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Subject: report on PhD thesis
Name of candidate: Ms Júlia Amorós Binefa
Title of thesis: Real-time optimal quantum control for atomic magnetometers with decoherence
Supervisers: Konrad Banaszek and Jan Kołodyński

To whom it may concern,

Ms Júlia Amorós Binefa's dissertation is devoted to the understanding of the fundamental limitations in parameter estimation for open quantum systems, and the development of estimation and control techniques for atomic magnetometers subjected to realistic decoherence noise. This topic lies at the overlap between quantum metrology and control theory, a highly active and interdisciplinary area at the forefront of current efforts towards developing sensing devices with quantum-enhanced precision. Currently, there is significant interest in improving the performance of atomic magnetometers and reaching the ultimate limits set by the laws of quantum mechanics. Ms Binefa's thesis makes a significant step in this direction and, in my view, is likely to become an important reference for practitioners and young researchers starting out in this field. The thesis' background sections offers a very clear, pedagogical, and rigorous introduction to a range of topics which are not part of the standard physics curriculum, including filtering theory, statistics and stochastic control. Building on this, the research part contains highly detailed original work on the theory, simulation and methodology of real-time estimation in open system dynamics. This work appeared in 3 publications, was presented at several international conferences and attracted significant attention in the community. In my experience, this is one of the most substantial and sophisticated dissertations I have refereed so far. I am particularly impressed by the derivation of the classical simulation bound and the discovery that this bound is achievable in certain conditions. I would also like to commend the mastery of highly technical derivations which evidence an unusual level of scientific maturity for this career stage.

The thesis is structured in 5 chapters together with introduction and conclusions. Chapters 1 to 3 lay the theoretical groundwork for the rest of the thesis, which gives a detailed account of the candidate's original results on real-time optimal estimation in magnetometry.

Chapter 1 provides a thorough overview of the mathematical and physical background underpinning the thesis. The chapter starts by introducing fundamental probabilistic concepts such as joint probability distribution, conditional distribution, expectation and covariance, and continues by discussing Gaussian variables, Markov, Poisson and Wiener processes. A large portion of this chapter is dedicated to introducing the framework of stochastic calculus. This includes the definition of the Ito integral, the Ito lemma and Ito product rule, and the concept of a stochastic differential equation (SDE). In spite of the technical nature of these topics, the presentation strikes a remarkable balance between keeping a rigorous mathematical style and providing the reader with intuitive understanding. This is achieved by including numerous detailed examples which illustrate how these tools are used in practice, including solving the stochastic differential equation of the geometric Brownian motion and the Ornstein-Uhlenbeck process. The classical part of the chapter ends with a brief discussion on numerical methods for solving SDEs and a careful and detailed treatment of time-discretisation for linear Gaussian systems. The second part of the chapter reviews a selection of concepts in quantum mechanics which play an important role in the thesis: oscillators, open systems and master equations, the total angular momentum, coherent spin states, the Wigner distribution and spin squeezing. This chapter demonstrates a significant breath of knowledge, a high level of mathematical skill and ability to carry out complicated calculations, which is apparent throughout the thesis.

Chapter 2 deals with Bayesian statistical inference and its applications in control theory. Section 2.1 introduces the concepts of prior and posterior distribution, the maximum-likelihood, maximum a-posteriori and the minimum mean square estimators. Section 2.2 reviews the frequentist and Bayesian Cramer-Rao bounds. As a minor remark, I note that the likelihood notation p(theta|y) on page 65 should perhaps be avoided as it may be confused with the posterior distribution. The general principles of Bayesian filtering theory are laid out in section 2.3; in particular, the author outlines a 3 steps procedure for computing the filter (conditional distribution of the unobserved process given the observed one) which consists of recursively predicting the next hidden state given past observed trajectory and updating the distribution conditional on the new observation. This recipe is then applied to the paradigmatic model of linear systems with Gaussian noise, which gives rise to the Kalman filter. Sections 2.4 to 2.7 provide an exhaustive treatment of the Kalman filter, starting from discrete time and uncorrelated noise, through to the case of continuous-time with correlated noise and the extended Kalman filter. As other background sections in the thesis, this clear and detailed derivation is a model of dissertation writing which I would happily recommend to starting researchers in the field. The last part of this chapter explores an important area in stochastic control which specialises to linear systems. By formulating control as an optimisation problem, one obtains an explicit solution of the optimal control problem in the form of the linear quadratic regulator (LQR) which prescribes the feedback control as a gain applied to the current state. When applied to stochastic linear systems, the LQR is combined with the Kalman filter to provide the optimal solution to the linear quadratic control problem. Although a full derivation of this result is beyond the scope of the thesis, the author presents a compelling and intuitive argument which skirts some of the technical issues. For the reader, it would perhaps be useful to point out more explicitly which parts of the presented arguments require a more careful analysis in a rigorous derivation.

Chapter 3 reviews the theory of continuously monitored quantum systems, which is the mathematical language used in deriving the subsequent original results. The presentation starts with the fundamental model of system-bath interaction and a detailed account of the different approximations leading to the quantum stochastic description of open quantum systems dynamics. The quantum filtering equation describing the evolution of an open system, conditional on the stochastic trajectory of measurement outcomes, is derived for both counting and homodyne measurements, starting from a discrete time setup and then taking the limit to continuous time. The general theory is then specialised to the setup of an atomic ensemble coupled with an off-resonant, polarized probe beam via the Faraday interaction. The chapter ends with a discussion of measurement based feedback methods, distinguishing between Bayesian feedback and Markovian feedback.

Chapter 4 investigates general bounds on the precision of estimating dynamical parameters of open quantum systems. Generically, the ultimate precision limit is given by the quantum Fisher information of the output field, and its achievability by means of realistic (time-local) measurements has been a subject of several recent investigations. In this thesis, the context is more specific and relates to frequency estimation in the presence of atomic dephasing and field fluctuations. The discrete-time model of the state evolution is described in section 4.1 and features alternating Kraus maps for measurements and in-between evolutions consisting of the composition of two Lindblad maps corresponding to parameter encoding and feedback. In section 4.2 this model is further specialised to collective dephasing together with local dephasing depending on unknown (possibly time dependent) frequency parameter. The key point is that this dynamics can be seen as a mixture of unitary transformations where the parameter is encoded in the weights of the different unitaries. This allows one to obtain a general bound (often called classical simulation) on the precision which is independent of the measurement and feedback details. While the principle is simple, the computation of the upper bound is highly non- trivial and constitutes one of key original results of the thesis, culminating with the formula for the Fisher information on page 150, for discrete time. The analysis is then extended to continuous time and the limits of infinite time, infinite prior and/or no field fluctuations are investigated. One of the upshots of this analysis is to exclude the possibility of attaining superclassical scaling in time of atom numbers in the presence of dephasing.

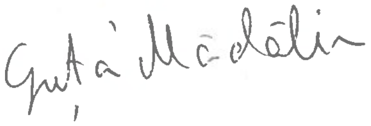
Chapter 5 is the most substantial part of the thesis, which combines all the tools presented in the background sections to tackle the real world problem of estimating a time-dependent magnetic field. After spelling out the model including noise contributions from dephasing, the law of the time-dependent frequency parameter, and the action of the controller, the author investigates different methods to simulate the dynamics and their range of validity. By exploiting permutational symmetry, it is shown that one can perform exact simulations for a moderate number of atoms. On the other hand, in a certain time regime the dynamics can be approximated by a linear Gaussian one for large atomic ensembles, cf. eqs. 5.16-5.19. In this case the (deterministic) equation governing the spin variance can be solved explicitly and as a result, it is shown that squeezing occurs when the collective dephasing is weak compared to the measurement strength. In the absence of decoherence and field fluctuations, the estimation precision is shown to have non-classical scaling in a finite time regime, cf. eqs. 5.45-5.46. The steady state regime of the linear system is then analysed and it is found that the classical simulation bound is achieved for pure Wiener noise processes. However, here (page 175) I have some difficulties interpreting the results: eq. 5.46 has a term which is exponentially growing in time, which is however negligible for large sizes. It should be clarified in what sense this is a steady state regime, and for what parameters we can consider that the Gaussian approximation is valid in the long time limit (for instance, it seems to be necessary that the omega bar in eq. 5.4 is equal to zero). The LG analysis is continued in section 5.4.1 where the main result is that the setup achieves the classical simulation bound for large ensembles. Since this is an important finding, it would be nice to provide an intuitive understanding of the mechanism behind the attainment and how it relates to squeezing, measurement strength etc.

Going beyond the linear (small rotations) approximation, the author then investigates the co-moving Gaussian approximation in which the spin vector rotates around the sphere but local Gaussian approximations can be made at each time. This model is validated by comparing with the results of exact simulations. Following the original proposal in reference [55], the author considers a modified setting in which feedback control is used to stabilise the collective atomic spin around direction x, thus providing useful information about the unknown parameter. To my understanding, the control is a combination of a naive field compensation control designed to remove fast rotations and bring the system in the linear regime, and the optimal linear quadratic compensator suitable for such regime. The results are presented in section 5.4.2. Firstly, different estimation and control schemes are compared in exact simulations with 200 atoms with constant magnetic field. As expected, the best performance is obtained by combining the extended Kalman filter with the linear quadratic compensator for control. A similar analysis is then performed for larger number of atoms. Subsequently, the case of fluctuating fields is analysed. Interestingly, it is found that for realistic parameters, the estimation error does not attain the CS bound due to the small value of the measurement strength, but when a small amount of collective dephasing is present, the error is close to the quantum limit. This chapter contains a wealth of interesting results and the reader may find is harder to navigate compared to the previous ones. I think a high level summary of key results at the beginning of the relevant sections would improve the accessibility and provide a better understanding of the various scenarios considered here.

In conclusion, I consider that the present manuscript is of high scientific value and brings significant original contributions to the field, which have already appeared in scientific publications or are currently under review. The thesis is very well written, presents a broad and highly informative overview of the field, showing a high degree of maturity and technical skill. I did not find any significant conceptual or computational errors, and my recommendations are mostly aimed at improving presentation and accessibility.

Based on the above considerations, I unreservedly recommend that the candidate is awarded the doctoral title.

Sincerely,



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