr{H}$ be an n-dimensional Hilbert space and let $\bigwedge^k \mathscr{H}$ be the antisymmetric tensor product of k copies of $\mathscr{H}$. It can be isometrically embedded in $\mathscr{H}^{\otimes k}$ by

$$\iota : \bigwedge^k \mathscr{H} \ni e_{a_1} \wedge e_{a_2} \wedge \ldots \wedge e_{a_k} \mapsto \frac{1}{\sqrt{k!}} \sum_{\sigma \in S_k} \operatorname{sgn}\left(\sigma\right) e_{a_{\sigma(1)}} \otimes e_{a_{\sigma(2)}} \otimes \ldots \otimes e_{a_{\sigma(k)}} \in \mathscr{H}^{\otimes k}. \tag{A.2}$$

We then define the quantum channel Φ from $\bigwedge^k \mathscr{H}$ to $\mathscr{H}$, given by

$$\Phi : \mathrm{B}\left(\bigwedge^k \mathscr{H}\right) \ni x \mapsto \left(\underbrace{\mathrm{Tr} \otimes \ldots \otimes \mathrm{Tr}}_{k-1} \otimes \mathrm{id}\right)\left(\iota x \iota^\dagger\right) \in \mathrm{B}\left(\mathscr{H}\right). \tag{A.3}$$

We may now define a quantum Kneser graph as the complement of the confusability graph of Φ.

To see that this definition is sensible, we first observe that

$$\Phi\left(\left|\bigwedge_{i=1}^k e_{a_i}\right\rangle\left\langle\bigwedge_{i=1}^k e_{a_i}\right|\right) = \frac{1}{k}\sum_{i=1}^k |e_{a_i}\rangle\langle e_{a_i}|. \tag{A.4}$$

In other words, a restriction of Φ to "classical" states is effectively the classical channel F from Remark A.1, whose confusability graph's complement is a Kneser graph.

To say more about quantum graphs defined in this way, we need to find the Kraus representation of Φ. We start by writing down the isometric embedding ι as

$$\iota = \frac{1}{\sqrt{k!}} \sum_{1 \le a_1 < a_2 < \ldots < a_k \le n} \sum_{\sigma \in S_k} \operatorname{sgn}\left(\sigma\right)\left(\bigotimes_{i=1}^k \left|e_{a_{\sigma(i)}}\right\rangle\right)\left\langle\bigwedge_{i=1}^k e_{a_i}\right|. \tag{A.5}$$

Additionally, we note that

$$\mathrm{Tr}\left(x\right) = \sum_{a=1}^n \langle e_a | x e_a\rangle, \tag{A.6}$$

which lets us conclude that the Kraus operators of Φ are of the form

$$\begin{aligned} K_{a_1,a_2,\ldots,a_k} &= \left(\left(\bigotimes_{i=1}^{k-1}\langle e_{a_i}|\right) \otimes \mathrm{id}\right)\iota \\ &= \frac{1}{\sqrt{k!}} \sum_{1 \le b_1 < b_2 < \ldots < b_k \le n} \sum_{\sigma \in S_k} \operatorname{sgn}\left(\sigma\right)\left(\prod_{i=1}^{k-1}\left\langle e_{a_i} \middle| e_{b_{\sigma(i)}}\right\rangle\right)\left|e_{b_{\sigma(k)}}\right\rangle\left\langle\bigwedge_{i=1}^k e_{b_i}\right|, \end{aligned} \tag{A.7}$$

where $a_i \in \overline{1,n}$. The operator system $\mathcal{S}$ representing the confusability graph of Φ is therefore spanned by operators of the form

$$K^{\dagger}_{a_1,a_2,\ldots,a_k} K_{a'_1,a'_2,\ldots,a'_k} = \frac{1}{k!} \sum_{\substack{1\le b_1<b_2<\ldots<b_k\le n \\ 1\le b'_1<b'_2<\ldots<b'_k\le n \\ \sigma,\rho\in S_k}} \operatorname{sgn}(\sigma\rho) \left(\prod_{i=1}^{k-1} \left\langle e_{b_{\sigma(i)}} \middle| e_{a_i} \right\rangle \left\langle e_{a'_i} \middle| e_{b'_{\rho(i)}} \right\rangle \right) \left\langle e_{b_{\sigma(k)}} \middle| e_{b'_{\rho(k)}} \right\rangle \left| \bigwedge_{i=1}^{k} e_{b_i} \right\rangle \left\langle \bigwedge_{i=1}^{k} e_{b'_i} \right|. \tag{A.8}$$

Let us denote its orthogonal complement in $\mathrm{B}\left(\bigwedge^k \mathscr{H}\right)$ by $\mathcal{V}_q$.

Definition A.2. A quantum Kneser graph $\mathcal{K}_{n,k}$ is the operator subspace $\mathcal{V}_q \subset \mathrm{B}\left(\bigwedge^k \mathscr{H}\right)$, where $\dim(\mathscr{H}) = n$.

Quantum Kneser graphs defined in this way share several properties with their classical counterparts. For example, they are symmetric (as $\mathcal{V}_q^* = \mathcal{V}_q$) and irreflexive (since $\mathcal{V}_q$ is the orthogonal complement of the operator system $\mathcal{S}$, which contains $\mathbb{1}$). To investigate the relation between quantum and classical Kneser graphs in more detail, we denote by $\mathcal{V}_c(\tilde{e})$, where $\{\tilde{e}_a\}_{i=1}^n$ is an orthonormal basis in $\mathscr{H}$, the operator subspace corresponding to a classical Kneser graph, spanned by operators of the form

$$\left| \bigwedge_{i=1}^{k} \tilde{e}_{c_i} \right\rangle \left\langle \bigwedge_{i=1}^{k} \tilde{e}_{c'_i} \right|, \tag{A.9}$$

where $\left\{c_i \middle| i \in \overline{1,k}\right\}$ and $\left\{c'_i \middle| i \in \overline{1,k}\right\}$ are disjoint. The following proposition shows the relation between $\mathcal{V}_q$ and $\mathcal{V}_c(\tilde{e})$.

Proposition A.3. *For any choice of $\tilde{e}$ we have $\mathcal{V}_c(\tilde{e}) \subset \mathcal{V}_q$.*

Proof. Since $\mathcal{V}_q$ does not depend on the choice of basis, it suffices to show the proposition for $\tilde{e} = e$. The Hilbert-Schmidt inner product of $K^{\dagger}_{a_1,a_2,\ldots,a_k} K_{a'_1,a'_2,\ldots,a'_k}$ and $\left|\bigwedge_{i=1}^k e_{c_i}\right\rangle\left\langle\bigwedge_{i=1}^k e_{c'_i}\right|$ is equal to

$$\frac{1}{k!} \sum_{\substack{1\le b_1<b_2<\ldots<b_k\le n \\ 1\le b'_1<b'_2<\ldots<b'_k\le n \\ \sigma,\rho\in S_k}} \operatorname{sgn}(\sigma\rho) \left(\prod_{i=1}^{k-1} \left\langle e_{b_{\sigma(i)}} \middle| e_{a_i} \right\rangle \left\langle e_{a'_i} \middle| e_{b'_{\rho(i)}} \right\rangle \right) \left\langle e_{b_{\sigma(k)}} \middle| e_{b'_{\rho(k)}} \right\rangle$$
$$\left\langle \bigwedge_{i=1}^{k} e_{c_i} \middle| \bigwedge_{i=1}^{k} e_{b_i} \right\rangle \left\langle \bigwedge_{i=1}^{k} e_{b'_i} \middle| \bigwedge_{i=1}^{k} e_{c'_i} \right\rangle. \tag{A.10}$$

We may assume without loss of generality that $1 \le c_1 < c_2 < \ldots < c_k \le n$ and $1 \le c'_1 < c'_2 < \ldots < c'_k \le n$, so $\left\langle \bigwedge_{i=1}^k e_{c_i} \middle| \bigwedge_{i=1}^k e_{b_i} \right\rangle \left\langle \bigwedge_{i=1}^k e_{b'_i} \middle| \bigwedge_{i=1}^k e_{c'_i} \right\rangle = \prod_{i=1}^k \delta_{b_i,c_i}\delta_{b'_i,c'_i}$, the inner product is therefore equal to

$$\frac{1}{k!} \sum_{\sigma,\rho\in S_k} \operatorname{sgn}(\sigma\rho) \left(\prod_{i=1}^{k-1} \left\langle e_{c_{\sigma(i)}} \middle| e_{a_i} \right\rangle \left\langle e_{a'_i} \middle| e_{c'_{\rho(i)}} \right\rangle \right) \left\langle e_{c_{\sigma(k)}} \middle| e_{c'_{\rho(k)}} \right\rangle. \tag{A.11}$$

However, since $\left\{c_i \middle| i \in \overline{1,k}\right\}$ and $\left\{c'_i \middle| i \in \overline{1,k}\right\}$ are disjoint, $\left\langle e_{c_{\sigma(k)}} \middle| e_{c'_{\rho(k)}} \right\rangle = 0$ for any choice of ρ and σ. This means that $\mathcal{V}_c(e) \subset \mathcal{S}^{\perp} = \mathcal{V}_q$, hence also $\mathcal{V}_c(\tilde{e}) \subset \mathcal{V}_q$. □

Since $\mathcal{V}_q$ does not depend on the choice of basis, we can say that

$$\iota^{\dagger} U^{\otimes k} \iota \mathcal{V}_q \iota^{\dagger} \left(U^{\dagger}\right)^{\otimes k} \iota = \mathcal{V}_q \tag{A.12}$$

for any unitary operator $U \in \mathrm{B}(\mathscr{H})$. This leads us to formulate the following conjecture.

Conjecture A.4. $\mathcal{V}_q$ is spanned by all possible operator subspaces corresponding to classical Kneser graphs, that is

$$\mathcal{V}_q = \operatorname{span}\left\{ \iota^\dagger U^{\otimes k} \iota \mathcal{V}_c(\tilde{e}) \iota^\dagger \left(U^\dagger\right)^{\otimes k} \iota \middle| U \in \mathrm{B}(\mathscr{H}), U^\dagger U = \mathrm{id} \right\}. \tag{A.13}$$

The conjecture effectively claims that quantum Kneser graphs (in the sense discussed in this Appendix) are obtained by imposing unitary symmetry on the corresponding classical Kneser graphs.

It may be interesting to analyze a few concrete examples of a quantum Kneser graph in this sense. In the first such example, $k = 1$, which corresponds to a complete graph in the classical case. The second example is the simplest non-trivial case (that is, one that does not boil down to either a complete graph or an empty graph), obtained by taking $n = 4$ and $k = 2$.

Example A.5. If $k = 1$, then $\bigwedge^k \mathscr{H} = \mathscr{H}$ and $\Phi : \mathrm{B}(\mathscr{H}) \to \mathrm{B}(\mathscr{H})$ is the identity channel. This channel has the Kraus decomposition

$$\Phi(x) = \mathbb{1} x \mathbb{1}, \tag{A.14}$$

where $\mathbb{1}$ denotes the identity of $\mathrm{B}(\mathscr{H})$. This means that $\mathcal{S}$ is spanned by $\mathbb{1}$, therefore

$$\mathcal{V}_q = \mathcal{S}^\perp = \operatorname{span}\{\mathbb{1}\}^\perp, \tag{A.15}$$

which corresponds exactly to the irreflexive complete graph.

Example A.6. Let $\mathscr{H}$ be spanned by $\{e_a\}_{a=1}^4$ and denote $e_{a,b} = e_a \wedge a_b \in \bigwedge^2 \mathscr{H}$. By applying Equation (A.8), we see that the operator system $\mathcal{S}$ representing the confusability graph of Φ is spanned by the following operators: $|e_{1,2}\rangle\langle e_{1,2}| + |e_{1,3}\rangle\langle e_{1,3}| + |e_{1,4}\rangle\langle e_{1,4}|$, $|e_{1,2}\rangle\langle e_{1,2}| + |e_{2,3}\rangle\langle e_{2,3}| + |e_{2,4}\rangle\langle e_{2,4}|$, $|e_{1,3}\rangle\langle e_{1,3}| + |e_{2,3}\rangle\langle e_{2,3}| + |e_{3,4}\rangle\langle e_{3,4}|$, $|e_{1,4}\rangle\langle e_{1,4}| + |e_{2,4}\rangle\langle e_{2,4}| + |e_{3,4}\rangle\langle e_{3,4}|$, $|e_{1,3}\rangle\langle e_{2,3}| + |e_{1,4}\rangle\langle e_{2,4}|$, $|e_{2,3}\rangle\langle e_{1,3}| + |e_{2,4}\rangle\langle e_{1,4}|$, $-|e_{1,2}\rangle\langle e_{2,3}| + |e_{1,4}\rangle\langle e_{3,4}|$, $-|e_{2,3}\rangle\langle e_{1,2}| + |e_{3,4}\rangle\langle e_{1,4}|$, $-|e_{1,2}\rangle\langle e_{2,4}| - |e_{1,3}\rangle\langle e_{3,4}|$, $-|e_{2,4}\rangle\langle e_{1,2}| - |e_{3,4}\rangle\langle e_{1,3}|$, $|e_{1,2}\rangle\langle e_{1,3}| + |e_{2,4}\rangle\langle e_{3,4}|$, $|e_{1,3}\rangle\langle e_{1,2}| + |e_{3,4}\rangle\langle e_{2,4}|$, $|e_{1,2}\rangle\langle e_{1,4}| - |e_{2,3}\rangle\langle e_{3,4}|$, $|e_{1,4}\rangle\langle e_{1,2}| - |e_{3,4}\rangle\langle e_{2,3}|$, $|e_{1,3}\rangle\langle e_{1,4}| + |e_{2,3}\rangle\langle e_{2,4}|$ and $|e_{1,4}\rangle\langle e_{1,3}| + |e_{2,4}\rangle\langle e_{2,3}|$. Then $\mathcal{V}_q$, the orthogonal complement of $\mathcal{S}$, has the following orthonormal basis (with respect to the standard Hilbert–Schmidt inner product):

- $f_1 = \frac{1}{2}\left(|e_{1,2}\rangle\langle e_{1,2}| - |e_{1,3}\rangle\langle e_{1,3}| - |e_{2,4}\rangle\langle e_{2,4}| + |e_{3,4}\rangle\langle e_{3,4}|\right)$,
- $f_2 = \frac{1}{2\sqrt{3}}\left(|e_{1,2}\rangle\langle e_{1,2}| + |e_{1,3}\rangle\langle e_{1,3}| - 2|e_{1,4}\rangle\langle e_{1,4}| - 2|e_{2,3}\rangle\langle e_{2,3}| + |e_{2,4}\rangle\langle e_{2,4}| + |e_{3,4}\rangle\langle e_{3,4}|\right)$,
- $f_3 = |e_{1,2}\rangle\langle e_{3,4}|$,
- $f_4 = |e_{3,4}\rangle\langle e_{1,2}|$,
- $f_5 = |e_{1,3}\rangle\langle e_{2,4}|$,
- $f_6 = |e_{2,4}\rangle\langle e_{1,3}|$,
- $f_7 = |e_{1,4}\rangle\langle e_{2,3}|$,
- $f_8 = |e_{2,3}\rangle\langle e_{1,4}|$,
- $f_9 = \frac{1}{\sqrt{2}}\left(|e_{1,3}\rangle\langle e_{2,3}| - |e_{1,4}\rangle\langle e_{2,4}|\right)$,
- $f_{10} = \frac{1}{\sqrt{2}}\left(|e_{2,3}\rangle\langle e_{1,3}| - |e_{2,4}\rangle\langle e_{1,4}|\right)$,
- $f_{11} = \frac{1}{\sqrt{2}}\left(|e_{1,2}\rangle\langle e_{2,3}| + |e_{1,4}\rangle\langle e_{3,4}|\right)$,
- $f_{12} = \frac{1}{\sqrt{2}}\left(|e_{2,3}\rangle\langle e_{1,2}| + |e_{3,4}\rangle\langle e_{1,4}|\right)$,
- $f_{13} = \frac{1}{\sqrt{2}}\left(|e_{1,2}\rangle\langle e_{2,4}| - |e_{1,3}\rangle\langle e_{3,4}|\right)$,
- $f_{14} = \frac{1}{\sqrt{2}}\left(|e_{2,4}\rangle\langle e_{1,2}| - |e_{3,4}\rangle\langle e_{1,3}|\right)$,
- $f_{15} = \frac{1}{\sqrt{2}}\left(|e_{1,2}\rangle\langle e_{1,3}| - |e_{2,4}\rangle\langle e_{3,4}|\right)$,
- $f_{16} = \frac{1}{\sqrt{2}}\left(|e_{1,3}\rangle\langle e_{1,2}| - |e_{3,4}\rangle\langle e_{2,4}|\right)$,
- $f_{17} = \frac{1}{\sqrt{2}}\left(|e_{1,2}\rangle\langle e_{1,4}| + |e_{2,3}\rangle\langle e_{3,4}|\right)$,
- $f_{18} = \frac{1}{\sqrt{2}}\left(|e_{1,4}\rangle\langle e_{1,2}| + |e_{3,4}\rangle\langle e_{2,3}|\right)$,
- $f_{19} = \frac{1}{\sqrt{2}}\left(|e_{1,3}\rangle\langle e_{1,4}| - |e_{2,3}\rangle\langle e_{2,4}|\right)$,
- $f_{20} = \frac{1}{\sqrt{2}}\left(|e_{1,4}\rangle\langle e_{1,3}| - |e_{2,4}\rangle\langle e_{2,3}|\right)$.

The adjacency matrix of this graph (on the quantum set defined by $\mathrm{B}\left(\bigwedge^2 \mathscr{H}\right)$ with the standard, unnormalized trace as δ-form) is then given by

$$A\left(x\right) = 6\sum_{i=1}^{20} f_i x f_i^*. \tag{A.16}$$

We may now calculate that $A\left(\mathbb{1}\right) = 20\mathbb{1}$ (where $\mathbb{1}$ is the unit of $\mathrm{B}\left(\bigwedge^2 \mathscr{H}\right)$), which means that this graph is regular, though its degree — 20 — is much higher than its classical counterpart's, which is equal to 1.

Appendix B. Quantum twin vertices and distinguishing numbers

Here, we present some results obtained as part of a joint work with Arkadiusz Bochniak, Paweł Kasprzak and Piotr Sołtan [3], in which we investigate the properties of the quantum Mycielski construction, defined previously by Bochniak and Kasprzak [4].

Definition B.1. Let $\mathcal{G} = (\mathrm{C}(\mathcal{G}), \psi_{\mathcal{G}}, A_{\mathcal{G}})$ be a quantum graph. Then the Mycielskian of $\mathcal{G}$ of order $r-1$, denoted by $\mu_{r-1}(\mathcal{G}) = \left(\mathrm{C}(\mu_{r-1}(\mathcal{G})), \psi_{\mu_{r-1}(\mathcal{G})}, A_{\mu_{r-1}(\mathcal{G})}\right)$, is defined by:

$$\mathrm{C}(\mu_{r-1}(\mathcal{G})) = \mathbb{C} \oplus \bigoplus_{k=1}^{r} \mathrm{C}(\mathcal{G}), \tag{B.1}$$

$$\psi_{\mu_{r-1}(\mathcal{G})}\left(\begin{bmatrix}\lambda \\ x_1 \\ \vdots \\ x_r\end{bmatrix}\right) = \frac{1}{1+r\delta_{\mathcal{G}}^2}\left(\lambda + \delta_{\mathcal{G}}^2 \sum_{k=1}^{r} \psi(x_k)\right), \tag{B.2}$$

$$A_{\mu_{r-1}(\mathcal{G})}\left(\begin{bmatrix}\lambda \\ x_1 \\ x_2 \\ \vdots \\ x_{r-1} \\ x_r\end{bmatrix}\right) = \begin{bmatrix}\delta_{\mathcal{G}}^2 \psi_{\mathcal{G}}(x_r) \\ A_{\mathcal{G}}(x_1 + x_2) \\ A_{\mathcal{G}}(x_1 + x_3) \\ \vdots \\ A_{\mathcal{G}}(x_{r-2} + x_r) \\ \lambda \mathbb{1} + A_{\mathcal{G}}(x_{r-1})\end{bmatrix}. \tag{B.3}$$

Additionally, we denote $\mu(\mathcal{G}) = \mu_1(\mathcal{G})$.

For classical graphs, this coincides with the classical Mycielski construction, which allows us to obtain graphs with low clique numbers and arbitrarily high chromatic numbers. We also denote by $\iota_0 : \mathbb{C} \to \mathrm{C}(\mu_{r-1}(\mathcal{G}))$ and $\iota_k : \mathrm{C}(\mathcal{G}) \to \mathrm{C}(\mu_{r-1}(\mathcal{G}))$ (for $k \in \overline{1,r}$) the isometric embeddings onto the corresponding direct summands. These allow us to express the adjacency operators of quantum Mycielskians by the following formula:

$$A_{\mu_{r-1}(\mathcal{G})} = \delta_{\mathcal{G}} \iota_r \psi_{\mathcal{G}}^{\dagger} \iota_0^{\dagger} + \delta_{\mathcal{G}} \iota_0 \psi_{\mathcal{G}} \iota_r^{\dagger} + \iota_1 A_{\mathcal{G}} \iota_1^{\dagger} + \sum_{k=1}^{r-1}\left(\iota_k A_{\mathcal{G}} \iota_{k+1}^{\dagger} + \iota_{k+1} A_{\mathcal{G}} \iota_k^{\dagger}\right). \tag{B.4}$$

Additionally, we also define $\eta_{\mathcal{G}} = \psi_{\mathcal{G}}^{\dagger}$ and identify elements of algebras with operators of left multiplication by those elements, allowing us to treat the adjoint $\dagger$ and the involution $*$ interchangeably.

Having defined quantum Mycielskians, we may now proceed to discussing their symmetries, distinguishing numbers and twin vertices. Those notions rely on quantum groups, so to understand them, we first need to recall Definition 1.3. We can now define quantum symmetry groups of quantum graphs.

Definition B.2. The quantum symmetry group of a quantum graph $\mathcal{G} = (\mathrm{C}(\mathcal{G}), \psi, A)$ is the compact quantum group $\mathrm{QAut}(\mathcal{G})$ equipped with a ψ-preserving action

$$\rho : \mathrm{C}(\mathcal{G}) \to \mathrm{C}(\mathcal{G}) \otimes \mathrm{C}(\mathrm{QAut}(\mathcal{G})) \tag{B.5}$$

(which means that $(\psi \otimes \mathrm{id})\rho = \psi(\cdot)\mathbb{1}$) which satisfies $\rho A = (A \otimes \mathrm{id})\rho$ and is universal among all ψ-preserving compact quantum group actions

$$\alpha : \mathrm{C}(\mathcal{G}) \to \mathrm{C}(\mathcal{G}) \otimes \mathrm{C}(\mathbb{G}) \tag{B.6}$$

such that $\alpha A = (A \otimes \mathrm{id})\alpha$.

The definition encodes the fact that the symmetry group of a graph must preserve its structure, given by the adjacency matrix. We can now define the quantum distinguishing number of a quantum graph.

Definition B.3. Let $\mathcal{G} = (\mathrm{C}(\mathcal{G}), \psi, A)$ be a quantum graph and $\mathfrak{P} = \{P_i\}_{i=1}^{c} \subset \mathrm{C}(\mathcal{G}) \otimes \mathsf{M}$ for some C*-algebra M — a partition of unity (called a quantum c-labeling). The quantum group $\mathrm{QAut}(\mathcal{G})_{\mathfrak{P}}$ of automorphisms preserving $\mathfrak{P}$ is a compact quantum group described by a

C*-algebra $\mathrm{C}\left(\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}\right)$ and a quantum group action $\rho : \mathrm{C}\left(\mathcal{G}\right) \to \mathrm{C}\left(\mathcal{G}\right) \otimes \mathrm{C}\left(\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}\right)$ such that

$$\rho A = \left(A \otimes \mathrm{id}\right)\rho, \tag{B.7}$$

$$\left(\psi \otimes \mathrm{id}\right)\rho = \psi\left(\cdot\right)\mathbb{1}, \tag{B.8}$$

$$\forall i \in \overline{1,c} : \left(\rho \otimes \mathrm{id}\right)\left(P_i\right) = \left(\mathrm{id} \otimes \sigma\right)\left(P_i \otimes \mathbb{1}\right), \tag{B.9}$$

which is universal among all quantum group actions satisfying those conditions (here, $\sigma : \mathsf{M} \otimes \mathrm{C}\left(\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}\right) \to \mathrm{C}\left(\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}\right) \otimes \mathsf{M}$ is the tensor flip map).

The quantum distinguishing number of $\mathcal{G}$ is the smallest number c for which there exists a quantum c-labeling $\mathfrak{P}$ such that $\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}$ is trivial.

We may also define quantum twin vertices in the following way.

Definition B.4. Let $\mathcal{G} = \left(\mathrm{C}\left(\mathcal{G}\right), \psi, A\right)$ be a quantum graph and let $\mathbb{G}$ be a compact quantum group. We say that $\mathcal{G}$ has $\mathbb{G}$-twin vertices if there exist 2 different actions

$$\rho_1, \rho_2 : \mathrm{C}\left(\mathcal{G}\right) \to \mathrm{C}\left(\mathcal{G}\right) \otimes \mathrm{C}\left(\mathbb{G}\right) \tag{B.10}$$

such that $\rho_1 A = \rho_2 A$.

We say that $\mathcal{G}$ has quantum twin vertices if the above condition is satisfied for some $\mathbb{G}$. If $\mathbb{G}$ is classical, we say that $\mathcal{G}$ has classical twin vertices.

It is worth noting that, if both $\mathcal{G}$ and $\mathbb{G}$ are classical, this definition coincides with the definition of twin vertices in classical graph theory (namely, vertices that have the same sets of neighboring vertices).

Additionally, we define the quantum isomorphism of quantum graphs in the following way, given by Brannan [6].

Definition B.5. Quantum graphs $\mathcal{G}$ and $\mathcal{F}$ are quantum isomorphic if there exists a non-zero $*$-algebra $\mathcal{O}\left(\mathrm{QAut}\left(\mathcal{F}\right), \mathrm{QAut}\left(\mathcal{G}\right)\right)$ generated by elements $\{p_{ij}\}$ of a unitary matrix subject to relations making the map

$$\rho_{\mathcal{F},\mathcal{G}} : e_j \mapsto \sum_i f_i \otimes p_{ij} \tag{B.11}$$

(where $\{e_i\}$ and $\{f_i\}$ are orthonormal bases of $\mathrm{C}\left(\mathcal{G}\right)$ and $\mathrm{C}\left(\mathcal{F}\right)$, respectively) a unital $*$-homomorphism satisfying the condition $\rho_{\mathcal{F},\mathcal{G}} A_{\mathcal{G}} = \left(A_{\mathcal{F}} \otimes \mathrm{id}\right)\rho_{\mathcal{F},\mathcal{G}}$.

We can now give several results we obtained about the quantum Mycielskians.

Proposition B.6. *Let $\mathcal{G}$ and $\mathcal{F}$ be quantum isomorphic. Then $\mu_{r-1}(\mathcal{G})$ and $\mu_{r-1}(\mathcal{F})$ are quantum isomorphic for any $r \geq 1$.*

Proof. Let

$$p = \left[p_{ij}\right]_{i,j} \in \mathrm{B}\left(L^2\left(\mathcal{G}\right), L^2\left(\mathcal{F}\right)\right) \otimes \mathcal{O}\left(\mathrm{QAut}\left(\mathcal{F}\right), \mathrm{QAut}\left(\mathcal{G}\right)\right) \tag{B.12}$$

be the unitary matrix defining $\mathcal{O}\left(\mathrm{QAut}\left(\mathcal{F}\right), \mathrm{QAut}\left(\mathcal{G}\right)\right)$ as above (remember the latter is non-zero) and let

$$\rho_{\mathcal{F},\mathcal{G}} \colon \mathrm{C}\left(\mathcal{G}\right) \ni e_j \mapsto \sum_i f_i \otimes p_{ij} \in \mathrm{C}\left(\mathcal{F}\right) \otimes \mathcal{O}\left(\mathrm{QAut}\left(\mathcal{F}\right), \mathrm{QAut}\left(\mathcal{G}\right)\right) \tag{B.13}$$

be the corresponding unital $*$-homomorphism intertwining the quantum adjacency matrices. Since $\mathcal{G}$ and $\mathcal{F}$ are quantum isomorphic, their quantum automorphism groups are monoidally equivalent, so, from [9], $\delta_{\mathcal{G}} = \delta_{\mathcal{F}}$. For the remainder of the proof we will denote this value by δ. The maps introduced above can be represented as

$$p\left(\xi \otimes a\right) = \sum_{i,j} |f_i\rangle\langle e_j|\xi\rangle \otimes p_{ij} a \tag{B.14}$$

and

$$\rho_{\mathcal{F},\mathcal{G}}\left(\xi\right) = p\left(\xi \otimes \mathbb{1}_{\mathcal{O}\left(\mathrm{QAut}\left(\mathcal{F}\right),\mathrm{QAut}\left(\mathcal{G}\right)\right)}\right). \tag{B.15}$$

We define an element $\widehat{p} \in \mathrm{B}\left(L^2\left(\mu_{r-1}\left(\mathcal{G}\right)\right), L^2\left(\mu_{r-1}\left(\mathcal{F}\right)\right)\right) \otimes \mathcal{O}\left(\mathrm{QAut}\left(\mathcal{F}\right), \mathrm{QAut}\left(\mathcal{G}\right)\right)$:

$$\widehat{p} = \varpi_{\mathcal{F},0}\varpi_{\mathcal{G},0}^* + \sum_{k=1}^{r} \varpi_{\mathcal{F},k} p \varpi_{\mathcal{G},k}^*, \tag{B.16}$$

where $\varpi_{\mathcal{G},j} = \iota_{\mathcal{G},j} \otimes \mathbb{1}_{\mathcal{O}(\mathrm{QAut}(\mathcal{F}),\mathrm{QAut}(\mathcal{G}))}$ and $\varpi_{\mathcal{F},j} = \iota_{\mathcal{F},j} \otimes \mathbb{1}_{\mathcal{O}(\mathrm{QAut}(\mathcal{F}),\mathrm{QAut}(\mathcal{G}))}$, with $\iota_{\mathcal{G},j}$ and $\iota_{\mathcal{F},j}$ (for $j \in \overline{1,r}$) denoting the isometric embeddings of $L^2\left(\mathcal{G}\right)$ and $L^2\left(\mathcal{F}\right)$ into $L^2\left(\mu_{r-1}\left(\mathcal{G}\right)\right)$ and $L^2\left(\mu_{r-1}\left(\mathcal{F}\right)\right)$, respectively.

First, notice that

$$\widehat{p}\widehat{p}^* = \varpi_{\mathcal{F},0}\varpi_{\mathcal{F},0}^* + \sum_{k=1}^{r} \varpi_{\mathcal{F},k} p p^* \varpi_{\mathcal{F},k}^* = \sum_{j=0}^{r} \varpi_{\mathcal{F},j}\varpi_{\mathcal{F},j}^* = \mathrm{id}_{L^2(\mu_{r-1}(\mathcal{F}))}, \tag{B.17}$$

$$\widehat{p}^*\widehat{p} = \varpi_{\mathcal{G},0}\varpi_{\mathcal{G},0}^* + \sum_{k=1}^{r} \varpi_{\mathcal{G},k} p p^* \varpi_{\mathcal{G},k}^* = \sum_{j=0}^{r} \varpi_{\mathcal{G},j}\varpi_{\mathcal{G},j}^* = \mathrm{id}_{L^2(\mu_{r-1}(\mathcal{G}))}, \tag{B.18}$$

so that $\widehat{p}$ is unitary.

Next, we check the intertwining property. First, we compute

$$\begin{aligned}
\widehat{p}A_{\mu_{r-1}(\mathcal{G})} = &\left(\varpi_{\mathcal{F},0}\varpi_{\mathcal{G},0}^* + \sum_{k=1}^{r} \varpi_{\mathcal{F},k} p \varpi_{\mathcal{G},k}^*\right) \\
&\left(\delta\varpi_{\mathcal{G},r}\eta_{\mathcal{G}}\varpi_{\mathcal{G},0}^* + \delta\varpi_{\mathcal{G},0}\eta_{\mathcal{G}}^*\varpi_{\mathcal{G},r}^* + \varpi_{\mathcal{G},1}A_{\mathcal{G}}\varpi_{\mathcal{G},1}^* + \sum_{k=1}^{r-1}\left(\varpi_{\mathcal{G},k}A_{\mathcal{G}}\varpi_{\mathcal{G},k+1}^* + \varpi_{\mathcal{G},k+1}A_{\mathcal{G}}\varpi_{\mathcal{G},k}^*\right)\right) \\
&= \delta\varpi_{\mathcal{F},0}\eta_{\mathcal{G}}^*\varpi_{\mathcal{G},r}^* + \delta\varpi_{\mathcal{F},r}p\eta_{\mathcal{G}}\varpi_{\mathcal{G},0}^* + \varpi_{\mathcal{F},1}pA_{\mathcal{G}}\varpi_{\mathcal{G},1}^* \\
&\quad + \sum_{j=1}^{r-1}\left(\varpi_{\mathcal{F},j}pA_{\mathcal{G}}\varpi_{\mathcal{G},j+1}^* + \varpi_{\mathcal{F},j+1}pA_{\mathcal{G}}\varpi_{\mathcal{G},j}^*\right),
\end{aligned} \tag{B.19}$$

and similarly,

$$\begin{aligned}
A_{\mu_{r-1}(\mathcal{F})}\widehat{p} = &\ \delta\varpi_{\mathcal{F},0}\eta_{\mathcal{F}}^*p\varpi_{\mathcal{G},r}^* + \delta\varpi_{\mathcal{F},r}\eta_{\mathcal{F}}\varpi_{\mathcal{G},0}^* + \varpi_{\mathcal{F},1}A_{\mathcal{F}}p\varpi_{\mathcal{G},1}^* \\
&+ \sum_{j=1}^{r-1}\left(\varpi_{\mathcal{F},j}A_{\mathcal{F}}p\varpi_{\mathcal{G},j+1}^* + \varpi_{\mathcal{F},j+1}A_{\mathcal{F}}p\varpi_{\mathcal{G},j}^*\right).
\end{aligned} \tag{B.20}$$

Since $p\eta_{\mathcal{G}} = \eta_{\mathcal{F}}$, we also have $\eta_{\mathcal{G}}^* = \eta_{\mathcal{F}}^*p$. This shows that indeed the intertwining property holds. Finally, the map

$$\rho_{\mu_{r-1}(\mathcal{F}),\mu_{r-1}(\mathcal{G})}\left(\begin{bmatrix}\lambda\\ \xi_1\\ \vdots\\ \xi_r\end{bmatrix}\right) = \widehat{p}\left(\begin{bmatrix}\lambda\\ \xi_1\\ \vdots\\ \xi_r\end{bmatrix} \otimes \mathbb{1}_{\mathcal{O}(\mathrm{QAut}(\mathcal{F}),\mathrm{QAut}(\mathcal{G}))}\right) \tag{B.21}$$

is a unital $*$-homomorphism $\mathrm{C}\left(\mu_{r-1}\left(\mathcal{G}\right)\right) \to \mathrm{C}\left(\mu_{r-1}\mathcal{F}\right) \otimes \mathcal{O}\left(\mathrm{QAut}\left(\mathcal{F}\right), \mathrm{QAut}\left(\mathcal{G}\right)\right)$ by construction. Since the algebra $\mathcal{O}\left(\mathrm{QAut}\left(\mu_{r-1}\left(\mathcal{F}\right)\right), \mathrm{QAut}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)\right)$ is universal for such homomorphisms, it has a non-zero homomorphic image, so it is itself non-zero. It follows that $\mu_{r-1}\left(\mathcal{G}\right)$ and $\mu_{r-1}\left(\mathcal{F}\right)$ are quantum isomorphic. □

Proposition B.7. *Let $\mathcal{G}$ be a quantum graph. Then $\mathrm{QAut}\left(\mathcal{G}\right)$ acts faithfully on quantum Mycielskians of $\mathcal{G}$.*

More precisely, for $r \geq 2$ we define the linear map $\Phi_r : \mathrm{C}\left(\mu_{r-1}\left(\mathcal{G}\right)\right) \to \mathrm{C}\left(\mu_{r-1}\left(\mathcal{G}\right)\right) \otimes \mathrm{C}\left(\mathrm{QAut}\left(\mathcal{G}\right)\right)$ by

$$\Phi_r\left(x\right) = \left(\iota_0\iota_0^*\left(x\right) \otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) + \sum_{k=1}^{r}\left(\iota_k \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}\left(\iota_k^*\left(x\right)\right). \tag{B.22}$$

Then Φ_r is a unital $*$-homomorphism and denoting by Δ the comultiplication on $\mathrm{C}(\mathrm{QAut}(\mathcal{G}))$ *we have*

$$\left(\Phi_r \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\Phi_r = \left(\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))} \otimes \Delta\right)\Phi_r, \tag{B.23}$$

$$\Phi_r A_{\mu_{r-1}(\mathcal{G})} = \left(A_{\mu_{r-1}(\mathcal{G})} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\Phi_r, \tag{B.24}$$

Moreover, the set $\left\{\left(\omega \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\Phi_r(x) \middle| x \in \mathrm{C}(\mu_{r-1}(\mathcal{G})), \omega \in \mathrm{C}(\mu_{r-1}(\mathcal{G}))^*\right\}$ *generates the* C^**-algebra* $\mathrm{C}(\mathrm{QAut}(\mathcal{G}))$.

Proof. We first establish that Φ_r is a unital $*$-homomorphism. We have

$$\mathbb{1}_{\mathrm{C}(\mu_{r-1}(\mathcal{G}))} = \sqrt{\tfrac{1}{1+r\delta_{\mathcal{G}}^2}}\left(\iota_0(1) + \delta_{\mathcal{G}}\sum_{k=1}^{r}\iota_k\left(\mathbb{1}_{\mathrm{C}(\mathcal{G})}\right)\right). \tag{B.25}$$

Hence

$$\begin{aligned}
&\Phi_r\left(\mathbb{1}_{\mathrm{C}(\mu_{r-1}(\mathcal{G}))}\right) \\
&\quad= \sqrt{\tfrac{1}{1+r\delta_{\mathcal{G}}^2}}\left(\iota_0(1)\otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))} + \delta_{\mathcal{G}}\sum_{k,l=1}^{r}\left(\iota_k \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}\left(\iota_k^*\iota_l\left(\mathbb{1}_{\mathrm{C}(\mathcal{G})}\right)\right)\right) \\
&\quad= \sqrt{\tfrac{1}{1+r\delta_{\mathcal{G}}^2}}\left(\iota_0(1)\otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))} + \delta_{\mathcal{G}}\sum_{k=1}^{r}\left(\iota_k \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\left(\mathbb{1}_{\mathrm{C}(\mathcal{G})}\otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\right) \\
&\quad= \sqrt{\tfrac{1}{1+r\delta_{\mathcal{G}}^2}}\left(\iota_0(1) + \delta_{\mathcal{G}}\sum_{k=1}^{r}\iota_k\left(\mathbb{1}_{\mathrm{C}(\mathcal{G})}\right)\right)\otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))} = \mathbb{1}_{\mathrm{C}(\mu_{r-1}(\mathcal{G}))}\otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}.
\end{aligned} \tag{B.26}$$

To prove the multiplicativity of Φ_r we temporarily denote by $m^{(0)}$, $m^{(1)}$, $m^{(2)}$ and $m^{(3)}$ the multiplication maps on $\mathbb{C}$, $\mathrm{C}(\mathcal{G})$, $\mathrm{C}(\mu_{r-1}(\mathcal{G}))$ and $\mathrm{C}(\mathrm{QAut}(\mathcal{G}))$ respectively (the last one defined only on the algebraic tensor product) and recall from [4] that

$$m^{(2)} = \sqrt{1+r\delta_{\mathcal{G}}^2}\left(\iota_0 m^{(0)}\left(\iota_0^*\otimes\iota_0^*\right) + \delta_{\mathcal{G}}^{-1}\sum_{k=1}^{r}\iota_k m^{(1)}\left(\iota_k^*\otimes\iota_k^*\right)\right). \tag{B.27}$$

Then note that for $s\in\{1,\dots,r\}$ the map $\sqrt{\tfrac{\delta_{\mathcal{G}}^2}{1+r\delta_{\mathcal{G}}^2}}\iota_s$ is multiplicative, which means that

$$m^{(2)}\left(\iota_s\otimes\iota_s\right) = \sqrt{\tfrac{1+r\delta_{\mathcal{G}}^2}{\delta_{\mathcal{G}}^2}}\iota_s m^{(1)}. \tag{B.28}$$

Hence for $(p,q)\neq(0,0)$, with χ denoting the flip map, by (B.28) we have

$$\begin{aligned}
\Phi_r m^{(2)}\left(\iota_p(a)\otimes\iota_q(b)\right) &= \Phi_r\left(\delta_{p,q}\sqrt{\tfrac{1+r\delta_{\mathcal{G}}^2}{\delta_{\mathcal{G}}^2}}\iota_p(ab)\right) \\
&= \delta_{p,q}\sqrt{\tfrac{1+r\delta_{\mathcal{G}}^2}{\delta_{\mathcal{G}}^2}}\left(\iota_p\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}(ab) \\
&= \delta_{p,q}\sqrt{\tfrac{1+r\delta_{\mathcal{G}}^2}{\delta_{\mathcal{G}}^2}}\left(\iota_p\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\left(m^{(1)}\otimes m^{(3)}\right)\left(\mathrm{id}_{\mathrm{C}(\mathcal{G})}\otimes\chi\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\left(\rho_{\mathcal{G}}\otimes\rho_{\mathcal{G}}\right)(a\otimes b) \\
&= \delta_{p,q}\left(m^{(2)}\otimes m^{(3)}\right)\left(\iota_p\otimes\iota_p\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \\
&\qquad\left(\mathrm{id}_{\mathrm{C}(\mathcal{G})}\otimes\chi\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\left(\rho_{\mathcal{G}}\otimes\rho_{\mathcal{G}}\right)(a\otimes b) \\
&= \delta_{p,q}\left(m^{(2)}\otimes m^{(3)}\right)\left(\mathrm{id}_{\mathrm{C}(\mu_{r-1}(\mathcal{G}))}\otimes\chi\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \\
&\qquad\left(\iota_p\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\otimes\iota_p\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\left(\rho_{\mathcal{G}}\otimes\rho_{\mathcal{G}}\right)(a\otimes b) \\
&= \left(m^{(2)}\otimes m^{(3)}\right)\left(\mathrm{id}_{\mathrm{C}(\mu_{r-1}(\mathcal{G}))}\otimes\chi\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\left(\Phi_r\otimes\Phi_r\right)\left(\iota_p(a)\otimes\iota_q(b)\right)
\end{aligned} \tag{B.29}$$

Similarly we compute that $\Phi_r m^{(2)} = \left(m^{(2)}\otimes m^{(3)}\right)\left(\mathrm{id}_{\mathrm{C}(\mu_{r-1}(\mathcal{G}))}\otimes\chi\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\left(\Phi_r\otimes\Phi_r\right)$ on simple tensors of the form $\iota_0(\lambda)\otimes\iota_0(\lambda')$ for any $\lambda,\lambda'\in\mathbb{C}$. Since $\mathrm{C}(\mu_{r-1}(\mathcal{G}))\otimes\mathrm{C}(\mu_{r-1}(\mathcal{G}))$

is spanned by elements of the form $\iota_k(a) \otimes \iota_l(b)$ with $k, l \in \{0, \ldots, r\}$, these computations show that Φ_r is multiplicative.

The fact that Φ_r is a $*$-map follows immediately from the fact that all maps involved in the definition of Φ_r are $*$-maps.

To check (B.23) we note that for $\lambda \in \mathbb{C}$ we have $\Phi_r(\iota_0(\lambda)) = \iota_0(\lambda) \otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}$ which immediately shows that $\left(\Phi_r \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \Phi_r(\iota_0(\lambda)) = \left(\mathrm{id}_{\mathrm{C}(\mu_{r-1}(\mathcal{G}))} \otimes \Delta\right) \Phi_r(\iota_0(\lambda))$ and similarly, since for any $k \in \{1, \ldots, r\}$ we have $\Phi_r \iota_k = \left(\iota_k \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}$, one finds that

$$\begin{aligned}
\left(\Phi_r \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \Phi_r(\iota_k(a)) &= \left(\Phi_r \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \left(\iota_k \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(a) \\
&= \left(\iota_k \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \left(\rho_{\mathcal{G}} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(a) \\
&= \left(\iota_k \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \left(\mathrm{id}_{\mathrm{C}(\mathcal{G})} \otimes \Delta\right) \rho_{\mathcal{G}}(a) \\
&= \left(\mathrm{id}_{\mathrm{C}(\mu_{r-1}(\mathcal{G}))} \otimes \Delta\right) \left(\iota_k \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(a) \\
&= \left(\mathrm{id}_{\mathrm{C}(\mu_{r-1}(\mathcal{G}))} \otimes \Delta\right) \Phi_r(\iota_k(a)) \qquad \text{(B.30)}
\end{aligned}$$

for any $a \in \mathrm{C}(\mathcal{G})$.

We now move on to the proof of (B.24). To do this, we recall:

$$A_{\mu_{r-1}(\mathcal{G})} = \delta_{\mathcal{G}} \iota_r \eta_{\mathcal{G}} \iota_0^* + \delta_{\mathcal{G}} \iota_0 \eta_{\mathcal{G}}^* \iota_r^* + \iota_1 A_{\mathcal{G}} \iota_1^* + \sum_{k=1}^{r-1} \left(\iota_k A_{\mathcal{G}} \iota_{k+1}^* + \iota_{k+1} A_{\mathcal{G}} \iota_k^*\right), \qquad \text{(B.31)}$$

where $\eta_{\mathcal{G}}$ is the unit map $\mathbb{C} \to \mathrm{C}(\mathcal{G})$. It follows from this equation that

$$A_{\mu_{r-1}(\mathcal{G})} \iota_0 = \delta_{\mathcal{G}} \iota_r \eta_{\mathcal{G}}, \qquad \text{(B.32)}$$

$$A_{\mu_{r-1}(\mathcal{G})} \iota_1 = \iota_1 A_{\mathcal{G}} + \iota_2 A_{\mathcal{G}}, \qquad \text{(B.33)}$$

$$A_{\mu_{r-1}(\mathcal{G})} \iota_l = \iota_{l-1} A_{\mathcal{G}} + \iota_{l+1} A_{\mathcal{G}}, \qquad \text{(B.34)}$$

$$A_{\mu_{r-1}(\mathcal{G})} \iota_r = \delta_{\mathcal{G}} \iota_0 \eta_{\mathcal{G}}^* + \iota_{r-1} A_{\mathcal{G}} \qquad \text{(B.35)}$$

for $l \in \overline{2, r-1}$. Using this we check that for $\lambda \in \mathbb{C}$

$$\begin{aligned}
\left(A_{\mu_{r-1}(\mathcal{G})} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \Phi_r(\iota_0(\lambda)) &= A_{\mu_{r-1}(\mathcal{G})} \iota_0(\lambda) \otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))} = \delta_{\mathcal{G}} \iota_r(\eta_{\mathcal{G}}(\lambda)) \otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))} \\
&= \delta_{\mathcal{G}} \lambda \iota_r\left(\mathbb{1}_{\mathrm{C}(\mathcal{G})}\right) \otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))} = \delta_{\mathcal{G}} \lambda \left(\iota_r \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}\left(\mathbb{1}_{\mathrm{C}(\mathcal{G})}\right) \\
&= \delta_{\mathcal{G}} \lambda \Phi_r\left(\iota_r\left(\mathbb{1}_{\mathrm{C}(\mathcal{G})}\right)\right) = \delta_{\mathcal{G}} \Phi_r\left(\iota_r\left(\lambda \mathbb{1}_{\mathrm{C}(\mathcal{G})}\right)\right) = \Phi_r\left(A_{\mu_{r-1}(\mathcal{G})} \iota_0(\lambda)\right). \qquad \text{(B.36)}
\end{aligned}$$

Similarly for $a \in \mathrm{C}(\mathcal{G})$ we have

$$\begin{aligned}
\left(A_{\mu_{r-1}(\mathcal{G})} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \Phi_r(\iota_1(a)) &= \left(A_{\mu_{r-1}(\mathcal{G})} \iota_1 \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(a) \\
&= \left(\iota_1 A_{\mathcal{G}} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(a) + \left(\iota_2 A_{\mathcal{G}} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(a) \\
&= \left(\iota_1 \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(A_{\mathcal{G}} a) + \left(\iota_2 \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(A_{\mathcal{G}} a) \\
&= \Phi_r(\iota_1(A_{\mathcal{G}} a) + \iota_2(A_{\mathcal{G}} a)) = \Phi_r\left(A_{\mu_{r-1}(\mathcal{G})} \iota_1(a)\right), \qquad \text{(B.37)}
\end{aligned}$$

as well as

$$\begin{aligned}
\left(A_{\mu_{r-1}(\mathcal{G})} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \Phi_r(\iota_l(a)) &= \left(A_{\mu_{r-1}(\mathcal{G})} \iota_l \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(a) \\
&= \left(\iota_{l-1} A_{\mathcal{G}} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(a) + \left(\iota_{l+1} A_{\mathcal{G}} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(a) \\
&= \left(\iota_{l-1} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(A_{\mathcal{G}} a) + \left(\iota_{l+1} \otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right) \rho_{\mathcal{G}}(A_{\mathcal{G}} a) \\
&= \Phi_r(\iota_{l-1}(A_{\mathcal{G}} a) + \iota_{l+1}(A_{\mathcal{G}} a)) = \Phi_r\left(A_{\mu_{r-1}(\mathcal{G})} \iota_l(a)\right) \qquad \text{(B.38)}
\end{aligned}$$

for $l=2,\dots,r-1$, and finally using the fact that $\eta_{\mathcal{G}}^*=\psi$ and that ψ is $\rho_{\mathcal{G}}$-invariant we arrive at

$$\begin{aligned}\left(A_{\mu_{r-1}(\mathcal{G})}\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\Phi_r\left(\iota_r\left(a\right)\right) &= \left(A_{\mu_{r-1}(\mathcal{G})}\iota_r\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}\left(a\right)\\ &= \delta_{\mathcal{G}}\left(\iota_0\eta_{\mathcal{G}}^*\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}\left(a\right)+\left(\iota_{r-1}A_{\mathcal{G}}\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}\left(a\right)\\ &= \delta_{\mathcal{G}}\left(\iota_0\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\left(\psi\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}\left(a\right)+\left(\iota_{r-1}\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}\left(A_{\mathcal{G}}a\right)\\ &= \delta_{\mathcal{G}}\left(\iota_0\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\left(\psi\left(a\right)\otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)+\left(\iota_{r-1}\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}\left(A_{\mathcal{G}}a\right)\\ &= \delta_{\mathcal{G}}\left(\iota_0\left(\eta_{\mathcal{G}}^*a\right)\otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)+\left(\iota_{r-1}\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}\left(A_{\mathcal{G}}a\right)\\ &= \Phi_r\left(\delta_{\mathcal{G}}\iota_0\left(\eta_{\mathcal{G}}^*a\right)+\iota_{r-1}\left(a\right)\right)=\Phi_r\left(A_{\mu_{r-1}(\mathcal{G})}\iota_r\left(a\right)\right).\end{aligned}\tag{B.39}$$

Finally, we note that if $\{e_1,\dots,e_{\dim \mathrm{C}(\mathcal{G})}\}$ is an orthonormal basis of $\mathrm{C}\left(\mathcal{G}\right)$ then for any $k\in\{1,\dots,r\}$

$$\Phi_r\left(\iota_k\left(e_j\right)\right)=\left(\iota_k\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G}}\left(e_j\right)=\sum_{i=1}^{r}\iota_k\left(e_i\right)\otimes u_{i,j}\tag{B.40}$$

showing that $\left\{\left(\omega\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\Phi_r\left(x\right)\middle|\, x\in \mathrm{C}\left(\mu_{r-1}\left(\mathcal{G}\right)\right),\omega\in \mathrm{C}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)^*\right\}$ generates the C*-algebra $\mathrm{C}\left(\mathrm{QAut}\left(\mathcal{G}\right)\right)$. ☐

Lemma B.8. *Let $\mathfrak{P}=\{P_1,\dots,P_n\}$ be a partition of unity in $\mathrm{C}\left(\mathcal{G}\right)\otimes \mathrm{B}\left(\mathscr{H}\right)$, and take $r\geq 2$. Then*

$$Q_0=\iota_0\iota_0^*\otimes \mathbb{1}_{\mathrm{B}(\mathscr{H})},\tag{B.41}$$

$$Q_a=\sum_{k=1}^{r}(\iota_k\otimes \mathbb{1}_{\mathrm{B}(\mathscr{H})})P_a\left(\iota_k^*\otimes \mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)\tag{B.42}$$

for $a\in\overline{1,n}$ form a partition of unity $\mathfrak{P}_\mu^{r-1}$ in $\mathrm{C}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)\otimes \mathrm{B}\left(\mathscr{H}\right)$ (which we refer to as the partition of unity induced by $\mathfrak{P}$).

Proof. First, notice that Q_0 and Q_a with $a\in\overline{1,n}$ are projections. Indeed, we immediately notice that both Q_0 and Q_a with $a\in\overline{1,n}$ are idempotent and self-adjoint. Finally, we note that

$$\begin{aligned}\sum_{Q\in\mathfrak{P}_\mu^{r-1}}Q=Q_0+\sum_{a=1}^{n}Q_a &= \left(\iota_0\otimes \mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)\left(\iota_0^*\otimes \mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)+\sum_{k=1}^{r}\sum_{a=1}^{n}\left(\iota_k\otimes \mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)P_a\left(\iota_k^*\otimes \mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)\\ &= \left(\iota_0\iota_0^*+\sum_{k=1}^{r}\iota_k\iota_k^*\right)\otimes \mathbb{1}_{\mathrm{B}(\mathscr{H})}=\mathbb{1}_{\mathrm{C}(\mu_{r-1}(G))\otimes \mathrm{B}(\mathscr{H})}.\end{aligned}\tag{B.43}$$

☐

Corollary B.9. *Let $\mathcal{G}$ be a quantum graph and $\mathfrak{P}=\{P_1,\dots,P_n\}\subset \mathrm{C}\left(\mathcal{G}\right)\otimes \mathrm{B}\left(\mathscr{H}\right)$ be a partition of unity. Then for any $r\geq 2$, the compact quantum group $\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}$ is embedded as a closed quantum subgroup of $\mathrm{QAut}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)_{\mathfrak{P}_\mu^{r-1}}$.*

Proof. Let $\Phi_{r,\mathscr{H}}$ be the homomorphism from Proposition B.7 extended onto $\mathrm{C}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)\otimes \mathrm{B}\left(\mathscr{H}\right)$, that is for $x\in \mathrm{C}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)\otimes \mathrm{B}\left(\mathscr{H}\right)$ we define

$$\begin{aligned}\Phi_{r,\mathscr{H}}\left(x\right) &= \left(\iota_0\iota_0^*\otimes \mathrm{id}_{\mathrm{B}(\mathscr{H})}\right)\left(x\right)\otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\\ &\quad+\sum_{k=1}^{r}\left(\iota_k\otimes \mathrm{id}_{\mathrm{B}(\mathscr{H})}\otimes \mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G},\mathscr{H}}\left(\iota_k^*\otimes \mathrm{id}_{\mathrm{B}(\mathscr{H})}\right)\left(x\right).\end{aligned}\tag{B.44}$$

Further, we note that

$$\Phi_{r,\mathscr{H}}\left(Q_0\right)=\left(\iota_0\iota_0^*\otimes \mathrm{id}_{\mathrm{B}(\mathscr{H})}\right)\left(\iota_0\iota_0^*\otimes \mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)\otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}=Q_0\otimes \mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))},\tag{B.45}$$

and

$$\Phi_{r,\mathscr{H}}\left(Q_a\right)$$
$$= \sum_{k,l=1}^{r}\left(\iota_k\otimes\mathrm{id}_{\mathrm{B}(\mathscr{H})}\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G},\mathscr{H}}\left(\iota_k^*\otimes\mathrm{id}_{\mathrm{B}(\mathscr{H})}\right)\left(\iota_l\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)P_a\left(\iota_l^*\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)$$
$$= \sum_{k=1}^{r}\left(\iota_k\otimes\mathrm{id}_{\mathrm{B}(\mathscr{H})}\otimes\mathrm{id}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right)\rho_{\mathcal{G},\mathscr{H}}\left(P_a\right)\left(\iota_k^*\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\otimes\mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}\right). \quad \text{(B.46)}$$

If $\rho_{\mathcal{G},\mathscr{H}}\left(P_a\right) = P_a\otimes\mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}$, then the above expression simplifies into $Q_a\otimes\mathbb{1}_{\mathrm{C}(\mathrm{QAut}(\mathcal{G}))}$, so that $\Phi_{r,\mathscr{H}}$ induces a homomorphism

$$\Phi_{r,\mathscr{H}}^{\mathfrak{P}}:\mathrm{C}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)\otimes\mathrm{B}\left(\mathscr{H}\right)\longrightarrow\mathrm{C}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)\otimes\mathrm{B}\left(\mathscr{H}\right)\otimes\mathrm{C}\left(\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}\right). \quad \text{(B.47)}$$

By the universal property of $\mathrm{QAut}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)$ there exists a unique homomorphism

$$\pi_r^{\mathfrak{P}}:\ \mathrm{C}\left(\mathrm{QAut}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)\right)\longrightarrow\mathrm{C}\left(\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}\right) \quad \text{(B.48)}$$

such that $\Phi_{r,\mathscr{H}}^{\mathfrak{P}} = \left(\mathrm{id}\otimes\pi_r^{\mathfrak{P}}\right)\rho_{\mu_{r-1}(\mathcal{G}),\mathscr{H}}$. Moreover, $\pi_r^{\mathfrak{P}}$ restricts itself to a well-defined homomorphism $\pi_r^{\mathfrak{P},\mathfrak{P}_\mu^{r-1}}:\ \mathrm{C}\left(\mathrm{QAut}\left(\mu_{r-1}\left(\mathcal{G}\right)\right)_{\mathfrak{P}_\mu^{r-1}}\right)\to\mathrm{C}\left(\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}\right)$ and this homomorphism is compatible with comultiplications (cf. [17, Proposition 4.7]) and is surjective by the last statement of Proposition B.7. ☐

Theorem B.10. *Let* $\mathcal{G} = \left(\mathbb{X}_{\mathcal{G}},\psi_{\mathcal{G}},A_{\mathcal{G}}\right)$ *be a quantum graph that does not have quantum twin vertices. Let* $\mathfrak{P}$ *be a partition of unity in* $\mathrm{C}\left(\mathcal{G}\right)\otimes\mathrm{B}\left(\mathscr{H}\right)$ *with some (separable) Hilbert space* $\mathscr{H}$*. Then* $\mathrm{QAut}\left(\mu\left(\mathcal{G}\right)\right)_{\mathfrak{P}_\mu}\cong\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}$.

Proof. Let us first recall that $\mathrm{QAut}\left(\mathcal{G}\right)_{\mathfrak{P}}$ can be viewed as a closed quantum subgroup of $\mathrm{QAut}\left(\mu\left(\mathcal{G}\right)\right)_{\mathfrak{P}_\mu}$ (cf. Corollary B.9). We shall prove that the two quantum groups are in fact identical.

In what follows $\mathrm{K}\left(\mathscr{H}\right)$ denotes the algebra of compact operators on $\mathscr{H}$ and $\mathrm{M}\left(\cdot\right)$ is the multiplier functor. More information about those concepts and their use can be found in introductory sections of [24] and [23].

So let us consider a partition of unity $\mathfrak{P}\subset\mathrm{C}\left(\mathcal{G}\right)\otimes\mathrm{B}\left(\mathscr{H}\right)$. Note that the partition $\mathfrak{P}_\mu\subset\mathrm{C}\left(\mu\left(\mathcal{G}\right)\right)\otimes\mathrm{B}\left(\mathscr{H}\right)$ assigned to $\mathfrak{P}$ contains the projection $P_\bullet\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\in\mathrm{C}\left(\mu\left(\mathcal{G}\right)\right)\otimes\mathrm{B}\left(\mathscr{H}\right)$ where $P_\bullet = \begin{bmatrix}1\\0\\0\end{bmatrix}\in\mathrm{C}\left(\mu\left(\mathcal{G}\right)\right)$. Let $\rho_{\mu(\mathcal{G}),\mathscr{H}}$ be the amplification of the canonical action of $\mathrm{QAut}\left(\mu\left(\mathcal{G}\right)\right)_{\mathfrak{P}_\mu}$ on $\mathrm{C}\left(\mu\left(\mathcal{G}\right)\right)$. The direct sum structure of $\mathrm{C}\left(\mu\left(\mathcal{G}\right)\right)$ gives rise to the "matrix structure" of $\rho_{\mu(\mathcal{G}),\mathscr{H}} = \left[\rho_{ij}\right]_{i,j=0,1,2}$ where using notation $\mathrm{C}\left(\mathbb{X}_0\right) = \mathbb{C}$, $\mathrm{C}\left(\mathbb{X}_1\right) = \mathrm{C}\left(\mathcal{G}\right)$ and $\mathrm{C}\left(\mathbb{X}_2\right) = \mathrm{C}\left(\mathcal{G}\right)$ we define $\rho_{ij}:\mathrm{C}\left(\mathbb{X}_j\right)\otimes\mathrm{K}\left(\mathscr{H}\right)\to\mathrm{M}\left(\mathrm{C}\left(\mathbb{X}_i\right)\otimes\mathrm{K}\left(\mathscr{H}\right)\otimes\mathrm{C}\left(\mathrm{QAut}\left(\mu\left(\mathcal{G}\right)\right)_{\mathfrak{P}_\mu}\right)\right)$ by the formula

$$\rho_{ij} = \left(\iota_j^*\otimes\mathrm{id}_{\mathrm{B}(\mathscr{H})}\otimes\mathrm{id}_{\mathrm{C}\left(\mathrm{QAut}(\mu(\mathcal{G}))_{\mathfrak{P}_\mu}\right)}\right)\rho_{\mu(\mathcal{G}),\mathscr{H}}\left(\iota_i\otimes\mathrm{id}_{\mathrm{B}(\mathscr{H})}\right). \quad \text{(B.49)}$$

Remembering that

$$\rho_{\mu(\mathcal{G}),\mathscr{H}}\iota_0\left(\lambda\right) = \sqrt{1+2\delta_{\mathcal{G}}^2}\,\lambda\left(P_\bullet\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\otimes\mathbb{1}_{\mathrm{C}\left(\mathrm{QAut}(\mu(\mathcal{G}))_{\mathfrak{P}_\mu}\right)}\right) \quad \text{(B.50)}$$

for every $\lambda\in\mathbb{C}$ and using $\iota_j^*\left(P_\bullet\right) = 0$ for $j\in\{1,2\}$ we get $\rho_{j0} = 0$ and

$$\rho_{00}\left(\lambda\right) = \lambda\left(P_\bullet\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\otimes\mathbb{1}_{\mathrm{C}\left(\mathrm{QAut}(\mu(\mathcal{G}))_{\mathfrak{P}_\mu}\right)}\right) \quad \text{(B.51)}$$

for every $\lambda\in\mathbb{C}$. Using the $*$-condition of the $*$-homomorphism $\rho_{\mu(\mathcal{G}),\mathscr{H}}$ we get $\rho_{01} = 0 = \rho_{02}$.

Having the above facts established we can now prove that $\rho_{12} = \rho_{21} = 0$. Indeed, since the adjacency matrix $A_{\mu(\mathcal{G})}$ is of the form

$$A_{\mu(\mathcal{G})} = \delta_{\mathcal{G}}\iota_2\eta_{\mathcal{G}}\iota_0^*+\delta_{\mathcal{G}}\iota_0\eta_{\mathcal{G}}^*\iota_2^*+\iota_1A_{\mathcal{G}}\iota_1^*+\iota_1A_{\mathcal{G}}\iota_2^*+\iota_2A_{\mathcal{G}}\iota_1^*, \quad \text{(B.52)}$$

we get

$$A_{\mu(\mathcal{G})}\left(\iota_0(1)\right)=\delta_{\mathcal{G}}\iota_2\left(\mathbb{1}_{\mathrm{C}(\mathcal{G})}\right) \tag{B.53}$$

and therefore

$$\left(\iota_1^*\otimes\mathrm{id}_{\mathrm{B}(\mathscr{H})}\otimes\mathrm{id}_{\mathrm{C}(\mu(\mathcal{G}))}\right)\rho_{\mu(\mathcal{G}),\mathscr{H}}\left(A_{\mu(\mathcal{G})}\left(\iota_0(1)\right)\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)=\delta_{\mathcal{G}}\rho_{12}\left(\mathbb{1}_{\mathrm{C}(\mathcal{G})}\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right). \tag{B.54}$$

A similar computation but with the amplified adjacency matrix $A_{\mu(\mathcal{G})}$ to the left of the action $\rho_{\mu(\mathcal{G}),\mathscr{H}}$ shows that the left hand side of the above equality vanishes, and consequently $\rho_{12}(\mathbb{1})=0$. This implies that $\rho_{12}=\rho_{21}=0$.

Using the vanishing of $\rho_{ij},\rho_{0j},\rho_{i0}$ for $i,j\in\{1,2\}$ proven so far and the equality

$$\rho_{\mu(\mathcal{G}),\mathscr{H}}\left(A_{\mu(\mathcal{G})}\iota_2(x)\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)=\left(A_{\mu(\mathcal{G})}\otimes\mathrm{id}_{\mathrm{K}(\mathscr{H})}\otimes\mathrm{id}_{\mathrm{C}(\mu(\mathcal{G}))}\right)\rho_{\mu(\mathcal{G}),\mathscr{H}}\left(\iota_2(x)\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right) \tag{B.55}$$

which holds for every $x\in\mathrm{C}(\mathcal{G})$ we easily conclude that $\rho_{11}\left(A_{\mathcal{G}}\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)=\rho_{22}\left(A_{\mathcal{G}}\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)$. "Deamplifying" the later equality an referring to the assumed lack of quantum twins in $\mathcal{G}$ we conclude that $\rho_{11}=\rho_{22}$.

Summarizing we have so far proved that the canonical action of $\mathrm{QAut}\left(\mu(\mathcal{G})\right)_{\mathfrak{P}_\mu}$ on $\mathrm{C}\left(\mu(\mathcal{G})\right)$ is induced by the action of $\mathrm{QAut}(\mathcal{G})$. Moreover using the fact that $\rho_{\mu(\mathcal{G})}$ preserves $\mathfrak{P}_\mu$ we see that in fact it is induced by the action of $\mathrm{QAut}(\mathcal{G})_{\mathfrak{P}}$ and therefore we conclude that $\mathrm{QAut}(\mathcal{G})_{\mathfrak{P}}$ and $\mathrm{QAut}(\mu\mathcal{G})_{\mathfrak{P}_\mu}$ are indeed isomorphic. □

Theorem B.11. *Let $\mathcal{G}$ be a quantum graph without quantum twin vertices and denote by $D(\mathcal{G})$ the quantum distinguishing number of $\mathcal{G}$. Then $D(\mu(\mathcal{G}))\leq D(\mathcal{G})+1$.*

Proof. Let $\mathfrak{P}=\{P_1,\ldots,P_{D(\mathcal{G})}\}\subset\mathrm{C}(\mathcal{G})\otimes\mathrm{B}(\mathscr{H})$ be a quantum $D(\mathcal{G})$-distinguishing labeling of $\mathcal{G}$, so that $\mathrm{C}\left(\mathrm{QAut}(\mathcal{G})_{\mathfrak{P}}\right)=\mathbb{C}$, and consider the induced partition of unity $\mathfrak{P}_\mu$ on $\mathrm{C}(\mu(\mathcal{G}))\otimes\mathrm{B}(\mathscr{H})$, that is $\mathfrak{P}_\mu=\{Q_0,Q_1,\ldots,Q_{D(\mathcal{G})}\}$ with

$$\begin{aligned}&Q_0=\iota_0\iota_0^*\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})},\\&Q_a=\left(\iota_1\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)P_a\left(\iota_1^*\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)+\left(\iota_2\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right)P_a\left(\iota_2^*\otimes\mathbb{1}_{\mathrm{B}(\mathscr{H})}\right),a=1,\ldots,D(\mathcal{G}).\end{aligned} \tag{B.56}$$

By Theorem B.10, $\mathfrak{P}_\mu$ is a quantum $(D(\mathcal{G})+1)$-labeling of $\mu(\mathcal{G})$. □

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