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Kinetic theory and hydrodynamics of nearly-integrable gases



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Summary

Integrable one-dimensional quantum gases have attracted significant attention. One could argue that this has happened for two primary reasons: analytical access to strongly correlated, non-perturbative physics and the availability of numerous experimental realizations. Recently, the field has shifted toward non-equilibrium dynamics, driven by theoretical breakthroughs such as generalized Gibbs ensemble (GGE) and generalized hydrodynamics (GHD).

The presence of integrability-breaking effects in experiments, such as external trapping or additional long-range interactions, calls for the development of perturbative techniques to tackle the dynamics of nearly-integrable gases. The external trapping occurs generically in the experiments, while the long-range interactions are especially prominent in dipolar quantum gases of elements with large magnetic dipole moments, such as dysprosium. The theoretical treatment of such systems is the main topic of this thesis.

The general goal was to provide a framework for understanding of two classes of non-equilibrium phenomena in nearly-integrable gases. The first class of problems which we tackle is related to thermalization or relaxation more broadly. In the context of gases, the description of relaxation is typically provided by kinetic theory, with the cornerstone being the BBGKY (Born-Bogoliubov-Green-Kirkwood-Yvon) hierarchy for correlation functions. The hierarchy is the starting point for further approximations and upon truncation, leads to collision operators such as the Boltzmann or Landau equations. Such collision kernels provide the time scales for relaxation and the fixed points of the dynamics.

In this thesis, a BBGKY-like framework was adapted to nearly-integrable systems. The resulting hierarchy can be efficiently truncated. A kinetic equation describing thermalization was derived and successfully tested against numerical simulations and experimental data for thermalization rates in dipolar gas experiments. The geometry of such experiments, in which 1D gases are placed in a 2D lattice, motivated the study of integrability breaking through interactions between two tubes. In such a scenario, interesting athermal stationary states were found, and an understanding through the kinematics of two-body scattering in 1D Galilean systems was proposed.

The second part of this contribution is related to the hydrodynamics of nearly-integrable gases, that is the effective theory of dynamics emerging at large spatiotemporal scales. The structure of hydrodynamics is intimately tied to the conservation laws present in the system. The set of conserved charges changes drastically upon integrability breaking, and so does the hydrodynamics.

Building on the frameworks of the GGE and GHD, we have shown that at sufficiently large space-time scales, the hydrodynamic universality class of nearly-integrable gases changes from GHD to a one-dimensional version of the Navier-Stokes (NS) equations. Transport coefficients – viscosity and thermal conductivity were computed explicitly. This goal was achieved by a suitable generalization of the classic derivation of NS hydrodynamics from the Boltzmann equation through the Chapman-Enskog (ChE) expansion. The logic of the ChE method was applied also to a two-tube system, where athermal states were found. Surprisingly, the Burgers equation was uncovered there in the large scale dynamics of the density.

Finally, we considered the problem of anomalous transport in 1D non-integrable gases. Using nonlinear fluctuating hydrodynamics, the well-established framework to tackle problems in this field, we found a mechanism leading to the suppression (with the inverse of the integrability breaking parameter) of anomalies in transport. We formulated a general prediction that nearly-integrable gases are characterized by three different hydrodynamic

universality classes, which occur in different space-time regimes.

Keywords: hydrodynamics, kinetic theory, integrable systems, one-dimensional quantum gases, thermalization, transport

Streszczenie

Całkowalne jednowymiarowe gazy kwantowe cieszą się dużym zainteresowaniem głównie z dwóch powodów: umożliwiają analityczny wgląd w silnie skorelowaną, nieperturbacyjną fizykę oraz są powszechnie realizowane w eksperymentach. W ostatnim czasie uwaga badaczy przesunęła się w stronę dynamiki nierównowagowej, co zostało spowodowane przez przełomy teoretyczne, takie jak uogólniony zespół Gibbsa (GGE) oraz uogólniona hydrodynamika (GHD).

Obecność czynników łamiących całkowalność w eksperymentach, takich jak zewnętrzne pułapkowanie czy dodatkowe oddziaływania dalekozasięgowe, wymaga jednak rozwoju metod perturbacyjnych do opisu dynamiki gazów prawie całkowalnych. Pułapkowanie zewnętrzne występuje w laboratoriach praktycznie zawsze, natomiast oddziaływania dalekozasięgowe są szczególnie widoczne w dipolowych gazach kwantowych pierwiastków o dużych magnetycznych momentach dipolowych, takich jak dysproz. Teoretyczny opis takich układów stanowi główny temat niniejszej rozprawy.

Głównym celem pracy było stworzenie ram teoretycznych dla zrozumienia dwóch klas zjawisk nierównowagowych w gazach niemal całkowalnych. Pierwsza klasa zagadnień dotyczy szeroko pojętej relaksacji i termalizacji. W kontekście gazów opis relaksacji opiera się zazwyczaj na teorii kinetycznej, której punktem wyjścia jest hierarchia BBGKY (Borna-Bogoliubowa-Greena-Kirkwooda-Yvona) dla funkcji korelacji. Hierarchia ta służy do wprowadzania dalszych przybliżeń i po obcięciu prowadzi do operatorów zderzeń, takich jak równania Boltzmanna czy Landaua, które wyznaczają skale czasowe relaksacji oraz punkty stałe dynamiki.

W dysertacji zaadaptowano strukturę typu BBGKY do układów prawie całkowalnych, co pozwoliło na jej efektywne obcięcie. Wyprowadzone w ten sposób równanie kinetyczne opisujące termalizację zostało pomyślnie zweryfikowane za pomocą symulacji numerycznych oraz danych doświadczalnych dotyczących szybkości termalizacji w gazach dipolowych. Ponieważ w eksperymentach gazy jednowymiarowe umieszcza się często w sieciach dwuwymiarowych, zbadano również łamanie całkowalności poprzez oddziaływania między dwiema tubami. W takim scenariuszu odkryto interesujące atermalne stany stacjonarne, których mechanizm wyjaśniono poprzez kinematykę rozpraszania dwucząstkowego w jednowymiarowych układach z symetrią Galileusza.

Druga część pracy poświęcona jest hydrodynamice gazów niemal całkowalnych, czyli efektywnej teorii dynamiki wyłaniającej się w dużych skalach czasoprzestrzennych. Struktura hydrodynamiki jest ściśle powiązana z występującymi w układzie prawami zachowania. Zestaw zachowanych ładunków zmienia się drastycznie po złamaniu całkowalności, co pociąga za sobą również fundamentalną zmianę hydrodynamiki.

Opierając się na strukturach GGE i GHD wykazano, że na dostatecznie dużych skalach czasoprzestrzennych klasa uniwersalności hydrodynamicznej gazów prawie całkowalnych zmienia się z GHD na jednowymiarową wersję równań Naviera-Stokesa (NS). Współczynniki transportu – lepkość oraz przewodnictwo cieplne – zostały obliczone w sposób jawny. Cel ten osiągnięto poprzez odpowiednie uogólnienie klasycznego wyprowadzenia hydrodynamiki NS z równania Boltzmanna przy użyciu rozwinięcia Chapmana-Enskog (ChE). Logikę metody ChE zastosowano także do opisu układu dwutubowego wykazującego stany atermalne, gdzie niespodziewanie odkryto równanie Burgersa w wieloskalowej dynamice gęstości.

Na koniec rozważono problem transportu anormalnego w jednowymiarowych gazach niecałkowalnych. Wykorzystując nieliniową fluktuującą hydrodynamikę, czyli ugruntowaną metodę w tej dziedzinie, znaleziono mechanizm prowadzący do tłumienia anomalii transportowych wraz z odwrotnością parametru łamania całkowalności. Sformułowano także

ogólne przewidywanie, według którego gazy prawie całkowne charakteryzują się trzema różnymi klasami uniwersalności hydrodynamiki, ujawniającymi się w różnych reżimach czasoprzestrzennych.

Słowa kluczowe: hydrodynamika, teoria kinetyczna, układy całkowne, jednowymiarowe gazy kwantowe, termalizacja, transport

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Publication list

This thesis is based on the following publications:

- I. **M. Łebek**, M. Panfil, R. M. Konik “*Prethermalization in coupled one-dimensional quantum gases*”, [SciPost Phys. 17, 007 \(2024\)](#)
- II. **M. Łebek**, M. Panfil “*Navier-Stokes equations for nearly-integrable quantum gases*”, [Phys. Rev. Lett. 134, \(Editors’ Suggestion\) 010405 \(2025\)](#)
- III. **M. Łebek**, M. Panfil “*Chapman-Enskog theory and crossover between diffusion and superdiffusion for nearly-integrable quantum gases*”, [SciPost Phys. 18, 186 \(2025\)](#)
- IV. S. Santra, **M. Łebek**, M. Panfil “*Crossover from generalized to conventional hydrodynamics in nearly-integrable systems under relaxation time approximation*”, [arXiv:2603.02158](#) (under review in Physical Review E)
- V. L. Biagetti, **M. Łebek**, M. Panfil, J. De Nardis “*Generalised Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy for near-integrable dynamics*”, [\(accepted in Physical Review X\)](#)
- VI. P. Lisiak, **M. Łebek**, M. Panfil “*Burgers equation from non-thermal stationary states in nearly-integrable gases*”, [arXiv:2512.09726](#) (under review in Physical Review E)

In addition, the following works beyond the scope of this dissertation were completed during the course of the author’s doctoral studies:

1. J. Kopyciński, **M. Łebek**, W. Górecki, K. Pawłowski “*Ultrawide dark solitons and droplet-soliton coexistence in a dipolar Bose gas with strong contact interactions*”, [Phys. Rev. Lett. 130, 0434010 \(2023\)](#)
2. M. Marciniak, **M. Łebek**, J. Kopyciński, W. Górecki, R. Ołdziejewski, K. Pawłowski “*Super-Tonks-Girardeau Quench in the Extended Bose-Hubbard Model*”, [Phys. Rev. A 108, 043304 \(2023\)](#)
3. J. Kopyciński, B. Tuzemen, W. Górecki, K. Pawłowski, **M. Łebek** “*Propagation properties and stability of dark solitons in weakly interacting Bose-Bose droplets*”, [J. Phys. B: At. Mol. Opt. Phys. 57 035302 \(2024\)](#)
4. **M. Łebek**, J. Kopyciński, W. Górecki, R. Ołdziejewski, K. Pawłowski “*Phase diagram and elementary excitations of strongly interacting droplets with non-local interactions*”, [J. Phys. B: At. Mol. Opt. Phys. 58 065301 \(2025\)](#)

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1 Introduction

This thesis concerns problems related to non-equilibrium dynamics of gases. The questions addressed are quite natural and easy to state. How does the gas reach thermal equilibrium? How can we describe its large-scale motion and capture phenomena such as dissipation or sound propagation? How can these phenomena be derived from the microscopic dynamics of particles given by Newton's laws or the Schrödinger equation?

With the simplicity of these fundamental questions comes the difficulty of providing an answer. Solely due to computational complexity, it is indeed extremely hard to study a system of N interacting degrees of freedom, where we typically desire N to be extremely large and of order $N \sim 10^{23}$. Being far from equilibrium, where much of the universality encoded in thermodynamics is lost, makes the problem even more difficult.

Remarkably, as one can read in textbooks on non-equilibrium statistical mechanics and kinetic theory [1–4], it is possible to find satisfactory answers to these questions by connecting the microscopic dynamics of particles to macroscopic phenomena known from our everyday lives. This was achieved through decades of progress in kinetic theory, mainly for weakly interacting and dilute gases. Many of the mysteries related to the nature of the thermalization process or the origins of hydrodynamics were resolved. The central theme behind these achievements is the successive reduction of complexity by discarding irrelevant degrees of freedom, see Fig. 1.

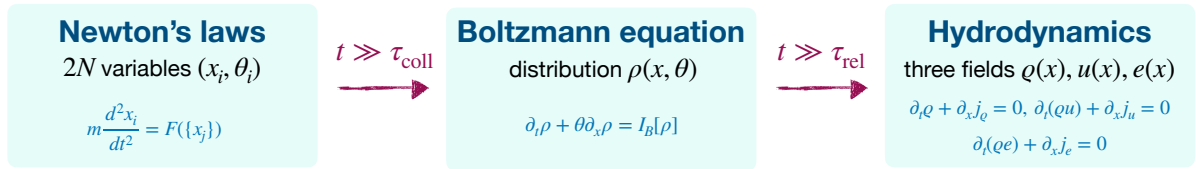


Figure 1: Successive reduction of complexity with one-dimensional classical gas as an example. Exact description in Newton's picture involves $2N$ positions x_i and velocities θ_i . In the Boltzmann theory, applicable at times larger than the characteristic time between two collisions $t \gg \tau_{\text{coll}}$, the degrees of freedom are reduced to single-particle distribution $\rho(x, \theta)$ on the phase space. At even larger times, much bigger than the relaxation time scale τ_{rel} , the gas can be described by just three hydrodynamic fields $\varrho(x), u(x), e(x)$.

Firstly, the extremely complex dynamics of an interacting gas was understood with the approach developed by Boltzmann in the second half of the 19th century. His framework pictured the dynamics of a dilute gas as a series of two-body collisions. These processes are encoded in the *Boltzmann collision integral*, which links two-body (or few-body more generally) scattering physics with processes happening on the scale of the whole gas, such as thermalization. A remarkable reduction of complexity was achieved. From there, characteristic relaxation timescales could be extracted and connected to microscopic details of interactions as encoded in the interaction potential between particles.

The key results of Boltzmann were leveraged by Chapman and Enskog forty years after his discovery. With the collision integral as the starting point, they managed to reduce the complexity even further and derive the hydrodynamic description of the gas, arriving at the famous Navier-Stokes equations, previously known and understood only phenomenologically. Not only did they recover their universal form but they also determined transport coefficients, such as viscosities and thermal conductivity in terms of the scattering data encapsulated in the Boltzmann collision kernel. Later, an alternative approaches has appeared, such as the methods of Grad [5] and Resibois [6], all leading to the same expressions for transport coefficients.

The main goal of this thesis is to complete such a program for a new class of systems. The most well-known examples, for which this has been achieved are dilute or weakly interacting gases and hard-sphere systems [1]. In this thesis we focus on *nearly-integrable one-dimensional*

gases. By this we mean weakly perturbed integrable models. Integrable gases include models such as the Lieb-Liniger [7] model or the one-dimensional hard rod gas [8].

The thesis has the following structure. It starts with an introduction aimed at guiding the reader through the most important concepts relevant to the achieved results. We detail the subsequent sections of the introduction shortly. After the introduction, we present the articles comprising the thesis in Sec. 2. Each article is preceded by a short summary explaining its scientific context and the most important findings. The thesis closes with Sec. 3 containing the summary and conclusions.

We now describe the contents of the introduction. In Sec. 1.1, we introduce the object of study of the thesis, namely nearly-integrable gases. We briefly discuss the mathematical frameworks behind their theoretical description as well as the experimental setups in which they naturally arise. Subsequently, we outline the main frameworks used to treat their dynamics, including generalized Gibbs ensembles in Sec. 1.2, generalized hydrodynamics in Sec. 1.3, and Fermi's golden rule approach to integrability breaking in Sec. 1.4 and 1.5. We also touch upon the hydrodynamics of conventional (non-integrable) one-dimensional gases and fluids in Sec. 1.6 and 1.7. The purpose of this introduction is to set the stage for the results presented in Articles I–VI, constituting new contributions to the topic of weak integrability breaking achieved throughout this PhD project.

1.1 Preliminaries: integrable and nearly-integrable gases

The experimental progress of recent decades has made it possible to create quantum gases with a remarkable degree of control over the interactions [9] and trapping potentials [10]. In particular, external potentials were used to confine ultracold gases into tight cylindrical or pancake traps, offering the possibility to study matter in lower dimensions. In this work, we will focus on one-dimensional gases, see Fig. 2(a). Many-body physics in one dimension is special due to the existence of so-called interacting integrable models [11–13]. Such models are interesting for a number of reasons, which will become clear shortly.

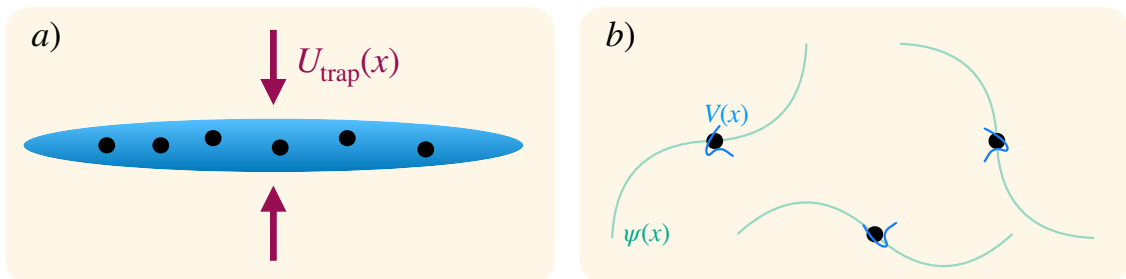


Figure 2: The LL model is the effective description of ultra cold gases in one dimension. Panel a): One-dimensionality is reached by restricting the motion of the atoms by an transversal, external confining potential $U_{\text{trap}}(x)$. Panel b): At very low temperatures, characteristic length scales of wavefunctions $\psi(x)$ are much larger than the typical range of Van der Waals interaction potential. In such a regime, the interparticle potential can be approximated by a delta function $V(x) = c\delta(x)$, as in the LL model.

Importantly, these models are by no means exotic, as an integrable model appears as an effective description of many one-dimensional gases when the temperature is very low [14], see Fig. 2(b). Intuitively, this is due to a simplification of the interaction potential: when the characteristic length scales of the particle wavefunctions are much larger than the range of the interatomic potential, the whole complex shape of the potential can be replaced by a delta function. A gas of such contact-interacting bosonic particles in one dimension is known as the

Lieb-Liniger (LL) model [7, 15]. The Hamiltonian is given by

$$H_{\text{LL}} = -\frac{1}{2} \sum_i \partial_{x_i}^2 + c \sum_{i < j} \delta(x_i - x_j). \quad (1)$$

Here and in what follows, we set the mass of the particles to unity and $\hbar = 1$. This model, being a prominent example of an integrable system, has a number of special features. Firstly, it is exactly solvable by the method of the Bethe Ansatz (BA) [16, 17]. This, in essence, means that we can find all the eigenstates of the system, which take the form of the Bethe wavefunction (below, S_N denotes the set of all permutations of N elements)

$$\Psi_{\theta_1, \dots, \theta_N}(x_1, \dots, x_N) \propto \sum_{P \in S_N} A(P) \exp[i(\theta_{P_1} x_1 + \dots + \theta_{P_N} x_N)] \quad (2)$$

parametrized by N quasimomenta (or rapidities) $\theta_1, \dots, \theta_N$. Quasimomenta fulfill transcendental equations, known as the Bethe equations

$$\theta_j L = 2\pi I_j - 2 \sum_l \arctan\left(\frac{\theta_j - \theta_l}{c}\right), \quad (3)$$

where I_j are different integers (for N odd) or half integers (for N even). The set of distinct $\{I_n\}$ with $n = 1, \dots, N$ specifies the state and should be understood as a generalization of quantum numbers to the many-body case. The energy of an eigenstate is simply given by $E_{\theta_1, \dots, \theta_N} = \sum_i \frac{1}{2} \theta_i^2$. The coefficients in the wavefunctions are

$$A(P) = \prod_{i < j} \left(1 + \frac{ic}{\theta_{P_i} - \theta_{P_j}}\right). \quad (4)$$

The wavefunction (2) is defined in the region $x_1 < \dots < x_N$, but can be straightforwardly extended to the whole space $\mathbb{R}^N$ by replacing the factor $c \rightarrow c \operatorname{sgn}(x_j - x_i)$ in (4).

Secondly, integrable gases are characterized by an infinite number of local, extensive conserved charges Q_n

$$[H, Q_n] = 0. \quad (5)$$

These conservation laws exist in addition to the standard (for Galilean gases) conservation of particle number $N \equiv Q_0$, momentum $P \equiv Q_1$ and energy $H \equiv Q_2$. As we will see later, the existence of such additional conserved quantities has dramatic consequences for the dynamics.

The BA structure is also found in other integrable models. In the context of quantum gases, the most important counterpart of the LL model is the fermionic Yang-Gaudin model [18, 19] of a delta-function interacting Fermi gas. The additional (spin) degree of freedom introduces more complexity into the equations through the existence of another set of parameters in addition to quasimomenta. Another interesting example is the quantum version of hard rod gas [20], for which exact, closed formulas for correlations in an interacting quantum many-body system [21, 22] were obtained recently.

Integrable models of gases also exist in the realm of classical physics. Here, the most relevant model is the hard rod gas (Tonks gas) [8], which represents a system of segments with a diameter a that cannot move through each other. The hard rod model is widely considered testbed [23, 24] for studying the generalized hydrodynamics of integrable systems, which will be described later. Its main advantage is the simplicity of the hard rod interaction, which allows for efficient numerical simulation and some rigorous mathematical treatments [25, 26]. While there is no BA in classical physics, quite surprisingly, classical integrable system share a lot of structural similarities with their quantum counterparts. For instance, they too feature an infinite number of conserved charges. Moreover, on the level of thermodynamic Bethe ansatz description and generalized hydrodynamics, detailed later, these classes of models are equivalent [27].

For the purpose of this thesis, it is important to mention yet another example of an integrable system, namely non-interacting gases. Very well studied in basic courses, they too formally fall into the structure of BA integrable models. It is obvious that their eigenstates are given by wavefunction of the form (2) (for instance, for free bosons we have $A(P) = 1$). They also admit an infinite set of conserved quantities. The case of a non-interacting integrable model will be especially important for setting up the study of nearly-integrable systems. For such models, many perturbative results were obtained in the past. One of the aims of this thesis is to generalize them to the interacting case.

Let us close this short discussion of integrable systems with a remark, that the list of integrable Hamiltonians is still not closed and there are active efforts in order to make it longer. Among some of the research directions aimed at finding systems with additional conservation laws are the so called $T\bar{T}$ deformations [28–31], the free fermions in disguise [32] and integrable quantum circuits [33–37]. It is also worth to mention exploring integrability in open setups [38–41] and devising numerical algorithms in order to find local integrals of motion [42–45] or detecting them on quantum computers [46].

In experiments, it is difficult to realize an integrable Hamiltonian exactly and we often deal with nearly-integrable systems, see Fig. 3. It is necessary to account for factors like external confinement, losses, couplings between nearby 1D systems, and deviations from the contact form of the interparticle potential. Many of these effects can be modeled by a small perturbation of an integrable Hamiltonian

$$H = H_{\text{int}} + V. \quad (6)$$

A generic perturbation V of an integrable Hamiltonian breaks its integrability. The model is no longer solvable, and most of the charges Q_n are conserved only approximately. Due to the ubiquity of integrability-breaking effects, it is crucial to understand how these can be modeled on a theoretical level. In other words, one seeks a perturbative formalism developed around interacting, integrable theories.

Perhaps the most common type of integrability breaking, an external potential, is quite well understood, both at the level of equilibrium and non-equilibrium physics. It corresponds to the perturbation $V = \int dx U(x) \varrho(x)$, where $\varrho(x)$ is the density operator and $U(x)$ is the shape of the external trap. At the level of equilibrium physics, it is typically accounted for by the so-called local density approximation [47–51]. Out of equilibrium, it gives rise to force terms in hydrodynamic equations for integrable models [52–56].

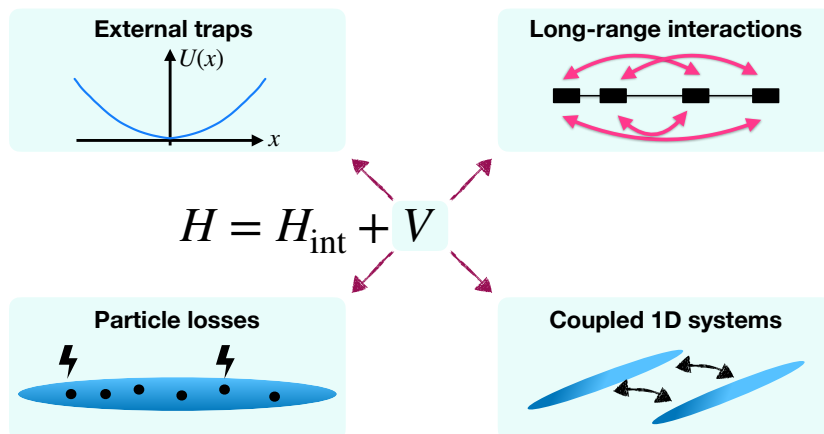


Figure 3: Different types of integrability breaking relevant for the experiments with one-dimensional quantum gases.

Particle losses, on the other hand, can be effectively modeled by a kinetic approach, and the dynamics of a gas with losses can be studied by a relevant Boltzmann-like equation [57–59].

In this thesis, we would like to study another type of integrability breaking introduced by weak long-range interactions, which occur naturally, for example, in ultracold dipolar gases. We will describe the specific details of the dipolar interaction potential a bit later; for now, let us focus on generic properties of such systems described by the following perturbation

$$V = \frac{1}{2} \int dx dy V(x-y) \varrho(x) \varrho(y). \quad (7)$$

On the other hand, when we think about integrability breaking through long-range coupled 1D systems, cf. Fig. 3, we have

$$V_{\text{inter}} = \int dx dy V(x-y) \varrho_1(x) \varrho_2(y), \quad (8)$$

where $\varrho_i(x)$ is particle density in i -th tube. In our studies, we will often assume the interaction potential takes the form

$$V(x-y) = \frac{V_0}{\xi} \mathcal{V}\left(\frac{x-y}{\xi}\right), \quad (9)$$

where V_0 measures the strength of the potential, ξ is its characteristic range, and $\mathcal{V}(x)$ is the function giving the shape of the potential. Let us also note that the Hamiltonian with the above perturbation exactly conserves particle number, momentum, and energy. At the same time, it breaks all the higher charges.

There are two examples of nearly-integrable Hamiltonian (6) with integrability breaking perturbation (7) that were studied extensively in the past. The first one is the case of a non-interacting integrable Hamiltonian with $H_{\text{int}} = H_{\text{free}}$. The *weakly interacting gas* was the subject of intensive studies both in equilibrium statistical mechanics and in kinetic theory. On the side of equilibrium approaches, the central framework was probably the Mayer expansion [60, 61], which should be understood as a perturbative scheme for calculating the equilibrium thermodynamic quantities such as free energy or correlation functions.

On the side of dynamics, the weakly interacting gas was described by the famous Vlasov, Landau, and Lennard-Balescu equations [3, 62]. These kinetic equations can be derived from a more general framework, namely the BBGKY hierarchy of correlation functions. The weakness of the interaction allows for the truncation of the hierarchy. The kinetic description led to derivations of the Navier-Stokes equations through perturbation theory [3, 6] or methods like Chapman-Enskog [2] or Grad expansions [5, 62].

The second important example of a nearly-integrable Hamiltonians (6) with (7) is the case of a *long-range interacting hard rod gas*, in which H_{int} coincides with the Hamiltonian of a classical gas of hard rods with diameter a and thus $H_{\text{int}} = H_{\text{hr}}$. This model attracted a lot of attention in the 1960s, mainly on the side of equilibrium studies [63–66]. It provided a basis for an exact derivation of the Van der Waals equation of state [67]. The one-dimensional model is often referred to as the Kac model. The Kac model involves the potential (9) in the exponential form $\mathcal{V}(x) = e^{-|x|}$ in the long-range mean field limit $\xi \rightarrow \infty$. The works addressed not only standard thermodynamic quantities like the free energy [63], but also correlation functions [64, 66].

Lastly, one can also take a more theoretical viewpoint in order to motivate the study of near-integrable models, namely that of the perturbation theory. Often, it is the only available tool to study the effects of the interactions in many-body systems. Perturbation theory relies on the exact solvability of the model around which it is performed. While in three dimensions the available options are scarce and mainly consist of non-interacting theories, much more is possible in one dimension, see Fig. 4. Due to the existence of interacting integrable models, a larger portion of the model space can be covered, provided suitable perturbative techniques are developed around such theories.

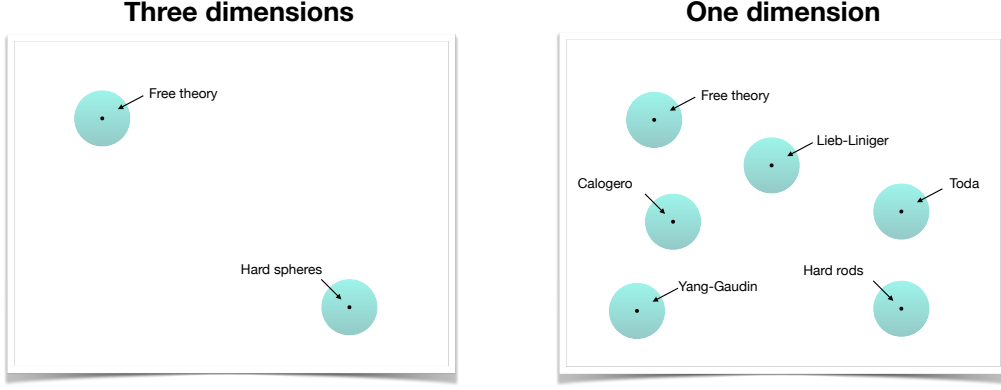


Figure 4: Cartoon of model spaces in 3D and 1D many-body physics. Interacting integrable systems can be used as exactly solvable points, around which perturbation theory is applied.

In what follows, we describe approaches to treat the dynamics of the integrable and nearly-integrable systems. However, before moving on to these descriptions, we discuss the main features of one-dimensional dipolar quantum gases. This is an important experimental realization of a nearly-integrable system in which the integrability is broken through the weak long-range interactions.

1.1.1 One-dimensional dipolar quantum gases

Certain elements, such as dysprosium or erbium have a large magnetic dipole moment. This means that the dipole-dipole forces are strong and one has to include the effects of interaction with a long-range potential, which behaves as $\sim 1/r^3$ at large distances. In general, two identical dipoles interact with a dipolar interaction potential given by

$$V_{\text{DDI}}(\vec{r}) = \frac{\mu_0 \mu^2}{4\pi} \frac{1 - 3(\hat{r} \cdot \hat{B})^2}{r^3}, \quad (10)$$

where $\hat{B}$ is the direction of a strong external magnetic field, typically used in experiments to align the atoms along a fixed direction and μ is the dipole moment.

In one-dimensional experiments, the motion of atoms is restricted through transverse harmonic confinement with a frequency ω . In such 1D setup, the potential (10) gets altered. On a technical level, this results from integrating out the transverse frozen degrees of freedom in the three-dimensional two-body Schrödinger equation with the potential (10). One then obtains a new, one-dimensional equation with an effective potential given by [68, 69]

$$V_{\text{DDI}}^{\text{1D}}(z) = V(\theta_{\text{DDI}}) \left[v_{\text{DDI}}^{\text{1D}}(u) - \frac{8}{3} \delta(u) \right], \quad u = \frac{z}{l_{\perp}}, \quad (11)$$

where $l_{\perp} = \sqrt{\hbar/(m\omega_{\perp})}$ is the transverse oscillator length. The prefactor is

$$V(\theta_{\text{DDI}}) = \frac{\mu_0 \mu^2}{4\pi} \frac{1 - 3 \cos^2 \theta_{\text{DDI}}}{4 l_{\perp}^3}, \quad (12)$$

and the function $v_{\text{DDI}}^{\text{1D}}$ is

$$v_{\text{DDI}}^{\text{1D}}(u) = -2|u| + \sqrt{2\pi} (1 + u^2) e^{u^2/2} \text{erfc}(|u|/\sqrt{2}). \quad (13)$$

In the formula (11) θ_{DDI} is the angle between the atoms and the tube axis. It can be used as an interaction-tuning parameter, as explained in Fig. 5(a). The contact term in the expression (11) can be absorbed into the LL coupling c . The rest plays the role of a long range potential of the form (7) with an interaction range ξ equal to the transversal oscillator length scale $l_{\perp}$.

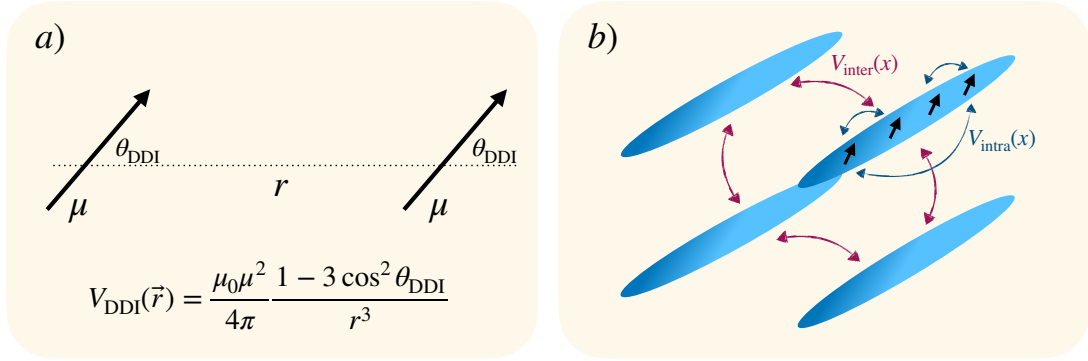


Figure 5: Panel *a*): Two interacting dipoles and the definition of the angle θ_{DDI} in (12). The angle influences the strength and the sign of the dipolar coupling. For the so-called magic angle $\theta_{\text{DDI}}^* = \arccos(1/\sqrt{3})$ the interaction vanishes, whereas for $\theta_{\text{DDI}} < \theta_{\text{DDI}}^*$ it is attractive and for $\theta_{\text{DDI}} > \theta_{\text{DDI}}^*$ it is repulsive. The angle θ_{DDI} provides a natural way to tune the strength of integrability breaking in experiments. Panel *b*): The geometry of the experimental setup [70]. Dysprosium atoms are confined in a lattice of one-dimensional tubes. An external magnetic field aligns them along a given direction.

Dipolar gases have been realized in many experiments [70–72]. In the context of non-equilibrium dynamics, our reference point will be the work [70], where thermalization induced by integrability-breaking through dipole-dipole interactions was investigated. The geometry of the experiment involved a whole lattice of one dimensional tubes, see Fig. 5(b). Thus, the integrability was broken through both the inter- and intratube dipolar interactions. Note that effects of intratube interactions are captured by (7), whereas the intertube interactions give rise to integrability breaking term (8).

In an experimental context, the way to tune the strength of the dipolar interaction is through the external magnetic field angle θ_{DDI} . The key measured quantity was the thermalization rate as a function of θ_{DDI} . In Article V a theory was developed to explain this results, a comparison with the experimental data is provided therein.

We now proceed to discuss the non-equilibrium dynamics of gases, having in mind mainly integrable and nearly-integrable systems. We start by reviewing ideas particularly relevant for the description of integrable models. An important role in this context is played by the concepts of the Thermodynamic Bethe Ansatz and Generalized Gibbs Ensembles.

1.2 Thermodynamic Bethe Ansatz and Generalized Gibbs Ensembles

The Thermodynamic Bethe Ansatz (TBA) framework was first introduced in the seminal paper [73] in the context of the LL model thermodynamics. Let us review the main features of these results.

The central idea lies in taking the thermodynamic limit of the Bethe equations (3). To do so, it is useful to introduce the concepts of particles and holes. We start with the Bethe equations (3); recall that the state is specified by a set of quantum numbers $\{I_n\}$. Consider the function $\theta(x)$ defined as a solution to the following equation (here, $\{\theta_l\}$ denote quasimomenta fulfilling Bethe equations (3))

$$L\theta(x) = 2\pi x - 2 \sum_l \arctan \left(\frac{\theta(x) - \theta_l}{c} \right). \quad (14)$$

Notice that the quasimomenta θ_i of the particles are fixed by the conditions $\theta(I_i/L) = \theta_i$, $I_i \in \{I_n\}$. On the other hand, in a similar way, we can define the quasimomenta of holes θ_i^h as $\theta(I_i^h/L) = \theta_i^h$, $I_i^h \in \{I_n\}^c$, where $\{I_n\}^c$ represents the unoccupied values of quantum numbers.

We can thus define the total density of holes and particles $\rho_t(x)$ as

$$\rho_t(\theta(x)) = \frac{d\theta(x)}{dx}. \quad (15)$$

Differentiating (14) and taking the thermodynamic limit $N, L \rightarrow \infty$ with $N/L = \text{const.}$ we find

$$\rho_t(\theta) = \frac{1}{2\pi} + \int d\theta' \mathcal{T}(\theta - \theta') \rho_p(\theta'), \quad (16)$$

where for the LL model $\mathcal{T}(\theta) = \frac{c}{\pi} \frac{1}{c^2 + \theta^2}$. Here, we defined the density of particles $\rho_p(\theta)$ (and analogously, the density of holes $\rho_h(\theta)$) as

$$L\rho_p(\theta)d\theta = \text{number of particles in } [\theta, \theta + d\theta], \quad L\rho_h(\theta)d\theta = \text{number of holes in } [\theta, \theta + d\theta]. \quad (17)$$

Obviously, the distributions fulfill $\rho_t(\theta) = \rho_p(\theta) + \rho_h(\theta)$.

In the TBA formalism, instead of quantum numbers $\{I_n\}$, the distributions $\rho_p(\theta)$ and $\rho_h(\theta)$ are used to parametrize the system's eigenstates. It is clear, however, that the distributions are not in one-to-one correspondence with the eigenstates of the system. Indeed, there are many eigenstates (quantum numbers) which give rise to the same distributions. The number of ways to put $L\rho_p(\theta)d\theta$ into $L\rho_t(\theta)d\theta$ boxes is

$$\frac{[L\rho_t(\theta)d\theta]!}{[L\rho_h(\theta)d\theta]![L\rho_p(\theta)d\theta]!}. \quad (18)$$

Such a collection of eigenstates, described by the same distributions $\rho_p(\theta)$ and $\rho_h(\theta)$, is sometimes referred to as *macrostate* and it is natural to expect that one can associate a non-zero entropy with it. The relevant entropy was computed in [73] and is called the Yang-Yang entropy. It reads

$$S = L \int d\theta [\rho_t(\theta) \log \rho_t(\theta) - \rho_p(\theta) \log \rho_p(\theta) - \rho_h(\theta) \log \rho_h(\theta)], \quad (19)$$

where we set the Boltzmann constant to unity $k_B = 1$. Note that this entropy can be understood as the extensive part of the logarithm of (18).

Macrostates are very useful when it comes to the description of non-equilibrium dynamics and relaxation. As the simplest case, imagine that an LL gas is coupled to a heat bath with a fixed temperature T and chemical potential μ . The system is expected to relax to a stationary state corresponding to the minimum of the free energy (in what follows $\beta \equiv 1/T$)

$$G_{\text{th}} = \beta E + \mu N - S. \quad (20)$$

The minimization condition yields

$$\epsilon(\theta) = \mu + \beta\theta^2 - \int d\theta' \mathcal{T}(\theta - \theta') \log(1 + e^{-\epsilon(\theta')}), \quad (21)$$

where we define the pseudoenergy $\epsilon(\theta)$ via $e^{\epsilon(\theta)} = \frac{\rho_h(\theta)}{\rho_p(\theta)}$. The algorithm to finding the thermal state with temperature T and chemical potential μ is as follows. Firstly, equation (21) is solved numerically for the pseudoenergy. From this, one computes the occupation function $n(\theta) \equiv \rho_p(\theta)/\rho_t(\theta) = 1/(1 + e^{\epsilon(\theta)})$. Knowledge of $n(\theta)$ allows for the numerical solution of (16), which finally gives $\rho_p(\theta)$. An exemplary solution to the TBA equations along these lines is presented in Fig. 6.

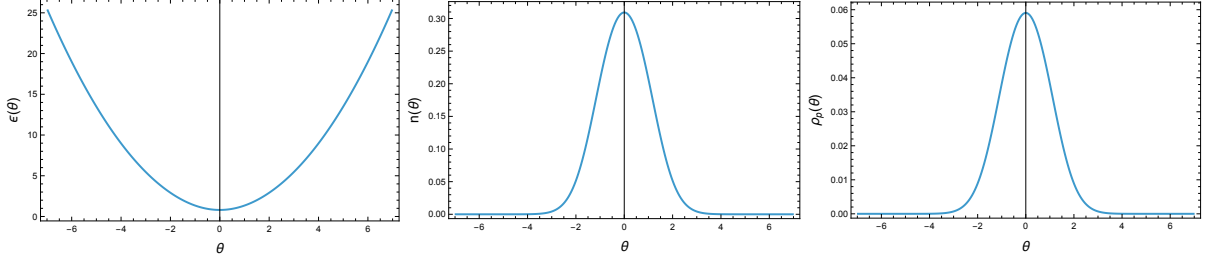


Figure 6: Thermal state of the LL model with $c = 1$. The chemical potential and inverse temperature are $\beta = \mu = 1$. From the left to the right: pseudoenergy $\epsilon(\theta)$, occupation function $n(\theta)$ and density of quasiparticles $\rho_p(\theta)$.

So far we have seen the usefulness of the TBA framework in determining the states which maximize the entropy in the presence of a heat bath with a fixed temperature. Its real power, however, becomes more visible when understanding the stationary states reached under the dynamics of an *isolated system*. In this context, the second property of integrable systems, namely the existence of additional conserved charges, becomes extremely important.

Imagine that a system is initialized in some arbitrary initial state $|\Psi_{\text{ini}}\rangle$ and we let it evolve under the integrable Hamiltonian dynamics. It should be clear that the stationary state reached by the system cannot be the one, which minimizes the free energy given by (20). This is because such a state in principle has different values of charges Q_n , which are conserved throughout the dynamics. The simplest modification to include these constraints is to incorporate Lagrange multipliers for the additional charges into the entropy maximization [74, 75]. In other words, one considers the free energy in the form

$$G_{\text{GGE}} = \sum_n \beta_n Q_n - S, \quad (22)$$

where β_n are the Lagrange multipliers, sometimes called generalized temperatures. The charge expectation values can be represented as

$$Q_n = \int d\theta \rho_p(\theta) \hbar_n(\theta), \quad (23)$$

where $\hbar_n(\theta)$ are single-particle charge eigenvalues. For instance, the energy is characterized by $\hbar_n(\theta) = \frac{1}{2}\theta^2$. We will assume that the set $\{\hbar_n(\theta)\}$ is complete. The equation for the pseudoenergy then modifies to

$$\epsilon(\theta) = \epsilon_0(\theta) - \int d\theta' \mathcal{T}(\theta - \theta') \log(1 + e^{-\epsilon(\theta')}), \quad (24)$$

with the bare pseudoenergy $\epsilon_0(\theta) = \sum_n \beta_n \hbar_n(\theta)$ specifying the state. The generalization of the free energy to the case with an infinite number of conserved charges is at the heart of the idea behind generalized Gibbs ensembles (GGEs). The problem of relaxation to the GGE, given formally by the density matrix

$$\rho_{\text{GGE}} \propto e^{-\sum \beta_n Q_n} \quad (25)$$

in quantum models, was studied extensively over the last twenty years. The typical protocol, often called *integrable quantum quench* consists of preparing the system in the ground state of some integrable Hamiltonian and rapidly changing the interaction parameter (for instance, the LL parameter c). The relaxation dynamics that follows has been the subject of many theoretical [76–82] and experimental [83, 84] investigations.

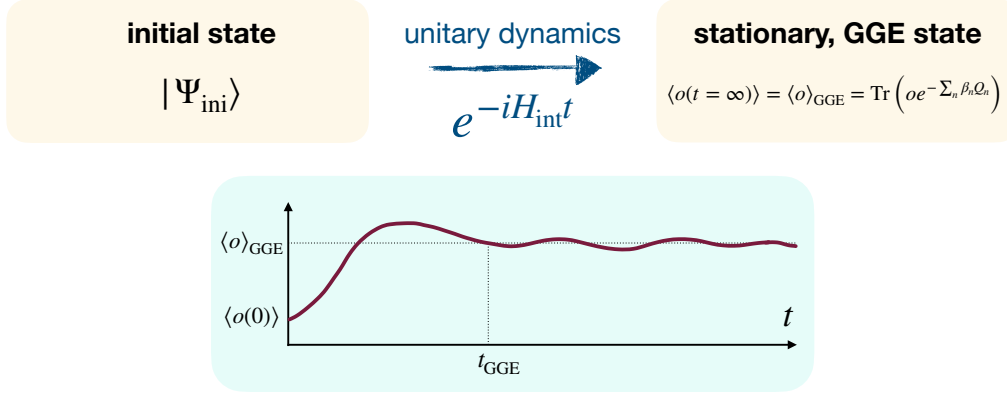


Figure 7: Relaxation to the stationary state in integrable system. Relaxation happens over a time scale t_{GGE} and expectation values of local observables o in the stationary state are given by expectation values computed with GGE density matrix (25).

What will be a basing assumption in our investigations is that *integrable systems generically relax to GGEs*. This means that the expectation values of local observables reached in stationary states are given by the GGE density matrix (25). The Lagrange parameters $\{\beta_n\}$ are fixed by the values of the conserved charges Q_n in the initial state $|\Psi_{\text{ini}}\rangle$ and are in a one-to-one correspondence with them. A similar correspondence holds between the Lagrange multipliers $\{\beta_n\}$ and the distribution $\rho_p(\theta)$. The different parametrizations of the GGE states are used interchangeably, depending on the context.

While quenches in integrable theories are the most relevant for our investigations, the literature on quenches defined more broadly, as protocols in which the non-equilibrium dynamics is driven by changing the parameters of the Hamiltonian is very vast. In particular, we mention here quenches in conformal field theories [85–87] and quenches across the continuous phase transition [88–94] as important examples.

In our presentation, the TBA formalism has appeared as a consequence of the BA description of the LL model. However, the framework is more universal and encompasses not only BA-solvable quantum models (such as the Yang-Gaudin gas) but also, surprisingly, classical integrable systems of many-body physics. In classical realm, one also defines GGEs with a similar interpretation of maximal entropy states constrained by the values of infinite number of conserved charges. In fact, the universal framework requires only two non-universal quantities from a specific integrable model. Firstly, the scattering kernel $\mathcal{T}(\theta, \theta')$ that encodes the details of interactions in the theory. Secondly, the particles' statistics. Many theories of integrable systems are formulated for arbitrary integrable models, making use of the universality in the TBA.

The assumption of relaxation to the GGE is a building block of some of the most successful theories describing the non-equilibrium dynamics of integrable and nearly-integrable models. Perhaps the most well-known is the hydrodynamic theory, dubbed *generalized hydrodynamics*, to which we now turn.

1.3 Generalized hydrodynamics

The development of GGE laid the groundwork for the hydrodynamic theory of many-body integrable systems [27, 95–100], currently known as *generalized hydrodynamics* (GHD). The theory received a considerable attention in the recent years and has been verified in many experiments [101–106]. While it is specific to systems with infinite conserved charges, it is based (with some notable exceptions, see Sec. 1.3.2) on standard hydrodynamic assumptions, known from more familiar types of hydrodynamic theories such as the Navier-Stokes equations. Let us start by reviewing these assumptions, before delving into the details of the GHD formalism.

1. **Conserved quantities** - hydrodynamics works only for systems characterized by conservation of extensive and local charges Q_n

$$Q_n = \int dx q_n(x, t). \quad (26)$$

Hydrodynamic equations can then be seen as continuity equations for the charge densities $q_n(x, t)$

$$\partial_t q_n + \partial_x j_n = 0, \quad (27)$$

where $j_n(x, t)$ are the currents associated with the charges. The crucial step in building a hydrodynamic theory is to determine how the currents depend on the charge densities. This is necessary in order to close the equations and make them well-defined.

2. **Local equilibrium assumption** - we assume that locally, the system is in a state that maximizes entropy under the constraints imposed by the values of the conserved charges. In the context of integrable systems, this means that the system is locally described by a GGE. On the other hand, for a more standard non-integrable gas and the Navier-Stokes equations this means that the state is locally thermal.
3. **Separation of length and time scales** - this assumption clarifies the meaning of locality in the previous assumption. We assume that the densities of the conserved charges vary over length scales l_h , which are much larger than the microscopic correlation length of the gas $l_h \gg l_{mic}$. Likewise, we assume that the dynamics of q_n occurs over time scales t_h , much longer than typical relaxation time to the maximal entropy state $t_h \gg t_{rel}$. In the context of GHD microscopic relaxation time is the relaxation time to GGE $t_{rel} = t_{GGE}$, see Fig. 7.
4. **Gradient expansion of the currents** - the dependence of the currents on the charge densities takes the form of a *gradient expansion*, namely

$$j_n(x, t) = j_n^{Eul}(\{q_k(x, t)\}) - \frac{1}{2} \sum_m D_{nm}(\{q_k(x, t)\}) \partial_x q_m(x, t) + \dots, \quad (28)$$

where D_{nm} denotes the diffusion matrix. Truncating this gradient expansion at the k -th order corresponds to k -th order hydrodynamics. Typically, the zeroth order theory is referred to as *Euler hydrodynamics*. At this order, the dynamics is reversible and no entropy is produced. This changes at the next order, the so-called *diffusive* scale. While in principle one can carry out this expansion to arbitrary order, problems such as instabilities at small length scales or no guarantee of positive entropy production, appear at higher orders [107].

1.3.1 Euler scale GHD

In our discussion of the GHD, we will focus mostly on the theory at the Euler scale. At this order, the problem of finding the currents was solved in [95, 96] and the answer is particularly simple to state in the language of TBA. The local equilibrium Assumption 2 in the case of integrable systems is expressed as follows. One assumes that the system evolves on the manifold of local GGE states, that is, states characterized by the density matrix

$$\rho_{GHD} \propto \exp \left[- \int dy \sum_i \beta_i(y) q_i(y) \right]. \quad (29)$$

This state can also be understood through the division of the gas into the so-called *fluid cells*. Physical intuition behind such a division is presented in Fig. 8.

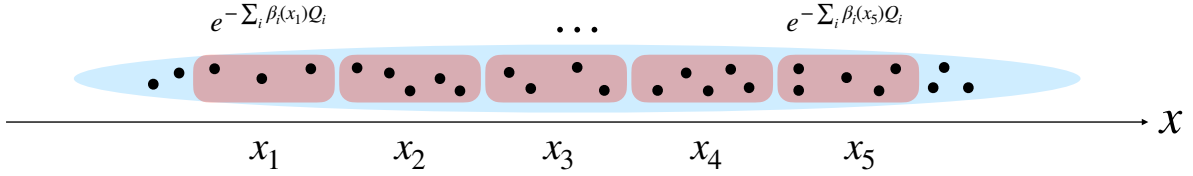


Figure 8: Separation of the gas into fluid cells centered around $x_1, x_2, \dots$ and the meaning of local GGE state (29). The state of a fluid cell at position x is described by a maximal entropy state characterized by a set of Lagrange multipliers $\beta_i(x)$. We assume that the fluid cells are large enough so that they are uncorrelated. The local GGE state is given by a product of states like $e^{-\sum_i \beta_i(x_1) Q_i}$. Continuum version of such a product is the local GGE state (29).

As described in Sec. 1.2, the local GGE state can be equally well described by the space-dependent quasimomentum distribution $\rho_p(x, \theta)$. This representation will be especially useful in the context of the GHD equations. The charge and current densities can be evaluated in the local GGE state [95, 96]

$$\langle q_n(x, t) \rangle = \int d\theta \mathcal{K}_n(\theta) \rho_p(x, \theta; t), \quad j_n^{\text{Eul}}(x, t) = \int d\theta \mathcal{K}_n(\theta) v^{\text{eff}}(x, \theta; t) \rho_p(x, \theta; t), \quad (30)$$

where v^{eff} is the so-called *effective velocity*, which crucially depends on the state of the system through the following integral equation

$$v^{\text{eff}}(\theta) = \theta + 2\pi \int d\theta' \mathcal{T}(\theta - \theta') \rho_p(\theta') \left(v^{\text{eff}}(\theta') - v^{\text{eff}}(\theta) \right). \quad (31)$$

According to the assumption 1, the equations of GHD are an infinite set of continuity equations. In such a form, the theory would not be very useful, as such a set is intractable in practical settings. However, there exists a much more useful representation in terms of the density of particles $\rho_p(x, \theta; t)$. Upon plugging (30) to the continuity equations (27) we find (recall the assumption that $\{\mathcal{K}_n\}$ form a complete set).

$$\partial_t \rho_p + \partial_x \left(v^{\text{eff}} \rho_p \right) = 0. \quad (32)$$

This is perhaps the most well-known form of the GHD equation. Even though it looks simple, the equation is highly nonlinear due to the state-dependent effective velocity. Already here we can observe that even though the equation is hydrodynamic in nature, it structurally looks similar to Liouville equation for free particles [62] known from kinetic theory, which reads

$$\partial_t \rho + \theta \partial_x \rho = 0. \quad (33)$$

We will build upon this similarity in Articles II, III and V.

1.3.2 Extensions

The GHD equation (32) can be extended in various directions. For instance, it can be easily generalized to describe integrability breaking through an external potential with integrability breaking term given by $V = \int dx U(x) \varrho(x)$. The equation is no longer of a continuity type, as a term $-\partial_x U \partial_\theta \rho_p$ appears on the right-hand side [52].

Especially surprising is the extension to the diffusive order, where the GHD starts to escape the standard paradigms of hydrodynamics [24, 37], more precisely, the Assumption 4. For a few years, the consensus of the community was that the proper extension of GHD to the diffusive scale amounts to computing the relevant diffusion matrix, as suggested by (28). The GHD equation on the diffusive scale was believed to be [108–110]

$$\partial_t \rho_p + \partial_x \left(v^{\text{eff}} \rho_p \right) = \frac{1}{2} \partial_x \left(\mathfrak{D} \partial_x \rho_p \right). \quad (34)$$

Recently, it was realized that the equation above is not the correct hydrodynamic equation at the diffusive scale. Let us emphasize here that now diffusive scale should be understood more broadly as compared to the definition in Assumption 4, namely as the scale including terms of the order $1/l_h^2$ in the equations. The breakdown of gradient expansion happens due to the appearance of long-range correlations generated dynamically throughout the evolution [111, 112] - a peculiar feature of integrable systems characterized by an infinite number of conserved quantities. The consequence of this is also a breakdown of local GGE state (29) assumption.

For a proper description of such correlations, one has to resort to an idea of a *correlated fluid cell ensemble* [37, 56] with a density matrix given by

$$\rho \propto \exp \left(- \int dx \beta_i(x) q_i(x) - \int_{x \neq x'} dx dx' \gamma_{i,j}(x, x') \delta q_i(x) \delta q_j(x') + \dots \right). \quad (35)$$

Such a state allows for correlations between fluid cells encoded in the coefficients $\gamma_{i,j}(x, x')$.

The correct GHD description at the diffusive scale in fact consists of two equations: one for the one-point function and the second, for the two-point correlations, for details see [37]. The equation for the one-point function is coupled to the two-point function, thus the structure resembles that of the BBGKY hierarchy for interacting particle systems. Let us point out that this analogy holds even though we still deal with a coarse-grained, hydrodynamic theory as opposed to the fully microscopic BBGKY hierarchy. This observation was pushed further in the development of a generalization of BBGKY hierarchy for nearly-integrable systems, which is one of the main of the results of the present thesis, see Article V.

The last extension of the GHD, in fact the most important for the purposes of this thesis, is the extension to include the effects of weak integrability breaking through long-range interactions. In such a case, an additional Boltzmann collision term appears in the equation. We discuss it in Sec. 1.5. To prepare the ground, we now turn to the derivation of the collision term, which arises due to integrability breaking already in homogeneous systems, in which the GHD dynamics is absent.

1.4 Fermi's Golden Rule approach to integrability breaking and thermalization

In this section, we shift our attention to the study of nearly-integrable systems whose integrability is broken through additional long-range interactions, i.e. perturbations of the type (7) and (8). To simplify the picture, we first consider a homogeneous setting, where no hydrodynamic flows take place.

The basic tool to perturbatively study the dynamics of a quantum many-body system is the time-dependent perturbation theory. In the leading order, such a framework relies on *Fermi's Golden Rule (FGR)*, which states, that the transition rate $\Gamma_{i \rightarrow f}$ from an initial state $|i\rangle$ to a final state $|f\rangle$ is given by the following formula

$$\Gamma_{i \rightarrow f} = 2\pi |\langle f | V | i \rangle|^2 \rho(E_f) \quad (36)$$

where $\rho(E_f)$ is the density of states at energy E_f .

The formalism of FGR was employed to study nearly-integrable quantum systems in the works [110, 113, 114]. The computation once again was based on the concept of the GGE - the FGR formula then dictates what the rates for jumping between different states on the GGE manifold are. The whole computation relied on the knowledge of matrix elements - so-called *form factors* of the perturbing operator. For the case of the perturbation (7) one needs to know the form factors $\langle f | \varrho | i \rangle$ of the density operator (form factors $\langle f | \varrho(x, t) \varrho | i \rangle$ are expressible with it). Such form factors were computed for the LL model in [115–120].

These studies explored what are the states $|f\rangle$, which lead to the largest matrix elements, in the thermodynamic limit. In our case of the potential (7), in order for $\langle f | \varrho | i \rangle$ to be large, the

states $|i\rangle$ and $|f\rangle$ should be, in a certain sense, close to each other. It turns out that the relevant states in this context can be understood using analogies to the particle-hole (ph) excitations familiar from the Fermi gas, see Fig. 9.

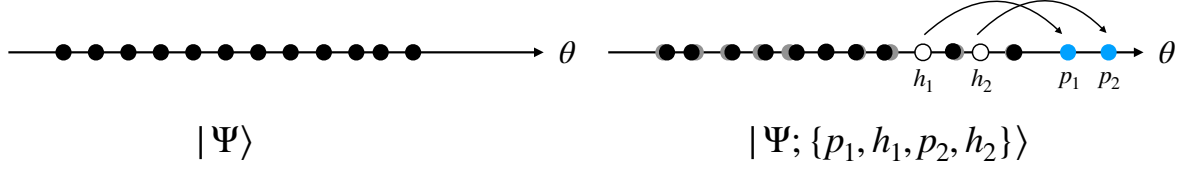


Figure 9: Pictorial representation of elementary excitations over an LL state. Left: An eigenstate $|\Psi\rangle$ specified by quasimomenta θ_i represented by black dots. Right: the same state, but with two ph excitations. A particle is removed (a hole is created) at rapidity h_i and a particle is then created at rapidity p_i . Due to the interactions in the theory, there is a backflow, meaning that all the other particles experience a small quasimomentum shift due to the creation of the ph excitation. This is indicated by the shift of the black dots from the gray ones, representing quasimomenta before making the ph excitation.

However, there are important differences between a non-interacting Fermi gas and an interacting, integrable model. Most importantly, the quasimomenta must fulfill the Bethe equations (3); thus, the creation of a ph excitation must be followed by a slight change in the other quasimomenta. In other words, the other particles must readjust their quasimomenta, and the excitations exhibit backflow effects. This is seen clearly in the expressions for the momentum and energy of an excitation. The naive expressions $p_i - h_i$ for the momentum and $\frac{1}{2}p_i^2 - \frac{1}{2}h_i^2$ for the energy are modified to $k(p_i) - k(h_i)$ and $\varepsilon(p_i) - \varepsilon(h_i)$, respectively. The functions $k(\theta)$ and $\varepsilon(\theta)$ are state-dependent functions known as the Dressed momentum and the Dressed energy

$$k(\theta) = \theta - \int d\theta' n(\theta') F(\theta'|\theta), \quad \varepsilon(\theta) = \frac{1}{2}\theta^2 - \int d\theta' n(\theta') F(\theta'|\theta)\theta'. \quad (37)$$

They are expressible in terms of the so-called backflow function $F(\theta|\mu)$, which accounts for the quasimomenta readjustment according to the Bethe equations. In the case of the LL model, it satisfies the following integral equation [12]

$$F(\theta|\mu) = \frac{\arctan((\theta - \mu)/c)}{\pi} + \int d\theta' \mathcal{T}(\theta - \theta') n(\theta') F(\theta'|\mu). \quad (38)$$

For later use, we also define another operator related to the backflow function, namely

$$\mathbb{F}(\theta, \mu) = \delta(\theta - \mu) + \partial_\theta (n(\theta) F(\theta|\mu)), \quad (39)$$

with which we can represent the Dressed momentum and energy as $k(\theta) = \mathbb{F} \cdot \theta$ and $\varepsilon(\theta) = \mathbb{F} \cdot (\frac{1}{2}\theta^2)$.

1.4.1 Kinematic picture of relaxation

To connect with the standard intuitions behind non-equilibrium dynamics in gases, one can associate a scattering process with the creation of ph excitations. Indeed, in such a process what happens is that the particles with quasimomenta $\{h_i\}$ change their quasimomenta to new values $\{p_i\}$. This is precisely what a scattering entails, see Fig. 10(a).

This observation transforms the question about the relevant matrix elements into a question of the corresponding scattering processes. As the dynamics conserve momentum and energy, the states before and after scattering should have exactly the same momentum and energy. This imposes a restriction on the outgoing momenta in the scattering process

$$k(p_1) + \dots k(p_n) = k(h_1) + \dots k(h_n), \quad \varepsilon(p_1) + \dots \varepsilon(p_n) = \varepsilon(h_1) + \dots \varepsilon(h_n). \quad (40)$$

These conservation laws already provide a significant insight into the dynamics. The leading order process in the dynamics of systems in three dimensions is two-body scattering. This was noted already by Boltzmann. In one dimension, the situation is different, see Fig. 10(b). The equations (40), specified to $n = 2$, have a very simple solution, namely

$$p_1 = h_2, \quad p_2 = h_1. \quad (41)$$

Obviously, such a process cannot change the state specified by the density of quasimomenta $\rho_p(\theta)$. This peculiar behaviour, which stems from the kinematic properties of scattering in 1D Galilean-invariant models is known as *kinetic blocking* [121, 122]. Two-body collisions are not sufficient to drive the system to a thermal state. A non-trivial reshuffling of the momenta happens only at the level of three-body scatterings, as the constraints (40) for $n = 3$ allow for non-trivial solutions for the outgoing quasimomenta.

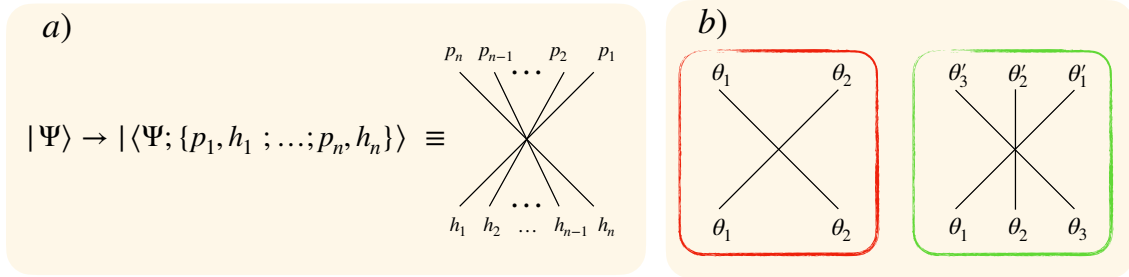


Figure 10: Panel a): We can associate an n -body scattering process with a matrix element involving n -ph excitation over the initial state. Panel b): Two-body scattering processes are not sufficient for thermalization in one dimension, as the only outcome is the exchange of quasimomenta. The leading processes correspond to the three colliding particles.

With the rates as computed from the FGR, one can write down a nonlinear master equation governing the dynamics of the state. Such an equation involves processes describing the scattering of three or more particles

$$\mathcal{I}[\rho_p] = \sum_{n \geq 3} \mathcal{I}^{n\text{ph}}[\rho_p]. \quad (42)$$

Thus, in order for such a description to be useful, a hierarchy between these scattering processes is needed to truncate this infinite sum. For the case of long-range interaction potential (7) such a hierarchy is provided by the large range of the interaction potential ξ (large compared to the microscopic length of the integrable model). The dominant process responsible for thermalization corresponds to $n = 3$. In this case, one finds the following structure

$$\partial_t \rho_p(p_1) = \mathcal{I}^{3\text{ph}}[\rho_p](p_1), \quad \mathcal{I}^{3\text{ph}} = \mathbb{F} \cdot \mathcal{I}_0^{3\text{ph}} \quad (43)$$

with the so-called bare collision integral given by

$$\begin{aligned} \mathcal{I}_0^{3\text{ph}}[\rho_p](p_1) = & \int dh_1 dh_2 dh_3 dp_2 dp_3 \mathcal{W}(\{h_1, h_2, h_3\} \rightarrow \{p_1, p_2, p_3\}) \delta(k) \delta(\varepsilon) \times \\ & [\rho_p(h_1) \rho_h(p_1) \rho_p(h_2) \rho_h(p_2) \rho_p(h_3) \rho_h(p_3) - (p_i \longleftrightarrow h_i)], \end{aligned} \quad (44)$$

where $\delta(k)\delta(\varepsilon)$ enforces the momentum and energy conservation laws (40). Moreover, the term $\mathcal{W}(\{h_1, h_2, h_3\} \rightarrow \{p_1, p_2, p_3\})$ is the rate associated with the three-particle scattering process computed from the FGR. The backflow operator in (43) accounts for the quasimomentum readjustment occurring in the gas due to the scattering process.

It is worth mentioning that one can show that the only fixed points of $\mathcal{I}^{3\text{ph}}$ are the thermal states, i.e., TBA states which minimize the free energy (20). Moreover, the dynamics given

by (43) is characterized by three conservation laws: particle number Q_0 , total momentum Q_1 and total energy Q_2 , as reflected in the following property of the collision kernel

$$\int d\theta \mathcal{K}_n(\theta) \mathcal{I}[\rho_p](\theta) = 0, \quad n = 0, 1, 2. \quad (45)$$

These two features will be crucial for determining the properties of the hydrodynamics discussed in what follows.

1.5 Inhomogeneous nearly-integrable gases: GHD-Boltzmann equation

After discussing thermalization and the collision integral in a homogeneous setting, we would like to study the dynamics of inhomogeneous states in nearly-integrable systems. In the preceding sections we also described GHD, which is the theory for the dynamics of inhomogeneous states in integrable systems. It is natural to expect that both of these frameworks should be relevant for inhomogeneous state dynamics under weak integrability breaking.

Once again, we adapt the fluid cell picture of Fig. 8 in which the gas is divided into cells comprising a large number of particles. Our central assumption is that for sufficiently smooth inhomogeneous states, the effects of integrability breaking can be described as a local collision integral acting within a single cell. In this way we consider the following equation [110, 123–127]

$$\partial_t \rho_p + \partial_x \left(v^{\text{eff}} \rho_p \right) = \frac{1}{2} \partial_x \left(\mathfrak{D} \partial_x \rho_p \right) + \mathcal{I}[\rho_p], \quad (46)$$

to which we will refer as the *GHD-Boltzmann equation*. The assumption of a locally acting collision integral is standard in kinetic theory for studying inhomogeneous, smoothly varying states. Let us note that the GHD-Boltzmann equation is very challenging to numerically solve in general. This is because performing a timestep requires the numerical evaluation of the collision integral (43) at all points in space, which is computationally very expensive. The complexity of the GHD-Boltzmann equation calls for simplifications for both practical and more fundamental reasons. Let us point out that as soon as the collision integral is considered (46) is no longer a hydrodynamic equation. At the same time it still features three conservation laws. Conserved are particle number, momentum and energy. Therefore, there should exist a simpler description of the gas, which comprises three continuity equations for these charges as per Assumption 1.

The main achievement of Articles II and III is the derivation of such equations from the GHD-Boltzmann equation. The reasoning was based on a generalization of the Chapman-Enskog (ChE) expansion, typically formulated for the standard Boltzmann equation describing a weakly interacting classical gas. Indeed, the GHD-Boltzmann equation (46) is structurally is not that far from the classic Boltzmann equation, here written for a 1D system

$$\partial_t \rho + \theta \partial_x \rho = \mathcal{I}_B[\rho]. \quad (47)$$

The ChE method essentially turns the limit of large spatiotemporal scales into a perturbative treatment of the GHD-Boltzmann equation. However, let us skip here the description of the method and refer the Reader to Article III, which presents an exhaustive exposition of it. We turn now to the equations which emerge from this expansion, namely the one-dimensional version of the Navier-Stokes equations.

1.6 Conventional hydrodynamics of 1D non-integrable gases

For a one-dimensional particle system in which only particle number, momentum, and energy are conserved, according to standard logic of hydrodynamics, one expects three continuity equations. With Galilean invariance, the relevant equations are the Euler incompressible equations at the Euler scale and the Navier-Stokes (NS) equations at the diffusive scale. The local equilibrium assumption states that the gas is locally described by a boosted thermal state. The standard

degrees of freedom used are the particle density $\varrho(x, t)$, the velocity field $u(x, t)$ and the internal energy $e(x, t)$, or temperature $T(x, t)$ fields. The NS equations are

$$\begin{aligned}\partial_t \varrho &= -\partial_x(\varrho u), & \partial_t(\varrho u) &= -\partial_x(\varrho u^2 + \mathcal{P}), \\ \partial_t(\varrho e) &= -\partial_x(u\varrho e + \mathcal{J}) - \mathcal{P}\partial_x u,\end{aligned}\tag{48}$$

where the hydrodynamic pressure $\mathcal{P}$ and heat current $\mathcal{J}$ are given by

$$\mathcal{P} = P - \zeta \partial_x u, \quad \mathcal{J} = -\kappa \partial_x T,\tag{49}$$

with ζ denoting the (bulk) viscosity, κ the thermal conductivity and P the thermodynamic pressure.

The universal NS equations require microscopic knowledge about the gas that we want to describe. First of all, we need to know its thermodynamics, as encoded in the thermodynamic pressure P . From a theoretical perspective this is already a very big challenge, as computation of the partition function of an interacting system is typically very difficult. It is even harder to determine the transport coefficients ζ and κ . These quantities involve the dynamical properties of the system, and we would like to point out two typical approaches using which they can be computed.

1.6.1 Transport coefficients and the link to kinetic equations

First of all, the transport coefficients can be computed from current-current correlation functions in equilibrium states through the Green-Kubo relations

$$\kappa = \frac{1}{T^2 L} \int_0^\infty d\tau \langle \delta J_Q(\tau) \delta J_Q(0) \rangle, \quad \zeta = \frac{1}{TL} \int_0^\infty d\tau \langle \delta J_P(\tau) \delta J_P(0) \rangle,\tag{50}$$

where L is the system size, J_Q and J_P are total heat and momentum currents, respectively. The advantage of this formulation is that such correlation functions are sometimes accessible from numerical simulations of homogeneous systems at equilibrium. This provides a way to compute transport coefficients in certain cases amenable to molecular dynamics simulations.

The second way to determine ζ and κ is through a kinetic theory computation. In cases, where a collision integral can be written and trusted, for instance for weakly interacting, dilute, or nearly-integrable gases, the transport coefficients can be computed from the Boltzmann kernel through the Chapman-Enskog expansion. For a weakly interacting gas, for instance, we get the following results

$$\kappa = \frac{1}{2} \int d\theta \rho_{\text{th}}(\theta) \theta^3 \phi(\theta), \quad \zeta = 0,\tag{51}$$

where $\rho_{\text{th}}(\theta)$ is the thermal distribution (Maxwell-Boltzmann distribution) at temperature T and density ϱ . The function $\phi(\theta)$ satisfies the following integral equation

$$\Gamma_B \phi = \chi, \quad \chi(\theta) = \rho_{\text{th}}(\theta) \frac{\theta}{T^2} \left(\frac{\theta^2}{2} - \frac{3T}{2} \right),\tag{52}$$

where $\Gamma_B = -(\delta I_B)/(\delta \rho)|_{\rho_{\text{th}}}$. We remind that here I_B denotes the Boltzmann collision integral. The generalization of these formulas to the case of nearly-integrable gases constitutes the central result of Article II.

1.6.2 Transport anomalies

The NS equations in one dimension famously break down when it comes to the description of certain observables, as noted already in [128–130]. This is seen most clearly in the equilibrium

correlation functions. For instance, the total current-current correlation functions develop long-time tails [131], which are non-integrable in time. Thus, the Green-Kubo formulas (50) for the transport coefficients are divergent.

Similar anomalies happen in the hydrodynamic correlation functions of the charge densities, for instance, the connected density-density correlation function $S(x, t) = \langle \varrho(x, t) \varrho(0, 0) \rangle_c$, measured on top of a thermal state. The linearized NS hydrodynamics then predicts the following formula

$$S_{\text{NS}}(x, t) = \varrho^2 \left[\frac{c_P - c_V}{c_P} \frac{1}{\sqrt{4\pi D_{\text{th}} t}} e^{-\frac{x^2}{4D_{\text{th}} t}} + \frac{c_V}{2c_P} \frac{1}{\sqrt{4\pi D_{\pm} t}} \left(e^{-\frac{(x-v_s t)^2}{4D_{\pm} t}} + e^{-\frac{(x+v_s t)^2}{4D_{\pm} t}} \right) \right]. \quad (53)$$

where c_V and c_P are the specific heats at constant volume and pressure, respectively. Moreover, v_s denotes the sound velocity of the gas, and the diffusion coefficients are

$$D_{\text{th}} = \frac{\kappa}{\varrho c_P}, \quad D_{\pm} = \frac{1}{2\varrho} \left[\left(\frac{1}{c_V} - \frac{1}{c_P} \right) \kappa + \zeta \right]. \quad (54)$$

The formula (53) describes three peaks in the correlator. At $x = 0$ there is a heat peak, which broadens diffusively with a constant D_{th} . The other two are sound peaks, which move with the sound velocity in opposite directions. In addition to that, they diffuse with a diffusion constant $D_{\pm}$. The formula (53) is not correct at the largest space-time scales. To correctly capture the correlator, one has to use the nonlinear fluctuating hydrodynamics.

The nonlinear fluctuating hydrodynamics (NLFH) [132, 133] is a well-established theory of correlations in 1D nonintegrable systems with momentum conservation. One of its main predictions is the anomalous broadening of the heat and sound peaks. Instead of a diffusive shape, the heat peak is described by a Levy- $\frac{5}{3}$ distribution with a dynamical exponent $z = 5/3$ instead of $z = 2$. Heat peaks, on the other hand, are described by the scaling function of the 1D Kardar-Parisi-Zhang (KPZ) equation with dynamical exponent with $z = 3/2$. Let us discuss the most important features and predictions of the NLFH.

1.7 Nonlinear fluctuating hydrodynamics for one-dimensional non-integrable gases

The main interest of the NLFH lies in studying fluctuations around the homogeneous, equilibrium state of a non-integrable one-dimensional gas, fluid, or anharmonic chain with three conservation laws. For the case of gases and fluids, these are the particle number, momentum, and energy.

One could argue that the natural starting point to describe the hydrodynamic phenomena close to equilibrium in such a case is the linearized NS equations. The picture that emerges from such a treatment is precisely that of (53). One finds two sound peaks and a heat peak, and they all spread diffusively. NLFH argues that this picture is not true at the large space-time scales. To capture the predicted anomalies, the linearized NS equations have to be supplemented by two ingredients. Firstly, *noise terms have to be added to the currents*. The noise represents degrees of freedom truncated when reducing the many-body dynamics to the hydrodynamic equations. Secondly, *nonlinearities in the current expansion around equilibrium cannot be disregarded*. We detail these modifications in what follows.

1.7.1 Noisy version of the 1D Navier-Stokes equations

In writing down the NLFH it is useful to change the degrees of freedom in the NS equations in a way that better exposes the structure of the continuity equations. We introduce the fields $\varrho_0, \varrho_1, \varrho_2$ defined as

$$\varrho = \varrho_0, \quad u = \frac{\varrho_1}{\varrho_0}, \quad e = \frac{\varrho_2}{\varrho_0} - \frac{\varrho_1^2}{2\varrho_0^2}, \quad (55)$$

whose dynamics is governed by the continuity equations (27) with the currents

$$\dot{q}_0 = q_1, \quad \dot{q}_1 = \frac{q_1^2}{q_0} + P - \zeta \partial_x (q_1/q_0), \quad \dot{q}_2 = \frac{q_1 q_2}{q_0} + \frac{q_1}{q_0} P - \zeta \frac{q_1}{q_0} \partial_x (q_1/q_0) - \kappa \partial_x T. \quad (56)$$

As we are close to equilibrium, we expand the currents around their equilibrium value. As mentioned earlier, we include the first nonlinear term and add a noise term. We get

$$\partial_t \delta q_\alpha + \partial_x \left(A_{\alpha\beta} \delta q_\beta + \frac{1}{2} H_{\beta\gamma}^\alpha \delta q_\beta \delta q_\gamma - \tilde{D}_{\alpha\beta} \partial_x \delta q_\beta + \tilde{B}_{\alpha\beta} \xi_\beta \right) = 0, \quad (57)$$

with $A_{\alpha\beta} = \frac{\partial \dot{q}_\alpha^{\text{NS}}}{\partial q_\beta}$, $H_{\beta\gamma}^\alpha = \frac{\partial^2 \dot{q}_\alpha^{\text{NS}}}{\partial q_\beta \partial q_\gamma}$, and the matrix $\tilde{D}$ involves terms with the transport coefficients ζ, κ . The noise term $\tilde{B}$ represents fluctuations of the currents satisfying

$$\langle \xi_\alpha(x, t) \rangle = 0, \quad \langle \xi_\alpha(x, t) \xi_\beta(x', t') \rangle = \delta_{\alpha\beta} \delta(x - x') \delta(t - t') \quad (58)$$

and its strength is set by $\tilde{B} \tilde{B}^T = \tilde{D} C + C \tilde{D}^T$. Moreover, we have

$$\langle \delta q_\alpha(x, t) \rangle = 0, \quad \langle \delta q_\alpha(x, t) \delta q_\beta(x', t) \rangle = C_{\alpha\beta} \delta(x - x'), \quad (59)$$

where $C_{\alpha\beta}$ is the susceptibility matrix given by the correlations at equilibrium. It is expressible in terms of the thermodynamic quantities of the fluid.

As we see in (57), NLFH is governed by a stochastic PDE. The approximate methods, employed to treat it [130, 132] yielded the following prediction for the density-density correlator

$$S_{\text{NLFH}}(x, t) = \varrho^2 \left[\frac{c_P - c_V}{c_P} \frac{1}{(\lambda_h t)^{3/5}} f_{\text{Levy}} \left(\frac{x}{(\lambda_h t)^{3/5}} \right) + \frac{c_V}{2c_P} \frac{1}{(\lambda_s t)^{2/3}} \left(f_{\text{KPZ}} \left(\frac{x - v_s t}{(\lambda_s t)^{2/3}} \right) + f_{\text{KPZ}} \left(\frac{x + v_s t}{(\lambda_s t)^{2/3}} \right) \right) \right]. \quad (60)$$

The two scaling functions f_{Levy} , f_{KPZ} are the Levy- $\frac{5}{3}$ distribution and the scaling function of the Prahöfer-Spohn solution to the 1D KPZ equation [134, 135]. The constants λ_h and λ_s are determined solely by thermodynamics of the gas. Therefore, the expression (60) interestingly does not depend on the diffusion matrix $\tilde{D}$. This means that it does not depend on the transport coefficients ζ, κ . The correlator (60) is entirely fixed by the thermodynamic quantities.

Historically, a considerable attention was given to the behavior of the total heat current-current correlation function at the thermal equilibrium [136–138]. Numerical simulations observed power law tails with an exponent of $-2/3$. Let us mention, that NLFH is also capable of studying such a correlator, and predicts precisely this value for the decay exponent. Another signature, often studied in numerics, appears in thermal conductivity, which becomes system-size dependent (and divergent in the thermodynamic limit).

1.7.2 Close to integrability

Anomalous hydrodynamics has been the subject of extensive numerical studies [139–142] by means of molecular dynamics of classical models. In the context of gases, a prominent role is played by the so-called alternating-mass hard-point gas (with particle masses of particles m and M). For identical masses $r = \frac{m}{M} = 1$, the model is integrable, but for any $r \neq 1$, it is not. Thus, it serves as an excellent model for studying transport phenomena near integrability.

While for parameters far from the integrable point ($r = 3$) the anomalous form of the correlator (60) was well confirmed [141], an interesting behavior was observed close to the integrable point $r = 1$. The studies focused mainly on the thermal conductivity and the current-current correlation function. Crucially, as observed in numerical simulations of the hard-point gas

in the nearly-integrable regime, *signatures of normal transport were observed for system sizes that are parametrically large with respect to the inverse of the integrability breaking parameter* [143–146]. A general conclusion was drawn that anomalies in transport start to appear at very large spatiotemporal scales in nearly-integrable systems. We will provide analytical explanation for these issues in Article III.

2 Results

The thesis consists of six research papers. Articles I, II and III, are already published in scientific journals, the Article V is accepted for publication and the other two are currently under the review.

The papers can be divided into two different groups, see Fig. 11. The first one corresponds to the articles related to relaxation or kinetic theory. The second one comprises hydrodynamics papers, with the main focus on the derivation of hydrodynamic theories from Boltzmann-like kinetic equations.

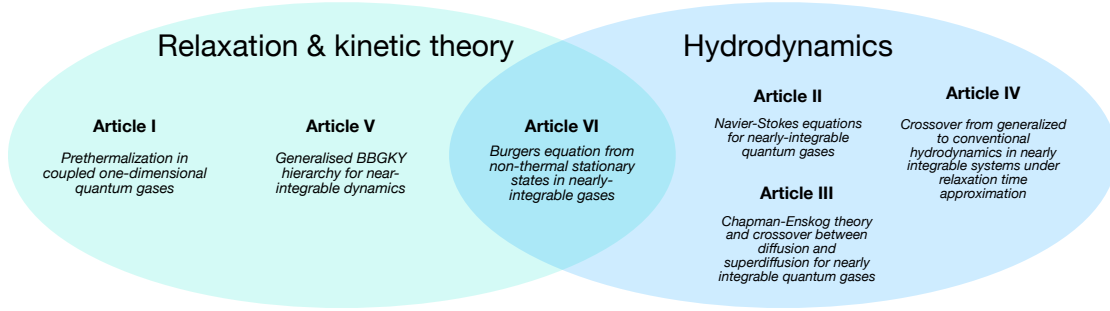


Figure 11: Articles as grouped by the main theme of the results obtained. Article VI discusses both kinematic origins of prethermalization and large scale hydrodynamic limit and thus, lies at the intersection of the two fields.

Before going into the full-text versions of the papers, let us provide a brief, general overview of the results:

- **Article I: *Prethermalization in coupled one-dimensional quantum gases*:** This work studies a setup of two coupled Lieb-Liniger gases. The gases interact via long-range interactions and are assumed to be in different states. In such a case, there exists non-trivial dynamics on the level of two-body scattering processes. Interestingly, this dynamics features non-thermal fixed points. The kinetic equation for such a setup stemming from the Fermi's Golden Rule, is solved numerically for different initial states and in different regimes of the Lieb-Liniger model. For large values of the Lieb-Liniger coupling, an explanation for the prethermal plateau in terms of additional conservation laws is proposed.
- **Article II: *Navier-Stokes equations for nearly-integrable quantum gases*:** We consider nearly-integrable gases with conserved particle number, momentum, and energy. Starting from the GHD equation supplemented with a Boltzmann-like collision integral, we derive the Navier-Stokes equations, which emerge in the limit of large space-time scales. The derivation is through perturbation theory (with the inverse length scale as a small parameter) applied to the GHD-Boltzmann equation. Viscosity and thermal conductivity are explicitly computed. For the setup of coupled Lieb-Liniger gases, transport coefficients are determined numerically.
- **Article III: *Chapman-Enskog theory and crossover between diffusion and superdiffusion for nearly-integrable quantum gases*:** In this work, a generalization of the Chapman-Enskog method is presented, which allows for a derivation of a fully nonlinear version of the Navier-Stokes equations, as opposed to the linearized theory analyzed in Article II. The equations are also extended to include an external potential. In the second part, transport anomalies in the Navier-Stokes equations are studied through the lens of nonlinear fluctuating hydrodynamics. Based on this, we show that due to the closeness

to integrability, anomalies are pushed to parametrically large (in the inverse of the integrability breaking parameter) space-time scales. We also demonstrate that transport in nearly-integrable gases is characterized by the existence of three distinct hydrodynamic regimes.

- **Article IV: *Crossover from generalized to conventional hydrodynamics in nearly-integrable systems under relaxation time approximation***: This work illustrates the crossover between generalized hydrodynamics and Navier-Stokes hydrodynamics occurring at large enough length scales. The collision integral is approximated through the relaxation time approximation, for which the transport coefficients and the relaxation time spectrum can be computed exactly. The GHD equation supplemented with such a collision integral is solved numerically, and a very good agreement is found with the Navier-Stokes equations. This demonstrates the correctness of the strategy taken in Article II.
- **Article V: *Generalised Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy for nearly-integrable dynamics***: In this work a more fundamental framework, as compared to a Boltzmann-like collision integral, is derived. The article presents a generalization of the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy for nearly-integrable systems. The hierarchy can be truncated efficiently with the range ξ of the integrability-breaking interaction potential. Upon truncation, the hierarchy is solved numerically and compared against molecular dynamics in a long-range interacting hard rod gas. A very good agreement is found. Next, a derivation of a collision integral is provided. The result exactly matches the computations based on Fermi's Golden Rule when specialized to the Lieb-Liniger model. Lastly, the results for relaxation rates are presented for the experimental setup of [70]. A reasonable agreement between theory and experiment is found.
- **Article VI: *Burgers equation from non-thermal stationary states in nearly-integrable gases***: This paper, similarly to the Article I, studies the setup of two coupled integrable gases. It explores the system-bath limit in which one of the systems is assumed to be much denser than the other. Firstly, a general study of the existence of non-thermal stationary states is conducted based on the kinematics of two-body processes. Next, the large space-time limit of the system's density dynamics is performed through a variant of the Chapman-Enskog expansion. The Burgers equation is found in the most general case. This picture is confirmed in *ab initio* simulations of toy particle models with stochastic two-body collisions.

The big picture of the dissertation is the adaptation of the methods of kinetic theory to nearly-integrable gases. Integrable gases feature stable quasiparticles for arbitrary interaction strength, which, upon the breaking of the integrability, start to interact with each other. As we show, the structures arising from these interactions in many ways resemble interactions in a weakly interacting, nearly-ideal gas. We build on that analogy to study thermalization, relaxation, and emergence of hydrodynamics. The main strategy lies in adapting classic frameworks from kinetic theory such as the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy or the Chapman-Enskog expansion.

2.1 Article I: *Prethermalization in coupled one-dimensional quantum gases*

- M. Łebek, M. Panfil, R. M. Konik “Prethermalization in coupled one-dimensional quantum gases”, *SciPost Phys.* 17, 007 (2024)

In certain many-body systems, a phenomenon called prethermalization occurs in the course of reaching the thermal state [147, 148]. For such systems, the relaxation process from a generic initial configuration features two distinct timescales: the prethermalization scale τ_{preth} and the thermalization scale τ_{th} . At the scale of τ_{preth} the system reaches a metastationary state, which is not thermal (i.e. it is not the state which maximizes the entropy under the constraints set by the conservation laws of the Hamiltonian). The thermalization happens on a much longer timescale τ_{th} . Typical dynamics of an observable in such a system is presented in Fig. 12(a).

In this work, we show that such a scenario takes place in a system of coupled Lieb-Liniger gases. More precisely, the geometry is that of two gases, which interact with each other through a long-range potential, see Fig. 12(b). There are no intra-species long-range interactions. Let us note that such setup can be realized with dipolar interactions, when the angle of the magnetic field is set to the magic angle θ_{DDI}^* .

The dynamics of the gases is determined by the collision integral stemming from FGR approach, similarly as described in Sec. 1.4. States of two tubes are assumed to be different. This turns out to have drastic effects on the dynamics. In such case, the dispersion relations in both tubes are different as well. In the consequence, the momentum-energy conservation laws (40) allow for non-trivial solutions already on the level of two-body scattering processes, see Fig. 12(c) (one particle per tube in scattering) and the solution (41) is no longer correct.

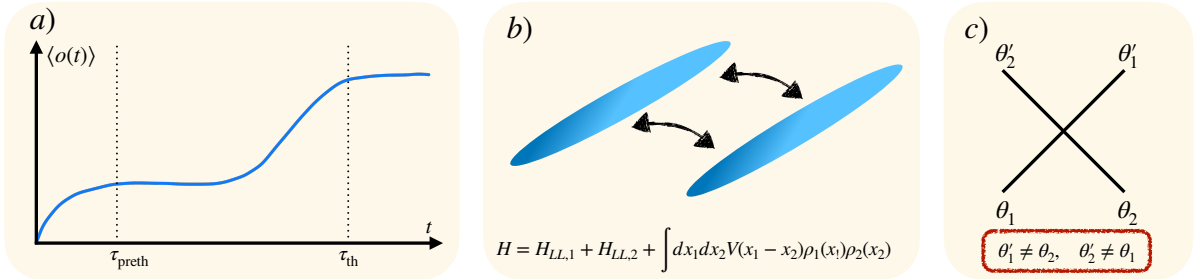


Figure 12: Panel a): dynamics of observable in a system featuring prethermal states. Panel b): the coupled LL models, which interact with potential $V(x_1 - x_2)$. Panel c): when the states of tubes are different, two-body scattering allows for non-trivial mixing of momenta, lifting the kinetic blocking.

Such two-body scattering processes are the leading processes in decompositions of the collision integral akin to (42). Interestingly, even though such processes lead to a non-trivial scattering, some constraints in kinematics are still visible. This is reflected in non-thermal stationary states predicted by the associated scattering integral.

As the main result of this work, we identify relations fulfilled by such states and show their existence by numerically solving the dynamics as given by the collision integral. These athermal states are metastationary states. The eventual thermalization happens on a longer time scale and through three-particle scatterings.

Typically, prethermalization is related to the presence of additional charges, which are conserved on the timescale of τ_{preth} . We construct such charges in the large coupling regime of the LL model. On the other hand, outside from this regime, we concluded that it is unlikely

that such charges exist. In view of this, it is interesting that prethermalization still happens. We attribute it to kinematic constraints in 1D scatterings, which possibly cannot be translated into the existence of additional local extensive charges. These observations are extended in Article [VI](#).

Candidate's contribution: I performed all numerical and the majority of the analytical computations. I contributed to the conceptualization of the work, participated in discussions and played a significant role in writing the manuscript. I prepared all the figures.

Prethermalization in coupled one-dimensional quantum gases

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Abstract

We consider the problem of the development of steady states in one-dimensional Bose gas tubes that are weakly coupled to one another through a density-density interaction. We analyze this development through a Boltzmann collision integral approach. We argue that when the leading order of the collision integral, where single particle-hole excitations are created in individual gases, is dominant, the state of the gas evolves first to a non-thermal fixed point, i.e. a prethermalization plateau. This order is dominant when a pair of tubes are inequivalent with, say, different temperatures or different effective interaction parameters, γ . When both tubes are in the strongly interacting regime we additionally characterize this non-thermal prethermalization plateau by constructing the quasi-conserved quantities that control the existence of this plateau as well as the associated generalized Gibbs ensemble.



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1 Introduction

How to understand the dynamics of a quantum system after the injection of energy is a paramount problem of modern many-body physics [1–3]. The injection of energy typically leads a system to arrive at a new steady state that is thermal in nature. The process by which this occurs in closed quantum systems is governed by the eigenstate thermalization hypothesis (ETH) [4–8]. This hypothesis asserts that quantum eigenstates of similar energy behave similarly with respect to all local observables, at least in the thermodynamic limit. However this hypothesis is not inviolate and there are models where ETH does not hold. One such class of ETH violating systems that has been extensively studied in the past few years support what are known as quantum scar states [9–12]. Quantum scar states have expectation values relative to some local observable that differs significantly from most other system eigenstates at the same energy. When a system is then in a scar state, it will experience atypical time dynamics relative to that observable and will not thermalize in the sense of ETH.

Another class of ETH violating systems are integrable models. Integrable models are characterized by an infinite set of conservation laws. The presence of these conservation laws means that their steady states are characterized by more complicated thermodynamic ensembles than Gibbs that take into account the presence of conserved charges beyond energy [8, 13]. Integrable models are ubiquitous in one-spatial dimension. Such models describe the dynamics of cold atomic Bose gases via the Lieb-Liniger model [14, 15], XXZ spin chains via the Heisenberg spin model [16], and itinerant interacting electron physics via the Hubbard model [17, 18] to name but a few.

Integrable models are to a certain degree platonic ideals. They only exist approximately in nature. Experimental realizations of one-dimensional Bose gases modelled by a Lieb-Liniger (LL) model involve arrays of one-dimensional tubes of the gas which see both intra- and intertube interactions beyond the interactions present in the LL Hamiltonian [19–21]. Similarly, quantum material realizations of one-dimensional spin chains are always in fact one-dimensional atomic chains embedded in a three dimensional matrix with interchain interactions, perhaps small but nonetheless, present [22, 23]. Finally, also material realizations of low-dimensional Hubbard models typically require the reduction of complicated multi-orbital physics down to effective one-band models [24–27].

The interactions that go beyond those present in integrable Hamiltonians will almost always break integrability. While an integrable model will not thermalize, a quasi-integrable model will (apart from cases when quantum scars are present). However it then becomes a

question of timescales. Rather than a non-thermal steady state persisting for all time, it instead picks up a finite, but perhaps long, lifetime. This persistent non-thermal steady state is known as a prethermalization plateau [28, 29]. This prethermalization plateau can usually be characterized by conserved quantities other than energy. However these conserved quantities are not necessarily the same as in the parent, unperturbed, integrable model.

It is the aim of this paper to present a scenario relevant to experiments in one-dimensional cold atomic Bose gases where long-live prethermalization plateaus occur. In typical realizations of one-dimensional Bose gases, the atoms in the gas are placed in an laser-induced array of quasi-one-dimensional transversal confining potentials, i.e. the gas forms of matrix of tubes [19–21]. The atoms in each tube are typically confined to the lowest transverse level of the tube but are free to move laterally along the tube. However the gas in each tube is not entirely independent of its neighbours in the array. Typically intertube density-density interactions are present. Such interactions in gases with dipolar couplings are tunable and can even be made to be of the same order as intratube interactions [19].

In Ref. [30] a framework for analyzing the thermalizing effects of intertube interactions was developed in the form of a Boltzmann collision integral. In Ref. [30], the tubes of gas were considered to be identical. A key feature of this work was then that the thermalizing interactions involve processes of the simultaneous creation of three particle-hole pairs in any two tubes: one pair in one tube and two pairs in a second tube, so-called (2,1) or (1,2) processes. Processes involving the production of a single pair in two identical tubes, (1,1) processes, led to no change in the state of the tubes, a simple consequence of energy-momentum conservation.

If however the two tubes are inequivalent, either by virtue of having a different density of atoms or a different effective interaction parameter, (1,1) processes are dominant at shorter time scales and do lead to an evolution in the state of the gas in the tube. However here the end point of this evolution is athermal. Because we cannot ignore (1,2)/(2,1) processes, this athermal state arrived at by the creation of single particle-hole pairs is temporary – it is in fact a prethermalization plateau. Nonetheless it is long-lived if the intertube interactions, which set the scale of the evolution, are weak. It is the aim of this paper to describe in detail the non-equilibrium evolution coming from tube heterogeneity and the resulting athermal state.

While the focus of this analysis takes place in the context of cold atomic systems, the analysis has elements of universality. A very similar set of considerations, for example, would apply to thermalizing $S_z S_z$ interchain interactions in spin chain materials. The universality arises because in the thermodynamic limit the matrix elements governing the creation of particle hole pairs (and their spin chain equivalents) have the same low energy form [31, 32].

The paper is organized as follows. In Section 2 we outline the basics of the Lieb-Liniger model including how to describe its thermodynamic states and its conserved quantities. In the next section, Section 3, we present the Boltzmann collision framework by which we analyze the dynamics of two tubes of gas due to intertube interactions. As part of this we present a general analysis of what constitutes a steady state in this framework and what are its conservation laws. We do this in complete generality considering all orders of particle-hole creation (i.e. (n,m)) processes. As part of this, we show that only energy, momentum, and particle number survive as conserved quantities and the resulting steady state is thermal.

In section 4 we perform the same analysis but only consider (1,1) processes. Here we show that with only (1,1) processes taken into account, there exist stationary states there are athermal. We then demonstrate by numerically solving the problem that indeed the system does not thermalize when only (1,1) processes are operative. In Section 5 we offer an explanation for the athermality by showing that the (1,1) dynamics has extra conserved charges beyond the energy, momentum and particle number. The new conserved charges are combination of the ultra-local charges of unperturbed Lieb-Liniger model from both tubes. We construct these charges for systems with strong intra-tube interactions, dubbed deformed Tonks-Girardeau

gas. We define it precisely in Section 5. We also formulate the Generalized Gibbs Ensemble for the coupled system essentially extending its applicability to perturbed integrable models. Finally, in Section 6, we provide a summary and concluding remarks.

More technical aspects of our work are relegated to the Appendices. Among them, in Appendix E, we demonstrate that the construction of the charges from the deformed Tonks-Girardeau gases does not extend to the full Lieb-Liniger model at arbitrary interaction strengths. This leaves open a question about the precise characterisation of the mechanism by which the system does not thermalize in the latter case.

2 Summary of the Lieb-Liniger model

We start with reviewing the relevant aspects of the Lieb-Liniger (LL) model [14, 15], which corresponds to a single gas tube. The LL model describes N bosons on a ring with length L , interacting with a repulsive contact potential. The Hamiltonian of the system reads

$$\hat{H}_{LL} = \int_0^L dx \left(-\hat{\psi}^\dagger(x) \partial_x^2 \hat{\psi}(x) + c \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\psi}(x) \right), \quad (1)$$

where $c \geq 0$ is the coupling constant and $\hat{\psi}(x)$ is the canonical bosonic field operator. The model is exactly solvable with the standard Bethe ansatz technique [33–35]. Eigenstates of (1) are parametrized by N real numbers, called quasimomenta, which are determined from the Bethe equations. In the thermodynamic limit $N, L \rightarrow \infty$ with $N/L = \text{const}$, the system is parametrized by the density of particles, $\rho_p(\lambda)$, and the total density of states, $\rho_t(\lambda)$, with λ the quasi-momentum or rapidity. These functions are related through the following integral equation [36],

$$\rho_t(\lambda) = \frac{1}{2\pi} + \int d\lambda' T(\lambda - \lambda') \rho_p(\lambda'), \quad (2)$$

with the kernel given by function, $T(\lambda)$:

$$T(\lambda) = \frac{c}{\pi} \frac{1}{c^2 + \lambda^2}. \quad (3)$$

In addition to ρ_t and ρ_p , we define the density of holes, ρ_h , as

$$\rho_h(\lambda) = \rho_t(\lambda) - \rho_p(\lambda). \quad (4)$$

It is sometimes convenient to express thermodynamic quantities in terms of ρ_h rather than ρ_p .

The LL model is integrable and so is characterized by an infinite number of conserved charges, $\hat{I}_n$, that commute with the Hamiltonian (1) and have the property

$$[\hat{I}_n, \hat{I}_m] = 0, \quad n, m = 0, 1, 2, \dots \quad (5)$$

There are multiple bases of these charges, both ultra-local [37] and semi-local [38–40]. In this work we will focus on the ultra-local representation despite their less than stellar behaviour in the UV [41]. Expectation values of the ultra-local conserved charges on thermodynamic states are functionals of the distribution, $\rho_p(\lambda)$, and take the particularly simple form

$$\langle \hat{I}_n \rangle = L \int d\lambda \rho_p(\lambda) \lambda^n. \quad (6)$$

The first operators in the sequence, $\hat{I}_n$, correspond to particle number, total momentum and total energy, $n = 0, 1, 2$, respectively:

$$N = L \int d\lambda \rho_p(\lambda), \quad P = L \int d\lambda \lambda \rho_p(\lambda), \quad E = L \int d\lambda \lambda^2 \rho_p(\lambda). \quad (7)$$

The presence of conserved charges $n > 2$ beyond these basic three means that the model can support non-thermal states.

We now turn to characterizing the thermodynamics and elementary excitations of the LL model. The entropy of a thermodynamic state of the gas is given by the Yang-Yang formula [36]

$$S = L \int d\lambda \left[(\rho_p(\lambda) + \rho_h(\lambda)) \log(\rho_p(\lambda) + \rho_h(\lambda)) - \rho_p(\lambda) \log \rho_p(\lambda) - \rho_h(\lambda) \log \rho_h(\lambda) \right]. \quad (8)$$

To describe the dynamics in the system we also need information about its elementary excitations. The coupling between the tubes that we consider, conserves number of particles in each tube. Therefore the relevant excitations takes form of particle-hole (ph) excitations in the system [15, 42]. The excitations are labelled by two quasimomenta p and h . The momentum carried by an excitation is $k = k(p) - k(h)$ and the corresponding energy is $\omega = \omega(p) - \omega(h)$. Here, the functions $k(p)$ and $\omega(p)$ are the so-called dressed momentum and energy. These are state-dependent functions and read

$$k(\lambda) = \lambda - \int d\mu n(\mu) F(\mu|\lambda), \quad (9a)$$

$$\omega(\lambda) = \lambda^2 - 2 \int d\mu \mu n(\mu) F(\mu|\lambda), \quad (9b)$$

where $n(\lambda) = \rho_p(\lambda)/\rho_t(\lambda)$ is the occupation function and $F(\mu|\lambda)$ is the backflow function [42] satisfying the following integral equation,

$$F(\lambda|\mu) = \frac{\theta(\lambda - \mu)}{2\pi} + \int d\lambda' T(\lambda - \lambda') n(\lambda') F(\lambda'|\mu), \quad (10)$$

with $\theta(\lambda) = 2 \arctan(\lambda/c)$.

For later use, we introduce notation for two dressing operations, following the notation of [43]. The first one, denoted Dr , is defined as

$$f_{\text{Dr}}(\lambda) = f(\lambda) - \int d\mu n(\mu) F(\mu|\lambda) n(\mu) f'(\mu). \quad (11)$$

Therefore $k(\lambda) = (\lambda)_{\text{Dr}}$ and $\omega(\lambda) = (\lambda^2)_{\text{Dr}}$. The second one, denoted dr , is defined as

$$f_{\text{dr}}(\lambda) = f(\lambda) + \int d\mu T(\mu, \lambda) n(\mu) f_{\text{dr}}(\mu). \quad (12)$$

The two dressing operations are related however through

$$(f_{\text{Dr}})' = (f')_{\text{dr}}. \quad (13)$$

3 Coupled Lieb-Liniger gases

In this section, we present a framework to study the dynamics of two coupled LL gases. The system is governed by the following Hamiltonian,

$$\hat{H} = \hat{H}_{LL,1} + \hat{H}_{LL,2} + \int dx_1 dx_2 A(x_1 - x_2) \hat{\rho}_1(x_1) \hat{\rho}_2(x_2). \quad (14)$$

The Hamiltonians $\hat{H}_{LL,i}$ describe LL gases with couplings c_i . The two gases are coupled by a long-range interaction potential $A(x_1 - x_2)$. By $\hat{\rho}_i(x) = \hat{\psi}_i^\dagger(x)\hat{\psi}_i(x)$, we denote the density operator in the i -th gas. The density-density interaction term $\delta\hat{H}$ breaks the integrability of the system $\hat{H} = \hat{H}_{\text{int}} + \delta\hat{H}$, where $\hat{H}_{\text{int}} = \hat{H}_{LL,1} + \hat{H}_{LL,2}$ is integrable. To describe its consequences on the gases' dynamics, we turn to an effective theory of relaxation developed in [30, 44]. Specifically in [30] the problem of the thermalization of two equivalent gases initialized in a non-thermal state was analyzed in quantitative detail. On the level of Fermi's golden rule approximation, the dynamics may be formulated in terms of Boltzmann-like equations governing the evolution of the rapidity distributions:

$$\begin{aligned}\partial_t \rho_{p,1}(\lambda) &= \tau^{-1} Q[\rho_{p,1}, \rho_{p,2}](\lambda), \\ \partial_t \rho_{p,2}(\lambda) &= \tau^{-1} Q[\rho_{p,2}, \rho_{p,1}](\lambda),\end{aligned}\tag{15}$$

where τ is the characteristic time scale for the evolution and $Q[\rho_{p,1}, \rho_{p,2}, \lambda]$ is the dressed scattering integral defined through

$$\begin{aligned}Q[\rho_{p,1}, \rho_{p,2}] &= \mathbb{F}_1 \cdot Q_0[\rho_{p,1}, \rho_{p,2}], \\ Q[\rho_{p,2}, \rho_{p,1}] &= \mathbb{F}_2 \cdot Q_0[\rho_{p,2}, \rho_{p,1}],\end{aligned}\tag{16}$$

where $Q_0[\rho_{p,1}, \rho_{p,2}]$ is bare scattering integral defined below in Eq. (22) and $\mathbb{F}_i(\lambda, \mu)$ reads

$$\mathbb{F}_i(\lambda, \mu) = \delta(\lambda - \mu) + \partial_\lambda(n_i(\lambda)F_i(\lambda|\mu)),\tag{17}$$

and implements the dressing operation, Dr , defined in Eqn. (11). We also define

$$(\mathbb{F} \cdot f)(\lambda) = \int d\mu \mathbb{F}(\lambda, \mu) f(\mu).\tag{18}$$

The first term in Eqn. (17) represents a direct scattering process while the second term gives the so-called backflow, i.e., the effect, due to interactions, that the creation of a particle-hole excitation has on the distribution of the remaining particles and holes in the gas.

The time scale, τ , appearing in (15) is determined by the parameters of the model [30]

$$\tau^{-1} = \frac{2A_0^2 m}{\hbar^3},\tag{19}$$

where m is mass of the particles and A_0 sets the strength of the inter-tube interaction,

$$A(x) = A_0 \frac{a(x/x_r)}{x_r}, \quad \int dx \frac{a(x/x_r)}{x_r} = 1.\tag{20}$$

Here x_r is the characteristic range of the potential. We will mostly work in momentum space, and so we define the Fourier transform of the normalized part:

$$\tilde{A}(k) = \int dx e^{ikx} \frac{a(x/x_r)}{x_r} = \tilde{a}(k/k_r), \quad \tilde{a}(k) = \int dx e^{ikx} a(x),\tag{21}$$

where $k_r = x_r^{-1}$ is the characteristic momentum exchanged between the tubes as a result of the coupling between them.¹

¹For all the simulations in the paper, we use the Gaussian potential $a(x) = 1/(\sqrt{\pi}) \exp(-x^2)$ with $x_r = 11$.

3.1 (n,m) ph processes

The bare collision integral $Q_0[\rho_{p,1}; \rho_{p,2}]$ contains contributions from processes involving n particle-holes (ph) excitations in the first tube and m ph excitations in the second tube:

$$Q_0[\rho_{p,1}, \rho_{p,2}] = \sum_{n,m} Q_0^{(n,m)}[\rho_{p,1}, \rho_{p,2}]. \quad (22)$$

The (n, m) contribution takes the following form [30],

$$Q_0^{(n,m)}[\rho_{p,1}, \rho_{p,2}](\lambda) = \int d\mathbf{p}^n d\mathbf{h}^n \sum_{i=1}^n \delta(\lambda - p_i) \tilde{A}^2(k_1(\mathbf{p}, \mathbf{h})) |F_1(\mathbf{p}, \mathbf{h})|^2 \\ \times \left(J_1^n(\mathbf{p}, \mathbf{h}) S_2^m(-k_1(\mathbf{p}, \mathbf{h}), -\omega_1(\mathbf{p}, \mathbf{h})) - (\mathbf{h} \leftrightarrow \mathbf{p}) \right), \quad (23)$$

where $\mathbf{p} = \{p_i\}_{i=1}^n$ and $\mathbf{h} = \{h_i\}_{i=1}^n$ denote rapidities of particles and holes, respectively. Here we have introduced the notation

$$d\mathbf{p}^n d\mathbf{h}^n = \frac{1}{(n!)^2} \prod_{i=1}^n dp_i dh_i, \quad J_1^n(\mathbf{p}, \mathbf{h}) = \prod_{i=1}^n \rho_{p,1}(h_i) \rho_{h,1}(p_i). \quad (24)$$

Moreover by $F(\mathbf{p}, \mathbf{h})$ we denote the density operator form factor [45–48]

$$F(\mathbf{p}, \mathbf{h}) = \langle \rho_p | \hat{\rho}(0) | \rho_p; \mathbf{p}, \mathbf{h} \rangle, \quad (25)$$

where the state $|\rho_p; \mathbf{p}, \mathbf{h}\rangle$ corresponds to the thermodynamic state $|\rho_p\rangle$ with ph excitations $\mathbf{p}$ and $\mathbf{h}$ on top of it. Finally, $S^m(k, \omega)$ is the m -particle contribution to the dynamic structure factor [46]. In Eqn. (23), we denote the total momentum and energy carried by excitations $\mathbf{h} \rightarrow \mathbf{p}$ in the first tube by $k_1(\mathbf{p}, \mathbf{h})$ and $\omega_1(\mathbf{p}, \mathbf{h})$:

$$k_1(\mathbf{p}, \mathbf{h}) = \sum_{i=1}^n (k_1(p_i) - k_1(h_i)), \quad \omega_1(\mathbf{p}, \mathbf{h}) = \sum_{i=1}^n (\omega_1(p_i) - \omega_1(h_i)). \quad (26)$$

For completeness, we also write down the bare collision integral for the second tube,

$$Q_0^{(n,m)}[\rho_{p,2}, \rho_{p,1}](\lambda) = \int d\mathbf{p}^n d\mathbf{h}^n \sum_{i=1}^n \delta(\lambda - p_i) \tilde{A}^2(k_2(\mathbf{p}, \mathbf{h})) |F_2(\mathbf{p}, \mathbf{h})|^2 \\ \times \left(J_2^n(\mathbf{p}, \mathbf{h}) S_1^m(-k_2(\mathbf{p}, \mathbf{h}), -\omega_2(\mathbf{p}, \mathbf{h})) - (\mathbf{h} \leftrightarrow \mathbf{p}) \right), \quad (27)$$

which amounts to a permutation of tube indices.

The form of the collision integral presented in Eqn. (23) is not the most convenient for our purposes. We thus derive now an alternative representation of it, expressing the dynamical structure factor contributions in terms of quasi-particle densities. The m -particle contribution to the dynamic structure factor reads

$$S^m(k, \omega) = (2\pi)^2 \int d\mathbf{p}^m d\mathbf{p}^m \prod_{i=1}^m \rho_p(h_i) \rho_h(p_i) |F(\mathbf{p}, \mathbf{h})|^2 \delta(k - k(\mathbf{p}, \mathbf{h})) \delta(\omega - \omega(\mathbf{p}, \mathbf{h})), \quad (28)$$

with the understanding the integrals require, in general, regulation [48]. Plugging the above representation into (23) leads to the desired formulation:

$$Q_0^{(n,m)}[\rho_{p,1}, \rho_{p,2}](\lambda) = (2\pi)^2 \int d\mathbf{p}_1^n d\mathbf{h}_1^n d\mathbf{p}_2^m d\mathbf{h}_2^m \sum_{j=1}^n \delta(\lambda - p_{j,1}) \tilde{A}^2(k_1(\mathbf{p}_1, \mathbf{h}_1)) |F_1(\mathbf{p}_1, \mathbf{h}_1)|^2 \\ \times |F_2(\mathbf{p}_2, \mathbf{h}_2)|^2 \delta(k_1(\mathbf{p}_1, \mathbf{h}_1) + k_2(\mathbf{p}_2, \mathbf{h}_2)) \delta(\omega_1(\mathbf{p}_1, \mathbf{h}_1) + \omega_2(\mathbf{p}_2, \mathbf{h}_2)) \\ \times \left(J_1^n(\mathbf{p}_1, \mathbf{h}_1) J_2^m(\mathbf{p}_2, \mathbf{h}_2) - (\mathbf{h} \leftrightarrow \mathbf{p}) \right), \quad (29)$$

and similarly for the second tube,

$$Q_0^{(n,m)}[\rho_{p,2}, \rho_{p,1}](\lambda) = (2\pi)^2 \int d\mathbf{p}_1^m d\mathbf{h}_1^m d\mathbf{p}_2^n d\mathbf{h}_2^n \sum_{j=1}^n \delta(\lambda - p_{j,2}) \tilde{A}^2(k_2(\mathbf{p}_2, \mathbf{h}_2)) |F_1(\mathbf{p}_1, \mathbf{h}_1)|^2 \\ \times |F_2(\mathbf{p}_2, \mathbf{h}_2)|^2 \delta(k_1(\mathbf{p}_1, \mathbf{h}_1) + k_2(\mathbf{p}_2, \mathbf{h}_2)) \delta(\omega_1(\mathbf{p}_1, \mathbf{h}_1) + \omega_2(\mathbf{p}_2, \mathbf{h}_2)) \\ \times (J_1^m(\mathbf{p}_1, \mathbf{h}_1) J_2^n(\mathbf{p}_2, \mathbf{h}_2) - (\mathbf{h} \leftrightarrow \mathbf{p})). \quad (30)$$

This representation is the most explicit. It is convenient for the numerical evaluation of the collision integral and for analyzing its small momentum limit. From the particle-hole symmetry of density operator form factors [46] and Eqn. (29), it is easy to find yet another representation,

$$Q_0^{(n,m)}[\rho_{p,1}, \rho_{p,2}](\lambda) = (2\pi)^2 \int d\mathbf{p}_1^n d\mathbf{h}_1^n d\mathbf{p}_2^m d\mathbf{h}_2^m \tilde{A}^2(k_1(\mathbf{p}_1, \mathbf{h}_1)) |F_1(\mathbf{p}_1, \mathbf{h}_1)|^2 |F_2(\mathbf{p}_2, \mathbf{h}_2)|^2 \\ \times \delta(k_1(\mathbf{p}_1, \mathbf{h}_1) + k_2(\mathbf{p}_2, \mathbf{h}_2)) \delta(\omega_1(\mathbf{p}_1, \mathbf{h}_1) + \omega_2(\mathbf{p}_2, \mathbf{h}_2)) J_1^n(\mathbf{p}_1, \mathbf{h}_1) \\ \times J_2^m(\mathbf{p}_2, \mathbf{h}_2) \left(\sum_{i=1}^n \delta(\lambda - p_{i,1}) - \sum_{i=1}^n \delta(\lambda - h_{i,1}) \right). \quad (31)$$

Obviously, analogous formula for $Q_0^{(n,m)}[\rho_{p,2}, \rho_{p,1}](\lambda)$ may be easily derived as well. This representation is especially convenient for our discussion of the conservation laws in Section 3.3 below.

Crucially the (n, m) ph processes possess a hierarchy of importance. For the situations considered here the most relevant contribution is due to (1,1) scattering. Processes involving more ph pairs contribute much less significantly. In particular, let us note that in the extreme limit of infinite interactions in both tubes $c_{1,2} = \infty$ we have $Q^{(n,m)} = 0$ for $m, n > 1$ and only the (1,1) dynamics contributes. This can be seen on the level of the higher ph density form factors which go as $1/c^{n+m-2}$ and so are zero when more than one ph pair is considered [46] in the infinite interaction limit. The hierarchy between different contributions is also clearly visible in the small momentum expansion of scattering integrals, which is explained in detail in the Appendix A. Assuming that the dynamics is driven mainly by the processes involving small momentum transfer between the tubes, we show there that

$$Q_0^{(n,m)} \sim k_r^{n+m} \times (1 + \mathcal{O}(k_r^2)). \quad (32)$$

Noting that k_r is a small parameter (the case of long-range interactions) we clearly see that the lowest processes are the most important ones. Furthermore because the subleading corrections involve k_r^2 , the hierarchy of the importance is as following:

1. The leading order (1,1) processes going as k_r^2 .
2. The leading (2,1) and (1,2) processes, of order k_r^3 .
3. The subleading (1,1) processes and the leading order of higher processes like (3,1) or (2,2), all of order k_r^4 .

As we have observed in [30] the processes (2,1) and (1,2) thermalize the system. There we considered the situation of identical states in both tubes, and in such a case, the scattering integral for (1,1) processes vanishes. But in the general case, the physics at short timescales is determined by the leading (1,1) processes. We analyze these in Section 4. In the remainder of this section, we discuss the properties of the dynamics generated by (15) for arbitrary processes. We address specifically the character of stationary states and the existence of conserved charges.

3.2 Stationary states

In this subsection we look for stationary states satisfying

$$Q[\rho_{p,1}, \rho_{p,2}](\lambda) = Q[\rho_{p,2}, \rho_{p,1}](\lambda) = 0. \quad (33)$$

First, we note that when analyzing the equation above, we may simplify the problem and focus on the bare collision integral Q_0 . This is because the invertible operation of dressing is linear and the vanishing of Q_0 implies the vanishing of Q . Let us start with the time evolution of the first tube. The bare scattering integral can vanish as a whole or because the integrand is identically zero. The second condition is stronger and implies cancellation of each excitation contributing to the scattering integral with its particle-hole exchanged counterpart. From the representation in Eqn. (29), we observe that the dynamics on the (n, m) level is proportional to the following factor

$$\mathcal{J}^{(n,m)}(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) = J_1^n(\mathbf{p}_1, \mathbf{h}_1) J_2^m(\mathbf{p}_2, \mathbf{h}_2) - J_1^n(\mathbf{h}_1, \mathbf{p}_1) J_2^m(\mathbf{h}_2, \mathbf{p}_2). \quad (34)$$

The remaining factors under the integral are strictly positive, thus the state in the first tube is stationary if

$$\mathcal{J}^{(n,m)}(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) = 0, \quad (35)$$

holds for all rapidities $\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2$ satisfying the momentum-energy conservation laws

$$\sum_{i=1}^n (k_1(p_{i,1}) - k_1(h_{i,1})) + \sum_{j=1}^m (k_2(p_{j,2}) - k_2(h_{j,2})) = 0, \quad (36)$$

$$\sum_{i=1}^n (\omega_1(p_{i,1}) - \omega_1(h_{i,1})) + \sum_{j=1}^m (\omega_2(p_{j,2}) - \omega_2(h_{j,2})) = 0. \quad (37)$$

For stationarity, Eqn. (35) needs to be fulfilled on all (n, m) levels. The same factors, $\mathcal{J}^{(n,m)}$ are present in the collision integral for the second tube and so stationarity in tube 1 implies stationarity in tube 2.

To understand the relevant structure in Eqn. (35), it is useful to reparametrize the system introducing the pseudo energy $\epsilon(\lambda)$ defined as [36, 49]

$$e^{\epsilon(\lambda)} = \frac{\rho_h(\lambda)}{\rho_p(\lambda)}. \quad (38)$$

We may then rewrite $\mathcal{J}^{(n,m)}(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2)$ as

$$\mathcal{J}^{(n,m)}(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) = J_1^n(\mathbf{h}_1, \mathbf{p}_1) J_2^m(\mathbf{h}_2, \mathbf{p}_2) (\exp[\epsilon_1(\mathbf{p}_1, \mathbf{h}_1) + \epsilon_2(\mathbf{p}_2, \mathbf{h}_2)] - 1), \quad (39)$$

and the condition in Eqn. (35) translates to

$$\sum_{i=1}^n (\epsilon_1(p_{i,1}) - \epsilon_1(h_{i,1})) + \sum_{j=1}^m (\epsilon_2(p_{j,2}) - \epsilon_2(h_{j,2})) = 0. \quad (40)$$

Importantly, the state of the system with two tubes in thermal states [36] with the same inverse temperature β is stationary. This is because in such a case, the pseudo energies, here generalized to boosted thermal states, read [36, 42]

$$\epsilon_1(\lambda) = -\beta\mu_1 + \kappa k_1(\lambda) + \beta\omega_1(\lambda), \quad \epsilon_2(\lambda) = -\beta\mu_2 + \kappa k_2(\lambda) + \beta\omega_2(\lambda), \quad (41)$$

where $\mu_{1,2}$ are the chemical potentials. For $\epsilon_{1,2}$ as in Eqn. (41), Eqn. (40) follows directly from the conservation of energy (37) and momentum (36). For all the numerical examples considered later, we set $\kappa = 0$.

The dynamics restricted to $(1, 1)$ processes are special and feature stationary states that are not necessarily thermal. This may happen when we consider two identical tubes (this case was analyzed before in [30]), in the Tonks-Girardeau limit in both tubes ($c_{1,2} = \infty$), or, finally in the small momentum limit of the collision integral. This last scenario provides the setting of our observed prethermalization plateaus is the central result of this work. We discuss that case in Section 4 in detail.

In Appendix B we provide evidence that the three types of athermal stationary states of the $(1, 1)$ processes are the only ones. However we cannot assert that these are definitively the only possibilities. Moreover for higher order (n, m) processes the condition for the stationarity would seem to exclude any possibility of athermal states. We conjecture once all processes are taken into account that the only stationary states are then in fact thermal.

3.3 Conservation laws

The evolution in Eqn. (15) is controlled by a presence of conservation laws. In particular the particle number in each tube, the total momentum, and the total energy do not change in time. In order to see this, let us consider the change in the particle number with respect to time

$$\begin{aligned} \partial_t N_1 / L &= \int d\lambda \partial_t \rho_{p,1}(\lambda) = \int d\lambda Q[\rho_{p,1}, \rho_{p,2}](\lambda) \\ &= \int d\lambda Q_0[\rho_{p,1}, \rho_{p,2}](\lambda) + \int d\mu d\lambda \partial_\lambda (n(\lambda) F(\lambda|\mu)) Q_0[\rho_{p,1}, \rho_{p,2}](\mu) = 0. \end{aligned} \quad (42)$$

In the last step, we have used the explicit form of (23) together with ph symmetry of the form factor to observe that the first integral vanishes. The second integral vanishes upon integration by parts with the boundary terms vanishing as for physical states the filling function decays to zero at large rapidities. Of course, similar arguments lead to $\partial_t N_2 = 0$. Note that this conservation law holds at all levels (n, m) .

Now, let us consider the total energy

$$E_{tot} = E_1 + E_2 = L \int d\lambda \lambda^2 (\rho_{p,1}(\lambda) + \rho_{p,2}(\lambda)), \quad (43)$$

and show that it is conserved in time. We consider its derivative with respect to time

$$\begin{aligned} \partial_t E_{tot} / L &= \int d\lambda \lambda^2 (Q[\rho_{p,1}, \rho_{p,2}](\lambda) + Q[\rho_{p,2}, \rho_{p,1}](\lambda)) \\ &= \int d\lambda d\mu \lambda^2 (\mathbb{F}_1(\lambda, \mu) Q_0[\rho_{p,1}, \rho_{p,2}](\mu) + \mathbb{F}_2(\lambda, \mu) Q_0[\rho_{p,2}, \rho_{p,1}](\mu)). \end{aligned} \quad (44)$$

We may shift the dressing operation from the collision integral Q to λ^2 . Integration over λ then gives

$$\int d\lambda \lambda^2 \mathbb{F}_i(\lambda, \mu) = \omega_i(\mu). \quad (45)$$

We thus get

$$\partial_t E_{tot} / L = \int d\mu (\omega_1(\mu) Q_0[\rho_{p,1}, \rho_{p,2}](\mu) + \omega_2(\mu) Q_0[\rho_{p,2}, \rho_{p,1}](\mu)). \quad (46)$$

We observe now that the following relation holds for arbitrary (n, m)

$$\int d\mu (\omega_1(\mu) Q_0^{(n,m)}[\rho_{p,1}, \rho_{p,2}](\mu) + \omega_2(\mu) Q_0^{(m,n)}[\rho_{p,2}, \rho_{p,1}](\mu)) = 0. \quad (47)$$

One can establish this by employing the representation in Eqn. (31) into the equation above. From the integration over μ , we find terms that exactly correspond to $\omega_1(\mathbf{p}_1, \mathbf{h}_1) + \omega_2(\mathbf{p}_2, \mathbf{h}_2)$ which is enforced to be zero by the Dirac delta inside $Q_0^{(n,m)}$. Finally, summing this relation above over n and m gives $\partial_t E_{tot} = 0$. Similar arguments lead to the conclusion that the total momentum

$$P_{tot} = P_1 + P_2, \quad (48)$$

is conserved as well.

In the remaining part of this section, we propose the following ansatz for additional conserved quantities:

$$\mathcal{I} = L \int d\lambda (\rho_{p,1}(\lambda) f_1(\lambda) + \rho_{p,2}(\lambda) f_2(\lambda)), \quad (49)$$

where we allowed $f_i(\lambda)$'s, the single particle eigenvalues of the supposed charges, to be different in both tubes. We ask whether there exist functions $f_1(\lambda)$ and $f_2(\lambda)$ such that $\partial_t \mathcal{I} = 0$. Explicitly, we write

$$\begin{aligned} \partial_t \mathcal{I} / L &= \int d\lambda \partial_t \rho_{p,1}(\lambda) f_1(\lambda) + \int d\lambda \partial_t \rho_{p,2}(\lambda) f_2(\lambda) \\ &= \int d\lambda (f_1(\lambda) Q[\rho_{p,1}, \rho_{p,2}](\lambda) + f_2(\lambda) Q[\rho_{p,2}, \rho_{p,1}](\lambda)) \\ &= \int d\lambda (f_{1,Dr}(\lambda) Q_0[\rho_{p,1}, \rho_{p,2}](\lambda) + f_{2,Dr}(\lambda) Q_0[\rho_{p,2}, \rho_{p,1}](\lambda)), \end{aligned} \quad (50)$$

where we have shifted the dressing operation from Q to f_i and used the Dressing operation from Eqn. (11). Similarly to the condition for the stationarity, $\partial_t \mathcal{I} = 0$ either because the expression vanishes as a whole or because the integrand is zero. In the latter case this leads to a condition on functions $f_i(\lambda)$, similar to Eqn. (40) which we will now derive.

Analogously to the case of conserved energy, in order to have $\partial_t \mathcal{I} = 0$, we require that

$$\int d\lambda (f_{1,Dr}(\lambda) Q_0^{(m,n)}[\rho_{p,1}, \rho_{p,2}](\lambda) + f_{2,Dr}(\lambda) Q_0^{(n,m)}[\rho_{p,2}, \rho_{p,1}](\lambda)) = 0, \quad (51)$$

should hold for all (n, m) . To write the result in a compact form we parametrise $Q_0^{(m,n)}$ as

$$\begin{aligned} Q_0^{(n,m)}(\lambda) &= \int d\mathbf{p}_1^n d\mathbf{h}_1^n d\mathbf{p}_2^m d\mathbf{h}_2^m G(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) \\ &\quad \times \delta(k_1 + k_2) \delta(\omega_1 + \omega_2) \left(\sum_{i=1}^n \delta(\lambda - p_{i,1}) - \sum_{i=1}^n \delta(\lambda - h_{i,1}) \right), \end{aligned} \quad (52)$$

where $G(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2)$ can be read off from Eqn. (31) but we do not need its explicit form. We only need to know that it is strictly positive. The result is

$$\begin{aligned} \partial_t \mathcal{I} &= \int d\mathbf{p}_1^n d\mathbf{h}_1^n d\mathbf{p}_2^m d\mathbf{h}_2^m G(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) \delta(k_1 + k_2) \delta(\omega_1 + \omega_2) \\ &\quad \times \left(\sum_{i=1}^n f_{1,Dr}(p_{i,1}) - f_{1,Dr}(h_{i,1}) + \sum_{j=1}^m f_{2,Dr}(p_{j,2}) - f_{2,Dr}(h_{j,2}) \right) = 0. \end{aligned} \quad (53)$$

From this expression we can read off a *necessary* condition for an existence of a conserved charge characterized locally in the rapidities. This translates to demanding that

$$\sum_{i=1}^n (f_{1,Dr}(p_{i,1}) - f_{1,Dr}(h_{i,1})) + \sum_{j=1}^m (f_{2,Dr}(p_{j,2}) - f_{2,Dr}(h_{j,2})) = 0, \quad (54)$$

is fulfilled for all $\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2$ satisfying the energy-momentum conservation laws (37) and (36). We observe a similar structure as in Eqn. (40) for the stationary state.² Therefore, by exactly the same arguments, we find no conserved charges of the form (49) apart from total momentum and energy. However, just as in the case of stationary state analysis, the situation will be different for (1, 1) processes which we discuss in details in the next two sections.

4 (1,1) dynamics in the small momentum limit

We have argued that due to the scaling with the characteristic momentum k_r , the processes involving small number of ph excitations are the most relevant. In this section, we study in detail the dynamics generated by (1, 1) processes solely. As we will show, in the small momentum limit of collision integral, such restricted dynamics are qualitatively different. In particular, the restricted dynamics feature non-thermal stationary states.

4.1 Stationary states

We start by reassessing the condition for the stationary state. In the small momentum limit it is convenient to introduce a center-of-mass coordinates for the rapidities,

$$\lambda_i = \frac{1}{2}(p_i + h_i), \quad \alpha_i = p_i - h_i, \quad i = 1, 2, \quad (55)$$

with α_i being small for small $k = k_i(p_i) - k_i(h_i)$. We analyze now the condition (40) together with the momentum-energy conservation laws (36) and (37) taking $(n, m) = (1, 1)$. After expanding in α , the energy-momentum conservation laws give

$$v_1(\lambda_1) = v_2(\lambda_2), \quad v(\lambda) \equiv \omega'(\lambda)/k'(\lambda), \quad (56)$$

$$\alpha_2 = -k'_1(\lambda_1)/k'_2(\lambda_2)\alpha_1. \quad (57)$$

On the other hand, expansion of (40) in small α_i together with (56) and (57) yield the following condition for the stationary state:

$$\frac{\epsilon'_1(\lambda_1)}{k'_1(\lambda_1)} = \frac{\epsilon'_2(\lambda_2)}{k'_2(\lambda_2)}, \quad v_1(\lambda_1) = v_2(\lambda_2). \quad (58)$$

Eqn. (58) sets the relation between pseudo energies in both tubes in the stationary state. The relation between velocities allows us to express λ_2 in terms of λ_1 .³ Therefore, the equation involving the pseudo energies has only one variable appearing and for any state $\epsilon_2(\lambda)$ it can be used to determine $\epsilon_1(\lambda)$ (up to a constant value). One may readily check that a state in which both tubes are in equally boosted thermal states with the same temperature, see eq. (41), fulfills (58) and therefore is a stationary state in terms of the small momentum limit of the (1, 1) dynamics. Note however, that in principle there might be other configurations, for which (58) is satisfied. In the remaining part of the paper, we will show that this is the case and the (1, 1) dynamics in the small momentum limit typically drive the system to such stationary but non-thermal states. This feature is particular to (1, 1) dynamics. The small momentum expansion of processes involving more ph excitations does not lead to the appearance of non-thermal stationary states.

²There is however a crucial difference between the two equations - the equation for the conserved charges has to be fulfilled dynamically at each time t . We will see the consequences of this in the simplest case of (1, 1) processes in Section 5.

³We are assuming here that the effective velocity is a monotonic function in λ .

It is difficult to write down a solution to Eqn. (58) for arbitrary interaction parameters as this requires, for a given $\epsilon_1(\lambda)$, solving a non-linear equation for $\epsilon_2(\lambda_2)$. The non-linearity appears because the functions, $k_2(\lambda_2)$ and $v_2(\lambda_2)$, depend themselves on $\epsilon'_2(\lambda_2)$ through the occupation function, $n_2(\lambda_2)$. Therefore, instead of constructing the solution, we will verify that the system, when evolved according to (1,1) dynamics in the small momentum limit, is described by the pseudo-energies $\epsilon_i(\lambda_i)$ obeying (58). We will also explicitly see that the state is non-thermal. These results are presented in Section 4.2. But to gain some insight, we will now study Eqn. (58) in two special cases where the analysis simplifies:

Tonks-Girardeau limit in both tubes

The Tonks-Girardeau limit is particularly simple as the dressings are then absent. From Eqn. (58) we then find

$$\epsilon'_1(\lambda) = \epsilon'_2(\lambda), \quad (59)$$

where we used that $k'(\lambda) = 1$ and $v(\lambda) = 2\lambda$ in the Tonks-Girardeau gas. This implies that pseudo-energies of both tubes in the stationary states are equal up to the chemical potentials: $\epsilon_2(\lambda) = \epsilon_1(\lambda) + \text{const.}$ Such solution obviously allows for a much wider class of configurations than (41).

Interestingly, this solution is valid also beyond the small momentum limit. This is a peculiarity of the Tonks-Girardeau gas where the conservations laws (36) and (37) at arbitrary momenta can be solved explicitly and yield $p_2 = h_1$ and $h_2 = p_1$. The condition (40) for the stationary state simplifies then to

$$\epsilon_1(p) - \epsilon_2(p) = \epsilon_1(h) - \epsilon_2(h), \quad (60)$$

and has the same solution as in the small momentum limit.

Deformed Tonks-Girardeau gas

Assuming the presence in the two tubes large couplings c_1 and c_2 , we may analyze (58) by expanding in $1/c_i$. Details here are shifted to Appendix D. Summarizing this calculation, we find that the allowed set of pseudo-energies implying stationarity extends beyond that corresponding to boosted thermal states. Indeed, a stationary state is given by any pair of functions $(\epsilon_1(\lambda), \epsilon_2(\lambda))$ satisfying,

$$\xi_2^{-1} \epsilon'_2(\xi_2 \lambda) = \xi_1^{-1} \epsilon'_1(\xi_1 \lambda) \times \left(1 + \mathcal{O}(1/c_i^3)\right), \quad \xi_i = 1 + \frac{2n_i}{c_i}, \quad (61)$$

which is a simple modification of the stationarity condition (59) in the Tonks-Girardeau gas. Here n_i is the density of particles in the i -th tube.

4.2 Numerical results

We now numerically solve the Boltzmann equation to demonstrate the main findings of our work. We study relaxation under dynamics restricted to (1,1) processes. We neglect the processes involving more ph pairs effectively considering Eqns. (15) with $Q_0 \equiv Q_0^{(1,1)}$. Moreover, we consider potentials narrow in the momentum space such that $Q_0^{(1,1)}$ is very well approximated by its low-momentum limit. With this, we may expect both non-thermal stationary states (58) as the final states of the evolution.

To display the athermal character of the stationary states we will compare them with the predictions of the standard Gibbs ensemble (GE). The bare pseudoenergies are then

$$\epsilon_{0,i}(\lambda) = \beta_{i,0} + \beta_2 \lambda^2, \quad (62)$$

with $\beta_{1,0}, \beta_{2,0}$ and β_2 fixed by the initial densities in both tubes and their total energy.

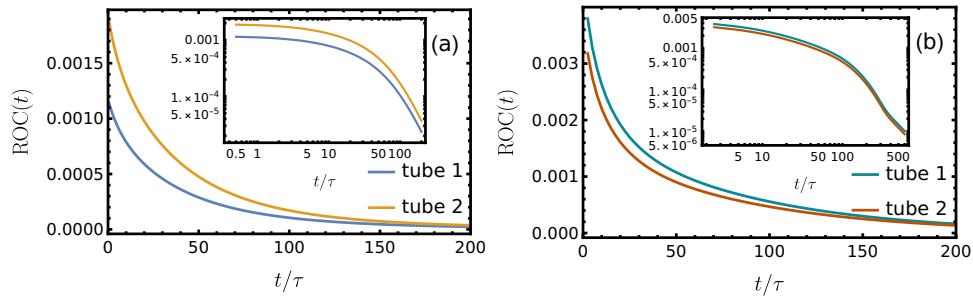


Figure 1: Rate of change of quasi-momentum distributions as a function of time, with Bragg-split states as initial states for the dynamics. Panel (a): intermediate regime, panel (b): strongly interacting regime. In both cases the systems reach the stationary state, with a different timescale for each tube. Insets: the same plots, but for longer times and in logarithmic scale.

Table 1: Parameters of the initial states used for the two cases of Bragg-split states explored in this paper.

case 1 (intermediate interactions)							case 2 (strong interactions)						
c_1	c_2	β_1	β_2	μ_1	μ_2	λ_0	c_1	c_2	β_1	β_2	μ_1	μ_2	λ_0
1.0	4.0	0.7	0.7	1.6	2.6	2.0	32.0	128.0	1.7	1.7	3.22	5.27	2.9

4.2.1 Bragg-split states

As initial states, we consider the experimentally relevant Bragg-split thermal distributions [19]. To probe both the intermediate and strong interaction regimes, we perform two computations: one with $c_1 = 1$ and $c_2 = 4$, and one with $c_1 = 32$, $c_2 = 128$. Concretely, we consider initial states of the following form:

$$\rho_{p,i}^{\text{ini}}(\lambda) = \frac{1}{2} \left(\rho_{\beta_i, \mu_i}^{\text{Gibbs}}(\lambda - \lambda_0) + \rho_{\beta_i, \mu_i}^{\text{Gibbs}}(\lambda + \lambda_0) \right), \quad (63)$$

with the parameters used summarized in Table 1. As the first probe of the relaxation, we investigate the rate of change of the quasiparticle distributions as a function of time defined in the following way

$$\text{ROC}(t) = \frac{\int d\lambda |\partial_t \rho_p(\lambda; t)|}{\int d\lambda \rho_p(\lambda; t)}. \quad (64)$$

The results are shown at Fig. 1. We observe that the dynamics becomes gradually slower with time and the system reaches a stationary state. The timescales of these processes are different in each tube. This is expected, since couplings and densities differ between the tubes.

The final, stationary state reached in the course of $(1, 1)$ dynamics is presented in Fig. 2. We compare in this figure the final quasiparticle densities $\rho_{p,i}$, $i = 1, 2$ to the initial distributions as well as to the predictions of the GE ensemble. We observe that the final, stationary state still resembles a Bragg-split distribution and is very far from the thermal GE state that one would expect from unconstrained dynamics. This is further confirmed as one inspects the Yang-Yang entropy as a function of time as shown in Fig. 3. The system relaxes to a state with a much lower entropy in comparison to the prediction computed on the thermal GE state.

Finally, we verify explicitly that the final state fulfills the stationarity condition (58). To do so, we define a function

$$H(\lambda_1) = \rho_{p,1}(\lambda_1) \rho_{h,1}(\lambda_1) \rho_{p,2}(\lambda_2) \rho_{h,1}(\lambda_2) \left(\epsilon'_1(\lambda_1) - \frac{k'_1(\lambda_1)}{k'_2(\lambda_2)} \epsilon'_2(\lambda_2) \right), \quad (65)$$

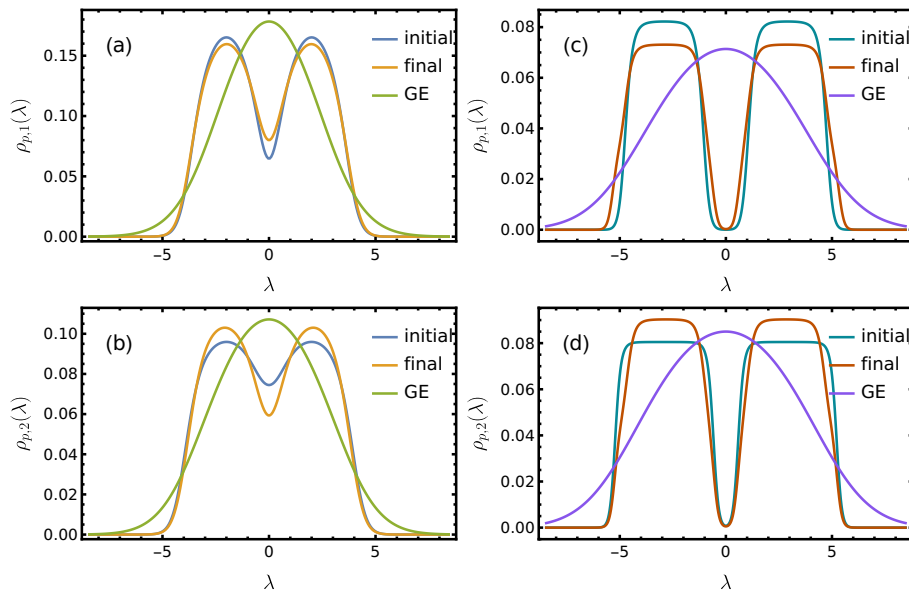


Figure 2: Initial, final and GE states for the dynamics with Bragg-split states as initial states. Panels (a) and (b) present distributions for the simulation in the intermediate regime, whereas in panels (c) and (d) we show the data for the strongly interacting regime. The final states are far from their respective GE states expected to appear as the final, thermal states in a full dynamics involving higher processes.

Table 2: Parameters used for the two cases of initial thermal states.

case 1 (intermediate interactions)						case 2 (strong interactions)					
c_1	c_2	β_1	β_2	μ_1	μ_2	c_1	c_2	β_1	β_2	μ_1	μ_2
2.0	4.0	0.4	0.7	2.7	4.2	32.0	128.0	0.4	0.7	9.0	9.6

where λ_2 is given by a solution to $v_1(\lambda_1) = v_2(\lambda_2)$ and plot it for several instances of time. If our stationary state is indeed a solution to the Eqn. (58), we should see function $H(\lambda_1)$ approaching zero at all λ for late times. We find that this is the case, see Fig. 4.

4.2.2 Two different thermal states

In the second scenario, to probe dynamics closer to an equilibrium state, we consider initial configurations corresponding to thermal states in both tubes. Specifically, the tubes are taken to have the same density $n_{1,2} = 1$ but different temperatures. The densities of the initial states read

$$\rho_{p,i}^{\text{ini}}(\lambda) = \rho_{\beta_i, \mu_i}^{\text{Gibbs}}(\lambda), \quad (66)$$

with the parameters used summarized in Table 2.

Despite the thermal starting point, the evolution for these states leads to final, non-thermal states that are nonetheless close to thermal states. This is illustrated in Fig. 5 where we compare the distributions $\rho_p(\lambda)$ in the initial and final states and in putative thermal equilibrium states consistent with the initial states. The lack of thermalization can be also witnessed in the evolution of the entropy, see Fig. 6. Here we see that the entropy draws close to the equilibrium thermal entropy but does not quite reach it.

The two kinds of initial states that we have considered show certain universal features of the dynamics generated by the (1,1) processes for long range interactions. We observe that these processes are not very efficient in redistributing the particles in the tubes. On a practical

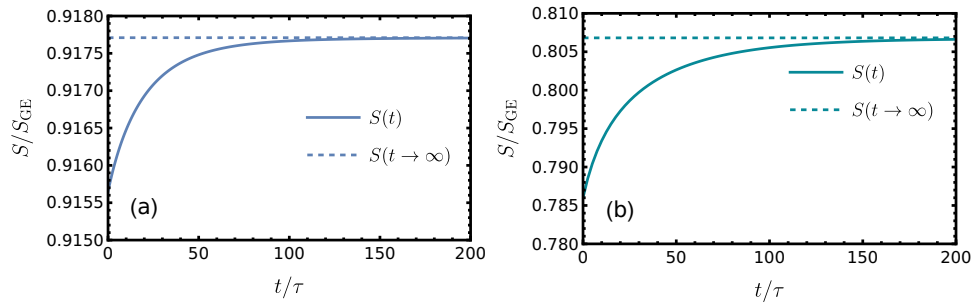


Figure 3: Yang-Yang entropy as a function of time for the dynamics of Bragg-split states. Panel (a): simulation in the intermediate regime, panel (b) dynamics in the strongly interacting regime. The entropy is normalized by the GE prediction for the final state. It reaches a value significantly smaller than 1, indicating that the dynamics is strongly constrained.

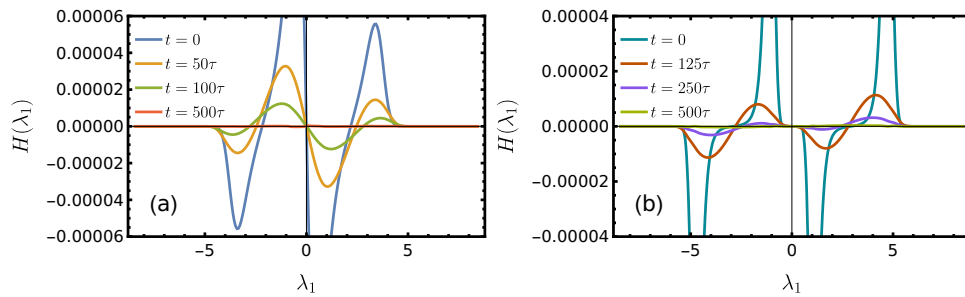


Figure 4: Function $H(\lambda_1)$ presented as a function of time. Panel (a): intermediate interactions, panel (b) strong interactions. We see that the function approaches zero for late times signalling that the state of the system in this limit fulfills (58).

level it results in small changes to the particles distributions with the stationary states very similar to the initial states. This is true for both Bragg-split and the thermal initial states. On the other hand, this raises a question on a mechanism behind this restricted dynamics. In the next section we show that in the case of strong intra-tube interactions this is caused by the existence of higher conserved charges.

5 Strongly interacting limit

In this section we consider the deformed Tonks-Girardeau gas. This is the strongly interacting limit of large c_i in which we keep corrections up to and including terms of order $1/c_i^2$. We will show that in such system there is an infinite family of conserved charges. These charges follow from solutions of eq. (54) and are linear combination of ultra-local charges (6) present in uncoupled tubes. Whereas the uncoupled system possesses two infinite families of charges, coupling the two tubes leads to halving the number of them. This leads to restricted (1,1) dynamics observed in the previous section.

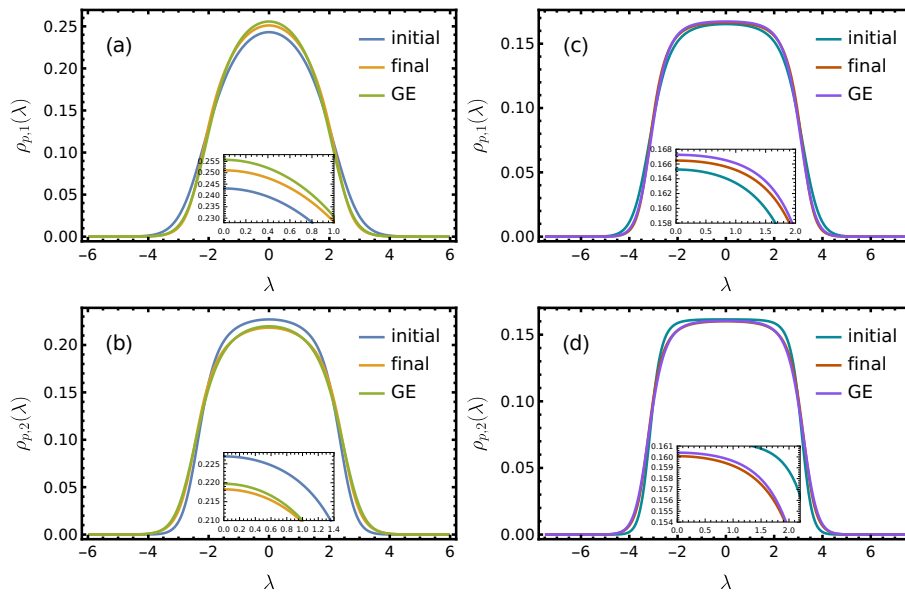


Figure 5: Initial, final and GE states for the dynamics with thermal states as initial states. Panels (a) and (b) present distributions for the simulation in the intermediate regime, whereas in panels (c) and (d) we show the data for strongly interacting regime. Final distributions are very close to the predictions of the GE, indicating that the system has almost thermalized. The small but real differences between the final and putative thermal distributions are clearly seen in the insets, where we have plotted the behavior of the distributions near $\lambda = 0$.

5.1 Conserved charges

We start by analyzing the sufficient condition (54) for an existence of a conserved charge. For the (1, 1) processes and in the small momentum limit it takes the form

$$\frac{(f'_1)_{\text{dr},1}(\lambda_1)}{\omega'_1(\lambda_1)} - \frac{(f'_2)_{\text{dr},2}(\lambda_2)}{\omega'_2(\lambda_2)} = 0, \quad v_1(\lambda_1) = v_2(\lambda_2), \quad (67)$$

where we have used the equality in Eqn. (13) between the two dressing procedures. We see that Eqn. (67) has exactly the same structure as Eqn. (58). Therefore, all solutions in the previous section for $\epsilon_{1,2}$ may be adapted for the case considered here. However there is a small caveat here that has far reaching consequences.

To illustrate the nature of the problem let us assume that we choose some function $f_1(\lambda)$ and solve for $f_2(\lambda)$. Because of the dressing procedures it is reasonable to expect that the resulting function depends on the distributions $\rho_{p,i}(\lambda)$. However, those evolve in time, whereas $f_2(\lambda)$ should be time-independent. Therefore, one possibility is that $f_2(\lambda)$ depends on the particles distributions through invariants of the evolution in each tube and the only such invariants are the total densities n_i of the particles.

This scenario is realized for the deformed Tonks-Girardeau gas. As we show in the Appendix D, Eq. (67) has a time independent solution only up to and including order $1/c_i^2$. The solution takes the form

$$\xi_2^{-1} f'_2(\xi_2 \lambda) = \xi_1^{-1} f'_1(\xi_1 \lambda). \quad (68)$$

Beyond that order it is impossible to find a pair of functions $(f_1(\lambda), f_2(\lambda))$ such that they are time independent and fulfil (67). Note that this solution has exactly the same structure as Eq. (61) for the stationary state of the deformed Tonks-Girardeau gas.

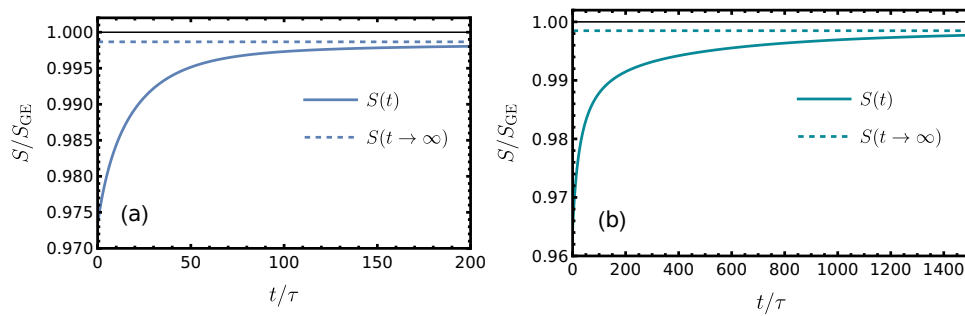


Figure 6: Yang-Yang entropy as a function of time, for the dynamics with initial thermal states. Panel (a): simulation in the intermediate regime, panel (b) dynamics in the strongly interacting regime. The entropy is normalized by the prediction of the GE ensemble for the final state. The final, stationary state is close to a thermal one, which is reflected in value of normalized entropy that is very close to 1.

From the solution in (68) we can construct a whole family of conserved charges based on the ultra-local conserved charges (6) present in a single Lieb-Liniger model. This results in the following conserved charges

$$\mathcal{I}_j = I_{1,j} + I_{2,j}, \quad I_{i,j} = L \int d\lambda f_{i,j}(\lambda) \rho_{p,i}(\lambda), \quad f_{i,j}(\lambda) = \xi_i^{2-j} \lambda^j. \quad (69)$$

We see that, due to the n_i dependent coefficients, the resulting expressions are in principle non-linear functions of densities $\rho_{p,i}$. However, since we are working at fixed density they are in fact linear. Even quantum mechanically, in a Hilbert space of fixed number of particles, the corresponding conserved charges would be linear operators. Interestingly, the rescaling of rapidities is the one found also in the $1/c$ expansion of ultra-local conserved charges [38].

The special feature of the dynamics of strongly interacting systems can be directly seen in the collision integral. In Appendix C we show that the collision integral with Tonks-Girardeau gas in both tubes obeys a symmetry relation

$$Q_0^{(1,1)}[\rho_{p,1}, \rho_{p,2}](\lambda) = -Q_0^{(1,1)}[\rho_{p,2}, \rho_{p,1}](\lambda). \quad (70)$$

This relation immediately implies that any functional of the form (49) with $f_1 = f_2$ is a conserved charge. In the deformed case this relation generalizes to

$$\xi_1^3 Q_0^{(1,1)}[\rho_{p,1}, \rho_{p,2}](\xi_1 \lambda) = -\xi_2^3 Q_0^{(1,1)}[\rho_{p,2}, \rho_{p,1}](\xi_2 \lambda) + \mathcal{O}(1/c^3), \quad (71)$$

and leads to conserved charges of the form (69) as shown in Appendix D.

The integral of motion shown in (69) is conserved only up to $1/c^2$, see Fig. 7. Furthermore, as we show in Appendix E, equation (67) has no solutions that would allow us straightforwardly to construct analytically conserved charges beyond that limit. However, the athermal stationary states do exist for arbitrary values of interaction parameters. This leaves open the question about the mechanism behind the lack of thermalization beyond the strongly interacting limit.

5.2 Generalized Gibbs ensemble

In the previous section we have observed that $(1,1)$ dynamics in the small momentum limit and for the deformed Tonks-Girardeau gases is characterized by additional conserved charges $\mathcal{I}_j$. We address now the problem of the final state reached by the system initialized in some

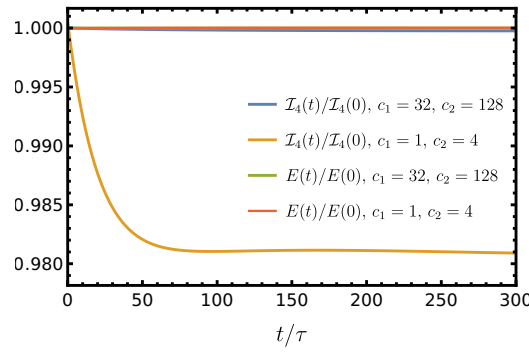


Figure 7: We display the first additional conserved charge $\mathcal{I}_4$ present for the deformed Tonks-Girardeau gas. It is conserved as well as the energies. Evaluating the same functional but in the system with intermediate interactions reveals that it is not anymore a conserved quantity. However the amplitude of its change is small, reflecting again the restricted dynamics of the $(1, 1)$ processes. The results shown here are for the Bragg-split states.

non-equilibrium configuration. One would expect relaxation to a Generalized Gibbs Ensemble (GGE) consistent with the presence of non-trivial conservation laws. In this section, we construct a GGE for our system and compare it with the properties of stationary state in Eqn. (58) found from the analysis of the collision integral. The whole construction is a generalization of Ref. [49, 50], where a single LL tube was analyzed.

The partition function of our system can be written as a functional integral over the inverse occupation [42],

$$Z = \text{const.} \int \mathcal{D}\left(\frac{\rho_{t,1}(\lambda)}{\rho_{p,1}(\lambda)}\right) \mathcal{D}\left(\frac{\rho_{t,2}(\lambda)}{\rho_{p,2}(\lambda)}\right) \delta(L\rho_{p,1}(\lambda) - N_1) \delta(L\rho_{p,2}(\lambda) - N_2) e^{-\mathcal{F}[\rho_{p,i}]}, \quad (72)$$

where

$$\begin{aligned} \mathcal{F} &= \beta_2 E + \sum_{j>2} \beta_j \mathcal{I}_j - S \\ &= L \int d\lambda \left[\beta_2 \lambda^2 (\rho_{p,1}(\lambda) + \rho_{p,2}(\lambda)) + \sum_{j>2} \beta_j (f_{1,j}(\lambda) \rho_{p,1}(\lambda) + f_{2,j}(\lambda) \rho_{p,2}(\lambda)) \right] \\ &\quad - L \sum_{i=1,2} \int d\lambda \left[\rho_{t,i}(\lambda) \ln(\rho_{t,i}(\lambda)) - \rho_{p,i}(\lambda) \ln(\rho_{p,i}(\lambda)) - \rho_{h,i}(\lambda) \ln(\rho_{h,i}(\lambda)) \right]. \end{aligned} \quad (73)$$

In the partition function we have introduced δ -functions that enforce that we are working at constant particle number in each of the tubes. Here we also see the influence that the system's conserved quantities, the energy E and the additional charges, $\mathcal{I}_j$, $j = 3, 4, \dots$, preserved under the $(1, 1)$ dynamics, have on the partition function. The form of $f_{1/2,j}$ is not arbitrary but assumed to satisfy Eqn. (68). β_2 is the temperature for the combined system's energy while $\beta_{j>2}$ are the generalized chemical potentials for the higher order charges. Note that the energy and $\mathcal{I}_{j>2}$ are charges defined involving degrees of freedom for both tubes and so β_j , with $j = 2, \dots$ do not differentiate between the tubes.

To evaluate the partition function in the thermodynamic limit we use the method of steepest descents [42]. The conserved charges are nonlinear functionals of $\rho_{p,i}$ and their variation with respect to $\rho_{p,i}$ has two contributions

$$\frac{\delta \mathcal{I}_{i,j}}{\delta \rho_{p,k}(\lambda)} = \delta_{ik} L (f_{i,j}(\lambda) + A_{i,j}), \quad A_{i,j} = \int d\mu \frac{\delta f_{i,j}(\mu)}{\delta \rho_{p,i}(\lambda)} \rho_{p,i}(\mu) = 2(2-j) \frac{I_{i,j}}{L \zeta_i c_i}. \quad (74)$$

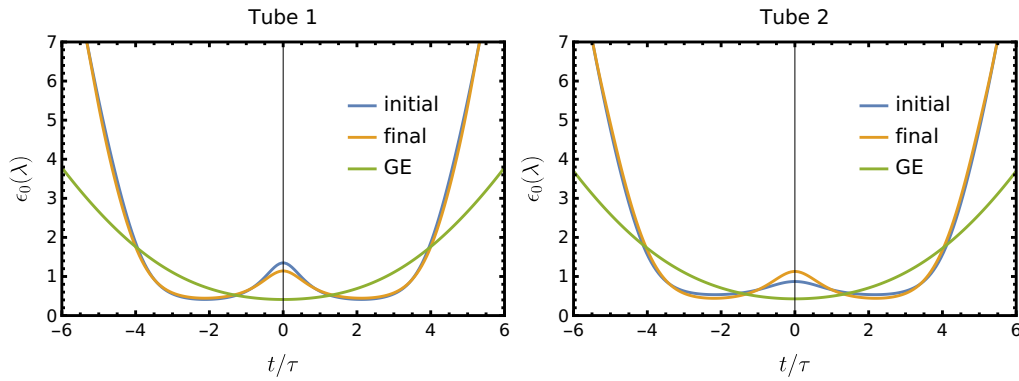


Figure 8: The pseudo-energies for the initial, final and GE states for the Bragg-split dynamics with $c_1 = 1$, $c_2 = 4$.

In writing this expression we have used eq. (69) for $I_{i,j}$ for which $f_{i,j}(\lambda) = \xi_i^{2-j} \lambda^j$.

We introduce now chemical potentials $\beta_{i,0}$ as Lagrange multipliers fixing the total densities of particles in both tubes. Introducing $\epsilon_i(\lambda) = \log(\rho_{h,i}(\lambda)/\rho_{p,i}(\lambda))$, the extremum condition for the generalized free energy is

$$\epsilon_i(\lambda) = \epsilon_{i,0}(\lambda) - \int d\lambda' T_i(\lambda - \lambda') \log[1 + e^{-\epsilon(\lambda')}] , \quad (75)$$

$$\epsilon_{i,0}(\lambda) = \beta_{i,0} + \sum_{j>2} \beta_j A_{i,j} + \beta_2 \lambda^2 + \sum_j \beta_j f_{i,j}(\lambda) , \quad (76)$$

$$\int d\lambda \rho_{p,i} = n_i , \quad (77)$$

which should be supplemented with the equation for $\rho_{t,i}(\lambda)$,

$$\rho_{t,i}(\lambda) = \frac{1}{2\pi} + \int d\lambda' T(\lambda - \lambda') n_i(\lambda') \rho_{t,i}(\lambda') , \quad (78)$$

which connects $\epsilon_i(\lambda)$ with $\rho_{p,i}(\lambda)$. We observe that while formulas (75) and (76) bear structural similarity to the standard TBA equations, there exists a significant distinction. In a standard formulation, the TBA equations are explicit, meaning that given a set of the chemical potentials, they directly yield $\epsilon(\lambda)$ from which expectation values of the conserved charges can be computed. Here, the relation is implicit, because $\epsilon(\lambda)$ depends itself on the value of the conserved charges through $A_{i,j}$. Nevertheless, the dependence on $A_{i,j}$ enters with a factor $1/c_i$ which allows for a consistent perturbative treatment. For instance, the $1/c_i$ correction to $\epsilon(\lambda)$ comes from the Tonks-Girardeau values of the conserved charges.

The implicit nature of the found GGE is also visible in the expectation values of the charges. The formulas for the particle number and energy are standard and given by: [49]

$$\langle N_i \rangle_{GGE} = \frac{L}{2\pi} \int d\lambda \frac{1}{1 + e^{\epsilon_i(\lambda)}} \frac{\partial \epsilon_i(\lambda)}{\partial \beta_{i,0}} , \quad \langle E_{tot} \rangle_{GGE} = \frac{L}{2\pi} \sum_{i=1,2} \int d\lambda \frac{1}{1 + e^{\epsilon_i(\lambda)}} \frac{\partial \epsilon_i(\lambda)}{\partial \beta_2} , \quad (79)$$

whereas for the other charges,

$$\langle \mathcal{I}_j \rangle_{GGE} = \sum_{i=1,2} \left[\frac{L}{2\pi} \int d\lambda \frac{1}{1 + e^{\epsilon_i(\lambda)}} \frac{\partial \epsilon_i(\lambda)}{\partial \beta_j} + A_{i,j} \langle N_i \rangle_{GGE} \right] . \quad (80)$$

The values of generalized chemical potentials in the final state are in principle determined from the equations $\langle \cdot \rangle_{ini} = \langle \cdot \rangle_{GGE}$ written for all conserved charges, where by $\langle \cdot \rangle_{ini}$ we denoted

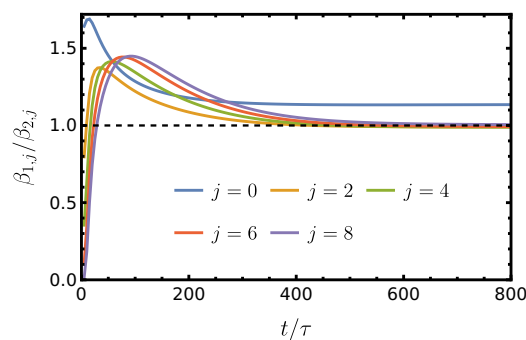


Figure 9: Time evolution of the generalized chemical potentials for the Bragg initial state with $c_1 = 32, c_2 = 128$. We plot the ratio $\beta_{1,j}/\beta_{2,j}$ of the potentials in the two tubes. The system equilibrates to $\beta_{1,j} = \beta_{2,j}$ $j \geq 2$ in agreement with the prediction of the GGE.

the expectation value in the initial state [13]. Unfortunately, solving these non-linear equations is a very difficult task. However, we can show that GGE states of the form (76) are stationary states in our dynamics. The proof of this statement is as follows. We note that

$$\epsilon'_i = (\epsilon'_{0,i})_{\text{dr},i}. \quad (81)$$

Dressing is a linear operation and therefore we can express ϵ'_i through the dressed single-particle eigenvalues, namely

$$\epsilon'_i = \beta_2(2\lambda)_{\text{dr},i} + \sum_j \beta_j (f'_{i,j})_{\text{dr},i}. \quad (82)$$

In the stationary state the pseudo-energies obey relation (58)

$$\frac{\epsilon'_1(\lambda_1)}{k'_1(\lambda_1)} - \frac{\epsilon'_2(\lambda_2)}{k'_2(\lambda_2)} = \sum_j \beta_j \left(\frac{(f'_{1,j})_{\text{dr},1}(\lambda_1)}{k'_1(\lambda_1)} - \frac{(f'_{2,j})_{\text{dr},2}(\lambda_2)}{k'_2(\lambda_2)} \right) = 0, \quad (83)$$

where in the last step we have used the conservation of the j -th conserved charged expressed in Eqn. (67). The athermal character of the pseudo-energies of the stationary states is shown in Fig. 8.

Finally, let us make the following observation. For a single tube Lieb-Liniger model there is a one-to-one relation between the state of the system and chemical potentials (generalized temperatures). We can now promote this relation to the two-tube case. The state of the system is then described by pseudo-energies $\epsilon_i(\lambda)$ characterized by the generalized temperatures $\beta_{i,j}$ which are potentially different for the two tubes, $\beta_{1,j} \neq \beta_{2,j}$. When the system evolves according to the Boltzmann equation this leads to a time dependence of the chemical potentials. The stationary state can be then understood as equalization of temperatures and higher chemical potentials, namely $\lim_{t \rightarrow \infty} (\beta_{1,j}(t) - \beta_{2,j}(t)) = 0$ for $j \geq 2$. We demonstrate it in Fig. 9.

6 Summary and conclusion

In this work we have studied the problem of the development of stationary states in a weakly perturbed integrable model. Our approach relies on a microscopic scattering integral in the framework of a Boltzmann equation. We applied this approach to the case of two weakly coupled Lieb-Liniger models, with the coupling taking the form of the density-density interactions,

a scenario relevant for the experiments with ultra-cold atomic gases. We found that for the long-range intertube interactions the leading processes described by the scattering integral do not thermalize the system, instead they lead to an athermal stationary state. To further understand the resulting time evolution we have solved numerically the Boltzmann equation for two very different initial states: i) the highly non-equilibrium Bragg split states and i) tubes characterized by thermal states at different temperatures. In both cases we have observed that time evolution is greatly restricted. The final stationary states resemble the initial states and do not evolve into thermal states. We also witnessed this arrested dynamics through studying the entropy production. There we saw that the system's entropy would evolve to a value lower than that of a thermal ensemble.

For the deformed Tonks-Girardeau gas we have shown that the restricted dynamics is caused by extra conserved charges beyond the particle number, momentum and energy. We have also shown that the athermal stationary state can be then characterized by the generalized Gibbs Ensemble. Interestingly, our construction of the conserved charges does not extend to the full Lieb-Liniger model.

In this work we have focused on revealing the structure of the athermal stationary states. An interesting question would be to combine the leading $(1, 1)$ with higher processes that do thermalize the system. This would provide an access to the full time evolution from the initial state, through the prethermalization plateau, all the way to the final thermal state. Numerical implementation of this problem requires a substantial effort that we delegate to future work.

Our work leads also to a more fundamental question whether the extra conserved charges exist at the quantum level, beyond the Boltzmann equation. Whereas their exact conservation is unlikely, they might correspond to 'slow variables' and therefore they remain important for the dynamics at intermediate timescales. This scenario is realized in the spin ladder systems as revealed recently [51].

Finally, the results for the case where the two tubes are initialized in thermal states at different temperatures shows that the evolution occurs entirely in the vicinity of thermal states. In such a situation it might be possible to develop an effective description of the dynamics which instead of looking at the full distribution of the rapidities would involve only examining the evolution of the temperatures and perhaps several of the higher chemical potentials. This would then allow the development of a sort of Newton's law of cooling for weakly perturbed integrable systems.

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A Scaling of the collision integral in the small momentum limit

In this appendix we show that $Q_0^{(m,n)} \sim k_r^{m+n}$ with k_r the characteristic momentum of the intertube potential. From the symmetry of Eqn. (29), we can replace sum of delta functions

by a single one, multiplied by n

$$Q_0^{(n,m)}[\rho_{p,1}, \rho_{p,2}](\lambda) = (2\pi)^2 n \int d\mathbf{p}_1^n d\mathbf{h}_1^n d\mathbf{p}_2^m d\mathbf{h}_2^m \delta(\lambda - p_{1,1}) \tilde{A}^2(k_1(\mathbf{p}_1, \mathbf{h}_1)) \\ \times |F_1(\mathbf{p}_1, \mathbf{h}_1)|^2 |F_2(\mathbf{p}_2, \mathbf{h}_2)|^2 \mathcal{J}^{(n,m)}(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) \\ \times \delta(k_1(\mathbf{p}_1, \mathbf{h}_1) + k_2(\mathbf{p}_2, \mathbf{h}_2)) \delta(\omega_1(\mathbf{p}_1, \mathbf{h}_1) + \omega_2(\mathbf{p}_2, \mathbf{h}_2)),$$

where we recall the definition of $\mathcal{J}^{(n,m)}$

$$\mathcal{J}^{(n,m)}(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) = J_1^n(\mathbf{p}_1, \mathbf{h}_1) J_2^m(\mathbf{p}_2, \mathbf{h}_2) - J_1^n(\mathbf{h}_1, \mathbf{p}_1) J_2^m(\mathbf{h}_2, \mathbf{p}_2). \quad (\text{A.1})$$

We express the collision integrals in the center of mass variables, namely,

$$\lambda_{i,j} = \frac{1}{2}(p_{i,j} + h_{i,j}), \quad \alpha_{i,j} = p_{i,j} - h_{i,j}. \quad (\text{A.2})$$

We make now the following observations. The form-factors of the density operator, due to their ph symmetry are even function of $\alpha_{i,j}$ and their leading order is $\alpha_{i,j}$ independent, i.e.,

$$F_1(\mathbf{p}_1, \mathbf{h}_1) = F_1(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1) + \mathcal{O}(\alpha_i^2). \quad (\text{A.3})$$

The difference of the density factors appearing through the J functions, see (24), is an odd function upon changing the sign of all $\alpha_{i,j}$. At the same time, the product of the density functions evaluated for a particle-hole excitation gives

$$\rho_p(\lambda - \alpha/2) \rho_h(\lambda + \alpha/2) = \rho_p(\lambda) \rho_h(\lambda) (1 + \epsilon'(\lambda) \alpha/2 + \mathcal{O}(\alpha^2)). \quad (\text{A.4})$$

Therefore we get

$$\mathcal{J}^{(n,m)}(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) = \prod_{i=1}^n \rho_{p,1}(\lambda_{i,1}) \rho_{h,1}(\lambda_{i,1}) \prod_{j=1}^m \rho_{p,2}(\lambda_{j,2}) \rho_{h,2}(\lambda_{j,2}) \\ \times \left(\sum_{i=1}^n \epsilon'_1(\lambda_{1,i}) \alpha_{1,i} + \sum_{i=1}^m \epsilon'_2(\lambda_{2,i}) \alpha_{2,i} + \mathcal{O}(\alpha_{i,j}^3) \right).$$

The scattering integral, up to the first two leading orders in $\alpha_{i,j}$, takes then the following form,

$$Q_0^{(n,m)}[\rho_{p,1}, \rho_{p,2}](\mu) = (2\pi)^2 n \int d\boldsymbol{\lambda}_1^n d\boldsymbol{\lambda}_2^m G(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2), \quad (\text{A.5})$$

where

$$G(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = |F_1(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1)|^2 |F_2(\boldsymbol{\lambda}_2, \boldsymbol{\lambda}_2)|^2 \prod_{i=1}^n \rho_{p,1}(\lambda_{i,1}) \rho_{h,1}(\lambda_{i,1}) \prod_{j=1}^m \rho_{p,2}(\lambda_{j,2}) \rho_{h,2}(\lambda_{j,2}), \quad (\text{A.6})$$

$$H(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \int d\boldsymbol{\alpha}_1^n d\boldsymbol{\alpha}_2^m \tilde{A}^2(k_1(\mathbf{p}_1, \mathbf{h}_1)) \delta(\mu - \lambda_{1,1} - \alpha_{1,1}/2) \delta(k_1(\mathbf{p}_1, \mathbf{h}_1) + k_2(\mathbf{p}_2, \mathbf{h}_2)) \\ \times \delta(\omega_1(\mathbf{p}_1, \mathbf{h}_1) + \omega_2(\mathbf{p}_2, \mathbf{h}_2)) \left(\sum_{i=1}^n \epsilon'_1(\lambda_{1,i}) \alpha_{1,i} + \sum_{i=1}^m \epsilon'_2(\lambda_{2,i}) \alpha_{2,i} \right). \quad (\text{A.7})$$

Consider first the simplest case of (1, 1) processes. Then we have to perform the following integral

$$H(\lambda_1, \lambda_2) = \int d\alpha_1 d\alpha_2 \tilde{A}^2(k_1(\lambda_1, \alpha_1)) \delta(\mu - \lambda_1 - \alpha_1/2) \delta(k_1(\lambda_1, \alpha_1) + k_2(\lambda_2, \alpha_2)) \\ \times \delta(\omega_1(\lambda_1, \alpha_1) + \omega_2(\lambda_2, \alpha_2)) (\epsilon'_1(\lambda_1) \alpha_1 + \epsilon'_2(\lambda_2) \alpha_2). \quad (\text{A.8})$$

For the momentum energy constraints we find

$$k(\lambda, \alpha) = \alpha k'(\lambda) + \mathcal{O}(\alpha^3), \quad \omega(\lambda, \alpha) = \alpha \omega'(\lambda) + \mathcal{O}(\alpha^3). \quad (\text{A.9})$$

Thus we have

$$\delta(k_1(\lambda_1, \alpha_1) + k_2(\lambda_2, \alpha_2)) = \delta(\alpha_1 k'_1(\lambda_1) + \alpha_2 k'_2(\lambda_2)), \quad (\text{A.10})$$

and similarly for the energy δ -function. Then we can write

$$\begin{aligned} & \delta(k_1(\lambda_1, \alpha_1) + k_2(\lambda_2, \alpha_2)) \delta(\omega_1(\lambda_1, \alpha_1) + \omega_2(\lambda_2, \alpha_2)) \\ &= \frac{\delta(\alpha_2 + \alpha_1 k'_1(\lambda_1)/k'_2(\lambda_2))}{|\alpha_1 k'_1(\lambda_1) k'_2(\lambda_2)|} \delta(v_1(\lambda_1) - v_2(\lambda_2)), \end{aligned} \quad (\text{A.11})$$

where $v(\lambda) = \omega'(\lambda)/k'(\lambda)$. We can perform now the integral over α_2 . The result is

$$\begin{aligned} H(\lambda_1, \lambda_2) &= \int d\alpha_1 \text{sgn}(\alpha_1) \tilde{A}^2(k_1(\lambda_1, \alpha_1)) \delta(\mu - \lambda_1 - \alpha_1/2) \\ &\quad \times \frac{\delta(v_1(\lambda_1) - v_2(\lambda_2))}{|k'_2(\lambda_2)|} \left(\frac{\epsilon'_1(\lambda_1)}{k'_1(\lambda_1)} - \frac{\epsilon'_2(\lambda_2)}{k'_2(\lambda_2)} \right). \end{aligned} \quad (\text{A.12})$$

The remaining integral over α_1 can be now also performed. The result is

$$H(\lambda_1, \lambda_2) = \text{sgn}(\mu - \lambda_1) \tilde{A}^2(k_1(\lambda_1, 2(\mu - \lambda_1))) \frac{\delta(v_1(\lambda_1) - v_2(\lambda_2))}{|k'_2(\lambda_2)|} \left(\frac{\epsilon'_1(\lambda_1)}{k'_1(\lambda_1)} - \frac{\epsilon'_2(\lambda_2)}{k'_2(\lambda_2)} \right). \quad (\text{A.13})$$

The remaining δ -function fixes uniquely λ_2 as a function of λ_1 . Therefore the scattering integral involves only one integral. The range of the integration over λ_1 is constrained by the function $\tilde{A}(k_1(\lambda_1, 2(\mu - \lambda_1)))$ which is peaked around $\lambda_1 = \mu$. The structure of the expression is then

$$Q_0^{(1,1)}[\rho_{p,1}, \rho_{p,2}](\mu) = \int d\lambda_1 \text{sgn}(\mu - \lambda_1) \tilde{A}^2(2(\mu - \lambda_1)k'(\lambda_1)) f(\lambda_1), \quad (\text{A.14})$$

where in $f(\lambda_1)$ we have gathered all the remaining factors. Now we will take advantage of properties of the interaction potential and use representation (21). We get

$$Q_0^{(1,1)}[\rho_{p,1}, \rho_{p,2}](\mu) \sim \int d\lambda_1 \text{sgn}(\mu - \lambda_1) \tilde{a}^2\left(\frac{2(\mu - \lambda_1)k'(\lambda_1)}{k_r}\right) f(\lambda_1). \quad (\text{A.15})$$

We assume now that function $\tilde{a}(k/k_r)$ is centered around 0 and symmetric. In the leading order in k_r we can approximate this expression by replacing $k'(\lambda_1)$ by $k'(\mu)$. We then change the variables to $x = 2(\mu - \lambda_1)k'(\mu)/k_r$. The result is

$$Q_0^{(1,1)}[\rho_{p,1}, \rho_{p,2}](\mu) \sim \frac{k_r}{2k'(\mu)} \int dx \text{sgn}(x) \tilde{a}^2(x) f\left(\mu - x \frac{k_r}{2k'(\mu)}\right). \quad (\text{A.16})$$

Expanding now in powers of x we find

$$Q_0^{(1,1)}[\rho_{p,1}, \rho_{p,2}](\mu) \sim \frac{k_r f(\mu)}{2k'(\mu)} \int dx \text{sgn}(x) \tilde{a}^2(x) + \frac{k_r^2 f'(\mu)}{(2k'(\mu))^2} \int dx |x| \tilde{a}(x) + \mathcal{O}(k_r^4). \quad (\text{A.17})$$

The first integral vanishes and the leading contribution comes from the second term which is proportional to the characteristic momentum k_r . The answer involves the derivative of

$f(\mu)$. This function involves all the density factors and form-factors. Its derivative is thus quite complicated and not very practical for numerical computations. We also note that the subleading corrections are of order k_r^4 and not k_r^3 which is important for the hierarchy of processes contributing to the Boltzmann equation as discussed in the main text.

We generalize now the analysis to the (n, m) case. We start by considering $H(\lambda_1, \lambda_2)$ and use the energy-momentum constraints to solve for $\alpha_{m,2}$ and $\lambda_{m,2}$ and the third δ -function to fix $\lambda_{1,1} = \mu - \alpha_{1,1}/2$. To this end we introduce the following notation

$$\bar{k} = \sum_{i=1}^n \alpha_{i,1} k'_1(\lambda_{i,1}) + \sum_{i=1}^{m-1} \alpha_{i,2} k'_2(\lambda_{i,2}), \quad (\text{A.18})$$

$$\bar{\omega} = \sum_{i=1}^n \alpha_{i,1} \omega'_1(\lambda_{i,1}) + \sum_{i=1}^{m-1} \alpha_{i,2} \omega'_2(\lambda_{i,2}), \quad (\text{A.19})$$

$$\bar{\epsilon} = \sum_{i=1}^n \alpha_{i,1} \epsilon'_1(\lambda_{i,1}) + \sum_{i=1}^{m-1} \alpha_{i,2} \epsilon'_2(\lambda_{i,2}), \quad (\text{A.20})$$

and $\bar{v} = \bar{\omega}/\bar{k}$. Then

$$\delta(k_1(\mathbf{p}_1, \mathbf{h}_1) + k_2(\mathbf{p}_2, \mathbf{h}_2)) \delta(\omega_1(\mathbf{p}_1, \mathbf{h}_1) + \omega_2(\mathbf{p}_2, \mathbf{h}_2)) = \frac{\delta(\alpha_{m,2} - \bar{\alpha}_{m,2}) \delta(v_2(\lambda_{m,2}) - \bar{v})}{k'(\lambda_{m,2}) |\bar{k}|}, \quad (\text{A.21})$$

with $\bar{\alpha}_{m,2} = \bar{k}/k'_2(\lambda_{m,2})$. This gives

$$H(\lambda_1, \lambda_2) = \int d\alpha_1^n d\alpha_2^{m-1} \tilde{A}^2(k_1(\mathbf{p}_1, \mathbf{h}_1)) \frac{\delta(v_2(\lambda_{1,2}) - \bar{v})}{k'(\lambda_{1,2})} \left(\frac{\bar{\epsilon}}{\bar{k}} - \frac{\epsilon'_2(\lambda_{1,2})}{k'_2(\lambda_{1,2})} \right) \text{sgn}(\bar{k}). \quad (\text{A.22})$$

We now introduce new variables $x_{i,j} = \alpha_{i,j} k'(\lambda_{i,j})/k_r$ for all i, j except for $i = j = 1$ where we write instead $x_{1,1} = \alpha_{1,1} k'(\mu)/k_r$. We now find for $H(\lambda_1, \lambda_2)$,

$$H(\lambda_1, \lambda_2) = k_r^{n+m-1} \frac{\delta(v_2(\lambda_{1,2}) - \bar{v})}{\prod_{i,j} k'(\lambda_{i,j})} \int d\mathbf{x}_1^n d\mathbf{x}_2^{m-1} \text{sgn}(\sum_{i,j} x_{i,j}) \tilde{a}^2 \left(\sum_i x_{i,1} \right) \left(\frac{\bar{\epsilon}}{\bar{k}} - \frac{\epsilon'_2(\lambda_{m,2})}{k'_2(\lambda_{m,2})} \right). \quad (\text{A.23})$$

In writing the argument of $\tilde{a}$ we have neglected contributions of order $x_{1,1}^2$ as they do not contribute to the integral in the first two leading orders. The ratio $\bar{\epsilon}/\bar{k}$ is the following

$$\frac{\bar{\epsilon}}{\bar{k}} = \frac{\sum x_{i,j} \epsilon'_j(\lambda_{i,j})/k'_j(\lambda_{i,j}) + x_{1,1} \epsilon'_1(\mu - \alpha_{1,1}/2)/k'_1(\mu)}{\sum x_{i,j} + x_{1,1} k'_1(\mu - \alpha_{1,1}/2)/k'_1(\mu)}, \quad (\text{A.24})$$

with $\alpha_{1,1} = k_r x_{1,1}/k'_1(\mu)$. Thus, at the leading order in k_r it is k_r independent and invariant under simultaneous change of sign of all the $x_{i,j}$. As a result, the integral over all the $x_{i,j}$'s vanishes. The leading contribution comes then from including the correction due to $\alpha_{1,1}$. Therefore, the whole contribution to the scattering integral is of the order k_r^{n+m} as stated in the main text.

B Absence of non-thermal stationary states beyond the small momentum limit

In the main body of the text we demonstrated that if the dynamics of the two tubes of gas are controlled by $(1, 1)$ processes in the small momentum limit, a prethermalization plateau

is arrived at that is characterized by a set of non-trivial conservation laws. In this appendix, we argue that this is in general a special case, that if we consider $(1, 1)$ processes outside a small momentum approximation, thermalization occurs. We note up front that there are two exceptions to this general rule:

- The Tonks-Girardeau (TG) limit. The thermalization that occurs due to the higher momentum $(1, 1)$ processes happens because of non-linearities in the dispersion relation induced by interactions. But at $c = \infty$ these interactions are absent and the argument for a non-trivial prethermalization plateau continues to hold even if higher momentum processes are accounted for. Moreover we can conclude that TG gases do not thermalize at all as higher-order ph processes are absent. Though we have broken the integrability at the level of a single tube, new exact integrals of motion have appeared for the two-tube case.
- The two gases are identical. In this case the $(1, 1)$ processes induces no dynamics whatsoever and the gas does not evolve. Thermalization however will still happen on account of higher order (n, m) processes.

We will comment on these two special cases further.

So let us turn to $(1, 1)$ processes at arbitrary momentum. The condition for the stationary state, $Q_0^{(1,1)} = 0$, translates to demanding that

$$\epsilon_1(p_1) - \epsilon_1(h_1) + \epsilon_2(p_2) - \epsilon_2(h_2) = 0, \quad (\text{B.1})$$

for all p_1, h_1, p_2, h_2 fulfilling the momentum-energy constraints

$$\begin{aligned} k_1(p_1) - k_1(h_1) + k_2(p_2) - k_2(h_2) &= 0, \\ \omega_1(p_1) - \omega_1(h_1) + \omega_2(p_2) - \omega_2(h_2) &= 0. \end{aligned} \quad (\text{B.2})$$

Here functions $k_i(\lambda)$ and $\omega_i(\lambda)$ are determined from the $\epsilon_i(\lambda)$ through the TBA equations. We start by analyzing the structure of Eqns. (B.1) and (B.2).

The constraints (B.2) can be solved for p_1 and h_1 . We assume that the solution is unique and takes the form $p_1 = f(p_2, h_2)$ and $h_1 = g(p_2, h_2)$ for two functions f and g . Because of the symmetry of the constraints upon replacing particles with holes the two functions are not independent and we have $g(p_2, h_2) = f(h_2, p_2)$. Therefore, a generic solution to the constraints can be written as

$$p_1 = f(p_2, h_2), \quad h_1 = f(h_2, p_2). \quad (\text{B.3})$$

We define now a function $F(p_2, h_2)$,

$$F(p_2, h_2) \equiv \epsilon_1(f(p_2, h_2)) - \epsilon_1(f(h_2, p_2)). \quad (\text{B.4})$$

The condition (B.1) for the stationary state in these terms is simply

$$F(p_2, h_2) = \epsilon_2(p_2) - \epsilon_2(h_2). \quad (\text{B.5})$$

To obtain a general solution here we assume that $f(p_2, h_2)$ is independent of one of its variables, which we choose to be p_2 . This simplifies the structure of F to

$$F(p_2, h_2) = \epsilon_1(f(h_2)) - \epsilon_1(f(p_2)). \quad (\text{B.6})$$

The equation for the stationary states separates now in two parts depending only on p_2 and h_2 respectively. This leads to two independent equations of the same form which fixes the

relation between ϵ_1 and ϵ_2 up to an additive constant which is just a difference of chemical potentials. The result is

$$\epsilon_1(f(\lambda)) = \epsilon_2(\lambda) + \mu_1 - \mu_2. \quad (\text{B.7})$$

The function $f(\lambda)$ by construction satisfies

$$k_1(f(p_2)) - k_1(f(h_2)) + k_2(p_2) - k_2(h_2) = 0, \quad (\text{B.8})$$

$$\omega_1(f(p_2)) - \omega_1(f(h_2)) + \omega_2(p_2) - \omega_2(h_2) = 0. \quad (\text{B.9})$$

Assuming the monotonicity of the dressed momenta k_i , it is straightforward to solve the first equation for f . The result is $f = k_1^{-1} \circ k_2$. The second equation provides now a constraint between the functions k_i and ω_i themselves. It reads

$$\omega_2 \circ k_2^{-1} = \omega_1 \circ k_1^{-1}. \quad (\text{B.10})$$

Putting it all together we find the following set of equations

$$\begin{aligned} \epsilon_1 \circ k_1^{-1}(p) - \mu_1 &= \epsilon_2(p) \circ k_2^{-1}(p) - \mu_2, \\ \omega_1 \circ k_1^{-1}(p) &= \omega_2 \circ k_2^{-1}(p). \end{aligned} \quad (\text{B.11})$$

This is a sought after rewriting of equations (B.1) and (B.2). For a given function ϵ_2 , the TBA equations determine k_2 and ω_2 . Thus the right hand sides of the above two equations are known. The equations then have to be solved for ϵ_1 such that TBA integral equations defining k_1 and ω_1 in terms of ϵ_1 are self-consistent.

We show now three special cases where we can solve the above:

- *Thermal states:* For a thermal state in the second tube, $\epsilon_2(\lambda) = \mu_2 + \beta \omega_2(\lambda)$, we have a chain of following transformations

$$\epsilon_1 \circ k_1^{-1}(p) - \mu_1 = \epsilon_2 \circ k_2^{-1}(p) - \mu_2 \quad (\text{B.12})$$

$$= \beta \omega_2 \circ k_2^{-1}(p) \quad (\text{B.13})$$

$$= \beta \omega_1 \circ k_1^{-1}(p). \quad (\text{B.14})$$

Therefore the solution is $\epsilon_1(\lambda) = \mu_1 + \beta \omega_1(\lambda)$ and we know that this choice is consistent with the TBA integrals defining ω_1 and k_1 .

- *Identical gases in identical states:* If the two tubes of gas are identical in all respects, then any choice of $\epsilon_1 = \epsilon_2$ (both thermal and athermal) satisfies the above conditions. Because $k_1 = k_2$ and $\omega_1 = \omega_2$, any scattering process must have $p_1 = h_2$ and $p_2 = h_1$. By symmetry under exchange of particles and holes, $Q^{1,1}$ vanishes identically and at this order, the gas does not evolve from its initial state.
- *Tonks-Girardeau gas:* When both tubes are in the strongly interacting limit, $c_i = \infty$, then there is no dressing and the second equality in (B.11) is trivially fulfilled. At the same time the first equations simplifies to $\epsilon_1(\lambda) - \mu_1 = \epsilon_2(\lambda) - \mu_2$. Therefore the stationary state is given by the pseudo-energies equal up to the chemical potentials which can be different in both tubes.

We cannot prove that other solutions are impossible, but we were unable to find additional possibilities. Therefore we expect that in general the higher momentum contributions to the $(1, 1)$ processes lead to the thermalization of the system.

Further support of this contention is provided by considering an expansion of the stationary state condition in the momentum transferred between the states. To this end we expand eqs. (B.1) and (B.2) in $\alpha_i = p_i - h_i$. At leading order we obtain the following constraint:

$$\begin{aligned} v_1(\lambda_1) &= v_2(\lambda_2), \\ \epsilon'_1(\lambda_1) &= \frac{k'_1(\lambda_1)}{k'_2(\lambda_2)} \epsilon'_2(\lambda_2), \end{aligned} \quad (\text{B.15})$$

where $\lambda_i = \frac{1}{2}(p_i + h_i)$. We already know that these constraints admit solutions that are athermal. This was our main result in the main body of the text. At next order in α_i , $\mathcal{O}(\alpha_i^3)$, we obtain additional constraints that need to be satisfied for stationarity:

$$\begin{aligned} v_2(\lambda_1) &= \left(1 - \frac{k'_1(\lambda_1)^3 k''_2(\lambda_2)}{k'_2(\lambda_2)^3 k''_1(\lambda_1)} \right)^{-1} \left(\frac{\omega'''_1(\lambda_1)}{k'''_1(\lambda_1)} - \frac{k'_1(\lambda_1)^3 \omega'''_2(\lambda_2)}{k'_2(\lambda_2)^3 k'''_1(\lambda_1)} \right), \\ \epsilon'''_1(\lambda_1) &= \left(\frac{k'''_1(\lambda_1)}{k'_2(\lambda_2)} - \frac{k'_1(\lambda_1)^3}{k'_2(\lambda_2)^4} k'''_2(\lambda_2) \right) \epsilon'_2(\lambda_2) + \frac{k'_1(\lambda_1)^3}{k'_2(\lambda_2)^3} \epsilon'''_2(\lambda_2). \end{aligned} \quad (\text{B.16})$$

It seems unlikely that both constraints (B.15) and (B.16) can be satisfied simultaneously, beyond the specialized circumstances discussed above. A similar line of reasoning may be straightforwardly extended to processes involving more ph pairs.

C Tonks-Girardeau gas

In this Appendix we study in details the Boltzmann equation for the Tonks-Girardeau gas, $c \rightarrow \infty$. In this limit the (1,1) processes are the only processes because the form-factors with larger number of particle-hole pairs vanish. Also in this limit the dressings are absent what significantly reduces the complexity of the problem and allows us to write the scattering integral in a simple form.

We start with (29) adopted to (1,1) processes. Using that in the Tonks-Girardeau (TG) gas the form-factors of single particle-hole excitations are identically 1 we obtain

$$\begin{aligned} Q^{(1,1)}[12](\lambda) &= (2\pi)^2 \int dp_1 dh_1 dp_2 dh_2 \delta(\lambda - p_1) \tilde{A}^2(k_1(p_1, h_1)) \\ &\quad \times \delta(k_1(p_1, h_1) + k_2(p_2, h_2)) \delta(\omega_1(p_1, h_1) + \omega_2(p_2, h_2)) \\ &\quad \times (J_1(p_1, h_1) J_2(p_2, h_2) - J_1(h_1, p_1) J_2(h_2, p_2)), \end{aligned} \quad (\text{C.1})$$

In this and next Appendix we shorten the notation from $Q^{(1,1)}[\rho_{p,1}, \rho_{p,2}]$ to $Q^{(1,1)}[12]$. The momentum and energy in TG gas are given by $k(\lambda) = \lambda$ and $\omega(\lambda) = \lambda^2$ respectively. This allows us to solve the kinematic constraint $k(p_1, h_1) + k(p_2, h_2) = 0$ and $\omega(p_1, h_1) + \omega(p_2, h_2) = 0$. In terms of the center-of-mass rapidities $\lambda_i = (p_i + h_i)/2$ and $\alpha_i = p_i - h_i$ we find

$$\delta(k(p_1, h_1) + k(p_2, h_2)) \delta(\omega(p_1, h_1) + \omega(p_2, h_2)) = \frac{\delta(\alpha_1 + \alpha_2) \delta(\lambda_1 - \lambda_2)}{2\alpha_i}. \quad (\text{C.2})$$

The Jacobian of transformation from (p_i, h_i) to (λ_i, α_i) is 1 and the collision integral, after evaluating integrals over λ_2 and α_2 with the help of the Dirac δ -functions, becomes

$$Q_0^{(1,1)}[12](\lambda) = (2\pi)^2 \int d\lambda_1 d\alpha_1 \delta(\lambda - p_1) \frac{\tilde{A}^2(\alpha_1)}{2|\alpha_1|} (J_1(p_1, h_1) J_2(p_2, h_2) - J_1(h_1, p_1) J_2(h_2, p_2)). \quad (\text{C.3})$$

Evaluating now the integral over λ_1 and dropping the index of α_1 gives

$$Q^{(1,1)}[12](\lambda) = (2\pi)^2 \int d\alpha (J_1(\lambda, \lambda - \alpha)J_2(\lambda - \alpha, \lambda) - J_1(\lambda + \alpha, \lambda)J_2(\lambda, \lambda + \alpha)) \frac{\tilde{A}^2(\alpha)}{2|\alpha|}. \quad (\text{C.4})$$

This is the final expression for the collision integral in the TG gas. It shows that the dynamics is driven by the difference in particles distributions with the rate controlled by the coupling potential.

The collision integral $Q^{(1,1)}[21](\lambda)$ for the second tube comes from exchanging the indices 1 and 2 and $Q^{(1,1)}[21](\lambda) = -Q^{(1,1)}[12](\lambda)$. This implies that the following combination

$$\int d\lambda (\rho_{p,1}(\lambda) + \rho_{p,2}(\lambda)) f(\lambda), \quad (\text{C.5})$$

is a conserved charge for any $f(\lambda)$ such that the integral exists.

We can also find the stationary state. From Eqn. (C.3) the condition is

$$J_1(\lambda, \lambda - \alpha)J_2(\lambda - \alpha, \lambda) - J_1(\lambda - \alpha, \lambda)J_2(\lambda, \lambda - \alpha) = 0, \quad (\text{C.6})$$

which gives $\epsilon_1(\lambda) = \epsilon_2(\lambda) + \text{const}$ in agreement with Eqn. (59) from the main text.

In the limit of small momentum transfer between the tubes, the collision integral can be further evaluated by expanding the integrand in α . The density factors have the following small α expansion

$$J_i(\lambda + \alpha, \lambda) = \rho_{p,i}(\lambda)\rho_{h,i}(\lambda) \left(1 + \frac{\rho'_{h,i}(\lambda)}{\rho_{h,i}(\lambda)}\alpha + \frac{1}{2} \frac{\rho''_{h,i}(\lambda)}{\rho_{h,i}(\lambda)}\alpha^2 \right), \quad (\text{C.7})$$

$$J_i(\lambda, \lambda + \alpha) = \rho_{p,i}(\lambda)\rho_{h,i}(\lambda) \left(1 + \frac{\rho'_{p,i}(\lambda)}{\rho_{p,i}(\lambda)}\alpha + \frac{1}{2} \frac{\rho''_{p,i}(\lambda)}{\rho_{p,i}(\lambda)}\alpha^2 \right). \quad (\text{C.8})$$

After substituting these expansions, we find that the leading term vanishes identically under the integral while the term linear in α vanishes after integrating over α ($\tilde{A}(\alpha)$ is an even function). Therefore, the leading contribution comes from order α^2 and is given by

$$Q_0^{(1,1)}[12](\lambda) = (2\pi)^2 \left(\prod_{i=1,2} \rho_{p,i}(\lambda)\rho_{h,i}(\lambda) \right) G_{1,2}(\lambda) \int d\alpha \frac{|\alpha|\tilde{A}^2(\alpha)}{4}, \quad (\text{C.9})$$

with

$$G_{1,2}(\lambda) = \frac{\rho''_{p,1}(\lambda)}{\rho_{p,1}(\lambda)} - \frac{\rho''_{h,1}(\lambda)}{\rho_{h,1}(\lambda)} - \frac{\rho''_{p,2}(\lambda)}{\rho_{p,2}(\lambda)} + \frac{\rho''_{h,2}(\lambda)}{\rho_{h,2}(\lambda)} + 2 \left(\frac{\rho'_{p,1}(\lambda)\rho'_{h,2}(\lambda)}{\rho_{p,1}(\lambda)\rho_{h,2}(\lambda)} - \frac{\rho'_{h,1}(\lambda)\rho'_{p,2}(\lambda)}{\rho_{h,1}(\lambda)\rho_{p,2}(\lambda)} \right). \quad (\text{C.10})$$

We can now extract the k_r dependence of the collision integral. We write $\tilde{A}(\alpha) = \tilde{a}(\alpha/k_r)$ according to Eqn. (21) and change the integration variable to find that $Q_0^{(1,1)}[12](\lambda) \sim k_r^2$ in agreement with the general result presented in Appendix A.

D Deformed Tonks-Girardeau gas

In this Appendix we consider the deformed Tonks-Girardeau, the $c \rightarrow \infty$ limit in which we keep terms $1/c$ up to and including order $1/c^2$. In practice, we will keep also terms of higher orders. This will be useful for Appendix E to illustrate a qualitative difference between the deformed Tonks-Girardeau gas and systems with weaker interactions.

TBA for the deformed Tonks-Girardeau gas

We start with the relation between the filling function $n(\theta)$ and the particles' density $\rho_p(\lambda)$. First, we use the defining relation (2) between $\rho_p(\lambda)$ and $\rho_t(\lambda)$,

$$\rho_t(\lambda) = \frac{1}{2\pi} + \int d\mu T(\lambda - \mu) \rho_p(\mu). \quad (\text{D.1})$$

The kernel $T(\lambda - \mu)$, defined in Eqn. (3), in the large c limit expands to

$$T(\lambda - \mu) = \frac{1}{\pi c} \left(1 - \left(\frac{\lambda - \mu}{c} \right)^2 + \left(\frac{\lambda - \mu}{c} \right)^4 \right) + \mathcal{O}(1/c^7). \quad (\text{D.2})$$

Substituting this expansion in Eqn. (D.1) and integrating term by term gives

$$\rho_t(\lambda) = \frac{1}{2\pi} \left(1 + \frac{2n}{c} - \frac{2}{c^3} (\lambda^2 n + e) + \frac{2}{c^5} (\lambda^4 n + 6\lambda^2 e + q_4) + \mathcal{O}(1/c^7) \right), \quad (\text{D.3})$$

where we defined

$$q_i = \int d\mu \mu^i \rho_p(\mu), \quad (\text{D.4})$$

with $n = q_0$ and $e = q_2$. The filling function then follows

$$n(\lambda) = \frac{\rho_p(\lambda)}{\rho_t(\lambda)} = 2\pi \rho_p(\lambda) \left(1 - \frac{2n}{c} + \frac{4n^2}{c^2} + \frac{2(n\lambda^2 + e - 4n^3)}{c^3} + \mathcal{O}(1/c^4) \right). \quad (\text{D.5})$$

Consider now the back-flow function. It obeys the following equation (we reproduce here Eqn. (10))

$$F(\lambda|\mu) = \frac{\theta(\lambda - \mu)}{2\pi} + \int d\lambda' T(\lambda - \lambda') n(\lambda') F(\lambda'|\mu), \quad (\text{D.6})$$

and in the large c expansion can be solved iteratively. This procedure yields

$$F(\lambda|\mu) = \frac{\lambda - \mu}{\pi c} - \frac{(\lambda - \mu)^3}{3\pi c^3} - \frac{\mu}{(\pi c)^2} \int d\lambda' n(\lambda') \left(1 + \frac{1}{\pi c} \int d\lambda' n(\lambda') \right) + \mathcal{O}(1/c^4). \quad (\text{D.7})$$

The integrals over the filling function can be evaluated. We will need only the first two leading corrections

$$\int d\lambda n(\lambda) = 2\pi n \left(1 - \frac{2n}{c} + \frac{4n^2}{c^2} \right) + \mathcal{O}(1/c^3). \quad (\text{D.8})$$

With this expression the back-flow function simplifies to

$$F(\lambda|\mu) = \frac{\lambda - \mu}{\pi c} - \frac{2n\mu}{\pi c^2} - \frac{(\lambda - \mu)^3}{3\pi c^3} + \mathcal{O}(1/c^4). \quad (\text{D.9})$$

With the knowledge of the back-flow function we can now compute the Dressed momentum and energy. We recall the formulas (9a) and (9b) from which we obtain

$$k(\lambda) = \lambda \left(1 + \frac{2n}{c} \right) - \frac{2\lambda}{c^3} \left(\frac{\lambda^2 n}{3} + e \right) + \mathcal{O}(1/c^4), \quad (\text{D.10})$$

$$\omega(\lambda) = \lambda^2 \left(1 + \frac{4e}{c^3} \right) + \text{const} + \mathcal{O}(1/c^4). \quad (\text{D.11})$$

The final ingredient of the TBA that we need is the dressing operation. This is defined in Eqn. (12). Expanding again the kernel produces

$$f_{\text{dr}}(\lambda) = f(\lambda) + \frac{1}{\pi c} \int d\mu \left(1 - \left(\frac{\lambda - \mu}{c} \right)^2 + \left(\frac{\lambda - \mu}{c} \right)^4 + \dots \right) n(\mu) f_{\text{dr}}(\mu), \quad (\text{D.12})$$

which is solved by

$$f_{\text{dr}}(\lambda) = f(\lambda) + \frac{1}{\pi c} \frac{\int d\mu n(\mu) f(\mu)}{1 - \frac{1}{\pi c} \int d\mu n(\mu)} - \frac{1}{\pi c^3} \int d\mu (\lambda - \mu)^2 n(\mu) f(\mu) + \mathcal{O}(1/c^5). \quad (\text{D.13})$$

Expressing now the filling function $n(\lambda)$ by $\rho_p(\lambda)$ using Eqn (D.5), we find the final formula

$$f_{\text{dr}}(\lambda) = f(\lambda) + \frac{2}{c} \int d\mu \rho_p(\mu) f(\mu) - \frac{2}{c^3} \int d\mu (\lambda - \mu)^2 \rho_p(\mu) f(\mu) + \mathcal{O}(1/c^4). \quad (\text{D.14})$$

This result implies the following expansions for $k' = (1)_{\text{dr}}$ and $\omega' = (2\lambda)_{\text{dr}}$,

$$k'(\lambda) = 1 + \frac{2n}{c} - \frac{2}{c^3} (\lambda^2 n + e) + \mathcal{O}(1/c^4), \quad (\text{D.15})$$

$$\omega'(\lambda) = 2\lambda \left(1 + \frac{4e}{c^3} + \mathcal{O}(1/c^4) \right), \quad (\text{D.16})$$

which are in agreement with (D.10) and (D.11). The effective velocity then follows

$$v(\lambda) = \frac{\omega'(\lambda)}{k'(\lambda)} = 2\lambda \left(1 - \frac{2n}{c} + \frac{4n^2}{c^2} + \frac{2n\lambda^2 + 6e - 8n^3}{c^3} + \mathcal{O}(1/c^4) \right). \quad (\text{D.17})$$

Here, we summarize the formulas for the deformed Tonks-Girardeau gas which will be important in the computation of the collision integral. For the deformed Tonks-Girardeau gas (neglecting contributions of order $1/c^3$ and higher) we find

$$k(\lambda) = \xi k_{\text{TG}}(\lambda), \quad (\text{D.18})$$

$$\omega(\lambda) = \omega_{\text{TG}}(\lambda) + \text{const}, \quad (\text{D.19})$$

where we defined

$$\xi = 1 + \frac{2n}{c}. \quad (\text{D.20})$$

We also note that the back-flow function of the deformed Tonks-Girardeau gas has a simple form

$$F(\lambda|\mu) = \frac{\lambda - \xi\mu}{\pi c} + \mathcal{O}(1/c^3). \quad (\text{D.21})$$

Collision integral

We compute now the collision integral for $(1, 1)$ processes in the deformed Tonks-Girardeau gas. This means including the corrections of order $1/c_i^2$. However, the computations simplify and their structure is more apparent if we include also contributions of higher orders. Specifically, we will often write ξ_i^{-1} and not expand it. This results in expressions that formally contain contributions of higher orders but are correct only up to and including orders $1/c_i^2$.

We define the rescaled center-of-mass coordinates

$$p_i = \xi_i \lambda_i + \alpha_i / (2\xi_i), \quad h_i = \xi_i \lambda_i - \alpha_i / (2\xi_i). \quad (\text{D.22})$$

The Jacobian of transformation from (p_i, h_i) to (λ_i, α_i) is 1. We express now the kinematic constraint appearing in the collision integral in terms of these new coordinates. To this end, we use (D.18) and (D.19) to find

$$\delta(k_1(p_1, h_1) + k_2(p_2, h_2)) \delta(\omega_1(p_1, h_1) + \omega_2(p_2, h_2)) = \frac{\delta(\alpha_1 + \alpha_2) \delta(\lambda_1 - \lambda_2)}{2|\alpha_1|}. \quad (\text{D.23})$$

We will use this expression in the formula for $Q_0^{(1,1)}[12](\lambda)$ which for the convenience we repeat here

$$\begin{aligned} Q^{(1,1)}[12](\lambda) = (2\pi)^2 \int dp_1 dh_1 dp_2 dh_2 \delta(\lambda - p_1) \tilde{A}^2(k_1(p_1, h_1)) |F(p_1, h_1)|^2 |F(p_2, h_2)|^2 \\ \times \delta(k_1(p_1, h_1) + k_2(p_2, h_2)) \delta(\omega_1(p_1, h_1) + \omega_2(p_2, h_2)) \\ \times (J_1(p_1, h_1) J_2(p_2, h_2) - J_1(h_1, p_1) J_2(h_2, p_2)). \end{aligned} \quad (\text{D.24})$$

Changing the integration variables from (p_i, h_i) to (λ_i, α_i) and performing the integrals over λ_2 and α_2 , with the help of the δ -functions from the kinematic constraints, sets $\lambda_2 = \lambda_1$ and $\alpha_2 = -\alpha_1$. The result is

$$\begin{aligned} Q_0^{(1,1)}[12](\lambda) = (2\pi)^2 \int d\lambda_1 d\alpha_1 \delta(\lambda - p_1) \frac{\tilde{A}^2(\alpha_1)}{2|\alpha_1|} |F(p_1, h_1)|^2 |F(p_2, h_2)|^2 \\ \times (J_1(p_1, h_1) J_2(p_2, h_2) - J_1(h_1, p_1) J_2(h_2, p_2)) + \mathcal{O}(1/c_i^3), \end{aligned} \quad (\text{D.25})$$

with

$$\begin{aligned} p_1 &= \xi_1 \lambda_1 + \alpha_1 / (2\xi_1), & h_1 &= \xi_1 \lambda_1 - \alpha_1 / (2\xi_1), \\ p_2 &= \xi_2 \lambda_1 - \alpha_1 / (2\xi_2), & h_2 &= \xi_2 \lambda_1 + \alpha_1 / (2\xi_2). \end{aligned} \quad (\text{D.26})$$

The analysis simplifies if we evaluate the collision integral at $\xi_1 \lambda$ instead of λ . We resolve then the remaining Dirac δ -function with the result

$$\begin{aligned} Q_0^{(1,1)}[12](\xi_1 \lambda) = (2\pi)^2 \xi_1^{-1} \int d\alpha \frac{\tilde{A}^2(\alpha)}{2|\alpha|} |F(p_1, h_1)|^2 |F(p_2, h_2)|^2 \\ \times (J_1(p_1, h_1) J_2(p_2, h_2) - J_1(h_1, p_1) J_2(h_2, p_2)) + \mathcal{O}(1/c_i^3), \end{aligned} \quad (\text{D.27})$$

where

$$\begin{aligned} p_1 &= \xi_1 \lambda, & h_1 &= \xi_1 \lambda - \alpha / \xi_1, \\ p_2 &= \xi_2 \lambda - \left(\frac{\xi_1}{\xi_2} + \frac{\xi_2}{\xi_1} \right) \frac{\alpha}{2\xi_1}, & h_2 &= \xi_2 \lambda + \left(\frac{\xi_1}{\xi_2} - \frac{\xi_2}{\xi_1} \right) \frac{\alpha}{2\xi_1}. \end{aligned} \quad (\text{D.28})$$

This is the final formula for the collision integral in the deformed Tonks-Girardeau gas valid at any momentum transfer between the tubes.

We can compute now the approximated collision integral valid at small momenta transfer between the tubes by expanding the integrand in small α . The computations are similar to the ones in the Tonks-Girardeau gas. The extra ingredient is the form factor which we first approximate in the small momentum limit by $F(p, h) = k'(\lambda) + \mathcal{O}(\alpha^2)$ and then use $k'(\lambda) = \xi + \mathcal{O}(1/c^3)$. Thus, the only effect of the form-factor is in rescaling of $Q_0^{(1,1)}[12](\lambda)$ by factor $\xi_1^2 \xi_2^2$. The final answer is

$$Q_0^{(1,1)}[12](\xi_1 \lambda) = (2\pi)^2 \xi_1^{-1} \xi_2^2 \left(\prod_{i=1,2} \rho_{p,i}(\xi_i \lambda) \rho_{h,i}(\xi_i \lambda) \right) G_{1,2}(\xi_1 \lambda) \int d\alpha \frac{|\alpha| \tilde{A}^2(\alpha)}{4} + \mathcal{O}(1/c_i^3), \quad (\text{D.29})$$

with

$$G_{1,2}(\xi_1\lambda) = \frac{\rho''_{p,1}(\xi_1\lambda)}{\rho_{p,1}(\xi_1\lambda)} - \frac{\rho''_{h,1}(\xi_1\lambda)}{\rho_{h,1}(\xi_1\lambda)} - \frac{\rho''_{p,2}(\xi_2\lambda)}{\rho_{p,2}(\xi_2\lambda)} + \frac{\rho''_{h,2}(\xi_2\lambda)}{\rho_{h,2}(\xi_2\lambda)} \\ + \left(\frac{\xi_1}{\xi_2} + \frac{\xi_2}{\xi_1}\right) \left(\frac{\rho'_{p,1}(\xi_1\lambda)\rho'_{h,2}(\xi_2\lambda)}{\rho_{p,1}(\xi_1\lambda)\rho_{h,2}(\xi_2\lambda)} - \frac{\rho'_{h,1}(\xi_1\lambda)\rho'_{p,2}(\xi_2\lambda)}{\rho_{h,1}(\xi_1\lambda)\rho_{p,2}(\xi_2\lambda)} \right) \\ + \left(\frac{\xi_1}{\xi_2} - \frac{\xi_2}{\xi_1}\right) \left(\frac{\rho'_{p,1}(\xi_1\lambda)\rho'_{p,2}(\xi_2\lambda)}{\rho_{p,1}(\xi_1\lambda)\rho_{p,2}(\xi_2\lambda)} - \frac{\rho'_{h,1}(\xi_1\lambda)\rho'_{h,2}(\xi_2\lambda)}{\rho_{h,1}(\xi_1\lambda)\rho_{h,2}(\xi_2\lambda)} \right). \quad (\text{D.30})$$

We observe the symmetry $G_{1,2}(\xi_1\lambda) = -G_{2,1}(\xi_2\lambda)$, which implies that

$$\xi_1^3 Q_0^{(1,1)}[12](\xi_1\lambda) = -\xi_2^3 Q_0^{(1,1)}[21](\xi_2\lambda) + \mathcal{O}(1/c_i^3). \quad (\text{D.31})$$

In the deformed Tonks-Girardeau gas to compute the collision integral we need to include the effect of the Dressing. The Dressing of the collision integral is given by the action of the operator, see Eqn. (17),

$$\mathbb{F}_i(\lambda, \mu) = \delta(\lambda - \mu) + \partial_\lambda(n_i(\lambda)F_i(\lambda|\mu)). \quad (\text{D.32})$$

In the deformed Tonks-Girardeau gas it becomes

$$\mathbb{F}_i(\lambda, \mu) = \delta(\lambda - \mu) + \frac{1}{\pi c} \partial_\lambda(\lambda n_i(\lambda)) - \frac{2\mu}{c} \partial_\lambda \rho_p(\lambda) + \mathcal{O}(1/c^3). \quad (\text{D.33})$$

The dressed scattering integral is then

$$Q^{(1,1)}[12](\lambda) = (\mathbb{F} \cdot Q_0)(\lambda) = Q_0^{(1,1)}[12](\lambda) + Q_1^{(1,1)}[12](\lambda) + \mathcal{O}(1/c_i^3), \quad (\text{D.34})$$

with

$$Q_1^{(1,1)}[12](\lambda) = \frac{1}{\pi c} \partial_\lambda(\lambda n_i(\lambda)) \int d\mu Q_0^{(1,1)}[12](\mu) - \partial_\lambda \rho_{p,1}(\lambda) \frac{2}{c_1} \int d\mu \mu Q_0^{(1,1)}[12](\mu). \quad (\text{D.35})$$

The first integral is 0 which reflects the conservation of the total particle number, see Eqn. (42).

Conserved charges

Consider now a candidate for an integral of motion,

$$\mathcal{I} = \int d\lambda (f_1(\lambda)\rho_{p,1}(\lambda) + f_2(\lambda)\rho_{p,2}(\lambda)), \quad (\text{D.36})$$

with $f_i(\lambda)$ an even function, otherwise the integral is identically zero. The time evolution of $\mathcal{I}$ has two contributions. The contribution from the bare collision integral Q_0 is

$$\dot{\mathcal{I}}_{\text{direct}} = \int d\lambda (f_1(\lambda)Q_0[12](\lambda) + f_2(\lambda)Q_0[21](\lambda)). \quad (\text{D.37})$$

We change the integration variable to $\lambda = \xi_1 \bar{\lambda}$ in the contribution from the first tube and to $\lambda = \xi_2 \bar{\lambda}$ in the contribution from the second tube. We obtain

$$\dot{\mathcal{I}}_{\text{direct}} = \int d\bar{\lambda} (\xi_1 f_1(\xi_1 \bar{\lambda})Q_0[12](\xi_1 \bar{\lambda}) + \xi_2 f_2(\xi_2 \bar{\lambda})Q_0[21](\xi_2 \bar{\lambda})). \quad (\text{D.38})$$

Using the symmetry (D.31) of the collision integral we find the following condition for $\dot{J}_0 = 0$,

$$\xi_1^{-2} f_1(\xi_1 \bar{\lambda}) = \xi_2^{-2} f_2(\xi_2 \bar{\lambda}). \quad (\text{D.39})$$

This implies that for arbitrary $n \geq 2$

$$f_i(\lambda) = \xi_i^{-n+2} \lambda^n, \quad (\text{D.40})$$

gives a quantity that does not evolve under the bare collision integral.

The second contribution to $\dot{Q}$ comes from the indirect collision integral $Q_1[12]$,

$$\dot{X}_{\text{indirect}} = \int d\lambda \left(f'_1(\lambda) \rho_{p,1}(\lambda) \frac{2}{c_1} \int d\mu \mu Q_0[12](\mu) + f'_2(\lambda) \rho_{p,2}(\lambda) \frac{2}{c_2} \int d\mu \mu Q_0[21](\mu) \right). \quad (\text{D.41})$$

This contribution vanishes because $\rho_{p,i}(\lambda)$ are even function whereas $f'_i(\lambda)$ is odd.

Stationary state

We consider now the equation for the stationary state. It takes the universal form, valid for any value of the interaction parameters,

$$\frac{\epsilon'_1(\lambda_1)}{\omega'_1(\lambda_1)} - \frac{\epsilon'_2(\lambda_2)}{\omega'_2(\lambda_2)} = 0, \quad v_1(\lambda_1) = v_2(\lambda_2). \quad (\text{D.42})$$

We start by solving $v_1(\lambda_1) = v_2(\lambda_2)$.

We use the large c expansion of the effective velocity given in (D.17). This results in the following equation

$$\xi_1^{-1} \lambda_1 = \xi_2^{-1} \lambda_2, \quad (\text{D.43})$$

We can now solve for $\epsilon'_2(\lambda_2)$. The result is

$$\epsilon'_2(\lambda_2) = \epsilon'_1(\xi_1 \xi_2^{-1} \lambda_2) \frac{\omega'_2(\lambda_2)}{\omega'_1(\xi_1 \xi_2^{-1} \lambda_2)} = \xi_1^{-1} \xi_2 \epsilon'_1(\xi_1 \xi_2^{-1} \lambda_2), \quad (\text{D.44})$$

where in the second step we used that $\omega'(\lambda) = 2\lambda$ in the deformed Tonks-Girardeau gas. The resulting equation can be presented in a more symmetric way as

$$\xi_2^{-1} \epsilon'_2(\xi_2 \lambda) = \xi_1^{-1} \epsilon'_1(\xi_1 \lambda). \quad (\text{D.45})$$

E Stationary states and the possibility of conserved charges beyond the deformed Tonks-Girardeau gas

In this Appendix we construct solutions to the stationary state equation in the $1/c$ expansion that go beyond the deformed Tonks-Girardeau limit by including corrections of order $\mathcal{O}(1/c^3)$. We, similarly, attempt the construction of the conserved charges to order $\mathcal{O}(1/c^3)$. Here, however, we show that it is not possible in our framework to find solutions to this order.

Stationary state

We start with the stationary state equation. We use the large c expansion of the effective velocity given in (D.17). This results in the following equation

$$\xi_1^{-1} \lambda_1 (1 + 3A_1 + B_1 \lambda_1^2) = \xi_2^{-1} \lambda_2 (1 + 3A_2 + B_2 \lambda_2^2), \quad (\text{E.1})$$

where

$$A_i = \frac{2e_i}{c_i^3}, \quad B_i = \frac{2n_i}{c_i^3}. \quad (\text{E.2})$$

Its perturbative solution is

$$\lambda_1 = \frac{\xi_1}{\xi_2} \lambda_2 (1 + 3(A_2 - A_1) + \lambda_2^2(B_2 - B_1)) + \mathcal{O}(1/c_i^4). \quad (\text{E.3})$$

We can now solve for $\epsilon'_2(\lambda_2)$. The result is

$$\epsilon'_2(\lambda_2) = \epsilon'_1(\lambda_1(\lambda_2)) \frac{\omega'_2(\lambda_2)}{\omega'_1(\lambda_1(\lambda_2))}. \quad (\text{E.4})$$

The ratio of the energies, using Eqn. (D.16), becomes

$$\frac{\omega'_2(\lambda_2)}{\omega'_1(\lambda_1(\lambda_2))} = \frac{\xi_2}{\xi_1} (1 + A_1 - A_2 + (B_1 - B_2)\lambda_2^2). \quad (\text{E.5})$$

We find

$$\begin{aligned} \epsilon'_2(\lambda_2) = & \frac{\xi_2}{\xi_1} \epsilon'_1\left(\frac{\xi_1}{\xi_2} \lambda_2\right) + 2\left(\frac{e_1}{c_1^3} - \frac{e_2}{c_2^3} + \left(\frac{n_1}{c_1^3} - \frac{n_2}{c_2^3}\right) \lambda_2^2\right) (\epsilon'_1(\lambda_2) - \lambda_2 \epsilon''_1(\lambda_2)) \\ & - 4\left(\frac{e_1}{c_1^3} - \frac{e_2}{c_2^3}\right) \lambda_2 \epsilon''_1(\lambda_2), \end{aligned} \quad (\text{E.6})$$

where we kept the ratio ξ_1/ξ_2 to simplify the formula. This expression for $\epsilon'_2(\lambda_2)$ has to be complemented with expressions for n_2 and e_2 . Together with a chemical potential of the second tube (which is a free parameter) they form a closed system that has to be solved.

Conserved charges

We turn now our attention to conserved charges. As discussed in the main text, the relevant equation is

$$\frac{(f'_1)_{\text{dr},1}(\lambda_1)}{\omega'_1(\lambda_1)} - \frac{(f'_2)_{\text{dr},2}(\lambda_2)}{\omega'_2(\lambda_2)} = 0, \quad v_1(\lambda_1) = v_2(\lambda_2). \quad (\text{E.7})$$

This equation has the same structure as the stationary state equation. Therefore, we can immediately write down the solution

$$\begin{aligned} (f'_2)_{\text{dr},2}(\lambda_2) = & \frac{\xi_2}{\xi_1} (f'_1)_{\text{dr},1}\left(\frac{\xi_1}{\xi_2} \lambda_2\right) + 2\left(\frac{e_1}{c_1^3} - \frac{e_2}{c_2^3} + \left(\frac{n_1}{c_1^3} - \frac{n_2}{c_2^3}\right) \lambda_2^2\right) (f'_1(\lambda_2) - \lambda_2 f''_1(\lambda_2)) \\ & - 4\left(\frac{e_1}{c_1^3} - \frac{e_2}{c_2^3}\right) \lambda_2 f''_1(\lambda_2) + \mathcal{O}(1/c_i^4). \end{aligned} \quad (\text{E.8})$$

Note that in writing the terms of order $1/c_i^3$ we have neglected the dressing as its effect for those terms is of order $1/c_i^4$. Unlike for the stationary state, this equation has to be fulfilled at every time. However while the state of the system evolves so do the densities $\rho_{p,i}(\lambda)$ and hence the dressings change. Therefore, it is not clear whether we can find time-independent functions $f_i(\lambda)$ such that this equation holds. To verify this as a first step we rewrite it in terms of the bare quantities. To simplify the considerations we assume that $f_1(\lambda)$ is an even function. Then, according to Eqn. (D.14),

$$(f'_1)_{\text{dr},1}(\lambda) = f'_1(\lambda) + \frac{4\langle f'_1 \rangle_1 \lambda}{c_1^3} + \mathcal{O}(1/c_1^5), \quad \langle f \rangle_i = \int d\mu \rho_{p,i}(\mu) \mu f(\mu). \quad (\text{E.9})$$

We note that the inverse operation is simply

$$f'_1(\lambda) = (f'_1)_{\text{dr},1}(\lambda) - \frac{4\langle (f'_1)_{\text{dr},1} \rangle_1 \lambda}{c_1^3} + \mathcal{O}(1/c_1^5). \quad (\text{E.10})$$

We can now express $(f'_2)_{\text{dr},2}$ in terms of bare f'_1 . The result is

$$\begin{aligned} (f'_2)_{\text{dr},2}(\lambda_2) &= \frac{\xi_2}{\xi_1} f'_1\left(\frac{\xi_1}{\xi_2} \lambda_2\right) + \frac{4\lambda_2 \langle f'_1 \rangle_1}{c_1^3} - 4\left(\frac{e_1}{c_1^3} - \frac{e_2}{c_2^3}\right) \lambda_2 f''_1(\lambda_2) \\ &\quad + 2\left(\frac{e_1}{c_1^3} - \frac{e_2}{c_2^3} + \left(\frac{n_1}{c_1^3} - \frac{n_2}{c_2^3}\right) \lambda_2^2\right) (f'_1(\lambda_2) - \lambda_2 f''_1(\lambda_2)). \end{aligned} \quad (\text{E.11})$$

We observe that $(f'_2)_{\text{dr},2}$ is an odd function of λ_2 . Finally we need to undress it to find bare f'_2 . To achieve this we use the inverse operation (E.10)

$$\begin{aligned} f'_2(\lambda_2) &= \frac{\xi_2}{\xi_1} f'_1\left(\frac{\xi_1}{\xi_2} \lambda_2\right) + 4\lambda_2 \left(\frac{\langle f'_1 \rangle_1}{c_1^3} - \frac{\langle f'_1 \rangle_2}{c_2^3}\right) - 4\left(\frac{e_1}{c_1^3} - \frac{e_2}{c_2^3}\right) \lambda_2 f''_1(\lambda_2) \\ &\quad + 2\left(\frac{e_1}{c_1^3} - \frac{e_2}{c_2^3} + \left(\frac{n_1}{c_1^3} - \frac{n_2}{c_2^3}\right) \lambda_2^2\right) (f'_1(\lambda_2) - \lambda_2 f''_1(\lambda_2)). \end{aligned} \quad (\text{E.12})$$

We can check whether this formula works for the total energy, which is always a conserved charge. Choosing $f_1(\lambda) = \lambda^2$ we observe that the second line vanishes. In the same time $\langle f'_1 \rangle_1 = 2e_1$ and $\langle f'_1 \rangle_2 = 2e_2$. This makes the second and the third term of the first line cancel out. We correctly find $f'_2(\lambda_2) = 2\lambda_2$.

For this equation to determine a conserved charge, the right hand side must be independent of time when $\rho_{p,i}$ evolves according to the Boltzmann equation. The densities n_i are conserved and therefore the time dependence enters only at order $1/c_i^3$ through e_i and $\langle f'_1 \rangle_i$. The time dependent part is

$$4\lambda_2 \left(\frac{\langle f'_1 \rangle_1}{c_1^3} - \frac{\langle f'_1 \rangle_2}{c_2^3}\right) + 2\left(\frac{e_1}{c_1^3} - \frac{e_2}{c_2^3}\right) (f'_1(\lambda_2) - 3\lambda_2 f''_1(\lambda_2)). \quad (\text{E.13})$$

It is a function of λ_2 and t . The time dependence enters through the averages $\langle \cdot \rangle_i$ and through the e_i which are not conserved individually (only the total energy $e_1 + e_2$ is conserved). For this combination to vanish there are two options. First, $f_i(\lambda)$ are constant functions - this corresponds to the total density which indeed is conserved. Second, $f'_1(\lambda_2) - 3\lambda_2 f''_1(\lambda_2)$ must be a linear function of λ_2 . This leads us again to the total energy. There are no other solutions and we conclude that beyond order $1/c^2$ it is not possible to find solutions to (E.7).

References

- [1] A. Polkovnikov, K. Sengupta, A. Silva and M. Vengalattore, *Colloquium: Nonequilibrium dynamics of closed interacting quantum systems*, Rev. Mod. Phys. **83**, 863 (2011), doi:[10.1103/RevModPhys.83.863](https://doi.org/10.1103/RevModPhys.83.863).
- [2] C. Gogolin and J. Eisert, *Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems*, Rep. Prog. Phys. **79**, 056001 (2016), doi:[10.1088/0034-4885/79/5/056001](https://doi.org/10.1088/0034-4885/79/5/056001).
- [3] L. D'Alessio, Y. Kafri, A. Polkovnikov and M. Rigol, *From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics*, Adv. Phys. **65**, 239 (2016), doi:[10.1080/00018732.2016.1198134](https://doi.org/10.1080/00018732.2016.1198134).

- [4] J. M. Deutsch, *Quantum statistical mechanics in a closed system*, Phys. Rev. A **43**, 2046 (1991), doi:[10.1103/PhysRevA.43.2046](https://doi.org/10.1103/PhysRevA.43.2046).
- [5] M. Srednicki, *Chaos and quantum thermalization*, Phys. Rev. E **50**, 888 (1994), doi:[10.1103/PhysRevE.50.888](https://doi.org/10.1103/PhysRevE.50.888).
- [6] M. Srednicki, *Thermal fluctuations in quantized chaotic systems*, J. Phys. A: Math. Gen. **29**, L75 (1996), doi:[10.1088/0305-4470/29/4/003](https://doi.org/10.1088/0305-4470/29/4/003).
- [7] M. Srednicki, *The approach to thermal equilibrium in quantized chaotic systems*, J. Phys. A: Math. Gen. **32**, 1163 (1999), doi:[10.1088/0305-4470/32/7/007](https://doi.org/10.1088/0305-4470/32/7/007).
- [8] M. Rigol, V. Dunjko and M. Olshanii, *Thermalization and its mechanism for generic isolated quantum systems*, Nature **452**, 854 (2008), doi:[10.1038/nature06838](https://doi.org/10.1038/nature06838).
- [9] S. Moudgalya, B. A. Bernevig and N. Regnault, *Quantum many-body scars and Hilbert space fragmentation: A review of exact results*, Rep. Prog. Phys. **85**, 086501 (2022), doi:[10.1088/1361-6633/ac73a0](https://doi.org/10.1088/1361-6633/ac73a0).
- [10] M. Serbyn, D. A. Abanin and Z. Papić, *Quantum many-body scars and weak breaking of ergodicity*, Nat. Phys. **17**, 675 (2021), doi:[10.1038/s41567-021-01230-2](https://doi.org/10.1038/s41567-021-01230-2).
- [11] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn and Z. Papić, *Weak ergodicity breaking from quantum many-body scars*, Nat. Phys. **14**, 745 (2018), doi:[10.1038/s41567-018-0137-5](https://doi.org/10.1038/s41567-018-0137-5).
- [12] A. J. A. James, R. M. Konik and N. J. Robinson, *Nonthermal states arising from confinement in one and two dimensions*, Phys. Rev. Lett. **122**, 130603 (2019), doi:[10.1103/PhysRevLett.122.130603](https://doi.org/10.1103/PhysRevLett.122.130603).
- [13] M. Rigol, V. Dunjko, V. Yurovsky and M. Olshanii, *Relaxation in a completely integrable many-body quantum system: An ab initio study of the dynamics of the highly excited states of 1d lattice hard-core bosons*, Phys. Rev. Lett. **98**, 050405 (2007), doi:[10.1103/PhysRevLett.98.050405](https://doi.org/10.1103/PhysRevLett.98.050405).
- [14] E. H. Lieb and W. Liniger, *Exact analysis of an interacting Bose gas. I. The general solution and the ground state*, Phys. Rev. **130**, 1605 (1963), doi:[10.1103/PhysRev.130.1605](https://doi.org/10.1103/PhysRev.130.1605).
- [15] E. H. Lieb, *Exact analysis of an interacting Bose gas. II. The excitation spectrum*, Phys. Rev. **130**, 1616 (1963), doi:[10.1103/PhysRev.130.1616](https://doi.org/10.1103/PhysRev.130.1616).
- [16] H. Bethe, *Zur Theorie der Metalle*, Z. Phys. **71**, 205 (1931), doi:[10.1007/bf01341708](https://doi.org/10.1007/bf01341708).
- [17] E. H. Lieb and F. Y. Wu, *Absence of Mott transition in an exact solution of the short-range, one-band model in one dimension*, Phys. Rev. Lett. **20**, 1445 (1968), doi:[10.1103/PhysRevLett.20.1445](https://doi.org/10.1103/PhysRevLett.20.1445).
- [18] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper and V. E. Korepin, *The one-dimensional Hubbard model*, Cambridge University Press, Cambridge, UK, ISBN 9780521802628 (2005), doi:[10.1017/cbo9780511534843](https://doi.org/10.1017/cbo9780511534843).
- [19] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan and B. L. Lev, *Thermalization near integrability in a dipolar quantum Newton's cradle*, Phys. Rev. X **8**, 021030 (2018), doi:[10.1103/PhysRevX.8.021030](https://doi.org/10.1103/PhysRevX.8.021030).

- [20] W. Kao, K.-Y. Li, K.-Y. Lin, S. Gopalakrishnan and B. L. Lev, *Topological pumping of a 1D dipolar gas into strongly correlated prethermal states*, Science **371**, 296 (2021), doi:[10.1126/science.abb4928](https://doi.org/10.1126/science.abb4928).
- [21] T. Kinoshita, T. Wenger and D. S. Weiss, *A quantum Newton's cradle*, Nature **440**, 900 (2006), doi:[10.1038/nature04693](https://doi.org/10.1038/nature04693).
- [22] R. Coldea, D. A. Tennant, E. M. Wheeler, E. Wawrzynska, D. Prabhakaran, M. Telling, K. Habicht, P. Smeibidl and K. Kiefer, *Quantum criticality in an Ising chain: Experimental evidence for emergent E_8 symmetry*, Science **327**, 177 (2010), doi:[10.1126/science.1180085](https://doi.org/10.1126/science.1180085).
- [23] B. Lake, D. A. Tennant, C. D. Frost and S. E. Nagler, *Quantum criticality and universal scaling of a quantum antiferromagnet*, Nat. Mater. **4**, 329 (2005), doi:[10.1038/nmat1327](https://doi.org/10.1038/nmat1327).
- [24] H. Fujisawa, T. Yokoya, T. Takahashi, S. Miyasaka, M. Kibune and H. Takagi, *Angle-resolved photoemission study of Sr_2CuO_3* , Phys. Rev. B **59**, 7358 (1999), doi:[10.1103/PhysRevB.59.7358](https://doi.org/10.1103/PhysRevB.59.7358).
- [25] R. Neudert, M. Knupfer, M. S. Golden, J. Fink, W. Stephan, K. Penc, N. Motoyama, H. Eisaki and S. Uchida, *Manifestation of spin-charge separation in the dynamic dielectric response of one-dimensional Sr_2CuO_3* , Phys. Rev. Lett. **81**, 657 (1998), doi:[10.1103/PhysRevLett.81.657](https://doi.org/10.1103/PhysRevLett.81.657).
- [26] C. Kim, A. Y. Matsuura, Z.-X. Shen, N. Motoyama, H. Eisaki, S. Uchida, T. Tohyama and S. Maekawa, *Observation of spin-charge separation in one-dimensional $SrCuO_2$* , Phys. Rev. Lett. **77**, 4054 (1996), doi:[10.1103/PhysRevLett.77.4054](https://doi.org/10.1103/PhysRevLett.77.4054).
- [27] Z. Chen et al., *Anomalously strong near-neighbor attraction in doped 1D cuprate chains*, Science **373**, 1235 (2021), doi:[10.1126/science.abf5174](https://doi.org/10.1126/science.abf5174).
- [28] M. Kollar, F. A. Wolf and M. Eckstein, *Generalized Gibbs ensemble prediction of prethermalization plateaus and their relation to nonthermal steady states in integrable systems*, Phys. Rev. B **84**, 054304 (2011), doi:[10.1103/PhysRevB.84.054304](https://doi.org/10.1103/PhysRevB.84.054304).
- [29] J. Berges, S. Borsányi and C. Wetterich, *Prethermalization*, Phys. Rev. Lett. **93**, 142002 (2004), doi:[10.1103/PhysRevLett.93.142002](https://doi.org/10.1103/PhysRevLett.93.142002).
- [30] M. Panfil, S. Gopalakrishnan and R. M. Konik, *Thermalization of interacting quasi-one-dimensional systems*, Phys. Rev. Lett. **130**, 030401 (2023), doi:[10.1103/PhysRevLett.130.030401](https://doi.org/10.1103/PhysRevLett.130.030401).
- [31] J. De Nardis, D. Bernard and B. Doyon, *Diffusion in generalized hydrodynamics and quasi-particle scattering*, SciPost Phys. **6**, 049 (2019), doi:[10.21468/scipostphys.6.4.049](https://doi.org/10.21468/scipostphys.6.4.049).
- [32] J. De Nardis, B. Doyon, M. Medenjak and M. Panfil, *Correlation functions and transport coefficients in generalised hydrodynamics*, J. Stat. Mech.: Theory Exp. 014002 (2022), doi:[10.1088/1742-5468/ac3658](https://doi.org/10.1088/1742-5468/ac3658).
- [33] M. Takahashi, *Thermodynamics of one-dimensional solvable models*, Cambridge University Press, Cambridge, UK, ISBN 9780521551434 (1999), doi:[10.1017/CBO9780511524332](https://doi.org/10.1017/CBO9780511524332).
- [34] M. Gaudin, *The Bethe wavefunction*, Cambridge University Press, Cambridge, UK, ISBN 9781107045859 (2014), doi:[10.1017/CBO9781107053885](https://doi.org/10.1017/CBO9781107053885).

- [35] F. Franchini, *An introduction to integrable techniques for one-dimensional quantum systems*, Springer, Cham, Switzerland, ISBN 9783319484860 (2017), doi:[10.1007/978-3-319-48487-7](https://doi.org/10.1007/978-3-319-48487-7).
- [36] C. N. Yang and C. P. Yang, *Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction*, J. Math. Phys. **10**, 1115 (1969), doi:[10.1063/1.1664947](https://doi.org/10.1063/1.1664947).
- [37] B. Davies and V. E. Korepin, *Higher conservation laws for the quantum non-linear Schroedinger equation*, doi:[10.1016/0378-4371\(90\)90126-D](https://doi.org/10.1016/0378-4371(90)90126-D).
- [38] T. Palmai and R. M. Konik, *Quasilocal charges and the generalized Gibbs ensemble in the Lieb-Liniger model*, Phys. Rev. E **98**, 052126 (2018), doi:[10.1103/PhysRevE.98.052126](https://doi.org/10.1103/PhysRevE.98.052126).
- [39] F. H. L. Essler, G. Mussardo and M. Panfil, *Generalized Gibbs ensembles for quantum field theories*, Phys. Rev. A **91**, 051602 (2015), doi:[10.1103/PhysRevA.91.051602](https://doi.org/10.1103/PhysRevA.91.051602).
- [40] F. H. L. Essler, G. Mussardo and M. Panfil, *On truncated generalized Gibbs ensembles in the Ising field theory*, J. Stat. Mech.: Theory Exp. 013103 (2017), doi:[10.1088/1742-5468/aa53f4](https://doi.org/10.1088/1742-5468/aa53f4).
- [41] J. De Nardis, B. Wouters, M. Brockmann and J.-S. Caux, *Solution for an interaction quench in the Lieb-Liniger Bose gas*, Phys. Rev. A **89**, 033601 (2014), doi:[10.1103/PhysRevA.89.033601](https://doi.org/10.1103/PhysRevA.89.033601).
- [42] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, Cambridge, UK, ISBN 9780521373203 (1993), doi:[10.1017/CBO9780511628832](https://doi.org/10.1017/CBO9780511628832).
- [43] P. Ruggiero, P. Calabrese, B. Doyon and J. Dubail, *Quantum generalized hydrodynamics*, Phys. Rev. Lett. **124**, 140603 (2020), doi:[10.1103/PhysRevLett.124.140603](https://doi.org/10.1103/PhysRevLett.124.140603).
- [44] J. Durnin, M. J. Bhaseen and B. Doyon, *Nonequilibrium dynamics and weakly broken integrability*, Phys. Rev. Lett. **127**, 130601 (2021), doi:[10.1103/PhysRevLett.127.130601](https://doi.org/10.1103/PhysRevLett.127.130601).
- [45] J. De Nardis and M. Panfil, *Density form factors of the 1D Bose gas for finite entropy states*, J. Stat. Mech.: Theory Exp. P02019 (2015), doi:[10.1088/1742-5468/2015/02/P02019](https://doi.org/10.1088/1742-5468/2015/02/P02019).
- [46] J. De Nardis and M. Panfil, *Particle-hole pairs and density-density correlations in the Lieb-Liniger model*, J. Stat. Mech.: Theory Exp. 033102 (2018), doi:[10.1088/1742-5468/aab012](https://doi.org/10.1088/1742-5468/aab012).
- [47] A. Cortés Cubero and M. Panfil, *Thermodynamic bootstrap program for integrable QFT's: Form factors and correlation functions at finite energy density*, J. High Energy Phys. **01**, 104 (2019), doi:[10.1007/JHEP01\(2019\)104](https://doi.org/10.1007/JHEP01(2019)104).
- [48] M. Panfil, *The two particle-hole pairs contribution to the dynamic correlation functions of quantum integrable models*, J. Stat. Mech.: Theory Exp. 013108 (2021), doi:[10.1088/1742-5468/abd30c](https://doi.org/10.1088/1742-5468/abd30c).
- [49] J. Mossel and J.-S. Caux, *Generalized TBA and generalized Gibbs*, J. Phys. A: Math. Theor. **45**, 255001 (2012), doi:[10.1088/1751-8113/45/25/255001](https://doi.org/10.1088/1751-8113/45/25/255001).
- [50] J.-S. Caux and R. M. Konik, *Constructing the generalized Gibbs ensemble after a quantum quench*, Phys. Rev. Lett. **109**, 175301 (2012), doi:[10.1103/PhysRevLett.109.175301](https://doi.org/10.1103/PhysRevLett.109.175301).

- [51] J. Pawłowski, M. Panfil, J. Herbrych and M. Mierzejewski, *Long-living prethermalization in nearly integrable spin ladders*, Phys. Rev. B **109**, L161109 (2024), doi:[10.1103/PhysRevB.109.L161109](https://doi.org/10.1103/PhysRevB.109.L161109).

2.2 Article II: *Navier-Stokes equations for nearly-integrable quantum gases*

- M. Łebek, M. Panfil “*Navier-Stokes equations for nearly-integrable quantum gases*”, *Phys. Rev. Lett.* 134, (*Editors’ Suggestion*) 010405 (2025)

Derivation of the NS equations from the Boltzmann equation is arguably one of the biggest achievements of kinetic theory. The first theory successful in this respect was provided by Sidney Chapman and David Enskog. They developed a formalism that identified a small parameter in the Boltzmann equation called the *Knudsen number* which allows for its perturbative treatment. In the decades after, different formalisms have appeared, such as the Grad’s moment expansion. Arguably the simplest one was that of Resibois, who adapted the methods of the perturbation theory known from the ordinary quantum mechanics. The price to be paid for the simplification is that the method is capable of deriving only the linearized version of the NS equations, which describe the physics close to a uniform, thermal state. For the full, nonlinear case one still has to resort to Chapman-Enskog method, which will be the subject of the Article III.

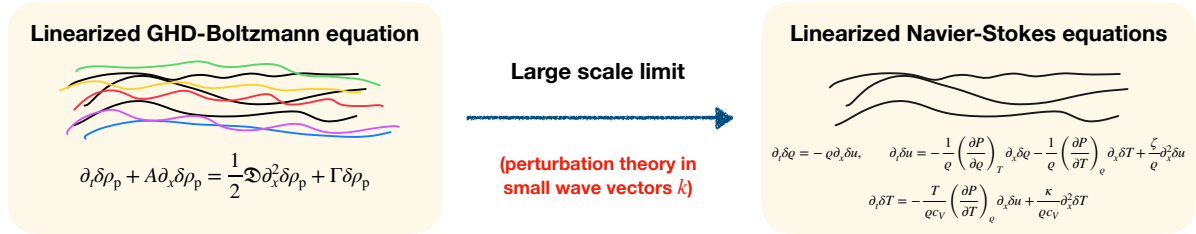


Figure 13: Main result: at large space-time scales the number of degrees of freedom can be effectively reduced to the three hydrodynamic fields and we recover linearized NS equations out of linearized Boltzmann equation.

In this work, we start with the linearized GHD Boltzmann equation. By adapting the methods developed by Resibois to this equation, we recovered the linearized NS equations. The viscosity and thermal conductivity were determined from integral equations involving the collision integral and the thermodynamics of the integrable model as described by the thermodynamic Bethe Ansatz. Our findings exactly recover earlier results upon applying to the case of non-interacting integrable model.

Generally, the thermodynamic quantities that enter the NS description are those of the unperturbed integrable model. This is in complete analogy to the NS equations for a weakly interacting ideal gas, in which all standard methods give the thermodynamics of an ideal gas. However, an important difference appears on the level of transport coefficients. We show that these formally have two contributions: one coming from the collision integral and the other stemming from the diffusion operator $\mathfrak{D}$. Interestingly, in a stark contrast to the weakly interacting gas, for which the viscosity ζ always vanishes, we show that interactions in an integrable theory can give rise to a finite viscosity.

The theory is illustrated with a concrete collision integral for coupled LL gases. The difference is that in this work the same state in both tubes was assumed, leading to the suppression of the prethermalization mechanism described in Article I. The leading order processes in such a case are three-particle collisions that thermalize the system. The viscosities and thermal conductivities were computed numerically for a wide range of parameters.

Candidate’s contribution: I carried out most of the analytical calculations, including nearly the entire supplementary material. I proposed the idea to generalize the approach of Resibois to

recover the NS equations. I also contributed to writing the manuscript and I am the corresponding author.

Navier-Stokes Equations for Nearly Integrable Quantum Gases

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The Navier-Stokes equations are paradigmatic equations describing hydrodynamics of an interacting system with microscopic interactions encoded in transport coefficients. In this work we show how the Navier-Stokes equations arise from the microscopic dynamics of nearly integrable 1D quantum many-body systems. We build upon the recently developed hydrodynamics of integrable models to study the effective Boltzmann equation with collision integral taking into account the non-integrable interactions. We compute the transport coefficients and find that the resulting Navier-Stokes equations have two regimes, which differ in the viscous properties of the fluid. We illustrate the method by computing the transport coefficients for an experimentally relevant case of coupled 1D cold-atomic gases.

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Hydrodynamics is a universal theory describing phenomena of transport and applicable to systems of sizes ranging from nuclear through biological, up to cosmological scales [1–3]. When viewed as an emergent theory, it assumes a huge reduction of the degrees of freedom present at the microscopic scale. The high-energy degrees of freedom are effectively integrated out which leads to dissipation effects present in the Navier-Stokes (NS) equations and captured by the transport coefficients. These are non-universal quantities depending on the details of a microscopic theory. In the present work we show how the NS equations arise from quantum many-body dynamics for systems that are nearly integrable. We also derive expressions for the transport coefficients which treat the integrable part of the interactions in an exact and non-perturbative way.

Recent years have witnessed important developments in the field of nonequilibrium dynamics of isolated quantum many-body systems. The program was especially intriguing in $1 + 1$ dimensions where interacting integrable models showcase behaviors differing from those of generic systems. Nonetheless, the rich methods of quantum integrability allowed for powerful descriptions [4–9]. Importantly, these developments were fueled by experimental progress in creating and manipulating cold-atomic systems [10–17]. It resulted in an understanding that typically quantum integrable systems relax to a generalized Gibbs ensemble (GGE) [18–20] which includes, beside the energy, also the additional conserved charges present in such theories.

The presence of additional conservation laws affects also the hydrodynamics of integrable models. The resulting generalized hydrodynamics (GHD) [21–25] takes a form of

a coupled continuity equations for an extensive number of conserved charge densities. The dispersive processes are also taken into account as a diffusion intrinsic to the integrable dynamics [26–28]. As the hydrodynamic picture assumes that locally the system is in equilibrium, it requires a separation of timescales between small t_{GGE} of the local thermalization to the GGE and large t_{GHD} associated with the flow created by spatial inhomogeneities.

This formalism can be extended beyond purely integrable models by capturing the additional interactions through the Boltzmann collision integral [13,14,29–36]. Building on that result one finds that there is an additional timescale, that we call t_{NS} , associated with the integrability breaking terms. In this work we show that for models with Galilean invariance and with the new interaction term which conserves only the particle number, momentum, and energy this results in the NS equations. We also find that there are two contributions to the transport coefficients. They originate from two types of collisions present in the system. First are integrable collisions captured by the diffusion term of the GHD. Second are the integrability breaking collisions captured by the Boltzmann collision integral. This is summarized in Fig. 1. Our approach to the problem generalizes the classic kinetic theory derivations of hydrodynamics through Chapman-Enskog expansion [37].

Generalized hydrodynamics—The GHD is a theory of hydrodynamics for integrable models, such as integrable spin chains or 1D Bose gases, where an extensive number of local conservation laws are present. Whereas the theory is fairly new [21,22] (with earlier developments in [38]), it has been studied quite extensively (e.g., see the collections of reviews [23]) and was verified numerically [26,39–41] and experimentally [11,14,42]. Recently it was also shown how to extend its applicability, by the logic of *Boltzmann collision integral*, to the cases when the integrability is weakly broken [14,29,30,43–49].

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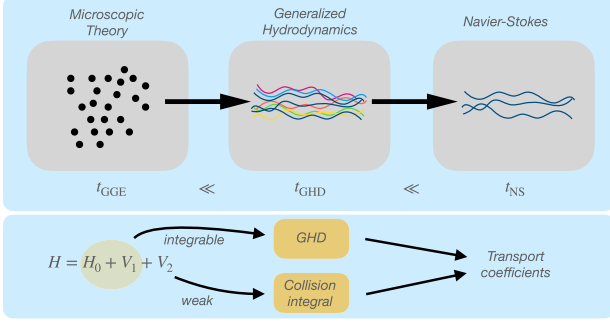


FIG. 1. Top panel: Typically quantum integrable systems locally relax to the GGE with some characteristic timescale t_{GGE} . When the system is globally inhomogeneous further evolution is described by the GHD and occurs at a timescale associated with dynamics of spatial gradients of conserved densities. Finally, in the presence of integrability breaking terms and at larger timescales, the energy dissipates from higher conserved charges to the 3 elementary ones resulting in the NS equations. Bottom panel: Our method separates the full Hamiltonian into integrable Hamiltonian and weak integrability breaking term V_2 . The transport coefficients have then two contributions, one from the interactions present in the integrable theory (integrable collisions) and the second from the integrability breaking term (nonintegrable collisions).

The GHD assumes a local relaxation of the system to the GGE [50,51] which in its microcanonical picture is described by the rapidity distribution $\rho_p \equiv \rho_p(\lambda; x, t)$ slowly varying in space and time. The rapidity distribution evolves through ballistic propagation of particles accompanied by integrable scattering processes leading to diffusive spreading and nonintegrable scatterings captured by the collision integral, as described by the GHD equation

$$\partial_t \rho_p + \partial_x (v_\rho \rho_p) + \mathbb{f} \partial_\lambda \rho_p = \frac{1}{2} \partial_x (\mathfrak{D}_\rho \partial_x \rho_p) + \mathcal{I}[\rho_p], \quad (1)$$

with effective velocity v_ρ , the diffusion kernel $\mathfrak{D}_\rho$, and the Boltzmann collision integral $\mathcal{I}[\rho_p]$. The effective velocity fulfills the integral equation [22,25] $v_\rho(\lambda) = \lambda + 2\pi \int d\lambda' \mathcal{T}(\lambda - \lambda') \rho_p(\lambda') [v_\rho(\lambda') - v_\rho(\lambda)]$, where $\mathcal{T}(\lambda)$ is the model-dependent scattering kernel. The diffusion kernel [26,28] can be found in the Appendix A. We also consider external potential $U(x)$ and associated force term $\mathbb{f} = -\partial_x U$ [52–55]. The index ρ signifies that the effective velocity and diffusion kernel are functionals of ρ_p and are space-time dependent. This renders the GHD equations highly nonlinear. Furthermore, the Boltzmann integral can be evaluated from the matrix elements of the perturbing operator in the spirit of Fermi's golden rule [29,30]. However, for the large scale dynamics the crucial aspects are their collision invariants.

In what follows we quite generally assume that there are 3 collision invariants corresponding to the total particle

number, total momentum, and energy such that

$$\int d\lambda \mathcal{R}_j(\lambda) \mathcal{I}[\rho_p](\lambda) = 0, \quad j = 0, 1, 2, \quad (2)$$

where $\mathcal{R}_j(\lambda)$ are single-particle eigenvalues of the conserved charges of the underlying integrable model,

$$q_j = \langle \mathcal{R}_j \rangle \equiv \int d\lambda \mathcal{R}_j(\lambda) \rho_p(\lambda). \quad (3)$$

Functions $\mathcal{R}_j(\lambda)$ for a Galilean invariant theory are $\mathcal{R}_j(\lambda) = \lambda^j / j!$. The 3 collisions invariants determine the nature of the stationary state of (1) to be a homogeneous thermal (boosted) state and the corresponding $\rho_p(\lambda)$ follows from the standard construction of the thermodynamic Bethe ansatz [56,57], which we recall in the Appendix B.

Macroscopic conservation laws—The 3 collision invariants lead to macroscopic conservation laws. These are conveniently formulated using hydrodynamic fields

$$u(x, t) = \frac{q_1(x, t)}{q_0(x, t)}, \quad e(x, t) = \frac{q_2(x, t)}{q_0(x, t)} - \frac{1}{2} u(x, t)^2, \quad (4)$$

and we additionally rename the particle density field $q(x, t) = q_0(x, t)$. We will refer to the set of hydrodynamic fields as $\{q_\alpha\} = q, u, e$. To this end, we multiply the GHD Eq. (1) by $\mathcal{R}_j(\lambda)$, integrate over λ and use the property (2). Following this procedure for $j = 0, 1, 2$ we obtain

$$\begin{aligned} \partial_t q &= -\partial_x (qu), & \partial_t (qu) &= -\partial_x (qu^2 + \mathcal{P}) + \mathbb{f}q, \\ \partial_t (qe) &= -\partial_x (uqe + \mathcal{J}) - \mathcal{P} \partial_x u, \end{aligned} \quad (5)$$

with $\mathcal{P}(x, t)$ local pressure and $\mathcal{J}(x, t)$ local heat current. The pressure and the heat current have two contributions. One intrinsic to the diffusion term present in the GHD and given by $\mathcal{P}_\mathfrak{D} = \mathcal{R}_1 \mathfrak{D}_\rho \partial_x \rho_p / 2$ and $\mathcal{J}_\mathfrak{D} = \mathcal{R}_2 \mathfrak{D}_\rho \partial_x \rho_p / 2$. The second one resembles the standard expressions from the kinetic theory,

$$\mathcal{P}_v = \langle (\lambda - u)v(\lambda) \rangle, \quad \mathcal{J}_v = \frac{1}{2} \langle (\lambda - u)^2 (v(\lambda) - u) \rangle. \quad (6)$$

The derivation of the conservation laws together with the formulas for $\mathcal{P}$ and $\mathcal{J}$ is elementary and presented in [58]. We only note that in deriving the first conservation law the identity $\langle v \rangle = \langle \mathcal{R}_1 \rangle$, a consequence of the Galilean invariance, is operatorial. This property establishes the velocity field u a relevant hydrodynamic degree of freedom.

The equations (5) are not closed because the pressure and heat current are functions of the full distribution ρ_p rather than just its 3 moments encoded through fields q, u and e . To circumvent this problem, in the phenomenological treatment one assumes the following expressions

$$\mathcal{P} = P - \zeta \partial_x u, \quad \mathcal{J} = -\kappa \partial_x T, \quad (7)$$

in terms of the hydrodynamic fields. Here $P = P(q, e)$ is the hydrostatic pressure, ζ is the bulk viscosity, and κ is the thermal conductivity, respectively. The temperature field and the energy field are related through the thermodynamic identity $T^{-1} = (\partial s / \partial e)_q$, where s is thermodynamic entropy per particle [50,56,57]. This leaves the theory with two phenomenological constants ζ and κ called *transport coefficients*. In this way we obtain the one-dimensional version of NS equations [59–61], for the fields $q(x, t)$, $u(x, t)$, and $e(x, t)$. In the following we confirm this phenomenological picture and provide explicit formulas for the transport coefficients.

Chapman-Enskog method—The standard procedure to justify the phenomenological picture and to turn the hydrodynamic equations into a closed set is the Chapman-Enskog (ChE) method. In the process it also provides a systematic computation of the transport coefficients from the microscopic theory. Originally it was formulated in the context of classical kinetic theory [1,37]. Here we adapt it to the GHD case. The ChE method assumes the time evolution of ρ_p is determined by the hydrodynamic fields, namely, $\rho_p[\lambda, x|q_\alpha(x, t)]$, called *normal solutions*. With this assumption the GHD equations (1) can be rewritten as

$$\sum_\alpha \frac{\partial \rho_p}{\partial q_\alpha} \partial_t q_\alpha + \partial_x (v_\rho \rho_p) = \frac{1}{2} \partial_x (\mathfrak{D}_\rho \partial_x \rho_p) + \mathcal{I}[\rho_p]. \quad (8)$$

As the external potential does not influence the determination of the transport coefficients we set it here to zero [65]. In what follows it will be useful to have a compact notation for the conservation laws. We write $\partial_t q_\alpha = D_\alpha[\{q_\beta\}]$ where $D_\alpha[\{q_\beta\}]$ can be read out from Eq. (5). With this we find the ChE equation

$$\sum_\alpha \frac{\partial \rho_p}{\partial q_\alpha} D_\alpha + \partial_x (v_\rho \rho_p) = \frac{\delta \mathfrak{D}}{2} \partial_x (\mathfrak{D}_\rho \partial_x \rho_p) + \frac{1}{\delta \mathcal{I}} \mathcal{I}[\rho_p], \quad (9)$$

which is an equation for ρ_p given the $\{q_\alpha\}$ and contains only spatial derivatives. The effective velocity and diffusion are now determined fully from the knowledge of $\{q_\alpha\}$. In writing Eq. (9) we rescaled x and t by hydrodynamic scales t_h, l_h introducing two small parameters $\delta \mathfrak{D}$ and $\delta \mathcal{I}$ into the equation. They can be understood as generalizations of Knudsen numbers, i.e., ratios of characteristic diffusion (collision integral) length scales to l_h . In the ChE method we treat the problem order by order in δ 's writing

$$\rho_p = \sum_{m,n=0}^{\infty} \delta \mathfrak{D}^m \delta \mathcal{I}^n \rho_p^{(m,n)}, \quad (10)$$

from which follow similar expansions for $\mathcal{P}$ and $\mathcal{J}$. Truncating the expansions of $\mathcal{P}$, $\mathcal{J}$ leads to the closure

of (5). The resulting hydrodynamics depends on the truncation order. At the leading order Eq. (9) gives $\mathcal{I}[\rho_p^{(0,0)}] = 0$. The solution is the locally varying boosted thermal state and this is the Euler scale hydrodynamics. Specifically, explicit computations show that the hydrodynamic pressure $\mathcal{P}$ reduces then to the hydrostatic pressure $\mathcal{P}^{(0,0)} = P(q, e)$ [9,56,58], and the heat current vanish identically $\mathcal{J}^{(0,0)} = 0$. The hydrodynamic equations at the Euler scale are

$$\begin{aligned} \partial_t q &= -\partial_x (qu), & \partial_t (qu) &= -\partial_x (qu^2 + P), \\ \partial_t (qe) &= -\partial_x (uqe) - P \partial_x u, \end{aligned} \quad (11)$$

and are fully determined from the thermodynamics of the system encoded in the equation of state $P = P(q, e)$. We thus partially confirm the phenomenological picture of Eq. (7) in which $\mathcal{P}$ and $\mathcal{J}$, beside the hydrostatic pressure P , contain derivative terms which do not enter at the Euler scale. This renders the Euler scale dynamic reversible as can be also witnessed by the conservation of entropy. Indeed, simple computations show that the thermodynamic entropy density $\mathbf{s}(x, t)$ obeys the continuity equation $\partial_t \mathbf{s} + \partial_x (v \mathbf{s}) = 0$ [58].

At the first orders in both parameters $\delta \mathfrak{D}, \delta \mathcal{I}$ we recover the full NS hydrodynamics and find explicit expressions for the transport coefficients. As announced earlier there are two contributions. The one from the GHD diffusion amounts to evaluating $\mathcal{P}_\mathfrak{D}$ and $\mathcal{J}_\mathfrak{D}$ on a boosted thermal state. The second contribution is due to the collision integral and is more technically involved. The transport coefficients that we find are $\zeta = \zeta_\mathcal{I} + \zeta_\mathfrak{D}$ and $\kappa = \kappa_\mathcal{I} + \kappa_\mathfrak{D}$ with

$$\begin{aligned} \zeta_\mathcal{I} &= \hbar_1 \mathcal{B} \phi_\zeta, & \zeta_\mathfrak{D} &= \beta \hbar_1 \mathfrak{D} \mathcal{C} \hbar_1 / 2, \\ \kappa_\mathcal{I} &= \hbar_2 \mathcal{B} \phi_\kappa, & \kappa_\mathfrak{D} &= \beta^2 \hbar_2 \mathfrak{D} \mathcal{C} \hbar_2 / 2, \end{aligned} \quad (12)$$

where $\mathcal{B}$ and $\mathcal{C}$ are kernels of the current and charge susceptibility matrices of the integrable model [62,63], explicitly written in the Appendix A. Additionally, $\beta \equiv 1/T$ and functions $\phi_{\kappa, \zeta}(\lambda)$ are found from the following integral equations

$$\Gamma \phi_{\kappa, \zeta} = \chi_{\kappa, \zeta}, \quad (13)$$

where $\Gamma = -(\delta \mathcal{I} / \delta \rho_p) \mathcal{C}$ evaluated in a thermal state, and

$$\begin{aligned} \chi_\zeta &= \sqrt{\beta q} \mathcal{B} h_1 - \sqrt{\beta / \kappa_T} \mathcal{C} (h_0 + \sqrt{c_P / c_V - 1} h_2), \\ \chi_\kappa &= \beta \sqrt{q c_V} \mathcal{B} (h_2 - \sqrt{c_P / c_V - 1} h_0). \end{aligned} \quad (14)$$

Isothermal compressibility κ_T and specific heats c_P, c_V are those of the unperturbed, integrable model. As we show in [58], thermodynamic quantities are represented as explicit functionals of the underlying thermal state with density q

and temperature T . Functions $h_n(\lambda)$ are charges constructed with Gram-Schmidt orthonormalization of $\mathcal{H}_n(\lambda)$ using *hydrodynamic scalar product* $(g|h) \equiv gCh$. The integral equations (13) are supplemented with additional conditions $(\mathcal{H}_i|\phi_{\kappa,\zeta}) = 0$, $i = 0, 1, 2$, which render their solutions unique. Equations (13) naturally generalize ChE integral equations [37] to interacting, nearly integrable models. In the process we confirm that there are no other gradient terms in $\mathcal{P}$ and $\mathcal{J}$ beside the two given in the phenomenological expression (7).

Both contributions to the transport coefficients have a feature known from the standard kinetic theory [37,64]. Namely, they are a simple generalization of coefficients found in a linearized hydrodynamics. Therefore, in what follows we focus on the linearized theory, which also has the advantage of revealing the physical ingredients and defer the further computations within the ChE method to a longer publication [65].

Linearized hydrodynamics—We linearize the GHD equations (1) around the homogeneous thermal state such that $\rho_p(x, t, \lambda) = \rho_{th}(\lambda) + \delta\rho_p(x, t, \lambda)$,

$$\partial_t \delta\rho_p + \mathcal{A}_{th} \partial_x \delta\rho_p = \frac{1}{2} \mathfrak{D}_{th} \partial_x^2 \delta\rho_p + \left(\frac{\delta \mathcal{I}}{\delta \rho_p} \right)_{th} \delta\rho_p, \quad (15)$$

where $\mathcal{A}$ is the kernel of flux Jacobian matrix [25,62]. The linearized GHD equation (15) leads to an eigenequation for perturbations of conserved charges $\delta q_n(x, t) = \int d\lambda h_n(\lambda) \delta\rho_p(\lambda; x, t)$, for more details see the Appendix C. We then consider a time evolution of a single mode $\delta q_n(x, t) \sim \exp[ikx - \Lambda(k)t] \delta q_n(k)$. For small k a standard perturbation theory methods apply and we find two sound modes and a heat mode (thermal mode) with dispersions [we neglect $\mathcal{O}(k^3)$ terms]

$$\Lambda_{\pm}(k) = \pm i v_s k + \frac{1}{2Q} \left[\left(\frac{1}{c_V} - \frac{1}{c_P} \right) \kappa + \zeta \right] k^2, \\ \Lambda_{th}(k) = \frac{\kappa}{Q c_P} k^2, \quad (16)$$

which agree with dispersion relations of linearized NS equations. The sound velocity is equal to thermodynamic formula $v_s = 1/\sqrt{\rho \kappa_S}$, where κ_S is the adiabatic compressibility of the integrable model calculated for the thermal background state ρ_{th} . The transport coefficients ζ and κ are given by (12) computed with the same state ρ_{th} . The details are referred to [58].

The corresponding eigenmodes are related to the hydrodynamic fields q , u and T

$$\delta q = \rho \sqrt{T \kappa_T} \delta q_0, \quad \delta u = \sqrt{T/\rho} \delta q_1, \quad \delta T = T/\sqrt{\rho c_V} \delta q_2, \quad (17)$$

where we took $q(x, t) = \rho + \delta q(x, t)$, $u(x, t) = 0 + \delta u(x, t)$ and $T(x, t) = T + \delta T(x, t)$. Therefore the first 3 orthonormalized charges are equivalent to the

hydrodynamic degrees of freedom. Expressing heat and sound modes in terms of δq , δu , δT we recover precisely the modes of NS equations [58].

Application to the Lieb-Liniger model—We now apply the introduced formalism to a situation relevant to experiments with cold atomic gases. We consider a setup of two 1D Bose gases, described by the integrable Lieb-Liniger (LL) model [66], coupled by a long-range interaction which breaks the integrability. The integrable part of the Hamiltonian is $H_{int} = H_1 + H_2$ with

$$H_i = \int dx \psi_i^\dagger(x) \left(-\frac{1}{2} \partial_x^2 + \frac{c}{2} \psi_i^\dagger(x) \psi_i(x) \right) \psi_i(x), \quad (18)$$

where we work in the units $\hbar = m = 1$ and $\psi_i(x), \psi_i^\dagger(x)$ are canonical bosonic fields. The integrability breaking coupling between the tubes is given by $V_2 = \int dx dy V(x-y) \hat{\rho}_1(x) \hat{\rho}_2(y)$, with $\hat{\rho}_i(x) = \psi_i^\dagger(x) \psi_i(x)$ and $V(x-y)$ interaction potential. In an experimental realizations with cold-atomic gases it originates from dipole-dipole interactions and its strength is tunable [12,16].

Of special interest is the case when the two tubes are initially in the same state. Then they stay the same throughout the dynamics and the system can be described by a single equation (1). The collision integral based on form-factor approach [67–69] and the resulting relaxation for the homogeneous initial state was studied in [30]. For the computation of the transport coefficients we need $\mathcal{I}$ linearized around a thermal state, cf. (13). The derivation is presented in [58]. The result for the matrix elements of Γ in the basis $\{h_j\}$ is

$$\Gamma_{nm} = \frac{2\pi}{\tau} \int d\lambda_1 d\lambda_2 g_{nm}(\lambda_1, \lambda_2) \left(\mathcal{T}^{dr}(\lambda_1, \lambda_2) \right)^2 |v_{12}|, \quad (19)$$

with the integration measure $d\lambda = \rho_p(\lambda)[1 - n(\lambda)]d\lambda$, the filling function $n(\lambda)$, $v_i = v(\lambda_i)$, $v_{12} = v_1 - v_2$, and

$$g_{nm}(\lambda_1, \lambda_2) = \int d\lambda [k'(\lambda)]^2 \frac{\mathcal{S}[h_n^{Dr}]\mathcal{S}[h_m^{Dr}]}{(v - v_1)^2(v - v_2)^2}. \quad (20)$$

Here $\mathcal{T}$ is the differential scattering kernel, for the LL model $\pi \mathcal{T}(\lambda - \lambda') = c/[c^2 + (\lambda - \lambda')^2]$ and dr and Dr are dressing operations [70], $k(\lambda)$ is the Dressed momentum while $\mathcal{S}[\varphi] \equiv \mathcal{S}[\varphi](\lambda, \lambda_1, \lambda_2)$ is

$$\mathcal{S}[\varphi] = \frac{\varphi'}{k'(v_1 - v_2)} + \frac{\varphi'_1}{k'_1(v_2 - v)} + \frac{\varphi'_2}{k'_2(v - v_1)}. \quad (21)$$

The timescale τ is related to the strength of inter-tube coupling $\tau^{-1} = \tilde{V}^2 m / (4\hbar^3)$, where $\tilde{V}^2 = \int dk \tilde{V}^2(k) k^2$ and $\tilde{V}(k)$ is Fourier transform of $V(x)$.

The transport coefficients are then determined from (12) and the numerical solution is shown in Fig. 2. We focus on

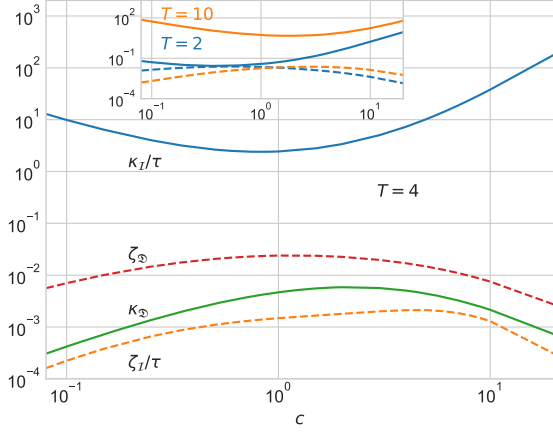


FIG. 2. Transport coefficients in the log-log scale for the coupled LL models as a function of intratube interaction coupling c . In the main plot we show separate contributions to the thermal conductivity and viscosity from integrable and nonintegrable collisions. In the inset we show the total thermal conductivity and viscosity for $\tau = 10^{-2}$. The viscosity is nonzero across different values of temperature and couplings.

the strong coupling regime of the LL model in the nomenclature of [71] where both interaction and thermal fluctuations are important. We find that there are two competing behaviors. When intratube interactions dominate the resulting fluid has relatively small viscosity and large thermal conductivity. Instead, when intertube interactions dominate then both viscosity and thermal conductivity are of the same order. In the limit of noninteracting integrable models, $c \rightarrow 0$ or $c^{-1} \rightarrow 0$, the bulk viscosity vanishes and thermal conductivity diverges [58] as expected from the classical kinetic theory of weakly interacting models [59].

Outlook—In this work we have shown that the NS equations universally arise for 1D quantum systems with weakly broken integrability and with Galilean invariance, assuming the GHD equation supplemented with the collision integral. We have provided explicit equations (13) for the transport coefficients linking the effective hydrodynamic description with the exact microscopic thermodynamics of the integrable model. Unlike the standard approach in the kinetic theory, we treat an important part of the interactions nonperturbatively. One consequence of this is a nonzero viscosity, which in the derivation of the NS equations based on the free theory, is identically zero [37,60]. The resulting fluid has two regimes. One dominated by the integrability breaking collisions with relatively small viscosity, and the second, dominated by the integrable collisions with thermal conductivity and viscosity of the same order.

It is well known that transport in one dimension with conserved momentum becomes anomalous upon introducing noise to the system [72–75]. The linearized framework with $U(x) = 0$ presented here together with initial results

for GHD with noise [36] open possibilities for further explorations in this direction.

Recently, the NS equations were also found for the low temperature dynamics of the GHD without an explicit integrability breaking terms [76]. The effective projection to the three lowest conserved densities is attributed there to the low temperatures and holds at intermediate timescales. The transport coefficients are then of the form $\zeta_{\mathfrak{D}}$ and $\kappa_{\mathfrak{D}}$ taken in the $T \rightarrow 0$ limit, namely, they originate from the integrable collisions.

Our results apply also to classical integrable Galilean-invariant models, whose dynamics is described by (1). In such cases collision integrals may follow from BBGKY equations [77] or be constructed phenomenologically, as in the relaxation time approximation [31]. The methods introduced here can be also directly generalized to models without the Galilean invariance, e.g., spin chains.

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- [1] J.R. Dorfman, H. van Beijeren, and T.R. Kirkpatrick, *Contemporary Kinetic Theory of Matter* (Cambridge University Press, Cambridge, England, 2021), [10.1017/9781139025942](https://doi.org/10.1017/9781139025942).
- [2] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics* (Oxford University Press, New York, 2013), [10.1093/acprof:oso/9780198528906.001.0001](https://doi.org/10.1093/acprof:oso/9780198528906.001.0001).
- [3] S. Vogel, *Life in Moving Fluids: The Physical Biology of Flow*, Princeton Paperbacks (Princeton University Press, Princeton, NJ, 1996).
- [4] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, *Adv. Phys.* **65**, 239 (2016).
- [5] F. H. L. Essler and M. Fagotti, *J. Stat. Mech.* (2016) 064002.
- [6] J.-S. Caux, *J. Stat. Mech.* (2016) 064006.
- [7] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, *J. Phys. B* **51**, 112001 (2018).
- [8] M. Borsi, B. Pozsgay, and L. Pristiyák, *J. Stat. Mech.* (2021) 094001.
- [9] I. Bouchoule and J. Dubail, *J. Stat. Mech.* (2022) 014003.
- [10] T. Kinoshita, T. Wenger, and D. S. Weiss, *Nature (London)* **440**, 900 (2006).
- [11] M. Schemmer, I. Bouchoule, B. Doyon, and J. Dubail, *Phys. Rev. Lett.* **122**, 090601 (2019).
- [12] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B.L. Lev, *Phys. Rev. X* **8**, 021030 (2018).
- [13] F. Cataldini, F. Møller, M. Tajik, J. Sabino, S.-C. Ji, I. Mazets, T. Schweigler, B. Rauer, and J. Schmiedmayer, *Phys. Rev. X* **12**, 041032 (2022).
- [14] F. Møller, C. Li, I. Mazets, H.-P. Stimming, T. Zhou, Z. Zhu, X. Chen, and J. Schmiedmayer, *Phys. Rev. Lett.* **126**, 090602 (2021).

- [15] C. Li, T. Zhou, I. Mazets, H.-P. Stimming, F. S. Møller, Z. Zhu, Y. Zhai, W. Xiong, X. Zhou, X. Chen, and J. Schmiedmayer, *SciPost Phys.* **9**, 058 (2020).
- [16] K.-Y. Li, Y. Zhang, K. Yang, K.-Y. Lin, S. Gopalakrishnan, M. Rigol, and B. L. Lev, *Phys. Rev. A* **107**, L061302 (2023).
- [17] J. M. Wilson, N. Malvania, Y. Le, Y. Zhang, M. Rigol, and D. S. Weiss, *Science* **367**, 1461 (2020).
- [18] M. Rigol, V. Dunjko, and M. Olshanii, *Nature (London)* **452**, 854 (2008).
- [19] E. Ilievski, J. De Nardis, B. Wouters, J.-S. Caux, F. H. L. Essler, and T. Prosen, *Phys. Rev. Lett.* **115**, 157201 (2015).
- [20] T. Langen, S. Erne, R. Geiger, B. Rauer, T. Schweigler, M. Kuhnert, W. Rohringer, I. E. Mazets, T. Gasenzer, and J. Schmiedmayer, *Science* **348**, 207 (2015).
- [21] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, *Phys. Rev. Lett.* **117**, 207201 (2016).
- [22] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, *Phys. Rev. X* **6**, 041065 (2016).
- [23] A. Bastianello, B. Bertini, B. Doyon, and R. Vasseur, *J. Stat. Mech.* (2022) 014001.
- [24] B. Doyon, S. Gopalakrishnan, F. Møller, J. Schmiedmayer, and R. Vasseur, *arXiv:2311.03438*.
- [25] B. Doyon, *SciPost Phys. Lect. Notes* **18** (2020).
- [26] J. De Nardis, D. Bernard, and B. Doyon, *Phys. Rev. Lett.* **121**, 160603 (2018).
- [27] S. Gopalakrishnan, D. A. Huse, V. Khemani, and R. Vasseur, *Phys. Rev. B* **98**, 220303(R) (2018).
- [28] J. D. Nardis, D. Bernard, and B. Doyon, *SciPost Phys.* **6**, 049 (2019).
- [29] J. Durnin, M. J. Bhaseen, and B. Doyon, *Phys. Rev. Lett.* **127**, 130601 (2021).
- [30] M. Panfil, S. Gopalakrishnan, and R. M. Konik, *Phys. Rev. Lett.* **130**, 030401 (2023).
- [31] J. Lopez-Piqueres, B. Ware, S. Gopalakrishnan, and R. Vasseur, *Phys. Rev. B* **103**, L060302 (2021).
- [32] J. Lopez-Piqueres, S. Gopalakrishnan, and R. Vasseur, *J. Phys. A* **55**, 234005 (2022).
- [33] J. Lopez-Piqueres and R. Vasseur, *Phys. Rev. Lett.* **130**, 247101 (2023).
- [34] J.-S. Caux, B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, *SciPost Phys.* **6**, 070 (2019).
- [35] I. Bouchoule, B. Doyon, and J. Dubail, *SciPost Phys.* **9**, 044 (2020).
- [36] A. Bastianello, J. D. Nardis, and A. D. Luca, *Phys. Rev. B* **102**, 161110(R) (2020).
- [37] P. Résibois and M. De Leener, *Classical Kinetic Theory of Fluids* (John Wiley & Sons, New York, 1977).
- [38] L. Bonnes, F. H. L. Essler, and A. M. Läuchli, *Phys. Rev. Lett.* **113**, 187203 (2014).
- [39] B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, *Phys. Rev. Lett.* **119**, 195301 (2017).
- [40] B. Doyon and H. Spohn, *J. Stat. Mech.* (2017) 073210.
- [41] L. Biagetti, G. Cecile, and J. D. Nardis, *Phys. Rev. Res.* **6**, 023083 (2024).
- [42] N. Malvania, Y. Zhang, Y. Le, J. Dubail, M. Rigol, and D. S. Weiss, *Science* **373**, 1129 (2021).
- [43] A. J. Friedman, S. Gopalakrishnan, and R. Vasseur, *Phys. Rev. B* **101**, 180302(R) (2020).
- [44] A. Bastianello, A. D. Luca, and R. Vasseur, *J. Stat. Mech.* (2021) 114003.
- [45] J. Pawłowski, M. Panfil, J. Herbrych, and M. Mierzejewski, *Phys. Rev. B* **109**, L161109 (2024).
- [46] M. Łebek, M. Panfil, and R. M. Konik, *SciPost Phys.* **17**, 007 (2024).
- [47] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, *Phys. Rev. Lett.* **115**, 180601 (2015).
- [48] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, *Phys. Rev. B* **94**, 245117 (2016).
- [49] G. del Vecchio del Vecchio, A. D. Luca, and A. Bastianello, *SciPost Phys.* **12**, 060 (2022).
- [50] J. Mossel and J.-S. Caux, *J. Phys. A* **45**, 255001 (2012).
- [51] J.-S. Caux and R. M. Konik, *Phys. Rev. Lett.* **109**, 175301 (2012).
- [52] B. Doyon and T. Yoshimura, *SciPost Phys.* **2**, 014 (2017).
- [53] A. Bastianello, A. D. Luca, B. Doyon, and J. D. Nardis, *Phys. Rev. Lett.* **125**, 240604 (2020).
- [54] J. Durnin, A. D. Luca, J. D. Nardis, and B. Doyon, *J. Phys. A* **54**, 494001 (2021).
- [55] X. Cao, V. B. Bulchandani, and J. E. Moore, *Phys. Rev. Lett.* **120**, 164101 (2018).
- [56] C. N. Yang and C. P. Yang, *J. Math. Phys. (N.Y.)* **10**, 1115 (1969).
- [57] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, England, 1999), 10.1017/CBO9780511524332.
- [58] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevLett.134.010405> for a detailed exposition of computations behind the presented results.
- [59] R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (John Wiley & Sons, New York, 1975).
- [60] H. Spohn, *Large Scale Dynamics of Interacting Particles*, Theoretical and Mathematical Physics (Springer, Berlin, Heidelberg, 2012).
- [61] S. Chakraborti, S. Ganapa, P. L. Krapivsky, and A. Dhar, *Phys. Rev. Lett.* **126**, 244503 (2021).
- [62] B. Doyon and H. Spohn, *SciPost Phys.* **3**, 039 (2017).
- [63] J. De Nardis, B. Doyon, M. Medenjak, and M. Panfil, *J. Stat. Mech.* (2022) 014002.
- [64] P. Résibois, *J. Stat. Phys.* **2**, 21 (1970).
- [65] M. Łebek and M. Panfil, *arXiv:2410.23209*.
- [66] E. H. Lieb and W. Liniger, *Phys. Rev.* **130**, 1605 (1963).
- [67] J. D. Nardis and M. Panfil, *J. Stat. Mech.* (2015) P02019.
- [68] J. D. Nardis and M. Panfil, *J. Stat. Mech.* (2018) 033102.
- [69] M. Panfil, *J. Stat. Mech.* (2021) 013108.
- [70] P. Ruggiero, P. Calabrese, B. Doyon, and J. Dubail, *Phys. Rev. Lett.* **124**, 140603 (2020).
- [71] K. V. Kheruntsyan, D. M. Gangardt, P. D. Drummond, and G. V. Shlyapnikov, *Phys. Rev. Lett.* **91**, 040403 (2003).
- [72] H. van Beijeren, *Phys. Rev. Lett.* **108**, 180601 (2012).
- [73] O. Narayan and S. Ramaswamy, *Phys. Rev. Lett.* **89**, 200601 (2002).
- [74] D. Forster, D. R. Nelson, and M. J. Stephen, *Phys. Rev. A* **16**, 732 (1977).
- [75] H. Spohn, *J. Stat. Phys.* **154**, 1191 (2014).
- [76] A. Urichuk, S. Scopa, and J. De Nardis, *Phys. Rev. Lett.* **132**, 243402 (2024).
- [77] L. Biagetti, M. Łebek, M. Panfil, and J. De Nardis, *arXiv:2408.00593*.

End Matter

Appendix A: Integral kernels—For a linear operator $\mathcal{O}$ we introduce the following notation for its action on arbitrary functions ψ, φ

$$(\mathcal{O}\psi)(\lambda) = \int d\mu \mathcal{O}(\lambda, \mu) \psi(\mu), \quad (\text{A1})$$

$$\psi \mathcal{O} \varphi = \int d\lambda d\mu \mathcal{O}(\lambda, \mu) \psi(\lambda) \varphi(\mu), \quad (\text{A2})$$

and use the standard notation for kernel multiplication

$$(\mathcal{O}_1 \mathcal{O}_2)(\lambda, \mu) = \int d\nu \mathcal{O}_1(\lambda, \nu) \mathcal{O}_2(\nu, \mu). \quad (\text{A3})$$

The kernels for operators $\mathcal{A}, \mathcal{B}, \mathcal{C}$ appearing in the main text are as follows

$$\mathcal{A} = (1 - nT)^{-1} v(1 - nT), \quad (\text{A4})$$

$$\mathcal{B} = (1 - nT)^{-1} \rho_p f v(1 - Tn)^{-1}, \quad (\text{A5})$$

$$\mathcal{C} = (1 - nT)^{-1} \rho_p f(1 - Tn)^{-1}. \quad (\text{A6})$$

where f is the statistical factor [25] (for fermionic statistics $f = 1 - n$). In the definitions above ρ_p, n, f, v should be viewed as diagonal kernels and we have $(T\psi)(\lambda) = \int d\lambda' T(\lambda - \lambda') \psi(\lambda')$. Lastly, the diffusion kernel is

$$\mathfrak{D} = (1 - nT)^{-1} \rho_{\text{tot}}^{-1} \tilde{\mathfrak{D}} \rho_{\text{tot}}^{-1} (1 - nT), \quad (\text{A7})$$

with $\tilde{\mathfrak{D}}(\theta, \alpha) = \delta(\theta - \alpha) w(\theta) - W(\theta, \alpha)$ and

$$W(\theta, \alpha) = \rho_p(\theta) f(\theta) \left(T^{\text{dr}}(\theta, \alpha) \right)^2 |v(\theta) - v(\alpha)|, \quad (\text{A8})$$

where $w(\theta) = \int d\alpha W(\alpha, \theta)$ and $T^{\text{dr}}(\lambda, \lambda')$ fulfills

$$T^{\text{dr}}(\lambda, \lambda') = T(\lambda - \lambda') + \int d\lambda'' T(\lambda - \lambda'') n(\lambda'') T^{\text{dr}}(\lambda'', \lambda'). \quad (\text{A9})$$

Appendix B: Boosted thermal states in the thermodynamic Bethe ansatz—We discuss here the basics of the thermodynamic Bethe ansatz (TBA) [50,51] for a model with fermionic statistics (for generic case see [58]) and give a precise meaning of boosted thermal states within this framework. The result of the TBA is that averages (of local operators or products thereof) in the grand canonical ensemble or generalized Gibbs ensemble are thermodynamically equal to expectation values computed in an eigenstate with

rapidity distribution ρ_p . This rapidity distribution follows from the solving the TBA equations that we now summarize. We start with bare pseudoenergy ϵ_0 which takes the following form:

$$\epsilon_0(\lambda, x, t) = \sum_n \mathcal{E}_n(\lambda) \beta_n. \quad (\text{B1})$$

The pseudoenergy ϵ and total density of states ρ_{tot} is then determined from the generalized TBA equations

$$\epsilon(\lambda) = \epsilon_0(\lambda) - \int d\mu T(\lambda - \mu) \log \left(1 + e^{-\epsilon(\mu)} \right), \quad (\text{B2})$$

$$\rho_{\text{tot}}(\lambda) = \frac{1}{2\pi} + \int d\mu T(\lambda - \mu) n(\mu) \rho_{\text{tot}}(\mu). \quad (\text{B3})$$

Solving the first equation yields $\epsilon(\lambda)$, which enters the filling function

$$n(\lambda) = \frac{1}{1 + e^{\epsilon(\lambda)}}, \quad (\text{B4})$$

and in turn allows us to solve the second equation. The total density ρ_{tot} determines then the particles' density according to $\rho_p(\lambda) = n(\lambda) \rho_{\text{tot}}(\lambda)$.

For thermal state $\bar{\rho}_p(\lambda)$ with chemical potential μ and temperature T one has $\bar{\beta}_0 = -\mu/T$, $\bar{\beta}_1 = 0$, $\bar{\beta}_2 = 1/T$, $\bar{\beta}_{n>2} = 0$. The Galilean invariance implies that after boosting by velocity u the state is a TBA state with $\rho_p(\lambda) = \bar{\rho}_p(\lambda - u)$ and with Lagrange multipliers $\beta_0 = \bar{\beta}_0 + \frac{1}{2} u^2 \bar{\beta}_2$, $\beta_1 = -u \bar{\beta}_2$, $\beta_2 = \bar{\beta}_2$, $\beta_{n>2} = 0$.

Appendix C: Further information on the linearized equations—We elaborate here on the computations involving the linearized Boltzmann equation. As indicated in the main text the structure of the resulting equations is simpler if we work in the orthonormal basis of the conserved charges with respect to the hydrodynamic scalar product $(g|h)$. Standard computations using the methods of thermodynamic Bethe ansatz yield, see [58],

$$\partial_t \delta q_n + A_{nm} \partial_x \delta q_m - D_{nm} \partial_x^2 \delta q_m = -\Gamma_{nm} \delta q_m, \quad (\text{C1})$$

where we assume a summation over the repeated indices. Moreover, A_{nm} , D_{nm} , and Γ_{nm} are now interpreted as matrix elements of operators associated with effective velocity, diffusion, and collision integral evaluated in the orthonormal basis. We observe that $D_{0n} = D_{n0} = 0$, which is a consequence of Markovianity of the GHD diffusion operator [26,58], while $\Gamma_{jn} = 0$ for $j, n = 0, 1, 2$ which reflects the collision invariants (2). The linearized GHD equations (C1) can be now solved perturbatively in

gradients of δq_n . This is the easiest to formulate by going to the Fourier space and considering a time evolution of a single mode $\delta q_n(x, t) \sim \exp[ikx - \Lambda(k)t]\delta q_n(k)$. This leads to the eigenvalue problem

$$(\Gamma_{nm} + ikA_{nm} + k^2D_{nm})\delta q_m(k) = \Lambda(k)\delta q_n(k), \quad (\text{C2})$$

with $\Lambda(k)$ the dispersion relations for different modes. Among them there are 3 gapless modes related to the 3 conservation laws respected by the collision integral. The problem can be treated analytically in

small k limit using standard perturbation theory around the spectrum of Γ operator. The three zero modes of Γ yield gapless dispersion relations (16) of heat and sound modes. At the first order in k the only nontrivial coefficient is sound velocity $v_s = (A_{1,0}^2 + A_{2,1}^2)^{1/2}$, which agrees with the standard thermodynamic formula $v_s = 1/\sqrt{\rho\chi_s}$, see [58]. At the second order in k after lengthy calculation we recover subleading corrections as in (16) with transport coefficients which are determined from (12). These derivations are presented in [58].

Supplemental Material: Navier-Stokes equations for nearly integrable quantum gases

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In the Supplemental Material we provide computations supporting the results presented in the main text.

In Sec. I we introduce the basic ingredients of the GHD equation: effective velocity, diffusion kernel and susceptibility matrices along with properties of these objects relevant in the present context. In Sec. II we derive hydrodynamic conservation laws for the three conserved charges and introduce dynamic pressure and heat current. Next, in Sec. III we analyze the basic properties of thermal boosted state of integrable models. We show that for such states the heat current vanishes and dynamic pressure becomes the hydrostatic pressure. In Sec. IV we derive expressions for thermodynamic quantities such as sound velocity and specific heats in terms of hydrodynamic matrices.

The following sections concern the linearized hydrodynamics and computation of transport coefficients. First, in Sec. V we present a derivation of a linearized transport equation for local charge densities. In Sec. VI we linearize Navier-Stokes equations and find the dispersion relations of heat and sound modes. We use these results in Sec. VII where we treat the eigenproblem perturbatively and identify expressions for bulk viscosity and thermal conductivity. Additionally, we give numerical results for the transport coefficients. Later in Sec. VIII we show that the results found in the perturbation theory can be formulated as generalized Chapman-Enskog integral equations, which are stated in the main text.

Finally, in Sec. IX we analyze the collision integral for two Lieb-Liniger gases coupled by the density-density interactions.

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I. EFFECTIVE VELOCITY, DIFFUSION OPERATOR AND SUSCEPTIBILITY MATRICES

We consider a Galilean-invariant model with energy $E(\lambda) = \frac{1}{2}\lambda^2$ and momentum $p(\lambda) = \lambda$. We set the mass to unity. As in the main text we introduce functions $\mathcal{H}_j(\lambda) = \lambda^j/j!$ related to ultra-local charges. The effective velocity [1] is given by $v(\lambda) = (E')^{\text{dr}}(\lambda)/(p')^{\text{dr}}(\lambda)$ where the dressing operation of an arbitrary function $g(\lambda)$ is implemented by the following integral equation

$$g^{\text{dr}}(\lambda) = g(\lambda) + \int d\lambda' \mathcal{T}(\lambda - \lambda') n(\lambda') g^{\text{dr}}(\lambda'), \quad (\text{I.1})$$

and $\mathcal{T}(\lambda - \lambda')$ is a symmetric scattering kernel, which depends on the model. For instance, in the case of Lieb-Liniger model [2] it reads $\mathcal{T}(\lambda) = \frac{c}{\pi} \frac{1}{c^2 + \lambda^2}$, where c is the coupling constant. We also defined the occupation function $n(\lambda) = \rho_p(\lambda)/\rho_{\text{tot}}(\lambda)$, where total density of states $\rho_{\text{tot}}(\lambda)$ is

$$\rho_{\text{tot}}(\lambda) = \frac{1}{2\pi} + \int d\lambda' \mathcal{T}(\lambda - \lambda') \rho_p(\lambda'). \quad (\text{I.2})$$

It can be shown that the effective velocity fulfils the following integral equation [1]

$$v(\lambda) = \mathcal{H}_1(\lambda) + 2\pi \int d\lambda' \mathcal{T}(\lambda - \lambda') \rho_p(\lambda') (v(\lambda') - v(\lambda)). \quad (\text{I.3})$$

With this, it is straightforward to check the identity used in the main text

$$\int d\lambda v(\lambda) \rho_p(\lambda) = \int d\lambda \mathcal{H}_1(\lambda) \rho_p(\lambda). \quad (\text{I.4})$$

Indeed, we have

$$\int d\lambda v(\lambda) \rho_p(\lambda) = \int d\lambda \mathcal{H}_1(\lambda) \rho_p(\lambda) + 2\pi \int d\lambda d\lambda' \mathcal{T}(\lambda - \lambda') \rho_p(\lambda) \rho_p(\lambda') (v(\lambda') - v(\lambda)) = \int d\lambda \mathcal{H}_1(\lambda) \rho_p(\lambda), \quad (\text{I.5})$$

where the integral $\int d\lambda d\lambda' (\dots)$ vanished due to the symmetry of the kernel $\mathcal{T}(-\lambda) = \mathcal{T}(\lambda)$.

In addition to the Euler-scale term involving the effective velocity, there is also the diffusion term [3] $\frac{1}{2}\partial_x (\mathfrak{D}\partial_x \rho_p)$ with $(\mathfrak{D}\partial_x \rho_p)(\theta) = \int d\alpha \mathfrak{D}(\theta, \alpha) \partial_x \rho_p(\alpha)$. The diffusion kernel is

$$\mathfrak{D} = (1 - n\mathcal{T})^{-1} \rho_{\text{tot}}^{-1} \tilde{\mathfrak{D}} \rho_{\text{tot}}^{-1} (1 - n\mathcal{T}), \quad \tilde{\mathfrak{D}}(\theta, \alpha) = \delta(\theta - \alpha) w(\theta) - W(\theta, \alpha), \quad (\text{I.6})$$

and

$$W(\theta, \alpha) = \rho_p(\theta) f(\theta) (\mathcal{T}^{\text{dr}}(\theta, \alpha))^2 |v(\theta) - v(\alpha)|, \quad w(\theta) = \int d\alpha W(\alpha, \theta), \quad (\text{I.7})$$

where $f(\theta)$ denotes statistical factor [1] defined in (IV.5), which depends on the particles' statistics. The dressed scattering kernel fulfills

$$\mathcal{T}^{\text{dr}}(\lambda, \lambda') = \mathcal{T}(\lambda - \lambda') + \int d\lambda'' \mathcal{T}(\lambda - \lambda'') n(\lambda'') \mathcal{T}^{\text{dr}}(\lambda'', \lambda'). \quad (\text{I.8})$$

Note also that our definition of $\tilde{\mathfrak{D}}$ differs from the standard one [3] by a factor of $\rho_{\text{tot}}^2(\theta)$.

From the Galilean invariance follows that diffusion kernel has left eigenvector with zero eigenvalue [3],

$$\int d\theta \mathfrak{D}(\theta, \alpha) = 0. \quad (\text{I.9})$$

This implies that the local density of particles is conserved by the diffusion term. Moreover from Markovianity of the diffusion kernel, there is a corresponding right eigenvector with zero eigenvalue, namely

$$\int d\alpha \mathfrak{D}(\theta, \alpha) (\mathcal{C} \mathcal{H}_0)(\alpha) = 0, \quad (\text{I.10})$$

where $\mathcal{C}$ is the charge susceptibility matrix defined as [1, 4]

$$\mathcal{C} = (1 - n\mathcal{T})^{-1} \rho_p f (1 - \mathcal{T}n)^{-1}. \quad (\text{I.11})$$

We have used here the fact [1] that $\rho_{\text{tot}}(\lambda) = \mathcal{H}_0^{\text{dr}}(\lambda)/(2\pi)$. For later use, we also introduce here the current susceptibility matrix $\mathcal{B}$ [1, 4]

$$\mathcal{B} = (1 - n\mathcal{T})^{-1} \rho_p f v (1 - \mathcal{T}n)^{-1}, \quad (\text{I.12})$$

we will discuss *matrix elements* of the operators $\mathcal{B}$ and $\mathcal{C}$ in more detail in Sec. IV.

II. HYDRODYNAMIC CONSERVATION LAWS

We derive here the conservation laws for the 3 hydrodynamic modes. We adopt notation from the main text for the expectation values of charges

$$\mathcal{Q}_n = \int d\lambda \mathcal{K}_n(\lambda) \rho_p(\lambda). \quad (\text{II.1})$$

We start with the GHD equation including the diffusion, force term $\mathfrak{f} = -\partial_x U(x)$ and the Boltzmann scattering integral

$$\partial_t \rho_p + \partial_x (v \rho_p) + \mathfrak{f} \partial_\lambda \rho_p = \frac{1}{2} \partial_x (\mathfrak{D}_\rho \partial_x \rho_p) + \mathcal{I}[\rho_p]. \quad (\text{II.2})$$

To obtain the conservation laws we multiply the GHD equation by $\mathcal{K}_j(\lambda)$ and integrate over λ .

- Mass conservation: for $j = 0$ we obtain

$$\partial_t \langle \mathcal{Q}_0 \rangle + \partial_x \langle \mathcal{Q}_1 \rangle = 0, \quad (\text{II.3})$$

where we used that both the diffusion and the collision term conserve the particles' density. This equation reflects the conservation of particles' density. We have also used that

$$\int d\lambda v(\lambda) \rho_p(\lambda) = \int d\lambda \mathcal{K}_1(\lambda) \rho_p(\lambda), \quad (\text{II.4})$$

- Momentum conservation: for $j = 1$ we obtain

$$\partial_t \langle \mathcal{Q}_1 \rangle + \partial_x \langle \mathcal{Q}_2 \rangle = -\partial_x \mathcal{P}_\mathfrak{D} + \mathfrak{f} \mathcal{Q}, \quad (\text{II.5})$$

where we defined

$$\mathcal{P}_\mathfrak{D} = -\frac{1}{2} \int d\lambda d\mu \mathcal{K}_1(\lambda) \mathfrak{D}_\rho(\lambda, \mu) \partial_x \rho_p(\mu). \quad (\text{II.6})$$

The average in the second term can be decomposed into two contributions: the convective flow and the pressure term. To this end we define the velocity field

$$u = \frac{\langle \mathcal{Q}_1 \rangle}{\langle \mathcal{Q}_0 \rangle}, \quad (\text{II.7})$$

and introduce rapidity variable ξ defined with respect to u , $\lambda = u + \xi$. From the definition of u it then follows that

$$\int d\xi \xi \rho_p(u + \xi) = 0, \quad (\text{II.8})$$

which is easily shown by transforming back to rapidity λ . We then have

$$\langle \mathcal{Q}_1 v \rangle = \int d\lambda v(\lambda) \lambda \rho_p(\lambda) = \int d\xi (u + \xi) v(u + \xi) \rho_p(u + \xi) = \frac{\langle \mathcal{Q}_1 \rangle^2}{\langle \mathcal{Q}_0 \rangle} + \mathcal{P}_v, \quad (\text{II.9})$$

where we defined the pressure by

$$\mathcal{P}_v = \int d\xi \xi v(u + \xi) \rho_p(u + \xi) = \int d\lambda (\lambda - u) v(\lambda) \rho_p(\lambda). \quad (\text{II.10})$$

We will later see that for a system locally in the thermal equilibrium state this reduces to the hydrostatic pressure P .

Summing up, the continuity equation reads

$$\partial_t \langle \mathcal{Q}_1 \rangle + \partial_x \left(\frac{\langle \mathcal{Q}_1 \rangle^2}{\langle \mathcal{Q}_0 \rangle} + \mathcal{P} \right) = \mathfrak{f} \mathcal{Q}, \quad (\text{II.11})$$

with $\mathcal{P} = \mathcal{P}_v + \mathcal{P}_\mathfrak{D}$.

- Energy conservation: Finally, for $j = 2$ we find

$$\partial_t \langle \varphi_2 \rangle + \partial_x \langle \varphi_2 v \rangle = -\partial_x (u \mathcal{P}_{\mathfrak{D}} + \mathcal{J}_{\mathfrak{D}}) + \mathfrak{f} \varrho u, \quad (\text{II.12})$$

where we have defined

$$u \mathcal{P}_{\mathfrak{D}} + \mathcal{J}_{\mathfrak{D}} = -\frac{1}{2} \int d\lambda d\mu \mathcal{K}_2(\lambda) \mathfrak{D}_{\rho}(\lambda, \mu) \partial_x \rho_{\text{p}}(\mu). \quad (\text{II.13})$$

The average on the left hand side of (II.12) can be again divided into various contributions. Transforming to ξ variable we have

$$\langle \varphi_2 v \rangle = \frac{1}{2} \int d\xi (u + \xi)^2 v(u + \xi) \rho_{\text{p}}(u + \xi) = \frac{1}{2} u^2 \langle \varphi_1 \rangle + u \mathcal{P}_v + \frac{1}{2} \int d\xi \xi^2 v(u + \xi) \rho_{\text{p}}(u + \xi). \quad (\text{II.14})$$

The remaining integral can be still divided into two contributions according to

$$\frac{1}{2} \int d\xi \xi^2 v(u + \xi) \rho_{\text{p}}(u + \xi) = \frac{1}{2} u \int d\xi \xi^2 \rho_{\text{p}}(u + \xi) + \frac{1}{2} \int d\xi \xi^2 (v(u + \xi) - u) \rho_{\text{p}}(u + \xi). \quad (\text{II.15})$$

The first integral, by going back to λ variable, becomes

$$\frac{1}{2} \int d\xi \xi^2 \rho_{\text{p}}(u + \xi) = \langle \varphi_2 \rangle - \frac{1}{2} \frac{\langle \varphi_1 \rangle^2}{\langle \varphi_0 \rangle}, \quad (\text{II.16})$$

whereas the second defines the $\mathcal{J}_v$ contribution to the heat current

$$\mathcal{J}_v = \frac{1}{2} \int d\xi \xi^2 (v(u + \xi) - u) \rho_{\text{p}}(u + \xi) = \frac{1}{2} \int d\lambda (\lambda - u)^2 (v(\lambda) - u) \rho_{\text{p}}(\lambda). \quad (\text{II.17})$$

The total heat current is $\mathcal{J} = \mathcal{J}_v + \mathcal{J}_{\mathfrak{D}}$. Finally we obtain

$$\partial_t \langle \varphi_2 \rangle + \partial_x \left(\frac{\langle \varphi_1 \rangle \langle \varphi_2 \rangle}{\langle \varphi_0 \rangle} + \frac{\langle \varphi_1 \rangle}{\langle \varphi_0 \rangle} \mathcal{P} + \mathcal{J} \right) = \mathfrak{f} \varrho u. \quad (\text{II.18})$$

Summarizing we obtain the following set of equations

$$\partial_t \langle \varphi_0 \rangle + \partial_x \langle \varphi_1 \rangle = 0, \quad \partial_t \langle \varphi_1 \rangle + \partial_x \left(\frac{\langle \varphi_1 \rangle^2}{\langle \varphi_0 \rangle} + \mathcal{P} \right) = \mathfrak{f} \varrho, \quad (\text{II.19})$$

$$\partial_t \langle \varphi_2 \rangle + \partial_x \left(\frac{\langle \varphi_1 \rangle \langle \varphi_2 \rangle}{\langle \varphi_0 \rangle} + \frac{\langle \varphi_1 \rangle}{\langle \varphi_0 \rangle} \mathcal{P} + \mathcal{J} \right) = \mathfrak{f} \varrho u. \quad (\text{II.20})$$

The continuity equations written above are for moments of the particles distribution. More standard form is found if we express them in terms of the velocity field u and internal energy field e defined by

$$e = \frac{\langle \varphi_2 \rangle}{\langle \varphi_0 \rangle} - \frac{1}{2} u^2. \quad (\text{II.21})$$

Introducing also $\varrho(x, t) = \langle \varphi_0 \rangle$ we find

$$\partial_t \varrho + \partial_x (\varrho u) = 0, \quad \partial_t (\varrho u) + \partial_x (\varrho u^2 + \mathcal{P}) = \mathfrak{f} \varrho, \quad (\text{II.22})$$

$$\partial_t \left(\varrho e + \frac{1}{2} u^2 \varrho \right) + \partial_x \left(u \varrho e + \frac{1}{2} u^3 \varrho + u \mathcal{P} + \mathcal{J} \right) = \mathfrak{f} \varrho u. \quad (\text{II.23})$$

This can be still simplified using that

$$\frac{1}{2} \partial_t (u^2 \varrho) = \frac{1}{2} u^2 \partial_x (\varrho u) - u \partial_x (\varrho u^2 + \mathcal{P}) + \mathfrak{f} \varrho u. \quad (\text{II.24})$$

Substituting this into the energy conservation law we obtain the final set of equations

$$\partial_t \varrho + \partial_x (\varrho u) = 0, \quad \partial_t (\varrho u) + \partial_x (\varrho u^2 + \mathcal{P}) = \mathfrak{f} \varrho, \quad (\text{II.25})$$

$$\partial_t (\varrho e) + \partial_x (u \varrho e + \mathcal{J}) + \mathcal{P} \partial_x u = 0, \quad (\text{II.26})$$

reported in the main text.

III. THERMAL BOOSTED STATE

In this Section we study thermal boosted states. We show that a thermal boosted state characterized by chemical potentials β_0 , β_1 and β_2 can be related to a thermal state with zero momentum and appropriately chosen $\bar{\beta}_0$, $\bar{\beta}_2$. Moreover, we will show that heat current $\mathcal{J}_v$ vanishes in such case and that the pressure $\mathcal{P}_v$ of a thermal boosted state is equal to the hydrostatic pressure P .

We start with the free energy density [1]

$$\mathfrak{f} = \frac{1}{2\pi} \int d\lambda F[\epsilon(\lambda)], \quad (\text{III.1})$$

where function $F[\epsilon(\lambda)]$ depends on the particle' statistics. For example, for fermionic case relevant for Lieb-Liniger model [5] $F[\epsilon(\lambda)] = -\log(1 + e^{-\epsilon})$. The pseudoenergy $\epsilon(\lambda)$ is determined from equation

$$\epsilon(\lambda) = \epsilon_0(\lambda) + \int d\lambda' \mathcal{T}(\lambda' - \lambda) F[\epsilon(\lambda')], \quad (\text{III.2})$$

where for thermal boosted state the bare pseudoenergy reads

$$\epsilon_0(\lambda) = \beta_0 \hbar_0(\lambda) + \beta_1 \hbar_1(\lambda) + \beta_2 \hbar_2(\lambda). \quad (\text{III.3})$$

In a thermal state, on the other hand $\beta_0 = -\mu/T$, $\beta_1 = 0$, $\beta_2 = 1/T$ with μ, T equal to chemical potential and temperature, respectively (we set $k_B = 1$).

We move now to discuss properties of a thermal boosted state, let $\rho_p(\lambda)$ be its distribution function. Then its first moment is non-zero,

$$u = \frac{\int d\lambda \lambda \rho_p(\lambda)}{\int d\lambda \rho_p(\lambda)}. \quad (\text{III.4})$$

Define $\bar{\rho}_p(\lambda) = \rho_p(\lambda + u)$. We will show that $\bar{\rho}_p(\lambda)$ is a thermal state and find expressions for $\bar{\beta}_0$ and $\bar{\beta}_2$ in terms of the chemical potentials of the boosted state. This can be argued on the basis of the Galilean invariance of the theory or computed explicitly. We follow the latter path.

Consider the total density $\rho_{\text{tot}}(\lambda)$, it obeys the integral relation (I.2). Using the definition of $\bar{\rho}_p(\lambda)$ it is easy to show that the corresponding total density is given by $\bar{\rho}_{\text{tot}}(\lambda) = \rho_{\text{tot}}(\lambda + u)$. From this follows that the occupation functions and dressed pseudo-energies $\epsilon(\lambda)$ and $\bar{\epsilon}(\lambda)$ obey the same relation $\bar{\epsilon}(\lambda) = \epsilon(\lambda + u)$. Inspecting equation (III.2) we finally find that also $\bar{\epsilon}_0(\lambda) = \epsilon_0(\lambda + u)$. Writing this relation explicitly we have

$$\bar{\beta}_0 + \frac{1}{2} \bar{\beta}_2 \lambda^2 = \beta_0 + \beta_1(\lambda + u) + \frac{1}{2} \beta_2(\lambda + u)^2. \quad (\text{III.5})$$

Matching the terms with the same power of λ we find

$$\bar{\beta}_0 = \beta_0 + u\beta_1 + \frac{1}{2}u^2\beta_2, \quad \bar{\beta}_2 = \beta_2, \quad \beta_1 + u\beta_2 = 0. \quad (\text{III.6})$$

The third equality is always fulfilled and gives $u = -\beta_1/\beta_2$.

The last property follows from the observation that for $\lambda = u$ function $\rho_p(\lambda)$ has the maximum. On the other hand

$$\frac{\partial \rho_p(\lambda)}{\partial \lambda} \sim \frac{\partial \epsilon(\lambda)}{\partial \lambda}. \quad (\text{III.7})$$

Function $\epsilon'(\lambda)$ obeys a linear integral equation

$$\epsilon'(\lambda) = \epsilon'_0(\lambda) + \int d\mu \mathcal{T}(\lambda - \mu) n(\mu) \epsilon'(\mu). \quad (\text{III.8})$$

which follows from (III.2) and the identity [1]

$$n = \frac{dF}{d\epsilon} \quad (\text{III.9})$$

and because of its linearity it maps zero to zero hence we need to solve $\epsilon'_0(\lambda) = 0$. Using $\epsilon_0(\lambda) = \beta_0 + \beta_1\lambda + \frac{1}{2}\beta_2\lambda^2$ we readily find the relation between u and the chemical potentials.

We can also show using (I.3) that the effective velocity transforms as $\bar{v}(\lambda) + u = v(\lambda + u)$. This has a consequence that $v(u) = u$ where we used that $\bar{v}(0) = 0$.

With these expressions we can show that the hydrodynamic pressure of a thermal boosted state reduces to the hydrostatic pressure P , see [6] and the end of the present section. Hence the bulk viscosity vanishes in such case. We have

$$\mathcal{P}_v = \int d\xi \xi v(\xi + u) \rho_p(\xi + u) = \int d\xi \xi (\bar{v}(\xi) + u) \bar{\rho}_p(\xi) = \int d\xi \xi \bar{v}(\xi) \bar{\rho}_p(\xi) = P. \quad (\text{III.10})$$

For the heat current we have

$$\mathcal{J}_v = \frac{1}{2} \int d\xi \xi^2 (v(u + \xi) - u) \rho_p(u + \xi) = \frac{1}{2} \int d\xi \xi^2 \bar{v}(\xi) \bar{\rho}_p(\xi) = 0, \quad (\text{III.11})$$

because $\bar{v}(\xi)$ is an odd function while $\bar{\rho}_p(\xi)$ is even.

The entropy of the boosted state is equal to the entropy of the unboosted state. Recall the formula for the entropy density

$$s = \int d\lambda \rho_{\text{tot}}(\lambda) g(n(\lambda)), \quad (\text{III.12})$$

where function $g(n)$ depends on particles' statistics [1, 3]. For the fermionic models we have $g[n] = n \log n + (1 - n) \log(1 - n)$. In general $g = \epsilon(\lambda) n(\lambda) - F[\epsilon(\lambda)]$. Using now the relations between the particle distribution and filling function of the boosted and unboosted state we see that the entropies are indeed equal.

Expression for the hydrostatic pressure in terms of $v(\lambda)$

Let us close this section by further validating equation (III.10). Our aim is to show that indeed

$$P = \int d\lambda \lambda v(\lambda) \rho_p(\lambda) \quad (\text{III.13})$$

where $\rho_p(\lambda)$ is thermal state, is the hydrostatic pressure as defined in thermodynamics, see eq. (IV.10) of the next section. To this end we generalize the computations for the Lieb-Liniger model presented in [6].

We start by noting that $\rho_{\text{tot}}(\lambda) = \frac{1}{2\pi} \mathcal{H}_0^{\text{dr}}(\lambda)$ and $v(\lambda) = \mathcal{H}_1^{\text{dr}}(\lambda) / \mathcal{H}_0^{\text{dr}}(\lambda)$, hence

$$P = \frac{1}{2\pi} \int d\lambda \lambda n(\lambda) \mathcal{H}_1^{\text{dr}}(\lambda). \quad (\text{III.14})$$

Note that we have also shown at this point that $P = \mathcal{J}_1$, where $\mathcal{J}_1$ is the momentum current (compare with formula for current expectation value in [1]). To get $\mathcal{H}_1^{\text{dr}}(\lambda)$ we look at (III.8). For thermal state it reads

$$\epsilon'(\lambda) = \frac{1}{T} \mathcal{H}_1(\lambda) + \int d\lambda' \mathcal{T}(\lambda - \lambda') n(\lambda') \epsilon'(\lambda'). \quad (\text{III.15})$$

Comparing with the definition of dressing operation (I.1) we conclude that $\mathcal{H}_1^{\text{dr}}(\lambda) = T \epsilon'(\lambda)$. Moreover, using (III.9) and integrating by parts we get

$$P = -\frac{T}{2\pi} \int d\lambda F[\epsilon(\lambda)] = -Tf \quad (\text{III.16})$$

in agreement with (IV.10).

IV. THERMODYNAMICS AND HYDRODYNAMIC MATRICES

In this section we will relate the matrix elements of hydrodynamic matrices [4] A , B and C to the thermodynamics of the integrable model. This will be very helpful in linking results from linearized GHD with thermodynamic quantities, which appear naturally in the Navier-Stokes equations.

We start by defining the free energy flux [1], in addition to the free energy density f defined in (III.1),

$$\mathbf{g} = \frac{1}{2\pi} \int d\lambda E'(\lambda) F[\epsilon(\lambda)], \quad (\text{IV.1})$$

where the pseudoenergy $\epsilon(\lambda)$ satisfies (III.2).

For a generic GGE state, the bare pseudoenergy includes higher conserved charges [7]. The simplest choice being the ultra-local charges $\mathcal{Q}_j$,

$$\epsilon_0(\lambda) = \sum_{j=0}^{\infty} \beta_j \mathcal{H}_j(\lambda). \quad (\text{IV.2})$$

The expectation values of the charges in the GGE state follow then from derivatives of the free energy density with respect to the generalized temperatures. In a similar fashion, the expectation values of the currents are given by the derivatives of the free energy flux,

$$\mathcal{Q}_i = \frac{\partial f}{\partial \beta_i}, \quad \mathcal{J}_i = \frac{\partial \mathbf{g}}{\partial \beta_i}. \quad (\text{IV.3})$$

The hydrodynamic matrices $\mathcal{B}$ and $\mathcal{C}$ are susceptibilities of the currents and charges

$$\mathcal{B}_{i,j} = -\frac{\partial \mathcal{J}_i}{\partial \beta_j} = \int d\lambda \rho_p(\lambda) f(\lambda) v(\lambda) \mathcal{H}_i^{\text{dr}}(\lambda) \mathcal{H}_j^{\text{dr}}(\lambda), \quad \mathcal{C}_{i,j} = -\frac{\partial \mathcal{Q}_i}{\partial \beta_j} = \int d\lambda \rho_p(\lambda) f(\lambda) \mathcal{H}_i^{\text{dr}}(\lambda) \mathcal{H}_j^{\text{dr}}(\lambda). \quad (\text{IV.4})$$

The matrices defined above are clearly symmetric. Above we defined

$$f(\lambda) = -\frac{d^2 F/d\epsilon^2}{dF/d\epsilon}, \quad (\text{IV.5})$$

which is called the statistical factor [1]. For instance, in the case of fermionic statistics one finds $f(\theta) = 1 - n(\theta)$. We also note that for Galilean invariant systems there is an useful relation $\mathcal{B}_{1,0} = \mathcal{C}_{1,1}$.

The hydrodynamic matrices depend on the choice of the basis for the ultra-local charges. For the thermodynamics it is natural to work with matrix elements of the ultralocal charges $\mathcal{H}_n(\lambda)$ as defined above. Instead, for the hydrodynamics, it is more convenient to work with the orthonormalized basis obtained from the ultra-local charges through the Gram-Schmidt method with the inner product (V.6). For $h_n(\lambda)$ denoting the single-particle eigenvalues of the resulting charges, the hydrodynamic matrices are

$$\mathcal{B}_{i,j} = -\frac{\partial \mathcal{J}_i}{\partial \beta_j} = \int d\lambda \rho_p(\lambda) f(\lambda) v(\lambda) h_i^{\text{dr}}(\lambda) h_j^{\text{dr}}(\lambda), \quad \mathcal{C}_{i,j} = -\frac{\partial \mathcal{Q}_i}{\partial \beta_j} = \int d\lambda \rho_p(\lambda) f(\lambda) h_i^{\text{dr}}(\lambda) h_j^{\text{dr}}(\lambda) = \delta_{i,j}. \quad (\text{IV.6})$$

In the remainder of this section we will relate matrix elements of $\mathcal{B}$ and $\mathcal{C}$ to thermodynamic quantities. We focus on the Gibbs state with a thermal $\epsilon_0(\lambda)$ function,

$$\epsilon_0(\lambda) = -\frac{\mu}{T} \mathcal{H}_0(\lambda) + \frac{1}{T} \mathcal{H}_2(\lambda). \quad (\text{IV.7})$$

We also introduce the notation

$$\varrho = \mathcal{Q}_0 = \int d\lambda \mathcal{H}_0(\lambda) \rho_p(\lambda), \quad \mathbf{e} = \mathcal{Q}_2 = \int d\lambda \mathcal{H}_2(\lambda) \rho_p(\lambda), \quad \mathbf{s} = \int d\lambda \rho_{\text{tot}}(\lambda) g[n(\lambda)], \quad (\text{IV.8})$$

for particle, energy and entropy density, respectively. Moreover, by

$$\mathbf{s} = \varrho^{-1} \mathbf{s}, \quad \mathbf{e} = \varrho^{-1} \mathbf{e}, \quad (\text{IV.9})$$

we denote entropy and energy per particle. For Gibbs state the free energy f takes the form of the grand potential per unit length divided by temperature and can be written as

$$f = -\frac{\mu}{T} \varrho + \frac{1}{T} \mathbf{e} - \mathbf{s} = -\frac{P}{T}. \quad (\text{IV.10})$$

With this, we compute various thermodynamic quantities. We start with the sound velocity.

A. Sound velocity

The sound velocity is related to the adiabatic compressibility κ_S and defined as (we work with unit mass)

$$v_s^2 = \left(\frac{\partial P}{\partial \varrho} \right)_s = \frac{1}{\varrho \kappa_S}, \quad \kappa_S = \frac{1}{\varrho} \left(\frac{\partial \varrho}{\partial P} \right)_s. \quad (\text{IV.11})$$

To compute the relevant derivative, we realize that P, ϱ and s depend only on two thermodynamic parameters, which specify the state: μ, T . Equivalently, they depend on β_0, β_2 . We may thus write

$$ds = \left(\frac{\partial s}{\partial \beta_0} \right)_{\beta_2} d\beta_0 + \left(\frac{\partial s}{\partial \beta_2} \right)_{\beta_0} d\beta_2, \quad (\text{IV.12})$$

and similarly for $d\varrho$ and dP . To keep the entropy fixed $d\beta_0$ and $d\beta_2$ must be related through

$$d\beta_0 = - \frac{\left(\frac{\partial s}{\partial \beta_2} \right)_{\beta_0}}{\left(\frac{\partial s}{\partial \beta_0} \right)_{\beta_2}} d\beta_2, \quad (\text{IV.13})$$

which leads to the following formula

$$\left(\frac{\partial \varrho}{\partial P} \right)_s = \frac{\left(\frac{\partial \varrho}{\partial \beta_0} \right)_{\beta_2} \left(\frac{\partial s}{\partial \beta_2} \right)_{\beta_0} - \left(\frac{\partial \varrho}{\partial \beta_2} \right)_{\beta_0} \left(\frac{\partial s}{\partial \beta_0} \right)_{\beta_2}}{\left(\frac{\partial P}{\partial \beta_0} \right)_{\beta_2} \left(\frac{\partial s}{\partial \beta_2} \right)_{\beta_0} - \left(\frac{\partial P}{\partial \beta_2} \right)_{\beta_0} \left(\frac{\partial s}{\partial \beta_0} \right)_{\beta_2}}. \quad (\text{IV.14})$$

The derivatives of the entropy can be computed from the expression for free energy (IV.10). The entropy per particle equals to

$$s = \frac{\beta_0 \varrho + \beta_2 \mathbf{e} - \mathbf{f}}{\varrho}, \quad (\text{IV.15})$$

and the two required derivatives are

$$\left(\frac{\partial s}{\partial \beta_0} \right)_{\beta_2} = \frac{\mathcal{C}_{0,0}(\beta_2 \mathbf{e} - \mathbf{f}) - \mathcal{C}_{2,0} \beta_2 \varrho}{\varrho^2}, \quad \left(\frac{\partial s}{\partial \beta_2} \right)_{\beta_0} = \frac{\mathcal{C}_{2,0}(\beta_2 \mathbf{e} - \mathbf{f}) - \mathcal{C}_{2,2} \beta_2 \varrho}{\varrho^2}. \quad (\text{IV.16})$$

We use now the relation $\mathcal{J}_1 = P$, valid for Galilean-invariant models, see (III.14). Together with $P = -T\mathbf{f} = -\mathbf{f}/\beta_2$ they yield

$$\mathcal{B}_{1,0} = - \left(\frac{\partial P}{\partial \beta_0} \right)_{\beta_2} = \frac{\varrho}{\beta_2} = T\varrho, \quad \mathcal{B}_{2,1} = - \left(\frac{\partial P}{\partial \beta_2} \right)_{\beta_0} = \frac{\beta_2 \mathbf{e} - \mathbf{f}}{\beta_2^2}, \quad (\text{IV.17})$$

which gives

$$\left(\frac{\partial s}{\partial \beta_0} \right)_{\beta_2} = \frac{1}{\mathcal{B}_{1,0}^2} (\mathcal{C}_{0,0} \mathcal{B}_{2,1} - \mathcal{C}_{2,0} \mathcal{B}_{1,0}), \quad \left(\frac{\partial s}{\partial \beta_2} \right)_{\beta_0} = \frac{1}{\mathcal{B}_{1,0}^2} (\mathcal{C}_{2,0} \mathcal{B}_{2,1} - \mathcal{C}_{2,2} \mathcal{B}_{1,0}). \quad (\text{IV.18})$$

We now go back to (IV.14) and (IV.11) to find the final expression for the adiabatic compressibility

$$\kappa_S = T \frac{\mathcal{C}_{2,2} \mathcal{C}_{0,0} - \mathcal{C}_{2,0}^2}{\mathcal{B}_{1,0}^2 \mathcal{C}_{2,2} + \mathcal{B}_{2,1}^2 \mathcal{C}_{0,0} - 2\mathcal{B}_{1,0} \mathcal{B}_{2,1} \mathcal{C}_{2,0}}, \quad (\text{IV.19})$$

from which the final expression for the sound velocity follows

$$v_s = \sqrt{\frac{1}{\mathcal{B}_{1,0}} \frac{\mathcal{B}_{1,0}^2 \mathcal{C}_{2,2} + \mathcal{B}_{2,1}^2 \mathcal{C}_{0,0} - 2\mathcal{B}_{1,0} \mathcal{B}_{2,1} \mathcal{C}_{2,0}}{\mathcal{C}_{2,2} \mathcal{C}_{0,0} - \mathcal{C}_{2,0}^2}}}. \quad (\text{IV.20})$$

We have analytically confirmed that this general formula reduces to known formulas for free fermions $v_s^{\text{Fermi}} = \sqrt{3T \frac{f_{3/2}(e^{\mu/T})}{f_{1/2}(e^{\mu/T})}}$ where $f_\alpha(x)$ are Fermi functions and for hard-rods [8] $v_s^{\text{HR}} = \sqrt{3T/(1-a\varrho)^2}$, where a is rod's length. For the Lieb-Liniger model we confirmed with derivatives computed numerically from TBA.

B. Specific heats, isothermal compressibility and additional identities

We calculate in a similar fashion several more thermodynamic quantities, which are useful for purposes of this work. **Isothermal compressibility:** The isothermal compressibility is defined as

$$\kappa_T = \frac{1}{\varrho} \left(\frac{\partial \varrho}{\partial P} \right)_T, \quad (\text{IV.21})$$

and standard manipulations give

$$\kappa_T = \frac{1}{\varrho} \frac{\left(\frac{\partial \varrho}{\partial \beta_0} \right)_{\beta_2}}{\left(\frac{\partial P}{\partial \beta_0} \right)_{\beta_2}} = \frac{1}{\varrho} \frac{\mathcal{C}_{0,0}}{\mathcal{B}_{1,0}}. \quad (\text{IV.22})$$

Specific heat at constant volume: The definition is

$$c_V = \left(\frac{\partial e}{\partial T} \right)_\varrho = \frac{1}{\varrho} \left(\frac{\partial \mathbf{e}}{\partial T} \right)_\varrho. \quad (\text{IV.23})$$

Expressing dT with $d\beta_0$ and $d\beta_2$ we obtain

$$c_V = \frac{1}{\varrho} \frac{1 - \left(\frac{\partial \mathbf{e}}{\partial \beta_0} \right)_{\beta_2} \left(\frac{\partial \varrho}{\partial \beta_2} \right)_{\beta_0} + \left(\frac{\partial \mathbf{e}}{\partial \beta_2} \right)_{\beta_0} \left(\frac{\partial \varrho}{\partial \beta_0} \right)_{\beta_2}}{-T^2 \left(\frac{\partial \varrho}{\partial \beta_0} \right)_{\beta_2}}, \quad (\text{IV.24})$$

which can now be expressed with the hydrodynamic matrices as

$$c_V = \frac{1}{T} \frac{\mathcal{C}_{2,2}\mathcal{C}_{0,0} - \mathcal{C}_{2,0}^2}{\mathcal{C}_{0,0}\mathcal{B}_{1,0}}. \quad (\text{IV.25})$$

Specific heat at constant pressure: The definition is

$$c_P = T \left(\frac{\partial s}{\partial T} \right)_P. \quad (\text{IV.26})$$

Proceeding in an analogous way as in the previous cases we find

$$c_P = \frac{1}{T} \frac{\mathcal{B}_{1,0}^2 \mathcal{C}_{2,2} + \mathcal{B}_{2,1}^2 \mathcal{C}_{0,0} - 2\mathcal{B}_{1,0}\mathcal{B}_{2,1}\mathcal{C}_{2,0}}{\mathcal{B}_{1,0}^3}. \quad (\text{IV.27})$$

As a check for our expressions, we also observe that the two compressibilities and two specific heats are not independent from each other. Instead, they obey the thermodynamic identity

$$\kappa_S = \frac{c_V}{c_P} \kappa_T. \quad (\text{IV.28})$$

One can easily verify that this identity holds for the expressions given in (IV.19), (IV.22), (IV.25) and (IV.27).

Expressions for the matrix elements of A : The sound velocity may be rewritten with A matrix elements in GS-orthonormalized basis. These matrix elements are defined in (V.14) and appear explicitly in the equation (VII.2), from which dispersion relations are determined. It is thus useful to write thermodynamic quantities in terms of A matrix in this basis

$$A_{1,0} = \sqrt{\frac{\mathcal{B}_{1,0}}{\mathcal{C}_{0,0}}} = \frac{1}{\sqrt{\varrho \kappa_T}}, \quad A_{2,1} = \frac{\mathcal{B}_{2,1}\mathcal{C}_{0,0} - \mathcal{B}_{1,0}\mathcal{C}_{2,0}}{\sqrt{\mathcal{C}_{0,0}\mathcal{B}_{1,0}(\mathcal{C}_{2,2}\mathcal{C}_{0,0} - \mathcal{C}_{2,0}^2)}} = \frac{T}{\varrho^2 c_V} \left(\frac{\partial P}{\partial T} \right)_\varrho = \frac{1}{\sqrt{\varrho \kappa_T}} \sqrt{c_P/c_V - 1}. \quad (\text{IV.29})$$

Above we have used a general thermodynamic identity [9]

$$c_P - c_V = \frac{T}{\varrho^2} \frac{\left(\frac{\partial P}{\partial T} \right)_\varrho^2}{\left(\frac{\partial P}{\partial \varrho} \right)_T}. \quad (\text{IV.30})$$

In the perturbation theory, we will find that speed of sound becomes

$$v_s = \sqrt{A_{1,0}^2 + A_{2,1}^2}, \quad (\text{IV.31})$$

which agrees with (IV.20). Moreover, we observe that

$$\frac{A_{2,1}^2}{A_{1,0}^2} = c_P \left(\frac{1}{c_V} - \frac{1}{c_P} \right) = \frac{c_P}{c_V} - 1. \quad (\text{IV.32})$$

Additional identities: For analysis of the structure of heat and sound modes the following identities are useful

$$\left(\frac{\partial P}{\partial T} \right)_e = \frac{1}{T^2} \frac{\mathcal{B}_{2,1} \mathcal{C}_{0,0} - \mathcal{B}_{1,0} \mathcal{C}_{2,0}}{\mathcal{C}_{0,0}}, \quad \frac{1}{\varrho} \left(\mathbf{e} + P - T \left(\frac{\partial P}{\partial T} \right)_e \right) = \frac{\mathcal{C}_{2,0}}{\mathcal{C}_{0,0}}. \quad (\text{IV.33})$$

They can be derived with the methods presented above. Lastly, let us comment that the method presented in this section gives thermodynamic quantities as explicit functionals (that is, combinations of matrix elements of hydrodynamic matrices) of the thermal rapidity distribution $\rho_p(\lambda)$. This representation is convenient and may be useful for applications which go beyond the scope of this work.

V. DERIVATION OF THE LINEARIZED EQUATION FOR CHARGE PERTURBATIONS

We again invoke the GHD equation supplemented with the collision integral term, this time with $U(x) = 0$,

$$\partial_t \rho_p + \partial_x (v_\rho \rho_p) = \frac{1}{2} \partial_x (\mathfrak{D}_\rho \partial_x \rho_p) + \mathcal{I}[\rho_p]. \quad (\text{V.1})$$

Our aim here is to first linearize this equation and then rewrite it as a continuity equation for the densities of local charges. To this end we linearize it around a stationary uniform thermal state $\rho(x, t) = \rho_{\text{th}} + \delta \rho_p(x, t)$ getting equation

$$\partial_t \delta \rho_p + \partial_x ((1 - n\mathcal{T})^{-1} v_\rho (1 - n\mathcal{T}) \delta \rho_p) = \frac{1}{2} \mathfrak{D}_{\text{th}} \partial_x^2 \delta \rho_p + \left(\frac{\delta \mathcal{I}}{\delta \rho_p} \right)_{\text{th}} \delta \rho_p. \quad (\text{V.2})$$

The second term in the equation above (linearized expression for effective velocity term) was derived in [4].

Motivated by the structure of Γ matrix that we consider (see Sec. IX), we wish to change the degrees of freedom from a perturbation of quasiparticle distribution to a perturbation of the bare pseudoenergy ϵ_0 . To establish a relation between them, we start with

$$\rho_p = n \rho_{\text{tot}}, \quad \delta \rho_p = \delta n \rho_{\text{tot}} + n \delta \rho_{\text{tot}}. \quad (\text{V.3})$$

The perturbation occupation function may be related to the perturbation of pseudoenergy via (III.9), hence $\delta n = -f n \delta \epsilon$. Moreover from (I.2) we have $\delta \rho_{\text{tot}} = \mathcal{T} \delta \rho_p$ and from (III.2) we get $\delta \epsilon = (1 - \mathcal{T}n)^{-1} \delta \epsilon_0$. Combining, we find the following relation

$$\delta \rho_p = -(1 - n\mathcal{T})^{-1} \rho_p f \delta \epsilon = -(1 - n\mathcal{T})^{-1} \rho_p f (1 - \mathcal{T}n)^{-1} \delta \epsilon_0 = -\mathcal{C} \delta \epsilon_0. \quad (\text{V.4})$$

where the charge susceptibility $\mathcal{C}$ operator was introduced in (I.11) and the operator $(1 - \mathcal{T}n)^{-1}$ implements the dressing (I.1). Function $\delta \epsilon_0(\lambda)$ can be expanded as

$$\delta \epsilon_0(\lambda) = \sum_m (h_m | \delta \epsilon_0) h_m(\lambda), \quad (\text{V.5})$$

where $h_m(\lambda)$ form an orthonormal set of functions with respect to the hydrodynamic scalar product $(\cdot | \cdot)$, see also [1],

$$(h|g) = \int d\lambda \rho_p(\lambda) f(\lambda) h^{\text{dr}}(\lambda) g^{\text{dr}}(\lambda). \quad (\text{V.6})$$

In practice, h_m are constructed by the Gram-Schmidt orthonormalization of the ultra-local charges with single particle eigenvalue $\mathcal{H}_m(\lambda) = \lambda^m / m!$. The most important first three functions are

$$h_0(\lambda) = \frac{1}{\sqrt{\mathcal{C}_{0,0}}} \mathcal{H}_0(\lambda), \quad h_1(\lambda) = \frac{1}{\sqrt{\mathcal{C}_{1,1}}} \mathcal{H}_1(\lambda), \quad h_2(\lambda) = \sqrt{\frac{\mathcal{C}_{0,0}}{\mathcal{C}_{2,2} \mathcal{C}_{0,0} - \mathcal{C}_{2,0}^2}} \left(\mathcal{H}_2(\lambda) - \frac{\mathcal{C}_{2,0}}{\mathcal{C}_{0,0}} \mathcal{H}_0(\lambda) \right). \quad (\text{V.7})$$

The expansion coefficients are directly related to the change in the expectation value of the m -th conserved charge, $(h_m|\delta\epsilon_0) = -\delta q_m$, where

$$\delta q_m = \int d\lambda h_m(\lambda) \delta \rho_p(\lambda). \quad (\text{V.8})$$

This relation follows from the following identity [1] (here dot-product is $h \cdot g = \int d\lambda h(\lambda)g(\lambda)$)

$$(1 - n\mathcal{T})^{-1} n g \cdot h = g \cdot n(1 - \mathcal{T}n)^{-1} h = g \cdot n h^{\text{dr}}, \quad (\text{V.9})$$

which follows from the symmetry of the kernel $\mathcal{T}$. With the help of this identity we find

$$\begin{aligned} \delta q_m &= - \int d\lambda h_m(\lambda) (1 - n\mathcal{T})^{-1} \rho_p f (1 - \mathcal{T}n)^{-1} \delta \epsilon_0 \\ &= - \int d\lambda \rho_p(\lambda) f(\lambda) h_m^{\text{dr}}(\lambda) \delta \epsilon_0^{\text{dr}}(\lambda) = -(h_m|\delta \epsilon_0), \end{aligned} \quad (\text{V.10})$$

where we used in the second step with $h = h_m$ and $g = \rho_p f \delta \epsilon_0^{\text{dr}}/n$. This allows us to write

$$\delta \rho_p = (1 - n\mathcal{T})^{-1} \rho_p f (1 - \mathcal{T}n)^{-1} \sum_m h_m \delta q_m. \quad (\text{V.11})$$

Functions h_n allow us to reformulate the GHD equations as equations for the charge perturbations. We multiply the GHD equation (V.2) by $h_n(\lambda)$, plug (V.11) and integrate over λ taking advantage of the orthonormality of functions h_n . Moreover, we introduce Γ operator

$$\Gamma = \frac{\delta \mathcal{I}}{\delta \epsilon_0} = - \frac{\delta \mathcal{I}}{\delta \rho_p} \mathcal{C}. \quad (\text{V.12})$$

The result is

$$\partial_t \delta q_n + \sum_m A_{nm} \partial_x \delta q_m = \sum_m D_{nm} \partial_x^2 \delta q_m - \sum_m \Gamma_{nm} \delta q_m, \quad (\text{V.13})$$

with

$$A_{nm} = B_{nm} = \int d\lambda \rho_p(\lambda) f(\lambda) v(\lambda) h_n^{\text{dr}}(\lambda) h_m^{\text{dr}}(\lambda), \quad (\text{V.14})$$

$$D_{nm} = \frac{1}{2} \int d\lambda d\mu h_n^{\text{dr}}(\lambda) \rho_{\text{tot}}^{-1}(\lambda) \tilde{\mathfrak{D}}_{\text{th}}(\lambda, \mu) \rho_{\text{tot}}^{-1}(\mu) \rho_p(\mu) f(\mu) h_m^{\text{dr}}(\mu), \quad (\text{V.15})$$

$$\Gamma_{nm} = \int d\lambda d\mu h_n(\lambda) \Gamma(\lambda, \mu) h_m(\mu). \quad (\text{V.16})$$

Furthermore, we can take into account that Γ operator hides the Dressing operation [10–12]. The Dressing operation is defined through (where $F(\mu|\lambda)$ is backflow function),

$$f^{\text{Dr}}(\lambda) = f(\lambda) - \int d\mu n(\mu) F(\mu|\lambda) f'(\mu), \quad (\text{V.17})$$

such that the collision integral is (note that operator $\mathbb{F}$ effectively implements Dressing)

$$\Gamma(\lambda, \mu) = \int d\nu d\nu' \mathbb{F}(\lambda, \nu) \Gamma_0(\nu, \nu') \mathbb{F}(\mu, \nu'), \quad \mathbb{F}(\lambda, \nu) = \delta(\lambda - \nu) + \partial_\lambda (n(\lambda) F(\lambda|\nu)), \quad (\text{V.18})$$

where Γ_0 is the kernel of linearized bare collision integral. This Dressing operation may be shifted to $h_n(\lambda)$ so finally we get

$$\Gamma_{nm} = \int d\lambda d\mu h_n^{\text{Dr}}(\lambda) \Gamma_0(\lambda, \mu) h_m^{\text{Dr}}(\mu). \quad (\text{V.19})$$

Let us comment on general properties of the matrix elements listed above. Matrix elements A_{nm} are non-zero where n, m are of different parity, instead Γ_{nm} and D_{nm} are non-zero only when n, m have the same parity. We also have $\Gamma_{nm} = 0$ for $n, m = 0, 1, 2$, which reflects the 3 collision invariants, whereas $D_{0,n} = D_{n,0} = 0$ which follows from the Markovianity of the diffusion kernel discussed in Section I.

For the spectrum of Γ operator we assume that it is non-negative and gapped. The eigenvectors correspond to either even or odd functions of rapidities. We confirm these properties for the collision integral analyzed in Sec. IX.

VI. LINEARIZED NAVIER-STOKES EQUATIONS AND TRANSPORT COEFFICIENTS

We linearize the Navier-Stokes equations, which can be understood as (II.25) with the phenomenological expressions

$$\mathcal{P} = P - \zeta \partial_x u, \quad \mathcal{J} = -\kappa \partial_x T, \quad (\text{VI.1})$$

for the dynamic pressure and heat current. In this section, we do not specify ζ and κ . Our aim is to focus on general structure of N-S equations and find relation between the spectrum of the hydrodynamic modes and the transport coefficients.

We start by introducing the temperature field to the Navier-Stokes equations. For this, we note the following general thermodynamic equality, which can be derived from basic principles [9]

$$\delta T = \frac{1}{c_V} \delta e - \frac{1}{c_V} \left(\frac{P}{\varrho^2} - \frac{T}{\varrho^2} \left(\frac{\partial P}{\partial T} \right)_\varrho \right) \delta \varrho. \quad (\text{VI.2})$$

One-dimensional version of Navier-Stokes equations [13, 14] for the fields $\varrho(x, t)$, $u(x, t)$ and $T(x, t)$ reads then

$$\begin{aligned} \partial_t \varrho &= -\partial_x(\varrho u), \\ \partial_t u &= -u \partial_x u - \frac{1}{\varrho} \left(\frac{\partial P}{\partial \varrho} \right)_T \partial_x \varrho - \frac{1}{\varrho} \left(\frac{\partial P}{\partial T} \right)_\varrho \partial_x T + \frac{\zeta}{\varrho} \partial_x^2 u, \\ \partial_t T &= -\frac{T}{\varrho c_V} \left(\frac{\partial P}{\partial T} \right)_\varrho \partial_x u - u \partial_x T + \frac{\kappa}{\varrho c_V} \partial_x^2 T + \frac{\zeta}{\varrho c_V} (\partial_x u)^2, \end{aligned} \quad (\text{VI.3})$$

where κ, ζ are thermal conductivity and bulk viscosity, respectively. Linearizing around a homogenous thermal state

$$\varrho(x, t) = \varrho + \delta \varrho(x, t), \quad u(x, t) = 0 + \delta u(x, t), \quad T(x, t) = T + \delta T(x, t), \quad (\text{VI.4})$$

one gets

$$\begin{aligned} \partial_t \delta \varrho &= -\varrho \partial_x \delta u, \\ \partial_t \delta u &= -\frac{1}{\varrho} \left(\frac{\partial P}{\partial \varrho} \right)_T \partial_x \delta \varrho - \frac{1}{\varrho} \left(\frac{\partial P}{\partial T} \right)_\varrho \partial_x \delta T + \frac{\zeta}{\varrho} \partial_x^2 \delta u, \\ \partial_t \delta T &= -\frac{T}{\varrho c_V} \left(\frac{\partial P}{\partial T} \right)_\varrho \partial_x \delta u + \frac{\kappa}{\varrho c_V} \partial_x^2 \delta T. \end{aligned} \quad (\text{VI.5})$$

Moving to the k -space and anticipating the time-dependence as

$$\delta \varrho(x, t) = \int dk e^{ikx - \Lambda t} \delta \varrho(k), \quad \delta u(x, t) = \int dk e^{ikx - \Lambda t} \delta u(k), \quad \delta T(x, t) = \int dk e^{ikx - \Lambda t} \delta T(k), \quad (\text{VI.6})$$

we get an eigenproblem with non-Hermitian matrix. As the matrix is non-Hermitian the left- and right-eigenvectors are different. This problem be cured by changing the variables as

$$\delta \varrho = \sqrt{T \varrho / (\partial P / \partial \varrho)_T} \delta \tilde{\varrho} = \varrho \sqrt{T \kappa_T} \delta \tilde{\varrho}, \quad \delta u = \sqrt{T / \varrho} \delta \tilde{u}, \quad \delta T = T / \sqrt{\varrho c_V} \delta \tilde{T}. \quad (\text{VI.7})$$

In these variables, the relevant matrix becomes symmetric. Explicitly, when looking for eigenvalues Λ we will deal with

$$\det \begin{pmatrix} -\Lambda & i \frac{1}{\sqrt{\varrho \kappa_T}} k & 0 \\ i \frac{1}{\sqrt{\varrho \kappa_T}} k & -\Lambda + \frac{\zeta}{\varrho} k^2 & i \sqrt{\frac{T}{\varrho^2 c_V}} \left(\frac{\partial P}{\partial T} \right)_\varrho k \\ 0 & i \sqrt{\frac{T}{\varrho^2 c_V}} \left(\frac{\partial P}{\partial T} \right)_\varrho k & -\Lambda + \frac{\kappa}{\varrho c_V} k^2 \end{pmatrix} = 0. \quad (\text{VI.8})$$

We use now the general thermodynamic identities (IV.28), (IV.30) and $\frac{c_P}{c_V} \left(\frac{\partial P}{\partial \varrho} \right)_T = v_s^2$ [9] and find the following dispersion relations

$$\Lambda_\pm(k) = \pm i v_s k + \frac{1}{2\varrho} \left[\left(\frac{1}{c_V} - \frac{1}{c_P} \right) \kappa + \zeta \right] k^2 + O(k^3), \quad \Lambda_{\text{th}}(k) = \frac{\kappa}{\varrho c_P} k^2 + O(k^3). \quad (\text{VI.9})$$

These dispersion relations correspond to two sound modes and one heat mode, respectively. We observe that in the case of vanishing bulk viscosity $\zeta = 0$, the ratio of the quadratic coefficients is $(c_P/c_V - 1)/2$ and therefore is universal. Depends only on the thermodynamics of the underlying gas but not on the integrability breaking terms. Vanishing bulk viscosity is an expected result for an ideal gas perturbed with weak interactions [13]. We will see that in our setup the bulk viscosity vanishes in the Tonks-Girardeau ($c \rightarrow \infty$) and in $c \rightarrow 0$ limits, see Sec. VIII.

In addition to the eigenvalues, we can find eigenvectors. In the limit of $k \rightarrow 0$ we have

$$\delta_{\pm}(k \rightarrow 0) = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{c_V}{c_P}} \delta \tilde{\varrho} \pm \delta \tilde{u} + \sqrt{\frac{c_P - c_V}{c_P}} \delta \tilde{T} \right), \quad \delta_{\text{th}}(k \rightarrow 0) = -\sqrt{\frac{c_P - c_V}{c_P}} \delta \tilde{\varrho} + \sqrt{\frac{c_V}{c_P}} \delta \tilde{T}. \quad (\text{VI.10})$$

The general structure revealed in this section will be our reference point for perturbative treatment of (V.13). It will allow us to identify Navier-Stokes hydrodynamics appearing in GHD equation supplemented with a collision term.

VII. PERTURBATION THEORY

As discussed in the main text the linearized problem (V.13) is solved by Fourier transform followed by perturbative solution in k . The dynamics of a single mode is

$$\delta q_n(x, t) \sim e^{ikx - \Lambda(k)t} \delta q_n(k). \quad (\text{VII.1})$$

Substituting this expression to the linearized GHD equations we obtain the following spectral problem

$$(\Gamma_{nm} + ikA_{nm} + k^2 D_{nm}) \delta q_m(k) = \Lambda(k) \delta q_m(k), \quad (\text{VII.2})$$

where summation over the repeated indices is assumed. We approach the problem perturbatively assuming that k is small. The spectrum consists then of gapped and gapless modes. The value of the gap is determined by the smallest non-zero eigenvalue of Γ . The gapped modes decay in time and become irrelevant at sufficiently large time scales. On the other hand, there are also gapless modes that are related to the collision invariants of Γ . In the logic of the perturbation theory in k the spectrum of the gapless modes has then the following structure

$$\Lambda(k) = ak + \omega k^2 + \mathcal{O}(k^3), \quad (\text{VII.3})$$

and importantly the linear coefficients depend only on A and therefore are universal, whereas the second order coefficients depend on both the collision integral and the GHD diffusion. The dependence on Γ enters through the second-order perturbation theory from coupling of the zero-modes with the dispersive ones. On the other hand, the dependence on D , comes directly from the first order perturbation theory because of the k^2 factor standing in front of it.

For the purpose of perturbation theory calculations it is convenient to introduce bra-ket notation such that $\Gamma_{nm} = \langle h_n | \Gamma | h_m \rangle$ and similarly for the other operators. More concretely,

$$\langle f | g \rangle = \int d\lambda f(\lambda) g(\lambda), \quad K | f \rangle = \int d\lambda K(\cdot, \lambda) f(\lambda), \quad (\text{VII.4})$$

with matrix elements understood in this notation as

$$A_{nm} = \langle h_n | A | h_m \rangle = \langle h_n | \mathcal{A} \mathcal{C} | h_m \rangle, \quad D_{nm} = \langle h_n | D | h_m \rangle = \frac{1}{2} \langle h_n | \mathfrak{D} \mathcal{C} | h_m \rangle, \quad \Gamma_{nm} = \langle h_n | \Gamma | h_m \rangle. \quad (\text{VII.5})$$

Where $\mathcal{A} = (1 - n\mathcal{T})^{-1}v(1 - n\mathcal{T})$ [1, 4] and operators $\mathcal{C}, \mathfrak{D}$ are discussed in Sec. I. Operator Γ is discussed in Sec. V with a concrete example in IX. Moreover, for the basis h_n introduced in Sec. V, the orthonormality condition is $\langle h_n | \mathcal{C} | h_m \rangle = \delta_{n,m}$.

With this notation, $|h_i\rangle$ with $i = 0, 1, 2$ are eigenstates of Γ with zero eigenvalues. We denote by $|g_\beta\rangle$ the eigenstates with non-zero eigenvalue $\gamma_\beta \neq 0$, $\Gamma |g_\beta\rangle = \gamma_\beta |g_\beta\rangle$. Whenever summation over β appears, it means summation over eigenvectors with non-zero eigenvalue. On the other hand, index α always corresponds to thermal and sound modes.

A. Perturbation theory at order k

At this order we can neglect the GHD diffusion. The linear part of the dispersion relation of the hydrodynamic modes is then found by solving the eigenproblem in the degenerate zero subspace of the collision integral. The problem

reduces to diagonalizing the following matrix

$$ikA_{\text{trunc}} = ik \begin{pmatrix} 0 & A_{1,0} & 0 \\ A_{1,0} & 0 & A_{2,1} \\ 0 & A_{2,1} & 0 \end{pmatrix}. \quad (\text{VII.6})$$

We find in total three modes with eigenvalues $\Lambda_{\pm} = \pm iv_s k$ (sound modes), and $\Lambda_{\text{th}} = 0$ (heat mode). The sound velocity given by the following formula, in agreement with thermodynamic sound velocity, see Sec. IV,

$$v_s = \sqrt{A_{1,0}^2 + A_{2,1}^2} = 1/\sqrt{\varrho \kappa_S}, \quad (\text{VII.7})$$

and associated eigenvectors are (normalization factor is $\mathcal{N} = \sqrt{1 + (A_{2,1}/A_{1,0})^2}$)

$$|h_{\pm}\rangle = \frac{1}{\sqrt{2}\mathcal{N}} \left(\frac{A_{1,0}}{A_{2,1}} |h_0\rangle \pm \mathcal{N} |h_1\rangle + |h_2\rangle \right), \quad |h_{\text{th}}\rangle = \frac{1}{\mathcal{N}} \left(-|h_0\rangle + \frac{A_{1,0}}{A_{2,1}} |h_2\rangle \right). \quad (\text{VII.8})$$

Therefore, in the first order of the perturbation theory, the degeneracy has been completely lifted. It is worth emphasising here that the found eigenvalues (yielding the speed of sound) are not eigenvalues of the full matrix A . These are eigenvalues of matrix A_{trunc} truncated to subspace of charges q_0, q_1, q_2 conserved by the perturbation Γ . Observe that eigenvectors (VII.8) may be written entirely in terms of the specific heats, using the relations between hydrodynamic matrices and thermodynamics derived in Sec. IV,

$$|h_{\pm}\rangle = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{c_V}{c_P}} |h_0\rangle \pm |h_1\rangle + \sqrt{\frac{c_P - c_V}{c_P}} |h_2\rangle \right), \quad |h_{\text{th}}\rangle = -\sqrt{\frac{c_P - c_V}{c_P}} |h_0\rangle + \sqrt{\frac{c_V}{c_P}} |h_2\rangle. \quad (\text{VII.9})$$

This result can be compared with known formulas for the modes in the classical three-dimensional ideal gas. For the ideal gas $c_P/c_V = 5/3$ and our formulas above reproduce results for sound and heat modes, when these values are substituted, see [13]. Furthermore using (VII.9), we also find useful relations for the sound modes in terms of the heat mode

$$|h_{\pm}\rangle = \sqrt{\frac{c_P - c_V}{2c_V}} |h_{\text{th}}\rangle + \sqrt{\frac{c_P}{2c_V}} |h_0\rangle \pm \frac{1}{\sqrt{2}} |h_1\rangle, \quad (\text{VII.10})$$

and heat mode in terms of ultra-local basis functions

$$|h_{\text{th}}\rangle = \frac{1}{\sqrt{\varrho T^2 c_P}} \left(|\mathcal{H}_2\rangle - \frac{\mathcal{B}_{2,1}}{\mathcal{B}_{1,0}} |\mathcal{H}_0\rangle \right). \quad (\text{VII.11})$$

Gapless modes are equal to the modes of Navier-Stokes equations

We complete considerations at this level of perturbation theory by noting that the found heat and sound modes coincide with the modes of Navier-Stokes equations (VI.10). For the purpose of this part, the bra-ket notation (which simplifies perturbation theory calculations) is not the most convenient. We therefore explicitly write sound and heat modes as

$$\delta q_{\pm} = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{c_V}{c_P}} \delta q_0 \pm \delta q_1 + \sqrt{\frac{c_P - c_V}{c_P}} \delta q_2 \right), \quad \delta q_{\text{th}} = -\sqrt{\frac{c_P - c_V}{c_P}} \delta q_0 + \sqrt{\frac{c_V}{c_P}} \delta q_2. \quad (\text{VII.12})$$

Already here we can see that these are exactly the modes of Navier-Stokes equation (VI.10), after the following identification is made

$$\delta q_0 = \delta \tilde{\varrho}, \quad \delta q_1 = \delta \tilde{u}, \quad \delta q_2 = \delta \tilde{T}. \quad (\text{VII.13})$$

We can also observe the truncated matrix (VII.6) expressed with thermodynamic quantities (IV.29) coincides with the relevant matrix (VI.8) for linearized Navier-Stokes equations, when terms of order k^2 are neglected (these terms do not influence the eigenvectors in $k \rightarrow 0$ limit)

$$ikA_{\text{trunc}} = ik \begin{pmatrix} 0 & \frac{1}{\sqrt{\varrho \kappa_T}} & 0 \\ \frac{1}{\sqrt{\varrho \kappa_T}} & 0 & \sqrt{\frac{T}{\varrho^2 c_V}} \left(\frac{\partial P}{\partial T} \right)_{\varrho} \\ 0 & \sqrt{\frac{T}{\varrho^2 c_V}} \left(\frac{\partial P}{\partial T} \right)_{\varrho} & 0 \end{pmatrix}. \quad (\text{VII.14})$$

The relation (VII.13) can also be justified from the very definition of the charges q_0, q_1, q_2 . To show this, we start with perturbations to ultra-local charges $\delta q_0, \delta q_1, \delta q_2$, see (V.7) and find

$$\delta q_0 = \sqrt{C_{0,0}} \delta q_0, \quad \delta q_1 = \sqrt{C_{1,1}} \delta q_1, \quad \delta q_2 = \sqrt{\frac{C_{2,2}C_{0,0} - C_{2,0}^2}{C_{0,0}}} \delta q_2 + \frac{C_{2,0}}{\sqrt{C_{0,0}}} \delta q_0. \quad (\text{VII.15})$$

Perturbations of ultra-local charges can be easily connected to perturbations of the fields $\delta \varrho, \delta u, \delta e$ via relations (II.7) and (II.21)

$$\delta \varrho = \delta q_0, \quad \delta u = \frac{1}{\varrho} \delta q_1, \quad \delta e = \frac{1}{\varrho} \delta q_2 - \frac{q_2}{\varrho^2} \delta q_0. \quad (\text{VII.16})$$

Lastly, to go to $\delta \varrho, \delta u, \delta T$ we need (VI.2). Combining all three transformations together with thermodynamic formulas from Sec.IV (in particular, (IV.33)), we find

$$\delta \varrho = \sqrt{T \varrho \left(\frac{\partial P}{\partial \varrho} \right)_T^{-1}} \delta q_0 = \varrho \sqrt{T \kappa_T} \delta q_0, \quad \delta u = \sqrt{T/\varrho} \delta q_1, \quad \delta T = T/\sqrt{\varrho c_V} \delta q_2. \quad (\text{VII.17})$$

These relations are obviously equivalent to (VII.13). Summing up, we have shown that perturbations of charges $\delta q_0, \delta q_1, \delta q_2$ are equal to perturbations of rescaled hydrodynamic fields.

B. Perturbation theory at order k^2

We move now to calculations at the k^2 order. As already seen in (VI.9), at this level we should expect transport coefficients ζ, κ appearing in corrections to the heat and sound modes. The perturbation theory will provide us with explicit expressions for transport coefficients.

The corrections of the order k^2 have two origins. There are second-order corrections coming from the coupling between the hydrodynamic modes and dispersive modes through the collision integral; we denote them $\omega_\alpha^{\text{coll}}$. There are also corrections coming from the GHD diffusion denoted $\omega_\alpha^{\text{diff}}$. Here $\alpha = \pm, \text{th}$ specifies one of the hydrodynamic modes. Thus, the general structure is

$$\Lambda_{\text{th}}(k) = (\omega_{\text{th}}^{\text{diff}} + \omega_{\text{th}}^{\text{coll}}) k^2 + \mathcal{O}(k^3), \quad (\text{VII.18})$$

$$\Lambda_{\pm}(k) = \pm i v_s k + (\omega_{\pm}^{\text{diff}} + \omega_{\pm}^{\text{coll}}) k^2 + \mathcal{O}(k^3).$$

The corrections due to the diffusion take a simple form. They are given by the first order perturbation theory in the diffusion operator $\omega_\alpha^{\text{diff}} = \langle h_\alpha | D | h_\alpha \rangle$. Because the diffusion operator is zero between eigenstates of different parity and $|h_0\rangle$ is its zero eigenvector (both left and right) there are only 2 non-zero matrix elements $D_{1,1}$ and $D_{2,2}$. In terms of them we find

$$\omega_{\text{th}}^{\text{diff}} = \frac{c_V}{c_P} D_{2,2}, \quad (\text{VII.19})$$

$$\omega_{\pm}^{\text{diff}} = \frac{1}{2} D_{1,1} + \frac{1}{2} \frac{c_P - c_V}{c_P} D_{2,2}.$$

Using the decomposition (VII.10) we also find

$$\omega_{\pm}^{\text{diff}} = \frac{c_P - c_V}{2c_V} \omega_{\text{th}}^{\text{diff}} + \frac{1}{2} D_{1,1}. \quad (\text{VII.20})$$

The second-order correction to the eigenvalues of the hydrodynamic modes come from the standard formula from the perturbation theory

$$\omega_\alpha^{\text{coll}} = \sum_{\beta} \frac{\langle h_\alpha | A | g_\beta \rangle \langle g_\beta | A | h_\alpha \rangle}{\gamma_\beta}, \quad \alpha = \pm, \text{th}, \quad (\text{VII.21})$$

where the sum does not include the kernel of operator Γ . Inspecting the formula we find $\omega_{\mp}^{\text{coll}} = \omega_{\mp}^{\text{coll}}$. This follows from the fact, that Γ operator has eigenfunctions which are either symmetric or antisymmetric functions of rapidities.

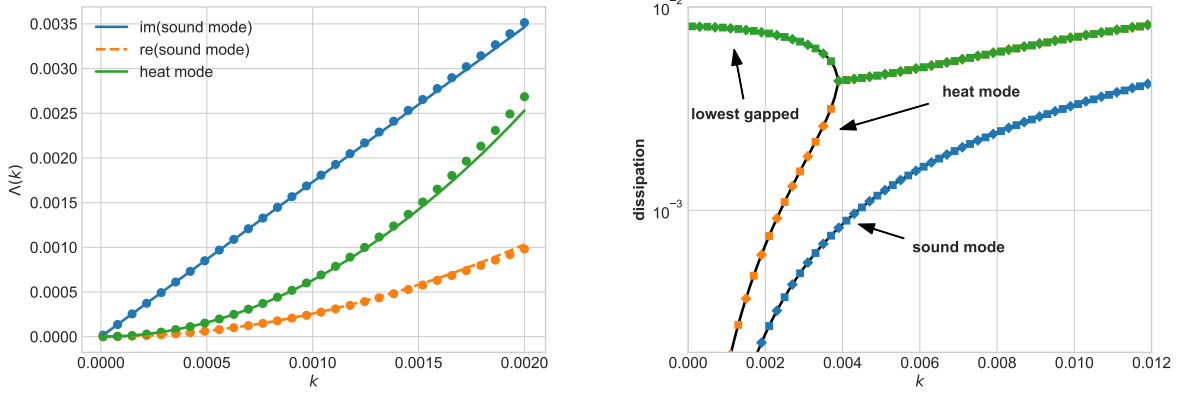


Figure 1: We present numerical solutions of (VII.2) for the coupled Lieb-Liniger model analyzed in the main text. The presented results are for $T = 2$, $n = 1$, $c = 2$ and $\tau = 0.01$. *Left*: small- k part of the spectrum of dispersion relations. For sufficiently small k , numerical results (points) agree with universal Navier-Stokes dispersion relations (VI.9) with transport coefficients (VII.23) (lines). At larger momenta we see deviations caused by the higher order corrections in k . We have multiplied the real parts of the dispersions by factor 5 to make them of the same order as the imaginary parts. *Right*: a numerical solution requires truncating the infinite system of equations to a finite one. Truncating to first 12 (squares), 18 (diamonds) and 24 (solid lines) charges yields results which differ on a scale below the resolution of the plot.

Note that action of A on $|h_0\rangle$ gives a mode proportional to $|h_1\rangle$ (this can be verified directly at the level of matrix elements (V.14)) and therefore the middle term from the decomposition (VII.10) does not contribute to $\omega_{\pm}^{\text{coll}}$ because $\langle g_{\beta}|h_1\rangle = 0$ from the definition of $|g_{\beta}\rangle$. In the expression for $\omega_{\pm}^{\text{coll}}$ there appear also mixed terms of the form $\langle h_{\text{th}}|A|g_{\beta}\rangle\langle g_{\beta}|A|h_1\rangle$. Such terms vanish because A is non-zero only for states of different parity. This allows us to write

$$\omega_{\pm}^{\text{coll}} = \frac{c_P - c_V}{2c_V}\omega_{\text{th}}^{\text{coll}} + \frac{1}{2}\omega_1^{\text{coll}}, \quad \omega_1^{\text{coll}} = \sum_{\beta} \frac{(\langle g_{\beta}|A|h_1\rangle)^2}{\gamma_{\beta}}. \quad (\text{VII.22})$$

C. Transport coefficients

We can finally read off the transport coefficients by comparing spectra (VI.9) of the linearized Navier-Stokes with the results of the perturbation theory. We find

$$\begin{aligned} \kappa^{\text{coll}} &= \varrho c_P \omega_{\text{th}}^{\text{coll}}, & \zeta^{\text{coll}} &= \varrho \omega_1^{\text{coll}}, \\ \kappa^{\text{diff}} &= \varrho c_V D_{2,2}, & \zeta^{\text{diff}} &= \varrho D_{1,1}. \end{aligned} \quad (\text{VII.23})$$

with $\kappa = \kappa^{\text{coll}} + \kappa^{\text{diff}}$ and $\zeta = \zeta^{\text{coll}} + \zeta^{\text{diff}}$.

We confirm these formulas on Fig.1, which presents small- k part of the spectrum found in numerical diagonalization of (VII.2). Our results agree with numerics in the small- k regime, as expected. The dispersion relations for larger number of modes and for higher values of k are presented in Fig.2. For small k the spectrum consists of Navier-Stokes modes and gapped modes. For high enough k , the spectrum is those of unperturbed GHD. This corresponds to operator Γ being negligible in the eigenproblem (VII.2), as compared to A and D . In between, there is a region which cannot be described neither by Navier-Stokes hydrodynamics, nor the pure diffusive GHD. Therefore, by numerically solving (VII.2) we are able to quantitatively capture the crossover from Navier-Stokes hydrodynamics to GHD.

Equations (VII.23) together with (VII.22) provide also an efficient way to evaluate the transport coefficients numerically. The procedure is the following. For a given thermal equilibrium state, we construct an orthonormal basis $\{|h_i\rangle\}$ using the hydrodynamic scalar product $(f|g)$. This allows us to compute the matrix elements A_{nm} , Γ_{nm} and D_{nm} . The matrix Γ_{nm} can be now diagonalized numerically. We obtain the spectrum γ_{β} but also expressions for $|g_{\beta}\rangle$ in terms of linear combinations of $|h_i\rangle$'s. This allows us to evaluate the matrix elements $\langle g_{\beta}|A|h_{\alpha}\rangle$ and $\omega_{\alpha}^{\text{coll}}$.

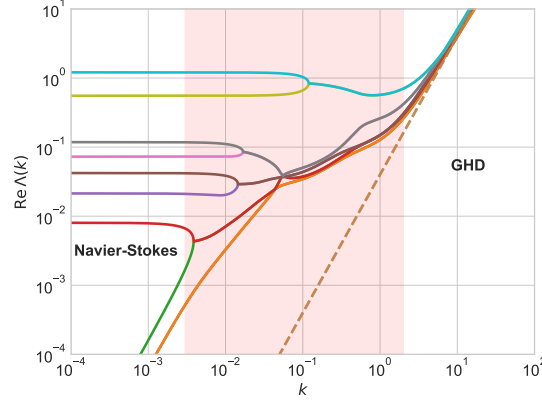


Figure 2: The real part of the dispersion relations for the lowest lying modes from the exact numerical solution of (VII.2). At small momenta, the spectrum is characterized by the 3 gapless modes (the orange curve is doubly degenerate) of the Navier-Stokes regime whereas at (relatively) large momenta by the modes of the GHD. The dashed lines are the gapless modes of the GHD without the integrability breaking terms. The joining and splitting of the modes defines an intermediate regime (shaded) where the dynamics cannot be reduced to either Navier-Stokes or unperturbed GHD.

The remaining ingredient is the heat capacity c_P which can be computed from the hydrodynamic matrices $\mathcal{B}$ and $\mathcal{C}$ according to (IV.27). The numerical solution requires truncating the infinite basis to a finite one. In Fig. 1 we show that it is sufficient to take around 20 basis functions.

Let us note that contributions κ^{diff} and ζ^{diff} can be also written as matrix elements in the ultra-local basis, as in the main text. Using relations (V.7), properties of the diffusion kernel and adopting notation of matrix elements of diffusion operator (V.14) as in the main text we find that indeed

$$\kappa^{\text{diff}} = \frac{1}{2T^2} \hbar_2 \mathcal{D} \mathcal{C} \hbar_2, \quad \zeta^{\text{diff}} = \frac{1}{2T} \hbar_1 \mathcal{D} \mathcal{C} \hbar_1. \quad (\text{VII.24})$$

Importantly $\kappa^{\text{diff}}, \zeta^{\text{diff}} \geq 0$, due to the positivity of operator $\mathcal{D} \mathcal{C}$ [3]. Moreover, it is easy to observe that $\omega_1^{\text{coll}}, \omega_{\text{th}}^{\text{coll}} \geq 0$ and hence $\kappa^{\text{coll}}, \zeta^{\text{coll}} \geq 0$. We thus conclude that transport coefficients found by us are non-negative, which assures that entropy production in the Navier-Stokes dynamics is positive [13].

In the next section we will confirm that κ^{coll} and ζ^{coll} found in the second-order perturbation theory (VII.23) can be written as generalized Chapman-Enskog expressions given in the main text.

VIII. INTEGRAL EQUATIONS FOR κ^{coll} AND ζ^{coll} TRANSPORT COEFFICIENTS

Expressions for the collisional transport coefficients involve sums over all non-zero eigenstates of the Γ operator. In the following, we will show how they can be rewritten as integral equations, which are identical to the integral equations found in the Chapman-Enskog procedure. We start with the thermal conductivity related to $\omega_{\text{th}}^{\text{coll}}$. As mentioned earlier for any β we have $\langle h_0 | A | g_\beta \rangle = 0$. Using explicit form (VII.11) of $|h_{\text{th}}\rangle$ in ultra-local basis we find

$$\langle h_{\text{th}} | A | g_\beta \rangle = \frac{1}{\sqrt{\varrho T^2 c_P}} \langle \hbar_2 | A | g_\beta \rangle, \quad (\text{VIII.1})$$

and thus we have

$$\omega_{\text{th}}^{\text{coll}} = \frac{1}{\sqrt{\varrho T^2 c_P}} \sum_{\beta} \frac{\langle \hbar_2 | A | g_\beta \rangle \langle g_\beta | A | h_{\text{th}} \rangle}{\gamma_{\beta}}. \quad (\text{VIII.2})$$

For the heat mode the sum can be easily extended to the whole space by noting that $A|h_{\text{th}}\rangle$ is orthogonal to the kernel of Γ . For $|h_0\rangle, |h_2\rangle$ it is clear from symmetry properties of A . For $|h_1\rangle$ we can show $\langle h_1 | A | h_{\text{th}} \rangle = 0$, which be

seen from explicit computation using (VII.11). This allows us to write

$$\omega_{\text{th}}^{\text{coll}} = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\varrho T^2 c_P}} \langle \mathcal{H}_2 | A \Gamma_\epsilon^{-1} A | h_{\text{th}} \rangle, \quad (\text{VIII.3})$$

where we used the completeness of the basis which implies that

$$\lim_{\epsilon \rightarrow 0} \sum_m |g_m\rangle \frac{1}{\gamma_m - \epsilon} \langle g_m| = \lim_{\epsilon \rightarrow 0} \frac{1}{\Gamma - \epsilon \mathcal{C}} \equiv \lim_{\epsilon \rightarrow 0} \Gamma_\epsilon^{-1}, \quad (\text{VIII.4})$$

where m extends now over all the eigenstates. Now, we can formally introduce vector $|\phi_\kappa\rangle$ such that,

$$\Gamma_\epsilon^{-1} A | h_{\text{th}} \rangle = \frac{T}{\sqrt{\varrho c_P}} |\phi_\kappa\rangle, \quad \omega_{\text{th}}^{\text{coll}} = \frac{1}{\varrho c_P} \langle \mathcal{H}_2 | A | \phi_\kappa \rangle. \quad (\text{VIII.5})$$

To find $|\phi_\kappa\rangle$ we can go to quasiparticle representation. We look for a function $\eta_\kappa(\lambda)$ which fulfills the following integral equation

$$\Gamma_\epsilon \phi_\kappa = (1 - n\mathcal{T})^{-1} \rho_P f \eta_\kappa, \quad (\text{VIII.6})$$

with function $\eta_\kappa(\lambda)$ related to heat mode and given by

$$\eta_\kappa(\lambda) = \frac{\sqrt{\varrho c_P}}{T} v(\lambda) h_{\text{th}}^{\text{dr}}(\lambda) = \frac{\sqrt{\varrho c_V}}{T} \left(h_2 - \sqrt{\frac{c_P - c_V}{c_V}} h_0 \right)^{\text{dr}}(\lambda). \quad (\text{VIII.7})$$

For $\epsilon \neq 0$, the operator Γ_ϵ is invertible and the solution to (VIII.6) is unique. However, we would like to explicitly take the limit $\epsilon \rightarrow 0$, which will invalidate uniqueness of the solution. This is clear from the existence of collision invariants of Γ - for any particular solution $\bar{\phi}_\kappa$ to the equation (VIII.6) with $\epsilon = 0$ the function

$$\bar{\phi}_\kappa + \sum_{i=0}^2 a_i \mathcal{H}_i \quad (\text{VIII.8})$$

is another solution (coefficients a_i are arbitrary). Therefore, before we set $\epsilon = 0$, let us multiply (VIII.6) with collision invariants $\mathcal{H}_i(\lambda)$ where $i = 0, 1, 2$ and integrate over $d\lambda$. This leads to the conditions

$$\langle \mathcal{H}_i | \mathcal{C} | \phi_\kappa \rangle = \langle \mathcal{H}_i | \phi_\kappa \rangle = 0, \quad (\text{VIII.9})$$

which we keep after taking $\epsilon \rightarrow 0$. The conditions above make the solution unique by fixing the constants a_i in the general solution.

In the main text we shorten the notation introducing $\chi_\kappa = (1 - n\mathcal{T})^{-1} \rho_P f \eta_\kappa$, which explicitly reads

$$\chi_\kappa = \frac{\sqrt{\varrho c_V}}{T} \mathcal{B}(h_2 - \sqrt{c_P/c_V - 1} h_0). \quad (\text{VIII.10})$$

The integral equation can be then written as $\Gamma \phi_\kappa = \chi_\kappa$, as in the main text. Finally, thermal conductivity κ^{coll} is given by

$$\kappa^{\text{coll}} = \int d\lambda \rho_P(\lambda) f(\lambda) v(\lambda) \mathcal{H}_2^{\text{dr}}(\lambda) \phi_\kappa^{\text{dr}}(\lambda) = \mathcal{H}_2 \mathcal{B} \phi_\kappa. \quad (\text{VIII.11})$$

We focus now on ζ^{coll} . We proceed similarly, with the exception that now $\langle g_\beta | A | h_1 \rangle \neq 0$, therefore we have to properly regularize matrix elements before we introduce representation of inverse Γ . We start with expression

$$\langle g_\beta | A | h_1 \rangle, \quad (\text{VIII.12})$$

which enters the sum for ζ^{coll} . We can add to it two zero terms, namely:

$$\langle g_\beta | A | h_1 \rangle + a \langle g_\beta | h_0 \rangle + b \langle g_\beta | h_2 \rangle. \quad (\text{VIII.13})$$

We want a, b to be such that the vector $|w_1\rangle = A|h_1\rangle + a|h_0\rangle + b|h_2\rangle$ is orthogonal to $|h_0\rangle, |h_1\rangle, |h_2\rangle$. We find $a = -A_{1,0}, b = -A_{2,1}$. Thus, we write (we have used here that $|h_1\rangle = 1/\sqrt{\varrho T} |\mathcal{H}_1\rangle$, see (V.7) and Sec.IV)

$$\omega_1^{\text{coll}} = \frac{1}{\sqrt{\varrho T}} \lim_{\epsilon \rightarrow 0} \langle \mathcal{H}_1 | A \Gamma_\epsilon^{-1} | w_1 \rangle. \quad (\text{VIII.14})$$

As previously, we introduce a vector $|\phi_\zeta\rangle$ such that $\Gamma_\epsilon^{-1}|w_1\rangle = \sqrt{T/\varrho}|\phi_\zeta\rangle$. In quasiparticle representation finding such vector corresponds to solving the following integral equation

$$\Gamma_\epsilon \phi_\zeta = (1 - n\mathcal{T})^{-1} \rho_p f \eta_\zeta, \quad (\text{VIII.15})$$

with function (we have used here (IV.29))

$$\eta_\zeta(\lambda) = \sqrt{\frac{\varrho}{T}} \left[v(\lambda) h_1^{\text{dr}}(\lambda) - \frac{1}{\sqrt{\varrho \kappa_T}} \left(h_0 + \sqrt{\frac{c_P - c_V}{c_V}} h_2 \right)^{\text{dr}}(\lambda) \right]. \quad (\text{VIII.16})$$

Similarly as for thermal conductivity we set $\epsilon \rightarrow 0$ by imposing additional conditions $(\mathcal{H}_i|\phi_\zeta) = 0$, $i = 0, 1, 2$. We shorten the notation and introduce $\chi_\zeta = (1 - n\mathcal{T})^{-1} \rho_p f \eta_\zeta$, which explicitly reads

$$\chi_\zeta = \sqrt{\frac{\varrho}{T}} \mathcal{B} h_1 - \frac{1}{\sqrt{T \kappa_T}} \mathcal{C} (h_0 + \sqrt{c_P/c_V - 1} h_2). \quad (\text{VIII.17})$$

The integral equation is then $\Gamma \phi_\zeta = \chi_\zeta$ and the bulk viscosity ζ^{coll} finally is

$$\zeta^{\text{coll}} = \int d\lambda \rho_p(\lambda) f(\lambda) v(\lambda) \mathcal{H}_1^{\text{dr}}(\lambda) \phi_\zeta^{\text{dr}}(\lambda) = \mathcal{H}_1 \mathcal{B} \phi_\zeta. \quad (\text{VIII.18})$$

In this way, we have shown that results from perturbation theory can be reformulated as generalized Chapman-Enskog integral equations, given in the main text. One of the advantages of integral equation approach is simple analysis of the non-interacting limit of the underlying integrable model. This is done in the next section.

Non-interacting limit of transport coefficients

In this section we look more closely on the limit of non-interacting integrable model with trivial dressings. Transport coefficients (VII.23) have contributions from collision integral and from diffusion operator. In the non-interacting limit, diffusion operator vanishes together with the associated contributions to the transport coefficients. We thus turn to contributions from collision integral, which in principle may be non-zero even in the limit of free integrable model. Obviously, we still consider non-zero integrability breaking term $\mathcal{I}$.

In the case of trivial dressings, the following symmetries appear on the level of hydrodynamic matrices

$$\mathcal{B}_{1,0} = 2\mathcal{C}_{2,0}, \quad \mathcal{B}_{2,1} = 2\mathcal{C}_{2,2}. \quad (\text{VIII.19})$$

Let us look how these affect expressions for $\eta_{\kappa,\zeta}(\lambda)$ given by (VIII.7), (VIII.16). It is convenient to represent these functions in the ultra-local basis first, we find

$$\eta_\kappa(\lambda) = \frac{1}{T^2} v(\lambda) \left(\mathcal{H}_2^{\text{dr}}(\lambda) - \frac{\mathcal{B}_{2,1}}{\mathcal{B}_{1,0}} \mathcal{H}_0^{\text{dr}}(\lambda) \right), \quad (\text{VIII.20})$$

$$\eta_\zeta(\lambda) = \frac{1}{T} \left(v(\lambda) \mathcal{H}_1^{\text{dr}}(\lambda) + \frac{\mathcal{B}_{2,1}\mathcal{C}_{2,0} - \mathcal{B}_{1,0}\mathcal{C}_{2,2}}{\mathcal{C}_{2,2}\mathcal{C}_{0,0} - \mathcal{C}_{2,0}^2} \mathcal{H}_0^{\text{dr}}(\lambda) - \frac{\mathcal{B}_{2,1}\mathcal{C}_{0,0} - \mathcal{B}_{1,0}\mathcal{C}_{2,0}}{\mathcal{C}_{2,2}\mathcal{C}_{0,0} - \mathcal{C}_{2,0}^2} \mathcal{H}_2^{\text{dr}}(\lambda) \right), \quad (\text{VIII.21})$$

where we have expressed thermodynamic quantities with hydrodynamic matrices, see Sec. IV. Moreover, we have represented GS-orthonormalized vectors (V.7) with ultra-local counterparts. Let us consider a limit, where dressing can be neglected and hence $v(\lambda) = \mathcal{H}_1(\lambda)$. Taking into account also the symmetries (VIII.19), we observe

$$\eta_\kappa(\lambda) \rightarrow \frac{1}{T^2} \mathcal{H}_1(\lambda) \left(\mathcal{H}_2(\lambda) - \frac{\mathcal{B}_{2,1}}{\mathcal{B}_{1,0}} \mathcal{H}_0(\lambda) \right), \quad (\text{VIII.22})$$

$$\eta_\zeta(\lambda) \rightarrow 0. \quad (\text{VIII.23})$$

The rate on which $\eta_\zeta(\lambda)$ approaches zero with vanishing coupling depends on the model. This might be important in cases where Γ vanishes in the free limit as well. When $\eta_\zeta(\lambda)$ approaches zero faster than Γ with the interaction coupling, the bulk viscosity tends to 0. This is the case for collision integral studied by us. On the other hand, as $\eta_\kappa(\lambda)$ remains finite in the non-interacting limit, the Γ operator, which vanishes in the free limit leads then to infinite thermal conductivity. This is a known result for non-interacting gas [13] and we observe it also for collision integral considered by us (see Fig. 2 in the main text).

IX. COLLISION INTEGRAL FOR THE COUPLED LIEB-LINIGER MODELS

In this supplementary material we derive the linearized collision integral for the two coupled Lieb-Liniger models. The tubes are coupled with intertube interaction potential $V_{\text{inter}}(x-y) = V_0 V(x-y)$. The resulting collision integral has all the necessary properties which, we have assumed about the Γ operator and will allow us to illustrate our general formalism in an experimentally relevant situation.

Generally speaking there are two contributions to the collision integral $\mathcal{I}$. The direct contributions comes from the very scattering events caused by the perturbing potential. But there is also an indirect contribution, that arises because a modification to the distribution of rapidities in one place (in the rapidity space) reshuffles rapidities elsewhere due to the interaction present in the Lieb-Liniger model. The two contributions together appear as a dressing of the bare scattering integral $\mathcal{I}_0$

$$\mathcal{I}[\rho_{\text{p},1}, \rho_{\text{p},2}](\lambda) = \tau_0^{-1} \int d\mu \mathbb{F}(\lambda, \mu) \mathcal{I}_0[\rho_{\text{p},1}, \rho_{\text{p},2}](\mu), \quad (\text{IX.1})$$

where $\mathbb{F}$ is backflow operator introduced in (V.18) and timescale $\tau_0^{-1} = 16E_F/(\hbar\pi^2) \times \gamma_{\text{inter}}^2 = \frac{2V_0^2 m}{\hbar^3}$ where E_F is Fermi energy of a single tube and $\gamma_{\text{inter}} = V_0 m/(\rho\hbar^2)$ where ρ is the density of the gas [11].

As discussed in [11, 15] in the homogeneous case when two tubes are in the same state the leading contribution to the collision integral comes from processes of type (1, 2) and (2, 1). These are processes in which 1 particle-hole pair is created in the first tube and 2 particle-hole pair is created in the second tube or *vice versa*, $\mathcal{I}[\rho_{\text{p}}] = \mathcal{I}^{(1,2)}[\rho_{\text{p}}] + \mathcal{I}^{(2,1)}[\rho_{\text{p}}] + (\dots)$, where $(\dots)$ stands for processes of higher order in momentum transferred between the tubes.

The bare collision integral for these processes is [11, 15]

$$\begin{aligned} \mathcal{I}_0[\rho_{\text{p}}](\lambda) = & \frac{(2\pi)^2}{(2!)^2} \int dp_{1,1} dh_{1,1} dp_{1,2} dh_{1,2} dp_2 dh_2 (\delta(\lambda - p_{1,1}) + \delta(\lambda - p_{1,2}) + \delta(\lambda - p_2)) \tilde{V}^2(k_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1)) \times \\ & |F_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1)|^2 |F_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2)|^2 \delta(k_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) + k_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2)) \delta(\omega_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) + \omega_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2)) \times \\ & (J_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) J_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2) - (\mathbf{h} \leftrightarrow \mathbf{p})), \end{aligned} \quad (\text{IX.2})$$

where $\tilde{V}(k)$ is Fourier transform of the interaction potential with $\tilde{V}(k) = \tilde{V}(-k)$, $F_{n\text{ph}}(\mathbf{p}, \mathbf{h})$ are density operator form factors and

$$k_{n\text{ph}}(\mathbf{p}, \mathbf{h}) = \sum_{i=1}^n (k(p_i) - k(h_i)), \quad \omega_{n\text{ph}}(\mathbf{p}, \mathbf{h}) = \sum_{i=1}^n (\omega(p_i) - \omega(h_i)). \quad (\text{IX.3})$$

where $k(\lambda) = \hbar_1^{\text{Dr}}(\lambda)$ and $\omega(\lambda) = \hbar_2^{\text{Dr}}(\lambda)$. Finally, J factors are

$$J_{n\text{ph}}(\mathbf{p}, \mathbf{h}) = \prod_{i=1}^n \rho_{\text{p}}(h_i) \rho_{\text{h}}(p_i). \quad (\text{IX.4})$$

We linearize now the collision integral around a thermal state. Any thermal state is a stationary state fulfilling the condition

$$J_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) J_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2) - J_{2\text{ph}}(\mathbf{h}_1, \mathbf{p}_1) J_{1\text{ph}}(\mathbf{h}_2, \mathbf{p}_2) = 0, \quad (\text{IX.5})$$

on a 4-dimensional manifold given by $\omega_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) + \omega_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2) = 0$ and $k_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) + k_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2) = 0$. To expand the collision integral around the thermal state it is useful to switch from the distribution functions to the pseudoenergy $\epsilon(\lambda)$ for which we have

$$\rho_{\text{p}}(\lambda) = \frac{\rho_{\text{tot}}(\lambda)}{1 + e^{\epsilon(\lambda)}}, \quad \rho_{\text{h}}(\lambda) = \frac{\rho_{\text{tot}}(\lambda)}{1 + e^{-\epsilon(\lambda)}}. \quad (\text{IX.6})$$

The ratio of the J -factors has then a simple form

$$\frac{J_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) J_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2)}{J_{2\text{ph}}(\mathbf{h}_1, \mathbf{p}_1) J_{1\text{ph}}(\mathbf{h}_2, \mathbf{p}_2)} = \exp(\epsilon(p_{1,1}) - \epsilon(h_{1,1}) + \epsilon(p_{1,2}) - \epsilon(h_{1,2}) + \epsilon(p_{2,1}) - \epsilon(h_{2,1})), \quad (\text{IX.7})$$

and the argument of the exponential vanishes for thermal states for which $\epsilon(\lambda) = \beta_2 \omega(\lambda) + \beta_0$.

To linearize the collision integral we consider a deviation from the thermal state $\epsilon \rightarrow \epsilon + \delta\epsilon$ and look at $\mathcal{I}[\epsilon + \delta\epsilon]$. To understand the structure of $\mathcal{I}[\epsilon + \delta\epsilon]$ we analyze first $\epsilon + \delta\epsilon$. From (III.8) we conclude that

$$\epsilon'(\lambda) = (\epsilon'_0)^{\text{Dr}}(\lambda). \quad (\text{IX.8})$$

Using now the relation [12] between large and small dressing operations $(f^{\text{Dr}})'(\lambda) = (f')^{\text{Dr}}(\lambda)$ and integrating we find

$$\epsilon(\lambda) = (\epsilon_0)^{\text{Dr}}(\lambda) + \text{const} \quad (\text{IX.9})$$

The same reasoning can be applied to the perturbed state, we get

$$\epsilon(\lambda) + \delta\epsilon(\lambda) + \mathcal{O}((\delta\epsilon_0)^2) = (\epsilon_0 + \delta\epsilon_0)^{\text{Dr} + \delta\text{Dr}}(\lambda) + \text{const}, \quad (\text{IX.10})$$

where by $\text{Dr} + \delta\text{Dr}$ we denoted Dressing implemented by the state with occupation function $n + \delta n$. The expression above may be expanded according to

$$\epsilon(\lambda) + \delta\epsilon(\lambda) = \epsilon_0^{\text{Dr} + \delta\text{Dr}}(\lambda) + (\delta\epsilon_0)^{\text{Dr} + \delta\text{Dr}} + \text{const} = \epsilon_0^{\text{Dr} + \delta\text{Dr}}(\lambda) + (\delta\epsilon_0)^{\text{Dr}}(\lambda) + \text{const} + \mathcal{O}((\delta\epsilon_0)^2). \quad (\text{IX.11})$$

With this, we evaluate $\mathcal{I}[\epsilon + \delta\epsilon]$. The result is

$$\mathcal{I}_0[\rho] = \Gamma_0[\rho_{\text{th}}](\delta\epsilon_0)^{\text{Dr}} + \mathcal{O}((\delta\epsilon)^2), \quad (\text{IX.12})$$

where

$$\begin{aligned} \Gamma_0(\lambda, \mu) = & \frac{(2\pi)^2}{8} \int dp_{1,1} dh_{1,1} dp_{1,2} dh_{1,2} dp_2 dh_2 G(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) \times \\ & \left(\delta(\lambda - p_{1,1}) - \delta(\lambda - h_{1,1}) + \delta(\lambda - p_{1,2}) - \delta(\lambda - h_{1,2}) + \delta(\lambda - p_2) - \delta(\lambda - h_2) \right) \times \\ & \left(\delta(\mu - p_{1,1}) - \delta(\mu - h_{1,1}) + \delta(\mu - p_{1,2}) - \delta(\mu - h_{1,2}) + \delta(\mu - p_2) - \delta(\mu - h_2) \right), \end{aligned} \quad (\text{IX.13})$$

with

$$\begin{aligned} G(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) = & \tilde{V}^2(k_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1)) |F_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1)|^2 |F_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2)|^2 J_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) J_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2) \times \\ & \delta(k_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) + k_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2)) \delta(\omega_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) + \omega_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2)). \end{aligned} \quad (\text{IX.14})$$

Note that $G(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) = G(\mathbf{h}_1, \mathbf{p}_1, \mathbf{h}_2, \mathbf{p}_2)$, namely function G is ph-symmetric. This implies that operator Γ_0 is symmetric, $\Gamma_0(\lambda, \mu) = \Gamma_0(\mu, \lambda)$ leading to $\Gamma_{mn} = \Gamma_{nm}$. It is also easy to observe that $\Gamma_{nn} \geq 0$, hence Γ is positive semi-definite.

What is more, Γ_0 has an additional symmetry. By performing change of variables in the integrals (changing sign in each one) we get $\Gamma_0(\lambda, \mu) = \Gamma_0(-\lambda, -\mu)$. This property leads to certain selection rules once we consider the action of Γ in the basis of ultra-local conserved charges.

The linearized collision integral has also three left eigenvectors with zero eigenvalues [15]. They reflect the fact that the total number of particles, the total momentum and the total energy are conserved by the perturbation,

$$\int d\lambda \mathcal{N}_0^{\text{Dr}}(\lambda) \Gamma_0(\lambda, \mu) = \int d\lambda \mathcal{N}_1^{\text{Dr}}(\lambda) \Gamma_0(\lambda, \mu) = \int d\lambda \mathcal{N}_2^{\text{Dr}}(\lambda) \Gamma_0(\lambda, \mu) = 0, \quad (\text{IX.15})$$

where to derive the last two properties we used conservation laws under the integral.

We consider now matrix elements of Γ in the basis of the ultra-local conserved charges. The matrix elements are defined according to eq. (V.14),

$$\Gamma_{nm} = \tau_0^{-1} \int d\lambda d\mu h_n^{\text{Dr}}(\lambda) \Gamma_0(\lambda, \mu) h_m^{\text{Dr}}(\mu). \quad (\text{IX.16})$$

From the construction of our basis (see Sec. V), functions h_n are either symmetric or antisymmetric (and dressings do not change this property). This implies that $\Gamma_{nm} = 0$ whenever functions h_n and h_m have different parity. According to the above discussion on the zero eigenvectors we also have $\Gamma_{nm} = 0$ for $n = 0, 1, 2$. Moreover, it is straightforward to observe that $\Gamma_{nm} = 0$ for $m = 0, 1, 2$ as well.

Small momentum limit

The matrix elements Γ_{nm} read explicitly

$$\begin{aligned} \Gamma_{nm} = & \tau_0^{-1} \times \frac{(2\pi)^2}{8} \int dp_{1,1} dh_{1,1} dp_{1,2} dh_{1,2} dp_2 dh_2 G(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2) \times \\ & \left(h_n^{\text{Dr}}(p_{1,1}) - h_n^{\text{Dr}}(h_{1,1}) + h_n^{\text{Dr}}(p_{1,2}) - h_n^{\text{Dr}}(h_{1,2}) + h_n^{\text{Dr}}(p_2) - h_n^{\text{Dr}}(h_2) \right) \times \\ & \left(h_m^{\text{Dr}}(p_{1,1}) - h_m^{\text{Dr}}(h_{1,1}) + h_m^{\text{Dr}}(p_{1,2}) - h_m^{\text{Dr}}(h_{1,2}) + h_m^{\text{Dr}}(p_2) - h_m^{\text{Dr}}(h_2) \right). \end{aligned} \quad (\text{IX.17})$$

We will find now a much simpler expression for them in the small momentum limit. To this end we express the integrand in terms of center of mass rapidities and the momentum transferred between the tubes.

First, we change the integration variables to the center-of-mass rapidities. These are defined through $\lambda = (p+h)/2$ and $\alpha = p-h$ for each particle-hole excitation. The Jacobian of the transformation from (p, h) to (λ, α) is 1 hence this amounts to simply replacing the integration measure by $d\lambda_{1,1} d\lambda_{1,2} d\lambda_2 d\alpha_1 d\alpha_2$. We then assume that the main contribution to the integral comes from the small momentum excitations. Therefore, the integrand can be expanded in small α 's. Specifically, the energy-momentum constraints become

$$k'(\lambda_{1,1})\alpha_{1,1} + k'(\lambda_{1,2})\alpha_{1,2} + k'(\lambda_2)\alpha_2 = 0, \quad (\text{IX.18a})$$

$$\omega'(\lambda_{1,1})\alpha_{1,1} + \omega'(\lambda_{1,2})\alpha_{1,2} + \omega'(\lambda_2)\alpha_2 = 0. \quad (\text{IX.18b})$$

Solving them for $\alpha_{1,1}$ and $\alpha_{1,2}$ gives

$$\bar{\alpha}_{1,1} = -\alpha_2 \frac{k'(\lambda_2)}{k'(\lambda_{1,1})} \frac{v(\lambda_2) - v(\lambda_{1,2})}{v(\lambda_{1,1}) - v(\lambda_{1,2})}, \quad \bar{\alpha}_{1,2} = -\alpha_2 \frac{k'(\lambda_2)}{k'(\lambda_{1,2})} \frac{v(\lambda_2) - v(\lambda_{1,1})}{v(\lambda_{1,2}) - v(\lambda_{1,1})}. \quad (\text{IX.19})$$

We can now use the following property of δ -functions

$$\delta(f(x, y))\delta(g(x, y)) = \frac{\delta(x - \bar{x})\delta(y - \bar{y})}{|\det J|}, \quad J = \begin{pmatrix} \partial_x f(x, y) & \partial_y f(x, y) \\ \partial_x g(x, y) & \partial_y g(x, y) \end{pmatrix}, \quad (\text{IX.20})$$

where $(\bar{x}, \bar{y})$ is a zero of $(f(x, y), g(x, y))$. This gives for the δ -functions implementing the conservation of energy-momentum the following expression

$$\delta(k_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) + k_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2)) \delta(\omega_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) + \omega_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2)) = \frac{\delta(\alpha_{1,1} - \bar{\alpha}_{1,1})\delta(\alpha_{1,2} - \bar{\alpha}_{1,2})}{k'(\lambda_{1,1})k'(\lambda_{1,2})|v(\lambda_{1,1}) - v(\lambda_{1,2})|}. \quad (\text{IX.21})$$

After these preparational steps we evaluate the integrals over $\alpha_{1,1}$ and $\alpha_{1,2}$ with the help of the Dirac δ -functions. This yields

$$\begin{aligned} \Gamma_{nm} = & \tau_0^{-1} \times \frac{(2\pi)^2}{8} \int d\lambda_{1,1} d\lambda_{1,2} d\lambda_2 d\alpha_2 \frac{G(\mathbf{p}_1, \mathbf{h}_1, \mathbf{p}_2, \mathbf{h}_2)}{k'(\lambda_{1,1})k'(\lambda_{1,2})|v(\lambda_{1,1}) - v(\lambda_{1,2})|} \times \\ & \left((h_n^{\text{Dr}})'(\lambda_{1,1})\bar{\alpha}_{1,1} + (h_n^{\text{Dr}})'(\lambda_{1,2})\bar{\alpha}_{1,2} + (h_n^{\text{Dr}})'(\lambda_2)\alpha_2 \right) \times \\ & \left((h_m^{\text{Dr}})'(\lambda_{1,1})\bar{\alpha}_{1,1} + (h_m^{\text{Dr}})'(\lambda_{1,2})\bar{\alpha}_{1,2} + (h_m^{\text{Dr}})'(\lambda_2)\alpha_2 \right). \end{aligned} \quad (\text{IX.22})$$

The second step is to evaluate the integral over α_2 which is proportional to the momentum transferred between the tubes. To this end we observe that the combinations of functions appearing in the last two lines of Γ_{nm} can be rewritten according to

$$\bar{\alpha}_{1,1}\varphi(\lambda_{1,1}) + \bar{\alpha}_{1,2}\varphi(\lambda_{1,2}) + \alpha_2\varphi(\lambda_2) = \frac{\alpha_2 k'(\lambda_2)}{v(\lambda_{1,1}) - v(\lambda_{1,2})} \mathcal{S}[\varphi], \quad (\text{IX.23})$$

where we introduced symbol $\mathcal{S}[\varphi]$ defined by

$$\mathcal{S}[\varphi] = \frac{\varphi'(\lambda_{1,1})}{k'(\lambda_{1,1})}(v(\lambda_{1,2}) - v(\lambda_2)) + \frac{\varphi'(\lambda_{1,2})}{k'(\lambda_{1,2})}(v(\lambda_2) - v(\lambda_{1,1})) + \frac{\varphi'(\lambda_2)}{k'(\lambda_2)}(v(\lambda_{1,1}) - v(\lambda_{1,2})). \quad (\text{IX.24})$$

We note that $\mathcal{S}[\varphi]$ is zero whenever any two of the 3 variables $\lambda_{1,1}$, $\lambda_{1,2}$ or λ_2 are equal. Moreover, it also vanishes identically for constant φ or $\varphi(\lambda) = k(\lambda), \omega(\lambda)$, and these functions naturally correspond to the collision invariants. In the limit of small momentum transfer between the tubes we can also simplify function G . The density factors and the form-factors are independent of α_2 in this limit. The density factors are $J_{\text{mph}}(\mathbf{p}, \mathbf{h}) = J_{\text{mph}}(\boldsymbol{\lambda}, \boldsymbol{\lambda})$, while the form-factors yield [16–18]

$$F_{1\text{ph}}(\mathbf{p}_2, \mathbf{h}_2) = k'(\lambda_2), \quad (\text{IX.25})$$

$$F_{2\text{ph}}(\mathbf{p}_1, \mathbf{h}_1) = \frac{2\pi k^2 \mathcal{T}^{\text{dr}}(\lambda_{1,1}, \lambda_{1,2})}{k'(\lambda_{1,1})k'(\lambda_{1,2})\bar{\alpha}_{1,1}\bar{\alpha}_{1,2}} = -2\pi \mathcal{T}^{\text{dr}}(\lambda_{1,1}, \lambda_{1,2}) \frac{(v(\lambda_{1,1}) - v(\lambda_{1,2}))^2}{(v(\lambda_2) - v(\lambda_{1,2}))(v(\lambda_2) - v(\lambda_{1,1}))}, \quad (\text{IX.26})$$

with $k = k'(\lambda_{1,1})\bar{\alpha}_{1,1} + k'(\lambda_{1,2})\bar{\alpha}_{1,2}$. Here, $\mathcal{T}^{\text{dr}}$ is the dressed integral kernel defined in (I.8). With this we find

$$\begin{aligned} \Gamma_{nm} = & \tau_0^{-1} \times \frac{(2\pi)^4}{8} \int d\lambda_{1,1} d\lambda_{1,2} d\lambda_2 d\alpha_2 \tilde{V}^2(k'(\lambda_2)\alpha_2) J_{2\text{ph}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1) J_{1\text{ph}}(\lambda_2, \lambda_2) \times \\ & \frac{\alpha_2^2 k'(\lambda_2)^4 \mathcal{T}^{\text{dr}}(\lambda_{1,1}, \lambda_{1,2})^2}{k'(\lambda_{1,1})k'(\lambda_{1,2})} \frac{|v(\lambda_{1,1}) - v(\lambda_{1,2})|}{(v(\lambda_2) - v(\lambda_{1,2}))^2 (v(\lambda_2) - v(\lambda_{1,1}))^2} \mathcal{S}[h_n^{\text{Dr}}] \mathcal{S}[h_m^{\text{Dr}}]. \end{aligned} \quad (\text{IX.27})$$

The integral over α_2 can be now performed. The result is the following

$$\begin{aligned} \Gamma_{nm} = & \tau_0^{-1} \times \frac{(2\pi)^4 \tilde{V}^2}{8} \int d\lambda_{1,1} d\lambda_{1,2} d\lambda_2 \frac{k'(\lambda_2) \mathcal{T}^{\text{dr}}(\lambda_{1,1}, \lambda_{1,2})^2 J_{2\text{ph}}(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1) J_{1\text{ph}}(\lambda_2, \lambda_2)}{k'(\lambda_{1,1})k'(\lambda_{1,2})} \times \\ & \frac{|v(\lambda_{1,1}) - v(\lambda_{1,2})|}{(v(\lambda_2) - v(\lambda_{1,2}))^2 (v(\lambda_2) - v(\lambda_{1,1}))^2} \mathcal{S}[h_n^{\text{Dr}}] \mathcal{S}[h_m^{\text{Dr}}], \end{aligned} \quad (\text{IX.28})$$

where $\tilde{V}^2 = \int dk \tilde{V}^2(k) k^2$. It is convenient to rewrite Γ_{nm} in the following form, which appears in the main text

$$\Gamma_{nm} = \frac{2\pi}{\tau} \int d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_2 \mathcal{T}^{\text{dr}}(\lambda_1, \lambda_2)^2 |v(\lambda_1) - v(\lambda_2)| g_{nm}(\lambda_1, \lambda_2), \quad (\text{IX.29})$$

where $\tau^{-1} = \frac{\tilde{V}_{\text{inter}}^2 m}{4\hbar^3}$ and $\tilde{V}_{\text{inter}}^2 = V_0^2 \tilde{V}^2$, moreover

$$g_{nm}(\lambda_1, \lambda_2) = \int d\boldsymbol{\lambda} (k'(\lambda))^2 \frac{\mathcal{S}[h_n^{\text{Dr}}] \mathcal{S}[h_m^{\text{Dr}}]}{(v - v_1)^2 (v - v_2)^2}. \quad (\text{IX.30})$$

where the measure is $d\boldsymbol{\lambda} = 2\pi \rho_p(\lambda) \rho_h(\lambda) / k'(\lambda) d\lambda = \rho_p(\lambda) f(\lambda) d\lambda$.

The apparent poles in Γ_{nm} at $\lambda_{1,1} = \lambda$ and $\lambda_{1,2} = \lambda$ are regularized by the expressions in the brackets which vanish in those limits. The double limit when all the λ 's are equal is additionally regularized (and vanishes) because of the factor $|v(\lambda_{1,1}) - v(\lambda_{1,2})|$ in the numerator.

Explicitly, we check (we shorten the notation and introduce $\tilde{\varphi} = \varphi'/k'$, moreover $v = v(\lambda)$ and $v_i = v(\lambda_i)$, similarly for $\tilde{\varphi}$)

$$\begin{aligned} \lim_{v_2 \rightarrow v} \frac{\mathcal{S}[\varphi]}{v_2 - v} &= \tilde{\varphi}_1 + \lim_{v_2 \rightarrow v} \frac{\tilde{\varphi}_2(v - v_1) + \tilde{\varphi}(v_1 - v_2)}{v_2 - v} \\ &= \tilde{\varphi}_1 - v_1 \lim_{v_2 \rightarrow v} \frac{\tilde{\varphi}_2 - \tilde{\varphi}}{v_2 - v} + \lim_{v_2 \rightarrow v} \frac{\tilde{\varphi}_2 v - \tilde{\varphi} v_2}{v_2 - v} \\ &= \tilde{\varphi}_1 - \tilde{\varphi} - (v_1 - v) \tilde{\varphi}'. \end{aligned} \quad (\text{IX.31})$$

The other apparent pole at $v = v_1$ is regularized in the same way. This a consequence of symmetry of $\mathcal{S}$ under a cyclic transformation of variables, $\mathcal{S}[\varphi](\lambda, \lambda_1, \lambda_2) = \mathcal{S}[\varphi](\lambda_2, \lambda, \lambda_1) = \mathcal{S}[\varphi](\lambda_2, \lambda, \lambda_1)$.

Finally, we also write down expression in large c expansion, where the collision integral simplifies

$$\begin{aligned} \Gamma_{nm} = & \frac{(2\pi)^2}{\tau} \frac{4}{c^2} \int d\lambda_{1,1} d\lambda_{1,2} d\lambda_2 \rho_p(\lambda_{1,1}) \rho_h(\lambda_{1,1}) \rho_p(\lambda_{1,2}) \rho_h(\lambda_{1,2}) \rho_p(\lambda_2) \rho_h(\lambda_2) |\lambda_{1,1} - \lambda_{1,2}| \times \\ & \left(h'_n(\lambda_{1,1}) \frac{1}{\lambda_2 - \lambda_{1,1}} + h'_n(\lambda_{1,2}) \frac{1}{\lambda_{1,2} - \lambda_2} - h'_n(\lambda_2) \left(\frac{1}{\lambda_2 - \lambda_{1,1}} + \frac{1}{\lambda_{1,2} - \lambda_2} \right) \right) \times \\ & \left(h'_m(\lambda_{1,1}) \frac{1}{\lambda_2 - \lambda_{1,1}} + h'_m(\lambda_{1,2}) \frac{1}{\lambda_{1,2} - \lambda_2} - h'_m(\lambda_2) \left(\frac{1}{\lambda_2 - \lambda_{1,1}} + \frac{1}{\lambda_{1,2} - \lambda_2} \right) \right). \end{aligned} \quad (\text{IX.32})$$

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- [1] B. Doyon, Lecture notes on Generalised Hydrodynamics, *SciPost Phys. Lect. Notes*, **18** (2020).
 - [2] E. H. Lieb and W. Liniger, Exact analysis of an interacting Bose gas. i. the general solution and the ground state, *Physical Review* **130**, 1605 (1963).
 - [3] J. De Nardis, D. Bernard, and B. Doyon, Hydrodynamic diffusion in integrable systems, *Phys. Rev. Lett.* **121**, 160603 (2018).
 - [4] B. Doyon and H. Spohn, Drude Weight for the Lieb-Liniger Bose Gas, *SciPost Phys.* **3**, 039 (2017).
 - [5] C. N. Yang and C. P. Yang, Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction, *Journal of Mathematical Physics* **10**, 1115 (1969).
 - [6] I. Bouchoule and J. Dubail, Generalized hydrodynamics in the one-dimensional bose gas: theory and experiments, *Journal of Statistical Mechanics: Theory and Experiment* **2022**, 014003 (2022).
 - [7] J. Mossel and J.-S. Caux, Generalized TBA and generalized Gibbs, *Journal of Physics A: Mathematical and Theoretical* **45**, 255001 (2012).
 - [8] J. R. Dorfman, H. van Beijeren, and T. R. Kirkpatrick, *Contemporary Kinetic Theory of Matter* (Cambridge University Press, 2021).
 - [9] P. Résibois and M. De Leener, *Classical Kinetic Theory of Fluids* (John Wiley & Sons, 1977).
 - [10] J. Durnin, M. J. Bhaseen, and B. Doyon, Nonequilibrium dynamics and weakly broken integrability, *Phys. Rev. Lett.* **127**, 130601 (2021).
 - [11] M. Panfil, S. Gopalakrishnan, and R. M. Konik, Thermalization of interacting quasi-one-dimensional systems, *Phys. Rev. Lett.* **130**, 030401 (2023).
 - [12] P. Ruggiero, P. Calabrese, B. Doyon, and J. Dubail, Quantum generalized hydrodynamics, *Phys. Rev. Lett.* **124**, 140603 (2020).
 - [13] R. Balescu, *Equilibrium and nonequilibrium statistical mechanics* (John Wiley & Sons, 1975).
 - [14] S. Chakraborti, S. Ganapa, P. Krapivsky, and A. Dhar, Blast in a one-dimensional cold gas: From newtonian dynamics to hydrodynamics, *Physical Review Letters* **126**, 244503 (2021).
 - [15] M. Lebek, M. Panfil, and R. M. Konik, Prethermalization in coupled one-dimensional quantum gases, *SciPost Physics* **17**, 007 (2024).
 - [16] J. D. Nardis and M. Panfil, Density form factors of the 1d bose gas for finite entropy states, *Journal of Statistical Mechanics: Theory and Experiment* **2015**, P02019 (2015).
 - [17] J. D. Nardis and M. Panfil, Particle-hole pairs and density-density correlations in the lieb-liniger model, *Journal of Statistical Mechanics: Theory and Experiment* **2018**, 033102 (2018).
 - [18] M. Panfil, The two particle-hole pairs contribution to the dynamic correlation functions of quantum integrable models, *Journal of Statistical Mechanics: Theory and Experiment* **2021**, 013108 (2021).

2.3 Article III: *Chapman-Enskog theory and crossover between diffusion and superdiffusion for nearly-integrable quantum gases*

- **M. Łebek**, M. Panfil “*Chapman-Enskog theory and crossover between diffusion and superdiffusion for nearly-integrable quantum gases*”, *SciPost Phys.* **18**, 186 (2025)

More than one hundred years ago, Sydney Chapman and David Enskog developed a theoretical framework to derive the NS equations. Nowadays, this method is known as the Chapman-Enskog expansion. Unlike the approach of Resibois described in Article II, it allows for the derivation of the fully nonlinear hydrodynamic equations. In addition to that, it is capable of incorporating an external potential into the dynamics. Such external potentials are ubiquitous in experiments with ultracold atoms.

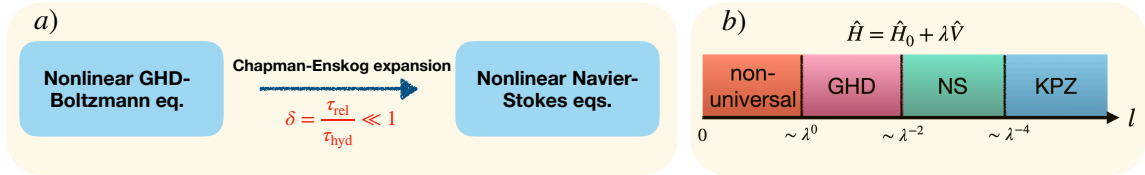


Figure 14: Panel a): the central result of the paper: derivation of the full nonlinear NS hydrodynamics from nonlinear GHD-Boltzmann equation. Panel b): three hydrodynamic regimes in nearly-integrable gases.

The key observation of the method lies in the identification of small parameters in the GHD-Boltzmann equation, see Fig. 14(a). One of these is typically known as the Knudsen number δ . It should be understood as the ratio of the microscopic relaxation timescale τ_{rel} (typical timescale over which the system relaxes to local thermal equilibrium) and the hydrodynamic timescale τ_{hyd} at which we probe the system and at which hydrodynamics is expected to occur. This observation is combined with the reduction of degrees of freedom from an infinite number to three hydrodynamic fields. Such a reduction is effectively possible thanks to the smallness of the Knudsen number.

The main contribution of the article is a systematic generalization of the ChE procedure to the GHD-Boltzmann equation. For this, an additional Knudsen number, $\delta_{\mathfrak{D}}$, had to be introduced, which can be understood as a ratio of the characteristic microscopic length scale $l_{\mathfrak{D}}$ related to the diffusion operator $\mathfrak{D}$ to the hydrodynamic length scale l_{hyd} . In our derivation, we recover exactly the same integral equations for transport coefficients as in Article II, thus finding an equivalence between the generalizations of the methods of Resibois and ChE. We also show how an external potential can be included in the emerging NS equations.

The second part of the article addresses slightly different issues. We concentrate there on the topic of transport in 1D gases and the anomalies therein, taking the derived nonlinear NS equations as the starting point. From there, we follow the standard procedure and add noise to the currents, which represent the degrees of freedom discarded in the ChE procedure. The fluctuating, nonlinear NS theory is analyzed using approximate methods to handle nonlinear stochastic partial differential equations. We follow the foundational paper [132].

The following picture emerges from this analysis. The transport in 1D non-integrable gases can be understood in terms of three normal modes: two sound peaks and a heat peak. Firstly, we note that in nearly-integrable regime, the transport coefficients (and thus, the diffusion constants governing spreading of the sound and heat modes) are parametrically large in the integrability-breaking parameter λ and scale as $D \sim \lambda^{-2}$. The key contribution of the second part

is the analysis of the impact of such large (for weak integrability breaking) diffusion constants on the equations governing the transport. The result is that the anomalies in transport happen at very large spatiotemporal scales. Thus, nearly-integrable gases display three hydrodynamic regimes, see Fig. 14(b).

Candidate's contribution: I proposed the idea of generalizing the ChE theory for nearly-integrable systems. I performed a significant part of the analytical calculations. Furthermore, I initiated the analysis of the crossover between conventional and anomalous hydrodynamics. I provided analytical arguments supporting its existence and conducted the numerical simulations. I contributed to writing the manuscript.

Chapman-Enskog theory and crossover between diffusion and superdiffusion for nearly integrable quantum gases

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Abstract

Integrable systems feature an infinite number of conserved charges and on hydrodynamic scales are described by generalised hydrodynamics (GHD). This description breaks down when the integrability is weakly broken and sufficiently large space-time-scales are probed. The emergent hydrodynamics depends then on the charges conserved by the perturbation. We focus on nearly-integrable Galilean-invariant systems with conserved particle number, momentum and energy. Basing on the Boltzmann approach to integrability-breaking we describe dynamics of the system with GHD equation supplemented with a collision term. The limit of large space-time-scales is addressed using Chapman-Enskog expansion adapted to the GHD equation. For length scales larger than $\sim \lambda^{-2}$, where λ is integrability-breaking parameter, we recover Navier-Stokes equations and find transport coefficients: viscosity and thermal conductivity. At even larger length scales, this description crosses over to Kardar-Parisi-Zhang universality class, characteristic to generic non-integrable one-dimensional fluids. Employing nonlinear fluctuating hydrodynamics we estimate this crossover length scale as $\sim \lambda^{-4}$.



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1 Introduction

Hydrodynamics is a universal theory of many-body systems out of equilibrium emerging at large space-time scales. It applies both to quantum and classical degrees of freedom and was proven extremely useful in systems as diverse as classical and quantum fluids [1–4], biological systems [5], relativistic matter [6], quantum circuits [7, 8] and electrically conducting fluids [9]. Within hydrodynamic approach, an enormous number of microscopic degrees of freedom is reduced to effective hydrodynamic fields, which follow the continuity equations written for densities of conserved quantities of the system.

From the perspective of the long history of hydrodynamics, the system which attracted the most attention was three-dimensional classical fluid (or gas), omnipresent in natural sciences. Basing on phenomenological arguments, it was established long time ago that such system can be well described by the famous Navier-Stokes (NS) equations [2, 3]. NS equations take a universal form and non-universal details of the specific fluid are fixed by *transport coefficients*. In the context of NS equations, these are of two types: the first is fixed solely by the thermodynamics of the fluid, whereas the second is related to dissipative effects and requires knowledge about the microscopic dynamics of the particles. Dissipative coefficients of the three-dimensional fluid are shear and bulk viscosities and thermal conductivity. An extremely important challenge related to the NS theory is microscopic derivation of these quantities.

The microscopic explanation of NS equations was provided by the kinetic theory, with the Boltzmann equation as the central object of the framework. The first method which successfully connected collisional dynamics of Boltzmann equation to NS hydrodynamics was formulated by S. Chapman and D. Enskog over one hundred years ago. It has uncovered not only the universal NS equations by the means of suitable perturbative expansion but also pro-

vided quantitative predictions for dissipative transport coefficients in terms of explicit integral equations, known today as Chapman-Enskog (ChE) integral equations.

Experimental [10–20] and theoretical progress [21–27] of the recent years gave rise to a large interest in dynamics of one-dimensional quantum integrable systems, such as spin chains or interacting bosonic and fermionic gases. Despite interactions between particles, integrable models possess an infinite number of conserved charges, displaying significantly different hydrodynamic behavior as compared to non-integrable counterparts. The relevant theory, which takes an infinite number of conservation laws into account, was developed in 2016 [28, 29] and is known under the name of generalized hydrodynamics (GHD) [30–32]. It was tested in cold-atomic experiments [16, 33] and nowadays GHD is the standard framework to tackle out-of-equilibrium dynamics of integrable systems.

Shortly after its formulation, the theory was extended in various directions. Notably, it was generalized to the case of external fields coupled to the charge densities [34, 35] including perhaps the most important case of external potential coupled to density of particles. Moreover, the GHD theory was later extended to include the effects of diffusive broadening of quasiparticles due to interactions [36–38]. For the comprehensive review of recent advances in GHD theory, see the series of articles [39–46] with an introduction given in [47].

Integrable systems are fine-tuned models and are never exactly realized in experiments. Thus, it is important to study the effects of weak integrability-breaking [12, 39, 48–61], such as additional weak interactions between particles [62–65], which make the model non-integrable. In the context of large space-time-scale dynamics, these can be treated as corrections to the GHD dynamics of the underlying integrable model. It is established now that under effects of weak integrability-breaking the relevant equation for dynamics of quasiparticles is GHD equation supplemented with Boltzmann-like collision term [39, 49, 63, 66], to which we will refer as GHD-Boltzmann equation. This feature renders the setup similar to the dynamics of classical Boltzmann equation, studied since a very long time.

The aim of the first part of this paper is to leverage this similarity and to adapt the ideas of Chapman and Enskog to a new physical context of Galilean-invariant nearly integrable quantum gases. In particular, as announced in a shorter paper [51], we derive the transport coefficients for such class of systems in terms of generalized ChE equations. Although GHD-Boltzmann equation differs from the standard Boltzmann counterpart by the presence of nonlinear effective velocity in the streaming term and by diffusion of quasiparticles, we show that a (generalized) ChE approach is capable of deriving the NS hydrodynamics from it, including the dissipative transport coefficients.

The results presented here parallel our developments reported in [51] where transport coefficients were derived from a linearized GHD-Boltzmann equation. Here, by generalizing the ChE method, we compute the transport coefficients beyond the linear approximation. Yet both approaches predict the same expressions for the transport coefficients, a feature recognized already in the classical kinetic theory [67] but a priori not guaranteed.

In the second part of our work, we address the important issue of the stability of NS equations due to interactions between hydrodynamic modes. Building on results of the first part, we employ the nonlinear fluctuating hydrodynamics (NLFH) to analyze the crossover between diffusive NS hydrodynamics and Kardar-Parisi-Zhang (KPZ) superdiffusion, which is expected to occur in generic non-integrable one-dimensional fluids [68–71] at largest length and time scales. Moreover, we estimate length scales on which different hydrodynamic descriptions (GHD, NS, KPZ) are valid.

The paper is organized as follows. Section 2 introduces the main ingredients of the framework that we use. These include Thermodynamic Bethe Ansatz (TBA), GHD-Boltzmann equation, collision integrals stemming from integrability-breaking, the role of Galilean invariance and length-scales relevant for the problem. In the following Section 3, we recall the ChE

theory for the classical Boltzmann equation. These first two sections serve as a basis for Section 4 in which we adapt the ChE method to the nearly integrable quantum gases described by the GHD-Boltzmann kinetic equation. In Section 5 we study equations of NLFH to analyze the crossover between NS diffusive and KPZ superdiffusive behavior of two-point functions. The paper finishes with conclusions presented in Sec. 6. Some additional formulas and short supplementary derivations are relegated to Appendices A and B.

2 Preliminaries

In this section we introduce the main objects used to derive the NS equations for nearly-integrable systems. We start with a short description of one-dimensional NS equations in 2.1. Next, we discuss the thermodynamics of quantum integrable models within the framework of the Thermodynamic Bethe Ansatz (TBA) and different representations of GHD-Boltzmann equation in 2.2. Consequences of Galilean invariance on the TBA quantities are derived in 2.3. After that in 2.4, we move to the discussion of the most important properties of collision integral in systems with conserved particle number, momentum and energy. In 2.5 we then derive mesoscopic conservation laws: the starting point for ChE expansion, which will be motivated by identification of small parameters in 2.6, which discuss the relevant lengthscales for our problem.

2.1 Navier-Stokes equations

The NS equations are continuity equations for conserved densities and associated currents. In $(1+1)$ dimensions these are particle density $\varrho_0(= \varrho)$, momentum density ϱ_1 and energy density ϱ_2 . Here and in the following, we suppress the dependence on the space-time coordinates (x, t) unless its presence is required for the clarity. In the hydrodynamic treatment it is customary to use velocity u and internal energy e instead of ϱ_1 and ϱ_2 . Furthermore, the internal energy density e is often replaced by local temperature T on the basis of the assumption of local thermal equilibrium. We will refer to the sets $\{\varrho, u, e\}$ or $\{\varrho, u, T\}$ as hydrodynamic fields.

The NS equations for the hydrodynamic fields, in the presence of the external potential $U(x)$ confining the particles, are [1–3]

$$\begin{aligned}\partial_t \varrho &= -\partial_x(\varrho u), & \partial_t(\varrho u) &= -\partial_x(\varrho u^2 + \mathcal{P}) - (\partial_x U)\varrho, \\ \partial_t(\varrho e) &= -\partial_x(u\varrho e + \mathcal{J}) - \mathcal{P}\partial_x u.\end{aligned}\tag{1}$$

The equations, beside the hydrodynamic fields, involve also hydrodynamic pressure $\mathcal{P}$ and heat current $\mathcal{J}$. In the phenomenological picture they take the following form

$$\mathcal{P} = P - \zeta \partial_x u, \quad \mathcal{J} = -\kappa \partial_x T.\tag{2}$$

Here, we expressed the heat current $\mathcal{J}$ through a gradient of the temperature to conform with the standard formulation. The two transport coefficients are viscosity ζ and thermal conductivity κ , and the two relations in (2) are respectively the Newton's law for a viscous fluid (with shear viscosity absent in one spatial dimension) and the Fourier's law for the heat current. P denotes here the standard thermodynamic pressure. The abovementioned thermodynamic relation between e and T fields which together with (2) allows to close the equations is [2]

$$dT = \frac{1}{c_V} de - \frac{1}{c_V} \left(\frac{P}{\varrho^2} - \frac{T}{\varrho^2} \left(\frac{\partial P}{\partial T} \right)_\varrho \right) d\varrho,\tag{3}$$

where c_V is the specific heat at constant volume.

The NS equations (1) describe a Galilean invariant fluid. The density ϱ and temperature T are invariant (scalars) under Galilean boost, whereas velocity and internal energy transform in the standard way. Another invariants are pressure and transport coefficients. This implies that pressure $P \equiv P(\varrho, T)$ and transport coefficients depend only on the local density and temperature of the fluid $\zeta \equiv \zeta(\varrho, T)$, $\kappa \equiv \kappa(\varrho, T)$.

The central result of this work is derivation of the NS equations for nearly-integrable quantum gases described with a particle number, momentum and energy conserving Hamiltonian of the form

$$\hat{H} = \hat{H}_0 + \lambda \hat{V}, \quad \lambda \ll 1, \quad (4)$$

where $\hat{H}_0$ is an interacting integrable Hamiltonian and $\hat{V}$ is the integrability-breaking term. Using a generalization of ChE method we recover exactly (1) with thermodynamic quantities such as specific heats and pressure dictated by the thermodynamics of the integrable model and with the explicit expressions for the two transport coefficients.

The transport coefficients that we find are sums of two contributions $\zeta = \zeta_{\mathcal{I}} + \zeta_{\mathcal{D}}$ and $\kappa = \kappa_{\mathcal{I}} + \kappa_{\mathcal{D}}$. The contributions $\zeta_{\mathcal{D}}, \kappa_{\mathcal{D}}$ originate from collisions of quasiparticles of the integrable model. They are an effect of interactions present in H_0 and are straightforwardly computed from the methods of integrability.¹ On the other hand, $\zeta_{\mathcal{I}}, \kappa_{\mathcal{I}}$ are related to Boltzmann collision integral $\mathcal{I}$ which represents the effects of integrability-breaking due to $\hat{V}$. Similarly as in the classical kinetic theory, computation of $\zeta_{\mathcal{I}}, \kappa_{\mathcal{I}}$ requires solution of integral equations. Expressions for transport coefficients were reported in [51] and we present their derivation in the ChE formalism in Sec. 4. In the classical kinetic theory there is only one type of contributions to the transport coefficients originating from the collision integral $\mathcal{I}$ since H_0 represents then a non-interacting system.

Finally, we mention an alternative representation of the NS equations in terms of the local conserved densities $\{q_0, q_1, q_2\}$. These are related to the hydrodynamic fields through the following relations

$$\varrho = q_0, \quad u = \frac{q_1}{q_0}, \quad e = \frac{q_2}{q_0} - \frac{q_1^2}{2q_0^2}. \quad (5)$$

The NS equations (1) for the conserved densities are then

$$\begin{aligned} \partial_t q_0 + \partial_x q_1 &= 0, \\ \partial_t q_1 + \partial_x \left(\frac{q_1^2}{q_0} + \mathcal{P} \right) &= -(\partial_x U) q_0, \\ \partial_t q_2 + \partial_x \left(\frac{q_1 q_2}{q_0} + \frac{q_1}{q_0} \mathcal{P} + \mathcal{J} \right) &= -(\partial_x U) q_1. \end{aligned} \quad (6)$$

The two formulations of the NS equations will be useful in what follows.

2.2 Thermodynamic Bethe Ansatz and GHD-Boltzmann equation

In this section, we introduce the central object of our study – the GHD-Boltzmann equation. Together with it we introduce the main ingredients of the TBA. As we focus on Galilean invariant systems, the bare energy reads $E(\lambda) = \lambda^2/2$ and bare momentum is $p(\lambda) = \lambda$. We set the particle mass to unity $m = 1$.

The GHD equation, like a standard kinetic equation, is an equation for a (quasi-)particles' distribution $\rho_p(\lambda; x, t)$. The integrable structure provides a number of alternative representations of the state of the system. The two additional useful representations are through the

¹They are thermal matrix elements of the GHD diffusion operator $\mathcal{D}$.

local values of conserved charges $\{\varphi_n(x, t)\}$ and local (generalized) temperatures $\{\beta_n(x, t)\}$. The values of the former are given by

$$\varphi_n(x, t) = \int d\lambda \, \mathcal{H}_n(\lambda) \rho_p(\lambda; x, t), \quad (7)$$

where $\mathcal{H}_n(\lambda)$ is the single-particle contribution to the charge density $\varphi_n(x, t)$. For a Galilean-invariant theory the ultra-local charges are given by choosing

$$\mathcal{H}_n(\lambda) = \lambda^n / n!, \quad (8)$$

and charges are cumulants of the quasiparticles' distribution.

The second representation of the state is through the generalized temperatures and requires introducing the generalized TBA which we now quickly recall in the standard version applicable to homogeneous systems [72–75]. The initial object is the pseudoenergy ϵ_0 which takes the following form

$$\epsilon_0(\lambda) = \sum_n \mathcal{H}_n(\lambda) \beta_n. \quad (9)$$

The local distribution of quasiparticles is then determined from the generalized TBA equations

$$\epsilon(\lambda) = \epsilon_0(\lambda) + \int d\mu \, \mathcal{T}(\lambda - \mu) F[\epsilon(\mu)], \quad (10)$$

$$\rho_{\text{tot}}(\lambda) = \frac{1}{2\pi} + \int d\mu \, \mathcal{T}(\lambda - \mu) n(\mu) \rho_{\text{tot}}(\mu), \quad (11)$$

where $\mathcal{T}(\lambda)$ is the model-dependent scattering kernel. For instance, for the Lieb-Liniger model [76, 77] with coupling c it reads $\mathcal{T}(\lambda) = \frac{c}{\pi} \frac{1}{c^2 + \lambda^2}$. Function $F[\epsilon(\mu)]$ is related to the free energy density $f = \frac{1}{2\pi} \int d\lambda F[\epsilon(\lambda)]$ and depends on the statistics of particles [31]. For models with fermionic statistic of quasi-particles (as the Lieb-Liniger model) it reads $F[\epsilon] = -\log(1 + e^{-\epsilon})$, for bosonic $F[\epsilon] = \log(1 - e^{-\epsilon})$ and for classical particles $F[\epsilon] = -e^{-\epsilon}$. Solving the first equation yields $\epsilon(\lambda)$ which enters the filling function

$$n(\lambda) = \left. \frac{dF}{d\epsilon} \right|_{\epsilon(\lambda)}, \quad (12)$$

and in turn specifies the second equation. We note here that for the fermionic statistics $n(\lambda) = 1/(1 + e^{\epsilon(\lambda)})$. The total density ρ_{tot} determines then the particles' density according to $\rho_p(\lambda) = n(\lambda) \rho_{\text{tot}}(\lambda)$.

Generalization to the non-homogeneous systems relies on assumption that locally the state of the system is described by the TBA with generalized temperatures varying in space. Namely the pseudoenergy (9) is promoted to

$$\epsilon_0(\lambda, x, t) = \sum_n \mathcal{H}_n(\lambda) \beta_n(x, t), \quad (13)$$

and the outcome of the TBA is space-time dependent $\rho_p(\lambda, x, t)$.

We move now to the GHD-Boltzmann equation for nearly integrable systems. It reads [35, 39, 50]

$$\partial_t \rho_p + \partial_x (v \rho_p) = (\partial_x U) \partial_\theta \rho_p + \frac{1}{2} \partial_x (\mathfrak{D} \partial_x \rho_p) + \mathcal{I}[\rho_p], \quad (14)$$

with effective velocity v , an external potential U , the diffusion kernel $\mathfrak{D}$ and the Boltzmann collision integral $\mathcal{I}[\rho_p]$. The effective velocity for Galilean-invariant models fulfills the following integral equation [28, 31]

$$v(\lambda) = \mathcal{H}_1(\lambda) + 2\pi \int d\lambda' \, \mathcal{T}(\lambda - \lambda') \rho_p(\lambda') (v(\lambda') - v(\lambda)). \quad (15)$$

Before we describe the diffusion kernel $\mathfrak{D}$, we introduce some notation. For an arbitrary operator $\mathcal{O}$ we introduce the following notation for its action on arbitrary functions ψ, φ

$$(\mathcal{O}\psi)(\lambda) = \int d\mu \mathcal{O}(\lambda, \mu) \psi(\mu), \quad \psi \mathcal{O} \varphi = \int d\lambda d\mu \mathcal{O}(\lambda, \mu) \psi(\lambda) \varphi(\mu), \quad (16)$$

and use the standard notation for kernel multiplication

$$(\mathcal{O}_1 \mathcal{O}_2)(\lambda, \mu) = \int d\nu \mathcal{O}_1(\lambda, \nu) \mathcal{O}_2(\nu, \mu). \quad (17)$$

The diffusion kernel is then (below ρ_{tot} and n should be understood as diagonal kernels) [37]

$$\mathfrak{D} = (1 - n\mathcal{T})^{-1} \rho_{\text{tot}}^{-1} \tilde{\mathfrak{D}} \rho_{\text{tot}}^{-1} (1 - n\mathcal{T}), \quad \tilde{\mathfrak{D}}(\theta, \alpha) = \delta(\theta - \alpha) w(\theta) - W(\theta, \alpha), \quad (18)$$

and

$$W(\theta, \alpha) = \rho_p(\theta) f(\theta) (\mathcal{T}^{\text{dr}}(\theta, \alpha))^2 |v(\theta) - v(\alpha)|, \quad w(\theta) = \int d\alpha W(\alpha, \theta), \quad (19)$$

where $f(\theta)$ denotes statistical factor [31]

$$f(\theta) = - \frac{d^2 F / d\epsilon^2}{dF / d\epsilon} \bigg|_{\epsilon(\theta)}, \quad (20)$$

which depends on the particles' statistics (it reads $f = 1 - n$ for fermions, $f = 1 + n$ for bosons and $f = 1$ for classical particles). The dressed scattering kernel fulfills

$$\mathcal{T}^{\text{dr}}(\lambda, \lambda') = \mathcal{T}(\lambda - \lambda') + \int d\lambda'' \mathcal{T}(\lambda - \lambda'') n(\lambda'') \mathcal{T}^{\text{dr}}(\lambda'', \lambda'), \quad (21)$$

and in addition to that we define dressing of an arbitrary function $g(\lambda)$ as solution to the integral equation

$$g^{\text{dr}}(\lambda) = g(\lambda) + \int d\lambda' \mathcal{T}(\lambda - \lambda') n(\lambda') g^{\text{dr}}(\lambda'), \quad (22)$$

which implies that $g^{\text{dr}} = (1 - \mathcal{T}n)^{-1} g$.

The effective velocity and the diffusion operator depend on the state ρ_p . This renders the GHD-Boltzmann equations non-linear even in the absence of the collision integral. The diffusion term leads to the entropy production [37] and in the interplay with the external potential leads to a stationary thermal state even in the absence of the collision term [78]. For now, we set $U = 0$ and discuss the effect of the external potential in Section 4.4.

In what follows, we will present a slightly different parametrization of the GHD-Boltzmann equation (14). We start with the relation between the generalized temperatures and conserved charges and associated currents, which are described by the hydrodynamic matrices

$$\mathcal{C}_{ab} = - \frac{\partial \mathcal{Q}_a}{\partial \beta_b} = \hbar_a \mathcal{C} \hbar_b, \quad \mathcal{B}_{ab} = - \frac{\partial \dot{\mathcal{J}}_a}{\partial \beta_b} = \hbar_a \mathcal{B} \hbar_b, \quad (23)$$

where $\dot{\mathcal{J}}_a$ is the expectation value of the conserved current operator $\hat{\mathcal{J}}_a$ evaluated on the generalized TBA state [28],

$$\dot{\mathcal{J}}_a(x, t) = \int d\lambda \hbar_a(\lambda) v(\lambda; x, t) \rho_p(\lambda; x, t). \quad (24)$$

The kernels of the two operators introduced above are

$$\mathcal{C} = (1 - n\mathcal{T})^{-1} \rho_p f (1 - \mathcal{T}n)^{-1}, \quad \mathcal{B} = (1 - n\mathcal{T})^{-1} \rho_p f v (1 - \mathcal{T}n)^{-1}. \quad (25)$$

In the passing we note that hydrodynamic matrices (23) are symmetric. Moreover, the matrix $\mathcal{C}$ defines a *hydrodynamic inner product*, which we will use throughout the paper

$$(f|g) = f \mathcal{C} g. \quad (26)$$

After introducing the required ingredients we can now rewrite the GHD-Boltzmann equations. First, from the TBA equations it follows that [31, 51]

$$\frac{\partial \rho_p}{\partial \beta_a} = -\mathcal{C} \hbar_a, \quad \frac{\partial (v \rho_p)}{\partial \beta_a} = -\mathcal{B} \hbar_a. \quad (27)$$

These relations allow us to transform the GHD equations to equations for the generalized temperatures

$$\mathcal{C}_{ab} \partial_t \beta_b + \mathcal{B}_{ab} \partial_x \beta_b = \frac{1}{2} \partial_x ((\mathcal{D}\mathcal{C})_{ab} \partial_x \beta_b) - \hbar_a \mathcal{I}[\{\beta_b\}]. \quad (28)$$

We can also transform them into the continuity equations for the charges. The first two terms transform easily using again the hydrodynamic matrices,

$$\partial_t q_a = -\mathcal{C}_{ab} \partial_t \beta_b, \quad \partial_x j_a = -\mathcal{B}_{ab} \partial_x \beta_b. \quad (29)$$

The diffusive term of the GHD equation requires expressing the change to β_a due to a change to the conserved charges. Assuming the susceptibility matrix $\mathcal{C}_{ab}$ is invertible we have

$$\partial_x \beta_a = - \sum_b \mathcal{C}_{ab}^{-1} \partial_x q_b, \quad (30)$$

and

$$\partial_t q_a + \partial_x j_a = \frac{1}{2} \partial_x (\mathcal{D}_{ab} \partial_x q_b) + \hbar_a \mathcal{I}[\{q_b\}]. \quad (31)$$

The three representations of the GHD-Boltzmann equation given in eqs. (14), (28) and (31) will be important for the discussion ahead.

2.3 Galilean invariance and thermal boosted state

We discuss now the role of the Galilean invariance. Specifically, we introduce the thermal boosted states and the transformations of the hydrodynamic matrices under the Galilean boost.

Consider a homogeneous state of the system described by distribution ρ_p which as a whole moves with velocity $u \neq 0$, where

$$u = \frac{\int d\lambda \hbar_1(\lambda) \rho_p(\lambda)}{\int d\lambda \hbar_0(\lambda) \rho_p(\lambda)}. \quad (32)$$

This state can be understood as a Galilean boost of the state with zero total momentum and with distribution function $\bar{\rho}_p(\lambda)$ such that $\rho_p(\lambda) = \bar{\rho}_p(\lambda - u)$. If one distribution is a solution to the TBA equations so is the other one and the corresponding bare pseudo-energies obey the same type of relation $\epsilon_0(\lambda) = \bar{\epsilon}_0(\lambda - u)$. This relation is a consequence of the kernel $\mathcal{T}(\lambda - \mu)$ being a difference operator (which itself is required by the Galilean invariance). The transformation under the Galilean boost can be realized by introducing a boost operator b_u such that $\rho_p = b_u^{-1} \bar{\rho}_p b_u$ with $b_0 = 1$ and $b_{-u} = b_u^{-1}$. In the rapidity variable the operators are represented by $b_u = \exp(u \partial_\lambda)$.

As discussed above, the TBA offers 3 equivalent characterizations of the GGE states one of which is the distribution function ρ_p . Another one is provided by the generalized temperatures. The transformation laws for the generalized temperatures can be read off from the equality between the bare pseudo-energies $\epsilon_0(\lambda) = \bar{\epsilon}_0(\lambda - u)$. Expanding both sides of the equality in generalized temperatures we obtain

$$\sum_a \beta_a \mathcal{H}_a(\lambda) = \sum_a \bar{\beta}_a \mathcal{H}_a(\lambda - u). \quad (33)$$

Functions $\mathcal{H}_a(\lambda - u)$ expand in $\mathcal{H}_a(\lambda)$,

$$\mathcal{H}_a(\lambda - u) = \sum_b g_{ab}(-u) \mathcal{H}_b(\lambda), \quad (34)$$

where we introduced a transformation matrix $g_{ab}(-u)$ whose structure is a consequence of the monomial form of functions $\mathcal{H}_a(\lambda)$ defined in (8). From this it follows that

$$\beta_a = \sum_b \bar{\beta}_b g_{ba}(-u). \quad (35)$$

The transformation matrix $g(u)$ is a representation of the boost operator b_u in the space of functions $\mathcal{H}_a$ determining the local conserved charges. The matrix is lower triangular because $\mathcal{H}_a(\lambda - u)$ expands into $\mathcal{H}_b(\lambda)$ with $b \leq a$. We also have

$$g(0) = 1, \quad g^{-1}(u) = g(-u). \quad (36)$$

In principle, $g(u)$ is an infinite-dimensional matrix, but because of its triangular structure it can be truncated at a chosen order. For example, in the subspace of 3 lowest conserved charges, which will be relevant in deriving the NS equations, the matrix is

$$g(u) = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ \frac{1}{2}u^2 & u & 1 \end{pmatrix}, \quad (37)$$

and its action in this subspace is closed and is invertible.

The expectation values of the conserved charges transform in the similar way as $\mathcal{H}_a$,

$$q_a = \sum_b g_{ab}(u) \bar{q}_b, \quad (38)$$

where $\bar{q}_a$ and q_a are values before and after the boost respectively. We also note that the effective velocity transforms as

$$v(\lambda) = \bar{v}(\lambda - u) + u. \quad (39)$$

For the analysis of the transport coefficients the transformation properties of the hydrodynamic matrices are important. First we note that, at the level of operators, we have

$$\mathcal{C}(\mu, \lambda) = b_u^{-1} \bar{\mathcal{C}}(\mu, \lambda) b_u, \quad \mathcal{B}(\mu, \lambda) = b_u^{-1} \bar{\mathcal{B}}(\mu, \lambda) b_u + u b_u^{-1} \bar{\mathcal{C}}(\mu, \lambda) b_u, \quad (40)$$

with the operators acting now on both variables, $b_u \sim e^{u\partial_\mu} \otimes e^{u\partial_\lambda}$. This implies that

$$\begin{aligned} \mathcal{C}_{ab} &= \sum_{cd} g_{ac}(u) \bar{\mathcal{C}}_{cd}(g^T(u))_{db} = (g(u) \bar{\mathcal{C}} g^T(u))_{ab}, \\ \mathcal{B}_{ab} &= (g(u) \bar{\mathcal{B}} g^T(u))_{ab} + u (g(u) \bar{\mathcal{C}} g^T(u))_{ab}. \end{aligned} \quad (41)$$

Furthermore and independently of the boost, $\mathcal{C}_{a0} = \mathcal{B}_{a1}$. This follows from the fact [31] that $v(\lambda) = \mathcal{H}_1^{\text{dr}}(\lambda)/\mathcal{H}_0^{\text{dr}}(\lambda)$ and from definitions (25) of $\mathcal{C}, \mathcal{B}$ kernels.

Among all the states described by the generalized TBA a special role is played by thermal states and boosted thermal states. The former are described by non-zero $\bar{\beta}_0$ and $\bar{\beta}_2$ with all other chemical potentials zero. Boosting the thermal state by u the generalized temperatures become

$$\beta_0 = \bar{\beta}_0 + \frac{1}{2}u^2\bar{\beta}_2, \quad \beta_1 = -u\bar{\beta}_2, \quad \beta_2 = \bar{\beta}_2. \quad (42)$$

This implies that (inverse) temperature is an invariant of the thermal state under the Galilean boost. Another invariant is the particle's density q_0 . Therefore, a triple (q_0, q_1, β_2) provides a convenient parametrization of thermal boosted states. It transforms minimally under the Galilean boost by u to $(q_0, q_1 + u, \beta_2)$.

The particles distribution for a thermal state is an even function, $\rho_p(\lambda) = \rho_p(-\lambda)$. A consequence of this is a checker-board structure of the hydrodynamic matrices. In the subspace of the 3 first conserved charges

$$\bar{\mathcal{C}}_{ab} = \begin{pmatrix} \bar{\mathcal{C}}_{00} & 0 & \bar{\mathcal{C}}_{02} \\ 0 & \bar{\mathcal{C}}_{11} & 0 \\ \bar{\mathcal{C}}_{02} & 0 & \bar{\mathcal{C}}_{22} \end{pmatrix}, \quad \bar{\mathcal{B}}_{ab} = \begin{pmatrix} 0 & \bar{\mathcal{B}}_{01} & 0 \\ \bar{\mathcal{B}}_{01} & 0 & \bar{\mathcal{B}}_{12} \\ 0 & \bar{\mathcal{B}}_{12} & 0 \end{pmatrix}. \quad (43)$$

We observe that $\bar{\mathcal{C}}_{ab}$ is invertible as $\det \bar{\mathcal{C}} = \bar{\mathcal{C}}_{11}(\bar{\mathcal{C}}_{00}\bar{\mathcal{C}}_{22} - \bar{\mathcal{C}}_{02}^2) = T^4 \varrho^4 c_V \kappa_T$, where κ_T is isothermal compressibility of the integrable model and c_V is specific heat at constant volume. Thermodynamic stability conditions guarantee that $c_V, \kappa_T > 0$, hence $\det \bar{\mathcal{C}} > 0$. Above we have used formulas for thermodynamic quantities expressed with matrix elements of $\bar{\mathcal{C}}, \bar{\mathcal{B}}$, given in Appendix A.1. The hydrodynamic matrices for thermal boosted states follow then from transformations (41).

2.4 Collision integral

In this section we look more closely at the collision integral $\mathcal{I}[\rho_p]$ and discuss the most important properties of it. The collision integrals considered by us originate from integrability-breaking. In Ref. [35, 63] it was shown how such terms arise due to perturbative treatment at the level of Fermi's Golden Rule (FGR). However, the ChE method does not depend on the precise form of the collision integral. Instead, it relies on two general and physically motivated properties that we now discuss. We refer to [51] for an example of a collision integral in the system of two coupled Lieb-Liniger models. It can be explicitly checked that collision integrals derived in [35, 63] have the properties listed below.

First of all, a characteristic feature of the collision integral $\mathcal{I}[\rho_p]$ is a set of collision invariants, namely quantities conserved by it. The natural choice for the collision invariants are the particle number, momentum and energy and such choice yields the NS equations. For collision integral $\mathcal{I}$ respecting these three conservation laws we have

$$\mathcal{H}_a \mathcal{I} = \int d\theta \mathcal{H}_a(\theta) \mathcal{I}[\rho_p](\theta) = 0, \quad a = 0, 1, 2. \quad (44)$$

Another choice, for example, is to respect only the particle number, the resulting hydrodynamic equations are the non-linear diffusion equations. We will investigate such situation in a future work.

The second important feature of the collision integral are the stationary states ρ_p^{st} fulfilling

$$\mathcal{I}[\rho_p^{\text{st}}] = 0. \quad (45)$$

We will generally assume that the stationary states are TBA boosted thermal states described in the section above. This can be shown for classical Boltzmann collision kernel [2]. Moreover, in Ref. [35] it was demonstrated for collision integrals constructed from FGR that indeed the stationary states are TBA boosted thermal states with pseudoenergies of the form

$$\epsilon_0(\theta) = -\beta\mu + \nu\hbar_1(\theta) + \beta\hbar_2(\theta). \quad (46)$$

However, collision integrals might also possess non-thermal stationary states as discussed, for example, in [65]. Such situations are also beyond the scope of the present work.

In our calculations we will be mostly interested in the operator

$$\Gamma = -\frac{\delta\mathcal{I}}{\delta\rho_p}\mathcal{C}. \quad (47)$$

The conservation of energy, momentum and particle number by the collision integral implies that Γ has three zero eigenvectors

$$\int d\lambda \hbar_a(\lambda) \Gamma(\lambda, \mu) = 0, \quad a = 0, 1, 2. \quad (48)$$

In our work, we also assume that Γ beside these three zero eigenvalues, the rest of the spectrum is positive and gapped, with a gap $\gamma > 0$ providing a characteristic time-scale for thermalization in the system. These assumptions were confirmed in a concrete example of coupled Lieb-Liniger models investigated in [51].

2.5 Mesoscopic conservation laws

In this section we make the first steps towards recovering hydrodynamic equations from GHD-Boltzmann equation (14). We start by writing the equations for the dynamics of the first three charges. Existence of three collision invariants leads to three mesoscopic conservation laws, c.f. (31),

$$\partial_t q_a + \partial_x j_a = \frac{1}{2} \partial_x (\mathcal{D}_{ab} \partial_x q_b), \quad a = 0, 1, 2, \quad (49)$$

with $j_a(x, t)$ defined in (24).

This set of three equations is not closed. The equations depend on the full information of the state of the system, not only on the first three charge densities. Otherwise, these equations are similar to the NS equations in a proper representation. The ChE method allows us to effectively integrate out the information about the evolution of the other charges which are not conserved by the collision term. Because of that, they are decaying and lead to appearance of additional (beside the contribution from the GHD diffusion) dissipative terms.

It is helpful to write equations (49) more explicitly. First, we use the Markovianity property of the diffusion operator. It implies that $\hbar_0 \mathcal{D} = 0$ for Galilean-invariant systems [36] and the density obeys a conservation law without diffusive terms,

$$\partial_t q_0 + \partial_x j_0 = 0, \quad j_0 = q_1. \quad (50)$$

The other two continuity equations are

$$\begin{aligned} \partial_t q_1 + \partial_x j_1 &= -\partial_x \mathcal{P}_{\mathcal{D}}, \\ \partial_t q_2 + \partial_x j_2 &= -\partial_x (u \mathcal{P}_{\mathcal{D}} + \mathcal{J}_{\mathcal{D}}), \end{aligned} \quad (51)$$

where we defined

$$\mathcal{P}_{\mathcal{D}} = -\frac{1}{2} \int d\lambda d\mu \hbar_1(\lambda) \mathcal{D}(\lambda, \mu) \partial_x \rho_p(\mu) = -\hbar_1 \mathcal{D} \partial_x \rho_p / 2, \quad (52)$$

$$u \mathcal{P}_{\mathcal{D}} + \mathcal{J}_{\mathcal{D}} = -\frac{1}{2} \int d\lambda d\mu \hbar_2(\lambda) \mathcal{D}(\lambda, \mu) \partial_x \rho_p(\mu) = -\hbar_2 \mathcal{D} \partial_x \rho_p / 2, \quad (53)$$

anticipating the structure of the expressions.

The currents $\mathcal{J}_a$ involve two types of contributions. The first one depends only on the hydrodynamic fields, the second requires knowledge of the full rapidity distribution ρ_p via contributions to hydrodynamic pressure $\mathcal{P}_v$ and heat current $\mathcal{J}_v$. The result is

$$\mathcal{J}_1 = \frac{q_1^2}{q_0} + \mathcal{P}_v, \quad \mathcal{J}_2 = \frac{q_1 q_2}{q_0} + u \mathcal{P}_v + \mathcal{J}_v, \quad (54)$$

where we defined the contributions to pressure and heat current

$$\mathcal{P}_v = \int d\lambda (\lambda - u) v(\lambda) \rho_p(\lambda), \quad (55)$$

$$\mathcal{J}_v = \frac{1}{2} \int d\xi \xi^2 (v(u + \xi) - u) \rho_p(u + \xi) = \frac{1}{2} \int d\lambda (\lambda - u)^2 (v(\lambda) - u) \rho_p(\lambda). \quad (56)$$

The resulting equations are now

$$\begin{aligned} \partial_t q_0 + \partial_x q_1 &= 0, \\ \partial_t q_1 + \partial_x \left(\frac{q_1^2}{q_0} + \mathcal{P} \right) &= 0, \\ \partial_t q_2 + \partial_x \left(\frac{q_1 q_2}{q_0} + u \mathcal{P} + \mathcal{J} \right) &= 0. \end{aligned} \quad (57)$$

They take a form of the NS equations (1) with $\mathcal{P} = \mathcal{P}_v + \mathcal{P}_{\mathfrak{D}}$ and $\mathcal{J} = \mathcal{J}_v + \mathcal{J}_{\mathfrak{D}}$. Anticipating our findings, the ChE method approximates expressions for $\mathcal{P}$ and $\mathcal{J}$ such that they depend only on the hydrodynamic fields through Newton's and Fourier's laws (2), thus closing the set of equations.

2.6 Relevant lengthscales

In the GHD-Boltzmann equation we have (at least) 3 relevant scales in the system. There is a microscopic length-scale $l_{\mathfrak{D}}$, (given by the scattering shift of the integrable theory), there is a hydrodynamic length-scale l_h at which ρ_p varies in space and there is time-scale $\tau_{\mathcal{I}}$ specifying the rate of the integrability-breaking collisions. Introducing the latter into the GHD equations we obtain (below we defined $\mathcal{I} = \frac{1}{\tau_{\mathcal{I}}} \tilde{\mathcal{I}}$)

$$\partial_t \rho_p + \partial_x (v \rho_p) - \frac{1}{2} \partial_x (\mathfrak{D} \partial_x \rho_p) = \frac{1}{\tau_{\mathcal{I}}} \tilde{\mathcal{I}}[\rho_p], \quad (58)$$

with the following dimensions

$$[\rho_p] = [\tilde{\mathcal{I}}] = \frac{1}{m^2/s}, \quad [\tau_{\mathcal{I}}] = s, \quad [v] = m/s, \quad [\mathfrak{D}] = m^2/s. \quad (59)$$

We introduce also $\langle v \rangle$, a typical velocity of quasiparticles which is a characteristic of the state. Then $l_{\mathcal{I}} = \langle v \rangle \tau_{\mathcal{I}}$ is a distance travelled by a quasiparticle between the non-integrable collisions. We can now rewrite the GHD equations in a dimensionless form. To this end, we write

$$\tilde{v} = \frac{v}{\langle v \rangle}, \quad \tilde{\mathfrak{D}} = \frac{\mathfrak{D}}{\langle v \rangle l_{\mathfrak{D}}}, \quad (60)$$

which defines the dimensionless effective velocity and diffusion operator. Then, in the reduced variables $\tilde{t} = t/\tau_h$ and $\tilde{x} = x/l_h$, we find

$$\partial_{\tilde{t}} \rho_p + \partial_{\tilde{x}} (\tilde{v} \rho_p) - \frac{1}{2} \frac{\tau_{\mathfrak{D}}}{\tau_h} \partial_{\tilde{x}} (\tilde{\mathfrak{D}} \partial_{\tilde{x}} \rho_p) = \frac{\tau_h}{\tau_{\mathcal{I}}} \tilde{\mathcal{I}}[\rho_p]. \quad (61)$$

The two ratios appearing in the equation can be reinterpreted through length-scales. Introducing parameters which can be identified as generalizations of Knudsen numbers [2] from standard kinetic theory

$$\delta_{\mathcal{D}} = \frac{\tau_{\mathcal{D}}}{\tau_h} = \frac{l_{\mathcal{D}}}{l_h}, \quad \delta_{\mathcal{I}} = \frac{\tau_{\mathcal{I}}}{\tau_h} = \frac{l_{\mathcal{I}}}{l_h}, \quad (62)$$

the GHD-Boltzmann equation becomes

$$\partial_t \rho_p + \partial_x (\tilde{v} \rho_p) - \frac{1}{2} \delta_{\mathcal{D}} \partial_x (\tilde{\mathcal{D}} \partial_x \rho_p) = \frac{1}{\delta_{\mathcal{I}}} \tilde{\mathcal{I}}[\rho_p]. \quad (63)$$

We note that in order to be in the hydrodynamic regime we require $\delta_{\mathcal{D}} \ll 1$ and in order to treat the collision integral as a perturbation we need also $\delta_{\mathcal{I}} \ll 1$. The second condition guarantees that collisions occur in effectively homogeneous background and system thermalizes locally.

Lastly, it is useful to realize that almost identical set of mesoscopic conservation laws as (57) can be derived in the rescaled units. What happens is that we recover the same equations with a single difference that parameter $\delta_{\mathcal{D}}$ appears explicitly in expressions for $\mathcal{P}_{\mathcal{D}}$ and $\mathcal{J}_{\mathcal{D}}$

$$\mathcal{P}_{\mathcal{D}} = -\delta_{\mathcal{D}} \hbar_1 \mathcal{D} \partial_x \rho_p / 2, \quad u \mathcal{P}_{\mathcal{D}} + \mathcal{J}_{\mathcal{D}} = -\delta_{\mathcal{D}} \hbar_2 \mathcal{D} \partial_x \rho_p / 2. \quad (64)$$

In what follows we work with the rescaled units (but without using the *tilde* symbol explicitly) which introduces small parameters $\delta_{\mathcal{D}}, \delta_{\mathcal{I}}$. After truncating expansion at a given order, as a last step we will restore the original units, thus eliminating the generalized Knudsen numbers from the equations.

3 Chapman-Enskog theory for the Boltzmann equation

ChE expansion [2–4] was the first theory, which successfully connected microscopic dynamics of Boltzmann equation to the equations of hydrodynamics. It was developed by S. Chapman and D. Enskog more than one hundred years ago. Originally formulated for three-dimensional classical gas, it has uncovered not only the universal NS equations but also provided quantitative predictions for transport coefficients in terms of explicit integral equations. In this section we write a summary of this classic result. It provides context for the generalization of the method to quantum integrable theories developed in the next section.

We start with the Boltzmann equation written as

$$\partial_t \rho + v \partial_x \rho = I_B[\rho], \quad (65)$$

where $\rho(x, v)$ is one-particle distribution function on a phase space and $I_B[\rho]$ is the Boltzmann collision integral, a nonlinear functional of the ρ distribution. For the sake of this discussion, we do not discuss the of the operator I_B and consider one-dimensional system. The main idea behind ChE expansion is based on two important observations about the Boltzmann equation:

- The only stationary states of collision integral $I_B[\rho]$ are space-varying boosted thermal states. For classical gas these are given by boosted Maxwell-Boltzmann distributions

$$\rho_{\text{bth}}(x, v) = \frac{\varrho(x)}{\sqrt{2\pi T(x)}} \exp\left(-\frac{(v - u(x))^2}{2T(x)}\right), \quad (66)$$

where $\varrho(x), u(x), T(x)$ are density, velocity and temperature of the gas at the point x , respectively. We have set $m = 1$ and $k_B = 1$. The process of relaxation from a generic state towards local boosted thermal states happens on a characteristic *relaxation*

time-scale τ_{I_B} , which depends on the interaction potential between particles encoded in collision integral $I_B[\rho]$ and on the state itself.

We emphasize here that in general the states (66) are stationary states of collision integral term only and some dynamics is generated by the streaming term $v\partial_x\rho$. Global equilibrium states of (65) correspond to boosted thermal states with hydrodynamic fields uniform in space $\varrho(x), u(x), T(x) = \text{const}$, for which the streaming term vanishes as well.

- Collision integral conserves density of particles, momentum and energy. This is equivalent to the existence of three collision invariants $\mathcal{H}_0(v) = 1$, $\mathcal{H}_1(v) = v$, $\mathcal{H}_2(v) = \frac{1}{2}v^2$, such that

$$\int dv \mathcal{H}_a(v) I_B[\rho](v) = 0, \quad a = 0, 1, 2. \quad (67)$$

Naturally, collision invariants can be directly related to hydrodynamic fields $\varrho(x), u(x), e(x)$, which are given by

$$\varrho(x) = \langle \mathcal{H}_0 \rangle, \quad u(x) = \frac{\langle \mathcal{H}_1 \rangle}{\langle \mathcal{H}_0 \rangle}, \quad e(x) = \frac{\langle \mathcal{H}_2 \rangle}{\langle \mathcal{H}_0 \rangle} - \frac{1}{2}u(x)^2, \quad (68)$$

where $\langle \cdot \rangle$ denotes average over velocities with respect to ρ . What is more, the temperature field is simply given by $e(x) = \frac{1}{2}T(x)$, assuming that the system is in local thermodynamic equilibrium. We collectively denote the hydrodynamic fields as $\{\varrho_a(x, t)\}$.

The observations listed above point towards the existence of two characteristic timescales in the system. Starting from an initial distribution, on a timescale given by τ_{I_B} the system will *locally thermalize*, such that it can be described by hydrodynamic fields $\varrho(x), u(x)$ and $T(x)$. On a longer, hydrodynamic timescale τ_h the streaming term will be dominant and these fields will evolve according to the standard equations of hydrodynamics. In order to see their structure explicitly, we multiply the Boltzmann equation with collision invariants and integrate over dv , similarly as in Sec. 2.5. We recover three equations which read

$$\begin{aligned} \partial_t \varrho + \partial_x(\varrho u) &= 0, \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + \mathcal{P}) &= 0, \\ \partial_t(\varrho e) + \partial_x(u\varrho e + \mathcal{J}) + \mathcal{P}\partial_x u &= 0, \end{aligned} \quad (69)$$

with dynamical pressure and heat current

$$\mathcal{P} = \int dv (v - u) v \rho(v), \quad (70)$$

$$\mathcal{J} = \frac{1}{2} \int dv (v - u)^3 \rho(v). \quad (71)$$

The hydrodynamic equations presented above are not closed as the dynamical pressure and heat current depend on the whole distribution ρ , not only on its first three moments (or equivalently, on hydrodynamic fields). The closure of equation will be the central point of the ChE expansion procedure, which we now discuss.

The two observations about Boltzmann equation listed before can be turned into perturbative expansion. To formally motivate it, we start with estimating the order of terms in (65). We assume that the state varies at a hydrodynamic length-scale l_h and the particles have some characteristic velocity $\bar{v}$. Then the hydrodynamic timescale associated to the streaming term

is $\tau_h = l_h/\bar{v}$. In the dimensionless units $\tilde{x} = x/l_h$, $\tilde{t} = t/\tau_h$, $\tilde{v} = v/\bar{v}$ and explicitly extracting the timescale from collision term $\tilde{I}_B = \frac{1}{\tau_{I_B}} I_B$ we can write Boltzmann equation as

$$\partial_{\tilde{t}} \rho + \tilde{v} \partial_{\tilde{x}} \rho = \frac{1}{\delta} \tilde{I}_B[\rho], \quad (72)$$

where $\delta = \tau_{I_B}/\tau_h$ defines the so-called Knudsen number [2]. We wish to address hydrodynamic limit of slow (in comparison to microscopic scales) space and time variation, hence Knudsen number is considered to be small $\delta \ll 1$. The form of Boltzmann equation (72) suggests perturbative solution in δ and indeed, in the ChE method we essentially assume that

$$\rho(x, v; t) = \rho^{(0)}(x, v; t) \left(1 - \sum_{n=1}^{\infty} \delta^n \rho^{(n)}(x, v; t) \right). \quad (73)$$

However, the expansion cannot be considered to be fully systematic as it is supplemented by additional assumptions which we will discuss now. Before that and in what follows we drop tildes from the notation.

Firstly, the crucial aspect of the method is the assumption about the time evolution of distribution ρ . We assume that it depends on time through three time-dependent hydrodynamic fields collectively denoted $\{\varrho_a(x, t)\}$,

$$\rho(x, v; t) \rightarrow \rho(x, v | \{\varrho_a(x, t)\}). \quad (74)$$

Moreover, there is an assumption about expansion which can be formulated as the following conditions

$$\langle \mathcal{K}_a \rangle = \int dv \mathcal{K}_a(v) \rho^{(0)}(x, v), \quad 0 = \int dv \rho^{(0)}(x, v) \mathcal{K}_a(v) \rho^{(n)}(x, v) = (\mathcal{K}_a | \rho^{(n)}), \quad (75)$$

for $n \geq 1$, which means that the averages of collision invariants (and in consequence, hydrodynamic fields) are given solely by the zeroth order contribution to the distribution $\rho^{(0)}$. In other words, we essentially assume that the hydrodynamic fields are not expanded in δ and corrections $\{\rho^{(n)}\}_{n \geq 1}$ are orthogonal to collision invariants (in the sense of the hydrodynamic inner product $(\cdot | \cdot)$, see (26), note that for classical particles $\mathcal{C} = \rho$). However, we crucially assume that dynamical pressure and heat current (70) do admit expansion in δ

$$\mathcal{P} = \sum_{n=0}^{\infty} \delta^n \mathcal{P}^{(n)}, \quad \mathcal{J} = \sum_{n=0}^{\infty} \delta^n \mathcal{J}^{(n)}. \quad (76)$$

Expanding now the problem to different orders in δ yields hydrodynamic equations at different scales.

3.1 Euler scale

We proceed now to (72) and plug the expansion (73). At zero-th order in δ we find

$$I_B[\rho^{(0)}] = 0. \quad (77)$$

From the properties of the scattering integral this equation implies that the distribution is locally thermal

$$\rho^{(0)}(x, v; t) = \rho_{\text{bth}}(x, v | \{\varrho_a(t)\}). \quad (78)$$

With this solution, we may evaluate

$$\mathcal{P}^{(0)} = P = \varrho T, \quad \mathcal{J}^{(0)} = 0, \quad (79)$$

where by P we have denoted the hydrostatic pressure of ideal gas. Moreover, at this order of expansion the hydrodynamic equations take the form of *Euler equations*

$$\begin{aligned}\partial_t \varrho &= -\partial_x(\varrho u), \\ \partial_t u &= -u\partial_x u - \frac{T}{\varrho}\partial_x \varrho - \partial_x T, \\ \partial_t T &= -2T\partial_x u - u\partial_x T,\end{aligned}\tag{80}$$

which do not lead to entropy production in the system [3]. We point out that these are Euler equations with thermodynamics of ideal gas, even though interactions encoded in Boltzmann term were necessary for their emergence.

3.2 Navier-Stokes scale

We go now to the first order in δ . We find an equation

$$\partial_t \rho^{(0)}(x, v; t) + v\partial_x \rho^{(0)}(x, v; t) = [\Gamma_B \rho^{(1)}](x, v; t),\tag{81}$$

where $\Gamma_B = -(\delta I_B / \delta \rho)|_{\rho^{(0)}} \rho^{(0)}$. More explicitly we have

$$\sum_a \frac{\partial \rho^{(0)}}{\partial \varrho_a} G_a^{(0)}(\{\varrho_b\}) + v \sum_a \frac{\partial \rho^{(0)}}{\partial \varrho_a} \partial_x \varrho_a = [\Gamma_B \rho^{(1)}](x, v; t),\tag{82}$$

where derivatives $\frac{\partial \rho^{(0)}}{\partial \varrho_a}$ can be explicitly computed. Moreover, the operator $G_a^{(n)}(\{\varrho_b\})$ defines the dynamics of hydrodynamic field ϱ_a at n -th order in δ (dependence on δ enters via hydrodynamic pressure and heat current)

$$\partial_t \varrho_a = \sum_{n=0}^{\infty} \delta^n G_a^{(n)}(\{\varrho_b\}).\tag{83}$$

The expressions for $G_a^{(0)}(\{\varrho_b\})$ can be simply read off from (80). After short calculation we get

$$\rho^{(0)}(x, v) \left(\frac{u(x) - v}{T(x)^2} \left(\frac{3}{2} T(x) - \frac{(v - u(x))^2}{2} \right) \partial_x T \right) = [\Gamma_B \rho^{(1)}](x, v).\tag{84}$$

We evaluate this equation at $v = \xi + u$. In the relative ξ variable, the equation can be expressed with local unboosted thermal state $\bar{\rho}^{(0)}(\xi) = \rho^{(0)}(\xi + u)$ (corresponding to the same hydrodynamic fields $\varrho(x), T(x)$ but with $u(x) = 0$) and $\bar{\rho}^{(1)}(\xi) = \rho^{(1)}(\xi + u)$. We then find

$$\bar{\rho}^{(0)}(\xi) \frac{\xi}{T^2} \left(\frac{\xi^2}{2} - \frac{3}{2} T \right) \partial_x T = [\bar{\Gamma}_B \bar{\rho}^{(1)}](\xi).\tag{85}$$

Transformation between Eqs. (84) and (85) is in fact a Galilean boost and illustrates Galilean invariance of the theory and as a result of the transport coefficients.

Assuming that $\bar{\rho}^{(1)}(\xi) = \bar{\phi}(\xi) \partial_x T$, we find that function $\bar{\phi}(\xi)$ fulfills an integral equation

$$[\bar{\Gamma}_B \bar{\phi}](\xi) = \chi(\xi), \quad \chi(\xi) = \bar{\rho}^{(0)}(\xi) \frac{\xi}{T^2} \left(\frac{\xi^2}{2} - \frac{3}{2} T \right),\tag{86}$$

with additional conditions (75)

$$(\mathcal{H}_a | \bar{\phi}) = 0, \quad a = 0, 1, 2.\tag{87}$$

We note that the function $\bar{\phi}(\xi)$ is antisymmetric (this follows from antisymmetry of $\chi(\xi)$ and properties of $\bar{\Gamma}_B$) and compute corrections to hydrodynamic pressure and heat current finding

$$\mathcal{P}^{(1)} = 0, \quad \mathcal{J}^{(1)} = -\frac{1}{2} \int d\xi \bar{\rho}^{(0)}(\xi) \xi^3 \bar{\phi}(\xi) \partial_x T, \quad (88)$$

recovering Newton's and Fourier's laws (2) with

$$\eta = 0, \quad \kappa = \frac{1}{2} \int d\xi \bar{\rho}^{(0)}(\xi) \xi^3 \bar{\phi}(\xi). \quad (89)$$

We have thus found NS equations with vanishing bulk viscosity. This is a known result for a weakly perturbed ideal gas [2, 3]. In order to evaluate thermal conductivity κ one has to find $\bar{\phi}(\xi)$ which involves solving the integral equation (86).

The expansion in δ can be carried out to higher orders which leads to so-called Burnett hydrodynamic equations [2]. We close our discussion at the first order and point out that the found expressions for transport coefficients (89) can be understood as special cases of the general formulas (143) (or (A.18)) derived by us in this work. One can recover them by plugging non-interacting limits of hydrodynamic matrices, diffusion operator and thermodynamic quantities computed for ideal gas.

4 Chapman-Enskog theory for the GHD-Boltzmann equation

We apply now the ChE method to the GHD-Boltzmann equation. The procedure parallels the classical method presented in the previous section, which can be summarized as a two-step scheme:

1. Write suitable expansion of the state ρ_p in parameters which are small in hydrodynamic limit. Assume that time dependence of the state is given by the time-dependence of the conserved hydrodynamic fields, which are not expanded in the small parameter of the theory. Expand GHD-Boltzmann equation and solve it order-by-order finding corrections to the stationary state obtained at the leading order.
2. Evaluate corresponding corrections to the hydrodynamic pressure $\mathcal{P}$ and heat current $\mathcal{J}$. These after plugging into (57), yield hydrodynamic equations at subsequent orders (Euler scale, Navier-Stokes scale and so on) and provide expressions for transport coefficients.

However, there are two main differences with respect to the previous section: a) the thermodynamics of the underlying theory is not of a free system, b) there are two expansion parameters. In the general discussion presented in this section the second point will be especially important. More precisely as we will see the presence of the two expansion parameters leads to two types of contributions to the transport coefficients. Instead, the thermodynamics of the system will be important for the properties of the transport coefficients. Ultimately, it is the thermodynamics that is responsible for non-zero bulk viscosity unlike in the ideal gas. This we discuss in further sections.

4.1 General structure

Recall that the central assumption of the ChE method is that the time dependence of ρ_p enters through the values of hydrodynamic fields, which we again collectively denote as $\{\varrho_a(x, t)\}$.

This then leads to the NS equations formulated in terms of these fields. However, equivalently we can assume that the time dependence of ρ_p is governed by the expectation values of the conserved charges $\{\varphi_a(x, t)\}$. The resulting NS equations will be then automatically formulated in terms of them, or by using charge and current susceptibility matrices in terms of the corresponding chemical potentials. We will mainly use the second option as it provides a more symmetric presentation of the problem but occasionally we will also display results for the hydrodynamic fields as they are more standard.

We choose local conserved charges to control the time-dependence of ρ_p ,

$$\rho_p(\lambda; x, t) \rightarrow \rho_p(\lambda, x | \{\varphi_a(x, t)\}), \quad (90)$$

with $\rho_p(\lambda, x | \{\varphi_a(x, t)\})$ called *normal solutions*. Their time evolution is given by

$$\partial_t \rho_p = \sum_{a=0,1,2} \frac{\partial \rho_p}{\partial \varphi_a} \partial_t \varphi_a = \sum_{a=0,1,2} \frac{\partial \rho_p}{\partial \varphi_a} G_a(\{\varphi_b\}), \quad (91)$$

where we defined (below $a, b = 0, 1, 2$)

$$G_a(\{\varphi_b\}) = -\partial_x \mathcal{J}_a(\{\varphi_b\}) + \frac{\delta_{\mathfrak{D}}}{2} \partial_x \int d\lambda \mathcal{H}_a(\lambda) (\mathfrak{D} \partial_x \rho_p)(x, \lambda | \{\varphi_b\}), \quad (92)$$

in accordance with the time evolution of the conserved charges (31). $G_a(\{\varphi_b\})$ are nonlinear functionals of ρ_p and determine the dynamics of the hydrodynamic fields

$$\partial_t \varphi_a = G_a(\{\varphi_b\}). \quad (93)$$

The GHD-Boltzmann equation for the normal solutions becomes

$$\sum_{a=0,1,2} \frac{\partial \rho_p}{\partial \varphi_a} G_a(\varphi_b) + \partial_x (v \rho_p) = \frac{\delta_{\mathfrak{D}}}{2} \partial_x (\mathfrak{D} \partial_x \rho_p) + \frac{1}{\delta_{\mathcal{I}}} \mathcal{I}[\rho_p]. \quad (94)$$

This is an ordinary differential equation in x in which time t plays a role of a parameter. We assume that the normal solutions can be expanded as follows

$$\rho_p = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{\mathfrak{D}}^n \delta_{\mathcal{I}}^m \rho_p^{(n,m)}, \quad (95)$$

where $\rho_p^{(n,m)} \equiv \rho_p^{(n,m)}(x, \lambda | \{\varphi_a(x, t, \lambda)\})$. To make closer connection with the standard expansion we reparametrize the corrections as

$$\rho_p^{(n,m)} = -\mathcal{C}^{(0,0)} \epsilon_0^{(n,m)}, \quad (96)$$

where $\mathcal{C}^{(0,0)}$ is the kernel $\mathcal{C}$ evaluated on $\rho_p^{(0,0)}$. The reason for such change is the following. We would like to think of small corrections as perturbations to the bare pseudoenergy of the state, a quantity which has a natural expansion in ultra-local basis. Note that in the classical case the corrections are in fact of the form $-\rho^{(0)} \rho^{(n)}$, $n \geq 1$ which agree with our definition after realizing that for classical non-interacting gas $\mathcal{C}^{(0)} = \rho^{(0)}$. In what follows, we will drop the superscript from matrix $\mathcal{C}$.

A crucial point of the method is that we do not expand φ_a in $\delta_{\mathfrak{D}}, \delta_{\mathcal{I}}$. This implies that local values of the conserved charges are determined solely from $\rho_p^{(0,0)}$,

$$\begin{aligned} \varphi_a(x, t) &= \int d\lambda \mathcal{H}_a(\lambda) \rho_p^{(0,0)}(x, \lambda | \{\varphi_b\}), \\ 0 &= \int d\lambda \mathcal{H}_a(\lambda) (\mathcal{C} \epsilon_0^{(n,m)})(x, \lambda | \{\varphi_b\}) = (\mathcal{H}_a | \epsilon_0^{(n,m)}), \quad (n, m) \neq (0, 0). \end{aligned} \quad (97)$$

Which once again means that corrections $\epsilon_0^{(n,m)}$ are orthogonal to collision invariants in the sense of hydrodynamic scalar product. These equations form constraints (constitutive conditions) on the expansion of the normal solutions. The evolution operator for hydrodynamic fields is expanded as well

$$G_a(\{\varphi_b\}) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{\mathfrak{D}}^n \delta_{\mathcal{I}}^m G_a^{(n,m)}(\{\varphi_b\}). \quad (98)$$

The first equation (level 1 ChE approximation) in the hierarchy is

$$\mathcal{I}[\rho_p^{(0,0)}] = 0, \quad (99)$$

which implies that $\rho_p^{(0,0)}$ is a stationary state of the collision integral, i.e. thermal boosted state. The next ones, at level 2, are

$$\Gamma \epsilon_0^{(1,0)} = 0, \quad \sum_{a=0,1,2} \frac{\partial \rho_p^{(0,0)}}{\partial \varphi_a} G_a^{(0,0)}(\{\varphi_b\}) + \partial_x \left(v^{(0,0)} \rho_p^{(0,0)} \right) = \Gamma \epsilon_0^{(0,1)}, \quad (100)$$

we defined here $\Gamma = -\mathcal{I}^{(1)}\mathcal{C}$ where $\mathcal{I}^{(1)}$ is the functional derivative of $\mathcal{I}[\rho_p]$ evaluated at $\rho_p = \rho_p^{(0,0)}$. The first equation is solved by any function from the kernel of operator Γ . Its kernel contains collision invariants $\mathcal{N}_a$ with $a = 0, 1, 2$ and therefore any solution is of the form $\epsilon_0^{(1,0)} = \sum_{a=0}^2 c_a(x, t) \mathcal{N}_a(\lambda)$. However, according to the constitutive conditions (97) only the leading term in the ChE expansion contributes to the hydrodynamic fields. This enforces that $\epsilon_0^{(1,0)} = 0$. Alternatively, equation $\Gamma \epsilon_0^{(1,0)} = 0$ can be understood as coming from expanding

$$\mathcal{I}[\rho_p^{(0,0)} - \delta_{\mathfrak{D}} \mathcal{C} \epsilon_0^{(1,0)}] = 0. \quad (101)$$

On the other hand, this equation itself just states that the combination $\rho_p^{(0,0)} - \delta_{\mathfrak{D}} \mathcal{C} \epsilon_0^{(1,0)}$ is a stationary state. Hence, choosing $\epsilon_0^{(1,0)} \neq 0$ has an effect of shifting the stationary, thermal boosted state and therefore is redundant. The role of the constitutive conditions is to remove this redundancy.

The second equation of the ChE hierarchy at level 2 given in (100) determines $\epsilon_0^{(0,1)}$. Note that it does not involve explicitly the diffusion term of the GHD and is structurally similar to the standard ChE equation at level 2, c.f. (82).

As mentioned earlier, hydrodynamic equations at subsequent orders follow from evaluating corrections to pressure $\mathcal{P}$ and heat current $\mathcal{J}$. The same equations in $\{\varphi_0, \varphi_1, \varphi_2\}$ or $\{\beta_0, \beta_1, \beta_2\}$ variables follow now from evaluating $G_a(\{\varphi_b\})$ on the series expansion of ρ_p . To discuss a general structure of the resulting equations it is useful to decompose G_a in two contributions differing in the scaling parameters,

$$\begin{aligned} G_{a,v}(\{\varphi_b\}) &= - \int d\lambda \mathcal{N}_a(\lambda) \partial_x (v \rho_p(x, \lambda | \{\varphi_b\})) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{\mathfrak{D}}^n \delta_{\mathcal{I}}^m G_{a,v}^{(n,m)}(\{\varphi_b\}), \\ G_{a,\mathfrak{D}}(\{\varphi_b\}) &= \frac{\delta_{\mathfrak{D}}}{2} \int d\lambda \mathcal{N}_a(\lambda) \partial_x (\mathfrak{D} \partial_x \rho_p(x, \lambda | \{\varphi_b\})) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \delta_{\mathfrak{D}}^n \delta_{\mathcal{I}}^m G_{a,\mathfrak{D}}^{(n,m)}(\{\varphi_b\}). \end{aligned} \quad (102)$$

The hydrodynamic equations, up to and including the first order terms, are then

$$\partial_t \varphi_a = \underbrace{G_{a,v}^{(0,0)}[\rho_p^{(0,0)}]}_{\text{Euler}} + \underbrace{\delta_{\mathfrak{D}} G_{a,\mathfrak{D}}^{(1,0)}[\rho_p^{(0,0)}] + \delta_{\mathcal{I}} G_{a,v}^{(0,1)}[\epsilon_0^{(0,1)}]}_{\text{NS}}, \quad (103)$$

where in the brackets $[\cdot]$ we have indicated dependence on correction to the state (95).

In the following sections we will derive the Euler and NS hydrodynamics equations from the ChE equation. The two relevant functions are $\rho_p^{(0,0)}$ and $\epsilon_0^{(0,1)}$ because $\epsilon^{(1,0)} = 0$. Therefore, to simplify the notation, we write $\rho_p^{(0)} \equiv \rho_p^{(0,0)}$, $\epsilon_0^{(1)} \equiv \epsilon_0^{(0,1)}$ and $G_a^{(0)} \equiv G_a^{(0,0)}$. The ChE equations are then

$$\mathcal{I}[\rho_p^{(0)}] = 0, \quad \sum_{a=0,1,2} \frac{\partial \rho_p^{(0)}}{\partial \varphi_a} G_a^{(0)}(\varphi_b) + \partial_x \left(v^{(0)} \rho_p^{(0)} \right) = \Gamma \epsilon_0^{(1)}. \quad (104)$$

It is worth emphasising again that the GHD diffusion enters these equations only through $G_a(\{\varphi_b\})$. In the following two sections we discuss the equations (104) and (103) at the Euler and at the NS scales respectively.

There is also an alternative route to the NS equations which capitalizes on the knowledge of their form. Therefore, instead of computing G_a and rediscovering the NS equations, we can directly compute the hydrodynamic pressure $\mathcal{P}$ and heat current $\mathcal{J}$. Both quantities have an expansion in δ 's

$$\mathcal{P} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{\mathcal{D}}^n \delta_{\mathcal{I}}^m \mathcal{P}^{(n,m)}, \quad \mathcal{J} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \delta_{\mathcal{D}}^n \delta_{\mathcal{I}}^m \mathcal{J}^{(n,m)}, \quad (105)$$

and knowing the expansion of the normal solutions they can be evaluated order by order.

This offers a shortcut in deriving the hydrodynamics and occasionally we will take advantage of it. Note however, that obtaining the hydrodynamics by computing G_a is more systematic as it does not anticipate the structure of the resulting equations. The later, in the present context, is dictated by the Galilean invariance. This is specific to continuum non-relativistic systems and does not hold for example in the spin chains systems or relativistic theories.

4.2 Euler scale

Consider first of the equations in (104). This equation implies that $\rho_p^{(0)}$ is a stationary state of the collision integral. Because the collision integral acts only on the rapidity dependence of $\rho_p^{(0)}$, the parameters of the stationary state may vary in space and time. This implies that a stationary state of the collision integral is a boosted thermal state that can be characterized by 3 generalized temperatures $\{\beta_0, \beta_1, \beta_2\}$ or equivalently by hydrodynamic fields $\{\varrho, u, T\}$. With this, we can evaluate dynamic pressure and heat current at the leading order

$$\mathcal{P}^{(0,0)} = \mathcal{P}_v^{(0,0)} = P, \quad \mathcal{J}^{(0,0)} = \mathcal{J}_v^{(0,0)} = 0, \quad (106)$$

where P is the thermodynamic pressure from TBA. Computations involve some operations on TBA objects and are demonstrated in Supplemental Material of [51]. We note that once again that diffusion contributions do not enter at zeroth order due to $\delta_{\mathcal{D}}$ prefactor in (64). Plugging this into (57) we recover the standard Euler hydrodynamics

$$\partial_t \varrho = -\partial_x (\varrho u), \quad \partial_t (\varrho u) = -\partial_x (\varrho u^2 + P), \quad \partial_t (\varrho e) = -\partial_x (\varrho u e) - P \partial_x u. \quad (107)$$

We note now that the same equations can be derived in $\{\beta_0, \beta_1, \beta_2\}$ variables by finding operators $G_a^0(\{\varphi_b\})$ and expressing them in these variables using (27),

$$G_a^{(0)}(\{\varphi_b\}) = -\hbar_a \partial_x \left(v^{(0)} \rho_p^{(0)} \right) = \mathcal{B}_{ab} \partial_x \beta_b, \quad (108)$$

where we used eq. (27). This leads to the following representation of the Euler scale equations

$$\sum_{b=0}^2 (\mathcal{C}_{ab} \partial_t \beta_b + \mathcal{B}_{ab} \partial_x \beta_b) = 0, \quad a = 0, 1, 2, \quad (109)$$

which will be useful in the next section. We note that these equations have appeared previously in [79].

4.3 Navier-Stokes scale

At the NS scale the hydrodynamic equations receive two types of corrections. The first one is due to the GHD diffusion and follow from evaluating the diffusion operator on the stationary state $\rho_p^{(0)}$. The second type of corrections come from the collision integral and require solving the second ChE equation. We start with the former. In terms of the corrections to pressure and heat current we have

$$\mathcal{P}_{\mathfrak{D}}^{(1,0)} = -\hbar_1 \mathfrak{D} \mathcal{C} \partial_x \rho_p^{(0)} / 2, \quad u \mathcal{P}_{\mathfrak{D}}^{(1,0)} + \mathcal{J}_{\mathfrak{D}}^{(1,0)} = -\hbar_2 \mathfrak{D} \mathcal{C} \partial_x \rho_p^{(0)} / 2, \quad (110)$$

moreover, contributions to G_a at this level are

$$G_{a,\mathfrak{D}}^{(1,0)} = \frac{1}{2} \partial_x \left(\hbar_a \mathfrak{D} \partial_x \rho_p^{(0)} \right). \quad (111)$$

In both cases we evaluate the coefficients by transferring the spatial derivative from the distribution to the generalized temperatures using (27)

$$\partial_x \rho_p^{(0)} = -\mathcal{C} \sum_{b=0}^2 \hbar_b \partial_x \beta_b. \quad (112)$$

This reduces the problem to determining $\hbar_a \mathfrak{D} \mathcal{C} \hbar_b$ which in principle leads to 9 coefficients. However, the diffusion operator has left zero eigenvector $\hbar_0$ and right zero eigenvector $\mathcal{C} \hbar_0$. This simplifies the problem from determining 9 to 4 coefficients. Furthermore, from the Galilean invariance it follows that in fact there are only 2 independent coefficients. To demonstrate this we use the boost operators to write

$$\hbar_a \mathfrak{D} \mathcal{C} \hbar_b = \hbar_a b_u^{-1} \tilde{\mathfrak{D}} \tilde{\mathcal{C}} b_u \hbar_b, \quad (113)$$

from which we find

$$\begin{aligned} \hbar_1 \mathfrak{D} \mathcal{C} \hbar_1 &= \hbar_1 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_1, & \hbar_1 \mathfrak{D} \mathcal{C} \hbar_2 &= u \hbar_1 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_1, \\ \hbar_2 \mathfrak{D} \mathcal{C} \hbar_1 &= u \hbar_1 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_1, & \hbar_2 \mathfrak{D} \mathcal{C} \hbar_2 &= \hbar_2 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_2 + u^2 \hbar_1 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_1, \end{aligned} \quad (114)$$

which indeed involve only 2 independent coefficients $\hbar_1 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_1$ and $\hbar_2 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_2$ which depend on the local density $\varrho(x, t)$ and local temperature $T(x, t)$ characterizing the local thermal state $\rho_p^{(0)}$.

For the dynamic pressure and the heat current we get

$$\mathcal{P}_{\mathfrak{D}}^{(1,0)} = -\zeta_{\mathfrak{D}} \partial_x u, \quad \mathcal{J}_{\mathfrak{D}}^{(1,0)} = -\kappa_{\mathfrak{D}} \partial_x T, \quad (115)$$

with the corresponding transport coefficients

$$\zeta_{\mathfrak{D}} = \hbar_1 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_1 / (2T), \quad \kappa_{\mathfrak{D}} = \hbar_2 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_2 / (2T^2). \quad (116)$$

Moreover, expressing the G operators through the two independent coefficients $\hbar_1 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_1$ and $\hbar_2 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_2$, we find

$$\begin{aligned} G_{0,\mathfrak{D}}^{(1,0)}[\rho_p^{(0)}] &= 0, \\ G_{1,\mathfrak{D}}^{(1,0)}[\rho_p^{(0)}] &= \frac{1}{2} \partial_x \left(\hbar_1 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_1 \beta_2 \partial_x u \right), \\ G_{2,\mathfrak{D}}^{(1,0)}[\rho_p^{(0)}] &= -\frac{1}{2} \partial_x \left(\hbar_2 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_2 \partial_x \beta_2 \right) + \frac{1}{2} \partial_x \left(u \hbar_1 \tilde{\mathfrak{D}} \tilde{\mathcal{C}} \hbar_1 \beta_2 \partial_x u \right). \end{aligned} \quad (117)$$

In writing this expressions we have used that (it follows from (42))

$$\partial_x \beta_1 + u \partial_x \beta_2 = -\beta_2 \partial_x u, \quad (118)$$

which points to using $u = -\beta_1/\beta_2$ as a hydrodynamic field instead of β_1 .

The knowledge of $G_{a,\mathfrak{D}}^{(1,0)}[\rho_p^{(0)}]$ allows us to write down the equations for the hydrodynamic fields which beside the Euler scale include also the effect of the GHD diffusion. As we shall see, the structure of the contributions to $G_{a,\mathfrak{D}}^{(1,0)}[\rho_p^{(0)}]$ and $G_{a,v}^{(1)}[\epsilon_0^{(1)}]$ is the same. Therefore, we will discuss the resulting equations after computing the contribution from the collision term. This requires solving the ChE equation which we discuss next.

We turn now our attention to the second equation of (104) which determines $\epsilon_0^{(1)}$ and in turn determines $G_{a,v}^{(1)}[\epsilon_0^{(1)}]$. The equation is

$$\partial_t \rho_p^{(0)} + \partial_x (v^{(0)} \rho_p^{(0)}) = \Gamma \epsilon_0^{(1)}, \quad (119)$$

and

$$G_{a,v}^{(1)}[\epsilon_0^{(1)}] = \partial_x (\mathcal{H}_a \mathcal{B} \epsilon_0^{(1)}). \quad (120)$$

Recall that since $\rho_p^{(0)}$ is a thermal boosted state it is completely characterized by fields $\{\varrho_\beta\}$ or equivalently by the three chemical potentials β_a with $a = 0, 1, 2$. Equation (119) determines the correction to pseudoenergy $\epsilon_0^{(1)}$ which additionally has to fulfill the constitutive relations (97) which can be written as

$$0 = \mathcal{H}_a \mathcal{C} \epsilon_0^{(1)}, \quad a = 0, 1, 2. \quad (121)$$

We observe that the constitutive relation itself is enough to conclude that $G_{0,v}^{(1)}[\epsilon_0^{(1)}] = 0$. Indeed, we have

$$G_{0,v}^{(1)}[\epsilon_0^{(1)}] = \partial_x (\mathcal{H}_0 \mathcal{B} \epsilon_0^{(1)}) = \partial_x (\mathcal{H}_1 \mathcal{C} \epsilon_0^{(1)}) = 0. \quad (122)$$

Together with the first formula in (117) this implies that there is no diffusion of the density as expected in a single component fluid.

We now go back to the full problem (119) and start by expressing the left hand side by the derivatives of the chemical potentials

$$\partial_t \rho_p^{(0)} + \partial_x (v^{(0)} \rho_p^{(0)}) = - \sum_{b=0}^2 (\mathcal{C} \mathcal{H}_b \partial_t \beta_b + \mathcal{B} \mathcal{H}_b \partial_x \beta_b), \quad (123)$$

where we used (27). The chemical potentials evolve now according to the Euler scale equations (109).

The Euler scale hydrodynamics is a truncation of this set to 3 equations for $a = 0, 1, 2$. Therefore, this expression in general is non-zero and gives rise to finite $\epsilon_0^{(1)}$. Denote by $\mathcal{C}^{\text{Eul}}$ and $\mathcal{B}^{\text{Eul}}$ reductions of the susceptibility matrices $\mathcal{C}_{ab}$ and $\mathcal{B}_{ab}$ to a subspace of $a, b = 0, 1, 2$. The Euler scale hydrodynamics (109) gives

$$\partial_t \beta_a = - \sum_{b=0}^2 \mathcal{A}_{ab}^{\text{Eul}} \partial_x \beta_b, \quad (124)$$

where $\mathcal{A}^{\text{Eul}} = (\mathcal{C}^{\text{Eul}})^{-1} \mathcal{B}^{\text{Eul}}$ (invertibility of $\mathcal{C}^{\text{Eul}}$ is demonstrated in Sec. 2.3) and

$$\partial_t \rho_p^{(0)} + \partial_x (v^{(0)} \rho_p^{(0)}) = \sum_{b,c=0}^2 (\delta_{bc} \mathcal{B} \mathcal{H}_c - \mathcal{C} \mathcal{H}_b \mathcal{A}_{bc}^{\text{Eul}}) \partial_x \beta_c. \quad (125)$$

This gives then the following equation for $\epsilon_0^{(1)}$,

$$\sum_{b,c=0}^2 (\delta_{bc} \mathcal{B} \mathcal{H}_c - \mathcal{C} \mathcal{H}_b \mathcal{A}_{bc}^{\text{Eul}}) \partial_x \beta_c = \Gamma \epsilon_0^{(1)}. \quad (126)$$

The rapidity variable on the left hand side appears only in the terms in front of the spatial derivative. Therefore, the space-time and rapidity dependence factorize and the solution has to be of the following form

$$\epsilon_0^{(1)}(\lambda; x, t) = - \sum_{c=0}^2 w_c(\lambda) \partial_x \beta_c(x, t). \quad (127)$$

Substituting this expression to equation (126) we find that functions w_c obey

$$\Gamma w_c = \mathcal{B} \mathcal{H}_c - \sum_{b=0}^2 \mathcal{C} \mathcal{H}_b \mathcal{A}_{bc}^{\text{Eul}}, \quad c = 0, 1, 2. \quad (128)$$

Additionally, function $\epsilon_0^{(1)}$ obeys the constitutive relations (121) which translate into the following conditions on functions w_c ,

$$\mathcal{H}_a \mathcal{C} w_c = 0, \quad a, c = 0, 1, 2. \quad (129)$$

Equations (128) and (129) are the final equations determining the structure of $\epsilon_0^{(1)}$. They depend on the local thermal boosted state $\rho_p^{(0)}(\lambda; x, t)$. This in turn is characterized by the density ϱ , temperature T and boost u . From the Galilean invariance of the theory it follows that the collision integral is an invariant object. Therefore, by going to the reference frame in which fluid-cell at position (x, t) is at rest the equations are invariant,

$$\bar{\Gamma} \bar{w}_c = \bar{\mathcal{B}} \mathcal{H}_c - \sum_{b=0}^2 \bar{\mathcal{C}} \mathcal{H}_b \bar{\mathcal{A}}_{bc}^{\text{Eul}}, \quad \mathcal{H}_a \bar{\mathcal{C}} \bar{w}_c = 0, \quad a, c = 0, 1, 2, \quad (130)$$

where all the formulas are now evaluated for a thermal state with density ϱ and temperature T . The transformation rules for Γ and w_c are $\Gamma = b_u^{-1} \bar{\Gamma} b_u$ and $w_c = b_u^{-1} \bar{w}_d (g^T(u))_{dc}$ with a summation over d implied. The details are presented in Appendix A.2.

In a thermal state matrix $\bar{\mathcal{A}}^{\text{Eul}}$ has only four non-zero matrix-elements, c.f. (43),

$$\bar{\mathcal{A}}^{\text{Eul}} = \begin{pmatrix} 0 & \bar{\mathcal{A}}_{01}^{\text{Eul}} & 0 \\ \bar{\mathcal{A}}_{10}^{\text{Eul}} & 0 & \bar{\mathcal{A}}_{12}^{\text{Eul}} \\ 0 & \bar{\mathcal{A}}_{21}^{\text{Eul}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\bar{\mathcal{B}}_{12} \bar{\mathcal{C}}_{02} - \bar{\mathcal{B}}_{01} \bar{\mathcal{C}}_{22}}{\bar{\mathcal{C}}_{02}^2 - \bar{\mathcal{C}}_{00} \bar{\mathcal{C}}_{22}} & 0 \\ \frac{\bar{\mathcal{B}}_{01}}{\bar{\mathcal{C}}_{11}} & 0 & \frac{\bar{\mathcal{B}}_{12}}{\bar{\mathcal{C}}_{11}} \\ 0 & \frac{\bar{\mathcal{B}}_{12} \bar{\mathcal{C}}_{00} - \bar{\mathcal{B}}_{01} \bar{\mathcal{C}}_{02}}{\bar{\mathcal{C}}_{00} \bar{\mathcal{C}}_{22} - \bar{\mathcal{C}}_{02}^2} & 0 \end{pmatrix}. \quad (131)$$

This simplifies the structure of the equations (130). Writing them explicitly we find

$$\bar{\Gamma} \bar{w}_a = \bar{\mathcal{B}} \mathcal{H}_a - \bar{\mathcal{C}} \mathcal{H}_1 \bar{\mathcal{A}}_{1a}^{\text{Eul}}, \quad a = 0, 2, \quad (132)$$

$$\bar{\Gamma} \bar{w}_1 = \bar{\mathcal{B}} \mathcal{H}_1 - \bar{\mathcal{C}} (\mathcal{H}_0 \bar{\mathcal{A}}_{01}^{\text{Eul}} + \mathcal{H}_2 \bar{\mathcal{A}}_{21}^{\text{Eul}}). \quad (133)$$

Recall again that in the Galilean invariant systems $\mathcal{B} \mathcal{H}_0 = \mathcal{C} \mathcal{H}_1$. This implies that $\bar{\mathcal{B}}_{01} = \bar{\mathcal{C}}_{11}$ and $\bar{\mathcal{A}}_{10}^{\text{Eul}} = 1$. In the consequence $\bar{w}_0 = 0$. This reduces the problem to solving for two unknown functions. Notice also that $\bar{w}_1$ is an even function of the rapidity while $\bar{w}_2$ is odd.

Boosting the state in general mixes different contributions according to

$$w_a = b_u^{-1} \sum_{b=0}^2 \bar{w}_b g(u)_{ab}. \quad (134)$$

Thanks to the triangular structure of the transformation matrix and $\bar{w}_0 = 0$, the result is simple

$$w_0 = 0, \quad w_1 = b_u^{-1} \bar{w}_1, \quad w_2 = b_u^{-1} (\bar{w}_2 + u \bar{w}_1), \quad (135)$$

or more explicitly

$$w_0(\lambda) = 0, \quad w_1(\lambda) = \bar{w}_1(\lambda - u), \quad w_2(\lambda) = (\bar{w}_2(\lambda - u) + u \bar{w}_1(\lambda - u)). \quad (136)$$

We evaluate now the $G_{a,v}^{(1)}$ operators. As mentioned earlier $G_{0,v}^{(1)} = 0$. The other two evaluate as follows

$$G_{1,v}^{(1)} = -(\hbar_1 \mathcal{B} w_1 \partial_x \beta_1 + \hbar_1 \mathcal{B} w_2 \partial_x \beta_2), \quad G_{2,v}^{(1)} = -(\hbar_2 \mathcal{B} w_1 \partial_x \beta_1 + \hbar_2 \mathcal{B} w_2 \partial_x \beta_2). \quad (137)$$

We write now

$$\hbar_a \mathcal{B} w_c = \hbar_a b_u^{-1} (\bar{\mathcal{B}} + u \bar{\mathcal{C}}) b_u w_c = \hbar_a b_u^{-1} \bar{\mathcal{B}} b_u w_c, \quad (138)$$

where we used the constitutive relations. More explicitly, we find

$$\begin{aligned} \hbar_1 \mathcal{B} w_1 &= \hbar_1 \bar{\mathcal{B}} \bar{w}_1, & \hbar_1 \mathcal{B} w_2 &= u \hbar_1 \bar{\mathcal{B}} \bar{w}_1, \\ \hbar_2 \mathcal{B} w_1 &= u \hbar_1 \bar{\mathcal{B}} \bar{w}_1, & \hbar_2 \mathcal{B} w_2 &= \hbar_2 \bar{\mathcal{B}} \bar{w}_2 + u^2 \hbar_1 \bar{\mathcal{B}} \bar{w}_1, \end{aligned} \quad (139)$$

which is of the same structure as (114) and therefore the G operators takes also form analogous to (117).

This allows us to write the NS equations in $\{\beta_0, \beta_1, \beta_2\}$ variables

$$\sum_{b=0}^2 \left(\mathcal{C}_{ab} \partial_t \beta_b + \mathcal{B}_{ab} \partial_x \beta_b + \frac{1}{2} \partial_x (D_{ab}^{\text{NS}} \partial_x \beta_b) \right) = 0, \quad a = 0, 1, 2, \quad (140)$$

where the diffusion coefficients are $D^{\text{NS}} = \mathcal{D}\mathcal{C} + D^{\text{coll}}$ with

$$(\mathcal{D}\mathcal{C})_{ab} = \hbar_a \mathcal{D}\mathcal{C} \hbar_b, \quad D_{ab}^{\text{coll}} = 2 \hbar_a \mathcal{B} w_b, \quad (141)$$

expressed through in total 4 transport coefficients $\hbar_1 \bar{\mathcal{D}} \bar{\mathcal{C}} \hbar_1$, $\hbar_2 \bar{\mathcal{D}} \bar{\mathcal{C}} \hbar_2$, $\hbar_1 \bar{\mathcal{B}} \bar{w}_1$ and $\hbar_2 \bar{\mathcal{B}} \bar{w}_2$. In the Galilean invariant systems $D_{0b}^{\text{NS}} = D_{a0}^{\text{NS}} = 0$.

Instead, computing corrections to pressure (55) and heat current (56) we find

$$\mathcal{P}_v^{(1,0)} = -\zeta_{\mathcal{I}} \partial_x u, \quad \mathcal{J}_v^{(1,0)} = -\kappa_{\mathcal{I}} \partial_x T, \quad (142)$$

with

$$\zeta_{\mathcal{I}} = \hbar_1 \bar{\mathcal{B}} \bar{w}_1 / T, \quad \kappa_{\mathcal{I}} = \hbar_2 \bar{\mathcal{B}} \bar{w}_2 / T^2. \quad (143)$$

The Eq. (140) can be recast into the more canonical form (57). In order to do so, we apply (29) to the first two terms and rewrite transport coefficients D^{NS} using (114) and (139). As a result we recover exactly (57) with $\mathcal{P}$ and $\mathcal{J}$ found in this section.

Finally, let us summarize the main results of the method. As described above, we have found four non-trivial transport coefficients D^{NS} which enter the NS equations written for the dynamics of $\{\beta_0, \beta_1, \beta_2\}$. At the same time, we have recovered Newton's and Fourier laws, which appear when NS equations are expressed with hydrodynamic fields. Contributions to transport coefficients are computed straightforwardly from (116). On the other hand, contributions to the transport coefficients from the collision integral (143) are found from solution of integral equations (132). Note that these equations are slightly different than the equations given in the main text of [51]. The difference lies in using the orthonormal (with respect to hydrodynamic inner product (26)) basis $\{h_n\}_{n \in \mathbb{N}}$ in our earlier work as opposed to the ultralocal basis $\{\hbar_n\}_{n \in \mathbb{N}}$ in the present paper. The equivalence between the two representations is demonstrated in Appendix A.3.

4.4 External potential

We generalize now our results to systems with external potential $U(x)$ that couples to the density of particles. The GHD-Boltzman equation is

$$\partial_t \rho_p + \partial_x (v \rho_p) = \frac{1}{2} \partial_x (\mathfrak{D} \partial_x \rho_p) + \mathcal{I}[\rho_p] + (\partial_x U) \partial_\lambda \rho_p. \quad (144)$$

Following the same procedure as before, we start by deriving the mesoscopic conservation laws. They take the expected form, see eq. (6),

$$\begin{aligned} \partial_t q_0 + \partial_x q_1 &= 0, \\ \partial_t q_1 + \partial_x \left(\frac{q_1^2}{q_0} + \mathcal{P} \right) &= -(\partial_x U) q_0, \\ \partial_t q_2 + \partial_x \left(\frac{q_1 q_2}{q_0} + u \mathcal{P} + \mathcal{J} \right) &= -(\partial_x U) q_1, \end{aligned} \quad (145)$$

with $\mathcal{P}$ and $\mathcal{J}$ of the same form as discussed in Section 2.5.

The external potential in principle introduces a new length-scale to the problem. However, at the level of the conservation laws its effect enters only through the hydrodynamic fields. Because of that it can be treated exactly within the ChE method which we now demonstrate.

The GHD equation for the normal solutions takes the following form

$$\sum_{a=0,1,2} \frac{\partial \rho_p}{\partial q_a} G_a(q_b) + \partial_x (v \rho_p) = \frac{1}{2} \partial_x (\mathfrak{D} \partial_x \rho_p) + \mathcal{I}[\rho_p] - \mathfrak{f} \partial_\lambda \rho_p, \quad (146)$$

where $\mathfrak{f} = -\partial_x U$. Solution to this equation is a particle distribution function ρ_p^{ChE} determined by the local values of the 3 conserved charges. From the point of view of the GGE this state can be described by the 3 lowest chemical potentials. We emphasize that the state is not a boosted thermal state and hence the higher chemical potentials are not zero. However, they are not free parameters either, they are functions of the 3 lowest chemical potentials by the virtue of eq. (146).

Because of these relations, the susceptibility matrix of state described by ρ_p^{ChE} does not take the usual form. Indeed, the standard susceptibility matrix assumes that all the chemical potentials are free parameters. We define the ChE susceptibility matrix ($a, b = 0, 1, 2$)

$$C_{ab}^{\text{ChE}} = -\frac{\partial q_a}{\partial \beta_b}, \quad (147)$$

where q_a are now values of conserved charges in state ρ_p^{ChE} . As with the GGE susceptibility matrix, we can use the ChE susceptibility matrix to evaluate various expressions. For example,

$$\partial_\lambda \rho_p^{\text{ChE}} = -C^{\text{ChE}} \mathfrak{h}'_b \beta_b, \quad (148)$$

with the summation over the 3 independent chemical potentials. In a similar fashion we can evaluate the additional contribution to G_a from the external potential

$$\mathfrak{f} \int d\lambda \mathfrak{h}_a(\lambda) \partial_\lambda \rho_p(x, \lambda | \{q_b\}) = -\mathfrak{f} \mathfrak{h}_a C^{\text{ChE}} \mathfrak{h}'_b \beta_b. \quad (149)$$

Collecting then the terms proportional to $\mathfrak{f}$ in eq. (146) we find

$$-\mathfrak{f} \left(\sum_{a=0}^2 \frac{\partial \rho_p}{\partial q_a} \mathfrak{h}_a + 1 \right) C^{\text{ChE}} \mathfrak{h}'_b \beta_b, \quad (150)$$

where used the expression for derivative of the particle distribution with respect to the rapidity introduced above. On the other hand, assuming that the ChE susceptibility matrix is invertible, we have

$$\frac{\partial \rho_p}{\partial q_a} = \sum_{c=0}^2 \frac{\partial \rho_p}{\partial \beta_c} \frac{\partial \beta_c}{\partial q_a} = -\mathcal{C}^{\text{ChE}} \sum_{c=0}^2 \mathcal{C}_{ca}^{\text{ChE}} (\mathcal{C}_{ca}^{\text{ChE}})^{-1}, \quad (151)$$

and therefore

$$\sum_{a=0}^2 \frac{\partial \rho_p}{\partial q_a} \mathcal{C}_{aa}^{\text{ChE}} = -1. \quad (152)$$

Hence, the part proportional to $\mathfrak{f}$ drops from the equations for the normal solutions. Therefore, the external potential does not modify the transport coefficients of the NS equations.

5 Crossover between NS diffusion and KPZ superdiffusion

So far, we have shown the emergence of NS equations in nearly integrable quantum gases described by the Hamiltonian (4). However, transport in one-dimensional momentum conserving fluids is expected to be anomalous at the largest space-time scales [68–71], in contrast to diffusion predicted by NS hydrodynamics. Moreover, the so-called long-time tails [80] in autocorrelation functions are expected to occur, leading to the divergence of Green-Kubo formulas for transport coefficients, again as opposed to finite values found by us. On the other hand, there are numerical and analytical evidences that at intermediate space-time scale conventional NS hydrodynamics prevails [81–85]. In particular in [82], a classical system admitting a collision integral description was considered. At large length scales, anomalous transport was found, whereas for smaller scales the diffusive behavior with diffusion constants predicted by kinetic theory was confirmed.

In this section, we will address these issues. In particular, we will look at the behavior of the system across different length scales and uncover that while hydrodynamics indeed become anomalous at the largest length scales, there is a clear regime, where the NS description (with finite transport coefficients) is the correct hydrodynamic theory of the system. Existence of such a regime is essentially due to the fact, that we are close to the integrable model, which results in large transport coefficients, allowing diffusive terms to compete with anomalous contributions at finite, but small inverse length scales k . The general picture, which emerges from our analysis is presented in Fig. 1. We will present the estimation of different crossover regions in what follows. To start our analysis, let us estimate on which characteristic length scale the NS equations are expected to emerge. The collision integral for a nearly integrable system follows from Fermi's Golden Rule and scales with the small integrability-breaking parameter λ as

$$\mathcal{I} \sim \lambda^2, \quad (153)$$

leading to the scaling of the characteristic length scale $l_{\mathcal{I}}$ discussed in Sec. 2.6 as $l_{\mathcal{I}} \sim \lambda^{-2}$. The crossover from GHD to NS happens around the point, when the Knudsen number $\delta_{\mathcal{I}} = l_{\mathcal{I}}/l$ becomes of order one. This leads to crossover length scale

$$l_{\text{NS/GHD}}^* \sim \lambda^{-2}. \quad (154)$$

The same estimate can be inferred from the analysis of linearized Boltzmann equation done in our other work [51]. There, we found three gapless modes (two sound modes denoted by $\sigma = \pm 1$ and the heat mode denoted by 0) with dispersion relations that at small k behave as $\text{Im}(\omega_{\sigma,0}(k)) = D_{\sigma,0} k^2$, where the diffusion constants are

$$D_0 = \frac{\kappa}{\varrho c_p}, \quad D_{\sigma} = \frac{1}{2\varrho} \left[\left(\frac{1}{c_v} - \frac{1}{c_p} \right) \kappa + \zeta \right], \quad (155)$$

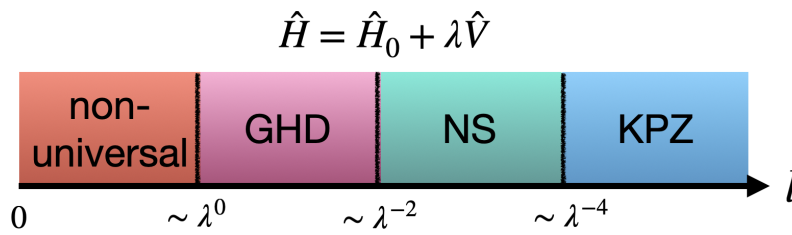


Figure 1: Hydrodynamic universality classes of the nearly integrable system across different length scales l . For $0 < l \lesssim l_0$, where $l_0 \sim \lambda^0$ is the microscopic length scale of integrable Hamiltonian $\hat{H}_0$, the hydrodynamics is not applicable and system behaves in non-universal way. At length scales $l_0 \ll l \ll l_{\text{NS/GHD}}^*$, where $l_{\text{NS/GHD}}^* \sim \lambda^{-2}$ the gas can be effectively described by GHD. For $l_{\text{NS/GHD}}^* \ll l \ll l_{\text{KPZ/NS}}^*$, where $l_{\text{KPZ/NS}}^* \sim \lambda^{-4}$ the gas is in NS hydrodynamic regime with transport coefficients computed in present work. Finally, at the largest length scales $l_{\text{KPZ/NS}}^* \ll l$ the system is characterized by KPZ anomalous transport, as expected for generic non-integrable one-dimensional fluids with momentum conservation.

where c_p is specific heat at constant pressure. As mentioned earlier both viscosity and thermal conductivity have two contributions

$$\kappa = \kappa_{\mathcal{D}} + \kappa_{\mathcal{I}}, \quad \zeta = \zeta_{\mathcal{D}} + \zeta_{\mathcal{I}}. \quad (156)$$

It is clear that transport coefficients stemming from $\mathcal{D}$ do not depend on integrability-breaking parameter λ , as visible from (116). On the other hand, the contributions from $\mathcal{I}$ scale as λ^{-2} , as can be examined from (143) and (132). Therefore, for λ small, contributions from $\mathcal{I}$ dominate over $\mathcal{D}$ contributions leading to an overall scaling of diffusion constants

$$D_{0,\sigma} \sim \lambda^{-2}. \quad (157)$$

We emphasize here that this scaling will be crucial also for the analysis of crossover between NS diffusion and KPZ superdiffusion.

The NS description breaks down, when imaginary part of frequencies become comparable to the gap γ of linearized Boltzmann kernel, proportional to λ^2 as the kernel itself. A comparison $\text{Im}(\omega_{\sigma,0}(k)) = D_{\sigma,0}k^2 \sim \gamma$ gives the same crossover length scale (154) as the simple argument with Knudsen number. At smaller length scales, the GHD theory applies [51]. Obviously, it inevitably has to break down to non-universal dynamics around $l \approx l_0$, where l_0 is microscopic length scale of integrable Hamiltonian $\hat{H}_0$.

Having established that NS equations occur for $l \gg l_{\text{NS/GHD}}^*$ we address the issue of KPZ anomalous transport emerging at the largest scales. Our point of interest will be the behavior of two-point functions of hydrodynamic fields close to the stationary, thermal state. The standard theory for such purpose in the context of one-dimensional non-integrable systems is the NLFH [71], to which we now turn.

5.1 Nonlinear fluctuating hydrodynamics for one-dimensional fluids

The framework of Ref. [71] is developed in the specific context of anharmonic chains. However, as noted in appendix of that work, the same theory can be equally well applied to one-dimensional fluids with conserved particle number, momentum and energy. The symmetries of those inter-mode couplings, which lead to long-lived perturbations are exactly the same [70]. We adapt the notation of [71] with only slight differences.

We start with NS equations written in the form (6) and with no external potential $U(x) = 0$. We consider small deviations from equilibrium as

$$q_0(x, t) \rightarrow q_0 + \delta q_0(x, t), \quad q_1(x, t) \rightarrow 0 + \delta q_1(x, t), \quad q_2(x, t) \rightarrow q_2 + \delta q_2(x, t), \quad (158)$$

and expand the NS currents

$$\begin{aligned} \dot{\mathcal{J}}_0^{\text{NS}} &= q_1, \\ \dot{\mathcal{J}}_1^{\text{NS}} &= \frac{q_1^2}{q_0} + P - \zeta \partial_x (q_1/q_0), \\ \dot{\mathcal{J}}_2^{\text{NS}} &= \frac{q_1 q_2}{q_0} + \frac{q_1}{q_0} P - \zeta \frac{q_1}{q_0} \partial_x (q_1/q_0) - \kappa \partial_x T, \end{aligned} \quad (159)$$

keeping terms linear and quadratic in deviations δq as well as linear in derivatives. Derivative $\partial_x T$ can be expressed with derivatives of charges via relations (3) and (5). The general structure of the resulting hydrodynamic balance equations is

$$\partial_t \delta q_\alpha + \partial_x \left(A_{\alpha\beta} \delta q_\beta + \frac{1}{2} H_{\beta\gamma}^\alpha \delta q_\beta \delta q_\gamma - \tilde{D}_{\alpha\beta} \partial_x \delta q_\beta + \tilde{B}_{\alpha\beta} \xi_\beta \right) = 0, \quad (160)$$

with $A_{\alpha\beta} = \frac{\partial \dot{\mathcal{J}}_\alpha^{\text{NS}}}{\partial q_\beta}$, $H_{\beta\gamma}^\alpha = \frac{\partial^2 \dot{\mathcal{J}}_\alpha^{\text{NS}}}{\partial q_\beta \partial q_\gamma}$, and matrix $\tilde{D}$ involves terms with transport coefficients ζ, κ . We have also added noise term $\tilde{B}$ representing fluctuations of the currents satisfying

$$\langle \xi_\alpha(x, t) \rangle = 0, \quad \langle \xi_\alpha(x, t) \xi_\beta(x', t') \rangle = \delta_{\alpha\beta} \delta(x - x') \delta(t - t'). \quad (161)$$

Moreover, we have

$$\langle \delta q_\alpha(x, t) \rangle = 0, \quad \langle \delta q_\alpha(x, t) \delta q_\beta(x', t) \rangle = C_{\alpha\beta} \delta(x - x'), \quad (162)$$

where $C_{\alpha\beta} \equiv \mathcal{C}_{\alpha\beta}$ is the susceptibility matrix, which fulfills $AC = CA^T$. The strength of the noise is fixed by fluctuation-dissipation relation $\tilde{D}C + C\tilde{D}^T = \tilde{B}\tilde{B}^T$. Equation (160) forms a starting point for the theory of NLFH. We will now recall main results of [71] for the correlation functions in that theory, which will allow us to understand the crossover between the NS and KPZ regimes.

We start with finding the normal modes (two sound modes denoted $\sigma = \pm 1$ and heat mode denoted by 0) by diagonalizing the matrix $A_{\alpha\beta}$ with transformation R such that $RCR^T = 1$ and $RAR^{-1} = \text{diag}(-c, 0, c)$, where c is sound velocity. Writing the equation in the normal mode basis $\vec{\phi} = R\delta\vec{q}$ we arrive at

$$\partial_t \phi_\alpha + \partial_x \left(c_\alpha \phi_\alpha + G_{\beta\gamma}^\alpha \phi_\beta \phi_\gamma - D_{\alpha\beta} \partial_x \phi_\beta + B_{\alpha\beta} \xi_\beta \right) = 0, \quad (163)$$

with $c_\alpha \in \{-c, 0, c\}$ and where matrices G, D, B follow from rotations of corresponding matrices $H, \tilde{D}, \tilde{B}$ with R . We are interested in two-point correlations of the fields ϕ_α

$$f_\alpha(x, t) = \langle \phi_\alpha(x, t) \phi_\alpha(0, 0) \rangle, \quad f_\alpha(x, 0) = \delta(x), \quad (164)$$

and analyze mode-coupling equations [71] giving approximate dynamics of $f_\alpha(x, t)$

$$\partial_t f_\alpha(x, t) = (-c_\alpha \partial_x + D_\alpha \partial_x^2) f_\alpha(x, t) + \int_0^t ds \int dy f_\alpha(x - y, t - s) \partial_y^2 M_{\alpha\alpha}(y, s), \quad (165)$$

where $D_\alpha = D_{\alpha\alpha}$ and memory kernel reads

$$M_{\alpha\alpha}(x, t) = 2 \sum_{\beta, \gamma=0, \pm 1} (G_{\beta\gamma}^\alpha)^2 f_\beta(x, t) f_\gamma(x, t). \quad (166)$$

It is expected that the sound modes will move with velocity c and therefore, after a sufficiently large time $f_\beta(x, t)f_\gamma(x, t) \approx 0$ for $\beta \neq \gamma$. Therefore, in this diagonal approximation

$$M_{\alpha\alpha}(x, t) \approx M_{\alpha}^{\text{dg}} = 2 \sum_{\gamma=0, \pm 1} (G_{\gamma\gamma}^{\alpha})^2 f_{\gamma}(x, t)^2. \quad (167)$$

In general, for one-dimensional fluids G couplings have the following symmetries, which will be important for the structure of mode-coupling equations

$$G_{\beta\gamma}^{\alpha} = G_{\gamma\beta}^{\alpha}, \quad G_{00}^0 = 0, \quad G_{\sigma\sigma}^0 = -G_{-\sigma-\sigma}^0 \neq 0, \quad G_{\alpha\beta}^{\sigma} = -G_{-\alpha-\beta}^{-\sigma} \neq 0. \quad (168)$$

We stress that G couplings follow from thermodynamics given by TBA and thus, do not depend on integrability-breaking parameter λ , as predicted by ChE. With this, we move to the analysis of sound-sound correlation functions through mode-coupling equation.

5.2 Sound-sound correlation function

Due to the fact that peak travels with c , the only term kept in memory kernel corresponds to the same sound mode – for others, the overlap $f_{\alpha} \partial_y^2 M_{\alpha\alpha}$ is small. Hence, only one coupling is relevant and we have

$$\partial_t f_{\sigma}(x, t) = (-c_{\sigma} \partial_x + D_{\sigma} \partial_x^2) f_{\sigma}(x, t) + 2(G_{\sigma\sigma}^{\sigma})^2 \int_0^t ds \int dy f_{\sigma}(x-y, t-s) \partial_y^2 f_{\sigma}(y, s)^2. \quad (169)$$

We go to Fourier space in position $f_{\sigma}(x, t) = \int dk e^{ikx} \hat{f}_{\sigma}(k, t)$, remove the ballistic part of the evolution by replacing $\hat{f}_{\sigma}(k, t) \rightarrow e^{-ikc_{\sigma}t} \hat{f}_{\sigma}(k, t)$ and find

$$\partial_t \hat{f}_{\sigma}(k, t) = -k^2 \left(D_{\sigma} \hat{f}_{\sigma}(k, t) + 2(G_{\sigma\sigma}^{\sigma})^2 \int_0^t ds \hat{f}_{\sigma}(k, t-s) \int dq \hat{f}_{\sigma}(q, s) \hat{f}_{\sigma}(k-q, s) \right). \quad (170)$$

To make some progress with this equation we consider solutions in the scaling form

$$\hat{f}_{\sigma}(k, t) = p_{\sigma}(\Lambda_{\sigma} k^{\nu_{\sigma}} t), \quad (171)$$

where

- NS diffusion is $\nu_{\sigma} = 2$, $p_{\sigma}(w) = e^{-w}$ and $\Lambda_{\sigma} = D_{\sigma} \sim O(\lambda^{-2})$,
- KPZ superdiffusion is $\nu_{\sigma} = 3/2$, $p_{\sigma}(w) = \hat{f}_{\text{mc}}(w)$ and $\Lambda_{\sigma} = 2\sqrt{2}G_{\sigma\sigma}^{\sigma} \sim O(1)$.

Here $\hat{f}_{\text{mc}}$ is solution to fixed-point equation (178), which turns out to be close to KPZ scaling function [71]. Introducing new integration variables $\mu = \Lambda_{\sigma} k^{\nu_{\sigma}} s$ and then $r = \mu(q/k)^{\nu_{\sigma}}$ we arrive at the following expression (scaling variable is $w = \Lambda_{\sigma} k^{\nu_{\sigma}} t$)

$$k^{\nu_{\sigma}} \Lambda_{\sigma} p'_{\sigma}(w) = -k^2 \left[D_{\sigma} p_{\sigma}(w) + \frac{2(G_{\sigma\sigma}^{\sigma})^2}{\nu_{\sigma} \Lambda_{\sigma}} k^{1-\nu_{\sigma}} \int_0^w d\mu \mu^{-1/\nu_{\sigma}} p_{\sigma}(w-\mu) \right. \\ \left. \times \int dr r^{\frac{1-\nu_{\sigma}}{\nu_{\sigma}}} p_{\sigma}(r) p_{\sigma}(\mu(1-(r/\mu)^{1/\nu_{\sigma}})^{\nu_{\sigma}}) \right]. \quad (172)$$

First, let us see what happens when the diffusive scaling function is assumed. We plug the corresponding values of ν_{σ} and Λ_{σ} getting

$$k^2 D_{\sigma} p'_{\sigma}(w) = -k^2 \left[D_{\sigma} p_{\sigma}(w) + \frac{(G_{\sigma\sigma}^{\sigma})^2}{D_{\sigma}} k^{-1} \int_0^w d\mu \mu^{-1/2} p_{\sigma}(w-\mu) \right. \\ \left. \times \int dr r^{-\frac{1}{2}} p_{\sigma}(r) p_{\sigma}(\mu(1-(r/\mu)^{1/2})^2) \right]. \quad (173)$$

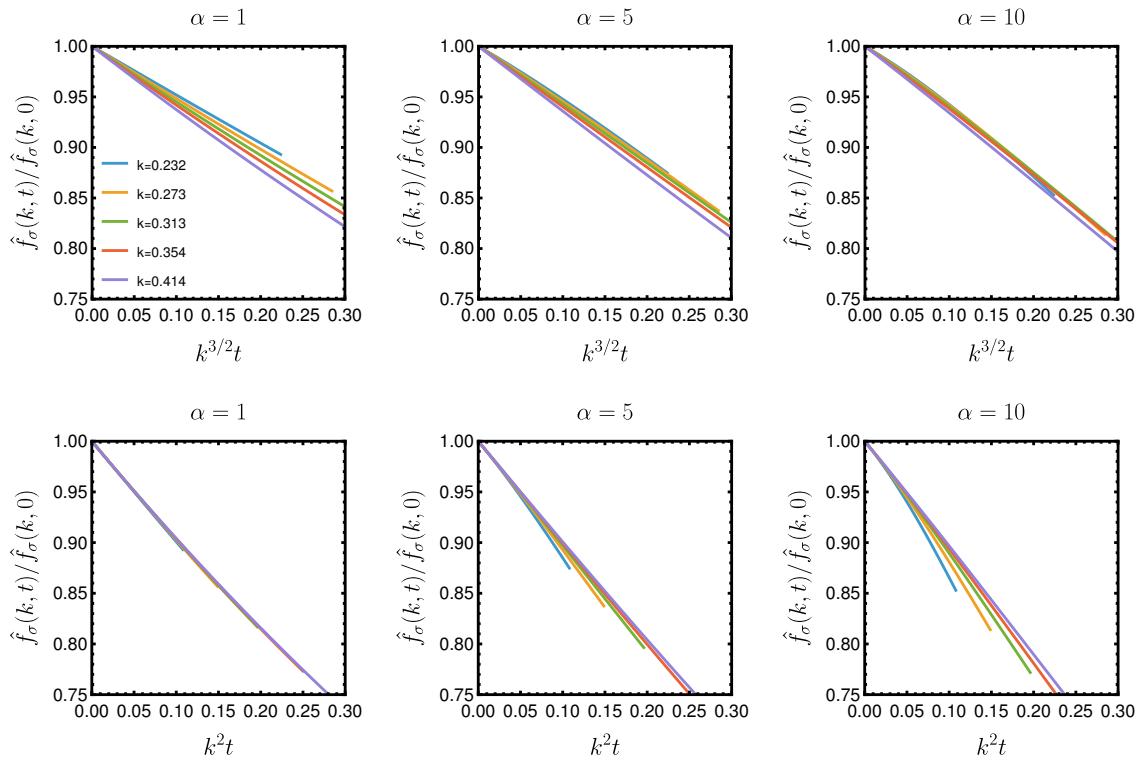


Figure 2: Dynamics of sound-sound correlation function $\hat{f}_\sigma(k, t)$. The initial state is a bimodal Gaussian distribution with variance defining characteristic inverse length scale of the state $k_c \approx 1/3$. In the equation (170), we rescaled k by k_c and time by $(k_c D_\sigma^2)^{-1}$ finding a single parameter $\alpha = 2(G_{\sigma\sigma}^\sigma)^2/(D_\sigma^2 k_c)$ controlling the relative strength of diffusive and mode-coupling terms in (170). Upper panel: correlator plotted versus anomalous scaling variable $k^{3/2}t$, lower panel: the same data against diffusive scaling variable k^2t . For smaller α , we recover diffusive behavior and good collapse to k^2t variable, whereas for large α , where mode-coupling term dominates, we recover anomalous scaling as predicted by our analysis.

We see that due to large diffusion constant ($D_\sigma \sim \lambda^{-2}$), at finite, but small k the first term on RHS may dominate the mode-coupling contribution (recall that G coupling do not depend on λ). The characteristic value of k , for which this happens may be found by comparing the two terms. This corresponds to $D_\sigma \approx 1/(D_\sigma k)$ leading to an estimate

$$k \sim \lambda^4. \quad (174)$$

Above this characteristic value of k , diffusion dominates and mode interactions can be neglected. In this regime we recover standard equation for diffusion, here written in scaling variable w

$$p'_\sigma(w) = -p_\sigma(w). \quad (175)$$

Similar analysis may be performed assuming the superdiffusive form of the scaling function. Let us estimate when the purely diffusive behavior will break down to superdiffusion. We plug the corresponding values of ν_σ and Λ_σ getting

$$k^{3/2} 2\sqrt{2} G_{\sigma\sigma}^\sigma p'_\sigma(w) = -k^2 \left[D_\sigma p_\sigma(w) \frac{2G_{\sigma\sigma}^\sigma}{3\sqrt{2}} k^{-1/2} \int_0^w d\mu \mu^{-2/3} p_\sigma(w-\mu) \right. \\ \left. \times \int dr r^{-1/3} p_\sigma(r) p_\sigma\left(\mu(1-(r/\mu)^{2/3})^{3/2}\right) \right]. \quad (176)$$

Again we can estimate the crossover k , below which we can drop the first term with diffusion as

$$D_\sigma \approx k^{-1/2}, \quad (177)$$

leading again to $k \sim \lambda^4$ consistently with the previous analysis. Below these values of k , mode-coupling dominates. Neglecting now the diffusive term, we find the following equation

$$p'_\sigma(w) = -\frac{1}{6} \int_0^w d\mu \mu^{-2/3} p_\sigma(w-\mu) \int dr r^{-\frac{1}{3}} p_\sigma(r) p_\sigma\left(\mu(1-(r/\mu)^{2/3})^{3/2}\right), \quad (178)$$

which gives a scaling function, which turns out to be close to KPZ scaling function [71].

In summary, we have found that above inverse length scales $k \sim \lambda^4$, the diffusion dominates over the mode-coupling term and sound-sound correlators are described by the standard diffusion equation with diffusion constant determined by transport coefficients found in this work (155). Therefore, we estimate the crossover length scale above which KPZ superdiffusion emerges as

$$l_{\text{KPZ/NS}}^* \sim \lambda^{-4}. \quad (179)$$

In principle, there are additional terms in mode-coupling equation, but they have a subleading role. We present a short analysis in Appendix B.1.

The conclusions from analytical analysis of mode-coupling equations can be supported by numerical solutions. In Fig. 2 we present dynamics of sound-sound correlation function for different values of parameter $\alpha = 2(G_{\sigma\sigma}^\sigma)^2/(D_\sigma^2 k_c)$, where k_c is characteristic inverse length-scale of initial state. For sufficiently small α , we observe diffusive behavior, whereas for large α we find the collapse using scaling variable $k^{3/2}t$.

In a similar fashion, one can analyze equation for heat-heat correlations, we shortly discuss that in Appendix B.2. One finds then, the mode-coupling terms are relevant only at even larger scales $\sim \lambda^{-6}$. Thus, we keep length scale $\sim \lambda^{-4}$ as a scale for the crossover between the NS and KPZ regimes.

5.3 Additional remarks

Let us conclude these results with a few more comments. The NLFH predicts long-time tails in current-current correlation functions [86]. Thus, transport coefficients given by Green-Kubo relations are infinite as the formulas involve integration from $t = 0$ to $t = \infty$ probing all the scales in the system. However, the *bare transport coefficients* [87], which enter the equations of NLFH via diffusion matrix D are finite and have to be finite in order to recover the KPZ superdiffusion [70]. Our work provides microscopic formulas for them as derived from ChE theory applied to the GHD-Boltzmann equation. Even though the formulas for correlators in KPZ regime do not depend on them, they are crucial for the NS dynamics in intermediate length scales (which in practice may be very large). Interestingly, as the KPZ correlation functions depend only on thermodynamic quantities fixed by TBA, namely G couplings and sound velocity, at the largest scale the dynamics of the system does not depend on the details of integrability-breaking perturbation.

It is also worth mentioning that Refs. [83–85] studying 1D nearly integrable systems found a normal transport across a large window of system sizes. This is in agreement with conclusions of our work, which is based on NLFH. The general picture of nearly-integrable dynamics are quasiparticles with parametrically (in integrability-breaking parameter λ) long relaxation times. From kinetic theory point of view this naturally leads to large bare transport coefficients, proportional to thermalization time. We emphasize that their large values are the only ingredients needed to arrive at our conclusions using NLFH. Thus, these observations point towards a more general implications of our analysis.

Lastly, we note that our framework incorporates external trapping, which breaks the momentum conservation. In such scenario, the transport is expected to be diffusive with finite thermal conductivity [88–91].

6 Conclusions and discussion

In this work, we have addressed a problem of non-equilibrium dynamics of nearly integrable quantum gases starting with the description of the system at the level of the generalized hydrodynamics. We have shown that, in the presence of integrability-breaking collision integral the problem can be studied by adopting the methods of kinetic theory. This allowed us to derive the NS equations, valid at large space-time-scales, together with the exact expressions for the transport coefficients.

The computations presented here complement findings of [51] where the NS equations were derived for a linearized theory and applied to a concrete setup of coupled Lieb-Liniger gases. Here instead, the full non-linear problem was analyzed by generalizing the ChE method known from the kinetic theory of dilute gases. Interestingly, we find that the transport coefficients in both approaches, linearized and ChE, yield the same expressions for the transport coefficients. We stress that derivation of the full, nonlinear hydrodynamics is of key importance to the study of crossover between NS diffusion and KPZ superdiffusion in our system. The theory of NLFH, on which our analysis is based, requires nonlinear contributions to the currents, which were absent in derivation presented in [51]. In addition to that, the ChE framework allowed us to incorporate the effects of external potential into the NS equations.

The resulting transport coefficients have two contributions reflecting the two types of interactions present in the system. There is an integrable interaction captured by the diffusion term of the unperturbed generalized hydrodynamics, and there is an integrability-breaking interaction described by the collision integral. Importantly, both contributions are non-perturbative in the strength of the integrable interactions and are valid beyond the dilute limit. As already pointed out in [51], the viscosity is generally non-zero. This should be contrasted with the kinetic theory of dilute gases which predicts always zero viscosity for 1d fluids.

One interesting direction in which the result of present work could be extended is the following. We could envision supplementing the NS equations by a dynamic equation for the slowest decaying mode [82]. This would extend the resulting description towards shorter timescales when more information on the integrable structure is relevant for the dynamics.

In the low temperature regime, the conventional hydrodynamics may emerge from GHD without integrability-breaking. In particular, the Euler-scale GHD becomes equivalent to conventional Euler hydrodynamics at zero temperature [79]. When the diffusive effects are incorporated into the GHD, the effective low-temperature description at early times is the NS hydrodynamics [92]. One then finds transport coefficients given by the contributions from integrable interactions $\kappa_{\mathfrak{D}}, \zeta_{\mathfrak{D}}$ solely.

Conventional hydrodynamics in one-dimensional momentum-conserving systems is known to be unstable at the largest space-time scales due to mode interactions [68–71]. The transport universality class changes then from diffusive to anomalous, KPZ class. The relevant theory of anomalous transport in such systems in the NLFH [71], which starts with 1D NS equations supplemented with noise terms. At the largest space-time scales, KPZ behavior of two-point functions is recovered.

We addressed this issue in Sec. 5, where we have shown the mode interactions are relevant only at the largest length scales $\sim \lambda^{-4}$, where λ is the integrability-breaking parameter. At smaller scales, the leading effects are captured by NS hydrodynamics with finite transport coefficients determined by us in this work. More precisely, we have found that nearly inte-

grable quantum gases are characterized by three different hydrodynamic regimes: GHD at the smallest (but still macroscopic) length scales, NS in the intermediate regime, and KPZ at the largest scales. The superdiffusive correlation functions in the KPZ regime are determined by the couplings fixed with the thermodynamics given by TBA and do not depend on the details of integrability-breaking perturbation.

A possible direction for future work is the comparison of the results from GHD-Boltzmann equation against NS hydrodynamics with transport coefficients computed in this work. As the numerical solution of GHD-Boltzmann equation suffers from costly computation of Boltzmann integrals, it seems natural to consider the so-called relaxation time approximation to collision integral. In Ref. [52] such equation was solved showing signatures of diffusive spreading characteristic to the NS dynamics. We also note that NS equations were proven useful in description of dynamics of 1d gases as recently demonstrated in Refs. [93, 94] comparing numerical simulations of blast waves initial conditions with NS hydrodynamics.

In this work, we have focused on a theory with Galilean invariance which determines the structure of the resulting hydrodynamic equations to be of the NS form. However, the developed methods are applicable to other systems described by the generalized hydrodynamics. The most natural and important are integrable spin chains whose non-equilibrium dynamics can be efficiently studied with tensor networks methods.

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A Chapman-Enskog theory: Additional formulas and results

A.1 Thermodynamics with hydrodynamic matrices

Starting from GHD-Boltzmann equation we naturally encounter matrix elements of $\mathcal{C}, \mathcal{B}$ matrices evaluated in thermal states. To recast the equations which follow from ChE expansion into universal NS form it is useful to connect these quantities to thermodynamics of the system. The calculations are presented in Supplemental Material of [51]. Here, we list the most relevant expressions which are formulas for adiabatic compressibility χ_s , isothermal compressibility χ_T , sound velocity c and specific heats at constant volume c_V and pressure c_P :

$$\chi_s = \frac{1}{\varrho} \left(\frac{\partial \varrho}{\partial P} \right)_s = T \frac{\bar{C}_{2,2} \bar{C}_{0,0} - \bar{C}_{2,0}^2}{\bar{B}_{1,0}^2 \bar{C}_{2,2} + \bar{B}_{2,1}^2 \bar{C}_{0,0} - 2 \bar{B}_{1,0} \bar{B}_{2,1} \bar{C}_{2,0}}, \quad (\text{A.1})$$

$$\chi_T = \frac{1}{\varrho} \left(\frac{\partial \varrho}{\partial P} \right)_T = \frac{1}{\varrho} \frac{\bar{C}_{0,0}}{\bar{B}_{1,0}}, \quad (\text{A.2})$$

$$c = \sqrt{\frac{1}{B_{1,0}} \frac{B_{1,0}^2 C_{2,2} + B_{2,1}^2 C_{0,0} - 2 B_{1,0} B_{2,1} C_{2,0}}{C_{2,2} C_{0,0} - C_{2,0}^2}}, \quad (\text{A.3})$$

$$c_V = \frac{1}{\varrho} \left(\frac{\partial e}{\partial T} \right)_\varrho = \frac{1}{T} \frac{\bar{C}_{2,2} \bar{C}_{0,0} - \bar{C}_{2,0}^2}{\bar{C}_{0,0} \bar{B}_{1,0}}, \quad (\text{A.4})$$

$$c_P = \left(\frac{\partial s}{\partial T} \right)_P = \frac{1}{T} \frac{\bar{B}_{1,0}^2 \bar{C}_{2,2} + \bar{B}_{2,1}^2 \bar{C}_{0,0} - 2\bar{B}_{1,0} \bar{B}_{2,1} \bar{C}_{2,0}}{\bar{B}_{1,0}^3}, \quad (\text{A.5})$$

$$\sqrt{c_P/c_V - 1} = \frac{\bar{B}_{2,1} \bar{C}_{0,0} - \bar{B}_{1,0} \bar{C}_{2,0}}{\bar{B}_{1,0} \sqrt{\bar{C}_{2,2} \bar{C}_{0,0} - \bar{C}_{2,0}^2}}, \quad \bar{B}_{1,0} = \varrho T. \quad (\text{A.6})$$

A.2 Galilean invariance of the Chapman-Enskog equations

In this Appendix, we demonstrate that the ChE equations are invariant under the Galilean boost. The equations, including the constitutive relations, are

$$\Gamma w_c = \mathcal{B} \hbar_c - \sum_{b=0}^2 \mathcal{C} \hbar_b \mathcal{A}_{bc}^{\text{Eul}}, \quad \hbar_a \mathcal{C} w_c = 0, \quad a, c = 0, 1, 2. \quad (\text{A.7})$$

To prove the invariance we start with the transformation rules of the (3×3) -dimensional hydrodynamic matrices at the Euler scale,

$$\mathcal{C}^{\text{Eul}} = g(u) \bar{\mathcal{C}}^{\text{Eul}} g^T(u), \quad \mathcal{B}^{\text{Eul}} = g(u) \bar{\mathcal{B}}^{\text{Eul}} g^T(u) + u g(u) \bar{\mathcal{C}}^{\text{Eul}} g^T(u), \quad (\text{A.8})$$

from which follows

$$(\mathcal{C}^{\text{Eul}})^{-1} = g^T(-u) (\bar{\mathcal{C}}^{\text{Eul}})^{-1} g(-u), \quad (\text{A.9})$$

and

$$\mathcal{A}^{\text{Eul}} = g^T(-u) \bar{\mathcal{A}}^{\text{Eul}} g^T(u) + u. \quad (\text{A.10})$$

This allows us to rewrite the right hand side as

$$\mathcal{B} \hbar_c - \sum_{b=0}^2 \mathcal{C} \hbar_b \mathcal{A}_{bc}^{\text{Eul}} = b_u^{-1} \left(\bar{\mathcal{B}} b_u \hbar_c - \sum_{b=0}^2 \bar{\mathcal{C}} b_u \hbar_b (g^T(-u) \bar{\mathcal{A}}^{\text{Eul}} g^T(u))_{bc} \right). \quad (\text{A.11})$$

We can now use that $b_u \sum_b \hbar_b (g^T(-u))_{bc} = \hbar_c$ and $b_u \hbar_c = \sum_b g_{cb}(u) \hbar_b$ to obtain

$$\mathcal{B} \hbar_c - \sum_{b=0}^2 \mathcal{C} \hbar_b \mathcal{A}_{bc}^{\text{Eul}} = b_u^{-1} \sum_{b=0}^2 \left(\bar{\mathcal{B}} \hbar_b - \sum_{d=0}^2 \bar{\mathcal{C}} \hbar_d (\bar{\mathcal{A}}^{\text{Eul}})_{db} \right) (g^T(u))_{bc}. \quad (\text{A.12})$$

Defining $\Gamma = b_u^{-1} \bar{\Gamma} b_u$ and $w_c = b_u^{-1} \bar{w}_d (g^T(u))_{dc}$ we find

$$\bar{\Gamma} \bar{w}_c = \bar{\mathcal{B}} \hbar_c - \sum_{b=0}^2 \bar{\mathcal{C}} \hbar_b \bar{\mathcal{A}}_{bc}^{\text{Eul}}, \quad \hbar_a \bar{\mathcal{C}} \bar{w}_c = 0, \quad a, c = 0, 1, 2, \quad (\text{A.13})$$

as reported in (130).

A.3 Equations for transport coefficients in the orthonormal basis

In this Appendix, we show explicitly that integral equations (132) are the same as equations presented in our earlier work [51], where a basis orthonormal with respect to the hydrodynamic inner product with matrix $\bar{\mathcal{C}}$, $(h_n | h_m) = \delta_{nm}$ was used.

Transport coefficients $\zeta_{\mathcal{I}}, \kappa_{\mathcal{I}}$ are determined from formulas (143) with functions $\bar{w}_1, \bar{w}_2$ determined from integral equations (132). The orthonormal basis $\{h_n\}_{n \in \mathbb{N}}$ is constructed using Gram-Schmidt procedure on the ultralocal basis $\{\hbar_n\}_{n \in \mathbb{N}}$. The first three elements are

$$h_0 = \frac{1}{\sqrt{\bar{\mathcal{C}}_{0,0}}} \hbar_0, \quad h_1 = \frac{1}{\sqrt{\bar{\mathcal{C}}_{1,1}}} \hbar_1, \quad h_2 = \sqrt{\frac{\bar{\mathcal{C}}_{0,0}}{\bar{\mathcal{C}}_{2,2} \bar{\mathcal{C}}_{0,0} - \bar{\mathcal{C}}_{2,0}^2}} \left(\hbar_2 - \frac{\bar{\mathcal{C}}_{2,0}}{\bar{\mathcal{C}}_{0,0}} \hbar_0 \right). \quad (\text{A.14})$$

Rewriting equations (132) and using thermodynamic expression from Appendix A.1 we find

$$\Gamma \bar{w}_1 = \sqrt{\varrho T} \bar{B} h_1 - \sqrt{T/\chi_T} \bar{C} (h_0 + \sqrt{c_P/c_V - 1} h_2), \quad (\text{A.15})$$

$$\Gamma \bar{w}_2 = T \sqrt{c_V} \bar{\varrho} \bar{B} (h_2 - \sqrt{c_P/c_V - 1} h_0), \quad (\text{A.16})$$

defining now

$$\phi_\zeta = \bar{w}_1/T, \quad \phi_\kappa = \bar{w}_2/T^2, \quad (\text{A.17})$$

we recover the equations given in the main text of [51]

$$\bar{\Gamma} \phi_\zeta = \sqrt{\beta \varrho} \bar{B} h_1 - \sqrt{\beta/\chi_T} \bar{C} (h_0 + \sqrt{c_P/c_V - 1} h_2), \quad (\text{A.18})$$

$$\bar{\Gamma} \phi_\kappa = \beta \sqrt{\varrho c_V} \bar{B} (h_2 - \sqrt{c_P/c_V - 1} h_0), \quad (\text{A.19})$$

and with the same transport coefficients. Indeed, expressions (143) become then

$$\zeta_{\mathcal{I}} = \hbar_1 \bar{B} \phi_\zeta, \quad \kappa_{\mathcal{I}} = \hbar_2 \bar{B} \phi_\kappa, \quad (\text{A.20})$$

as in [51].

B Mode-coupling equations: Additional results

B.1 Analysis of the subleading mode-coupling terms

In this part we investigate subleading corrections to sound-sound mode-coupling equation which stem from additional interactions. Before, we were considering memory kernel in the form

$$M_1(x, t) = 2(G_{11}^1)^2 f_1(x, t) f_1(x, t), \quad (\text{B.1})$$

now, we include more terms, i.e.

$$M_1(x, t) = 2 \left[(G_{11}^1)^2 f_1(x, t) f_1(x, t) + (G_{00}^1)^2 f_0(x, t) f_0(x, t) + (G_{-1-1}^1)^2 f_{-1}(x, t) f_{-1}(x, t) \right]. \quad (\text{B.2})$$

Let us look on the mode-coupling equation in Fourier space

$$\begin{aligned} \partial_t \hat{f}_1(k, t) &= (-ikc - k^2 D_1) \hat{f}_1(k, t) \\ &\quad - 2k^2 \int_0^t ds \hat{f}_1(k, t-s) \left[(G_{11}^1)^2 \int dq \hat{f}_1(q, s) \hat{f}_1(k-q, s) \right. \\ &\quad \left. + (G_{00}^1)^2 \int dq \hat{f}_0(q, s) \hat{f}_0(k-q, s) + (G_{-1-1}^1)^2 \int dq \hat{f}_{-1}(q, s) \hat{f}_{-1}(k-q, s) \right]. \end{aligned} \quad (\text{B.3})$$

We write the phases explicitly replacing

$$\hat{f}_1(k, t) \rightarrow e^{-ikct} \hat{f}_1(k, t), \quad \hat{f}_{-1}(k, t) \rightarrow e^{ikct} \hat{f}_1(k, t), \quad (\text{B.4})$$

and get

$$\begin{aligned} \partial_t \hat{f}_1(k, t) &= -k^2 \left[D_1 \hat{f}_1(k, t) + 2 \int_0^t ds \hat{f}_1(k, t-s) \right. \\ &\quad \times \left((G_{11}^1)^2 \int dq \hat{f}_1(q, s) \hat{f}_1(k-q, s) + (G_{00}^1)^2 e^{ikcs} \right. \\ &\quad \left. \left. \times \int dq \hat{f}_0(q, s) \hat{f}_0(k-q, s) + (G_{-1-1}^1)^2 e^{2ikcs} \int dq \hat{f}_1(q, s) \hat{f}_1(k-q, s) \right) \right]. \end{aligned} \quad (\text{B.5})$$

The idea here is to assume diffusive scaling of heat-heat and sound-sound correlators and estimate, under this assumption, the role of additional interactions. Thus we write

$$\hat{f}_0(k, t) = g_0((D_0 t)^{1/2} k), \quad \hat{f}_1(k, t) = p_1(D_1 k^2 t) = g_1((D_1 t)^{1/2} k), \quad (\text{B.6})$$

where, for convenience we introduced two representations of the same function through $p_1(w) = e^{-w}$ and $g_{0,1}(w) = e^{-w^2}$. In RHS of (B.5) we have three mode coupling terms and we already analyzed the first one in Sec. 5.2. For the second one we introduce $w = D_1 k^2 t$ change variables as $\bar{s} = ks$ and then $r = (D_0 \bar{s})^{1/2} k^{-1/2} q$ getting

$$\frac{2(G_{00}^1)^2}{D_0^{1/2} k^{1/2}} \int_0^{\frac{w}{D_1 k}} d\bar{s} \frac{e^{ic\bar{s}}}{\sqrt{\bar{s}}} p_1(w - D_1 k \bar{s}) \int dr g_0(r) g_0((D_0 \bar{s})^{1/2} k^{1/2} - r). \quad (\text{B.7})$$

We note now that due to our assumption $k \ll \lambda^2$ we also have $D_{0,1} k \ll 1$. This allows to extend the integration to ∞ and drop k -dependent terms in arguments of p_1 and g_0 getting

$$\frac{2(G_{00}^1)^2}{D_0^{1/2} k^{1/2}} \int_0^\infty d\bar{s} \frac{e^{ic\bar{s}}}{\sqrt{\bar{s}}} \int dr g_0(r) g_0(-r) p_1(w) := \frac{\mathfrak{C}_{00}^1}{D_0^{1/2} k^{1/2}} p_1(w), \quad (\text{B.8})$$

where $\mathfrak{C}_{00}^1 = \text{const.}$ Similar analysis may be performed for the third term, one finds

$$\frac{2(G_{-1-1}^1)^2}{D_1^{1/2} k^{1/2}} \int_0^\infty d\bar{s} \frac{e^{2ic\bar{s}}}{\sqrt{\bar{s}}} \int dr g_1(r) g_1(-r) p_1(w) := \frac{\mathfrak{C}_{-1-1}^1}{D_1^{1/2} k^{1/2}} p_1(w), \quad (\text{B.9})$$

thus we can write the full mode-coupling equation as

$$k^2 D_1 p_1'(w) = -k^2 D_1 \left[p_1(w) + \frac{(G_{11}^1)^2}{D_1^2 k} \int_0^w d\mu \mu^{-1/2} p_1(w - \mu) \int dr r^{-\frac{1}{2}} p_1(r) p_1(\mu(1 - (r/\mu)^{1/2})^2) \right. \\ \left. + \frac{\mathfrak{C}_{00}^1}{D_1 D_0^{1/2} k^{1/2}} p_1(w) + \frac{\mathfrak{C}_{-1-1}^1}{D_1^{3/2} k^{1/2}} p_1(w) \right]. \quad (\text{B.10})$$

We clearly see, that for inverse length scales much bigger than $\sim \lambda^4$ (which means $D_{0,1}^2 k \gg 1$) both the second term and also the two additional ones are negligible comparing to the first one, which gives standard diffusion. Actually, the two additional terms become comparable to diffusion at inverse lengths scales $\sim \lambda^6$ and thus are subleading corrections.

B.2 Heat-heat correlation function

The heat-heat correlator is characterized by a different exponent and is described by Levy- $\frac{5}{3}$ distribution, instead of KPZ scaling function. This is essentially due to the fact that for heat mode self-coupling is absent as $G_{00}^0 = 0$ [71] and one has to look for subleading terms in the memory kernel. Including the coupling to the sound-modes gives

$$\partial_t f_0(x, t) = D_0 \partial_x^2 f_0(x, t) + 2 \sum_\sigma (G_{\sigma\sigma}^0)^2 \int_0^t ds \int dy f_0(x - y, t - s) \partial_y^2 f_\sigma(y, s)^2. \quad (\text{B.11})$$

We start with the relevant equation in Fourier space. It reads (we write phase factors $e^{\pm i k c s}$ explicitly)

$$\partial_t \hat{f}_0(k, t) = -k^2 \left(D_0 \hat{f}_0(k, t) + 4(G_{\sigma\sigma}^0)^2 \int_0^t ds \cos(kcs) \hat{f}_0(k, t - s) \int dq \hat{f}_\sigma(q, s) \hat{f}_\sigma(k - q, s) \right). \quad (\text{B.12})$$

Once again we try with solution in scaling form

$$\hat{f}_0(k, t) = p_0(\Lambda_0 k^{\nu_0} t), \quad (\text{B.13})$$

where

- NS diffusion is $\nu_0 = 2$, $p_0(w) = e^{-w}$ and $\Lambda_0 = D_0 \sim O(\lambda^{-2})$,
- Levy- $\frac{5}{3}$ superdiffusion is $\nu_0 = 5/3$, $p_0(w) = e^{-|w|}$ and $\Lambda_\sigma = \Lambda_0^{\text{Levy}} \sim O(1)$.

Following similar calculation to the one presented for the case of sound-sound correlator and in [71] we find that there again is a crossover between normal diffusion and superdiffusion. This time however, we find that it happens at even larger scales

$$l_{\text{Levy/NS}}^{\text{heat}} \sim \lambda^{-6}. \quad (\text{B.14})$$

References

- [1] J. R. Dorfman, H. van Beijeren and T. R. Kirkpatrick, *Contemporary kinetic theory of matter*, Cambridge University Press, Cambridge, UK, ISBN 9781139025942 (2021), doi:[10.1017/9781139025942](https://doi.org/10.1017/9781139025942).
- [2] P. Résibois and M. De Leener, *Classical kinetic theory of fluids*, Wiley & Sons, New York, USA, ISBN 9780471716945 (1977).
- [3] R. Balescu, *Equilibrium and nonequilibrium statistical mechanics*, Wiley & Sons, New York, USA, ISBN 9780471046004 (1975).
- [4] S. Chapman and T. G. Cowling, *The mathematical theory of non-uniform gases*, Cambridge University Press, Cambridge, UK, ISBN 9780521408448 (1991).
- [5] S. Vogel, *Life in moving fluids: The physical biology of flow*, Princeton University Press, Princeton, USA, ISBN 9780691026169 (1996).
- [6] L. Rezzolla and O. Zanotti, *Relativistic hydrodynamics*, Oxford University Press, Oxford, UK, ISBN 9780198528906 (2013), doi:[10.1093/acprof:oso/9780198528906.001.0001](https://doi.org/10.1093/acprof:oso/9780198528906.001.0001).
- [7] V. Khemani, A. Vishwanath and D. A. Huse, *Operator spreading and the emergence of dissipative hydrodynamics under unitary evolution with conservation laws*, Phys. Rev. X **8**, 031057 (2018), doi:[10.1103/PhysRevX.8.031057](https://doi.org/10.1103/PhysRevX.8.031057).
- [8] T. Rakovszky, F. Pollmann and C. W. von Keyserlingk, *Diffusive hydrodynamics of out-of-time-ordered correlators with charge conservation*, Phys. Rev. X **8**, 031058 (2018), doi:[10.1103/PhysRevX.8.031058](https://doi.org/10.1103/PhysRevX.8.031058).
- [9] S. Galtier, *Introduction to modern magnetohydrodynamics*, Cambridge University Press, Cambridge, UK, ISBN 9781107158658 (2016), doi:[10.1017/CBO9781316665961](https://doi.org/10.1017/CBO9781316665961).
- [10] T. Kinoshita, T. Wenger and D. S. Weiss, *A quantum Newton's cradle*, Nature **440**, 900 (2006), doi:[10.1038/nature04693](https://doi.org/10.1038/nature04693).
- [11] F. Cataldini, F. Møller, M. Tajik, J. Sabino, S.-C. Ji, I. Mazets, T. Schweigler, B. Rauer and J. Schmiedmayer, *Emergent Pauli blocking in a weakly interacting Bose gas*, Phys. Rev. X **12**, 041032 (2022), doi:[10.1103/PhysRevX.12.041032](https://doi.org/10.1103/PhysRevX.12.041032).

- [12] F. Møller, C. Li, I. Mazets, H.-P. Stimming, T. Zhou, Z. Zhu, X. Chen and J. Schmiedmayer, *Extension of the generalized hydrodynamics to the dimensional crossover regime*, Phys. Rev. Lett. **126**, 090602 (2021), doi:[10.1103/PhysRevLett.126.090602](https://doi.org/10.1103/PhysRevLett.126.090602).
- [13] C. Li et al., *Relaxation of bosons in one dimension and the onset of dimensional crossover*, SciPost Phys. **9**, 058 (2020), doi:[10.21468/SciPostPhys.9.4.058](https://doi.org/10.21468/SciPostPhys.9.4.058).
- [14] K.-Y. Li, Y. Zhang, K. Yang, K.-Y. Lin, S. Gopalakrishnan, M. Rigol and B. L. Lev, *Rapidity and momentum distributions of one-dimensional dipolar quantum gases*, Phys. Rev. A **107**, L061302 (2023), doi:[10.1103/PhysRevA.107.L061302](https://doi.org/10.1103/PhysRevA.107.L061302).
- [15] J. M. Wilson, N. Malvania, Y. Le, Y. Zhang, M. Rigol and D. S. Weiss, *Observation of dynamical fermionization*, Science **367**, 1461 (2020), doi:[10.1126/science.aaz0242](https://doi.org/10.1126/science.aaz0242).
- [16] N. Malvania, Y. Zhang, Y. Le, J. Dubail, M. Rigol and D. S. Weiss, *Generalized hydrodynamics in strongly interacting 1D Bose gases*, Science **373**, 1129 (2021), doi:[10.1126/science.abf0147](https://doi.org/10.1126/science.abf0147).
- [17] D. Wei et al., *Quantum gas microscopy of Kardar-Parisi-Zhang superdiffusion*, Science **376**, 716 (2022), doi:[10.1126/science.abk2397](https://doi.org/10.1126/science.abk2397).
- [18] K. Yang, Y. Zhang, K.-Y. Li, K.-Y. Lin, S. Gopalakrishnan, M. Rigol and B. L. Lev, *Phantom energy in the nonlinear response of a quantum many-body scar state*, Science **385**, 1063 (2024), doi:[10.1126/science.adk8978](https://doi.org/10.1126/science.adk8978).
- [19] L. Dubois, G. Thémèze, F. Nogrette, J. Dubail and I. Bouchoule, *Probing the local rapidity distribution of a one-dimensional Bose gas*, Phys. Rev. Lett. **133**, 113402 (2024), doi:[10.1103/PhysRevLett.133.113402](https://doi.org/10.1103/PhysRevLett.133.113402).
- [20] Y. Guo, H. Yao, S. Ramanjanappa, S. Dhar, M. Horvath, L. Pizzino, T. Giamarchi, M. Landini and H.-C. Nägerl, *Observation of the 2D-1D crossover in strongly interacting ultracold bosons*, Nat. Phys. **20**, 934 (2024), doi:[10.1038/s41567-024-02459-3](https://doi.org/10.1038/s41567-024-02459-3).
- [21] A. Polkovnikov, K. Sengupta, A. Silva and M. Vengalattore, *Colloquium: Nonequilibrium dynamics of closed interacting quantum systems*, Rev. Mod. Phys. **83**, 863 (2011), doi:[10.1103/RevModPhys.83.863](https://doi.org/10.1103/RevModPhys.83.863).
- [22] L. D'Alessio, Y. Kafri, A. Polkovnikov and M. Rigol, *From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics*, Adv. Phys. **65**, 239 (2016), doi:[10.1080/00018732.2016.1198134](https://doi.org/10.1080/00018732.2016.1198134).
- [23] F. H. L. Essler and M. Fagotti, *Quench dynamics and relaxation in isolated integrable quantum spin chains*, J. Stat. Mech.: Theory Exp. 064002 (2016), doi:[10.1088/1742-5468/2016/06/064002](https://doi.org/10.1088/1742-5468/2016/06/064002).
- [24] J.-S. Caux, *The quench action*, J. Stat. Mech.: Theory Exp. 064006 (2016), doi:[10.1088/1742-5468/2016/06/064006](https://doi.org/10.1088/1742-5468/2016/06/064006).
- [25] E. Ilievski, M. Medenjak, T. Prosen and L. Zadnik, *Quasilocal charges in integrable lattice systems*, J. Stat. Mech.: Theory Exp. 064008 (2016), doi:[10.1088/1742-5468/2016/06/064008](https://doi.org/10.1088/1742-5468/2016/06/064008).
- [26] R. Vasseur and J. E. Moore, *Nonequilibrium quantum dynamics and transport: From integrability to many-body localization*, J. Stat. Mech.: Theory Exp. 064010 (2016), doi:[10.1088/1742-5468/2016/06/064010](https://doi.org/10.1088/1742-5468/2016/06/064010).

- [27] P. Calabrese, *Entanglement spreading in non-equilibrium integrable systems*, SciPost Phys. Lect. Notes 20 (2020), doi:[10.21468/SciPostPhysLectNotes.20](https://doi.org/10.21468/SciPostPhysLectNotes.20).
- [28] O. A. Castro-Alvaredo, B. Doyon and T. Yoshimura, *Emergent hydrodynamics in integrable quantum systems out of equilibrium*, Phys. Rev. X 6, 041065 (2016), doi:[10.1103/PhysRevX.6.041065](https://doi.org/10.1103/PhysRevX.6.041065).
- [29] B. Bertini, M. Collura, J. De Nardis and M. Fagotti, *Transport in out-of-equilibrium XXZ chains: Exact profiles of charges and currents*, Phys. Rev. Lett. 117, 207201 (2016), doi:[10.1103/PhysRevLett.117.207201](https://doi.org/10.1103/PhysRevLett.117.207201).
- [30] B. Doyon, S. Gopalakrishnan, F. Møller, J. Schmiedmayer and R. Vasseur, *Generalized hydrodynamics: A perspective*, Phys. Rev. X 15, 010501 (2025), doi:[10.1103/PhysRevX.15.010501](https://doi.org/10.1103/PhysRevX.15.010501).
- [31] B. Doyon, *Lecture notes on generalised hydrodynamics*, SciPost Phys. Lect. Notes 18 (2020), doi:[10.21468/SciPostPhysLectNotes.18](https://doi.org/10.21468/SciPostPhysLectNotes.18).
- [32] F. H. L. Essler, *A short introduction to generalized hydrodynamics*, Phys. A: Stat. Mech. Appl. 631, 127572 (2023), doi:[10.1016/j.physa.2022.127572](https://doi.org/10.1016/j.physa.2022.127572).
- [33] M. Schemmer, I. Bouchoule, B. Doyon and J. Dubail, *Generalized hydrodynamics on an atom chip*, Phys. Rev. Lett. 122, 090601 (2019), doi:[10.1103/PhysRevLett.122.090601](https://doi.org/10.1103/PhysRevLett.122.090601).
- [34] B. Doyon and T. Yoshimura, *A note on generalized hydrodynamics: Inhomogeneous fields and other concepts*, SciPost Phys. 2, 014 (2017), doi:[10.21468/SciPostPhys.2.2.014](https://doi.org/10.21468/SciPostPhys.2.2.014).
- [35] J. Durnin, A. De Luca, J. De Nardis and B. Doyon, *Diffusive hydrodynamics of inhomogeneous Hamiltonians*, J. Phys. A: Math. Theor. 54, 494001 (2021), doi:[10.1088/1751-8121/ac2c57](https://doi.org/10.1088/1751-8121/ac2c57).
- [36] J. De Nardis, D. Bernard and B. Doyon, *Hydrodynamic diffusion in integrable systems*, Phys. Rev. Lett. 121, 160603 (2018), doi:[10.1103/PhysRevLett.121.160603](https://doi.org/10.1103/PhysRevLett.121.160603).
- [37] J. De Nardis, D. Bernard and B. Doyon, *Diffusion in generalized hydrodynamics and quasi-particle scattering*, SciPost Phys. 6, 049 (2019), doi:[10.21468/SciPostPhys.6.4.049](https://doi.org/10.21468/SciPostPhys.6.4.049).
- [38] S. Gopalakrishnan, D. A. Huse, V. Khemani and R. Vasseur, *Hydrodynamics of operator spreading and quasiparticle diffusion in interacting integrable systems*, Phys. Rev. B 98, 220303 (2018), doi:[10.1103/PhysRevB.98.220303](https://doi.org/10.1103/PhysRevB.98.220303).
- [39] A. Bastianello, A. De Luca and R. Vasseur, *Hydrodynamics of weak integrability breaking*, J. Stat. Mech.: Theory Exp. 114003 (2021), doi:[10.1088/1742-5468/ac26b2](https://doi.org/10.1088/1742-5468/ac26b2).
- [40] J. De Nardis, B. Doyon, M. Medenjak and M. Panfil, *Correlation functions and transport coefficients in generalised hydrodynamics*, J. Stat. Mech.: Theory Exp. 014002 (2022), doi:[10.1088/1742-5468/ac3658](https://doi.org/10.1088/1742-5468/ac3658).
- [41] I. Bouchoule and J. Dubail, *Generalized hydrodynamics in the one-dimensional Bose gas: Theory and experiments*, J. Stat. Mech.: Theory Exp. 014003 (2022), doi:[10.1088/1742-5468/ac3659](https://doi.org/10.1088/1742-5468/ac3659).
- [42] A. C. Cubero, T. Yoshimura and H. Spohn, *Form factors and generalized hydrodynamics for integrable systems*, J. Stat. Mech.: Theory Exp. 114002 (2021), doi:[10.1088/1742-5468/ac2eda](https://doi.org/10.1088/1742-5468/ac2eda).

- [43] B. Buča, K. Klobas and T. Prosen, *Rule 54: Exactly solvable model of nonequilibrium statistical mechanics*, J. Stat. Mech.: Theory Exp. 074001 (2021), doi:[10.1088/1742-5468/ac096b](https://doi.org/10.1088/1742-5468/ac096b).
- [44] V. B. Bulchandani, S. Gopalakrishnan and E. Ilievski, *Superdiffusion in spin chains*, J. Stat. Mech.: Theory Exp. 084001 (2021), doi:[10.1088/1742-5468/ac12c7](https://doi.org/10.1088/1742-5468/ac12c7).
- [45] G. A. El, *Soliton gas in integrable dispersive hydrodynamics*, J. Stat. Mech.: Theory Exp. 114001 (2021), doi:[10.1088/1742-5468/ac0f6d](https://doi.org/10.1088/1742-5468/ac0f6d).
- [46] M. Borsi, B. Pozsgay and L. Pristiyák, *Current operators in integrable models: A review*, J. Stat. Mech.: Theory Exp. 094001 (2021), doi:[10.1088/1742-5468/ac0f6b](https://doi.org/10.1088/1742-5468/ac0f6b).
- [47] A. Bastianello, B. Bertini, B. Doyon and R. Vasseur, *Introduction to the special issue on emergent hydrodynamics in integrable many-body systems*, J. Stat. Mech.: Theory Exp. 014001 (2022), doi:[10.1088/1742-5468/ac3e6a](https://doi.org/10.1088/1742-5468/ac3e6a).
- [48] G. P. Brandino, J.-S. Caux and R. M. Konik, *Glimmers of a quantum KAM theorem: Insights from quantum quenches in one-dimensional Bose gases*, Phys. Rev. X **5**, 041043 (2015), doi:[10.1103/PhysRevX.5.041043](https://doi.org/10.1103/PhysRevX.5.041043).
- [49] J. Durnin, M. J. Bhaseen and B. Doyon, *Nonequilibrium dynamics and weakly broken integrability*, Phys. Rev. Lett. **127**, 130601 (2021), doi:[10.1103/PhysRevLett.127.130601](https://doi.org/10.1103/PhysRevLett.127.130601).
- [50] A. J. Friedman, S. Gopalakrishnan and R. Vasseur, *Diffusive hydrodynamics from integrability breaking*, Phys. Rev. B **101**, 180302 (2020), doi:[10.1103/PhysRevB.101.180302](https://doi.org/10.1103/PhysRevB.101.180302).
- [51] M. Łebek and M. Panfil, *Navier-Stokes equations for nearly integrable quantum gases*, Phys. Rev. Lett. **134**, 010405 (2025), doi:[10.1103/PhysRevLett.134.010405](https://doi.org/10.1103/PhysRevLett.134.010405).
- [52] J. Lopez-Piqueres, B. Ware, S. Gopalakrishnan and R. Vasseur, *Hydrodynamics of nonintegrable systems from a relaxation-time approximation*, Phys. Rev. B **103**, L060302 (2021), doi:[10.1103/PhysRevB.103.L060302](https://doi.org/10.1103/PhysRevB.103.L060302).
- [53] J. Lopez-Piqueres, S. Gopalakrishnan and R. Vasseur, *Integrability breaking in the rule 54 cellular automaton*, J. Phys. A: Math. Theor. **55**, 234005 (2022), doi:[10.1088/1751-8121/ac6b66](https://doi.org/10.1088/1751-8121/ac6b66).
- [54] J. Lopez-Piqueres and R. Vasseur, *Integrability breaking from backscattering*, Phys. Rev. Lett. **130**, 247101 (2023), doi:[10.1103/PhysRevLett.130.247101](https://doi.org/10.1103/PhysRevLett.130.247101).
- [55] B. Bertini, F. H. L. Essler, S. Groha and N. J. Robinson, *Thermalization and light cones in a model with weak integrability breaking*, Phys. Rev. B **94**, 245117 (2016), doi:[10.1103/PhysRevB.94.245117](https://doi.org/10.1103/PhysRevB.94.245117).
- [56] J. Pawłowski, M. Panfil, J. Herbrych and M. Mierzejewski, *Long-living prethermalization in nearly integrable spin ladders*, Phys. Rev. B **109**, L161109 (2024), doi:[10.1103/PhysRevB.109.L161109](https://doi.org/10.1103/PhysRevB.109.L161109).
- [57] A. Bastianello, J. De Nardis and A. De Luca, *Generalized hydrodynamics with dephasing noise*, Phys. Rev. B **102**, 161110 (2020), doi:[10.1103/PhysRevB.102.161110](https://doi.org/10.1103/PhysRevB.102.161110).
- [58] G. del Vecchio del Vecchio, A. De Luca and A. Bastianello, *Transport through interacting defects and lack of thermalisation*, SciPost Phys. **12**, 060 (2022), doi:[10.21468/SciPostPhys.12.2.060](https://doi.org/10.21468/SciPostPhys.12.2.060).

- [59] J.-S. Caux, B. Doyon, J. Dubail, R. Konik and T. Yoshimura, *Hydrodynamics of the interacting Bose gas in the quantum Newton cradle setup*, SciPost Phys. **6**, 070 (2019), doi:[10.21468/SciPostPhys.6.6.070](https://doi.org/10.21468/SciPostPhys.6.6.070).
- [60] I. Bouchoule, B. Doyon and J. Dubail, *The effect of atom losses on the distribution of rapidities in the one-dimensional Bose gas*, SciPost Phys. **9**, 044 (2020), doi:[10.21468/SciPostPhys.9.4.044](https://doi.org/10.21468/SciPostPhys.9.4.044).
- [61] A. Bastianello, V. Alba and J.-S. Caux, *Generalized hydrodynamics with space-time inhomogeneous interactions*, Phys. Rev. Lett. **123**, 130602 (2019), doi:[10.1103/PhysRevLett.123.130602](https://doi.org/10.1103/PhysRevLett.123.130602).
- [62] B. Bertini, F. H. L. Essler, S. Groha and N. J. Robinson, *Prethermalization and thermalization in models with weak integrability breaking*, Phys. Rev. Lett. **115**, 180601 (2015), doi:[10.1103/PhysRevLett.115.180601](https://doi.org/10.1103/PhysRevLett.115.180601).
- [63] M. Panfil, S. Gopalakrishnan and R. M. Konik, *Thermalization of interacting quasi-one-dimensional systems*, Phys. Rev. Lett. **130**, 030401 (2023), doi:[10.1103/PhysRevLett.130.030401](https://doi.org/10.1103/PhysRevLett.130.030401).
- [64] L. Biagetti, M. Lebek, M. Panfil and J. De Nardis, *Generalised BBGKY hierarchy for near-integrable dynamics*, (arXiv preprint) doi:[10.48550/arXiv.2408.00593](https://doi.org/10.48550/arXiv.2408.00593).
- [65] M. Lebek, M. Panfil and R. M. Konik, *Prethermalization in coupled one-dimensional quantum gases*, SciPost Phys. **17**, 007 (2024), doi:[10.21468/SciPostPhys.17.1.007](https://doi.org/10.21468/SciPostPhys.17.1.007).
- [66] T. Tanaka and Y. Nishida, *Thermal conductivity of a weakly interacting Bose gas in quasi-one-dimension*, Phys. Rev. E **106**, 064104 (2022), doi:[10.1103/PhysRevE.106.064104](https://doi.org/10.1103/PhysRevE.106.064104).
- [67] P. Résibois, *On linearized hydrodynamic modes in statistical physics*, J. Stat. Phys. **2**, 21 (1970), doi:[10.1007/BF01009709](https://doi.org/10.1007/BF01009709).
- [68] D. Forster, D. R. Nelson and M. J. Stephen, *Large-distance and long-time properties of a randomly stirred fluid*, Phys. Rev. A **16**, 732 (1977), doi:[10.1103/PhysRevA.16.732](https://doi.org/10.1103/PhysRevA.16.732).
- [69] O. Narayan and S. Ramaswamy, *Anomalous heat conduction in one-dimensional momentum-conserving systems*, Phys. Rev. Lett. **89**, 200601 (2002), doi:[10.1103/PhysRevLett.89.200601](https://doi.org/10.1103/PhysRevLett.89.200601).
- [70] H. van Beijeren, *Exact results for anomalous transport in one-dimensional Hamiltonian systems*, Phys. Rev. Lett. **108**, 180601 (2012), doi:[10.1103/PhysRevLett.108.180601](https://doi.org/10.1103/PhysRevLett.108.180601).
- [71] H. Spohn, *Nonlinear fluctuating hydrodynamics for anharmonic chains*, J. Stat. Phys. **154**, 1191 (2014), doi:[10.1007/s10955-014-0933-y](https://doi.org/10.1007/s10955-014-0933-y).
- [72] C. N. Yang and C. P. Yang, *Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interaction*, J. Math. Phys. **10**, 1115 (1969), doi:[10.1063/1.1664947](https://doi.org/10.1063/1.1664947).
- [73] J.-S. Caux and R. M. Konik, *Constructing the generalized Gibbs ensemble after a quantum quench*, Phys. Rev. Lett. **109**, 175301 (2012), doi:[10.1103/PhysRevLett.109.175301](https://doi.org/10.1103/PhysRevLett.109.175301).
- [74] J. Mossel and J.-S. Caux, *Generalized TBA and generalized Gibbs*, J. Phys. A: Math. Theor. **45**, 255001 (2012), doi:[10.1088/1751-8113/45/25/255001](https://doi.org/10.1088/1751-8113/45/25/255001).

- [75] M. Takahashi, *Thermodynamics of one-dimensional solvable models*, Cambridge University Press, Cambridge, UK, ISBN 9780521551434 (1999), doi:[10.1017/CBO9780511524332](https://doi.org/10.1017/CBO9780511524332).
- [76] E. H. Lieb and W. Liniger, *Exact analysis of an interacting Bose gas. I. The general solution and the ground state*, Phys. Rev. **130**, 1605 (1963), doi:[10.1103/PhysRev.130.1605](https://doi.org/10.1103/PhysRev.130.1605).
- [77] E. H. Lieb, *Exact analysis of an interacting Bose gas. II. The excitation spectrum*, Phys. Rev. **130**, 1616 (1963), doi:[10.1103/PhysRev.130.1616](https://doi.org/10.1103/PhysRev.130.1616).
- [78] A. Bastianello, A. De Luca, B. Doyon and J. De Nardis, *Thermalization of a trapped one-dimensional Bose gas via diffusion*, Phys. Rev. Lett. **125**, 240604 (2020), doi:[10.1103/PhysRevLett.125.240604](https://doi.org/10.1103/PhysRevLett.125.240604).
- [79] B. Doyon, J. Dubail, R. Konik and T. Yoshimura, *Large-scale description of interacting one-dimensional Bose gases: Generalized hydrodynamics supersedes conventional hydrodynamics*, Phys. Rev. Lett. **119**, 195301 (2017), doi:[10.1103/PhysRevLett.119.195301](https://doi.org/10.1103/PhysRevLett.119.195301).
- [80] B. J. Alder and T. E. Wainwright, *Decay of the velocity autocorrelation function*, Phys. Rev. A **1**, 18 (1970), doi:[10.1103/PhysRevA.1.18](https://doi.org/10.1103/PhysRevA.1.18).
- [81] L. Delacrétaz, A. L. Fitzpatrick, E. Katz and M. Walters, *Thermalization and hydrodynamics of two-dimensional quantum field theories*, SciPost Phys. **12**, 119 (2022), doi:[10.21468/SciPostPhys.12.4.119](https://doi.org/10.21468/SciPostPhys.12.4.119).
- [82] A. Miron, J. Cividini, A. Kundu and D. Mukamel, *Derivation of fluctuating hydrodynamics and crossover from diffusive to anomalous transport in a hard-particle gas*, Phys. Rev. E **99**, 012124 (2019), doi:[10.1103/PhysRevE.99.012124](https://doi.org/10.1103/PhysRevE.99.012124).
- [83] S. Chen, J. Wang, G. Casati and G. Benenti, *Nonintegrability and the Fourier heat conduction law*, Phys. Rev. E **90**, 032134 (2014), doi:[10.1103/PhysRevE.90.032134](https://doi.org/10.1103/PhysRevE.90.032134).
- [84] H. Zhao and W.-g. Wang, *Fourier heat conduction as a strong kinetic effect in one-dimensional hard-core gases*, Phys. Rev. E **97**, 010103 (2018), doi:[10.1103/PhysRevE.97.010103](https://doi.org/10.1103/PhysRevE.97.010103).
- [85] S. Lepri, R. Livi and A. Politi, *Too close to integrable: Crossover from normal to anomalous heat diffusion*, Phys. Rev. Lett. **125**, 040604 (2020), doi:[10.1103/PhysRevLett.125.040604](https://doi.org/10.1103/PhysRevLett.125.040604).
- [86] S. Lepri (ed.), *Thermal transport in low dimensions*, Springer, Cham, Switzerland, ISBN 9783319292595 (2016), doi:[10.1007/978-3-319-29261-8](https://doi.org/10.1007/978-3-319-29261-8).
- [87] K. Saito, M. Hongo, A. Dhar and S.-i. Sasa, *Microscopic theory of fluctuating hydrodynamics in nonlinear lattices*, Phys. Rev. Lett. **127**, 010601 (2021), doi:[10.1103/PhysRevLett.127.010601](https://doi.org/10.1103/PhysRevLett.127.010601).
- [88] T. Prosen and D. K. Campbell, *Normal and anomalous heat transport in one-dimensional classical lattices*, Chaos: Interdiscip. J. Nonlinear Sci. **15**, 015117 (2005), doi:[10.1063/1.1868532](https://doi.org/10.1063/1.1868532).
- [89] A. Dhar, *Heat transport in low-dimensional systems*, Adv. Phys. **57**, 457 (2008), doi:[10.1080/00018730802538522](https://doi.org/10.1080/00018730802538522).
- [90] A. Dhar, A. Kundu, J. L. Lebowitz and J. A. Scaramazza, *Transport properties of the classical Toda chain: Effect of a pinning potential*, J. Stat. Phys. **175**, 1298 (2019), doi:[10.1007/s10955-019-02284-6](https://doi.org/10.1007/s10955-019-02284-6).

- [91] P. Di Cintio, S. Iubini, S. Lepri and R. Livi, *Transport in perturbed classical integrable systems: The pinned Toda chain*, Chaos Solit. Fractals **117**, 249 (2018), doi:[10.1016/j.chaos.2018.11.003](https://doi.org/10.1016/j.chaos.2018.11.003).
- [92] A. Urichuk, S. Scopa and J. De Nardis, *Navier-Stokes equations for low-temperature one-dimensional quantum fluids*, Phys. Rev. Lett. **132**, 243402 (2024), doi:[10.1103/PhysRevLett.132.243402](https://doi.org/10.1103/PhysRevLett.132.243402).
- [93] S. Chakraborti, S. Ganapa, P. L. Krapivsky and A. Dhar, *Blast in a one-dimensional cold gas: From Newtonian dynamics to hydrodynamics*, Phys. Rev. Lett. **126**, 244503 (2021), doi:[10.1103/PhysRevLett.126.244503](https://doi.org/10.1103/PhysRevLett.126.244503).
- [94] S. Ganapa, S. Chakraborti, P. L. Krapivsky and A. Dhar, *Blast in the one-dimensional cold gas: Comparison of microscopic simulations with hydrodynamic predictions*, Phys. Fluids **33**, 087113 (2021), doi:[10.1063/5.0058152](https://doi.org/10.1063/5.0058152).

2.4 Article IV: Crossover from generalized to conventional hydrodynamics in nearly-integrable systems under relaxation time approximation

- S. Santra, M. Łebek, M. Panfil “Crossover from generalized to conventional hydrodynamics in nearly-integrable systems under relaxation time approximation”, [arXiv:2603.02158](https://arxiv.org/abs/2603.02158) (under review in Physical Review E)

While many works on the emergence of hydrodynamics address asymptotically large space-time scales, these in many cases can be out of reach of experimental setups. In our context, when the integrability is weakly broken, the deviations from the purely integrable dynamics (given by GHD) could be visible on space and time scales that are large in the sense of experimental capabilities.

This motivates the study not only of the asymptotic regime, in which the NS equations (and anomalous transport) emerge, but also of the whole crossover between generalized and conventional hydrodynamics. The present article is dedicated to such an investigation.

To tackle this problem, we consider a simplified version of the collision integral in the form of the relaxation time approximation, a standard approximation known from the kinetic theory, see Fig. 15(a). The collision kernel in such an approximation features a single characteristic relaxation time scale, on which the state relaxes exponentially to a local thermal equilibrium. Furthermore, thanks to the simplicity of the collision integral, the transport coefficients can be explicitly computed.

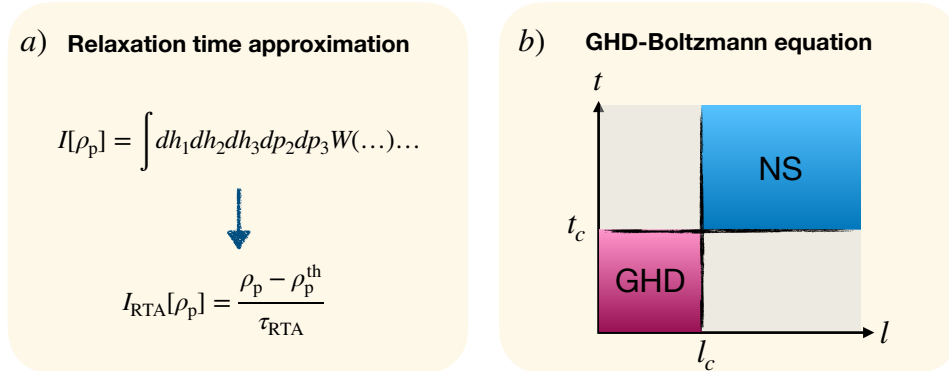


Figure 15: Panel a): We consider a simplified version of the collision integral by employing the relaxation time approximation. Panel b): In this work we study different space-time regimes of the GHD-Boltzmann equation. At sufficiently large times $t \gg t_c$ and length scales $l \gg l_c$, the NS hydrodynamics applies, whereas at small scales the correct description is provided by the GHD. In grey regions, none of these descriptions applies and one has to solve the full GHD-Boltzmann equation.

The crossover between different hydrodynamic regimes is analyzed through the numerical solution of the GHD-Boltzmann equation and a comparison with GHD and NS hydrodynamics. We pinpoint the relevant crossover length and time scales and confirm that at relatively short times (and length scales) the correct description is provided by GHD, whereas on larger scales, the NS equations become the correct theory, see Fig.15(b). We also examine hydrodynamic correlation functions, where the same scenario takes place.

Candidate's contribution: I led the theoretical analysis presented in Sec.4 (GHD-NS crossover). I also performed the numerical and analytical computations for the hydrodynamic correlation

functions. I contributed to writing of the article and assisted in determining the transport coefficients.

Crossover from generalized to conventional hydrodynamics in nearly integrable systems under relaxation time approximation

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Upon breaking the integrability, the equations of generalized hydrodynamics (GHD) are supplemented by a Boltzmann collision term. Such terms are typically complicated and stem from a perturbative treatment of integrability-breaking terms in the hamiltonian. In our work, we study a simplified version of the collision operator in a form of relaxation time approximation familiar from kinetic theory. We explicitly compute transport coefficients which characterize the Navier-Stokes (NS) hydrodynamic regime emerging at large space-time scales. We also thoroughly study the crossover between GHD and NS hydrodynamic descriptions, identifying relevant characteristic space-time scales for the transition. In particular, we show how the emergence of NS hydrodynamics is visible in dynamics of conserved and non-conserved charge densities, and in hydrodynamic two-point functions.

I. INTRODUCTION

Understanding the out-of-equilibrium behavior of interacting many-body quantum systems remains one of the most profound and active areas of research in theoretical physics. Generic isolated systems are expected to thermalize under non-equilibrium dynamics [1–4]. However, certain class of complex systems have been identified that appear to avoid thermalization even after long times due to their integrable structure. To describe the hydrodynamics of such systems, a theoretical framework known as generalized hydrodynamics (GHD) has been developed [5–10]. It captures the essential characteristics of integrable dynamics, namely the existence of infinitely many conservation laws and the presence of stable, ballistically propagating quasiparticles. This framework has been successful in explaining various phenomena, such as the lack of thermalization observed in the well-known Newton’s cradle experiment [11, 12], the derivation of exact expressions for Drude weights [13–16] and the accurate description of anomalous transport in strongly interacting spin chains [17–22], to name a few. Recent developments [11, 12, 23–28] in cold atomic systems, particularly in controlling and manipulating the interactions have fuelled the progress of GHD which by now has undergone several numerical and experimental verifications [29–37].

Integrable systems are fine-tuned and not exactly realized in experiments. Therefore, it is important to study nearly integrable systems where the integrability is weakly broken due to, for instance, additional interactions present among the particles. In such scenario a system is described by a Hamiltonian consisting of an integrable part, $\hat{H}_0$ and a weak non-integrable perturbation $\lambda \hat{V}$, where λ is a small parameter controlling the strength of the integrability breaking. If the perturbation $\hat{V}$ de-

scribes an additional, long-range interactions between the particles and thus destroys all but three of the conserved quantities of the underlying integrable model, it has been shown that such systems eventually thermalize at times $t \approx \mathcal{O}[(a\lambda)^{-2}]$ [38] where a is the parameter controlling strength of integrable interactions. At large space-time scales, the effect of integrability breaking can be incorporated as corrections to the original GHD of the underlying integrable model [26, 27, 39–48]. Specifically, in the weak integrability breaking regime, the quasiparticle distribution $\rho_p(\lambda, x, t)$ evolves according to the GHD equation modified with a collision integral $\mathcal{I}[\rho_p]$, given by [40, 43]

$$\partial_t \rho_p + \partial_x (v \rho_p) = \frac{1}{2} \partial_x (\mathfrak{D} \partial_x \rho_p) + \mathcal{I}[\rho_p], \quad (1)$$

where v is the effective velocity and $\mathfrak{D}$ represents the diffusion kernel [5, 7]. Evaluating the collision integral $\mathcal{I}[\rho_p]$ involves, either applying Fermi’s Golden Rule to the matrix elements of the integrability-breaking term [40, 43] or using the gBBGKY hierarchy [38].

In this work, rather than computing the collision integral directly from a given integrability breaking perturbation of the system’s Hamiltonian, we adopt an approximate form under the relaxation time approximation (RTA), as introduced in the GHD context in [44]:

$$\mathcal{I}_{\text{RTA}}[\rho_p] = \frac{\rho_p^{\text{th}} - \rho_p}{\tau}. \quad (2)$$

The fundamental assumption of RTA is that the quasiparticle distribution relaxes over a single characteristic timescale τ . Remarkably, this assumption has been found to work well in reproducing the numerical time evolution in nearly-integrable XXZ model [44].

Moreover, the RTA preserves the standard (conventional) conservation laws, making it particularly well-suited for studying hydrodynamic behavior. Consequently, one can argue that the RTA is capable of capturing the essential features of the crossover from generalized to conventional hydrodynamics—one of the central phenomena explored in this work.

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It is expected that at short timescales ($t \ll \tau$), the effect of the collision integral on the dynamics is negligible and the system evolves according to GHD. In contrast, at long timescales ($t \gg \tau$), the dynamics is completely described by conventional hydrodynamics where only the conserved quantities respected by the collision integral play a role. If the collision integral destroys all conserved quantities except for the particle number, momentum and energy, the long-time dynamics is governed by the universal Navier-Stokes (NS) equations which are nothing but the continuity equations for the density, momentum and the energy fields [49–53].

In this work we present a detailed analysis of the crossover between the GHD and the conventional hydrodynamics for nearly integrable quantum gases within the RTA. We start with evaluating the transport coefficients for the Navier-Stokes equations. This amounts to solving certain integral equations that depend on the collision integral. For the RTA collision integral, as we show, they admit analytical solutions and are expressed in terms of the Drude weights and other thermodynamic quantities computed for integrable theory. This is discussed in Section III and provides us with precise predictions of the conventional hydrodynamics. In the following Section IV, we consider the full GHD dynamics and show that the conventional hydrodynamics emerges at late times and identify relevant space and time scales for the crossover. Here, we focus on a linear regime where initially homogeneous system is perturbed locally in space. This gives us an analytical control on the dynamics from which the emergence of the conventional hydrodynamics can be derived and understood. This allows us also to identify the relevant degrees of freedom which separate into the three standard conservation laws and the decaying modes. In the final Section V, we apply our framework to study of the hydrodynamic correlation functions, in which a similar crossover between and GHD and conventional hydrodynamics is identified. Some of the technical detail of our work are to be found in the Appendices.

II. PRELIMINARIES

We provide here a short overview of GHD and introduce ingredients needed in the later parts. Nearly integrable systems under the RTA can be described by GHD-Boltzmann equation of the form [40, 43, 47]

$$\partial_t \rho_p + \partial_x (v_\rho \rho_p) = \frac{1}{2} \partial_x (\mathfrak{D}_\rho \partial_x \rho_p) + \mathcal{I}_{\text{RTA}}[\rho_p]. \quad (3)$$

The subscript ρ in the effective velocity v_ρ and the diffusion kernel $\mathfrak{D}_\rho$ indicates that both are functionals of the quasiparticle distribution $\rho_p(\lambda, x, t)$ and therefore depend on space and time. However, for notational simplicity, we have omitted the subscript ρ in (1) and will continue to do so hereafter. We consider Galilean-invariant models of particles with unit mass. While in this work we focus

on particles with fermionic statistics having in mind the Lieb-Liniger model, all the results can be easily generalized to arbitrary statistics by the means of changing statistical factor defined below and adjusting the expression for the free energy in terms of ρ_p . For such generalization, we refer to [8]. In what follows we introduce the necessary ingredients of Thermodynamic Bethe Ansatz (TBA) and GHD, which are needed to understand the structure of GHD-Boltzmann equation.

Setting the right-hand side of (3) to zero corresponds to Euler GHD which is time-reversible and describes purely ballistic transport. For a quasiparticle with rapidity λ , the effective velocity $v(\lambda, x, t)$ refers to its group velocity, renormalized due to interactions with other quasiparticles inside the local state at position x and time t . It can be obtained by solving the following linear integral equation [6, 8]:

$$v(\lambda) = \lambda + 2\pi \int d\lambda' \mathcal{T}(\lambda - \lambda') \rho_p(\lambda') [v(\lambda') - v(\lambda)]. \quad (4)$$

The two-body scattering kernel $\mathcal{T}(\lambda - \lambda')$ in (4) encodes the scattering shift due to a collision between two quasiparticles with rapidity λ and λ' . The function $\mathcal{T}(\lambda)$ explicitly depends on the underlying microscopic interactions and is therefore model-dependent. In the case of the Lieb-Liniger model, $\mathcal{T}$ takes a particularly simple form,

$$\mathcal{T}(\lambda) = \frac{1}{2\pi} \frac{2c}{c^2 + \lambda^2}, \quad (5)$$

where c denotes the strength of the δ -function interaction potential [54, 55].

In addition to the Euler-scale GHD, a diffusion term appears in the large scale dynamics of interacting integrable systems [7, 56, 57]. In such systems, diffusion originates from scatterings among quasiparticles, a feature absent in non-interacting integrable systems [58]. This diffusion term provides subleading corrections to the Euler-scale GHD and is responsible for positive entropy production as well as diffusive relaxation mechanisms [7, 59]. The diffusion operator in (3) acts as $\partial_x [\mathfrak{D} \partial_x \rho_p](\theta) = \int d\alpha \partial_x [\mathfrak{D}(\theta, \alpha) \partial_x \rho_p(\alpha)]$ and the diffusion kernel $\mathfrak{D}$ is given by

$$\mathfrak{D} = (1 - n\mathcal{T})^{-1} \rho_{\text{tot}}^{-1} \tilde{\mathfrak{D}} \rho_{\text{tot}}^{-1} (1 - n\mathcal{T}), \quad (6)$$

with $\tilde{\mathfrak{D}}(\lambda, \mu) = \delta(\lambda - \mu) w(\lambda) - W(\lambda, \mu)$. In our notation, $n(\lambda)$ denotes the occupation function which is related to the rapidity distribution $\rho_p(\lambda)$ through $\rho_p(\lambda) = n(\lambda) \rho_{\text{tot}}(\lambda)$, where ρ_{tot} is the total density of states. It satisfies the following integral equation [8]

$$\rho_{\text{tot}}(\lambda) = \frac{1}{2\pi} + \int d\lambda' \mathcal{T}(\lambda - \lambda') n(\lambda') \rho_{\text{tot}}(\lambda'). \quad (7)$$

The two auxiliary functions appearing in $\tilde{\mathfrak{D}}$ are

$$W(\lambda, \mu) = \rho_p(\lambda) f(\lambda) [\mathcal{T}^{\text{dr}}(\lambda, \mu)]^2 |v(\lambda) - v(\mu)|, \quad (8)$$

and $w(\mu) = \int d\lambda W(\lambda, \mu)$. The dressing operation applied to any function $g(\lambda)$ is defined as

$$g^{\text{dr}}(\lambda) = g(\lambda) + \int d\lambda' \mathcal{T}(\lambda - \lambda') n(\lambda') g^{\text{dr}}(\lambda'). \quad (9)$$

At this point, let us comment on the recent refinements [60] in the theory of GHD at the diffusive scale. The picture of a single equation with diffusion kernel (6) is not correct in the presence of long-range correlations [61]. Such correlations are dynamically generated in inhomogeneous interacting integrable systems and one has to solve a system of two equations (for one- and two-point function) in order to properly describe the dynamics. In our case, we will always work in the linear regime, where the single equation with diffusion kernel $\mathfrak{D}$ is valid [61].

Lastly, we discuss the Boltzmann kernel in RTA. The specific form of the collision integral (2) ensures that the system locally thermalizes to the Gibbs state over relaxation timescale τ . The state ρ_p^{th} in (2) is a nonlinear functional of ρ_p , determined by requiring that its conserved charges

$$\mathcal{Q}_n^{\text{th}}(x, t) = \int d\lambda \mathcal{H}_n(\lambda) \rho_p^{\text{th}}[\rho_p(\lambda, x, t)], \quad n = 0, 1, 2, \quad (10)$$

locally coincide with those of the state ρ_p . By

$$\mathcal{H}_n(\lambda) = \lambda^n / n!, \quad n \in \mathbb{N}, \quad (11)$$

we denote single particle eigenvalues of the *ultralocal charges*. In order to define thermal states we introduce more objects of TBA. In general, pseudoenergy $\epsilon(\lambda)$ fulfills the following integral equation

$$\epsilon(\lambda) = \epsilon_0(\lambda) - \int d\lambda' \mathcal{T}(\lambda - \lambda') \log(1 + e^{-\epsilon(\lambda')}), \quad (12)$$

where the bare pseudoenergy is given by

$$\epsilon_0(\lambda) = \sum_n \beta_n \mathcal{H}_n(\lambda). \quad (13)$$

A thermal boosted state is characterized by the first three chemical potentials, β_0 , β_1 and β_2 while the remaining ($n \geq 3$) are zero. From the pseudoenergy we can get the occupation function as $n(\lambda) = 1/(1 + e^{\epsilon(\lambda)})$, which is then used to solve the integral equation (7) for ρ_{tot} . Once $\rho_p^{\text{th}}(\lambda)$ is known, the expectation values of the conserved charges $\mathcal{Q}_n^{\text{th}}(x, t)$ can be computed using (10). For a state defined by the distribution ρ_p , the corresponding local thermal state is specified by the chemical potentials β_0, β_1 and β_2 which are adjusted so that the charges $\mathcal{Q}_n^{\text{th}}(x, t)$ coincide with $\mathcal{Q}_n(x, t)$ obtained from ρ_p .

How the variation in chemical potentials changes the charges is encoded in the charge-susceptibility matrix $\mathcal{C}$, with kernel given as [8, 13]

$$\mathcal{C} = (1 - n\mathcal{T})^{-1} \rho_p f(1 - n\mathcal{T}). \quad (14)$$

Its matrix elements, in the basis of the ultralocal charges, are given by

$$\mathcal{C}_{mn} = -\frac{\delta \mathcal{Q}_m}{\delta \beta_n} = \int d\lambda \rho_p(\lambda) f(\lambda) \mathcal{H}_m^{\text{dr}}(\lambda) \mathcal{H}_n^{\text{dr}}(\lambda). \quad (15)$$

We will also use the bra-ket notation for the matrix elements which makes the basis apparent,

$$\mathcal{C}_{mn} = \langle \mathcal{H}_m | \mathcal{C} | \mathcal{H}_n \rangle. \quad (16)$$

This is useful for introducing the hydrodynamic scalar product $(f|g) \equiv \langle f | \mathcal{C} | g \rangle$ and the special basis $\{h_n\}$ of *orthonormal charges* which diagonalize the hydrodynamic scalar product $(h_m | h_n) = \delta_{mn}$. They are constructed via the Gram-Schmidt (GS) orthonormalization of the ultralocal charges $\{\mathcal{H}_n(\lambda)\}$. Similar to the charge-susceptibility matrix $\mathcal{C}$, one can define an analogous matrix $\mathcal{B}$ that characterizes the response of the associated currents $\{\mathcal{J}_m\}$ to changes in the chemical potentials $\{\beta_n\}$. The matrix $\mathcal{B}$ is given by

$$\mathcal{B} = (1 - n\mathcal{T})^{-1} \rho_p f v(1 - n\mathcal{T}), \quad (17)$$

with elements in the basis of ultralocal charges

$$\mathcal{B}_{mn} = -\frac{\delta \mathcal{J}_m}{\delta \beta_n} = \int d\lambda \rho_p(\lambda) f(\lambda) v(\lambda) \mathcal{H}_m^{\text{dr}}(\lambda) \mathcal{H}_n^{\text{dr}}(\lambda). \quad (18)$$

Finally, we are interested in the linearized GHD dynamics where $\rho_p = \rho_p^{\text{th}} + \delta \rho_p$. Here ρ_p^{th} is a homogeneous thermal state and $\delta \rho_p$ is a small perturbation of quasiparticle density on top of it. Linearizing Eq. (1) about the thermal state ρ_p^{th} , one gets

$$\partial_t \delta \rho_p + \mathcal{A}_{\text{th}} \partial_x \delta \rho_p = \frac{1}{2} \mathfrak{D}_{\text{th}} \partial_x^2 \delta \rho_p + \left(\frac{\delta \mathcal{I}}{\delta \rho_p} \right)_{\text{th}} \delta \rho_p, \quad (19)$$

where the operator $\mathcal{A} = (1 - n\mathcal{T})^{-1} v_p (1 - n\mathcal{T})$ [13] and the subscript ‘th’ indicates that these quantities are evaluated in the homogeneous thermal state ρ_p^{th} . It is convenient to express the resulting dynamics in terms of perturbations to the orthonormal charges, $\delta q_n(x, t) = \int d\lambda h_n(\lambda) \delta \rho_p(\lambda, x, t)$. The equations are [52]

$$\partial_t \delta q_n + \sum_m A_{nm} \partial_x \delta q_m = \sum_m \mathcal{D}_{nm} \partial_x^2 \delta q_m - \sum_m \Gamma_{nm} \delta q_m. \quad (20)$$

The matrices A , $\mathcal{D}$ and Γ have the following elements,

$$A_{nm} = \int d\lambda \rho_p(\lambda) f(\lambda) v(\lambda) h_n^{\text{dr}}(\lambda) h_m^{\text{dr}}(\lambda), \quad (21)$$

$$\mathcal{D}_{nm} = \frac{1}{2} \int d\lambda d\mu h_n^{\text{dr}}(\lambda) \rho_{\text{tot}}^{-1}(\lambda) \tilde{\mathfrak{D}}_{\text{th}}(\lambda, \mu), \quad (22)$$

$$\times \rho_{\text{tot}}^{-1}(\mu) \rho_p(\mu) f(\mu) h_m^{\text{dr}}(\mu) \quad (23)$$

$$\Gamma_{nm} = \int d\lambda d\mu h_n(\lambda) \Gamma(\lambda, \mu) h_m(\mu), \quad (24)$$

where the collisional operator Γ is obtained from the functional derivative of the collision integral with respect to the quasiparticle density and takes the form

$$\Gamma = -\frac{\delta \mathcal{I}}{\delta \rho_p} \mathcal{C}. \quad (25)$$

As a last ingredient we introduce Drude weights which measures the ballistic, dissipationless part of transport in a system [13]. The matrix elements are given by (note that these are matrix elements in orthonormal basis)

$$D_{mn} = \int d\lambda \rho_p(\lambda) f(\lambda) v^2(\lambda) h_m^{\text{dr}}(\lambda) h_n^{\text{dr}}(\lambda). \quad (26)$$

We now turn to the analysis of transport coefficients in nearly integrable systems within the RTA.

III. NAVIER-STOKES EQUATIONS AND TRANSPORT COEFFICIENTS

The first three conserved charges ($n = 0, 1, 2$) in the Galilean invariant model correspond to particle number, momentum and energy. If the collision integral respects these three conservation laws i.e.,

$$\int d\lambda \mathcal{H}_n(\lambda) \mathcal{I}(\rho_p) = 0, \quad n = 0, 1, 2, \quad (27)$$

and breaks all the other then at large space-time scales the dynamics can be described by the Navier-Stokes equations [49–53],

$$\begin{aligned} \partial_t \varrho &= -\partial_x(\varrho u), & \partial_t(\varrho u) &= -\partial_x(\varrho u^2 + \mathcal{P}), \\ \partial_t(\varrho e) &= -\partial_x(u\varrho e + \mathcal{J}) - \mathcal{P}\partial_x u. \end{aligned} \quad (28)$$

The hydrodynamic fields are related to the local densities through the following relations [52]:

$$\varrho = \varrho_0, \quad u = \frac{\varrho_1}{\varrho_0}, \quad e = \frac{\varrho_2}{\varrho_0} - \frac{\varrho_1^2}{2\varrho_0^2}. \quad (29)$$

The dynamic pressure $\mathcal{P}$ and heat current $\mathcal{J}$ are determined by the Newton's law for a viscous fluid (with shear viscosity absent in one spatial dimension) and the Fourier's law for the heat current:

$$\mathcal{P} = P - \zeta \partial_x u, \quad \mathcal{J} = -\kappa \partial_x T. \quad (30)$$

Here, P denotes the standard thermodynamic pressure, T the local temperature and the two transport coefficients are viscosity ζ and thermal conductivity κ . They receive additive contributions which can be identified as one coming from the GHD diffusion, hence due to integrable interactions, and the other from the collision integral, hence due to integrability breaking interactions,

$$\zeta = \zeta_{\mathcal{D}} + \zeta_{\mathcal{I}}, \quad \kappa = \kappa_{\mathcal{D}} + \kappa_{\mathcal{I}}. \quad (31)$$

In Ref. [52], the contributions $\zeta_{\mathcal{D}}$ and $\kappa_{\mathcal{D}}$, which arise from the GHD diffusion and are independent of the details of the collision integral, have been thoroughly investigated. They have the following forms

$$\zeta_{\mathcal{D}} = \frac{1}{2T} \langle \mathcal{H}_1 | \mathcal{D} \mathcal{C} | \mathcal{H}_1 \rangle, \quad \kappa_{\mathcal{D}} = \frac{1}{2T^2} \langle \mathcal{H}_2 | \mathcal{D} \mathcal{C} | \mathcal{H}_2 \rangle. \quad (32)$$

On the other hand, the contributions $\kappa_{\mathcal{I}}$ and $\zeta_{\mathcal{I}}$ are due to the collision integral and depend on their precise structure. They capture dissipative effects of quasiparticle scattering and are given by

$$\zeta_{\mathcal{I}} = \langle \mathcal{H}_1 | \mathcal{B} | \phi_{\zeta} \rangle, \quad \kappa_{\mathcal{I}} = \langle \mathcal{H}_2 | \mathcal{B} | \phi_{\kappa} \rangle. \quad (33)$$

The functions ϕ_{ζ} and ϕ_{κ} are determined by integral equations involving the linearized collision operator

$$\Gamma | \phi_{\zeta, \kappa} \rangle = | \eta_{\zeta, \kappa} \rangle, \quad (34)$$

together with the relations $\langle h_n | \mathcal{C} | \phi_{\zeta, \kappa} \rangle = 0$ for $n = 0, 1, 2$. The source terms $\eta_{\zeta, \kappa}$ are given by [52]

$$\eta_{\zeta} = \sqrt{\frac{\varrho}{T}} \left[\mathcal{B} h_1 - \frac{1}{\sqrt{\varrho \kappa_T}} \mathcal{C} \left(h_0 + \sqrt{\frac{c_P - c_V}{c_V}} h_2 \right) \right], \quad (35)$$

$$\eta_{\kappa} = \frac{\sqrt{\varrho c_V}}{T} \mathcal{B} \left(h_2 - \sqrt{\frac{c_P - c_V}{c_V}} h_0 \right). \quad (36)$$

Here, κ_T represents the isothermal compressibility of the system while the specific heats at constant volume and constant pressure are denoted by c_V and c_P , respectively. All thermodynamic quantities are computed for the integrable model using TBA. The Γ operator can be diagonalised (it is real and symmetric), $\Gamma | \gamma_{\alpha} \rangle = \gamma_{\alpha} | \gamma_{\alpha} \rangle$ and the solution to the equation (34) is

$$| \phi_{\zeta, \kappa} \rangle = \sum_{\gamma_{\beta} \neq 0} \gamma_{\beta}^{-1} \langle g_{\beta} | \eta_{\zeta, \kappa} \rangle | g_{\beta} \rangle. \quad (37)$$

As we shown in Appendix A, it is the orthonormal basis $\{h_n\}$ that diagonalizes the RTA Γ operator. Consequently, it has three zero eigenvalues, corresponding to the three collision invariants, while all the other eigenvalues are equal τ^{-1} , a feature specific to the RTA collision integral. This gives

$$| \phi_{\zeta, \kappa} \rangle = \tau \sum_{n \neq 0, 1, 2} \langle h_n | \eta_{\zeta, \kappa} \rangle | h_n \rangle, \quad (38)$$

with the relations $\langle h_n | \mathcal{C} | \phi_{\zeta, \kappa} \rangle = 0$ for $n = 0, 1, 2$ fulfilled. The formulas for the transport coefficients are then

$$\zeta_{\mathcal{I}} = \tau \sum_{n \neq 0, 1, 2} \langle \mathcal{H}_1 | \mathcal{B} | h_n \rangle \langle h_n | \eta_{\zeta} \rangle, \quad (39)$$

$$\kappa_{\mathcal{I}} = \tau \sum_{n \neq 0, 1, 2} \langle \mathcal{H}_2 | \mathcal{B} | h_n \rangle \langle h_n | \eta_{\kappa} \rangle. \quad (40)$$

After straightforward algebraic manipulations, shown in Appendix B, these expressions can be simplified to

$$\zeta_{\mathcal{I}} = \varrho \tau \omega_1^{\text{coll}}, \quad \kappa_{\mathcal{I}} = \varrho c_V \tau \omega_{\text{th}}^{\text{coll}}, \quad (41)$$

with

$$\omega_1^{\text{coll}} = D_{11} - A_{10}^2 - A_{12}^2, \quad \omega_{\text{th}}^{\text{coll}} = D_{22} - A_{12}^2. \quad (42)$$

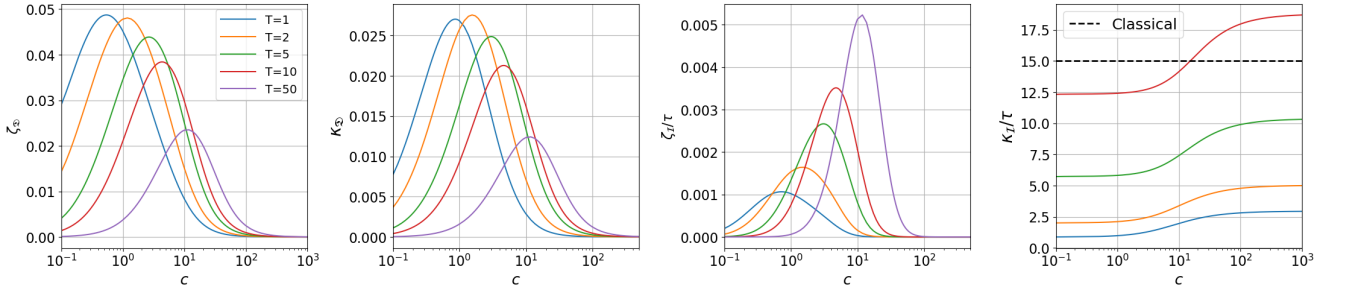


FIG. 1. Contributions to the transport coefficients as a function of the interaction strength c for several values of temperature T . The contributions $\zeta_{\mathfrak{D}}$ and $\kappa_{\mathfrak{D}}$ are computed using (32) while the contributions $\zeta_{\mathcal{I}}$ and $\kappa_{\mathcal{I}}$ are obtained from (41). Irrespective of the temperature T , all coefficients except $\kappa_{\mathcal{I}}$ show non-monotonic behavior as a function of c . Starting near zero for small c , they attain a maximum value at a finite value of c and eventually vanish at very large c . With increasing temperature T , the positions of these maxima shift toward larger values of c . In contrast, the $\kappa_{\mathcal{I}}$ displays monotonic behavior with respect to c , it begins at a finite value at $c \rightarrow 0$, increases monotonically with c and asymptotically approaches another finite value at $c \rightarrow \infty$. For reference, a dashed line corresponding to classical free particles, given by the formula $\kappa_{\mathcal{I}} = \frac{3}{2}\varrho T$, which is derived from simple adaptation of (42) is shown in the last panel for the temperature $T = 10$. For better visibility, we refrain from plotting the data for $\kappa_{\mathcal{I}}$ at temperature $T = 50$. In all figures, the density ϱ is set to 1.

Note that, the matrix elements A_{10} and A_{12} in (42) are explicitly known in terms of the thermodynamic parameters of the system [52]:

$$A_{10} = \frac{1}{\sqrt{\varrho \chi_T}}, \quad A_{12} = \frac{1}{\sqrt{\varrho \chi_T}} \sqrt{\frac{c_P - c_V}{c_V}}. \quad (43)$$

In Fig. 1, we plot the $\mathfrak{D}$ and $\mathcal{I}$ components of the thermal conductivity and the viscosity separately as a function of the interaction strength c . Apart from the $\kappa_{\mathcal{I}}$, all other contributions exhibit non-monotonic behaviour as a function of c : approaching zero in the limit $c \rightarrow 0$, they reach a maximum at some finite value of c and then vanish at $c \rightarrow \infty$. The limiting values of the $\mathfrak{D}$ components are already known, since at the two special limits ($c \rightarrow 0$ and $c \rightarrow \infty$) the system becomes non-interacting and therefore there are no diffusive corrections to the GHD equation [7]. The fact that the coefficient $\zeta_{\mathcal{I}}$ also vanishes in these two limits is shown explicitly in Appendix. C. Several curves are shown corresponding to different system temperature. In the strongly interacting regime i.e., when $c \gg 1$, the transport coefficient increases with increasing temperature. For smaller c , quantum effects become significant, and the temperature dependence is reversed. The $\kappa_{\mathcal{I}}$, on the other hand, behaves differently: it starts from a finite value at $c \rightarrow 0$, increases monotonically with c , and asymptotically approaches another value at $c \rightarrow \infty$. Irrespective of the value of c , its numerical value increases with increasing temperature.

In summary, we have obtained the expressions (32) and (41) for the transport coefficients ζ and κ in terms of the known thermodynamic matrices such as $\tilde{\mathfrak{D}}$, D and A . We have illustrated their behavior in Fig. 1. In the next section, we discuss the crossover from GHD to NS dynamics.

IV. GHD-NS CROSSOVER

In this section, we investigate how different quantities—both conserved and non-conserved evolve in a nearly integrable systems within the RTA framework. Specifically, we consider the following scenario: a small amount of charge, such as particles or energy, is deposited at a specific location and we observe how it spreads across the system over time. Since transport phenomena are typically described by small deviations from local equilibrium, it is sufficient to consider linear perturbations around a homogeneous thermal background. This approximation allows for a systematic study of how local disturbances propagate and relax over time.

We begin by discussing the space-time evolution of the conserved charges as governed by the NS equations. Subsequently, we address both the conserved and non-conserved charges using the GHD-Boltzmann equation.

To set up the stage, we start with recalling the basic facts about Navier-Stokes equations in the linear regime, which will serve us as a reference point in the study of GHD-Boltzmann dynamics.

A. Navier-Stokes hydrodynamics in linear regime

In what follows it will be convenient to change the degrees of freedom in NS equations (28) using the thermodynamic identity [62]

$$\delta T = \frac{1}{c_V} \delta e - \frac{1}{c_V \varrho^2} \left[P - T \left(\frac{\partial P}{\partial T} \right)_\varrho \right] \delta \varrho. \quad (44)$$

Our motivation here is to solve NS equations in the linear regime by considering perturbations around homoge-

neous thermal state. The relevant equations are [50]

$$\begin{aligned}\partial_t \delta \varrho &= -\varrho \partial_x \delta u, \\ \partial_t \delta u &= -\frac{1}{\varrho} \left(\frac{\partial P}{\partial \varrho} \right)_T \partial_x \delta \varrho - \frac{1}{\varrho} \left(\frac{\partial P}{\partial T} \right)_\varrho \partial_x \delta T + \frac{\zeta}{\varrho} \partial_x^2 \delta u, \\ \partial_t \delta T &= -\frac{T}{\varrho c_V} \left(\frac{\partial P}{\partial T} \right)_\varrho \partial_x \delta u + \frac{\kappa}{\varrho c_V} \partial_x^2 \delta T.\end{aligned}\quad (45)$$

They can be solved with Fourier-Laplace transform

$$\delta \varrho(x, t) = \int dk e^{ikx - \Lambda^{\text{NS}}(k)t} \delta \tilde{\varrho}(k, 0), \quad (46)$$

and similarly for $\delta u, \delta T$, which turns them into k -dependent 3×3 eigenproblem. To make the associated matrix symmetric, it is convenient to change the degrees of freedom to $\delta q_0, \delta q_1, \delta q_2$ defined as follows [52]

$$\delta \varrho = \varrho \sqrt{T \kappa_T} \delta q_0, \quad \delta u = \sqrt{T} \varrho \delta q_1, \quad \delta T = \frac{T}{\sqrt{\varrho c_V}} \delta q_2. \quad (47)$$

The eigenvalue problem then reads

$$\mathcal{M}_{nm}^{\text{NS}}(k) \delta \tilde{q}_m(k) = \Lambda^{\text{NS}}(k) \delta \tilde{q}_n(k), \quad n = 0, 1, 2, \quad (48)$$

with summation over $m = 0, 1, 2$ is implied and with $\mathcal{M}^{\text{NS}}(k)$ given by

$$\mathcal{M}^{\text{NS}}(k) = \begin{pmatrix} 0 & i \frac{1}{\sqrt{\varrho \kappa_T}} k & 0 \\ i \frac{1}{\sqrt{\varrho \kappa_T}} k & \frac{\zeta}{\varrho} k^2 & i \sqrt{\frac{T}{\varrho^2 c_V}} \left(\frac{\partial P}{\partial T} \right)_\varrho k \\ 0 & i \sqrt{\frac{T}{\varrho^2 c_V}} \left(\frac{\partial P}{\partial T} \right)_\varrho k & \frac{\kappa}{\varrho c_V} k^2 \end{pmatrix}. \quad (49)$$

Let us denote by $R^{\text{NS}}(k)$ the matrix which diagonalizes the problem: $(R^{\text{NS}})^{-1}(k) \mathcal{M}^{\text{NS}}(k) R^{\text{NS}}(k) = \Lambda^{\text{NS}}(k)$. We find two sound modes and a heat mode, with dispersion relations for small k given by

$$\begin{aligned}\Lambda_{\pm}^{\text{NS}}(k) &= \pm i v_s k + \frac{1}{2\varrho} \left[\left(\frac{1}{c_V} - \frac{1}{c_P} \right) \kappa + \zeta \right] k^2 + \mathcal{O}(k^3), \\ \Lambda_{\text{th}}^{\text{NS}}(k) &= \frac{\kappa}{\varrho c_P} k^2 + \mathcal{O}(k^3),\end{aligned}\quad (50)$$

where v_s denotes the speed of sound and can be derived from the thermodynamics of the system [50, 52]. The corresponding eigenstates (sound modes and heat mode) are encoded in the matrix $R^{\text{NS}}(k)$ in $k \rightarrow 0$ limit:

$$R^{\text{NS}}(k) = \begin{pmatrix} \frac{1}{\sqrt{2}} \sqrt{\frac{c_V}{c_P}} & -1 & \sqrt{\frac{c_P - c_V}{c_P}} \\ -\sqrt{\frac{c_P - c_V}{c_P}} & 0 & \sqrt{\frac{c_V}{c_P}} \\ \frac{1}{\sqrt{2}} \sqrt{\frac{c_V}{c_P}} & 1 & \sqrt{\frac{c_P - c_V}{c_P}} \end{pmatrix}. \quad (51)$$

Lastly, let us write the explicit dynamics of the fields $\delta \tilde{q}_n$. It is given by the following expression

$$\delta \tilde{q}_n(k, t) = \sum_{\sigma, p} (R^{\text{NS}})^{-1}_{n\sigma}(k) e^{-\Lambda_{\sigma}^{\text{NS}}(k)t} R_{\sigma p}^{\text{NS}}(k) \delta \tilde{q}_p(k, 0), \quad (52)$$

where the index σ runs over the eigenmodes $\sigma = +, -, \text{th}$, while the indices n and p run over $0, 1, 2$, labelling the charge perturbations.

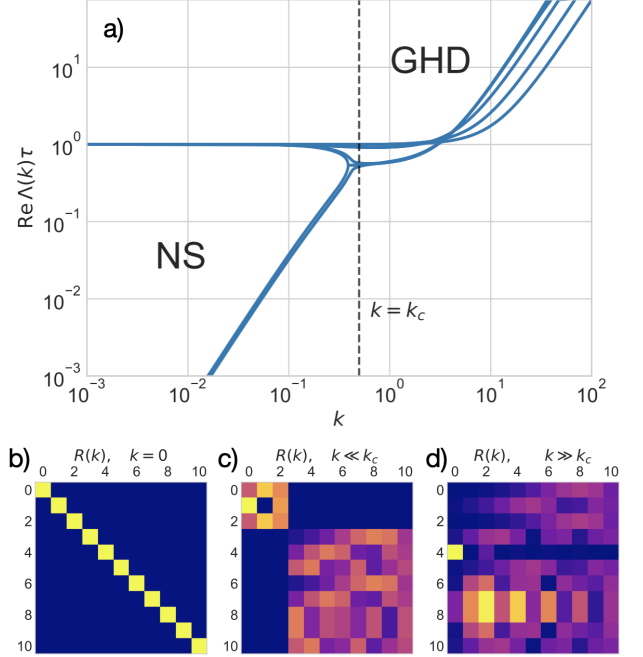


FIG. 2. Panel (a): real part of the spectrum of matrix $\mathcal{M}(k)$ as a function of k . We consider interaction coupling $c = 2$, temperature $T = 4.36$ and density $\varrho = 1$ with RTA timescale $\tau = 1$. For small momenta $k \ll k_c$ we observe gapless modes (in fact there are three such modes, the sound modes have degenerate $\text{Re}\Lambda(k)$) and gapped ones with a gap τ^{-1} . At larger momenta $k \gg k_c$ we find the modes of GHD. Panels (b)-(d): absolute value of rotation matrix $R(k)$ in three different regimes. For $k = 0$ we have $\mathcal{M} = \Gamma$ and the operator is diagonal. For k finite, but much smaller than k_c we observe a NS sector for the first three charges. The coefficients are those of standard sound and heat modes. For $k \gg k_c$ the modes of the systems are GHD modes, which are not simply related to the basis $\{h_n\}$. This is reflected in featureless R matrix.

B. GHD-Boltzmann equation in linear regime

We now turn to the linearized GHD-Boltzmann equation (20). It can be simplified using Fourier-Laplace transform, similarly to (46). This transforms Eq. (20) into k -dependent eigenvalue problem

$$\Lambda(k) \delta \tilde{q}_n(k) = \sum_m \mathcal{M}_{nm}(k) \delta \tilde{q}_m(k), \quad (53)$$

where the matrix elements of $\mathcal{M}(k)$ are given by

$$\mathcal{M}_{nm}(k) = (\Gamma_{nm} + ik A_{nm} + k^2 \mathcal{D}_{nm}). \quad (54)$$

Let $R(k)$ denote the matrix which diagonalizes the matrix $\mathcal{M}(k)$ such that $R^{-1}(k) \mathcal{M}(k) R(k) = \Lambda(k)$ with $\Lambda(k) = \text{diag}(\Lambda_1(k), \Lambda_2(k), \dots)$. The dynamics of charges, in complete analogy to (52), is then given by

$$\delta \tilde{q}_n(k, t) = \sum_{m, p} R_{nm}^{-1}(k) e^{-\Lambda_m(k)t} R_{mp}(k) \delta \tilde{q}_p(k, 0), \quad (55)$$

where $m, p = 0, 1, 2, \dots$ run over all charges (both conserved or non-conserved). The solution (55) represents the exact dynamics in linearized regime of GHD-Boltzmann equation. It involves the data from eigenvalue problem (53) in the form of eigenvalues $\Lambda(k)$ together with the rotation matrix $R(k)$ and the initial state $\delta\tilde{q}_p(k, 0)$. In what follows we will show how the properties of these objects can be used to understand and characterize the GHD-NS crossover, controlled by the (inverse) spatial scale k and time t . Even though the picture found by us is simpler in RTA, the intuitions presented here should hold for more general form of collision integral $\mathcal{I}[\rho_p]$. Indeed, all the qualitative features mentioned above were also found in the analysis for FGR collision integral describing coupled Lieb-Liniger models [52].

Firstly, it is shown in Appendix A that within the RTA, the orthonormalized charge basis $\{h_n(\lambda)\}$ diagonalizes the matrix Γ . The matrix element Γ_{nm} appearing in Eq. (24) simplifies to

$$\Gamma_{nm} = \delta_{nm} \begin{cases} 0, & n = 0, 1, 2, \\ \tau^{-1}, & n \neq 0, 1, 2. \end{cases} \quad (56)$$

Let us now analyze the eigenvalues of (54). For generic k it is not possible to find the spectrum analytically. However, for small k the operator $ikA_{nm} + k^2\mathcal{D}_{nm}$ can be treated as a perturbation to Γ_{nm} given by (56). In that regime of particular interest are corrections to the three zero modes, which for finite, but small k become the modes of NS hydrodynamics. Indeed, computing corrections up to the second order in k one finds eigenvalues exactly given by (50), meaning that $\Lambda_0(k) \approx \Lambda_{\text{th}}^{\text{NS}}(k)$, and $\Lambda_{1,2}(k) \approx \Lambda_{-,+}^{\text{NS}}(k)$ [52]. In the found expressions, the transport coefficients take exactly the form given by (32).

In the opposite limit of relatively large k , it is Γ_{nm} that becomes a negligible correction to $ikA_{nm} + k^2\mathcal{D}_{nm}$ and thus we recover the GHD spectrum. This simple analysis is the basic intuition behind explanation of the GHD-NS crossover.

An exemplary plot of real part of the eigenvalues is presented in Fig. 2(a). We clearly see these two distinct regimes. For very small momenta, there are gapless modes (which we identify as the NS normal modes) and the rest is clearly gapped. At larger k , we again notice modes with quadratic dispersions, this time given by GHD diffusion matrix.

Accordingly, we propose here the following criterion for the GHD-NS crossover. The crossover occurs when the NS dispersion relations (50) hit the line set by the RTA gap τ^{-1} . This leads to the following formula

$$k_c = \min \left[(\mathcal{D}_{\pm}\tau)^{-1/2}, (\mathcal{D}_{\text{th}}\tau)^{-1/2} \right], \quad (57)$$

where $\mathcal{D}_{\text{th}} = \frac{\kappa}{\varrho c_P}$ and $\mathcal{D}_{\pm} = \frac{1}{2\varrho} \left[\left(\frac{1}{c_V} - \frac{1}{c_P} \right) \kappa + \zeta \right]$. In Fig. 2(a), we see that this way of defining k_c qualitatively reproduces the crossover point found in numerical diagonalization of $\mathcal{M}(k)$. The crossover happens also

in the structure of normal modes, which is contained in the matrix $R(k)$. This is shown in Figs. 2(b)-(d). In particular note the structure in the $k \ll k_c$ regime in Fig. 2(c), where three conserved modes clearly decouple from higher charges.

C. Dynamics of small perturbations

Let us now introduce the initial state into the discussion and analyze the main features of the dynamics given by (55). The vector $R_{mp}(k)\delta\tilde{q}_p(k, 0)$, which enters the expression is a decomposition of the initial state in the eigenbasis of $\mathcal{M}(k)$. As we shall see, the emergence of the Navier-Stokes equations is driven by an interplay of two mechanisms. The first one is a *homogenisation*, namely the decay in time of large momentum modes with $k > k_c$. This occurs in the dynamics of all the charges. The second mechanism, that we call *thermalisation*, is a decay of charges broken by the collision integral and is important for the dynamics of the small momentum modes. To simplify the picture, we will first look at the dynamics of initial states which are localized in the momentum space such that $\{\delta\tilde{q}_p(k, 0)\}_{p \in \mathbb{N}} = 0$ for $|k|$ of order k_c and larger. After that we will look into dynamics of a generic initial state.

1. Initial states with $k_{\text{ini}} \ll k_c$

For such large scale states we are deep on the left of crossover point in Fig. 2(a) and the spectrum consists of three NS gapless modes, the rest is gapped. Crucially, the sum over p can be truncated due to the structure of matrix visible in Fig. 2(c), namely

$$R_{\sigma p}(k \ll k_c) \approx 0, \quad (58)$$

for $\sigma = +, -, \text{th}$ and $p > 2$ or $\sigma > 2$ and $p < 2$. We also observe that these matrix elements of R matrix coincide with those of R^{NS} . The decoupling in R matrix motivates studying the cases of conserved $n = 0, 1, 2$ and non-conserved $n > 2$ charge cases separately. For $n = 0, 1, 2$ we have

$$\delta\tilde{q}_n(k, t) \approx \sum_{\sigma, p} (R^{\text{NS}})_{n\sigma}^{-1}(k) e^{-\Lambda_{\sigma}^{\text{NS}}(k)t} R_{\sigma p}^{\text{NS}}(k) \delta\tilde{q}_p(k, 0). \quad (59)$$

Note that (59) is nothing else but the NS dynamics (52). Therefore, it is clear that we recover NS hydrodynamics in GHD-Boltzmann equation.

Next, we consider the $n > 2$ case. There we find just the exponentially decaying dynamics

$$\delta\tilde{q}_{n>2}(k, t) \approx e^{-t/\tau} \delta\tilde{q}_n(k, 0). \quad (60)$$

It is interesting to observe that in the $k_{\text{ini}} \ll k_c$ regime the conserved and non-conserved charges decouple and one does not have to be in the regime $t \gg \tau$ in order

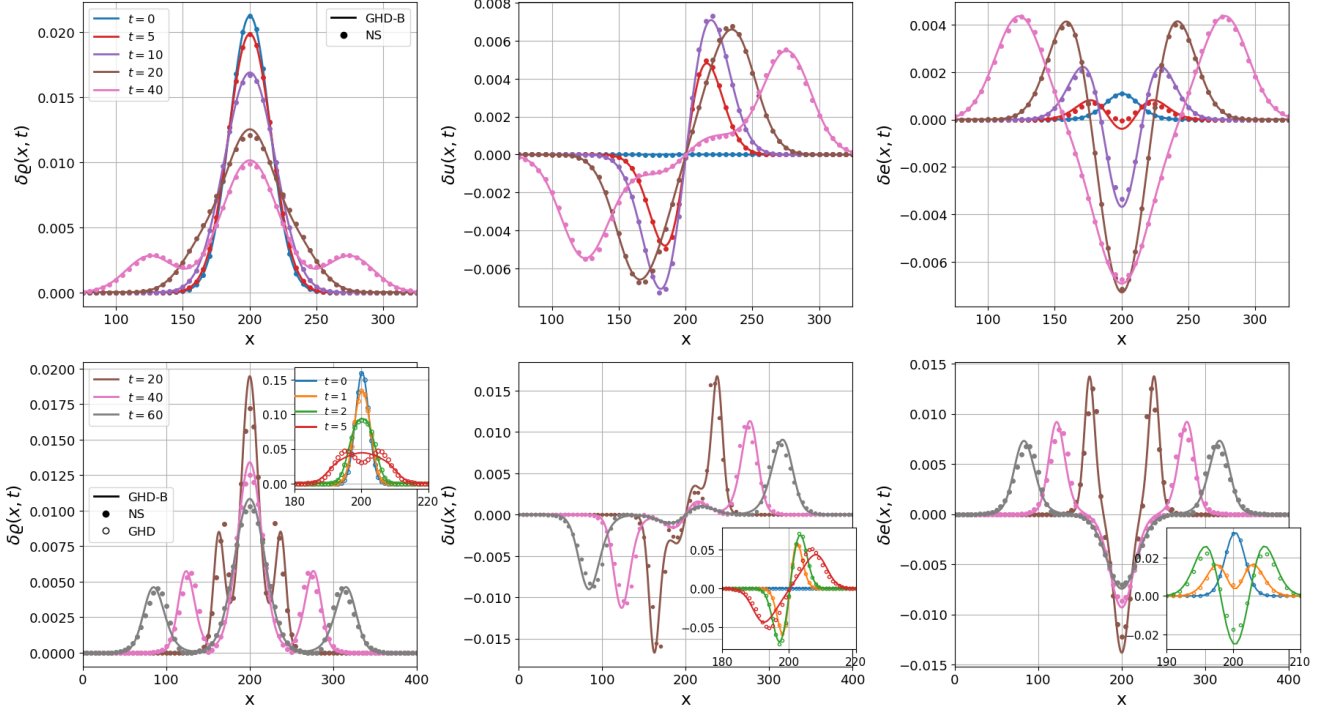


FIG. 3. Dynamics of the density, velocity and energy fields starting from an initial perturbation in the $\delta\rho$ and δe fields around a homogeneous thermal state ($\varrho^{\text{th}} = \varrho, u^{\text{th}} = 0, T^{\text{th}} = T$). The perturbation is evolved using the linearized NS equations, (45), with the transport coefficients ζ and κ computed from (31). These results are shown in the main plots as solid circles labeled “NS” at different times during the evolution. They are then compared with the corresponding fields obtained by solving the linearized GHD–Boltzmann equation, (20), shown as solid lines labeled “GHD-B.”. The agreement between the two at late times confirms that the large-scale space–time behavior of the conserved fields is accurately captured by the NS dynamics with correctly computed transport coefficients. The parameters used here are: $c = 2.0$ and $\tau = 2.0$, $\varrho = 1.0$, $T = 1.36$. In this setup, the crossover momentum k_c obtained from (57) is approximately 0.57, while the characteristic momentum of initial perturbation is $k_{\text{ini}} \approx 0.25$ (upper panels) and $k_{\text{ini}} \approx 1.9$ (lower panels). In the top row, where $k_{\text{ini}} < k_c$, the original dynamics (GHD–Boltzmann) match NS results even at short times $t \sim \tau$. In contrast, in the bottom row, where $k_{\text{ini}} > k_c$, it takes longer time ($t \sim 10\tau$) for the dynamics to converge to the corresponding NS fields. In the insets of the lower panels, we compare our GHD–Boltzmann data with the pure GHD results [i.e., without the collision term in (1)], shown as void circles labeled “GHD”. At very short times, $t \sim \tau$, the two results match, indicating that the effect of collision integral has not yet become significant. As time increases further, deviations between the two start to appear and at very large time NS dynamics start to appear which is shown in the main plots.

to see that the conserved fields with $n = 0, 1, 2$ obey NS hydrodynamics. This is demonstrated in the upper row of Fig. 3. Note however, that only at such times the perturbations of the higher charges has decayed and the state is fully characterized by the NS degrees of freedom.

2. Generic initial states

We now move to the discussion of more general states, which are not necessarily localized in the region $|k| \ll k_c$ and involve higher Fourier components. The components with $k \gg k_c$ decay on timescales given by GHD dispersion relations. Even though such modes are typically thought of as long-lived, we have to be aware that this is the statement for asymptotically small k . The timescales related with decay of $k \gg k_c$ GHD modes are still faster

than τ , see Fig 2(a). Thus at $t \gg \tau$ all Fourier components with $k > k_c$ decay. This puts us back into the setting analyzed in previous section and we still get the NS dynamics (59) for conserved fields with $n = 0, 1, 2$. The modes with $n > 2$ obviously decay to 0.

The dynamics (59) is thus the universal fate of any initial state in the GHD–Boltzmann framework. The NS hydrodynamization, that is the emergence of NS hydrodynamics happens due to the decay of higher charges as well as higher Fourier components. Eventually, at infinite time only $k = 0$ component remains and system becomes homogenous, as expected from hydrodynamics. This picture is confirmed numerically in Fig. 3, where we compare the corresponding fields obtained numerically by evolving the linearized GHD–Boltzmann equation (19) and the linearized NS equations (45). All coefficients entering the NS equations are evaluated from the thermodynam-

ics and collisional matrices as given by (31). We find excellent agreement between them for times $t \gg \tau$. On the other hand, for small times, the dynamics is well captured by GHD, see insets in the bottom row of Fig. 3.

V. HYDRODYNAMIC CORRELATION FUNCTIONS

In this section, we supplement the discussion of the dynamics by considering the problem of hydrodynamic correlation functions [63]. The central object of interest is the connected correlator of local charges

$$S_{o_1, o_2}(x, t) = \langle o_1(x, t) o_2(0, 0) \rangle_c, \quad (61)$$

where $o_{1,2}$ are densities of some local charges $O_{1,2}$. The initial value of correlation is given by $S_{o_1, o_2}(x, 0) = \delta(x) \langle f_{o_1} | \mathcal{C} | f_{o_2} \rangle$, see (62) for definition of $f_{o_{1,2}}$. The correlator of local fields can be accessed through the standard reasoning of linear response and in our setup we will build on the solution (55) in order to get it. For convenience, let us consider the following expectation value

$$o(x) = \int d\lambda f_o(\lambda) \rho_p(x, \lambda) \quad (62)$$

with $f_o(\lambda)$ giving the associated single particle eigenvalue. We assume that this quantity can be represented in the orthonormalized basis meaning that

$$f_o(\lambda) = \sum_n (h_n | f_o) h_n(\lambda) \equiv \sum_n \alpha_n^o h_n(\lambda), \quad (63)$$

then the correlator (61) can be expressed as

$$S_{o_1, o_2}(x, t) = \sum_{i,j} \alpha_i^{o_1} \alpha_j^{o_2} S_{q_i, q_j}(x, t). \quad (64)$$

In what follows, we will assume that the hydrodynamic state is described by space-dependent local GGE with density matrix given by

$$\rho_{\text{GGE}}(t) \propto e^{-\int dy \sum_n \beta_n(y, t) q_n(y, t)}. \quad (65)$$

For such states, the hydrodynamic correlation function can be computed as [8, 64]

$$S_{q_i, q_j}(x, t) = -\frac{\delta q_i(x, t)}{\delta \beta_j(0, 0)}. \quad (66)$$

To evaluate this, we can use the expression for charge dynamics given by (55). Using the definition of susceptibility matrix and the fact that we work in the basis, which is orthonormal with respect to inner product induced by it, we get the following

$$S_{q_i, q_j}(x, t) = \int dk e^{ikx} \sum_m R_{im}^{-1}(k) e^{-\Lambda_m(k)t} R_{mj}(k). \quad (67)$$

Correlator $S_{q_i, q_j}(x, t)$ is the simplest to analyze and at the same time in principle gives access to an arbitrary correlator of local fields as given in (61). Let us analyze different regimes of such object in what follows.

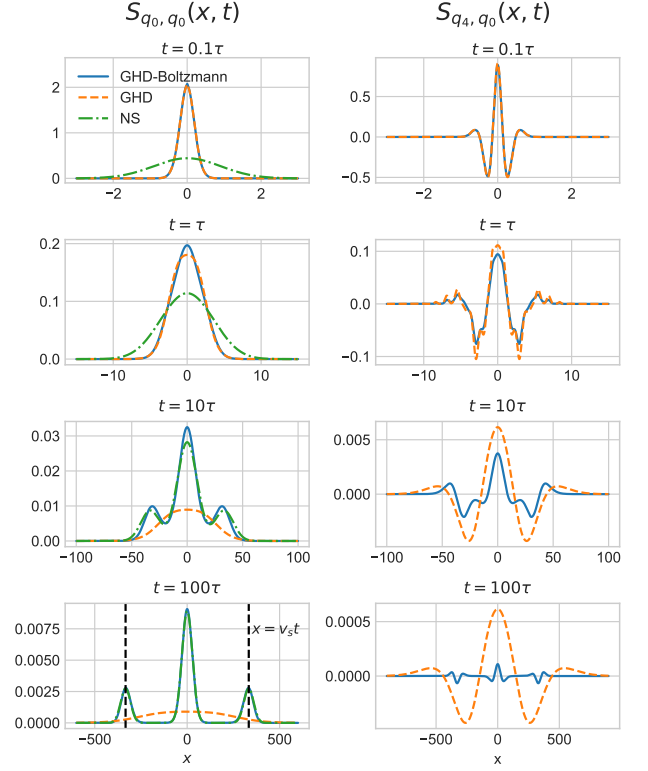


FIG. 4. Dynamics of correlation functions of conserved (left column) and non-conserved (right column) charges in GHD-Boltzmann equation. At late times, the density-density correlator exhibits NS behavior (69) with two sound modes propagating with sound velocity v_s and heat mode in the center. On the other hand, correlator of non-conserved charge $S_{q_4, q_0}(x, t)$ decays in time. At early times the dynamics in both cases is well described by the GHD without the collision term.

A. Regime $t \ll \tau$: generalized hydrodynamics

For small times $t \ll \tau$, we expect to recover the known GHD results [64]. It can be recovered by setting $\Gamma_{nm} = 0$ in the $\mathcal{M}_{nm}(k)$ matrix. However, we will use a more convenient formula for GHD correlator at the Euler scale, which reads

$$S_{q_i, q_j}^{\text{GHD, E}}(x, t) = \int d\lambda \delta[x - v(\lambda)t] \rho_p(\lambda) f(\lambda) h_i^{\text{dr}}(\lambda) h_j^{\text{dr}}(\lambda). \quad (68)$$

In principle, one can compute also the GHD correlation function on diffusive scale. However, it provides only a subleading corrections while introducing technical problems related to computing the exponential of diffusion operator $\mathfrak{D}$. In Fig. 4 we see that indeed in the small time regime the dynamics given by GHD-Boltzmann equation (67) is correctly described by correlations as predicted by GHD. At such short timescales, effects of integrability breaking are still not visible, therefore there are no qualitative differences between correlators involving conserved (non-conserved) charges. The difference is

visible at late times, to which we now turn.

B. Regime $t \gg \tau$: conventional fluctuating hydrodynamics

At late times the integral in (67) is dominated by contributions from small k modes. Thus the R matrix structure makes all the correlators involving non-conserved charge decay exponentially with the rate set by τ . This is demonstrated in the second column of Fig. 4. On the other hand, in the case where both charges belong to conserved subspace, the correlator features long-lived behavior. In fact, it approaches the well-known correlator of conventional fluid predicted by fluctuating hydrodynamics. There the hydrodynamic fields are random Gaussian variables with mean fixed by the deterministic dynamics and the variance controlled by the fluctuation-dissipation relation. Since dissipative processes are controlled by the transport coefficients, they set also the strength of the fluctuations. Thus, the transport coefficient are required to actually set up the fluctuating hydrodynamics. For the full exposition see [65, 66] or [53]. Here we write for reference the (normalized) density-density correlator

$$S_{q_0, q_0}^{\text{NS}}(x, t) = \frac{c_P - c_V}{c_P} \frac{1}{\sqrt{4\pi\mathcal{D}_{\text{th}}t}} e^{-\frac{x^2}{4\mathcal{D}_{\text{th}}t}} + \frac{c_V}{2c_P} \frac{1}{\sqrt{4\pi\mathcal{D}_{\pm}t}} \left(e^{-\frac{(x-v_{\text{st}})^2}{4\mathcal{D}_{\pm}t}} + e^{-\frac{-(x+v_{\text{st}})^2}{4\mathcal{D}_{\pm}t}} \right). \quad (69)$$

Obviously a similar expressions can be obtained for S_{q_i, q_j}^{NS} for $i, j = 0, 1, 2$. In Fig. 4 we show that indeed $S_{q_0, q_0}(x, t)$ becomes $S_{q_0, q_0}^{\text{NS}}(x, t)$ at times $t \gg \tau$, in the agreement with the general picture that NS hydrodynamics should be recovered at late times.

As discussed in [53], the diffusive form of correlators, such as (69), breaks down at the largest scales. This is the prediction of nonlinear fluctuating hydrodynamics, which incorporates nonlinear terms in the currents of NS equations and emphasizes their impact on correlation functions, which becomes dominant on large scales. In the present context, the nonlinearities become relevant at times of order $O(\tau^2/\tau_{\text{mic}})$, where τ_{mic} is characteristic microscopic timescale of integrable system. Since GHD Boltzmann equation implicitly assumes $\tau \gg \tau_{\text{mic}}$ in order for quasiparticle description to hold, there is a parametrically large window, on which linear fluctuating hydrodynamics is expected to provide the correct description.

VI. CONCLUSIONS

In this work, we have addressed two aspects of hydrodynamics under weak integrability breaking. Firstly, we have presented a computation of transport coefficients: viscosity and thermal conductivity in relaxation time approximation for collision integral. Such approximation is motivated by the typically complicated structure

of collision integrals stemming from FGR or gBBGKY. The complexity of such collision terms is due to the fact that three particle collisions are the leading thermalization processes in 1D Galilean systems, as opposed to two-particle scatterings in higher dimensions. Moreover, when the collision integral is not known, RTA can still be useful with timescale τ treated as a phenomenological parameter. The transport coefficients obtained within the RTA can be expressed entirely in terms of the TBA thermodynamics and the Drude weights of the underlying integrable model.

Secondly, we took a close look at the crossover between GHD of infinite conserved charges and NS theory with particle number, momentum and energy being the only conserved quantities. The control parameters of this crossover are space and time scales, which we identified in this work as k_c defined in (57) and characteristic RTA timescale τ . In the case of RTA, τ^{-1} equals the gap of $\mathcal{I}_{\text{RTA}}$. The explicit solution of dynamics in the linear regime allowed us to understand the behavior of charge densities. Depending on whether the considered charge is conserved by the perturbation or not, we observed a different dynamics. In the late time regime $t \gg \tau$ the conserved charge densities obey NS dynamics and non-conserved quantities decay exponentially, irrespectively of the initial state. We also characterized the influence of initial state on the dynamics and described the process of homogenisation understood here as a decay of the modes, which are not governed by NS hydrodynamics.

Lastly, we looked at hydrodynamic two-point functions of charge densities. For small times, GHD behavior was found, whereas in the late time regime for conserved charges we found results consistent with linear fluctuating hydrodynamics build upon the NS equations. On the other hand, correlators involving non-conserved charges decay exponentially on timescale set by τ . We also commented on the role of current nonlinearities for the correlation functions of conserved quantities. The effect of such terms is expelled to large space-time scales due to the proximity to integrable model assumed in this work.

Generalizing our RTA computations to other sets of conserved charges is a natural direction for future research. This was done already for the case of Galilean-invariant system with only particle number conservation in [67]. Thanks to its simplicity, the RTA collision integral can be also useful to study dynamics in non-linear Boltzmann equation, possibly with noise. Such studies could be useful to quantify the role of nonlinearities on various observables at the largest space-time scales.

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ence Centre, Poland, under the PRELUDIUM grant 2025/57/N/ST2/04184.

Appendix A: Computation of the matrix elements Γ_{nm}

In this section, within the relaxation time approximation, we present the computation of the eigenvalues and the corresponding eigenstates of the operator Γ appearing in (25) with elements given in (24). We first compute the functional derivative of the collision integral [given by (2)] with respect to the rapidity distribution,

$$\frac{\delta \mathcal{I}_{\text{RTA}}[\rho_p](\lambda)}{\delta \rho_p(\mu)} = -\frac{1}{\tau} \left(\delta(\lambda - \mu) - \frac{\delta \rho_p^{\text{th}}(\lambda)}{\delta \rho_p(\mu)} \right). \quad (\text{A1})$$

To compute the functional derivative of the thermal distribution ρ_p^{th} , we decompose it through variations of the charges:

$$\frac{\delta \rho_p^{\text{th}}(\lambda)}{\delta \rho_p(\mu)} = \sum_{m,n=0}^2 \underbrace{\frac{\delta \rho_p^{\text{th}}(\lambda)}{\delta \beta_n}}_{-\mathcal{C} \mathcal{R}_n(\lambda)} \underbrace{\frac{\delta \beta_n}{\delta \mathcal{Q}_m}}_{-\mathcal{C}_{nm}^{-1}} \underbrace{\frac{\delta \mathcal{Q}_m}{\delta \rho_p(\mu)}}_{\mathcal{R}_m(\mu)}, \quad (\text{A2})$$

where the derivatives are known (as indicated by the underbraces), see [53]. Using this result, (A2) simplifies

$$\begin{aligned} \frac{\delta \mathcal{I}_{\text{RTA}}[\rho_p](\lambda)}{\delta \rho_p(\mu)} &= \frac{1}{\tau} \left[\sum_{m,n=0}^2 (\mathcal{C} \mathcal{R}_n)(\lambda) (\mathcal{C}^{-1})_{nm} \mathcal{R}_m(\mu) - \delta(\lambda - \mu) \right]. \end{aligned}$$

The orthonormal charges are defined with respect to the hydrodynamic scalar product involving $\mathcal{C}$. This implies

$$\sum_{m,n=0}^2 \mathcal{R}_n(\lambda) (\mathcal{C}^{-1})_{nm} \mathcal{R}_m(\mu) = \sum_{m=0}^2 h_m(\lambda) h_m(\mu). \quad (\text{A3})$$

Thus, in the bra-ket notation the Γ operator in (25) can be written as

$$\Gamma = \frac{1}{\tau} \left(1 - \sum_{m=0,1,2} \mathcal{C} |h_m\rangle \langle h_m| \right) \mathcal{C}. \quad (\text{A4})$$

It is clearly diagonal in the orthonormal basis with eigenvalue 0 for $n = 0, 1, 2$ and eigenvalue τ^{-1} for $n > 2$.

Appendix B: Simplification of transport coefficients

In this section, we show how we get (41) from (39) and (40) of the main text. Here, we present the intermediate steps only for $\zeta_{\mathcal{I}}$. The corresponding steps for the $\kappa_{\mathcal{I}}$ proceed in the same manner. Note that the sum

in (39) runs over all basis states except 0, 1, 2. It can therefore be expressed as

$$\begin{aligned} \zeta_{\mathcal{I}} &= \tau \langle \mathcal{R}_1 | \mathcal{B} \left(\sum_n |h_n\rangle \langle h_n| - \sum_{n=0,1,2} |h_n\rangle \langle h_n| \right) | \eta_{\zeta} \rangle, \\ &= \tau \langle \mathcal{R}_1 | \mathcal{B} \mathcal{C}^{-1} | \eta_{\zeta} \rangle - \tau \sum_{n=0,1,2} \langle \mathcal{R}_1 | \mathcal{B} | h_n \rangle \langle h_n | \eta_{\zeta} \rangle. \end{aligned} \quad (\text{B1})$$

In going from the first line to the second line of (B1), we have used the resolution of the identity

$$1 = \sum_{\beta} |h_{\beta}\rangle \langle h_{\beta}| \mathcal{C}. \quad (\text{B2})$$

Using the relation between the ultralocal charge basis $\mathcal{R}_1$ and the corresponding orthonormal charge basis h_1 , namely $\mathcal{R}_1(\lambda) = \sqrt{\mathcal{C}_{11}} h_1(\lambda) = \sqrt{T\varrho} h_1(\lambda)$ [52], we can rewrite (B1) as

$$\zeta_{\mathcal{I}} = \tau \sqrt{T\varrho} \left(\langle h_1 | \mathcal{B} \mathcal{C}^{-1} | \eta_{\zeta} \rangle - \sum_{n=0,1,2} \langle h_1 | \mathcal{B} | h_n \rangle \langle h_n | \eta_{\zeta} \rangle \right). \quad (\text{B3})$$

From the expression of η_{ζ} in (36) and using the fact that $\langle h_m | \mathcal{B} | h_n \rangle$ is nonzero only when m and n are of different parities, we simplify (B3) to

$$\begin{aligned} \zeta_{\mathcal{I}} &= \tau \varrho \left(\langle h_1 | \mathcal{B} \mathcal{C}^{-1} \mathcal{B} | h_1 \rangle - \sum_{n=0,2} \langle h_1 | \mathcal{B} | h_n \rangle \langle h_n | \mathcal{B} | h_1 \rangle \right), \\ &= \tau \varrho \left(\langle h_1 | D | h_1 \rangle - \langle h_1 | \mathcal{B} | h_0 \rangle^2 - \langle h_1 | \mathcal{B} | h_2 \rangle^2 \right), \\ &= \varrho \tau \omega_1^{\text{coll}}. \end{aligned} \quad (\text{B4})$$

Note, the Drude weight is defined as $D \equiv \mathcal{B} \mathcal{C}^{-1} \mathcal{B}$ and the ω_1^{coll} is given by (42).

Appendix C: Limiting values of ω_1^{coll}

It is important to emphasize that the Lieb-Liniger model in the $c \rightarrow 0$ limit maps to free bosons while in the $c \rightarrow \infty$ limit it maps to free fermions. In this section, we explicitly show that ω_1^{coll} as given in (42) vanishes both in the $c \rightarrow 0$ and $c \rightarrow \infty$ limit. Note that (42) can also be written as

$$\omega_1^{\text{coll}} = \tau [D_{11} - v_s^2], \quad (\text{C1})$$

since $v_s^2 = A_{10}^2 + A_{21}^2$ where v_s denotes the speed of sound and can be expressed in terms of the known matrices:

$$v_s = \sqrt{\frac{1}{\mathcal{B}_{10}} \frac{\mathcal{B}_{10}^2 \mathcal{C}_{22} + \mathcal{B}_{21}^2 \mathcal{C}_{00} - 2\mathcal{B}_{10} \mathcal{B}_{20} \mathcal{C}_{21}}{\mathcal{C}_{22} \mathcal{C}_{00} - \mathcal{C}_{20}^2}}}. \quad (\text{C2})$$

The matrix elements $\mathcal{B}_{mn}$ and $\mathcal{C}_{mn}$ can be found from (15) and (18), respectively. The matrix element D_{11} in (C1) denotes the momentum-momentum Drude weight given in (26). Now, we show that D_{11} is exactly equal to v_s^2 both in the $c \rightarrow 0$ and $c \rightarrow \infty$ limits. We start with the $c \rightarrow \infty$ limit.

1. $c \rightarrow \infty$ limit

The scattering kernel, $\mathcal{T}$ in (5) vanishes in the $c \rightarrow \infty$ limit. Consequently, any dressed function $g^{\text{dr}}(\lambda)$ in (9), can be replaced by their original counterparts $g(\lambda)$. As a result all the matrix elements $\mathcal{B}_{mn}$ and D_{mn} can be expressed in terms of the charge-susceptibility matrix $\mathcal{C}$:

$$\mathcal{B}_{10} = 2\mathcal{C}_{20}, \mathcal{B}_{21} = 2\mathcal{C}_{22} \text{ and } D_{11} = 2\frac{\mathcal{C}_{22}}{\mathcal{C}_{20}}. \quad (\text{C3})$$

We compute v_s^2 from (C2) using the relations in (C3)

$$v_s^2 = \frac{1}{2\mathcal{C}_{20}} \frac{4\mathcal{C}_{20}^2\mathcal{C}_{22} + 4\mathcal{C}_{22}^2\mathcal{C}_{00} - 8\mathcal{C}_{20}\mathcal{C}_{22}}{\mathcal{C}_{22}\mathcal{C}_{00} - \mathcal{C}_{20}^2} = 2\frac{\mathcal{C}_{22}}{\mathcal{C}_{20}} = D_{11}. \quad (\text{C4})$$

Therefore, the collisional component of the viscosity vanishes exactly in the limit $c \rightarrow \infty$.

2. $c \rightarrow 0$ limit

In the $c \rightarrow 0$ limit, the scattering kernel, $\mathcal{T}(\lambda - \lambda')$ in (5) approximates the delta function $\delta(\lambda - \lambda')$. As a result, (9) is simplified to

$$g^{\text{dr}}(\lambda) = g(\lambda)[1 - n(\lambda)]^{-1}. \quad (\text{C5})$$

The integral equation for the pseudoenergy $\epsilon(\lambda)$ in (12) changes to an algebraic equation:

$$\epsilon(\lambda) = \frac{\lambda^2}{2} - \mu - \log(1 + e^{-\epsilon(\lambda)}), \quad (\text{C6})$$

from which ϵ can be solved as

$$\epsilon(\lambda) = \log\left(e^{\frac{\lambda^2}{2} - \mu} - 1\right). \quad (\text{C7})$$

The charge susceptibility matrix elements [in (15)] can then be written as

$$\mathcal{C}_{mn} = \int d\lambda \rho_p(\lambda) \hbar_m(\lambda) \hbar_n(\lambda) / (1 - n(\lambda)). \quad (\text{C8})$$

The statistical factor $[1 - n(\lambda)]$ can be related to the bosonic occupation function $n_b(\lambda)$ as

$$1 - n(\lambda) = 1 - \frac{1}{e^{\epsilon(\lambda)} + 1} = \frac{1}{1 + n_b(\lambda)} \quad (\text{C9})$$

with $n_b(\lambda) = 1/(e^{\epsilon(\lambda)} - 1)$. Thus the matrix elements $\mathcal{C}_{mn}$ in (C8) changes to

$$\mathcal{C}_{mn} = \int d\lambda \rho_p(\lambda) [1 + n_b(\lambda)] \hbar_m(\lambda) \hbar_n(\lambda). \quad (\text{C10})$$

As a result, the statistical factor $f(\lambda)$ in (15), (18) and (26) is replaced by $[1 + n_b(\lambda)]$ and all the calculations in the $c \rightarrow 0$ limit resemble that of the $c \rightarrow \infty$ limit. We recover the relation: $v_s^2 = D_{11}$ and the collisional contribution ω_1^{coll} vanishes in the limit $c \rightarrow 0$ as well.

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- [1] K. Sengupta, S. Powell, and S. Sachdev, *Phys. Rev. A* **69**, 053616 (2004).
 - [2] J. Berges, S. Borsányi, and C. Wetterich, *Phys. Rev. Lett.* **93**, 142002 (2004).
 - [3] M. Rigol, V. Dunjko, and M. Olshanii, *Nature* **452**, 854 (2008).
 - [4] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, *Rev. Mod. Phys.* **91**, 021001 (2019).
 - [5] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, *Phys. Rev. Lett.* **117**, 207201 (2016).
 - [6] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, *Phys. Rev. X* **6**, 041065 (2016).
 - [7] J. De Nardis, D. Bernard, and B. Doyon, *Phys. Rev. Lett.* **121**, 160603 (2018).
 - [8] B. Doyon, *SciPost Phys. Lect. Notes*, 18 (2020).
 - [9] A. Bastianello, B. Bertini, B. Doyon, and R. Vasseur, *Journal of Statistical Mechanics: Theory and Experiment* **2022**, 014001 (2022).
 - [10] B. Doyon, S. Gopalakrishnan, F. Möller, J. Schmiedmayer, and R. Vasseur, *Phys. Rev. X* **15**, 010501 (2025).
 - [11] T. Kinoshita, T. Wenger, and D. S. Weiss, *Nature* **440**, 900 (2006).
 - [12] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, *Phys. Rev. X* **8**, 021030 (2018).
 - [13] B. Doyon and H. Spohn, *SciPost Phys.* **3**, 039 (2017).
 - [14] E. Ilievski and J. De Nardis, *Phys. Rev. B* **96**, 081118 (2017).
 - [15] E. Ilievski and J. De Nardis, *Phys. Rev. Lett.* **119**, 020602 (2017).
 - [16] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, *Phys. Rev. B* **97**, 045407 (2018).
 - [17] M. Ljubotina, M. Žnidarič, and T. Prosen, *Nature Communications* **8**, 16117 (2017).
 - [18] R. J. Sánchez, V. K. Varma, and V. Oganesyan, *Phys. Rev. B* **98**, 054415 (2018).
 - [19] E. Ilievski, J. De Nardis, M. Medenjak, and T. c. v. Prosen, *Phys. Rev. Lett.* **121**, 230602 (2018).
 - [20] S. Gopalakrishnan and R. Vasseur, *Phys. Rev. Lett.* **122**, 127202 (2019).
 - [21] M. Ljubotina, M. Žnidarič, and T. c. v. Prosen, *Phys. Rev. Lett.* **122**, 210602 (2019).
 - [22] J. De Nardis, M. Medenjak, C. Karrasch, and E. Ilievski, *Phys. Rev. Lett.* **123**, 186601 (2019).
 - [23] M. Schemmer, I. Bouchoule, B. Doyon, and J. Dubail, *Phys. Rev. Lett.* **122**, 090601 (2019).
 - [24] J. M. Wilson, N. Malvania, Y. Le, Y. Zhang, M. Rigol, and D. S. Weiss, *Science* **367**, 1461 (2020).
 - [25] C. Li, T. Zhou, I. Mazets, H.-P. Stimming, F. S. Möller, Z. Zhu, Y. Zhai, W. Xiong, X. Zhou, X. Chen, and

- J. Schmiedmayer, *SciPost Phys.* **9**, 058 (2020).
- [26] F. Møller, C. Li, I. Mazets, H.-P. Stimming, T. Zhou, Z. Zhu, X. Chen, and J. Schmiedmayer, *Phys. Rev. Lett.* **126**, 090602 (2021).
- [27] F. Cataldini, F. Møller, M. Tajik, J. a. Sabino, S.-C. Ji, I. Mazets, T. Schweigler, B. Rauer, and J. Schmiedmayer, *Phys. Rev. X* **12**, 041032 (2022).
- [28] K.-Y. Li, Y. Zhang, K. Yang, K.-Y. Lin, S. Gopalakrishnan, M. Rigol, and B. L. Lev, *Phys. Rev. A* **107**, L061302 (2023).
- [29] B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, *Phys. Rev. Lett.* **119**, 195301 (2017).
- [30] B. Doyon and H. Spohn, *Journal of Statistical Mechanics: Theory and Experiment* **2017**, 073210 (2017).
- [31] L. Biagetti, G. Cecile, and J. De Nardis, *Phys. Rev. Res.* **6**, 023083 (2024).
- [32] N. Malvania, Y. Zhang, Y. Le, J. Dubail, M. Rigol, and D. S. Weiss, *Science* **373**, 1129 (2021).
- [33] M. Horvath, A. Bastianello, S. Dhar, R. Koch, Y. Guo, J.-S. Caux, M. Landini, and H.-C. Nägerl, (2025), [arXiv:2505.10550 \[cond-mat.quant-gas\]](#).
- [34] Y. Zeng, A. Bastianello, S. Dhar, Z. Wang, X. Yu, M. Horvath, G. E. Astrakharchik, Y. Guo, H.-C. Nägerl, and M. Landini, (2026), [arXiv:2602.17657 \[cond-mat.quant-gas\]](#).
- [35] L. Dubois, G. Thémèze, J. Dubail, and I. Bouchoule, *SciPost Phys.* **20**, 008 (2026).
- [36] L. Dubois, G. Thémèze, F. Nogrette, J. Dubail, and I. Bouchoule, *Phys. Rev. Lett.* **133**, 113402 (2024).
- [37] E. Charnay, A. Escoubet, F. Copie, S. Randoux, T. Bonnemain, A. Bastianello, and P. Suret, (2026), [arXiv:2601.21085 \[nlin.SI\]](#).
- [38] L. Biagetti, M. Lebek, M. Panfil, and J. D. Nardis, Generalised bbgky hierarchy for near-integrable dynamics (2025), [arXiv:2408.00593](#).
- [39] J.-S. Caux, B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, *SciPost Phys.* **6**, 070 (2019).
- [40] A. J. Friedman, S. Gopalakrishnan, and R. Vasseur, *Phys. Rev. B* **101**, 180302 (2020).
- [41] I. Bouchoule, B. Doyon, and J. Dubail, *SciPost Phys.* **9**, 044 (2020).
- [42] A. Bastianello, J. De Nardis, and A. De Luca, *Phys. Rev. B* **102**, 161110 (2020).
- [43] J. Durnin, M. J. Bhaseen, and B. Doyon, *Phys. Rev. Lett.* **127**, 130601 (2021).
- [44] J. Lopez-Piqueres, B. Ware, S. Gopalakrishnan, and R. Vasseur, *Phys. Rev. B* **103**, L060302 (2021).
- [45] J. Lopez-Piqueres, S. Gopalakrishnan, and R. Vasseur, *Journal of Physics A: Mathematical and Theoretical* **55**, 234005 (2022).
- [46] J. Lopez-Piqueres and R. Vasseur, *Phys. Rev. Lett.* **130**, 247101 (2023).
- [47] M. Panfil, S. Gopalakrishnan, and R. M. Konik, *Phys. Rev. Lett.* **130**, 030401 (2023).
- [48] M. Lebek, M. Panfil, and R. M. Konik, *SciPost Phys.* **17**, 007 (2024).
- [49] R. C. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics* (John Wiley & Sons, 1975).
- [50] P. M. V. Résibois and M. De Leener, *Classical Kinetic Theory of Fluids* (John Wiley & Sons, New York, 1977).
- [51] J. R. Dorfman, H. van Beijeren, and T. R. Kirkpatrick, *Contemporary Kinetic Theory of Matter* (Cambridge University Press, 2021).
- [52] M. Lebek and M. Panfil, *Phys. Rev. Lett.* **134**, 010405 (2025).
- [53] M. Lebek and M. Panfil, *SciPost Phys.* **18**, 186 (2025).
- [54] E. H. Lieb and W. Liniger, *Phys. Rev.* **130**, 1605 (1963).
- [55] C. N. Yang and C. P. Yang, *Journal of Mathematical Physics* **10**, 1115 (1969).
- [56] S. Gopalakrishnan, D. A. Huse, V. Khemani, and R. Vasseur, *Phys. Rev. B* **98**, 220303 (2018).
- [57] J. D. Nardis, D. Bernard, and B. Doyon, *SciPost Phys.* **6**, 049 (2019).
- [58] H. Spohn, *Journal of Mathematical Physics* **59**, 091402 (2018).
- [59] M. Panfil and J. Pawelczyk, *SciPost Physics Core* **1**, 10.21468/scipostphyscore.1.1.002 (2019).
- [60] F. Hübner, L. Biagetti, J. De Nardis, and B. Doyon, *Phys. Rev. Lett.* **134**, 187101 (2025).
- [61] B. Doyon, G. Perfetto, T. Sasamoto, and T. Yoshimura, *Phys. Rev. Lett.* **131**, 027101 (2023).
- [62] P. Résibois and M. De Leener, *Classical Kinetic Theory of Fluids* (John Wiley & Sons, 1977).
- [63] J. De Nardis, B. Doyon, M. Medenjak, and M. Panfil, *Journal of Statistical Mechanics: Theory and Experiment* **2022**, 014002 (2022).
- [64] B. Doyon, *SciPost Phys.* **5**, 054 (2018).
- [65] H. Spohn, *Journal of Statistical Physics* **154**, 1191 (2014).
- [66] C. B. Mendl and H. Spohn, *Phys. Rev. E* **90**, 012147 (2014).
- [67] P. Lisiak, M. Lebek, and M. Panfil, Burgers equation from non-thermal stationary states in nearly-integrable gases (2026), [arXiv:2512.09726](#).

2.5 Article V: Generalised BBGKY hierarchy for near-integrable dynamics

- L. Biagetti, M. **Łebek**, M. Panfil, J. De Nardis “Generalised Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy for near-integrable dynamics”, (accepted in [Physical Review X](#))

The exact dynamics of an interacting system of particles can be written as an infinite hierarchy of dynamical equations. These equations dictate the evolution of correlation functions in phase space. The dynamics of the single-particle function $\rho(x, \theta)$ is coupled to the two-particle function $g^{(2)}(x_1, \theta_1; x_2, \theta_2)$, which in turn is coupled to the three-particle function, etc. Such a hierarchy of equations is known as Born-Bogoliubov-Green-Kirkwood-Yvon (BBGKY) hierarchy. It is a fundamental tool used in kinetic theory, for instance, for the derivation of collision operators such as the Landau or Boltzmann kinetic equations. In case of these collision operators, the hierarchy can be truncated due to the weakness of the interaction or in the low-density limit, respectively.

In this work, we took up the challenge of deriving a BBGKY-like framework (dubbed GBBGKY) for nearly-integrable systems. Our aim was to provide a hierarchy for the dynamics of the correlation functions that requires neither weakness of interaction (as the interactions of the integrable part of Hamiltonian can be arbitrarily strong) nor a low-density limit, to be truncated. Instead, the small parameter occurring in our hierarchy is the inverse of the interaction range ξ^{-1} .

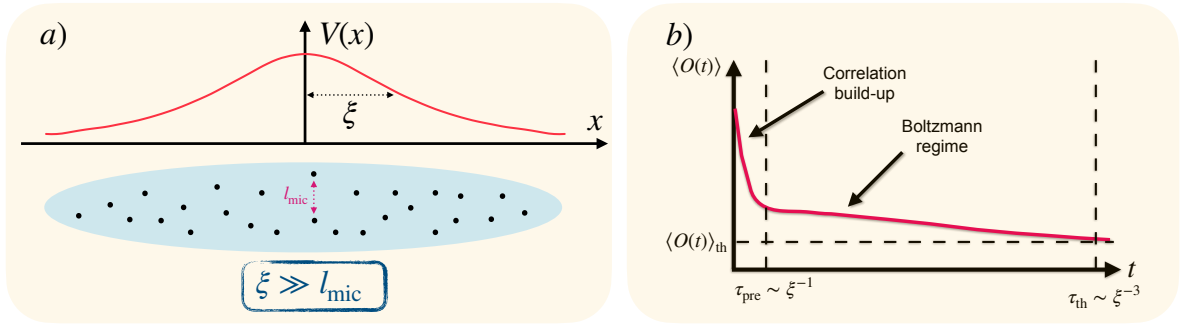


Figure 16: Panel a): the main assumption behind the GBBGKY - the separation of length scales between the microscopic correlation length l_{mic} and the range of the long-range interaction potential ξ . Panel b): two phases of a typical thermalization dynamics.

Unlike the original framework, GBBGKY is not a fully microscopic theory and includes assumptions typically present in hydrodynamic theories. It relies heavily on the separation of length scales: the range of the long-range potential ξ is assumed to be much larger than the microscopic correlation length l_{mic} of the unperturbed integrable Hamiltonian, see Fig. 16(a). Thus, the aforementioned truncation with respect to ξ^{-1} can be regarded as similar in spirit to the truncation of gradient expansions.

In our approach, the dynamics coming from the (in principle) strongly correlated, integrable Hamiltonian is treated non-perturbatively in the interaction coupling, albeit perturbatively in the length scale at which we are looking at the system. At large length scales, such dynamics can be treated with the apparatus of the TBA and GHD. The main achievement of the present work is to incorporate the effects of long-range interactions acting on such large scales and show, that these can be captured through a generalization of the BBGKY hierarchy.

Upon truncation, the derived hierarchy can be solved numerically. Furthermore, a simplification occurs at later times, when the hierarchy can be integrated into a single collision operator. The derivation of such an operator constitutes one of the main results of the paper. We show that it

naturally corresponds to a three-particle collision process in which particles interact both with the integrable part of interaction potential and the long-range coupling that breaks integrability. We prove that the collision integral exactly matches the FGR result, when specialized to the LL model.

Furthermore, the hierarchy is benchmarked against numerical simulations of a classical nearly-integrable model, namely, the long-range interacting hard rod gas. We find a perfect agreement and observe two characteristic regimes of the dynamics, see Fig. 16(b). In the phase I, the correlations evolve rapidly and the dynamics cannot be reduced to a collision operator and the equations of the (truncated) hierarchy have to be solved. In phase II, which we refer to as the kinetic regime, the evolution is well captured by the collision integral.

Our framework is used to compare against the experimental data of [70]. We specialize the results for the collision integral to nearly-integrable Lieb-Liniger models and compute from it the thermalization rate as a function of the magnetic field angle θ_{DDI} . We find a reasonable agreement between our theory and experiments.

Candidate's contribution: I participated in the derivation of the hierarchy and in the computations using the correlated fluid cell ensemble. I led the derivation of the collision operator from the hierarchy. I performed the analysis of the experimental setup, computed the relaxation timescales and prepared Fig. 8. I contributed to writing the manuscript.

Generalised BBGKY hierarchy for near-integrable dynamics

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We study quantum and classical many-body Hamiltonian systems that combine integrable contact interactions with generic long-range two-body potentials. We show that the dynamics of local observables can be cast into a generalized Bogoliubov–Born–Green–Kirkwood–Yvon (gBBGKY) hierarchy formulated in terms of the quasiparticle densities of the underlying integrable model and their correlations. Starting from an ansatz for the state at time t , which we call the *correlated fluid-cell ensemble*, we derive this hierarchy and prove that it reproduces exactly the time evolution of one- and multi-point correlation functions in perturbed integrable models, at all times. We validate these predictions against microscopic molecular-dynamics simulations, finding perfect agreement.

At late times, the one-particle distribution relaxes via a Boltzmann-type scattering integral encoding the interplay between integrable contact processes and long-range collisions, whereas higher-point correlations remain strongly non-thermal on thermalization time scales, indicative of a form of incomplete or *generalised thermalisation*. Focusing on long-range dipolar quantum gases, where the relevant matrix elements can be obtained explicitly, we show that our collision integral reduces exactly to the Fermi golden rule result and provide a complete theoretical account of the experimental observations of Tang *et al.* (Phys. Rev. X **8**, 021030 (2018)). More broadly, our framework extends the BBGKY program to regimes with strong local interactions, and applies to a wide class of experimentally relevant systems, from one-dimensional dipolar cold-atom gases to Lennard–Jones molecular fluids.

I. INTRODUCTION

The study of non-equilibrium many-body physics continues to reveal ever more intricate phenomena arising from complex interactions. Classical theories of chaos, turbulence, and active dynamics remain vibrant research areas, and in recent years, quantum many-body systems have also entered the non-equilibrium arena [1–4]. In low spatial dimensions, integrable models—distinguished by their stable quasiparticles—form a key class of dynamical systems. These quasiparticles govern transport properties, entanglement spreading [5–7], anomalous scrambling [8, 9], extended coherence times [10–12], and exotic hydrodynamic behavior [13–26]. Typical examples of interacting integrable systems include bosons, spins, or fermions with two-body contact interactions, such as the Lieb–Liniger gas [27, 28] and the Fermi–Hubbard chain [29]. This naturally raises the question: how do quasiparticles and the overall dynamics evolve when the Hamiltonian includes additional, longer-range interactions? Integrability breaking has attracted considerable interest [30–55], prompted by experiments [56–59] that report anomalously slow thermalization in quasi-one-dimensional settings, a challenge shared across molecular dynamics, plasma physics, and astrophysics [60–76].

Significant progress has been achieved for systems whose unperturbed integrable limit is free—where matrix elements of local operators are typically known, allowing a Boltzmann scattering approach via Fermi’s golden rule. However, for genuinely interacting integrable models, matrix elements depend nontrivially on both the state and the model, and existing methods accurately capture only late-time behavior, leaving short-time dynamics—often

of primary experimental interest—largely unexplored.

Despite advances in specific contexts [47, 48, 77], a general framework is still lacking to compute the dynamics of any integrable model perturbed by weak two-body or higher-order potentials. Developing such a framework would be both theoretically valuable and directly relevant to cold-atom experiments [58], where the precise form of interparticle potentials can be uncertain.

In kinetic-theory texts, the Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy is a classic tool [78–84], typically applied to free particles with weak or long-range interactions (as in plasma and astrophysical systems), truncating at two- or three-point correlations to describe thermalization and dissipation. Yet, to date, BBGKY framework has not been extended to unperturbed Hamiltonians that are interacting integrable models with well-defined quasiparticles.

Here, we generalize the BBGKY hierarchy to cases where the unperturbed (classical or quantum) theory features arbitrarily strong integrable contact interactions. Specifically, we consider a hard-core repulsion $V_a^{\text{int}}(r)$ of radius a supplemented by a long-range field $\varphi(r)$, leading to the two-body potential (see Fig. 1)

$$U(r) = V_a^{\text{int}}(r) + V(r), \quad V(r) = \frac{V_0}{\xi} \varphi(r/\xi), \quad (1)$$

where $V_0 > 0$ (< 0) describes repulsive (attractive) long-range interactions acting over the scale ξ . Notable examples include atomic dipolar systems [23, 58, 85–94], whose Hamiltonian in terms of the bosonic field $\psi(x)$ can be

written as

$$H = \int dx \left(\frac{1}{2m} |\partial_x \psi(x)|^2 + a |\psi(x)|^4 - \mu |\psi(x)|^2 \right) + \int dx dy V(x-x') |\psi(x)|^2 |\psi(x')|^2, \quad (2)$$

in which a local contact interaction is again supplemented by a long-range component. Our approach provides a fully predictive theory for generic near-integrable dynamics in both classical and quantum regimes, encompassing bosonic, fermionic, and spin-chain models. In particular, it unifies the description of prethermalization and full thermalization dynamics. As a byproduct, we re-derive the conventional BBGKY hierarchy using an alternative correlated fluid-cell ensemble picture, showing that the hydrodynamic-like assumption underlying BBGKY naturally emerges from this viewpoint.

II. BACKGROUND AND SUMMARY OF THE RESULTS

Integrable systems are characterized by stable quasi-particle excitations which, similarly to free systems, provide a complete description of local properties. In particular, a generic thermodynamic state is specified by the *quasiparticle momentum distribution*, or one-particle (quasi)momenta distribution, denoted by $\rho_\theta(x)$, where we use θ as a generic parametrization of the quasiparticle momenta $k(\theta)$. Under the unperturbed time evolution, the spatial integral of $\rho_\theta(x)$ is conserved.

Such systems arise in both classical and quantum physics [20, 95–104], typically in $1+1$ dimensions, where, in certain cases, scattering processes factorize. Generic perturbations, such as couplings between different one-dimensional channels or the addition of long-range interactions, break integrability. In this situation quasiparticles become unstable, and most distributions $\rho_\theta(x)$ are no longer stationary. Under weak integrability breaking, typically only thermal distributions remain stationary, as first recognized by Boltzmann.

The evolution from a generic out-of-equilibrium quasi-particle distribution toward a thermal state is the subject of *prethermalization* and kinetic theory. These frameworks have applications ranging from high-energy collisions and plasma physics to, more recently, quantum many-body dynamics in ultracold atomic systems [56, 58, 105–110]. A common approach models this evolution through a collision integral derived from Fermi's golden rule [30, 47, 48, 111], the perturbation induces transitions between quasiparticle states with rates determined by the corresponding matrix elements, or by classical cross sections. This approximation, however, neglects correlations that build up dynamically, and the computation of the relevant matrix elements is generally difficult. The BBGKY hierarchy provides a more comprehensive framework, as it allows one to track correlations up to arbitrary order. It also enables a systematic derivation of

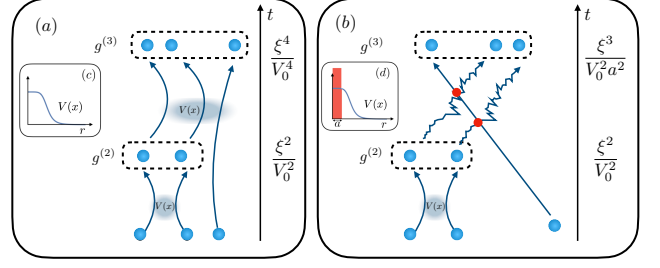


Figure 1. Schematic evolution of (a) BBGKY and (b) gB-BGKY for the two-point functions $g^{(2)}$, the two insets show, respectively, (c) the long-range potential for free particles and (d) the combination of long-range and short-range interactions in the interacting integrable case. In (a), two particles drawn from the momentum distribution $\rho(\theta)$ scatter via $V(r)$ and become correlated on the time scale ξ^2/V_0^2 , this is the *prethermalisation* stage where correlations build up but no extensive mixing occurs, they then scatter again and generate three-point correlations $g^{(3)}$, which control the *thermalisation* stage on time scales $(\xi/V_0)^4$. In (b), with integrable local interactions in homogeneous systems, the two-point correlation is generated by the large-scale interaction $V(r)$, while the three-point function is produced by a different and faster mechanism. Particles propagate and scatter through local interactions, shown as red circles, providing an effective diffusive bath for the two-point functions, and leading to the thermalisation of one and two-point functions, namely *generalised thermalisation*, on time scales $\xi^3/(V_0 a)^2$.

the collision integral without requiring explicit knowledge of all transition amplitudes, and it goes beyond the late-time approximation inherent in the Boltzmann approach [112, 113].

Concretely, we consider a model of particles in the continuum, the extension to lattice systems is immediate, with Hamiltonian, quantum or classical,

$$\hat{H} = \hat{H}_a^{\text{int}} + \frac{1}{2} \int dx dx' V(x-x') \hat{q}_0(x) \hat{q}_0(x'), \quad (3)$$

where $\hat{q}_0(x)$ is the local particle density operator, and $\hat{H}_a^{\text{int}}$ is an integrable Hamiltonian containing standard kinetic terms together with an integrable interaction $\hat{V}_a^{\text{int}}$. For example, in the Lieb–Liniger model, as in (2), the integrable interaction is $V^{\text{int}}(x-x') = c \delta(x-x')$, with $c = 0$ and $c \rightarrow \infty$ corresponding to free and hard-core bosons, effectively free fermions, respectively. We denote by a the strength of the integrable interaction. In the limit $a \rightarrow 0$, the integrable potential vanishes and the model reduces to free particle propagation,

$$\lim_{a \rightarrow 0} \hat{H}_a^{\text{int}} = H^{\text{free}} = \int dx \frac{(\hat{q}_1(x))^2}{2m}, \quad (4)$$

with $\hat{q}_1(x)$ the local momentum density. The integrable Hamiltonian $\hat{H}_a^{\text{int}}$ possesses an extensive family of conserved charges $q_i(x)$, with $i \in \mathbb{N}$, and $q_0(x) = \hat{q}_0(x)$. The

gBBGKY theory describes the time evolution of the expectation values of these densities and of their connected correlations.

To describe the system at finite macroscopic time t , we introduce the following ansatz. The state is given by the *correlated fluid cells ensemble*, which can be formulated either as a density matrix or as a phase-space density, for a precise definition see Eq. (14). This state can be viewed as an embedding of a generalized Gibbs ensemble, $\rho \propto \exp(-\sum_i \int dx \beta^i(x) q_i(x))$, determined by a set of generalized temperatures $\beta^i(x)$ associated with the local conserved quantities $\hat{q}_i(x)$. The correlated fluid cells ensemble additionally involves higher-rank Lagrange parameters $\beta_{(n)}^{i_1, \dots, i_n}(x_1, \dots, x_n)$, which encode information about long-range correlations in the system. When these higher-order terms are nonzero, the state is not simply a product of independent local stationary GGE cells. It is instead a collection of correlated fluid cells.

The reason to expect this form is as follows. The hydrodynamic postulates assert that, for macroscopic times $t \gg \tau_{\text{mic}}$, much larger than the microscopic time τ_{mic} , a generic interacting system relaxes locally giving rise to quasiparticle description. Fast degrees of freedom associated with non-conserved modes decay, and the evolution becomes confined to a reduced manifold of states. Since the slow modes are precisely the conserved densities and their products, we include them here in full generality.

A similar logic can be applied to standard BBGKY hierarchy. The only difference lies in the fact that $\tau_{\text{mic}} \rightarrow 0$ as there is no transient phase needed for local relaxation to GGE and emergence of quasiparticles. Thus, in the usual setting, where one starts from states without correlations, there is little practical need to make the structure of (14) explicit. Our formulation therefore *generalizes BBGKY to situations in which the stationary states of the unperturbed Hamiltonian already displays nontrivial local correlations* (in contrast to H^{free} , where local correlations are absent and the hierarchy simplifies substantially). We will thus refer to the conventional hierarchy as the *standard BBGKY*, and to the present extension as the *generalized BBGKY* (gBBGKY).

In principle, the full hierarchy of multi-point correlations encodes complete information about the state, but, to obtain a tractable description, it must be truncated at some finite order. Truncating the hierarchy at the level of three-point functions yields the explicit evolution equations in Eqs. (32)–(34). Equivalently, these equations can be rewritten in terms of quasiparticle densities $\rho_\theta(x, t)$ and their correlations by the replacement $i \rightarrow \theta$ and by identifying the local densities, or operator or phase-space densities, as $\langle q_i(x, t) \rangle \rightarrow \rho_\theta(x, t)$, which also recovers standard textbook results in the limit $a \rightarrow 0$.

In this work we show that *the presence of local interactions strongly affects the hierarchy*. The main new feature is the emergence of delta-function jumps in the evolution equations for multi-point correlations, which appear as discontinuities in spatial derivatives at coinciding points. These jumps encode ballistic, *non-linear*, namely interact-

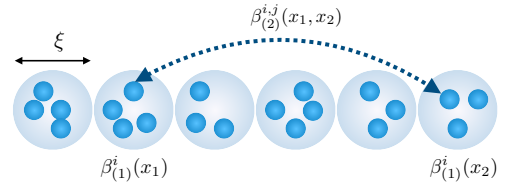


Figure 2. Schematic representation of the correlated fluid cells ensemble of Eq. (14). Fluid cells of size ξ can be correlated over long distances by the external potential V , or by the interplay of contact interactions and ballistic dynamics in the system.

ing, modes, and they persist even when integrability is weakly broken, since ballistic transport remains relevant at intermediate times, thereby driving the evolution of lower-order correlations. This has important physical consequences, see Fig. 1. For $a = 0$, correlations are generated solely by scattering mediated by the perturbing potential $V(x)$. In one dimension, two-body scattering alone cannot thermalize the system, so three-point correlations must be retained, leading to thermalization times scaling as V_0^{-4} , a phenomenon known as *kinetic blocking*. For $a \neq 0$, local integrable interactions allow quasiparticles to scatter via V and then interact again locally with a ballistically propagating particle. These processes generate three-point functions with coincident-point jumps, which in turn induce dissipative, diffusive contributions to the dynamics of the two-point correlations, and lead to effective thermalisation. As a result, thermalization of one and two-point functions, which we refer to as *incomplete or generalised thermalisation*, is reached at times of order $(a V_0)^{-2}$. Higher-point functions, such as three-point functions, remain far from thermal on these time scales, preserving remarkable *long-range correlations*, as a direct consequence of the interplay between local contact interactions and long-range interactions. Recent preliminary experiments [58] have observed this scaling, and our gBBGKY framework places it on firmer theoretical ground. Moreover, we show that truncation at third order reproduces the dynamics of one and two-point functions *very precisely and at all times*, as we demonstrate by comparison with exact numerical simulations.

Organisation of the paper

The paper is organized as follows. In Section III we derive the gBBGKY hierarchy, an infinite set of coupled equations for correlation functions of charges and currents. We then show how to truncate this hierarchy consistently, obtaining a closed set of equations for connected charge correlators. This requires neglecting correlations beyond a given order, and assuming a separation of scales between local integrable interactions and the long-range integrability-breaking potential. We also show that the generalized hierarchy reduces to the standard one when

the integrable interactions vanish. The resulting equations are later solved numerically and validated against numerical simulations.

To clarify the mechanisms behind, generalised, thermalization, in Section IV we derive a kinetic equation, dubbed the generalized Landau equation, from which the thermalization time scale quoted above follows. This derivation assumes a separation of time scales between the fast dynamics of correlations and the slow evolution of the one-body function.

In the next two sections we apply the gBBGKY and generalized Landau equations to two systems. First, in Section V the predictions of the preceding sections are tested against exact numerical simulations for classical hard rods, hard spheres, with long-range interactions, showing perfect agreement. Second, in Section VI we apply the framework to dipolar cold-atomic gases trapped in an array of quasi-one-dimensional tubes. This requires an extension of the methods to multicomponent systems. It also provides an opportunity to compare the generalized Landau equation, derived here under explicit assumptions, with a heuristic approach where the scattering integral is obtained from Fermi's golden rule. We find exact agreement between the two approaches in the regime where both apply.

In the final Section VII, we summarize our findings and outline future directions. More technical aspects are relegated to the appendices. In the accompanying Supplementary Materials we provide derivations of the generalised Landau equation and of the gBBGKY hierarchy for multicomponent systems.

III. THE gBBGKY HIERARCHY

We start by introducing the notation needed to set up the, generalised, BBGKY hierarchy. We first work with densities of conserved quantities, and later move to particles or quasiparticle occupation functions, as is customary in standard formulations of BBGKY.

We denote by $\hat{q}_i(x)$ the density of the conserved charge $\hat{Q}_i$, which commutes with the integrable Hamiltonian, $[\hat{H}_a^{\text{int}}, \hat{Q}_i] = 0$, noting that a suitable gauge must be chosen since these densities are defined up to a total derivative. The first standard conserved quantities are particle number $\hat{Q}_0 = \hat{N}$, momentum $\hat{Q}_1 = \hat{P}$, and unperturbed energy $\hat{Q}_2 = \hat{H}_a^{\text{int}}$, to which we refer as the *kinetic energy of quasiparticles*. Throughout the derivation we focus on the quantum formulation, yet all steps can be translated straightforwardly to classical models by replacing the com-

mutator with the Poisson bracket, $i[\cdot, \cdot] \rightarrow \{\cdot, \cdot\}$. We will often refer to the case of free classical particles, where the equations are simplest and essentially all steps are exact, and we also assume Galilean invariance, although violations can be incorporated without difficulty. We set $\hbar = 1$ and start from the Heisenberg equation for the charge densities,

$$\partial_t \hat{q}_i = i[\hat{H}, \hat{q}_i]. \quad (5)$$

We decompose the commutator and introduce currents $\hat{j}_i$ and generalized currents $\hat{j}_{i,k}$, defined by

$$i[\hat{H}_a^{\text{int}}, \hat{q}_i] + \partial_x \hat{j}_i = 0, \quad i[\hat{Q}_i, \hat{q}_k] + \partial_x \hat{j}_{i,k} = 0, \quad (6)$$

where the density does not generate any dynamics, $\hat{j}_{0,i} = 0$. To simplify the Heisenberg equation, we use locality of the charges,

$$i[\hat{q}_i(x), \hat{q}_0(x')] = \hat{j}_{i,0}(x) \delta'(x - x'). \quad (7)$$

Using this relation for commutators between local charges, the quantum generalisation of the classical Liouville equation reads [114, 115]

$$\begin{aligned} \partial_t \hat{q}_{i_1}(x_1) + \partial_{x_1} \hat{j}_{i_1}(x_1) = & -\frac{1}{2} \int dx_2 V'(x_1 - x_2) \times \\ & \times [\hat{q}_0(x_2) \hat{j}_{i_1,0}(x_1) + \hat{j}_{i_1,0}(x_1) \hat{q}_0(x_2)]. \end{aligned} \quad (8)$$

Similar equations to (8) can be written for multi-point operators involving arbitrary products of charge densities. Taking expectation values then generates the hierarchy of dynamical equations

$$\begin{aligned} \partial_t \langle q_{i_1}(x_1) \dots q_{i_n}(x_n) \rangle = & \\ - \sum_{k=1}^n \partial_{x_k} \langle q_{i_1}(x_1) \dots j_{i_k}(x_k) \dots q_{i_n}(x_n) \rangle - \sum_{k=1}^n \int dy \times & \\ V'(x_k - y) \langle q_{i_1}(x_1) \dots q_0(y) j_{i_k,0}(x_k) \dots q_{i_n}(x_n) \rangle. & \end{aligned} \quad (9)$$

To isolate the relevant parts of correlators we move to connected correlation functions. For $n = 2$ we set $\langle o_1 o_2 \rangle^c := \langle o_1 o_2 \rangle - \langle o_1 \rangle \langle o_2 \rangle$, whereas for $n = 3$ we define

$$\begin{aligned} \langle o_1 o_2 o_3 \rangle^c := & \langle o_1 o_2 o_3 \rangle - \langle o_1 o_2 \rangle^c \langle o_3 \rangle - \\ & \langle o_1 o_3 \rangle^c \langle o_2 \rangle - \langle o_2 o_3 \rangle^c \langle o_1 \rangle - \langle o_1 \rangle \langle o_2 \rangle \langle o_3 \rangle, \end{aligned} \quad (10)$$

and for $n > 3$ we proceed analogously. The first three layers of the hierarchy can then be written as

$$\partial_t \langle q_{i_1}(x_1) \rangle + \partial_{x_1} \langle j_{i_1}(x_1) \rangle = - \int dx_2 V'(x_1 - x_2) \langle j_{i_1,0}(x_1) \rangle \langle q_0(x_2) \rangle - \int dx_2 V'(x_1 - x_2) \langle j_{i_1,0}(x_1) q_0(x_2) \rangle^c, \quad (11)$$

$$\begin{aligned} & \partial_t \langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle^c + [\partial_{x_1} \langle j_{i_1}(x_1) q_{i_2}(x_2) \rangle^c]_{(1,2)} = \\ & = - \left[\int dx_3 V'(x_1 - x_3) \left(\langle j_{i_1,0}(x_1) \rangle \langle q_{i_2}(x_2) q_0(x_3) \rangle^c + \langle j_{i_1,0}(x_1) q_{i_2}(x_2) \rangle^c \langle q_0(x_3) \rangle + \langle j_{i_1,0}(x_1) q_{i_2}(x_2) q_0(x_3) \rangle^c \right) \right]_{(1,2)}, \end{aligned} \quad (12)$$

$$\begin{aligned} & \partial_t \langle q_{i_1}(x_1) q_{i_2}(x_2) q_{i_3}(x_3) \rangle^c + [\partial_{x_1} \langle j_{i_1}(x_1) q_{i_2}(x_2) q_{i_3}(x_3) \rangle^c]_{(1,2,3)} = \\ & - \left[\int dx_4 V'(x_1 - x_4) \left(\langle j_{i_1,0}(x_1) \rangle \langle q_{i_2}(x_2) q_{i_3}(x_3) q_0(x_4) \rangle^c + \langle j_{i_1,0}(x_1) q_{i_2}(x_2) q_{i_3}(x_3) \rangle^c \langle q_0(x_4) \rangle + \right. \right. \\ & \left. \left. + \langle j_{i_1,0}(x_1) q_{i_2}(x_2) \rangle^c \langle q_{i_3}(x_3) q_0(x_4) \rangle^c + \langle j_{i_1,0}(x_1) q_{i_3}(x_3) \rangle^c \langle q_{i_2}(x_2) q_0(x_4) \rangle^c + \langle j_{i_1,0}(x_1) q_{i_2}(x_2) q_{i_3}(x_3) q_0(x_4) \rangle^c \right) \right]_{(1,2,3)}, \end{aligned} \quad (13)$$

Here the symbol $[\dots]_{(i_1, \dots, i_n)}$ denotes the sum of the expression inside the square brackets over all *cyclic* permutations of the indices $(i_1, \dots, i_n)$.

When the integrable model reduces to a free theory, this hierarchy is equivalent to the standard BBGKY hierarchy, as discussed later in Section III E. A key difference is that the present formulation is expressed in terms of correlations of conserved charges and currents, whereas the standard BBGKY hierarchy is usually written directly for quasiparticle distributions. Nevertheless, the familiar structures are readily recognized. Truncating the hierarchy at the level of expectation values, namely setting the two-point functions to zero, yields the Vlasov equation [116–118]. Retaining two-body correlations instead leads, at large times, to either the Landau or the Boltzmann equation, depending on the truncation procedure. Here we keep three layers of the hierarchy because, due to kinetic blocking, the first two layers are not sufficient to go beyond the mean-field regime [68, 69]. Finally, as in the standard BBGKY case [78], the hierarchy can be represented diagrammatically. In Fig. 3 we show the diagrams corresponding to the first three layers written above.

The hierarchy of equations (11), (12) and (13) is not closed. First, as in the standard BBGKY hierarchy, the equation for an n -point function involves the $(n+1)$ -point function. Closing the system therefore requires assuming that connected correlation functions beyond a given order are negligible. Second, the hierarchy contains correlators involving currents, of the form $\langle \dots q_{i_j} j_k q_{i_{j'}} \dots \rangle^c$ and $\langle \dots q_{i_j} q_0 j_k q_{i_{j'}} \dots \rangle^c$. To obtain a consistent truncation scheme, these correlators must be expressed in terms of correlators involving charge densities only. In a hierarchy built on a non-interacting theory, current operators are linear functionals of the charge densities, so this step is straightforward. In the present setting, it requires identifying the relevant structure of correlations in integrable theories.

To achieve this, two ingredients are needed, the ensemble with respect to which expectation values are computed, and the associated fluid-cell averaging procedure. These steps are discussed in full generality below, and later specialized to homogeneous systems.

Before proceeding, we summarize the assumptions made throughout the manuscript:

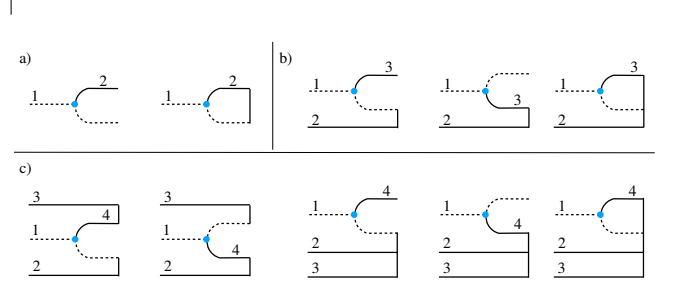


Figure 3. Diagrams representing the right-hand sides of the hierarchy equations (11), (12) and (13). Dashed lines represent the current operator $j_{i_1,0}(x_1)$, while solid lines represent charge operators $q_{i_k}(x_k)$, with k indicated by the number above the line. Vertical lines represent connected correlations between operators. Each diagram contains a single vertex associated with the potential $V(x - x')$, which couples a current to an additional operator q_0 . To translate a diagram into an expression, one takes products of connected correlation functions according to the vertical lines, multiplies by $V(x_1 - x_{m+1})$, where $m+1$ labels the extra charge operator, and integrates over x_{m+1} . To draw diagrams at the m -th level, one considers one dashed line from which an extra solid line emanates, together with $m-1$ additional solid lines, and then enumerates all possible connections, or partitions, between the $m+1$ lines. The case with no connections is excluded, as it corresponds to the disconnected part of the correlator.

- At large space-time distances the connected correlators scale as $\langle q_{i_1}(x_1) \dots q_{i_n}(x_n) \rangle^c \sim 1/\xi^{n-1}$, where ξ is the characteristic range of the long-range potential, and is much larger than the range a of the integrable interaction. This is discussed in Sec. III A.
- The state of the system is described by the correlated fluid cells ensemble (14). It encodes the expectation values of local charge densities and the correlations between them, as discussed in Sec. III A. Using this ensemble one also obtains equations for correlation functions involving currents, see Sec. III B.
- The resulting partial differential equations, obtained by imposing the previous points in Eqs. (11), (12) and (13), are expanded at large ξ , up to and including order ξ^{-3} , yielding the final hierarchy reported in Sec. III C.

A. The correlated fluid cells ensemble

To truncate the hierarchy and obtain a closed description of the system at time t , correlations must be retained only up to a chosen maximal order. This mirrors standard hydrodynamics, where a mesoscopic length ℓ separates microscopic fluctuations from macroscopic, observable fields. In the homogeneous setting considered here, the natural choice for this scale is the interaction length ξ of the potential. Although additional length scales may emerge in inhomogeneous systems, we assume that, in the present case, ξ is the only relevant spatial scale.

We postulate that, at macroscopic times, the system relaxes to the state described by the *correlated fluid cells density matrix*

$$\begin{aligned} \rho = \exp \Big(& - \int dx_1 \beta_{(1)}^i(x_1) q_i(x_1) \\ & - \int_{x_1 \neq x_2} d\vec{x} \beta_{(2)}^{i,j}(x_1, x_2) \delta q_i(x_1) \delta q_j(x_2) \\ & - \int_{x_1 \neq x_2 \neq x_3 \neq x_1} d\vec{x} \beta_{(3)}^{i,j,k}(x_1, x_2, x_3) \delta q_i(x_1) \delta q_j(x_2) \delta q_k(x_3) \\ & + \dots \Big). \end{aligned} \quad (14)$$

In the following, the ensemble average $\langle \dots \rangle$ is always taken with respect to density matrix (14). The first term corresponds to a family of *local* GGE states, one for each uncorrelated fluid cell at position x . Accordingly, we denote by $\langle \dots \rangle_1$ the average with respect to the zeroth order of the correlated fluid cells ensemble, namely the local GGE average given by eq. (14) with $\beta_{(n>1)} = 0$. In (14) we also introduced fluctuations around the local GGE, $\delta q_i(x) = q_i(x) - \langle q_i(x) \rangle_1$.

For the states defined by (14), the entropy is maximized independently in each cell, with additional constraints imposed by conservation laws. Setting all higher terms to zero amounts to neglecting nonlocal constraints, which implies that only short-range, thermal-like correlations are retained, decaying on distances much smaller than the scale ξ . As in the standard BBGKY hierarchy, the goal is instead to keep higher-order correlations in the description of the state. The state (14) therefore maximizes the entropy under constraints given not only by fixed one-point functions, but also by correlations of arbitrary order. As we will show in this section, the nonlocal generalized temperatures $\beta_{(n)}^{i_1, \dots, i_n}(x_1, \dots, x_n)$ are directly related to the connected n -point functions $\langle q_{i_1}(x_1) \cdots q_{i_n}(x_n) \rangle^c$, and, in particular, they display the same scaling with the length scale ξ .

For generic states, computing expectation values with respect to ensemble (14) is a difficult task, and, in close analogy with (g)BBGKY, it calls for a perturbative scheme. Here a natural truncation arises from the assumed separation of length scales. A key hypothesis for hydrodynamic states is that connected correlation func-

tions satisfy a large-deviation scaling

$$\langle q_{i_1}(x_1) \cdots q_{i_n}(x_n) \rangle^c \sim \frac{1}{\xi^{n-1}}, \quad (15)$$

which also implies $\beta_{(n)}^{i_1, \dots, i_n}(x_1, \dots, x_n) \sim \xi^{-n+1}$. Equivalently, as the order n increases, the magnitude of the connected correlator decreases by a factor $1/\xi$ for each additional field. This ensures that higher-order correlations become progressively smaller and can be neglected beyond a finite truncation order. This scaling is central for defining a controlled truncation scheme, both for the gBBGKY hierarchy and for the correlated fluid cells density matrix. Under the scaling (15), the density matrix (14) can be evaluated perturbatively in the parameter ξ^{-1} . As an example, we compute the two-point function using the state (14), expanded to first order in ξ^{-1} ,

$$\begin{aligned} \langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle^c = & \langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle_1^c + \\ & \int_{x_3 \neq x_4} dx_3 dx_4 \beta_{(2)}^{i_3, i_4}(x_3, x_4) \langle \delta q_{i_3}(x_3) \delta q_{i_4}(x_4) q_{i_1}(x_1) q_{i_2}(x_2) \rangle_1 \\ & + O(\xi^{-2}). \end{aligned} \quad (16)$$

Using the cumulant expansion for four-point functions and the locality of the GGE we obtain

$$\begin{aligned} \langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle^c = & \delta(x_1 - x_2) C_{i_1, i_2} + \\ & + 2\beta_{(2)}^{i_3, i_4}(x_1, x_2) C_{i_1, i_3} C_{i_2, i_4} + O(\xi^{-2}), \end{aligned} \quad (17)$$

where we used $\langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle_1^c = \delta(x_1 - x_2) C_{i_1, i_2}$, and where C_{i_1, i_2} denotes the local charge-charge susceptibility matrix. From (17) we define the long-range charge-charge correlation $g_{i_1, i_2}^{(2)}(x_1, x_2)$ through

$$g_{i_1, i_2}^{(2)}(x_1, x_2) = 2\beta_{(2)}^{i_3, i_4}(x_1, x_2) C_{i_1, i_3} C_{i_2, i_4} + O(\xi^{-2}), \quad (18)$$

so that

$$\langle q_{i_1}(x_1) q_{i_2}(x_2) \rangle^c = \delta(x_1 - x_2) C_{i_1, i_2}(x_1) + g_{i_1, i_2}^{(2)}(x_1, x_2). \quad (19)$$

Similarly, as shown in Appendix A, for the three-point function we find

$$\begin{aligned} \langle q_{i_1} q_{i_2} q_{i_3} \rangle^c = & \delta(x_1 - x_2) \delta(x_1 - x_3) C_{i_1, i_2, i_3}(x_1) + \\ & + [\delta(x_1 - x_2) C_{i_1, i_2, j} C^{j, k} g_{k, i_3}^{(2)}(x_1, x_3)]_{(1,2,3)} \\ & + g_{i_1, i_2, i_3}^{(3)}(x_1, x_2, x_3), \end{aligned} \quad (20)$$

with

$$\begin{aligned} g_{i_1, i_2, i_3}^{(3)}(x_1, x_2, x_3) = \\ = 3\beta_{(3)}^{j_1, j_2, j_3}(y, y') C_{i_1, j_1} C_{i_2, j_2} C_{i_3, j_3} + O(\xi^{-3}), \end{aligned} \quad (21)$$

and where C_{i_1, i_2, i_3} is the local GGE three-point function, and $C^{i, j}$ is the inverse covariance matrix, $C_{i, j} C^{j, k} = \delta_{i, k}$. We emphasize that the matrices C_{i_1, i_2} and C_{i_1, i_2, i_3} are known functions of the expectation values of the conserved

charge densities $\langle q_i \rangle_1$. It is also important to note that the functions $g^{(2)}(x_1, x_2)$, $g^{(3)}(x_1, x_2, x_3)$, and higher orders, as well as the generalized temperatures $\beta_{(n)}$, are smooth everywhere except at coinciding points, $x_i = x_j$, where discontinuities may appear but no divergences occur. For this reason, to avoid ill-defined expressions, in Eq. (14) the integrals are defined over configurations with non-coinciding points.

B. The expression for the currents in the ensemble

Importantly, once the state (14) is assumed, the averages of all local, or quasi-local, observables can be expressed through cumulant expansions in the charge densities.

In this section we use the state (14) to derive the cumulant expansion for the average current

$$\begin{aligned} \langle j_{i_1}(x_1) \rangle &= \langle j_{i_1}(x_1) \rangle_1 + \\ &+ \int_{x_2 \neq x_3} dx_2 dx_3 \beta_{(2)}^{i_2, i_3}(x_2, x_3) \langle q_{i_2}(x_2) q_{i_3}(x_3) j_{i_1}(x_1) \rangle_1^c \\ &+ O(\xi^{-2}). \end{aligned} \quad (22)$$

We now observe that, due to the locality of the GGE average, one has

$$\begin{aligned} \langle q_{i_2}(x_2) q_{i_3}(x_3) j_{i_1}(x_1) \rangle_1^c &= 0, \\ \text{for } (|x_2 - x_1| + |x_3 - x_1| > \varepsilon) \end{aligned} \quad (23)$$

where ε denotes a typical microscopic length scale of the model. Since the generalized temperature $\beta_{(2)}$ is in general discontinuous at coinciding points, due to fast microscopic-scale fluctuations encoded in the local GGE, this contribution must be treated with care. Condition (23) effectively localizes the integral on the r.h.s. of Eq. (22) to a neighborhood of size ε around the discontinuity. Within this region, the correlation function decomposes into its local parity-time ($\mathcal{PT}$) symmetric and antisymmetric components, $\beta_{(2)}^{i_1, i_2} = \beta_{(2), \text{sym}}^{i_1, i_2} + \beta_{(2), \text{anti}}^{i_1, i_2}$. Crucially, because the microscopic jump is inherently antisymmetric, the singularity is entirely confined to $\beta_{(2), \text{anti}}^{i_1, i_2}$. This implies that the symmetric component, defined locally as

$$\begin{aligned} \beta_{(2), \text{sym}}^{i_1, i_2}(x + \varepsilon, x - \varepsilon, t) &= \frac{1}{2} \left[\beta_{(2)}^{i_1, i_2}(x + \varepsilon, x - \varepsilon, t + \epsilon) + \right. \\ &\left. + \beta_{(2)}^{i_1, i_2}(x - \varepsilon, x + \varepsilon, t - \epsilon) \right], \end{aligned} \quad (24)$$

remains smooth even as $\epsilon \rightarrow 0$. Since charge and current density operators are $\mathcal{PT}$ -symmetric [119], they couple exclusively to this smooth symmetric part in Eq. (22). Consequently, the integral projects out the singular antisymmetric term, rendering the limit $\epsilon \rightarrow 0$ regular

$$\begin{aligned} \langle j_{i_1}(x_1) \rangle &= \langle j_{i_1}(x_1) \rangle_1 + \\ &+ \beta_{(2), \text{sym}}^{i_2, i_3}(x_1^+, x_1^-) \frac{\delta^2 \langle j_{i_1}(x_1) \rangle_1}{\delta \beta_{(1)}^{i_2} \delta \beta_{(1)}^{i_3}} + O(\xi^{-2}). \end{aligned} \quad (25)$$

This calculation follows analogously for the ensemble average of any $\mathcal{PT}$ -symmetric observable. Also, we used that in a local GGE correlation functions of local densities can be obtained by differentiation with respect to $\beta_{(1)}$. Applying (18) and the definition of the inverse susceptibility matrix, we obtain

$$\begin{aligned} \langle j_{i_1}(x_1) \rangle &= \langle j_{i_1}(x_1) \rangle_1 + \\ &+ \frac{1}{2} \frac{\delta^2 \langle j_{i_1}(x_1) \rangle_1}{\delta \langle q_{i_2} \rangle_1 \delta \langle q_{i_3} \rangle_1} g_{i_2, i_3}^{(2), \text{sym}}(x_1^+, x_1^-) + O(\xi^{-2}). \end{aligned} \quad (26)$$

In analogy, we define

$$g_{i_2, i_3}^{(2), \text{sym}}(x_1^+, x_1^-) = \frac{g_{i_2, i_3}^{(2)}(x_1^+, x_1^-) + g_{i_2, i_3}^{(2)}(x_1^-, x_1^+)}{2}, \quad (27)$$

where we dropped the splitting in time, since the correlation functions are always smooth for any t . Finally, introducing the notation

$$H_{i_1}^{i_2, i_3} \equiv \frac{\delta^2 \langle j_{i_1} \rangle_1}{\delta \langle q_{i_2} \rangle_1 \delta \langle q_{i_3} \rangle_1}, \quad (28)$$

we obtain the double-projection formula

$$\begin{aligned} \langle j_{i_1}(x_1) \rangle &= \langle j_{i_1}(x_1) \rangle_1 + \\ &+ \frac{1}{2} H_{i_1}^{i_2, i_3} g_{i_2, i_3}^{(2), \text{sym}}(x_1^+, x_1^-) + O(\xi^{-2}). \end{aligned} \quad (29)$$

Similarly, for the current-charge correlator we find

$$\begin{aligned} \langle j_i(x_1) q_j(x_2) \rangle^c &= A_i^k \langle q_k(x_1) q_j(x_2) \rangle^c + \\ &+ \frac{1}{2} H_i^{k, l} g_{k, l, j}^{(3), \text{sym}}(x_1^+, x_1^-, x_2) + O(\xi^{-3}), \end{aligned} \quad (30)$$

where we used the chain rule to introduce the flux Jacobian $A_i^k = \delta \langle j_i \rangle_1 / \delta \langle q_k \rangle_1$, and where $g_{k, l, j}^{(3), \text{sym}}(x_1^+, x_1^-, x_2)$ is defined in analogy with (27). This simplification is crucial, as it allows one to express the full hierarchy in terms of few-point charge correlators. We also note that the generalized currents $j_{i,0}$ appearing on the right-hand side of Eq. (11), (12) and (13) are linear in the charge density, therefore the cumulant expansion truncates after the first term,

$$\langle j_{i_1,0} q_{i_2} \dots \rangle^c = A_{i_1,0}^k \langle q_k q_{i_2} \dots \rangle^c, \quad (31)$$

with $A_{i_1,0}^k = \delta \langle j_{i_1,0} \rangle_1 / \delta \langle q_k \rangle_1$.

C. The gBBGKY hierarchy

In this section we make use of the assumptions and results introduced in Sec III A in order to write Eqs. (11), (12) and (13) as a hierarchy of equations for the smooth few point functions $g_{i_1, \dots, i_n}^{(n)}$. The formulas (19) and (20), a direct consequence of taking expectation values within the ensemble (14), reveal that the correlators, and thus the hierarchy, feature singular terms. Thus,

to understand the dynamics it is crucial to plug these expressions into the hierarchy alongside with the expression for the correlation functions containing the currents. To simplify the equations, we restrict to a *homogeneous*

setting, where the one-point functions are space independent, and two-point functions depend only on the relative distances. The equations (11), (12) and (13) reduce then to the following first layers of the gBBGKY hierarchy

$$\partial_t \langle q_{i_1} \rangle = - \int dx_2 V'(x_1 - x_2) A_{i_1,0}^k g_{k,0}^{(2)}(x_1, x_2), \quad (32)$$

$$\begin{aligned} \partial_t g_{i_1, i_2}^{(2)}(x_1, x_2) + \left[\partial_{x_1} A_{i_1}^j g_{j, i_2}^{(2)}(x_1, x_2) + \frac{1}{2} \partial_{x_1} H_{i_1}^{j, k} g_{j, k, i_2}^{(3), \text{sym}}(x_1^+, x_1^-, x_2) \right]_{(1,2)} = \\ - \left[V'(x_1 - x_2) A_{i_1,0}^j \left(\langle q_j \rangle C_{i_2,0} + g_{j,k}^{(2)}(x_1, x_2) C_{k,m}^{j,m} C_{m, i_2,0} \right) + \int dx_3 V'(x_1 - x_3) A_{i_1,0}^j \left(\langle q_j \rangle g_{i_2,0}^{(2)}(x_2, x_3) + g_{j, i_2,0}^{(3)} \right) \right]_{(1,2)} \\ - \delta(x_1 - x_2) \int dx_3 V'(x_1 - x_3) \left[- C_{i_1, i_2, j} C_{j, k}^{j, k} A_{k,0}^n + \left(A_{i_1,0}^j \delta_{i_2}^k + \delta_{i_1}^j A_{i_2,0}^k \right) C_{j, k, m} C_{m, n} \right] g_{n,0}^{(2)}(x_1, x_3), \end{aligned} \quad (33)$$

$$\begin{aligned} \partial_t g_{i_1, i_2, i_3}^{(3)}(x_1, x_2, x_3) + \left[\partial_{x_1} A_{i_1}^j g_{j, i_2, i_3}^{(3)}(x_1, x_2, x_3) + \frac{1}{2} \partial_{x_1} H_{i_1}^{j, k} g_{j, i_2}^{(2)}(x_1, x_2) g_{k, i_3}^{(2)}(x_1, x_3) \right]_{(1,2,3)} = \\ = \left[- \delta(x_1 - x_2) \left(A_{i_1}^k C_{k, i_2, m} C_{m, n}^{j, n} - C_{i_1, i_2, k} C_{k, m}^{j, m} A_m^n + H_{i_1}^{m, n} C_{m, i_2} \right) \partial_{x_1} g_{n, i_3}^{(2)}(x_1, x_3) \right]_{(1,2,3)} + O(V_0^2 \xi^2, V_0 \xi^{-3}), \end{aligned} \quad (34)$$

where, for brevity, in Eq. (34) we truncated higher-order terms in both parameters V_0 and $1/\xi$. Let us comment on the structure of this complex hierarchy of equations. Firstly, in homogeneous systems the evolution equation for the density, eq. (32), does not contain the Vlasov term and therefore all its dynamics is given by the evolution of the two-point functions, Eq. (33). Here, on the LHS are the kinetic term describing the ballistic transport of two-point function and its diffusive corrections. On the RHS we observe that the singular contribution to the two-point function creates a contact forcing proportional to V_0/ξ . Finally, the dynamics of the three-point function is written in (34) up to order $O(V_0/\xi)$. Again, its LHS include the kinetic terms for the ballistic transport of correlations, while the RHS behave as singular forcing, which is given the integrable dynamics, i.e. by the integrable contact interactions. In Appendix E more details are given in order to derive Eq. (32), (33) and (34). Finally, we observe that the assumption of large hydrodynamic length scale ξ is sufficient to have a well-defined truncation scheme for the hierarchy.

D. The hierarchy for the (quasi)particle occupations and correlations

In this section we express the hierarchy in terms of quasiparticles of the Hamiltonian $\hat{H}_a^{\text{int}}$. In particular, for integrable systems, the quasiparticles are characterized by Bethe rapidities θ , whose distribution function in the fluid cell (x, t) is denoted by $\rho_\theta(x, t)$ (in the rest of the text, we use Greek indices for rapidities, while Latin

ones for charge labels). The rapidities are related to the quasiparticles momentum through $k_\theta = k(\theta)$. The density $\rho_\theta(x, t)$ is defined through its relation with the hydrodynamic variables

$$\langle q_i \rangle(x, t) = \int d\theta \rho_\theta(x, t) h_i(\theta), \quad (35)$$

where $h_i(\theta)$ is the single-particle eigenvalue of the charge Q_i . With a complete set of charges, it is possible to invert this expression, having a bijection between the quasiparticle density $\rho_\theta(x, t)$ and the expectation values of the charge density $\langle q_i \rangle$. Hence, it is possible to recast the infinite tower of hydrodynamic equations for the conservation of charges, labelled by the integer i , to the single evolution of $\rho_\theta(x, t)$. We now define the dressing operator acting on a test function $h(\theta)$

$$h^{\text{dr}} = (1 - Tn)^{-1} \cdot h, \quad (36)$$

where the action of a linear operator is defined as $(A \cdot h)(\theta) = \int d\gamma A(\theta, \gamma) h(\gamma) = A_\theta^\gamma h_\gamma$, assuming the Einstein convention for the integration over repeated indexes. In particular $T = T_{\theta, \gamma}$ is the scattering shift of the model (which vanishes in the limit of zero contact interactions $a \rightarrow 0$), being independent on the state. The filling function defines the occupation of rapidities respect to the total density of states $\rho^\dagger = (k')^{\text{dr}}/2\pi$ and it is given by $n = 2\pi\rho/(k')^{\text{dr}}$. Moreover, we shall make extensive use of compact matrix notation, for example by introducing the rotation matrix

$$R_{\theta_1, \theta_2} = [1 - Tn]_{\theta_1, \theta_2} = \delta_{\theta_1, \theta_2} - T_{\theta_1, \theta_2} n_{\theta_2}, \quad (37)$$

which implements the dressing operation as $h^{\text{dr}} = R^{-1} \cdot h$. Crucially, it is also possible to express the density of currents, as

$$\langle j_i(x) \rangle = \int d\theta \rho_\theta(x) v_\theta^{\text{eff}}(x) h_i(\theta) + O(1/\xi), \quad (38)$$

with the effective velocity $v_\theta^{\text{eff}} = (h_2')^{\text{dr}}(\theta)/(h_1')^{\text{dr}}(\theta)$, where for Galilean particles with unit mass $h_2(\theta) = \theta^2/2$ and $h_1(\theta) = \theta$. Similarly, we can define quasiparticle expression for few point functions as

$$\langle q_{i_1} \dots q_{i_n} \rangle = \int \prod_{k=1}^n [d\theta_k h_{i_k}(\theta_k)] \langle \rho_{\theta_1} \dots \rho_{\theta_n} \rangle. \quad (39)$$

Another important object is the flux Jacobian, that in quasiparticles reads

$$A_i^j = \int d\theta_1 d\theta_2 h_i(\theta_1) R_{\theta_1, \gamma}^{-t} v_\gamma^{\text{eff}} R_{\gamma, \theta_2}^t h^j(\theta_2), \quad (40)$$

where h^j is defined as the inverse of h_j , such that $h_j(\theta) h^j(\theta') = \delta_{\theta, \theta'}$. For completeness we also recall $A_\theta^{\theta'} \equiv R_{\theta, \gamma}^{-t} v_\gamma^{\text{eff}} R_{\gamma, \theta'}^t$, with $R^t = 1 - nT$. In a similar manner we compute the second derivative of current as

$$H_i^{j,k} = \int d\theta d\alpha d\beta h_i(\theta) H_\theta^{\alpha, \beta} h^j(\alpha) h^k(\beta), \quad (41)$$

where we defined (from below onwards we shall use the notation $v_{\mu, \gamma} \equiv v_\mu^{\text{eff}} - v_\gamma^{\text{eff}}$)

$$H_\theta^{\alpha, \beta} = [R_{\theta, \gamma}^{-t} v_{\mu, \gamma} \frac{T_{\gamma, \mu}^{\text{dr}}}{\rho_\gamma^t} R_{\mu, \alpha}^t R_{\gamma, \beta}^t]_{(\alpha, \beta)}. \quad (42)$$

Analogously to what was done in the previous section, we consider the connected few point correlators and separately recognize the GGE contribution from the regular long range part. For the two-point function we have

$$\langle \rho_{\theta_1}(x_1) \rho_{\theta_2}(x_2) \rangle^c = \delta(x_1 - x_2) C_{\theta_1, \theta_2} + g_{\theta_1, \theta_2}^{(2)}(x_1, x_2), \quad (43)$$

where $C_{\theta_1, \theta_2} = R_{\theta_1, \gamma}^{-t} \rho_\gamma f_\gamma R_{\gamma, \theta_2}^{-1}$ is the susceptibility matrix in quasiparticle representation, and $g_{\theta_1, \theta_2}^{(2)}(x_1, x_2)$ is

the non-singular contribution. Also, the generic symbol $f_\theta = \{1, 1 - n_\theta, 1 + n_\theta, n_\theta^{-1}\}$ denotes the statistical factor (classical, fermionic, bosonic and wave statistics, respectively). Analogously, for the three-point functions, we can write

$$\begin{aligned} \langle \rho_{\theta_1}(x_1) \rho_{\theta_2}(x_2) \rho_{\theta_3}(x_3) \rangle^c &= \\ &= \delta(x_1 - x_2) \delta(x_1 - x_3) C_{\theta_1, \theta_2, \theta_3} \\ &+ \left[\delta(x_1 - x_2) C_{\theta_1, \theta_2}^\gamma g_{\gamma, \theta_3}^{(2)}(x_1, x_3) \right]_{(1, 2, 3)} \\ &+ g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3), \end{aligned} \quad (44)$$

where $C_{\theta_1, \theta_2, \theta_3}$ is the local three-point function in a GGE. We also introduce the tensor given by the functional derivative of the susceptibility respect to the local density

$$\begin{aligned} C_{\theta_1, \theta_2}^{\theta_3} &\equiv \frac{\delta C_{\theta_1, \theta_2}}{\delta \rho_{\theta_3}} = R_{\theta_1, \alpha}^{-t} \left[\delta_{\alpha, \beta} \delta_{\alpha, \theta_3} f_\alpha - \delta_{\alpha, \beta} \varepsilon n_\alpha R_{\alpha, \theta_3}^t \right. \\ &\left. + T_{\alpha, \beta}^{\text{dr}} \frac{R_{\alpha, \theta_3}^t}{\rho_\alpha^t} f_\beta \rho_\beta + f_\alpha \rho_\alpha T_{\alpha, \beta}^{\text{dr}} \frac{R_{\beta, \theta_3}^t}{\rho_\beta^t} \right] R_{\beta, \theta_2}^{-1}, \end{aligned} \quad (45)$$

with $\varepsilon = \{0, -1, 1\}$ respectively for classical, Fermi-Dirac or Bose-Einstein statistics and we also introduce the tensor giving the kinetic term of the three-point correlations

$$M_{\theta_1, \theta_2}^{\theta_3} = [R_{\theta_1, \mu}^{-t} R_{\theta_2, \gamma}^{-t} T_{\mu, \gamma}^{\text{dr}} \frac{\rho_\gamma f_\gamma}{\rho_\mu^t} v_{\mu, \gamma} R_{\mu, \theta_3}^t]_{(\theta_1, \theta_2)}, \quad (46)$$

which is proportional to the strength of the local interactions a (see Appendix E A for a complete derivation).

With this, we can write down the first three layers in the quasiparticle representation:

$$\partial_t \rho_{\theta_1} = \int dx_2 d\theta_2 V'(x_1 - x_2) \partial_{\theta_1} g_{\theta_1, \theta_2}^{(2)}(x_1, x_2), \quad (47)$$

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2}^{(2)}(x_1, x_2) &+ \left[\partial_{x_1} (A_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2)}(x_1, x_2)) + \frac{1}{2} \partial_{x_1} (H_{\theta_1}^{\gamma, \eta} g_{\gamma, \eta, \theta_2}^{(3), \text{sym}}(x_1^+, x_1^-, x_2)) \right]_{(1, 2)} = \\ &= \left[V'(x_1 - x_2) \int d\theta_3 \partial_{\theta_1} \left(\rho_{\theta_1} C_{\theta_2, \theta_3} + C_{\theta_2, \theta_3}^\gamma g_{\theta_1, \gamma}^{(2)}(x_1, x_2) \right) \right. \\ &+ \left. \int dx_3 d\theta_3 V'(x_1 - x_3) \partial_{\theta_1} \left(\rho_{\theta_1} g_{\theta_2, \theta_3}^{(2)}(x_2, x_3) + g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) \right) \right]_{(1, 2)} \\ &+ \delta(x_1 - x_2) \int dx_3 d\theta_3 V'(x_1 - x_3) \left(-C_{\theta_1, \theta_2}^\gamma \partial_\gamma + (\partial_{\theta_1} + \partial_{\theta_2}) C_{\theta_1, \theta_2}^\gamma \right) g_{\gamma, \theta_3}^{(2)}(x_1, x_3), \end{aligned} \quad (48)$$

$$\partial_t g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) + \left[\partial_{x_1} A_{\theta_1}^\gamma g_{\gamma, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) \right]_{(1,2,3)} = \left[\delta(x_1 - x_2) M_{\theta_1, \theta_2}^\gamma \partial_{x_1} g_{\gamma, \theta_3}^{(2)}(x_1, x_3) \right]_{(1,2,3)} + O(V_0^2 \xi^2, V_0 \xi^{-3}). \quad (49)$$

Let us comment on the kinetic terms for the two- and three-point functions in Eq. (48) and (49). The first terms with the matrix $A_\theta^\gamma = [R^{-1} v^{\text{eff}} R]_\theta^\gamma$ represent convective spreading with velocities associated to the different quasi-particles. While in an interacting model this matrix is non-diagonal (due to mode mixing), we shall see in the next section that the latter simply reduces to a diagonal matrix containing the velocities $v^{\text{br}}(\theta)$ as eigenvalues in the limit $a \rightarrow 0$. The second term in eq. (48) is instead non-zero only in interacting systems. The latter can be seen as an effective diffusion induced by the motion of the three-point functions for the two-point functions. Indeed, its form becomes more familiar in the small times limit, i.e. for states close to the local GGE. In this limit, we can indeed show the relation (see [120] for a complete derivation)

$$\lim_{t \rightarrow 0^+} H_{\theta_1}^{\gamma, \eta} g_{\gamma, \eta, \theta_2}^{(3), \text{sym}}(x_1^+, x_1^-, x_2; t) = \mathfrak{D}_{\theta_1}^\gamma \partial_{x_1} g_{\gamma, \theta_2}^{(2)}(x_1, x_2), \quad (50)$$

where $\mathfrak{D}_\theta^\gamma$ is indeed the diffusion matrix [119, 121, 122] associated with the local equilibrium state around which we are linearising. As the system evolves, the validity of this relation gets worse, due to the build-up of nontrivial long-range three-point correlations, yet this contribution is fundamental to thermalization of the two-point functions and, consequently, of the 1-point functions as we shall see in the next sections. In the limit $a \rightarrow 0$ this contribution vanishes, as there is indeed no internal (convective) diffusion $\mathfrak{D} \approx a^2$, as also confirmed by the analysis in the next section III E.

E. Recovering the standard BBGKY hierarchy in the limit $a \rightarrow 0$

The standard BBGKY hierarchy is recovered in the limit of non-interacting unperturbed systems, i.e. $\hat{H}_a^{\text{int}} \rightarrow \hat{H}^{\text{free}}$. It is important here to stress that the gBBGKY formalism is valid both for classical and quantum systems and difference between the two scenarios is fully incorporated into the statistical factor $f = f(n)$. Here, in order to compare with standard textbook expressions [78], we specialize to the classical case and set $f = 1$.

The limit $\hat{H}_a^{\text{int}} \rightarrow \hat{H}^{\text{free}}$ corresponds to zero scattering shift $T \rightarrow 0$. We now carefully analyze the effect of this limit on the objects defined in Section III D. Firstly, we observe that this limit trivialize the dressing operator as $R_{\theta, \theta'} \rightarrow \delta_{\theta, \theta'}$, and hence $h^{\text{dr}} \rightarrow h$. As a consequence, the effective velocity reduces to the particle's velocity $v_\theta^{\text{eff}} \rightarrow v_\theta^{\text{br}} = \theta$. In a similar way the flux Jacobian becomes $A_\theta^\gamma \rightarrow v_\theta^{\text{br}} \delta_{\theta, \gamma}$. Since non-interacting particles don't

exhibit non-trivial correlations, we also have

$$C_{\theta_1, \theta_2} \rightarrow \delta_{\theta_1, \theta_2} \rho_{\theta_1}, \quad C_{\theta_1, \theta_2}^{\theta_3} \rightarrow \delta_{\theta_1, \theta_2} \delta_{\theta_1, \theta_3}. \quad (51)$$

As a consequence we observe that the delta contribution appearing in Eq (48) is vanishing, being proportional to

$$\int d\gamma [\delta_{\theta_1, \theta_2} \delta_{\theta_1, \gamma} \partial_\gamma - (\partial_{\theta_1} + \partial_{\theta_2}) \delta_{\theta_1, \theta_2} \delta_{\theta_1, \gamma}] h_\gamma = 0. \quad (52)$$

We also observe that, in this limit, we have $H \rightarrow 0$ and $M \rightarrow 0$, both being proportional to T . We also stress that any higher-order hydrodynamic correction in the cumulant expansion is vanishing since non-interacting particles simply spread ballistically. Hence, in the non-interacting limit $T \rightarrow 0$, Eq. (47), (48) and (49) reduce to

$$\partial_t \rho_{\theta_1} = \int dx d\theta_2 V'(x) \partial_{\theta_1} g_{\theta_1, \theta_2}^{(2)}(x), \quad (53)$$

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2}^{(2)}(x_1, x_2) + \left[\partial_{x_1} (v_{\theta_1}^{\text{br}} g_{\theta_1, \theta_2}^{(2)}(x_1, x_2)) \right]_{(1,2)} = \\ = \left[V'(x_1 - x_2) \partial_{\theta_1} (\rho_{\theta_1} \rho_{\theta_2} + g_{\theta_1, \theta_2}^{(2)}) \right] + \\ + \int dx_3 d\theta_3 V'(x_1 - x_3) \partial_{\theta_1} \rho_{\theta_1} g_{\theta_2, \theta_3}^{(2)}(x_2, x_3) + \\ + \int dx_3 d\theta_3 V'(x_1 - x_3) \partial_{\theta_1} g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) \end{aligned} \quad (54)$$

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) + \\ + \left[\partial_{x_1} (v_{\theta_1}^{\text{br}} g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3)) \right]_{(1,2,3)} + \dots = 0, \end{aligned} \quad (55)$$

that perfectly coincide with the first equations of celebrated BBGKY hierarchy for a homogeneous system. It is also easy to prove that this equivalence is valid for any level of the hierarchy and that can be extended also in inhomogeneous settings. It is important to stress that, through the gBBGKY formalism, we provided a new alternative derivation of BBGKY in terms of correlated fluid cells that doesn't require to use the full system phase space distribution function.

IV. THE GENERALISED LANDAU EQUATION AND LATE-TIME DYNAMICS

The gBBGKY equations presented above are clearly complicated to analyze and to fully numerically simulate. However, to extract the thermalisation dynamics,

some approximations can be performed, leading to what is known as Landau equation, namely the Boltzmann equation in the limit of small momentum exchange, also known as the grazing collisions limit.

We shall perform a macroscopic rescaling of space $x \rightarrow \xi x$ and time $t \rightarrow \xi t$, and importantly, in what follows we will assume that the long-range coupling is weak by treating V_0 as a small parameter. Thus, scattering integral description have a narrower range of applicability than the hierarchy, which assumes only large ξ . However, as we show in Sec. V, the scattering integral turns out to be quantitatively predictive also for quite large values of V_0 , provided that $V_0/\xi \ll 1$. Nevertheless, this approach is sufficient to understand the main mechanisms behind the thermalisation of the system.

We start the derivation by considering the gBBGKY hierarchy written in terms of quasiparticles occupations and correlations, as introduced in Sec. III D. More precisely, in this section, we explicitly make use of homogeneity by expressing correlations as uniquely functions of relative distance. Hence, for the two-point function we have $g_{\theta_1, \theta_2}^{(2)}(x_1, x_2) = g_{\theta_1, \theta_2}^{(2)}(z = x_1 - x_2)$ and for the three-point function $g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) = g_{\theta_1, \theta_2, \theta_3}^{(3)}(z, z')$, with $z = x_1 - x_2$ and $z' = x_2 - x_3$.

Furthermore, we perform a decomposition of the two-point function $g^{(2)}$ into leading and subleading contributions in the $1/\xi$ expansion, namely

$$g_{\theta_1, \theta_2}^{(2)}(z) = \frac{1}{\xi} g_{\theta_1, \theta_2}^{(2),1}(z) + \frac{1}{\xi^2} g_{\theta_1, \theta_2}^{(2),2}(z). \quad (56)$$

We also explicitly factorize the scaling parameter of the three-point functions using $g^{(3)} \rightarrow g^{(3)}/\xi^2$. Now, using the definition of the interaction potential (1), the $1/\xi$ dependence is fully explicit. Plugging the decomposition into equation (48) for the two-point function and requiring equality both at the order $1/\xi$ and $1/\xi^2$ produces two equations:

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2}^{(2),1}(z) + \partial_z \left(A_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2),1}(z) - A_{\theta_2}^\gamma g_{\theta_1, \gamma}^{(2),1}(z) \right) = \\ V_0 \varphi'(z) \int d\gamma \left(\partial_{\theta_1} \rho_{\theta_1} C_{\gamma, \theta_2} - \partial_{\theta_2} \rho_{\theta_2} C_{\theta_1, \gamma} \right), \end{aligned} \quad (57)$$

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2}^{(2),2}(z) + \partial_z \left(A_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2),2}(z) - A_{\theta_2}^\gamma g_{\theta_1, \gamma}^{(2),2}(z) \right) = \\ - \frac{1}{2} \partial_z \left(H_{\theta_1}^{\gamma_1, \gamma_2} g_{\gamma_1, \gamma_2, \theta_2}^{(3), \text{sym}}(0^+, z) - H_{\theta_2}^{\gamma_1, \gamma_2} g_{\theta_1, \gamma_1, \gamma_2}^{(3), \text{sym}}(z, 0^+) \right). \end{aligned} \quad (58)$$

These equations are the effective evolution laws for $g^{(2),1}$ and $g^{(2),2}$. We can already observe that, meanwhile $g^{(2),1}$ is rapidly evolving due to the coupling with the density, the subleading correction $g^{(2),2}$ slowly builds up forced by the three-point function. We start with considering four equations (47), (57), (58) and (49). The goal is to reduce them to a single equation for quasiparticle distribution ρ_θ . We will follow a procedure in spirit similar to classic

derivations of Boltzmann scattering kernel from BBGKY hierarchy [68, 69]. The central idea here is the separation of timescales over which different correlations relax. For instance, one-point function is expected to relax on much longer time scale than two-point function. Schematically, the derivation will proceed as follows:

1. Integrate (57) in time expressing $g_{\theta_1, \theta_2}^{(2),1}(z, t)$ as explicit functional of $\rho_\theta(\tau)$ for all earlier times $\tau < t$.
2. Plug the result of previous step into (49) and again integrate in time expressing $g_{\theta_1, \theta_2, \theta_3}^{(3)}(z, z', t)$ as functional of $\rho_\theta(\tau)$.
3. Use this result to compute $g_{\theta_1, \theta_2, \theta_3}^{(3)}(z, z', t)$, plug into equation (58) and integrate in time expressing $g_{\theta_1, \theta_2}^{(2),2}(z, t)$ as functional of $\rho_\theta(\tau)$.
4. Plug this result into (47) and perform approximations related to separation of timescales which finally yield generalized Landau kinetic equation.

We elaborate on the derivation in what follows. We assume that at $t = 0$ the state is GGE state with $g^{(n)} = 0$ for $n \geq 2$. This assumption is not necessary as the system will eventually forget about the initial correlations, but simplifies the calculation which anyway leads to universal, late-time physics.

The first step consists of considering (57) and viewing it as a linear equation (we assume here that A_θ^γ is constant operator) for $g_{\theta_1, \theta_2}^{(2),1}(z)$ with a ρ -dependent source term on RHS given by

$$\mathcal{S}_{\theta_1, \theta_2}^{(2),1}[\rho](z) = V_0 \varphi'(z) \int d\gamma \left(\partial_{\theta_1} \rho_{\theta_1} C_{\gamma, \theta_2} - \partial_{\theta_2} \rho_{\theta_2} C_{\theta_1, \gamma} \right). \quad (59)$$

Thus, (57) can formally be solved using Green's function method (for details see Appendix E B), which gives

$$\begin{aligned} g_{\theta_1, \theta_2}^{(2),1}[\rho](z, t) = R_{\theta_1, \xi_1}^{-t} R_{\theta_2, \xi_2}^{-t} \int_0^t d\tau \int \frac{dk}{2\pi} \int dz' \times \\ e^{ik(z+z'-v_{\xi_1, \xi_2} \tau)} R_{\xi_1, \eta_1}^t R_{\xi_2, \eta_2}^t \mathcal{S}_{\eta_1, \eta_2}^{(2),1}[\rho](z', t - \tau). \end{aligned} \quad (60)$$

In this way we have expressed $g_{\theta_1, \theta_2}^{(2),1}(z, t)$ with ρ_θ at all earlier times. The structure of a linear equation with ρ -dependent source term turns out to be typical in this derivation and we encounter it also in the second step, which amounts to plugging (60) into the equation (49). The linear equation features a source term on the RHS which reads

$$\begin{aligned} \mathcal{S}_{\theta_1, \theta_2, \theta_3}^{(3)}[\rho](z, z') = \delta(z) M_{\theta_1, \theta_2}^\gamma \partial_z g_{\gamma, \theta_3}^{(2),1}(z + z') + \\ - \delta(z') M_{\theta_2, \theta_3}^\gamma \partial_z g_{\theta_1, \gamma}^{(2),1}(z) + \delta(z') M_{\theta_2, \theta_3}^\gamma \partial_{z'} g_{\theta_1, \gamma}^{(2),1}(z + z'). \end{aligned} \quad (61)$$

Using a similar technique as used to get (60) we explicitly find the functional $g_{\theta_1, \theta_2, \theta_3}^{(3)}[\rho](z, z', t)$. In the third

step we compute $g_{\theta_1, \theta_2, \theta_3}^{(3), \text{sym}}[\rho]$, and plug it to equation (58). Once again, this equation is integrated in time yielding $g_{\theta_1, \theta_2}^{(2), 1}[\rho](z, t)$. In the last step we insert the expressions for $g_{\theta_1, \theta_2}^{(2), 1}[\rho](z, t)$ and $g_{\theta_1, \theta_2}^{(2), 2}[\rho](z, t)$ into the equation for ρ_θ , which now becomes closed as all the higher correlations have been eliminated through time integration. We find that the equation has two contributions

$$\partial_t \rho_\theta = \frac{1}{\xi} \mathcal{I}^{(2), 1}[\rho](\theta) + \frac{1}{\xi^2} \mathcal{I}^{(2), 2}[\rho](\theta). \quad (62)$$

Restoring the microscopic time variable, the two terms are, respectively, proportional to V_0^2/ξ^2 and $a^2 V_0^2/\xi^3$. In sections IV A and IV B we derive them and show that $\mathcal{I}^{(2), 1}$ is vanishing, proving that the leading thermalization rate is $a^2 V_0^2/\xi^3$.

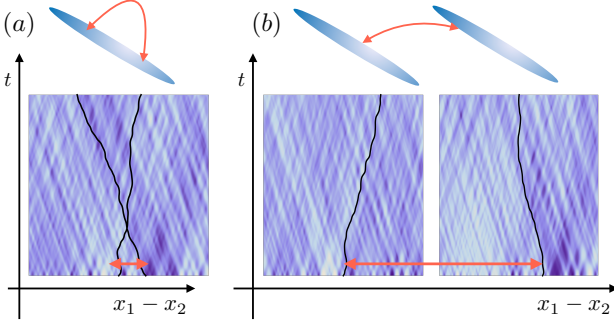


Figure 4. Schematic representation of the two thermalisation mechanisms leading to the two terms in eq. (66). On the *left* is represented the dynamics giving the *self* + *cross* term: two correlated particles at a distance ξ experience fluctuations due to convective waves carried by the other quasiparticles, giving cross diffusion, and they also scatter among each other, giving a self-diffusion to the two-point functions. On the *right*, the dynamics giving the *cross* term: two correlated particles never interact with contact interactions; therefore, they only experience the convective density waves through the system, giving standard diffusion to the two-point functions.

A. Kinetic blocking at times ξ^2/V_0^2

We start by considering the first term which reads

$$\mathcal{I}^{(2), 1}[\rho](\theta_1) = \int dz d\theta_2 V_0 \varphi'(z) \partial_{\theta_1} g_{\theta_1, \theta_2}^{(2), 1}[\rho](z), \quad (63)$$

with $g_{\theta_1, \theta_2}^{(2), 1}[\rho](z)$ given by (60). We perform an approximation which amounts to replacing $\rho_\theta(z, t - \tau)$ with $\rho_\theta(z, t)$ under the integral and extending the integration range in time to infinity. Next, we use an identity $\int_0^\infty d\tau e^{i\tau x} = \pi \delta(x) + i\mathcal{P}(\frac{1}{x})$, where $\mathcal{P}$ denotes principal value. The term with principal value vanishes due to the

symmetry of the integrand, the term with delta function yields (we also use $\int d\theta R_{\theta, \gamma}^{-t} = 2\pi \rho_\gamma^t$)

$$\mathcal{I}^{(2), 1}[\rho](\theta_1) = \int \frac{dk}{2\pi} |k| V_0^2 \hat{\varphi}^2(k) \partial_{\theta_1} R_{\theta_1, \gamma_1}^{-t} \rho_{\gamma_2}^t \delta(v_{\gamma_1, \gamma_2}) \times \\ R_{\gamma_1, \eta_1}^t R_{\gamma_2, \eta_2}^t \int d\gamma_3 (\partial_{\eta_1} \rho_{\eta_1} C_{\gamma_3, \eta_2} - \partial_{\eta_2} \rho_{\eta_2} C_{\eta_1, \gamma_3}), \quad (64)$$

where $\hat{\varphi}(k) = \int dx e^{ikx} \varphi(x)$. Crucially, the Dirac delta condition $v_{\gamma_1 \gamma_2} = 0$ implies $\gamma_1 = \gamma_2$ and hence, the integral *vanishes* identically

$$\mathcal{I}^{(2), 1}[\rho](\theta_1) = 0. \quad (65)$$

This is a feature of 1D systems known under the name of *kinetic blocking*. It is strictly related to the nature of two-particle scattering in 1D, which can only induce momentum exchanges between particles. The kinetic blocking can be avoided in systems with non-monotonous effective velocity, for instance in lattice. Then, so called Umklapp scatterings allow the system to thermalize due to two-body collisions [38, 39]. For the present discussion, however, we focus on systems with monotonous effective velocity. Another scenario involves systems of coupled 1D gases, as described in Section VI. If 1D gases have different densities or different strengths of integrable interactions, this opens a channel for scattering that redistribute the momenta, albeit not very effectively, leading only to prethermal stationary states [123]. Hence, here, in order to capture the thermalization phenomena we have to include higher-order terms, which describe three-particle scatterings. These processes are encoded in $\mathcal{I}^{(2), 2}[\rho](\theta)$, which we now analyze.

B. (generalised) Thermalization at times $\xi^3/(V_0 a)^2$

The derivation of $\mathcal{I}^{(2), 2}$ proceeds as sketched above and the approximations made along the way are of the same form as in derivation of $\mathcal{I}^{(2), 1}$. Namely, every time we time-integrate the equation for correlation function, we replace $\rho_{\theta_1}(z, t - \tau)$ with $\rho_{\theta_1}(z, t)$ under the integral and extend integration range in time to infinity. In order to obtain the collision term, three such integrations have to be performed leading to much more complex collision term as compared to (64). The details of this quite long calculation are exposed in [120]. Here we present the final result. We simplify the notation $\mathcal{I} \equiv \mathcal{I}^{(2), 2}$ and find two type of terms

$$\mathcal{I} = \mathcal{I}^{\text{cross}} + \mathcal{I}^{\text{self}}, \quad (66)$$

Both terms are given by $\mathcal{I}^\alpha = \partial_{\theta_1} R_{\theta_1, \gamma}^{-t} \mathcal{I}_0^\alpha(\gamma)$, where $\alpha \in \{\text{cross}, \text{self}\}$ and using the notation $v_{\theta_2, \theta_1} = v_{\theta_2}^{\text{eff}} - v_{\theta_1}^{\text{eff}}$, we have

$$\mathcal{I}_0^{\text{cross}}[\rho](\theta_1) = 2\pi V_0^2 \int dk k^2 \hat{\varphi}^2(k) \int d\theta_2 d\theta_3 \left[\frac{(\rho_{\theta_3}^t)^2}{\rho_{\theta_1}^t} (T_{\theta_2, \theta_1}^{\text{dr}})^2 \frac{|v_{\theta_2, \theta_1}|}{v_{\theta_3, \theta_2} v_{\theta_1, \theta_3}^2} - \rho_{\theta_1}^t (T_{\theta_3, \theta_2}^{\text{dr}})^2 \frac{|v_{\theta_3, \theta_2}|}{v_{\theta_1, \theta_3} v_{\theta_2, \theta_1}^2} \right] \mathcal{B}_{\theta_1, \theta_2, \theta_3}, \quad (67)$$

$$\begin{aligned} \mathcal{I}_0^{\text{self}}[\rho](\theta_1) = 2\pi V_0^2 \int d\theta_2 d\theta_3 \int dk k^2 \hat{\varphi}(k) \rho_{\theta_3}^t T_{\theta_1, \theta_2}^{\text{dr}} \frac{v_{\theta_2, \theta_1}}{v_{\theta_1, \theta_3}} \left[\hat{\varphi} \left(\frac{v_{\theta_2, \theta_1}}{v_{\theta_1, \theta_3}} k \right) \frac{\rho_{\theta_2}^t}{\rho_{\theta_1}^t} \frac{T_{\theta_3, \theta_1}^{\text{dr}}}{|v_{\theta_1, \theta_3}| |v_{\theta_3, \theta_2}|} - \right. \\ \left. - 2\hat{\varphi} \left(\frac{v_{\theta_2, \theta_1}}{v_{\theta_3, \theta_2}} k \right) \frac{T_{\theta_2, \theta_3}^{\text{dr}}}{v_{\theta_1, \theta_3} |v_{\theta_2, \theta_3}|} \right] \mathcal{B}_{\theta_1, \theta_2, \theta_3}, \end{aligned} \quad (68)$$

with the $\mathcal{B}$ tensor given by

$$\mathcal{B}_{\theta_1, \theta_2, \theta_3} = \left[\frac{v_{\theta_3, \theta_2}}{\rho_{\theta_1}^t} R_{\theta_1, \eta}^t \partial_\eta \rho_\eta \rho_{\theta_3} f_{\theta_3} \rho_{\theta_2} f_{\theta_2} \right]_{(\theta_1, \theta_2, \theta_3)}. \quad (69)$$

It can be shown that the derived collision integral has all the expected features: thermal states are stationary states, it conserves particle number, momentum and energy and satisfies H-theorem with positive entropy increase.

Restoring the microscopic times in the kinetic equation (62) and using that $T^{\text{dr}} \sim a$, we can finally observe that the scattering integral given by Eq (67) and Eq (68) predict a thermalization rate proportional to $(V_0/\xi)^2 a^2/\xi$. The scaling can be understood as following: two factors of V_0/ξ are necessary to form correlated two-point functions on large scales, then, two local scatterings with other particles carry a^2 , and the spatial inhomogeneity of the two-point functions, inducing its convective diffusion in space, carry another factor $1/\xi$. See also in Fig. 1: particles correlate due to the long-range potential and then diffuse in the convective flow carried by the sea of ballistic particles. This picture may look counter-intuitive, as, due to integrability breaking, one would naively expect the ballistic channels to decay at large times. However, this is not the case, as higher-order correlations preserve the ballistic structure to much longer times than ξ^3 , giving an effective bath of convective particles inducing density waves which is the main and only mechanism for particle diffusion in interacting integrable models, see for example [121, 124–129].

The reason for distinguishing the two terms is motivated by two different dynamical mechanisms, see Fig. 4 related to convective diffusion. The first term, the cross

term, originates from the diffusion of the two-point correlations where the two correlated particles never interact via contact interaction. This can be the case when for example two quasi-one dimensional systems interact with a long-range potential that only acts between the two different tubes. The second term, the self term, originates from the self-induced diffusion of the two-particles that can also interact via contact interaction: namely one of the two particles acts as a convective density wave for the other and vice-versa. As shown in [120], this scattering term is absent when the long-range interaction acts only between different tubes, but is present when it acts also on the same tube. In the situation where the long-range potential acts across different tubes and on the same one, the scattering integral from eq. (66) should include also the cross terms due to other tubes. We will discuss such scenarios in Section VI when we apply this formalism to experimental setups with dipolar quantum gases.

It is important to stress that, even if one-point functions thermalize on time scales of order $\xi^3/(V_0 a)^2$, *higher-point functions remain far from thermal*, which is why we have dubbed this process *generalized thermalization*, in contrast with standard thermalization. In particular, three-point functions exhibit long-range correlations that do not decay on these time scales and arise entirely from local integrable interactions, mirroring the behavior of two-point functions in fully integrable dynamics [126]. In order to derive the collision integral, in [120] we derive an explicit expression for the dynamics of the three-point functions given by eq. (49); here we report its large-time limit, namely in the regime where time is of the order $\xi^3(V_0 a)^{-2}$:

$$\begin{aligned} g_{\theta_1, \theta_2, \theta_3}^{(3)}(z, z', t \gg \xi^3(V_0 a)^{-2}) = \\ = \left[\frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_1, \gamma_2}} [\text{sgn}(z) + \text{sgn}(v_{\gamma_1, \gamma_2})] R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_1, \mu_2}^\beta \partial_1 g_{\beta, \mu_3}^{(2), 1} \left(\frac{v_{\gamma_2, \gamma_3}}{v_{\gamma_1, \gamma_2}} z - z' \right) \right] + \\ + \left[\frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_1, \gamma_3}} [\text{sgn}(z + z') + \text{sgn}(v_{\gamma_1, \gamma_3})] \times R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_1, \mu_3}^\beta \partial_1 g_{\beta, \mu_2}^{(2), 1} \left(\frac{v_{\gamma_3, \gamma_2}}{v_{\gamma_1, \gamma_3}} z + \frac{v_{\gamma_1, \gamma_2}}{v_{\gamma_1, \gamma_3}} z' \right) \right] + \\ + \left[\frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_2, \gamma_3}} [\text{sgn}(z') + \text{sgn}(v_{\gamma_2, \gamma_3})] R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_2, \mu_3}^\beta \partial_1 g_{\beta, \mu_1}^{(2), 1} \left(z + \frac{v_{\gamma_2, \gamma_1}}{v_{\gamma_2, \gamma_3}} z' \right) \right], \end{aligned} \quad (70)$$

where the symbol ∂_1 means that the derivative is per-

formed on the spatial argument of the function, the sym-

bol $M_{\theta_1, \theta_2}^{\theta_3} \sim a$ is defined in eq. (46) and the two-point functions $g_{\theta_1, \theta_2}^{(2),1}(z)$ are given by their asymptotic value in time. Given that the latter decays rapidly in z , the expression above shows that there exist distinct directions in the (z, z') plane, e.g., along lines $z' \simeq \frac{v_{\gamma_2, \gamma_3}}{v_{\gamma_1, \gamma_2}} z$ from the first term, along which long-range three-point correlations persist well beyond the thermalization time scales. Therefore, at times of order $\xi^3/(V_0 a)^2$, thermalization is incomplete (generalized), and the deviation from a standard thermal state can be diagnosed by higher-point functions.

C. Thermalization dynamics in the limit $a \rightarrow 0$.

In the limit $a \rightarrow 0$, the generalized Landau equation, i.e., the contributions (67) and (68), vanishes, as it is proportional to a^2 . As noted above, the thermalization mechanism encoded by the generalized Landau equation is present only for perturbed interacting integrable models. In the case $a = 0$, thermalization is driven solely by the long-range interaction. A kinetic equation for weak long-range potentials in this regime was derived in [68, 69]. Using the same approximations introduced in Sec. IV, the standard BBGKY equations governing thermalization, analogous to (57) and (58), read:

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2}^{(2),1}(x_1, x_2) + [v^{\text{br}}(\theta_1) \partial_{x_1} g_{\theta_1, \theta_2}^{(2),1}(x_1, x_2)]_{(1,2)} = \\ = [V_0 \varphi'(x_1 - x_2) \partial_{\theta_1} \rho_{\theta_1} \rho_{\theta_2}]_{(1,2)}, \quad (71) \end{aligned}$$

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2}^{(2),2}(x_1, x_2) + [v^{\text{br}}(\theta_1) \partial_{x_1} g_{\theta_1, \theta_2}^{(2),2}(x_1, x_2)]_{(1,2)} = \\ = \left[\int d\theta_3 dx_3 V_0 \varphi'(x_1 - x_3) \partial_{\theta_1} g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1, x_2, x_3) \right]_{(1,2)}. \quad (72) \end{aligned}$$

Similarly, the leading equation for the evolution of $g^{(3)}$ contains a forcing term proportional to the potential. As a consequence, the four nested equations for ρ , $g^{(2),1}$, $g^{(2),2}$, and $g^{(3)}$, which must be integrated to derive the scattering integral, have the interaction potential on the right hand side, yielding a thermalization rate proportional to $(V_0)^4/\xi^4$, as also demonstrated numerically in Fig. 7. In this case the picture is simpler, see Fig. 4, the thermalization rate is set first by the formation of two-point functions, namely $(V_0/\xi)^2$, and then by the formation of three-point functions, necessary for thermalisation, giving this way another factor of $(V_0/\xi)^2$ and contributing to the final thermalization rate $(V_0/\xi)^4$.

V. HARD SPHERES WITH LONG-RANGE INTERACTIONS

In order to numerically verify the gBBGKY hierarchy of equations, we apply it to a gas of classical impenetrable hard spheres (Tonks gas) with diameter a [130–133],

interacting with the long-range potential of Eq. (1). In particular, we consider the potential

$$V(z) = \frac{V_0}{\xi} \frac{1}{1 + |z/\xi|^3}. \quad (73)$$

which mimics the dipolar interactions emerging in one-dimensional cold atoms, as we shall see in the next section, as well as standard Lennard-Jones interacting potentials [60–64] (especially in the case of $V_0 < 0$). Hard spheres (or rods) gas in one dimension is integrable in the absence of other interactions, namely in the limit $V_0 = 0$, with a well-known hydrodynamic description [133–136] which share the same structure as quantum integrable systems, modulus changing the statistical factor [137], related to quasiparticles' statistics $f = 1$, and constant scattering shift $T_{\theta, \theta'} = -a/(2\pi)$. We focus on a homogeneous system with constant density of particles, and we compare gBBGKY prediction with exact microscopic simulations.

We initialize the time-evolution from a finite temperature states in which half of the particles is boosted with velocity θ_0 and half with velocity $-\theta_0$, $\rho_{\beta; \theta_0} \equiv \sum_{\sigma=\pm} e^{-\beta(\theta - \sigma\theta_0)^2}/Z$, and with $g_{\theta_1, \theta_2}^{(2)}(t=0) = 0$. The state is stationary under the unperturbed Hamiltonian $\hat{H}_a^{\text{int}}$ but it evolves non-trivially as long as V_0 is finite. As a probe of thermalisation dynamics, we compute the kurtosis of the (normalised) quasiparticle distribution and the two-point correlations. The kurtosis is expected to be zero in the thermal state, where quasiparticle distribution is Gaussian.

Solving the set of gBBGKY equations (47), (48) and (49) is numerically too expensive in terms of computational time and memory usage. In fact, it requires to save and perform operations on the three-point function $g_{\theta_1, \theta_2, \theta_3}^{(3)}(x_1 - x_2, x_2 - x_3)$, being, even in a homogeneous system, a five order tensor. In order to reduce the complexity of the numerical computation we separately solve the dynamics at small and large times. For small times $t < t_L$, we solve the gBBGKY hierarchy (47) and (48) by replacing the contribution proportional to H in (48) with the known Kubo diffusion term (explicitly known for integrable systems), as given by Eq. (50), and neglecting the remaining contribution from $g^{(3)}$. For large times $t \geq t_L$, we evolve the system by solving the Landau equation (66), using as initial state the gBBGKY solution obtained at time $t = t_L$. See Appendix C for additional informations about the numerical solution of gBBGKY hierarchy.

In Fig. 6 we show how gBBGKY reproduces the full molecular dynamics, predicting the evolution of Kurtosis as well as the evolution of two-point function. We can observe two regimes in the quasiparticle evolution: a fast evolution at short times $t < t_L$, driven by the change in the interaction energy due to the build-up of the density-density correlations in the system and the *kinetic regime* $t > t_L$, where the system slowly evolves toward the thermal state. In this regime, the kinetic energy is mostly constant and the dynamics is well described by the Landau equation, which well describes the

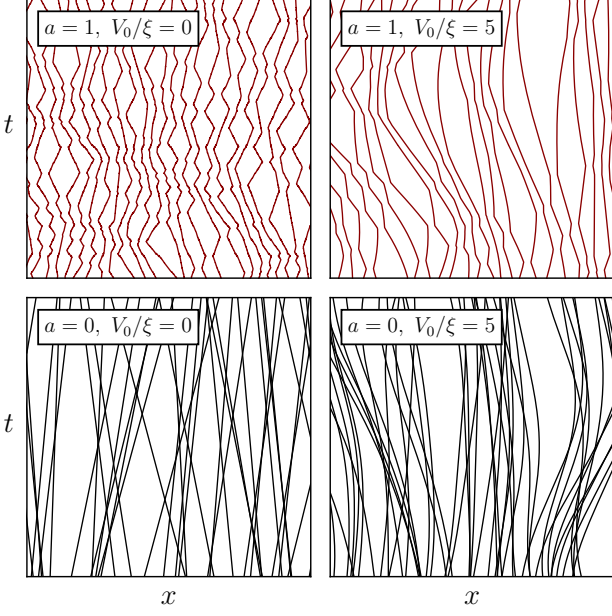


Figure 5. Examples of microscopic trajectories of hard spheres with long-range interaction given by the potential (73), both with $a = 0$ (free particles) and finite $a = 1$. In the latter case we can notice in the trajectories with finite V_0 the interplay of local scatterings and long-range interaction.

(generalized) thermalisation with rate $(aV_0)^2/\xi^2$, see Fig. 7, as predicted in the previous section.

We can observe the striking difference between the dynamics of point particles $a = 0$ and the ones with finite a already from a single trajectory snapshot, see Fig. 5: the dynamics with $a = 0$ and V_0 finite is given by smoothly long-range interacting particles, while the one with finite a is given both by long-range interactions and local scatterings, even at long times, where the thermalisation mechanisms start to kick in. The difference between these two cases is also highlighted in Fig. 6(a), where we show how the solution gBBGKY with $a = 0$ (dashed lines), i.e. BBGKY, is far from the hard spheres molecular dynamics from any arbitrary macroscopic small time. Interestingly, the short-time dynamics with $a = 0$ appears to be faster, compared to the interacting case with $a = 1$, namely in the presence of contact interaction the system appears to be jammed. However the scaling of the thermalisation, long-times, rates in the two cases shows the opposite behavior: V_0^4/ξ^4 in the free case and $(aV_0)^2/\xi^3$, resulting in a much faster (generalized) thermalisation in the interacting case $a > 0$.

VI. APPLICATION TO DIPOLAR ONE-DIMENSIONAL QUANTUM GASES

Integrability breaking due to additional long-range interactions arises naturally in quantum gases of atoms or

molecules with large magnetic moments [85]. In these systems, the basic Hamiltonian is given by Eq. (2). When confined to one dimension, the van der Waals interaction can be modeled effectively by a contact potential [138], and the gas is described by the integrable Lieb–Liniger (LL) Hamiltonian [27, 28]:

$$H_{\text{LL}} = -\frac{\hbar^2}{2m} \sum_i \partial_{x_i}^2 + c \sum_{i < j} \delta(x_i - x_j), \quad (74)$$

where m is the mass of the bosons and c the interaction strength.

If the atoms carry a nonzero magnetic moment, they also interact via a dipolar potential, which decays more slowly in space and must be treated separately. In the geometry considered by us (see Fig. 8), dipolar interactions couple both the atoms inside a tube (intratube interactions) as well as atoms in neighbouring tubes (intertube interactions). We discuss first the intratube potential. Upon integrating out the two “frozen” transverse degrees of freedom, the effective one-dimensional dipolar coupling becomes [139, 140]

$$V_{\text{DDI}}^{\text{1D}}(z) = V(\theta_{\text{DDI}}) \left[v_{\text{DDI}}^{\text{1D}}(u) - \frac{8}{3} \delta(u) \right], \quad u = \frac{z}{l_{\perp}}, \quad (75)$$

where $l_{\perp} = \sqrt{\hbar/(m\omega_{\perp})}$ is the transverse oscillator length. The prefactor is

$$V(\theta_{\text{DDI}}) = \frac{\mu_0 \mu^2}{4\pi} \frac{1 - 3 \cos^2 \theta_{\text{DDI}}}{4 l_{\perp}^3}, \quad (76)$$

and the dimensionless shape function is

$$v_{\text{DDI}}^{\text{1D}}(u) = -2|u| + \sqrt{2\pi} (1 + u^2) e^{u^2/2} \text{erfc}(|u|/\sqrt{2}). \quad (77)$$

Here μ is the dipole moment, and θ_{DDI} is the angle between the dipoles (aligned by an external field) and the tube axis; by tuning θ_{DDI} , one effectively controls the strength of the dipolar interaction.

Following Ref. [58], we consider ^{162}Dy atoms ($\mu = 9.93\mu_B$) loaded into a two-dimensional lattice of one-dimensional tubes (lattice spacing $a_l = 370$ nm), each with average density $\bar{\rho} = 0.8 \mu\text{m}^{-1}$ (see sketch in Fig. 8). Transversal confinement is provided by harmonic trap with oscillator length $l_{\perp} = 57$ nm. In addition to intratube dipolar interactions, atoms in different tubes (i, j) interact with ones in the tube $(0, 0)$ via

$$V_{\text{inter}}^{i,j}(x) = \frac{\mu_0 \mu^2}{4\pi} \frac{1 - 3(\hat{r} \cdot \hat{B})^2}{r^3}, \quad (78)$$

where $\vec{r} = (x, i a_l, j a_l)$ and $\hat{B} = (\cos \theta_{\text{DDI}}, 0, \sin \theta_{\text{DDI}})$ denotes the magnetic field direction.

At $t = 0$, a Bragg pulse splits the atoms into two momentum components $\pm 2\hbar k_D$, with $k_D = 2\pi/\lambda$ and $\lambda = 741$ nm. The gas then prethermalizes to a state described by a reference density $\rho_{\text{Init}}(\theta)$ (Fig. 8), which closely matches the experimental observations at the end of the first stage of the dynamics.

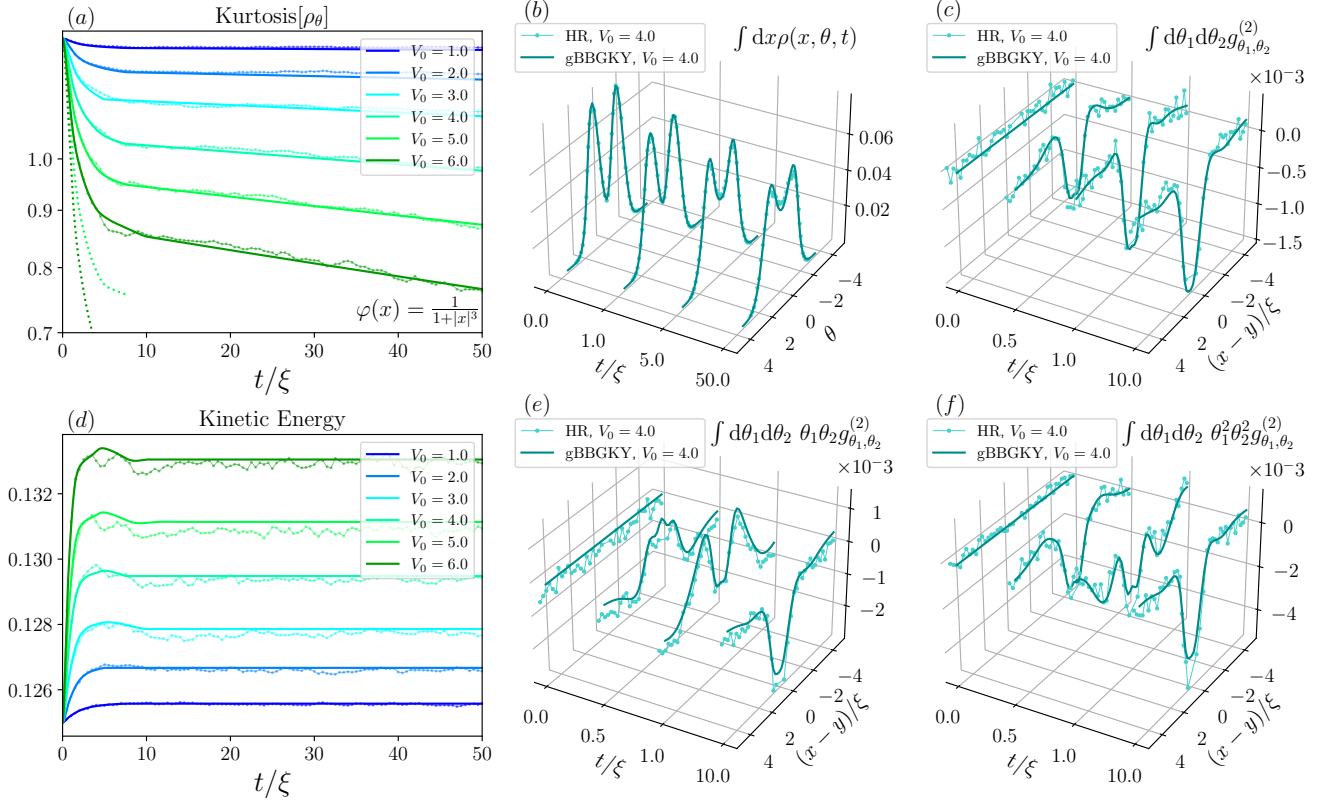


Figure 6. Numerical results for long-range interacting hard spheres (HR) with $a = 1$, $\xi = 20$ and $\varphi = 1/(1 + |x|^3)$, starting from the two boosted thermal states $\rho_{\beta; \theta_0}$ with $\beta = 2.5$, $\theta_0 = 1$, $\bar{\rho} = 0.2$. The system is homogeneous on a ring. In all the figures, the dots represent the molecular dynamics of the system, while the solid line represent the gBBGKY prediction. Figure (a) shows the dynamics of kurtosis of rapidity distribution, for $\xi = 20$ and with $V_0 = \{1, 2, \dots, 6\}$, as function of the rescaled time t/ξ . The numerical solution of gBBGKY hierarchy is detailed in Appendix C. In particular, for $t < t_L$ the solid line represents the solution of the two coupled equations (111) and (112), while for $t > t_L$ it shows the solution of generalised Landau equation (66). We observe a convincing agreement between molecular dynamics and theoretical predictions for all the values of integrability breaking potential V_0 . The additional dashed lines represent solutions of the standard BBGKY hierarchy, i.e. gBBGKY with $a = 0$, truncated to two first levels. As expected, this purely perturbative theory doesn't predict the correct system's evolution. Similarly, Fig. (d) shows the dynamics of Kinetic Energy $\int d\theta \rho_\theta \theta^2/2$. Fig. (b) shows the evolutions of distribution of rapidity ρ_θ for different times. Fig. (c, e, f) represent the evolution two-point correlation function $\int d\theta_1 d\theta_2 \theta_1^\alpha \theta_2^\alpha g_{\theta_1, \theta_2}^{(2)}$ respectively with $\alpha = \{0, 1, 2\}$. As expected, the gBBGKY is not only able to predict the evolution of both one and two-point functions.

Although the dipolar potential decays as $V_{\text{DDI}}^{\text{1D}}(x) \sim x^{-3}$ at large distances, it also contains a short-range component. To separate the integrable hard-core part from the genuine long-range contribution, we note that the LL coupling c has two parts: the van der Waals term

$$c_{\text{vdW}} = -\frac{2\hbar^2}{m a_{1\text{D}}}, \quad a_{1\text{D}} = -435 \text{ nm}, \quad (79)$$

and a dipolar term

$$c_{\text{DDI}}(\theta_{\text{DDI}}) = \frac{4V(\theta_{\text{DDI}})}{3}. \quad (80)$$

Hence $c(\theta_{\text{DDI}}) = c_{\text{vdW}} + c_{\text{DDI}}(\theta_{\text{DDI}})$.

To estimate the effect of the long-range potential we invoke the characteristic distance ξ . In the LL model the scattering shift defines an effective hard-sphere diameter

for the quasiparticle interaction [19, 141]

$$a(\theta - \theta') = \frac{2(c/m)}{(c/\hbar)^2 + (\theta - \theta')^2} \quad (81)$$

Estimating $\theta - \theta' \approx \hbar k_D/m$ (reflecting the initial momentum spread) and comparing with the dipolar length scales yields

$$\xi_{\text{intra}} = \frac{l_\perp}{a} = 0.81, \quad \xi_{\text{inter}} = \frac{a_l}{a} = 5.28. \quad (82)$$

Thus, intratube dipolar interactions can be absorbed into the contact coupling, while intertube interactions act as a truly long-range perturbation. This is consistent with the modeling of recent experimental results in [23, 94] where the authors show that intratube dipolar effects mostly renormalize the coupling c at equilibrium [23] and

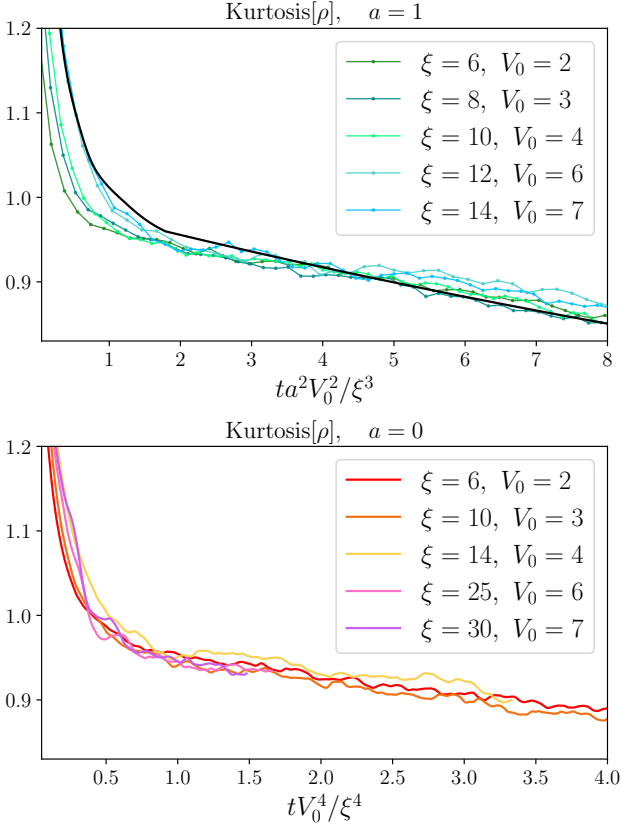


Figure 7. Numerical confirmation of the predicted thermalization rate $\propto a^2 V_0^2 / \xi^3$, when $a \neq 0$ and $\propto V_0^4 / \xi^4$, when $a = 0$. The two plots show the kurtosis of ρ_θ for different values of V_0 , ξ , in the case $a = 1$ (top) and $a = 0$ (bottom), respectively as function of the rescaled times $\bar{t} = ta^2 V_0^2 / \xi^3$ and $\bar{t} = tV_0^4 / \xi^4$. The dynamics firstly undergoes a fast small time evolution and then, for $\bar{t} > 1$, the late time thermalization dynamics. We show that, under this appropriate time scaling, at late time the dynamics induced by different potentials parameters perfectly collapse, confirming that the thermalization rate is $\propto a^2 V_0^2 / \xi^3$ for $a \neq 0$ and $\propto V_0^4 / \xi^4$ for $a = 0$. More precisely, in this figure we introduce a constant offset parameter that takes into account the differences in the small time behavior, in order to renormalize the starting point for the late time dynamics. The black curve represents the solution of gBBGKY for $\xi = 14$, $V_0 = 7$, being in good agreement with the numerical simulation.

during dynamics, slightly in contrast with the original interpretation in [58]. Moreover, we observe that summing the contributions to the Landau collision integral from tubes with indices $-2 \leq i, j \leq 2$ (we assume that each tube is in the same state) suffices to capture the relaxation rate, since more distant tubes have negligible effect.

To model the two-stage dynamics observed under the Bragg-split protocol (cf. the hard-rod gas), we adapt our theory to the experimental parameters. The key observable in [58] is the thermalization rate as a function of θ_{DDI} . We consider a measure of distance to (generalized) ther-

malization $\text{DT}(t)$ very similar to the one studied in [58]. We fit a Gaussian distribution $\tilde{\rho}_\theta(t)$ to the state $\rho_\theta(t)$ at each time t , then $\text{DT}(t)$ is computed as

$$\text{DT}(t) = \int d\theta (\rho_\theta(t) - \tilde{\rho}_\theta(t))^2, \quad (83)$$

measuring the distance of the state at time t from a Gaussian distribution. After the first phase of dynamics, an exponential decay is observed, with

$$\log \text{DT}(t) \approx -t/t_{\text{therm}} + \dots \quad (84)$$

We compute t_{therm} from the expression of the Landau collision integral, giving a very good agreement with the experimental rates, see Fig. 8, providing this way a first direct application of gBBGKY to an experimental setting. We should stress that in our approach, we track the Gaussianity of the rapidity distribution ρ_θ , whereas in [58] the momentum distribution is measured. However, these two are expected to be close due to high temperature of the state. This is especially clear in the final state ρ_{th} , see inset of Fig. 8 which is very well approximated by a Gaussian, just like final momentum distributions measured in [58].

What is more, our result can also be used to deduct important aspects of the thermalization rates. Indeed, a more conventional theoretical approach would linearize the collision integral around the thermal distribution ρ_{th} and study spectrum of the linearized collision operator. Equivalently, one can track perturbations in the conserved charges δq_n :

$$\partial_t \langle \delta q_i \rangle = -\Gamma_i^j \langle \delta q_j \rangle, \quad (85)$$

where matrix Γ is symmetric, positive semidefinite, and has three zero modes (particle number, momentum, energy). See [120] for the explicit expression and the derivation of matrix Γ . The first nonzero eigenvalue γ_{gap} sets $t_{\text{therm}} = \gamma_{\text{gap}}^{-1}$. However, this approach predicts thermalization times of order 10^3 s—far longer than experimentally accessible—as it captures the slowest decaying mode, whereas the current-state of the art experiments can only probe the fastest relaxation channels.

A. Fermi's Golden rule approach and check of the generalised Landau equation

The advantage of restricting to this case of perturbed Lieb-Liniger gas also offer us the possibility of giving a non-trivial check of the generalised Landau equation (66) and therefore of our gBBGKY via a different approach. Indeed, the dynamics of a quantum nearly integrable systems can be also addressed with the Fermi's Golden Rule (FGR), which can be used to write the Boltzmann equation [47, 48, 142]. In this section, we give a short review of these results and compare them to generalized Landau equation obtained from our hierarchy.

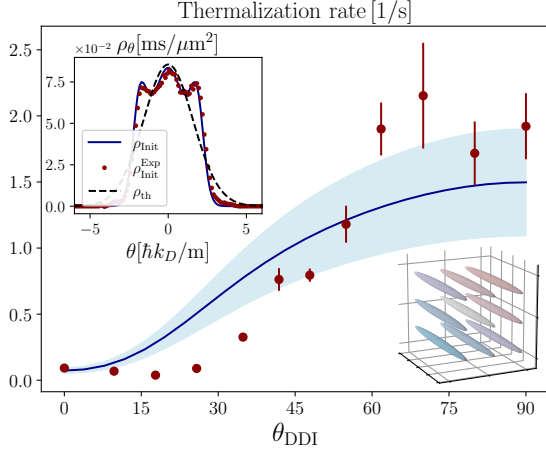


Figure 8. Thermalization rate (blue solid line) as defined in (84) for the experimental parameters of [58], obtained from gBBGKY, together with shaded area indicating uncertainty resulting from 13% density variation in the experiment. The bottom right inset shows the geometry of the quasi-one-dimensional tubes loaded with dipolar atoms. The top left inset shows the initial state ρ_{init} (solid line) compared with the experimentally measured initial state (dots) (determined from the data presented in Figs.2(c), 2(d) and 2(e) of [58]) together with the final, thermal state ρ_{th} (dashed line).

The FGR approach relies heavily on the knowledge of form factors, which are the matrix elements of the perturbing operator computed for integrable theory. In the case of integrability breaking perturbation in the form of additional long-range interactions, the relevant form factors are one- and two-point functions of the density operator. In general, these objects are not known, however in the limit of small excitations between the states, there are exact results in the LL model [143–147]. As we will see, the small excitation limit will be automatically guaranteed by the long-range limit of the integrability-breaking potential, rendering FGR Boltzmann kernel computable in the regime of interest of the present work.

Once again we assume that the Hamiltonian is $\hat{H} = \hat{H}_{\text{LL}} + \hat{V}$ where $\hat{V} = \int dx \hat{v}(x)$ is the perturbation:

$$\hat{v}(x) = \frac{1}{2} \int dy V(x-y) \hat{q}_0(x) \hat{q}_0(y). \quad (86)$$

The LL model has fermionic statistics and we define density of holes $\rho^h = \rho^t - \rho$. In homogeneous system, the dynamics is given by Boltzmann-like equation [47, 48]

$$\partial_t \rho_\theta = \mathcal{I}[\rho](\theta) = \int d\mathbf{p} d\mathbf{h} \mathbb{F}_{\theta, \mathbf{p}} B(\mathbf{p} \rightarrow \mathbf{h}) [\rho_{\mathbf{p}}^h \rho_{\mathbf{h}} - \rho_{\mathbf{h}}^h \rho_{\mathbf{p}}], \quad (87)$$

with $\rho^h(\mathbf{p}) = \prod_{p \in \mathbf{p}} \rho_p^h$, $\rho(\mathbf{h}) = \prod_{h \in \mathbf{h}} \rho_h$ and the measure involving different particle-hole (particle-hole) scattering

processes reads

$$d\mathbf{p} d\mathbf{h} = \sum_{N=0}^{\infty} \frac{1}{(N!)^2} \prod_{i=1}^N dp_i dh_i. \quad (88)$$

Apart from the density factors there are two ingredients in (87). Firstly, function $B(\mathbf{p} \rightarrow \mathbf{h})$ is related to particle-hole form-factors of the perturbing operator $\langle \rho | v | \rho; \mathbf{p}, \mathbf{h} \rangle$ [148]

$$B(\mathbf{p} \rightarrow \mathbf{h}) = (2\pi)^2 \delta(k) \delta(\varepsilon) |\langle \rho | v | \rho; \mathbf{p}, \mathbf{h} \rangle|^2. \quad (89)$$

Assuming particle-hole symmetry we have $B(\mathbf{p} \rightarrow \mathbf{h}) = B(\mathbf{h} \rightarrow \mathbf{p})$. The Dressed energy and momentum are

$$k_\theta = \theta - \int d\mu n_\mu F_{\mu, \theta}, \quad (90)$$

$$\varepsilon_\theta = \frac{\theta^2}{2} - \int d\mu \mu n_\mu F_{\mu, \theta},$$

where $F_{\mu, \theta}$ is the backflow function [149] of the underlying integrable model. Explicitly, the terms inside Dirac delta functions are then $k = \sum_{p \in \mathbf{p}} k_p - \sum_{h \in \mathbf{h}} k_h$, $\varepsilon = \sum_{p \in \mathbf{p}} \varepsilon_p - \sum_{h \in \mathbf{h}} \varepsilon_h$. For later use, we note that $k'_\lambda = 2\pi \rho_\lambda^t$ and $v_\lambda = \varepsilon'_\lambda / k'_\lambda$. The second ingredient in (87) is the backflow effect implemented by the term $\mathbb{F}_{\theta, \mathbf{p}}$ which reads

$$\mathbb{F}_{\theta, \mathbf{p}} = \sum_{p \in \mathbf{p}} \mathbb{F}_{\theta, p}, \quad \mathbb{F}_{\theta, p} = \delta_{\theta, p} + \partial_\theta (n_\theta F_{\theta, p}). \quad (91)$$

The main practical difficulty for evaluating the collision term is the resulting infinite sum over different numbers of particle-hole excitations. Indeed, writing the sum in integration measure explicitly, we have

$$\mathcal{I}[\rho](\theta) = \mathbb{F}_{\theta, \nu} \sum_{m=1}^{\infty} \mathcal{I}_0^{(m)}[\rho](\nu), \quad (92)$$

where we have factored out the dressing operator $\mathbb{F}$ defining bare collision integrals $\mathcal{I}_0^{(m)}$. Due to this structure, an additional small parameter is needed to effectively truncate the sum. Such situation will happen in our case of long-range interactions, where the potential allows only for small momentum transfer, which is equivalent to considering ξ as a large parameter. This allows us to truncate the sum to the leading term, which corresponds to 3-particle-hole ($m = 3$) excitations. The $m = 2$ term is zero, due to the kinetic blocking effect described in Sec. IV A.

From now on, we will assume that all transferred momenta $p_i - h_i$ are small (this approximation is discussed in the Supplementary Materials [120]). We therefore introduce new variables $\lambda_i = (p_i + h_i)/2$ and $q_i = k(p_i) - k(h_i)$ and obtain $m = 3$ collision integral in the form

$$\mathcal{I}_0^{(3)}[\rho](\theta) = \frac{1}{3!} \frac{1}{(2\pi)^3} \partial_\theta \int d\lambda d\mathbf{q} B(\lambda, \mathbf{q}) \times \left(\sum_{i=1}^3 \frac{q_i}{2k'_{\lambda_i}} \delta(\theta - \lambda_i) \right) \left(\sum_{i=1}^3 q_i \frac{\varepsilon'_{\lambda_i}}{k'_{\lambda_i}} \right), \quad (93)$$

with integration measure $d\lambda d\mathbf{q} = \prod_{i=1}^3 d\lambda_i d\mathbf{q}_i \rho_{\lambda_i} f_{\lambda_i}$ where for fermionic statistics $f = 1 - n$. Moreover, in that limit the back-flow operator becomes $\mathbb{F}_{\theta,p} = \frac{q}{2k_{\lambda}} \partial_{\theta} R_{\theta,\lambda}^{-1}$ and implements the usual dressing procedure. To move forward, we have to analyze now $m = 3$ form factors of the density-density operator. Consider form-factor of bi-local operator $\langle \rho | \hat{q}_0(x) \hat{q}_0(0) | \rho; \mathbf{p}, \mathbf{h} \rangle$. We plug the resolution of the identity between the two operators to obtain

$$\langle \rho | \hat{q}_0(x) \hat{q}_0(0) | \rho; \mathbf{p}, \mathbf{h} \rangle = \sum_{\alpha} \langle \rho | \hat{q}_0(x) | \rho; \alpha \rangle \langle \rho; \alpha | \hat{q}_0(0) | \rho; \mathbf{p}, \mathbf{h} \rangle. \quad (94)$$

The leading contribution to the composite form-factor comes then from bipartite partitions of the 3-particle-hole excitation $(\mathbf{p}, \mathbf{h})$. More explicitly, for $(\mathbf{p}, \mathbf{h}) = (p_1, p_2, p_3, h_1, h_2, h_3)$ there are 6 relevant partitions. Three of them are 1-particle-hole excitations: $\alpha = (p_1, h_1)$, $\alpha = (p_2, h_2)$ and $\alpha = (p_3, h_3)$. The other three are 2-particle-hole excitations $\alpha = (p_1, p_2, h_1, h_2)$, $\alpha = (p_2, p_3, h_2, h_3)$ and $\alpha = (p_1, p_3, h_1, h_3)$. Using reparametrization invariance of the thermodynamic form-factors [150], a property that an excitation common to the bra and ket can be absorbed into the thermodynamic state ρ , the contributions from 1-particle-hole and 2-particle-hole excitations are pairwise identical up to the x -dependent term. The expressions for the form-factors (assuming small momentum transfer) are [143, 145]

$$\langle \rho | \hat{q}_0(0) | \rho; p_1, h_1 \rangle = k'_{\lambda_1}, \quad (95)$$

$$\langle \rho | \hat{q}_0(0) | \rho; p_1, p_2, h_1, h_2 \rangle = 2\pi T_{\lambda_1, \lambda_2}^{\text{dr}} \frac{(q_1 + q_2)^2}{q_1 q_2}, \quad (96)$$

and lead to six relevant contributions that can be written as follows

$$\langle \rho | \hat{v} | \rho; \mathbf{p}, \mathbf{h} \rangle = (2\pi)^2 \left(\hat{V}(q_3) T_{\lambda_1 \lambda_2}^{\text{dr}} k'_{\lambda_3} \frac{(q_1 + q_2)^2}{q_1 q_2} \right)_{(1,2,3)}. \quad (97)$$

The factor B appearing in the collision integral is then

$$B(\boldsymbol{\lambda}, \mathbf{q}) = (2\pi)^4 \delta(q) \delta(vq) \left[\left(T_{\lambda_1 \lambda_2}^{\text{dr}} k'_{\lambda_3} \hat{V}(q_3) \frac{q_3^2}{q_1 q_2} \right)_{(1,2,3)} \right]^2, \quad (98)$$

where we defined $\delta(q) = \delta(q_1 + q_2 + q_3)$, $\delta(vq) = \delta(v_1 q_1 + v_2 q_2 + v_3 q_3)$ and used that $\hat{V}(k) = \hat{V}(-k)$. Crucially, expanding the square we find two types of contributions, which correspond one-to-one to the two contributions in eq. (66). The cross ones are proportional to $(\hat{V}(q_i))^2$, whereas the self terms involve $\hat{V}(q_i) \hat{V}(q_{j \neq i})$ factors. The collision integral has consequently two contributions, which turn out to be exactly equal to the two contributions of eq. (67) and (68), therefore giving an independent, and mathematically highly non-trivial, verification of the gBBGKY approach.

VII. CONCLUSIONS

In this work, we have extended the celebrated BBGKY hierarchy to systems with integrable contact interactions. We derived the hierarchy from a *correlated fluid-cell ensemble*, which enables accurate characterization of multi-point correlations both in the standard BBGKY framework (where the unperturbed Hamiltonian is free) and in more general settings.

For classical hard spheres with long-range interactions, we tested our theory against molecular-dynamics simulations and observed excellent agreement, even at relatively strong coupling. For quantum gases described by the Lieb–Liniger model with long-range interactions, we showed that the generalized Landau equation aligns with the Fermi golden rule approach, providing a rigorous validation of both our theory and the thermalization hypothesis. Finally, we established a theoretical foundation for the experimental results of Ref. [58], demonstrating that dynamics in these quasi-one-dimensional systems are encompassed by gBBGKY hierarchy. The latter, in the presence of strong contact interactions, accurately captures short-time dynamics during the *prethermal* regime, driven by the rapid build-up of correlations that our formalism tracks precisely. Conventional kinetic equations fail to describe this transient phase, which is often the only accessible regime in experiments and simulations. Thus, the gBBGKY provides both a conceptual framework and quantitative predictions for prethermal behavior.

At late times, gBBGKY overcomes the well-known kinetic blocking in one dimension. For perturbed free systems, thermalization is driven by three-body scatterings generated by the external potential V , yielding a rate that scales as V_0^4 . By contrast, with contact interactions, the combined effect of long-range interactions and convective diffusion among particles produces a thermalization rate of order $(Va)^2$. Importantly, such thermalization is not complete (*generalized thermalisation*) as only one- and two-point functions reach their canonical thermal values. Higher-order correlations remain far from thermal over the same window of times and retain their integrable structure, leading to finite long-range correlations that would be absent in a thermal Gibbs ensemble but which are instead correctly captured by the correlated fluid-cell ensemble.

Our method is broadly applicable: its central ingredient is a thermodynamic description of the unperturbed integrable model, yielding a universal framework for diverse classical and quantum systems. Unlike approaches based on Fermi's golden rule and explicit matrix elements, the gBBGKY hierarchy enables direct access to the Boltzmann scattering integrals and the time evolution of multi-point correlations in a wide range of perturbed integrable models. It can be applied to lattice systems—such as long-range spin chains and fermionic models [90, 151–154]—inspired by recent experiments. Additionally, it opens a path to connect kinetic theories of interacting waves (via wave turbulence) [59, 155] with general non-

linear wave systems [156–158]. We expect that the gB-BGKY hierarchy will be an important tool in further studies of thermalization and prethermalization across a wide range of physical settings, from low-dimensional cold-atom experiments to complex condensed-matter and optical systems with tunable interactions.

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APPENDIX

A. CORRELATION FUNCTIONS IN THE CORRELATED FLUID CELL ENSEMBLE

In this appendix we will derive the formulas for correlation functions in the ensemble (14). We will also connect the various ingredients of these correlators to Lagrange multipliers of the ensemble. The main idea behind this calculation consists on expanding the measure (14) in $1/\xi$ (according to (15)) in order to compute the corrections to the local GGE prediction. Firstly, we consider the equal

time two-point function

$$\begin{aligned} \langle q_i(x)q_j(x') \rangle^c &= \delta(x-x')C_{i,j} + \\ &+ \frac{1}{2} \int dy dy' \beta_{(2)}^{k,l}(y, y') \langle \delta q_k(y) \delta q_l(y') q_i(x) q_j(x') \rangle_1. \end{aligned} \quad (99)$$

In order to write the correlator in a cumulant expansion, we now use the relation

$$\begin{aligned} \langle \delta a \delta b \delta c \rangle &= \langle abcd \rangle^c + \langle ac \rangle^c \langle bd \rangle^c + \\ &+ \langle ad \rangle^c \langle bc \rangle^c + \langle acd \rangle^c \langle \delta b \rangle + \langle bcd \rangle^c \langle \delta a \rangle, \end{aligned} \quad (100)$$

where we observe that $\langle \delta q_i \rangle = 0$ and that $\langle abcd \rangle^c$ is higher order in ξ^{-1} . Hence, Eq. (99) reduces to

$$\begin{aligned} &= \delta(x-x')C_{i,j} + \\ &+ \int dy dy' \beta_{(2)}^{k,l}(y, y') \langle q_k(y) q_i(x) \rangle_1^c \langle q_l(y') q_j(x') \rangle_1^c \\ &= \delta(x-x')C_{i,j} + \beta_{(2)}^{k,l}(x, x') C_{i,k}(x) C_{j,l}(x') \\ &= \delta(x-x')C_{i,j} + g_{i,j}^{(2)}(x, x'), \end{aligned} \quad (101)$$

where in the last line we identified $g_{i,j}^{(2)}(x, x') \equiv \beta_{(2)}^{k,l}(x, x') C_{i,k}(x) C_{j,l}(x') + O(\xi^{-2})$. Also, in the last two lines, we explicitly computed the correlation functions on the local GGE.

We now compute the three-point function using the same technique

$$\begin{aligned} \langle q_i(x)q_j(x')q_k(x'') \rangle^c &= \delta(x-x')\delta(x-x'')C_{i,j,k}(x) + \frac{1}{2} \int dy dy' \beta_{(2)}^{m,n}(y, y') \langle \delta q_m(y) \delta q_n(y') q_i(x) q_j(x') q_k(x'') \rangle_1 \\ &+ \frac{1}{3} \int dy dy' dy'' \beta_{(3)}^{m,n,p}(y, y', y'') \langle \delta q_m(y) \delta q_n(y') \delta q_p(y'') q_i(x) q_j(x') q_k(x'') \rangle_1 \\ &= \delta(x-x')\delta(x-x'')C_{i,j,k}(x) + \left[\delta(x'-x'')\beta_{(2)}^{m,n}(x, x') C_{m,i}(x) C_{n,j,k}(x') \right]_{(i,j,k)} \\ &+ \beta_{(3)}^{m,n,p}(y, y') C_{i,m}(x) C_{j,n}(x') C_{k,p}(x'') \\ &= \delta(x-x')\delta(x-x'')C_{i,j,k}(x) + \left[\delta(x'-x'')g_{i,q}^{(2)}(x, x') C^{q,n}(x') C_{n,j,k}(x') \right]_{(i,j,k)} + g_{i,j,k}^{(3)}(x, x', x''). \end{aligned} \quad (102)$$

In particular, in the last line we defined $g_{i,j,k}^{(3)}(x, x', x'') = \beta_{(3)}^{m,n,p}(y, y') C_{i,m}(x) C_{j,n}(x') C_{k,p}(x'')$, and where we used the formula for $g^{(2)}$ derived in Eq. (101). Finally, we observe that the last line coincides with Eq. (20).

B. gBBGKY HIERARCHY AND GENERALISED LANDAU EQUATION FOR LONG-RANGE INTERACTING HARD SPHERES

In this section we explicitly write the main hydrodynamic functions for the hard spheres model that permits to explicitly write the gBBGKY hierarchy and the generalised Landau equation in this simpler case. The hard spheres gas is a classical system of billiard balls with radius a and unitary mass. The rapidities of the particles

in the system are defined as the set of particles velocities. The scattering shift is simply $T_{\theta_1, \theta_2} = -a/2\pi$. Hence, the rotation matrix is given by $R_{\theta_1, \theta_2} = \delta_{\theta_1, \theta_2} + a/2\pi 1_{\theta_1} n_{\theta_2}$, giving $\rho^\dagger = (1 + a\bar{\rho})/2\pi$ and $n_\theta = 2\pi\rho_\theta/(1 - a\bar{\rho})$, being $\bar{\rho} = \int d\theta \rho_\theta$. Since in this model the rotation matrix can be explicitly inverted, the dressing operator can be computed explicitly $R_{\theta_1, \theta_2}^{-1} = \delta_{\theta_1, \theta_2} - a 1_{\theta_1} \rho_{\theta_2}$, as well as the effective velocity

$$v_{\theta_1}^{\text{eff}} = \frac{\theta_1 - a \int d\theta_2 \theta_2 \rho_{\theta_2}}{1 - a\bar{\rho}}. \quad (103)$$

The susceptibility matrix for this system is given by

$$C_{\theta_1, \theta_2} = \delta_{\theta_1, \theta_2} \rho_{\theta_1} + a\bar{\rho}(a\bar{\rho} - 2)\rho_{\theta_1} \rho_{\theta_2}. \quad (104)$$

Hence, we can also compute its functional derivative with respect to the density

$$C_{\theta_1, \theta_2}^{\theta_3} = \delta_{\theta_1, \theta_3} \delta_{\theta_2, \theta_3} + a^2 \rho_{\theta_1} \rho_{\theta_2} 1_{\theta_3} + a(a\bar{\rho} - 2)\rho_{\theta_2} \delta_{\theta_1, \theta_3} + a(a\bar{\rho} - 2)\rho_{\theta_1} \delta_{\theta_2, \theta_3}. \quad (105)$$

This tensor is entering in the gBBGKY hierarchy as

$$\int d\theta_3 C_{\theta_2, \theta_3}^\gamma g_{\theta_1, \gamma}^{(2)}(x_1, x_2) = (1 - a\bar{\rho})^2 g_{\theta_1, \theta_2}^{(2)}(x_1, x_2) - 2a(1 - a\bar{\rho})\rho_{\theta_1} \int d\gamma g_{\theta_3, \theta_2}^{(2)}(x_1, x_2). \quad (106)$$

The flux Jacobian $A_{\theta_1}^{\theta_2}$ can be easily computed using the explicit relations for the dressing operator, the rotation matrix and the effective velocity. We also define $\mathcal{D}$ as the linear response diffusion matrix of the system, that for hard spheres reads

$$\mathcal{D}_{\theta_1}^{\theta_2} = a^2(1 - a\bar{\rho}) \left[\delta_{\theta_1, \theta_2} \int d\theta_3 \rho_{\theta_3} |v_{\theta_1, \theta_3}| - \rho_{\theta_1} |v_{\theta_1, \theta_2}| \right]. \quad (107)$$

Finally, we can explicitly write the generalised Landau equation for the hard spheres model. For simplicity, we write it considering systems with vanishing total momentum $\int d\theta_2 \theta_2 \rho_{\theta_2} = 0$

$$\begin{aligned} \mathcal{I}_{\theta_1}^{\text{cross}} &= \frac{1}{2\pi} a^2 (1 - a\bar{\rho})^3 \int dk k^2 (\hat{V}(k))^2 \\ &\times \int d\theta_2 d\theta_3 \left(\frac{|v_{\theta_3 \theta_1}^{\text{br}}|}{v_{\theta_2 \theta_3}^{\text{br}} (v_{\theta_1 \theta_2}^{\text{br}})^2} - \frac{|v_{\theta_2 \theta_3}^{\text{br}}|}{v_{\theta_1 \theta_2}^{\text{br}} (v_{\theta_3 \theta_1}^{\text{br}})^2} \right) \\ &\times \mathcal{B}_{\theta_1, \theta_2, \theta_3}^{\text{br}}, \end{aligned} \quad (108)$$

$$\begin{aligned} \mathcal{I}_{\theta_1}^{\text{self}} &= \frac{1}{2\pi} a^2 (1 - a\bar{\rho})^3 \int d\theta_3 d\theta_2 \int dk k^2 \hat{V}(v_{\theta_1 \theta_3}^{\text{br}} k) \\ &\times \left(\hat{V}(v_{\theta_1 \theta_2}^{\text{br}} k) \frac{v_{\theta_3 \theta_1}^{\text{br}} v_{\theta_1 \theta_2}^{\text{br}}}{v_{\theta_3 \theta_2}^{\text{br}}} - 2\hat{V}(v_{\theta_2 \theta_3}^{\text{br}} k) \frac{(v_{\theta_2 \theta_3}^{\text{br}})^2 v_{\theta_3 \theta_1}^{\text{br}}}{(v_{\theta_1 \theta_2}^{\text{br}})^2} \right) \\ &\times \mathcal{B}_{\theta_1, \theta_2, \theta_3}^{\text{br}}, \end{aligned} \quad (109)$$

where we defined the bare velocity difference as $v_{\theta_1, \theta_2}^{\text{br}} = \theta_1 - \theta_2$ and where

$$\begin{aligned} \mathcal{B}_{\theta_1, \theta_2, \theta_3}^{\text{br}} &\equiv v_{\theta_3 \theta_2}^{\text{br}} \partial_{\theta_1} \rho_{\theta_1} \rho_{\theta_3} \rho_{\theta_2} + \\ &+ v_{\theta_2 \theta_1}^{\text{br}} \rho_{\theta_1} \partial_{\theta_3} \rho_{\theta_3} \rho_{\theta_2} + v_{\theta_1 \theta_3}^{\text{br}} \rho_{\theta_1} \rho_{\theta_3} \partial_{\theta_2} \rho_{\theta_2}. \end{aligned} \quad (110)$$

C. ADDITIONAL DETAILS ON THE NUMERICAL SOLUTION OF gBBGKY HIERARCHY

In this section, we detail the numerical methods used to produce the results shown in Fig. 6. Let us start from the microscopic simulation of hard spheres system with long-range interactions. The molecular dynamics is solved using the algorithm introduced in [159], combining symplectic integration of equation of motion and the explicit solution of hard core collisions. More precisely, we proceeded setting a time interval Δt_S for the symplectic integrator and computing the minimal time before the first hard core interaction in the system $\Delta t_{\text{Int}} \approx \min_i \{(x_{i+1} - x_i)/(\theta_{i+1} - \theta_i)\}$. At any time, if $\Delta t_S > \Delta t_{\text{Int}}$, we perform the symplectic integration of equation of motion using an explicit Runge-Kutta method of order 8 implemented by the *DOP853* method in `scipy.integrate.solve_ivp` function with time step Δt_S . If, instead, $\Delta t_S < \Delta t_{\text{Int}}$, we perform the symplectic integration with time step Δt_{Int} , and then we explicitly compute the hard core collision between the two colliding rods, i.e. exchanging the colliding rods' velocities. In particular, we choose $\Delta t_S = 5 \times 10^{-3}$. This value guarantees the correct convergence of the numerical method for the set of parameters considered in the simulations. The results are averaged over $N_{\text{sym}} = 1000$ initial configurations. The initial state is a GGE state, see [141] for a detailed explanation.

We now discuss the solution of gBBGKY equations. In the results shown in Fig. 6, we split the dynamics in two stages, solving the system separately for $t < t_L$ and $t > t_L$. For the small time dynamics $t < t_L$, we solved the following two gBBGKY equations for a hard spheres system (see Appendix B for the explicit functions)

$$\partial_t \rho_{\theta_1} = \frac{1}{\xi} \int dx_2 d\theta_2 V'(x_1 - x_2) \partial_{\theta_1} g_{\theta_1, \theta_2}^{(2)}(x_1, x_2), \quad (111)$$

$$\begin{aligned}
& \partial_t g_{\theta_1, \theta_2}^{(2)}(x_1, x_2) + \\
& + \left[\partial_{x_1} A_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2)}(x_1, x_2) - \frac{1}{2} \partial_{x_1} \mathfrak{D}_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2)} \right]_{(1,2)} = \\
& = \int d\theta_3 \partial_{\theta_1} \left[\int dx_3 V'(x_1 - x_3) \rho_{\theta_1} g_{\theta_2, \theta_3}^{(2)}(x_2, x_3) \right. \\
& + V'(x_1 - x_2) \left(\rho_{\theta_1} C_{\theta_2, \theta_3} + C_{\theta_2, \theta_3}^\gamma g_{\theta_1, \gamma}^{(2)}(x_1, x_2) \right) \Big]_{(1,2)} \\
& + \delta(x_1 - x_2) \int dx_3 d\theta_3 V'(x_1 - x_3) \left(-C_{\theta_1, \theta_2}^\gamma \partial_\gamma \right. \\
& \quad \left. + (\partial_{\theta_1} + \partial_{\theta_2}) C_{\theta_1, \theta_2}^\gamma \right) g_{\gamma, \theta_3}^{(2)}(x_1, x_3),
\end{aligned} \tag{112}$$

using a 4-th order Runge-Kutta method with time interval = 0.02. In particular, we used a grid of 100 points in space and rapidity in the intervals, with $-10\xi < x < 10\xi$ and $-7 < \theta < 7$. The space interval has been chosen large enough to have vanishing correlations at the boundaries at any time, in order to avoid spurious reflective effects in the dynamics of $g^{(2)}$. For $t > t_L$, we instead solved the generalised Landau equation. The precise value of t_L has been considered as a "fitting" parameter, hence fixed in

the interval $\xi < t_L < \xi^2$, in order to accurately reproduce the molecular dynamics results.

For the data presented in Fig. 8 we use grid of 150 points in rapidity space with $-4.4\hbar k_d/m \leq \theta \leq 4.4\hbar k_d/m$. The initial state presented in inset of Fig. 8 is then evolved for a short time evolved using expression for inter-tube collision integral provided in SM. From that dynamics, we extract the rate of decay of log DT .

D. gBBGKY HIERARCHY FOR INHOMOGENEOUS SYSTEMS

In this section we write the first equations of gBBGKY hierarchy for inhomogeneous systems. The derivation follows exactly the same procedure introduced in Sec. III. For brevity reasons, we show here only the first two equations of the hierarchy. higher-order equations can be derived using the same technique. We first consider the set of generic equations (11) and (12). Hence, the next steps are to use the cumulant expansion for charge-currents correlators (see Sec. IIIB) and to factorize the singular component of few point functions (see Sec. III A). After these steps, the equations read

$$\partial_t \langle q_{i_1}(x_1) \rangle + \partial_{x_1} \left(A_{i_1}^k \langle q_k(x_1) \rangle + \frac{1}{2} H_{i_1}^{j,k} g_{j,k}^{(2),\text{sym}}(x_1^+, x_1^-) \right) = - \int dx_2 V'(x_1 - x_2) A_{i_1,0}^k \left(\langle q_k(x_1) \rangle \langle q_0(x_2) \rangle + g_{k,0}^{(2)}(x_1, x_2) \right), \tag{113}$$

$$\begin{aligned}
& \partial_t g_{i_1, i_2}^{(2)}(x_1, x_2) + \left[\partial_{x_1} A_{i_1}^j g_{j, i_2}^{(2)}(x_1, x_2) + \frac{1}{2} \partial_{x_1} H_{i_1}^{j,k} g_{j,k, i_2}^{(3),\text{sym}}(x_1^+, x_1^-, x_2) \right]_{(1,2)} = \\
& = -\delta(x_1 - x_2) \langle q_{i_1} q_{i_2} j_k^- \rangle_1^c C^{kl} \partial_{x_1} \langle q_l(x_1) \rangle - \left[V'(x_1 - x_2) A_{i_1,0}^j \left(\langle q_j(x_1) \rangle C_{i_2,0} + g_{j,k}^{(2)}(x_1, x_2) C^{k,m} C_{m, i_2,0} \right) + \right. \\
& + \int dx_3 V'(x_1 - x_3) A_{i_1,0}^j \left(\langle q_j(x_1) \rangle g_{i_2,0}^{(2)}(x_2, x_3) + \langle q_0(x_3) \rangle g_{j, i_2}^{(2)}(x_1, x_2) + g_{j, i_2,0}^{(3)} \right) \Big]_{(1,2)} \\
& - \delta(x_1 - x_2) \int dx_3 V'(x_1 - x_3) \left[-C_{i_1, i_2, j} C^{j,k} A_{k,0}^n + \left(A_{i_1,0}^j \delta_{i_2}^k + \delta_{i_1}^j A_{i_2,0}^k \right) C_{j,k,m} C^{m,n} \right] g_{n,0}^{(2)}(x_1, x_3).
\end{aligned} \tag{114}$$

with $j_i^- = j_i - A_i^j q_j$. In the last equation, we can observe few important differences with respect to the homogeneous case. Firstly, the dynamical equation for density (113) gets more involved, since now it non-trivial flow of particles are also allowed. It is important to observe that the first term on RHS is a generalization to the celebrated Vlasov forcing term. We also observe that Eq. (113) is coupled to two-point function also through the second term in currents cumulant expansion, in the exact way as for purely integrable evolution, see the recent results in [126]. The equation for two-point function (114), has an additional non-trivial contribution proportional to a Dirac delta function, originating from the complex propagation of initial local fluctuations, also already introduced in [126].

E. EXTRA DETAILS ON THE DERIVATION OF THE GBBGKY HIERARCHY

In this section we detail the main steps needed in order to derive Eq. (32), (33) and (34) from the hierarchy Eq. (11), (12) and (13) written in terms of complete correlation functions. Firstly, we specialize these equations to homogeneous systems, hence Eq. (32) gets particularly simple for q_i being independent on the space coordinate. In addition, the disconnected forcing term is vanishing because of the symmetry of the potential. Similarly, the singular contribution to two-point function is not entering in the equation since $\varphi'(0) = 0$. Eq. (33) describe the evolution of regular connected two-point function. This equation is derived from (12) by performing three main steps: expanding currents through hydrodynamic

projections, subtract disconnected component and finally separate singular and regular part of correlators. The most involved part of the calculation is related to the calculation of singular terms in RHS, i.e. proportional to the Dirac delta. Hence in this section we describe the main steps of their derivation. Firstly, we compute the contributions coming from the delta part in LHS of (12)

$$\begin{aligned} \delta(x_1 - x_2) \partial_t C_{i_1, i_2} + \left[\partial_{x_1} \delta(x_1 - x_2) A_{i_1}^j C_{j, i_2} \right] = \\ = \delta(x_1 - x_2) \int dx_3 V'(x_1 - x_3) \times \\ \times C_{i_1, i_2, j} C^{j, k} A_{k, 0}^n g_{n, 0}^{(2)}(x_1, x_3), \end{aligned} \quad (115)$$

where we used $\delta C_{i_1, i_2} / \langle \delta q_k \rangle_1 = C_{i_1, i_2, j} C^{j, k}$ and where we used Eq. (32). The additional singular contribution to the RHS of (12) comes from the singular part of the three-point function $\langle j_{i, 0}(x) q_j(y) q_0 \rangle^c$ for coinciding x, y . All the remaining delta contributions from RHS are integrated, i.e. appear as regular terms. We now focus on Eq. (34). The procedure to derive this equation follows the same logic as Eq. (33), but here we simply stop at the leading order terms in $O(V_0, 1/\xi)$. Firstly, we observe that in the case of three-point function the hydrodynamic projection formula is more involved already for Euler scales as, for any local observable $\hat{o}$, we have [160]

$$\begin{aligned} \langle o(x) q_j(y) q_k(z) \rangle^c = \frac{\delta \langle o(x) \rangle_1}{\delta \langle q_i(x) \rangle_1} \langle q_i(x) q_j(y) q_k(z) \rangle^c + \\ + \frac{\delta^2 \langle o(x) \rangle_1}{\delta \langle q_i(x) \rangle_1 \delta \langle q_m(z) \rangle_1} \langle q_m(x) q_k(z) \rangle^c \langle q_i(x) q_j(y) \rangle^c. \end{aligned} \quad (116)$$

More precisely, in this equation, the regular part of second term on RHS appears as higher-order in V_0 with respect to the regular part of the first one, hence we did not explicitly include in Eq. (34). Then, we proceed separating the singular part from it. In doing that we used the following simplifications coming from homogeneity: $\partial_t C_{i, j, k} \sim O(V_0^2)$, $\partial_x C_{i, j, k} = 0$.

A. Derivation of the equation for the three-point functions

In this section we derive the RHS of Eq (49). As a starting point, we consider the RHS of Eq. (34). In particular, it turns out to be writable in a more convenient way as

$$\begin{aligned} A_i^m C_{m, j, p} C^{p, q} - C_{i, j, m} C^{m, p} A_p^q + H_i^{m, q} C_{m, j} = \\ = \frac{\delta \langle j_i \rangle_1}{\delta \langle q_m \rangle_1} \frac{\delta}{\delta \langle q_q \rangle_1} \frac{\delta \langle q_m \rangle_1}{\delta \beta^j} - \frac{\delta}{\delta \langle q_p \rangle_1} \left[\frac{\delta \langle q_i \rangle_1}{\delta \beta^j} \right] \frac{\delta \langle j_p \rangle_1}{\delta \langle q_q \rangle_1} + \\ + \frac{\delta^2 \langle j_i \rangle_1}{\delta \langle q_m \rangle_1 \delta \langle q_q \rangle_1} \frac{\delta \langle q_m \rangle_1}{\delta \beta^j} = \\ = \frac{\delta^2 \langle j_i \rangle_1}{\delta \beta^j \delta \beta^p} \frac{\delta \beta^p}{\delta \langle q_q \rangle_1} - \frac{\delta}{\delta \langle q_p \rangle_1} \left[\frac{\delta \langle q_i \rangle_1}{\delta \beta^j} \right] \frac{\delta \langle j_p \rangle_1}{\delta \langle q_q \rangle_1} \\ = \langle q_i q_j j_p \rangle_1^c C^{p, q} - \langle q_i q_j q_m \rangle_1^c C^{m, p} A_p^q = \langle q_i q_j j_p \rangle_1^c C^{p, q}, \end{aligned} \quad (117)$$

where we used that $AC = CA^t$. Here we defined $j_i^- = j_i - A_i^j q_j$. Hence, as shown in [141], in quasiparticle picture it turns out to be equal to

$$\langle q_i q_j j_p^- \rangle_1^c C^{p, q} = \int d\theta_1 d\theta_2 h_i(\theta_1) h_j(\theta_2) M_{\theta_1, \theta_2}^{\theta_3} h^q(\theta_3), \quad (118)$$

with the M tensor defined in (46).

B. Solutions with Green's function method

The aim of this appendix is to justify the solution (60), which is crucial for derivation of generalized Landau kinetic equation. Our starting point is the following equation (we suppress upper indices for lighter notation)

$$\partial_t g_{\theta_1, \theta_2}(z) + \partial_z (A_{\theta_1}^\gamma g_{\gamma, \theta_2}(z) - A_{\theta_2}^\gamma g_{\theta_1, \gamma}(z)) = \mathcal{S}_{\theta_1, \theta_2}[\rho](z). \quad (119)$$

We start with moving to normal modes in order to diagonalize A operators $g_{\theta_1, \theta_2}(z) = R_{\theta_1, \xi_1}^{-t} R_{\theta_2, \xi_2}^{-t} \bar{g}_{\xi_1, \xi_2}(z)$ and similarly for $\mathcal{S}_{\theta_1, \theta_2}[\rho](z)$. We get a equation

$$\partial_t \bar{g}_{\theta_1, \theta_2}(z) + v_{\theta_1, \theta_2} \partial_z \bar{g}_{\theta_1, \theta_2}(z) = \bar{\mathcal{S}}_{\theta_1, \theta_2}[\rho](z), \quad (120)$$

for which the Green's function can be found with Fourier transform and reads

$$G_{\theta_1, \theta_2}(z, z'; t, t') = \delta(z + z' - v_{\theta_1, \theta_2} t) \theta_H(t - t'), \quad (121)$$

where θ_H is Heaviside step function. Hence, we find

$$\bar{g}_{\theta_1, \theta_2}(z) = \int dz' \int_0^t d\tau \int \frac{dk}{2\pi} e^{ik(z+z'-v_{\theta_1, \theta_2}\tau)} \times \bar{\mathcal{S}}_{\theta_1, \theta_2}[\rho](z', t - \tau). \quad (122)$$

Lastly, we rotate back to $g_{\theta_1, \theta_2}(z)$ to obtain the final result (60).

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- [1] P. Calabrese, F. H. Essler, and G. Mussardo, *J. Stat. Mech.* **2016**, 064001 (2016).
- [2] A. Polkovnikov, K. M. Kennes, and D. M. Basko, *Reviews of Modern Physics* **83**, 863 (2011).
- [3] M. Cheneau, P. Barmettler, D. Poletti, M. Endres, P. Schauss, T. Fukuhara, C. Gross, I. Bloch, C. Kolath, and S. Kuhr, *Nature* **481**, 484 (2012).
- [4] D. A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S. L. Sondhi, *Physical Review B* **88**, 014206 (2013).
- [5] M. Fagotti and P. Calabrese, *Phys. Rev. A* **78**, 010306 (2008).
- [6] V. Alba and P. Calabrese, *SciPost Phys.* **4**, 017 (2018).
- [7] V. Alba, B. Bertini, and M. Fagotti, *SciPost Phys.* **7**, 005 (2019).
- [8] V. Alba and P. Calabrese, *Phys. Rev. B* **100**, 115150 (2019).
- [9] J. Lopez-Piqueres, B. Ware, S. Gopalakrishnan, and R. Vasseur, *Phys. Rev. B* **104**, 104307 (2021).
- [10] P. Fendley, *Journal of Physics A: Mathematical and Theoretical* **49**, 30LT01 (2016).
- [11] P. R. Pasnoori, Y. Tang, J. Lee, J. H. Pixley, N. Andrei, and P. Azaria, *Spin fractionalization and zero modes in the spin- $\frac{1}{2}$ xxz chain with boundary fields* (2023).
- [12] A. Burshtein and M. Goldstein, *PRX Quantum* **5**, 020323 (2024).
- [13] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, *Phys. Rev. X* **6**, 041065 (2016).
- [14] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, *Phys. Rev. Lett.* **117**, 207201 (2016).
- [15] S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, *Phys. Rev. D* **85**, 085029 (2012).
- [16] P. Ruggiero, P. Calabrese, B. Doyon, and J. Dubail, *Phys. Rev. Lett.* **124**, 140603 (2020).
- [17] A. Lucas and K. C. Fong, *Journal of Physics: Condensed Matter* **30**, 053001 (2018).
- [18] N. Malvania, Y. Zhang, Y. Le, J. Dubail, M. Rigol, and D. S. Weiss, *Science* **373**, 1129–1133 (2021).
- [19] M. Schemmer, I. Bouchoule, B. Doyon, and J. Dubail, *Phys. Rev. Lett.* **122**, 090601 (2019).
- [20] A. Scheie, N. E. Sherman, M. Dupont, S. E. Nagler, M. B. Stone, G. E. Granroth, J. E. Moore, and D. A. Tennant, *Nature Physics* **17**, 726–730 (2021).
- [21] F. Moller *et al.*, *Physical Review Letters* **126**, 090602 (2021).
- [22] F. Cataldini *et al.*, *Physical Review X* **12**, 041032 (2022).
- [23] K.-Y. Li, Y. Zhang, K. Yang, K.-Y. Lin, S. Gopalakrishnan, M. Rigol, and B. L. Lev, *Phys. Rev. A* **107**, L061302 (2023).
- [24] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, *Phys. Rev. B* **97**, 045407 (2018).
- [25] V. B. Bulchandani, C. Karrasch, and J. E. Moore, *Proceedings of the National Academy of Sciences* **117**, 12713–12718 (2020).
- [26] L. Piroli, J. De Nardis, M. Collura, B. Bertini, and M. Fagotti, *Phys. Rev. B* **96**, 115124 (2017).
- [27] E. H. Lieb and W. Liniger, *Physical Review* **130**, 1605 (1963).
- [28] E. H. Lieb, *Physical Review* **130**, 1616 (1963).
- [29] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper, and V. E. Korepin, *The One-Dimensional Hubbard Model* (Cambridge University Press, 2005).
- [30] K. Mallayya, M. Rigol, and W. De Roeck, *Phys. Rev. X* **9**, 021027 (2019).
- [31] J. Lopez-Piqueres and R. Vasseur, *Phys. Rev. Lett.* **130**, 247101 (2023).
- [32] T. Goldfriend and J. Kurchan, *Phys. Rev. E* **102**, 022201 (2020).
- [33] A. J. Friedman, S. Gopalakrishnan, and R. Vasseur, *Phys. Rev. B* **101**, 180302 (2020).
- [34] A. Bastianello, A. De Luca, and R. Vasseur, *Journal of Statistical Mechanics: Theory and Experiment* **2021**, 114003 (2021).
- [35] P. Orlov, A. Tiutiakina, R. Sharipov, E. Petrova, V. Gritsev, and D. V. Kurlov, *Phys. Rev. B* **107**, 184312 (2023).
- [36] D. Szász-Schagrin, B. Pozsgay, and G. Takács, *SciPost Phys.* **11**, 037 (2021).
- [37] B. Pozsgay, R. Sharipov, A. Tiutiakina, and I. Vona, *SciPost Physics* **17**, 10.21468/scipostphys.17.3.075 (2024).
- [38] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, *Phys. Rev. Lett.* **115**, 180601 (2015).
- [39] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, *Physical Review B* **94**, 10.1103/physrevb.94.245117 (2016).
- [40] F. H. L. Essler, S. Kehrein, S. R. Manmana, and N. J. Robinson, *Phys. Rev. B* **89**, 165104 (2014).
- [41] J. Lopez-Piqueres, B. Ware, S. Gopalakrishnan, and R. Vasseur, *Phys. Rev. B* **103**, L060302 (2021).
- [42] F. M. Surace and O. Motrunich, *Phys. Rev. Res.* **5**, 043019 (2023).
- [43] M. Lebek, M. Panfil, and R. M. Konik, *SciPost Phys.* **17**, 007 (2024).
- [44] J. A. Scaramazza, P. Smacchia, and E. A. Yuzbashyan, *Phys. Rev. B* **99**, 054520 (2019).
- [45] A. Bastianello, *EPL (Europhysics Letters)* **125**, 20001 (2019).
- [46] G. del Vecchio del Vecchio, A. D. Luca, and A. Bastianello, *SciPost Physics* **12**, 060 (2022).
- [47] J. Durnin, M. J. Bhaseen, and B. Doyon, *Phys. Rev. Lett.* **127**, 130601 (2021).
- [48] M. Panfil, S. Gopalakrishnan, and R. M. Konik, *Phys. Rev. Lett.* **130**, 030401 (2023).
- [49] M. Lebek and M. Panfil, *SciPost Phys.* **18**, 186 (2025).
- [50] S. Lepri, R. Livi, and A. Politi, *Phys. Rev. Lett.* **125**, 040604 (2020).
- [51] L. Delacrétaz, A. L. Fitzpatrick, E. Katz, and M. Walters, *SciPost Physics* **12**, 119 (2022).
- [52] A. Miron, J. Cividini, A. Kundu, and D. Mukamel, *Physical Review E* **99**, 012124 (2019).
- [53] S. Chen, J. Wang, G. Casati, and G. Benenti, *Physical Review E* **90**, 032134 (2014).
- [54] H. Zhao and W. g. Wang, *Physical Review E* **97**, 010103 (2018).
- [55] S. Lepri, R. Livi, and A. Politi, *Physical Review Letters* **125**, 040604 (2020).
- [56] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, *Science* **337**, 1318–1322 (2012).
- [57] F. Møller, C. Li, I. Mazets, H.-P. Stimming, T. Zhou, Z. Zhu, X. Chen, and J. Schmiedmayer, *Phys. Rev. Lett.* **126**, 090602 (2021).
- [58] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, *Phys. Rev.*

- X **8**, 021030 (2018).
- [59] C. Colléaux, J. Skipp, J. Laurie, and S. Nazarenko, *Semi-local one-dimensional optical wave turbulence* (2024).
 - [60] S. Lepri, P. Sandri, and A. Politi, *Eur. Phys. J. B* **47**, 549 (2005).
 - [61] M. Mareschal and A. Amellal, *Phys. Rev. A* **37**, 2189 (1988).
 - [62] M. Eltohf, X. Wang, C. M. Griffin, and F. Robicheaux, *Phys. Rev. E* **110**, 014114 (2024).
 - [63] A. Parameshwaran and B. S. Gupta, *Phys. Rev. E* **111**, 025405 (2025).
 - [64] G. Benettin, G. Orsatti, and A. Ponno, *J. Stat. Phys.* **190**, 131 (2023).
 - [65] O. C. Eldridge and M. Feix, *The Physics of Fluids* **6**, 398–406 (1963).
 - [66] J. M. Dawson, *The Physics of Fluids* **7**, 419–425 (1964).
 - [67] J. L. Rouet and M. R. Feix, *Physics of Fluids B: Plasma Physics* **3**, 1830–1834 (1991).
 - [68] J.-B. Fouvry, B. Bar-Or, and P.-H. Chavanis, *Physical Review E* **100**, 052142 (2019).
 - [69] J.-B. Fouvry, P.-H. Chavanis, and C. Pichon, *Physical Review E* **102**, 052110 (2020).
 - [70] J. Berges, G. S. Denicol, M. P. Heller, and T. Preis, *Far from equilibrium hydrodynamics of nonthermal fixed points* (2025).
 - [71] J.-B. Fouvry and M. Roule, *Phys. Rev. E* **108**, 054108 (2023).
 - [72] G. B. Rybicki, *Astrophysics and Space Science* **14**, 56–72 (1971).
 - [73] J. Reidl, Charles J. and B. N. Miller, *The Astrophysical Journal* **332**, 619 (1988).
 - [74] L. Milanović, H. Posch, and W. Thirring, *Physical Review E* **57**, 2763–2775 (1998).
 - [75] T. Tsuchiya, N. Gouda, and T. Konishi, *Physical Review E* **53**, 2210–2216 (1996).
 - [76] B. N. Miller, K. R. Yawn, and V. P. Youngkins, Recent progress in studies of one-dimensional gravitating systems, in *Dynamical Systems, Plasmas and Gravitation* (Springer Berlin Heidelberg, 1996) p. 164–175.
 - [77] M. Lebek and M. Panfil, *Phys. Rev. Lett.* **134**, 010405 (2025).
 - [78] R. Balescu, *Equilibrium and Non-Equilibrium Statistical Mechanics* (Wiley-Interscience, New York, 1975).
 - [79] J. G. Kirkwood, *The Journal of Chemical Physics* **15**, 72 (1947).
 - [80] J. G. Kirkwood, *The Journal of Chemical Physics* **14**, 180 (1946).
 - [81] J. Yvon, *La théorie statistique des fluides et l'équation d'état*, Actual. Sci. & Indust. No. 203 (Hermann, Paris, 1935).
 - [82] S. Harris, *An introduction to the theory of the Boltzmann equation* (Courier Corporation, 2004).
 - [83] N. N. Bogoliubov and K. P. Gurov, *Journal of Experimental and Theoretical Physics (in Russian)* **17**, 614 (1947).
 - [84] V. I. Gerasimenko, (2021), [arXiv:2107.10872](https://arxiv.org/abs/2107.10872).
 - [85] L. Chomaz, I. Ferrier-Barbut, F. Ferlaino, B. Laburthe-Tolra, B. L. Lev, and T. Pfau, *Reports on Progress in Physics* **86**, 026401 (2023).
 - [86] C. Chen, G. Bornet, M. Bintz, G. Emperauger, L. Leclerc, V. S. Liu, P. Scholl, D. Barredo, J. Hauschild, S. Chatterjee, M. Schuler, A. M. Läuchli, M. P. Zaletel, T. Lahaye, N. Y. Yao, and A. Browaeys, *Nature* **616**, 691–695 (2023).
 - [87] M. Płodzień, G. Lochead, J. de Hond, N. J. van Druten, and S. Kokkelmans, *Phys. Rev. A* **95**, 043606 (2017).
 - [88] J. Rincón, M. Ganahl, and G. Vidal, *Phys. Rev. B* **92**, 115107 (2015).
 - [89] E. Orignac, S. De Palo, L. Salasnich, and R. Citro, *Phys. Rev. A* **109**, 043316 (2024).
 - [90] I. Frérot, P. Naldesi, and T. Roscilde, *Phys. Rev. B* **95**, 245111 (2017).
 - [91] L. Chomaz, S. Baier, D. Petter, M. J. Mark, F. Wächtler, L. Santos, and F. Ferlaino, *Phys. Rev. X* **6**, 041039 (2016).
 - [92] A. Recati and S. Stringari, *Nature Reviews Physics* **5**, 735–743 (2023).
 - [93] G. Emperauger, M. Qiao, C. Chen, F. Caleca, S. Bocini, M. Bintz, G. Bornet, R. Martin, B. Gély, L. Klein, D. Barredo, S. Chatterjee, N. Y. Yao, F. Mezzacapo, T. Lahaye, T. Roscilde, and A. Browaeys, *Physical Review X* **15**, 10.1103/qfnp-6dpz (2025).
 - [94] K. Yang, Y. Zhang, K.-Y. Li, K.-Y. Lin, S. Gopalakrishnan, M. Rigol, and B. L. Lev, *Science* **385**, 1063–1067 (2024).
 - [95] D. Bernard and B. Doyon, *Journal of Statistical Mechanics: Theory and Experiment* **2016**, 064005 (2016).
 - [96] B. Doyon, S. Gopalakrishnan, F. Møller, J. Schmiedmayer, and R. Vasseur, *Physical Review X* **15**, 10.1103/physrevx.15.010501 (2025).
 - [97] P. Schüttelkopf, M. Tajik, N. Bazhan, F. Cataldini, S.-C. Ji, J. Schmiedmayer, and F. Møller, *Characterising transport in a quantum gas by measuring drude weights* (2024).
 - [98] R. Vasseur and J. E. Moore, *Journal of Statistical Mechanics: Theory and Experiment* **2016**, 064010 (2016).
 - [99] M. L. Kerr and K. V. Kheruntsyan, *AAPPS Bulletin* **33**, 10.1007/s43673-023-00095-2 (2023).
 - [100] V. Alba, B. Bertini, M. Fagotti, L. Piroli, and P. Ruggiero, *Journal of Statistical Mechanics: Theory and Experiment* **2021**, 114004 (2021).
 - [101] F. H. Essler, *Physica A: Statistical Mechanics and its Applications* **631**, 127572 (2023), lecture Notes of the 15th International Summer School of Fundamental Problems in Statistical Physics.
 - [102] S. Scopa, P. Ruggiero, P. Calabrese, and J. Dubail, *Phys. Rev. A* **108**, 013324 (2023).
 - [103] P. Suret, S. Randoux, A. Gelash, D. Agafontsev, B. Doyon, and G. El, *Physical Review E* **109**, 10.1103/physreve.109.061001 (2024).
 - [104] A. Bastianello, A. Tikan, F. Copie, S. Randoux, and P. Suret, *Physical Review A* **113**, 10.1103/xz2w-hndp (2026).
 - [105] T. Kinoshita, T. Wenger, and D. S. Weiss, *Nature* **440**, 900 (2006).
 - [106] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, *Science* **349**, 842–845 (2015).
 - [107] W. Kao, K.-Y. Li, K.-Y. Lin, S. Gopalakrishnan, and B. L. Lev, *Science* **371**, 296–300 (2021).
 - [108] W. Morong, F. Liu, P. Becker, K. S. Collins, L. Feng, A. Kyprianidis, G. Pagano, T. You, A. V. Gorshkov, and C. Monroe, *Nature* **601**, E13–E13 (2022).
 - [109] S. Birnkammer, A. Bastianello, and M. Knap, *Nature Communications* **13**, 10.1038/s41467-022-35301-6 (2022).
 - [110] Y. Le, Y. Zhang, S. Gopalakrishnan, M. Rigol, and D. S. Weiss, *Nature* **618**, 494–499 (2023).

- [111] T. Young, J. LLoyd, and C. von Keyserlingk, [Breakdown of fermi's golden rule in 1d systems at non-zero temperature](#) (2025).
- [112] P. A. E. M. Janssen, [Journal of Physical Oceanography](#) **33**, 863–884 (2003).
- [113] J. Garnier, M. Lisak, and A. Picozzi, [Journal of the Optical Society of America B](#) **29**, 2229 (2012).
- [114] B. Doyon and T. Yoshimura, [SciPost Phys.](#) **2**, 014 (2017).
- [115] J. Durnin, A. D. Luca, J. D. Nardis, and B. Doyon, [Journal of Physics A: Mathematical and Theoretical](#) **54**, 494001 (2021).
- [116] A. A. Vlasov, [Sov. Phys. Usp.](#) **10**, 721 (1968).
- [117] W. Braun and K. Hepp, [Communications in Mathematical Physics](#) **56**, 101–113 (1977).
- [118] P. H. Chavanis, J. Sommeria, and R. Robert, [The Astrophysical Journal](#) **471**, 385–399 (1996).
- [119] J. D. Nardis, D. Bernard, and B. Doyon, [SciPost Phys.](#) **6**, 049 (2019).
- [120] See the Supplementary Material.
- [121] M. Medenjak, J. D. Nardis, and T. Yoshimura, [SciPost Phys.](#) **9**, 075 (2020).
- [122] S. Gopalakrishnan, D. A. Huse, V. Khemani, and R. Vasseur, [Phys. Rev. B](#) **98**, 220303 (2018).
- [123] M. Lebek, M. Panfil, and R. M. Konik, [SciPost Physics](#) **17**, 007 (2024).
- [124] i. c. v. Krajnik, J. Schmidt, V. Pasquier, E. Ilievski, and T. c. v. Prosen, [Phys. Rev. Lett.](#) **128**, 160601 (2022).
- [125] S. Gopalakrishnan, E. McCulloch, and R. Vasseur, [Proceedings of the National Academy of Sciences](#) **121**, 10.1073/pnas.2403327121 (2024).
- [126] F. Hübner, L. Biagetti, J. De Nardis, and B. Doyon, [Physical Review Letters](#) **134**, 10.1103/physrevlett.134.187101 (2025).
- [127] E. McCulloch, R. Vasseur, and S. Gopalakrishnan, [arXiv preprint arXiv:2407.03412](#) (2024), 5 pages, 4 figures.
- [128] T. Yoshimura and Žiga Krajnik, [arXiv preprint arXiv:2406.20091](#) (2024).
- [129] B. Doyon, [Hydrodynamic noise in one dimension: projected kubo formula and how it vanishes in integrable models](#) (2025).
- [130] H. Spohn, [Large Scale Dynamics of Interacting Particles](#) (Springer-Verlag, Berlin, 1991).
- [131] M. Flicker, [Journal of Mathematical Physics](#) **9**, 171 (1968).
- [132] A. Robledo and J. Rowlinson, [Molecular Physics](#) **58**, 711–721 (1986).
- [133] B. Doyon and H. Spohn, [Journal of Statistical Mechanics: Theory and Experiment](#) **2017**, 073210 (2017).
- [134] H. Spohn, Springer, Berlin (1991).
- [135] C. Boldrighini, R. L. Dobrushin, and Y. M. Sukhov, [Journal of Statistical Physics](#) **31**, 577 (1983).
- [136] C. Boldrighini and Y. M. Suhov, [Communications in Mathematical Physics](#) **189**, 577 (1997).
- [137] B. Doyon, [SciPost Physics Lecture Notes](#) 10.21468/sci-postphyslectnotes.18 (2020).
- [138] M. Olshanii, [Physical Review Letters](#) **81**, 938 (1998).
- [139] F. Deuretzbacher, J. C. Cremon, and S. M. Reimann, [Physical Review A](#) **81**, 063616 (2010).
- [140] F. Deuretzbacher, J. C. Cremon, and S. M. Reimann, [Physical Review A](#) **87**, 039903 (2013).
- [141] A. Urilyon, L. Biagetti, J. Kethepalli, and J. De Nardis, [Simulating generalised fluids via interacting wave packets evolution](#) (2025).
- [142] K. Mallayya and M. Rigol, [Phys. Rev. B](#) **104**, 184302 (2021).
- [143] J. De Nardis and M. Panfil, [Journal of Statistical Mechanics: Theory and Experiment](#) **2015**, P02019 (2015).
- [144] J. D. Nardis and M. Panfil, [SciPost Phys.](#) **1**, 015 (2016).
- [145] J. De Nardis and M. Panfil, [Journal of Statistical Mechanics: Theory and Experiment](#) **2018**, 033102 (2018).
- [146] A. C. Cubero and M. Panfil, [Journal of High Energy Physics](#) **2019**, 104 (2019).
- [147] M. Panfil, [Journal of Statistical Mechanics: Theory and Experiment](#) **2021**, 013108 (2021).
- [148] Note the additional missing factor of 2π with respect to [47].
- [149] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, [Quantum Inverse Scattering Method and Correlation Functions](#) (Cambridge University Press, 1993).
- [150] M. Panfil and R. M. Konik, [Journal of High Energy Physics](#) **2023**, 72 (2023).
- [151] C. Monroe, W. C. Campbell, L.-M. Duan, Z.-X. Gong, A. V. Gorshkov, P. W. Hess, R. Islam, K. Kim, N. M. Linke, G. Pagano, P. Richerme, C. Senko, and N. Y. Yao, [Rev. Mod. Phys.](#) **93**, 025001 (2021).
- [152] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt, and C. F. Roos, [Nature](#) **511**, 202–205 (2014).
- [153] B. Kloss and Y. Bar Lev, [Phys. Rev. A](#) **99**, 032114 (2019).
- [154] J. T. Schneider, S. J. Thomson, and L. Sanchez-Palencia, [Phys. Rev. B](#) **106**, 014306 (2022).
- [155] S. Nazarenko, [Wave Turbulence](#), Lecture Notes in Physics, Vol. 825 (Springer Berlin Heidelberg, 2011).
- [156] L. Fache, H. Damart, F. m. c. Copie, T. Bonnemain, T. Congy, G. Roberti, P. Suret, G. El, and S. Randoux, [Phys. Rev. Lett.](#) **134**, 147201 (2025).
- [157] A. Mucci, P. Suret, F. m. c. Copie, S. Randoux, R. Mullyadzhyanov, and A. Gelash, [Phys. Rev. Lett.](#) **134**, 193804 (2025).
- [158] L. Fache, F. Copie, P. Suret, and S. Randoux, [Perturbed nonlinear evolution of optical soliton gases: Growth and decay in integrable turbulence](#) (2025).
- [159] L. Biagetti, G. Cecile, and J. D. Nardis, [Physical Review Research](#) **6**, 023083 (2024).
- [160] B. Doyon, G. Perfetto, T. Sasamoto, and T. Yoshimura, [SciPost Phys.](#) **15**, 136 (2023).
- [161] J. De Nardis, D. Bernard, and B. Doyon, [Phys. Rev. Lett.](#) **121**, 160603 (2018).

Supplemental Material: Generalised BBGKY hierarchy for near-integrable dynamics

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In the Supplemental Material we provide additional computations supporting the results presented in the main text. The material is presented as follows

1. DERIVATION OF THE GENERALISED LANDAU EQUATION

In this section we derive in full details the generalised Landau equation from the gBBGKY hierarchy of equations introduced in the main text. As already introduced in the main text, we start considering the first three layers of the hierarchy and decompose the two-point function into leading $g^{(2),1}$ and subleading $g^{(2),2}$ contribution in $1/\xi$. Hence, according to this notation, we consider the following set of equations

$$\partial_t \rho_{\theta_1} = \frac{1}{\xi^2} \int dz d\theta_2 V_0 \varphi'(z) \partial_{\theta_1} (g_{\theta_1, \theta_2}^{(2),1}(z) + \frac{1}{\xi} g_{\theta_1, \theta_2}^{(2),2}(z)), \quad (1.1)$$

$$\partial_t g_{\theta_1, \theta_2}^{(2),1}(z) + \partial_z \left(A_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2),1}(z) - A_{\theta_2}^\gamma g_{\theta_1, \gamma}^{(2),1}(z) \right) = V_0 \varphi'(z) \int d\gamma \left(\partial_{\theta_1} \rho_{\theta_1} C_{\gamma, \theta_2} - \partial_{\theta_2} \rho_{\theta_2} C_{\theta_1, \gamma} \right), \quad (1.2)$$

$$\partial_t g_{\theta_1, \theta_2}^{(2),2}(z) + \partial_z \left(A_{\theta_1}^\gamma g_{\gamma, \theta_2}^{(2),2}(z) - A_{\theta_2}^\gamma g_{\theta_1, \gamma}^{(2),2}(z) \right) = -\frac{1}{2} \partial_z \left(H_{\theta_1}^{\gamma_1 \gamma_2} g_{\gamma_1, \gamma_2, \theta_2}^{(3), \text{sym}}(0^+, z) - H_{\theta_2}^{\gamma_1, \gamma_2} g_{\theta_1, \gamma_1, \gamma_2}^{(3), \text{sym}}(z, 0^+) \right), \quad (1.3)$$

$$\begin{aligned} \partial_t g_{\theta_1, \theta_2, \theta_3}^{(3)}(z, z') + \partial_z A_{\theta_1}^\gamma g_{\gamma, \theta_2, \theta_3}^{(3)}(z, z') + (\partial_{z'} - \partial_z) A_{\theta_2}^\gamma g_{\theta_1, \gamma, \theta_3}^{(3)}(z, z') - \partial_{z'} A_{\theta_3}^\gamma g_{\theta_1, \theta_2, \gamma}^{(3)}(z, z') = \\ = \delta(z) M_{\theta_1, \theta_2}^\gamma \partial_z g_{\gamma, \theta_3}^{(2),1}(z + z') - \delta(z') M_{\theta_2, \theta_3}^\gamma \partial_z g_{\theta_1, \gamma}^{(2),1}(z) + \delta(z') M_{\theta_2, \theta_3}^\gamma \partial_{z'} g_{\theta_1, \gamma}^{(2),1}(z + z'). \end{aligned} \quad (1.4)$$

The goal of this section is to integrate this set of equations in order to derive a unique closed kinetic equation for $\partial_t \rho$. In order to do that, we crucially observe that all these equations are now written in an analogous form: as a convection equation with an external driving force. Hence, the main idea behind the derivation is to impose separation of time scales for the evolution of ρ , $g^{(2),1}$, $g^{(2),2}$ and $g^{(2),3}$. More precisely, we impose that the particle density evolve much slower than the few point functions. In this way we can decouple the set of equations and solve them separately. Hence, using a standard Green's function technique, we find the following time evolution relation

$$g_{\theta_1, \theta_2}^{(2),1}(z, t) = R_{\theta_1, \sigma_1}^{-t} R_{\theta_2, \sigma_2}^{-t} \int_0^t d\tau \int \frac{dk}{2\pi} \int d\gamma \int dz' e^{ik(z+z'-v_{\sigma_1, \sigma_2} \tau)} R_{\sigma_1, \eta_1}^t R_{\sigma_2, \eta_2}^t V_0 \varphi'(z') \left[\partial_{\eta_1} \rho_{\eta_1} C_{\gamma, \eta_2} - \partial_{\eta_2} \rho_{\eta_2} C_{\eta_1, \gamma} \right]_{t-\tau}, \quad (1.5)$$

where we defined $v_{\xi_1, \xi_2} \equiv v_{\xi_1}^{\text{eff}} - v_{\xi_2}^{\text{eff}}$ and where the subscript $t - \tau$ indicates the time at which must be computed the function inside the brackets. We now perform an approximation which amounts to replacing $\rho_\theta(z, t - \tau)$ with $\rho_\theta(z, t)$ under the integral and extending the integration range in time to infinity. This simplification follows from the assumption of separation of scales. Next, we use the identity $\int_0^\infty d\tau e^{i\tau x} = \pi \delta(x) + i\mathcal{P}(\frac{1}{x})$, where $\mathcal{P}$ denotes principal value. The contribution coming from the Dirac delta function vanishes since the argument of the square bracket is antisymmetric with respect to η_1, η_2 . Hence the result reads

$$\lim_{t \rightarrow \infty} g_{\theta_1, \theta_2}^{(2),1}(z, t) = V_0 \varphi'(z') R_{\theta_1, \sigma_1}^{-t} R_{\theta_2, \sigma_2}^{-t} \frac{1}{v_{\sigma_1, \sigma_2}} R_{\sigma_1, \eta_1}^t R_{\sigma_2, \eta_2}^t \left(\partial_{\eta_1} \rho_{\eta_1} C_{\gamma, \eta_2} - \partial_{\eta_2} \rho_{\eta_2} C_{\eta_1, \gamma} \right), \quad (1.6)$$

where the principal value has been removed, for the function being analytic. Also, here we stress that the RHS is evaluated at time t . In an analogous way, we can now proceed integrating Eq. (1.4). More precisely, we use the same exact set of approximations in order to time integrate the evolution equation for $g^{(3)}$. In this case the result is slightly more complicated, since this equation is characterized by three forcing terms being proportional to space Dirac delta

functions

$$\begin{aligned}
g_{\theta_1, \theta_2, \theta_3}^{(3)}(z, z'; t) = & \frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_1, \gamma_2}} [\text{sgn}(z) - \text{sgn}(z - v_{\gamma_1, \gamma_2} t)] \times \\
& \times R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_1, \mu_2}^\beta \partial_1 g_{\beta, \mu_3}^{(2), 1} \left(\frac{v_{\gamma_2, \gamma_3}}{v_{\gamma_1, \gamma_2}} z - z'; t - \frac{z}{v_{\gamma_1, \gamma_2}} \right) + \\
& + \frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_1, \gamma_3}} [\text{sgn}(z + z') - \text{sgn}(z + z' - v_{\gamma_1, \gamma_3} t)] \times \\
& \times R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_1, \mu_3}^\beta \partial_1 g_{\beta, \mu_2}^{(2), 1} \left(\frac{v_{\gamma_3, \gamma_2}}{v_{\gamma_1, \gamma_3}} z + \frac{v_{\gamma_1, \gamma_2}}{v_{\gamma_1, \gamma_3}} z'; t - \frac{z + z'}{v_{\gamma_1, \gamma_3}} \right) + \\
& + \frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_2, \gamma_3}} [\text{sgn}(z') - \text{sgn}(z' - v_{\gamma_2, \gamma_3} t)] \times \\
& \times R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_2, \mu_3}^\beta \partial_1 g_{\beta, \mu_1}^{(2), 1} \left(z + \frac{v_{\gamma_2, \gamma_1}}{v_{\gamma_2, \gamma_3}} z'; t - \frac{z'}{v_{\gamma_2, \gamma_3}} \right), \tag{1.7}
\end{aligned}$$

where the symbol ∂_1 means that the derivative is performed on the spacial argument of the function. Now, we make use of the assumption of initial state null correlations, for which $g_{\theta_1, \theta_2}^{(2), 1}(z, t \leq 0) = 0$. Hence, the time argument in the correlation functions imposes

$$t - \frac{z'}{v_{\alpha, \beta}} > 0 \longrightarrow \text{sgn}(z' - v_{\alpha, \beta} t) = -\text{sgn}(v_{\alpha, \beta}). \tag{1.8}$$

In order to use this integrated correlator into evolution equation (1.3), we need to evaluate the result (1.7) and compute the point split symmetrization, reading

$$\begin{aligned}
g_{\theta_1, \theta_2, \theta_3}^{(3), \text{sym}}(0^+, z'; t) = & \frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} \frac{1}{|v_{\gamma_1, \gamma_2}|} R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t M_{\mu_1, \mu_2}^\beta \partial_1 g_{\beta, \theta_3}^{(2), 1}(-z'; t) \\
& + \frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_1, \gamma_3}} [\text{sgn}(z') + \text{sgn}(v_{\gamma_1, \gamma_3})] R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_1, \mu_3}^\beta \partial_1 g_{\beta, \mu_2}^{(2), 1} \left(\frac{v_{\gamma_1, \gamma_2}}{v_{\gamma_1, \gamma_3}} z'; t - \frac{z'}{v_{\gamma_1, \gamma_3}} \right) \\
& + \frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_2, \gamma_3}} [\text{sgn}(z') + \text{sgn}(v_{\gamma_2, \gamma_3})] R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_2, \mu_3}^\beta \partial_1 g_{\beta, \mu_1}^{(2), 1} \left(\frac{v_{\gamma_2, \gamma_1}}{v_{\gamma_2, \gamma_3}} z'; t - \frac{z'}{v_{\gamma_2, \gamma_3}} \right). \tag{1.9}
\end{aligned}$$

Here, we can already observe that the expression has two different kind of terms: the first one on RHS is simpler, and will generate the *cross* part of Landau equation, while the second and third lines will be part of the *self* contributions. Using the expression for the M tensor introduced in the main text, the effective forcing term appearing in RHS of Eq. (1.3) reads

$$\begin{aligned}
& \frac{1}{2} \partial_z \left(H_{\theta_1}^{\gamma_1, \gamma_2} g_{\gamma_1, \gamma_2, \theta_2}^{(3), \text{sym}}(0^+, z) - H_{\theta_2}^{\gamma_1, \gamma_2} g_{\theta_1, \gamma_1, \gamma_2}^{(3), \text{sym}}(z, 0^+) \right) = \\
& \frac{1}{4} \partial_z \left(-2\mathfrak{D}_{\theta_1, \beta} \partial_z g_{\beta, \theta_2}^{(2), 1}(z) - 2\mathfrak{D}_{\theta_2, \beta} \partial_z g_{\theta_1, \beta}^{(2), 1}(z) \right. \\
& + R_{\theta_1, \eta}^{-t} R_{\theta_2, \nu_1}^{-t} (\delta_{\sigma, \nu_2} \delta_{\eta, \nu_3} + \delta_{\eta, \nu_2} \delta_{\sigma, \nu_3}) (\delta_{\nu_1, \zeta_1} \delta_{\nu_2, \zeta_2} + \delta_{\nu_2, \zeta_1} \delta_{\nu_1, \zeta_2}) (\text{sgn}(z) + \text{sgn}(v_{\nu_1, \nu_2})) \times \\
& \times \frac{T_{\sigma, \eta}^{\text{dr}} T_{\zeta_1, \zeta_2}^{\text{dr}}}{\rho_\eta^t \rho_{\zeta_1}^t} \frac{v_{\sigma, \eta} v_{\zeta_1, \zeta_2}}{v_{\nu_1, \nu_2}} \rho_{\zeta_2} f_{\zeta_2} R_{\nu_3, \mu_3}^t R_{\zeta_1, \beta}^t \frac{v_{\nu_1, \nu_2}}{v_{\nu_2, \nu_3}} \partial_z g_{\beta, \mu_3}^{(2), 1} \left(\frac{v_{\nu_2, \nu_3}}{v_{\nu_1, \nu_2}} z; t - \frac{z}{v_{\nu_1, \nu_2}} \right) + \\
& + R_{\theta_1, \eta}^{-t} R_{\theta_2, \nu_1}^{-t} (\delta_{\sigma, \nu_2} \delta_{\eta, \nu_3} + \delta_{\eta, \nu_2} \delta_{\sigma, \nu_3}) (\delta_{\nu_1, \zeta_1} \delta_{\nu_3, \zeta_2} + \delta_{\nu_3, \zeta_1} \delta_{\nu_1, \zeta_2}) (\text{sgn}(z) + \text{sgn}(v_{\nu_1, \nu_3})) \times \\
& \times \frac{T_{\sigma, \eta}^{\text{dr}} T_{\zeta_1, \zeta_2}^{\text{dr}}}{\rho_\eta^t \rho_{\zeta_1}^t} \frac{v_{\sigma, \eta} v_{\zeta_1, \zeta_2}}{v_{\nu_1, \nu_3}} \rho_{\zeta_2} f_{\zeta_2} R_{\nu_2, \mu_2}^t R_{\zeta_1, \beta}^t \frac{v_{\nu_1, \nu_3}}{v_{\nu_3, \nu_2}} \partial_z g_{\beta, \mu_2}^{(2), 1} \left(\frac{v_{\nu_3, \nu_2}}{v_{\nu_1, \nu_3}} z; t - \frac{z}{v_{\nu_1, \nu_3}} \right) + \\
& - R_{\theta_2, \eta}^{-t} R_{\theta_1, \nu_3}^{-t} (\delta_{\sigma, \nu_1} \delta_{\eta, \nu_2} + \delta_{\eta, \nu_1} \delta_{\sigma, \nu_2}) (\delta_{\nu_1, \zeta_1} \delta_{\nu_3, \zeta_2} + \delta_{\nu_3, \zeta_1} \delta_{\nu_1, \zeta_2}) (\text{sgn}(z) + \text{sgn}(v_{\nu_1, \nu_3})) \times \\
& \times \frac{T_{\sigma, \eta}^{\text{dr}} T_{\zeta_1, \zeta_2}^{\text{dr}}}{\rho_\eta^t \rho_{\zeta_1}^t} \frac{v_{\sigma, \eta} v_{\zeta_1, \zeta_2}}{v_{\nu_1, \nu_3}} \rho_{\zeta_2} f_{\zeta_2} R_{\nu_2, \mu_2}^t R_{\zeta_1, \beta}^t \frac{v_{\nu_1, \nu_3}}{v_{\nu_1, \nu_2}} \partial_z g_{\beta, \mu_2}^{(2), 1} \left(\frac{v_{\nu_1, \nu_2}}{v_{\nu_1, \nu_3}} z; t - \frac{z}{v_{\nu_1, \nu_3}} \right) + \\
& - R_{\theta_2, \eta}^{-t} R_{\theta_1, \nu_3}^{-t} (\delta_{\sigma, \nu_1} \delta_{\eta, \nu_2} + \delta_{\eta, \nu_1} \delta_{\sigma, \nu_2}) (\delta_{\nu_2, \zeta_1} \delta_{\nu_3, \zeta_2} + \delta_{\nu_3, \zeta_1} \delta_{\nu_2, \zeta_2}) (\text{sgn}(z) + \text{sgn}(v_{\nu_2, \nu_3})) \times \\
& \times \frac{T_{\sigma, \eta}^{\text{dr}} T_{\zeta_1, \zeta_2}^{\text{dr}}}{\rho_\eta^t \rho_{\zeta_1}^t} \frac{v_{\sigma, \eta} v_{\zeta_1, \zeta_2}}{v_{\nu_2, \nu_3}} \rho_{\zeta_2} f_{\zeta_2} R_{\nu_1, \mu_1}^t R_{\zeta_1, \beta}^t \frac{v_{\nu_2, \nu_3}}{v_{\nu_2, \nu_1}} \partial_z g_{\beta, \mu_1}^{(2), 1} \left(\frac{v_{\nu_2, \nu_1}}{v_{\nu_2, \nu_3}} z; t - \frac{z}{v_{\nu_2, \nu_3}} \right) \Big). \tag{1.10}
\end{aligned}$$

In particular, we used the crucial relation

$$H_{\theta_1}^{\eta_1, \eta_2} R_{\eta_1, \gamma_1}^{-t} R_{\eta_2, \gamma_2}^{-t} \frac{1}{|v_{\gamma_1, \gamma_2}|} R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t M_{\mu_1, \mu_2}^\beta = 2(R_{\theta_1, \mu_1}^{-t} - R_{\theta_1, \mu_2}^{-t})(T_{\mu_2, \mu_1}^{\text{dr}})^2 \frac{\rho_{\mu_2} f_{\mu_2}}{(\rho_{\mu_1}^t)^2} |v_{\mu_2, \mu_1}| R_{\mu_1, \beta}^t = \mathfrak{D}_\theta^\beta, \quad (1.11)$$

with $\mathfrak{D}$ being the linear response diffusion matrix in integrable systems [119, 161], see [141] for an explicit derivation. The next step is to perform the large time limit $t \rightarrow \infty$ and use the solution (1.6) into (1.10)

$$\begin{aligned} -\mathcal{S}_{\theta_1, \theta_2}^{(2), 2}(z) &\equiv \lim_{t \rightarrow \infty} \frac{1}{2} \partial_z \left(H_{\theta_1}^{\gamma_1, \gamma_2} g_{\gamma_1, \gamma_2, \theta_2}^{(3), \text{sym}}(0^+, z) - H_{\theta_2}^{\gamma_1, \gamma_2} g_{\theta_1, \gamma_1, \gamma_2}^{(3), \text{sym}}(z, 0^+) \right) = \\ &- \frac{1}{2} V_0 \partial_z^2 \varphi(z) (\mathfrak{D}_{\theta_1, \beta} R_{\beta \alpha}^{-t} R_{\theta_2, \gamma}^{-t} + \mathfrak{D}_{\theta_2, \beta} R_{\theta_1, \alpha}^{-t} R_{\beta, \gamma}^{-t}) \frac{1}{v_{\alpha, \gamma}} (R_{\alpha, \eta}^t \partial_\eta \rho_\eta \rho_\gamma^t \rho_\gamma f_\gamma - \rho_\alpha^t \rho_\alpha f_\alpha R_{\gamma, \eta}^t \partial_\eta \rho_\eta) + \\ &+ \frac{1}{4} R_{\theta_1, \eta}^{-t} R_{\theta_2, \nu_1}^{-t} (\delta_{\sigma, \nu_2} \delta_{\eta, \nu_3} + \delta_{\eta, \nu_2} \delta_{\sigma, \nu_3}) (\delta_{\nu_1, \zeta_1} \delta_{\nu_2, \xi_2} + \delta_{\nu_2, \zeta_1} \delta_{\nu_1, \xi_2}) (\text{sgn}(z) + \text{sgn}(v_{\nu_1, \nu_2})) \times \\ &\times \frac{T_{\sigma, \eta}^{\text{dr}} T_{\zeta_1, \xi_2}^{\text{dr}}}{\rho_\eta^t \rho_{\zeta_1}^t} \frac{v_{\sigma, \eta} v_{\zeta_1, \xi_2}}{v_{\nu_1, \nu_2}} \frac{v_{\nu_1, \nu_2}}{v_{\nu_2, \nu_3}} \rho_{\xi_2} f_{\xi_2} V_0 \partial_z^2 \varphi \left(\frac{v_{\nu_2, \nu_3}}{v_{\nu_1, \nu_2}} z \right) \frac{1}{v_{\zeta_1, \nu_3}} (R_{\zeta_1, \gamma}^t \partial_\gamma \rho_\gamma \rho_{\nu_3}^t \rho_{\nu_3} f_{\nu_3} - \rho_{\zeta_1}^t \rho_{\zeta_1} f_{\zeta_1} R_{\nu_3, \gamma}^t \partial_\gamma \rho_\gamma) + \\ &+ \frac{1}{4} R_{\theta_1, \eta}^{-t} R_{\theta_2, \nu_1}^{-t} (\delta_{\sigma, \nu_2} \delta_{\eta, \nu_3} + \delta_{\eta, \nu_2} \delta_{\sigma, \nu_3}) (\delta_{\nu_1, \zeta_1} \delta_{\nu_3, \xi_2} + \delta_{\nu_3, \zeta_1} \delta_{\nu_1, \xi_2}) (\text{sgn}(z) + \text{sgn}(v_{\nu_1, \nu_3})) \times \\ &\times \frac{T_{\sigma, \eta}^{\text{dr}} T_{\zeta_1, \xi_2}^{\text{dr}}}{\rho_\eta^t \rho_{\zeta_1}^t} \frac{v_{\sigma, \eta} v_{\zeta_1, \xi_2}}{v_{\nu_1, \nu_3}} \frac{v_{\nu_1, \nu_3}}{v_{\nu_3, \nu_2}} \rho_{\xi_2} f_{\xi_2} V_0 \partial_z^2 \varphi \left(\frac{v_{\nu_3, \nu_2}}{v_{\nu_1, \nu_3}} z \right) \frac{1}{v_{\zeta_1, \nu_2}} (R_{\zeta_1, \gamma}^t \partial_\gamma \rho_\gamma \rho_{\nu_2}^t \rho_{\nu_2} f_{\nu_2} - \rho_{\zeta_1}^t \rho_{\zeta_1} f_{\zeta_1} R_{\nu_2, \gamma}^t \partial_\gamma \rho_\gamma) + \\ &- \frac{1}{4} R_{\theta_2, \eta}^{-t} R_{\theta_1, \nu_3}^{-t} (\delta_{\sigma, \nu_1} \delta_{\eta, \nu_2} + \delta_{\eta, \nu_1} \delta_{\sigma, \nu_2}) (\delta_{\nu_1, \zeta_1} \delta_{\nu_3, \xi_2} + \delta_{\nu_3, \zeta_1} \delta_{\nu_1, \xi_2}) (\text{sgn}(z) + \text{sgn}(v_{\nu_1, \nu_3})) \times \\ &\times \frac{T_{\sigma, \eta}^{\text{dr}} T_{\zeta_1, \xi_2}^{\text{dr}}}{\rho_\eta^t \rho_{\zeta_1}^t} \frac{v_{\sigma, \eta} v_{\zeta_1, \xi_2}}{v_{\nu_1, \nu_3}} \frac{v_{\nu_1, \nu_3}}{v_{\nu_1, \nu_2}} \rho_{\xi_2} f_{\xi_2} V_0 \partial_z^2 \varphi \left(\frac{v_{\nu_1, \nu_2}}{v_{\nu_1, \nu_3}} z \right) \frac{1}{v_{\zeta_1, \nu_2}} (R_{\zeta_1, \gamma}^t \partial_\gamma \rho_\gamma \rho_{\nu_2}^t \rho_{\nu_2} f_{\nu_2} - \rho_{\zeta_1}^t \rho_{\zeta_1} f_{\zeta_1} R_{\nu_2, \gamma}^t \partial_\gamma \rho_\gamma) + \\ &- \frac{1}{4} R_{\theta_2, \eta}^{-t} R_{\theta_1, \nu_3}^{-t} (\delta_{\sigma, \nu_1} \delta_{\eta, \nu_2} + \delta_{\eta, \nu_1} \delta_{\sigma, \nu_2}) (\delta_{\nu_2, \zeta_1} \delta_{\nu_3, \xi_2} + \delta_{\nu_3, \zeta_1} \delta_{\nu_2, \xi_2}) (\text{sgn}(z) + \text{sgn}(v_{\nu_2, \nu_3})) \times \\ &\times \frac{T_{\sigma, \eta}^{\text{dr}} T_{\zeta_1, \xi_2}^{\text{dr}}}{\rho_\eta^t \rho_{\zeta_1}^t} \frac{v_{\sigma, \eta} v_{\zeta_1, \xi_2}}{v_{\nu_2, \nu_3}} \frac{v_{\nu_2, \nu_3}}{v_{\nu_2, \nu_1}} \rho_{\xi_2} f_{\xi_2} V_0 \partial_z^2 \varphi \left(\frac{v_{\nu_2, \nu_1}}{v_{\nu_2, \nu_3}} z \right) \frac{1}{v_{\zeta_1, \nu_1}} (R_{\zeta_1, \gamma}^t \partial_\gamma \rho_\gamma \rho_{\nu_1}^t \rho_{\nu_1} f_{\nu_1} - \rho_{\zeta_1}^t \rho_{\zeta_1} f_{\zeta_1} R_{\nu_1, \gamma}^t \partial_\gamma \rho_\gamma). \end{aligned} \quad (1.12)$$

In the last step we used the relation for susceptibility matrix

$$\int d\theta_2 C_{\theta_1, \theta_2} = 2\pi R_{\theta_1, \gamma}^{-t} \rho_\gamma f_\gamma \rho_\gamma^t. \quad (1.13)$$

We now proceed by integrating Eq. (1.4) in time using the same Green's function technique as used before

$$\lim_{t \rightarrow \infty} g_{\theta_1, \theta_2}^{(2), 2}[\rho](z, t) = R_{\theta_1, \sigma_1}^{-t} R_{\theta_2, \sigma_2}^{-t} \int_0^\infty d\tau \int \frac{dk}{2\pi} \int dz' e^{ik(z+z'-v_{\sigma_1, \sigma_2}\tau)} R_{\sigma_1, \eta_1}^t R_{\sigma_2, \eta_2}^t \mathcal{S}_{\eta_1, \eta_2}^{(2), 2}[\rho](z', t), \quad (1.14)$$

where $\mathcal{S}_{\eta_1, \eta_2}^{(2), 2}[\rho]$ is defined in (1.12), and where, we again used the time scale separation assumption as $\mathcal{S}_{\eta_1, \eta_2}^{(2), 2}[\rho](z, t-\tau) \approx \mathcal{S}_{\eta_1, \eta_2}^{(2), 2}[\rho](z, t)$. Under this assumption, we can perform the time integral, finding

$$\lim_{t \rightarrow \infty} g_{\theta_1, \theta_2}^{(2), 2}[\rho](z, t) = R_{\theta_1, \sigma_1}^{-t} R_{\theta_2, \sigma_2}^{-t} \frac{1}{v_{\sigma_1, \sigma_2}} R_{\sigma_1, \eta_1}^t R_{\sigma_2, \eta_2}^t \mathcal{S}_{\eta_1, \eta_2}^{(2), 2}[\rho](z, t). \quad (1.15)$$

The final step of the derivation consists in plugging the solution (1.15) into the quasiparticle density evolution equation (1.1), together with the forcing tensor definition (1.12)

$$\partial_t \rho_{\theta_1} = -\frac{1}{\xi^2} \int dk dz d\theta_2 V_0 \hat{\varphi}(k) e^{-ikz} \partial_{\theta_1} R_{\theta_1, \sigma_1}^{-t} R_{\theta_2, \sigma_2}^{-t} \frac{1}{v_{\sigma_1, \sigma_2}} R_{\sigma_1, \eta_1}^t R_{\sigma_2, \eta_2}^t \mathcal{S}_{\eta_1, \eta_2}^{(2), 2}[\rho](z) = \frac{1}{\xi^2} \partial_{\theta_1} R_{\theta_1, \sigma_1}^{-t} (\mathcal{I}_{\sigma_1}^{\text{cross}} + \mathcal{I}_{\sigma_1}^{\text{self}}). \quad (1.16)$$

In the last step we also used that, as shown in the main text, the contribution from $g^{(2), 1}$ in Eq. (1.1) is vanishing. In particular, Eq. (1.18) already represent a kinetic equation, that is correctly depending only on quasiparticle occupation function. In the next steps, we simply leverage algebraic manipulations in order to recast it in a more explicit form. Firstly, we observe that Eq. (1.18) has two different type of contributions, that we call cross and self. Cross terms

involve the diffusion matrix $\mathfrak{D}$ and are proportional to the square product $T_{\alpha,\beta}^{\text{dr}} T_{\alpha,\beta}^{\text{dr}}$. Meanwhile, the self terms only show product between dressed scattering shift $T_{\alpha,\beta}^{\text{dr}} T_{\alpha,\gamma}^{\text{dr}}$. Firstly we compute the cross part the scattering integral

$$\begin{aligned} \mathcal{I}_{\theta_1}^{\text{cross}} = & \int dk k^2 V_0^2 \hat{\varphi}^2(z) \int d\theta_2 \frac{\rho_{\theta_2}^t}{v_{\theta_1, \theta_2}} R_{\theta_1, \eta_1}^t R_{\theta_2, \eta_2}^t (\mathfrak{D}_{\eta_1, \beta} R_{\beta, \alpha}^{-t} R_{\eta_2, \gamma}^{-t} + \mathfrak{D}_{\eta_2, \beta} R_{\eta_1, \alpha}^{-t} R_{\beta, \gamma}^{-t}) \times \\ & \times \frac{1}{v_{\alpha\gamma}} (R_{\alpha, \eta}^t \partial_\eta \rho_\eta \rho_\gamma^t \rho_\gamma f_\gamma - \rho_\alpha^t \rho_\alpha f_\alpha R_{\gamma, \eta}^t \partial_\eta \rho_\eta). \end{aligned} \quad (1.17)$$

We now use the explicit formula for diffusion matrix expressed in (1.11) to rewrite the term

$$\begin{aligned} \mathcal{I}_{\theta_1}^{\text{cross}} = & 2\pi \int dk k^2 \hat{V}^2(z) \int (\delta_{\theta_1, \mu_1} - \delta_{\theta_1, \mu_2}) (T_{\mu_2, \mu_1}^{\text{dr}})^2 \frac{\rho_{\mu_2} f_{\mu_2} \rho_{\theta_2}^t}{(\rho_{\mu_1}^t)^2} \frac{|v_{\mu_2, \mu_1}|}{v_{\mu_1, \theta_2} v_{\theta_1, \theta_2}} (R_{\mu_1, \eta}^t \partial_\eta \rho_\eta \rho_{\theta_2}^t \rho_{\theta_2} f_{\theta_2} - \rho_{\mu_1}^t \rho_{\mu_1} f_{\mu_1} R_{\theta_2, \eta}^t \partial_\eta \rho_\eta) \\ & + 2\pi \int dk k^2 \hat{V}^2(z) \int (\delta_{\theta_2, \mu_1} - \delta_{\theta_2, \mu_2}) (T_{\mu_2, \mu_1}^{\text{dr}})^2 \frac{\rho_{\mu_2} f_{\mu_2} \rho_{\theta_2}^t}{(\rho_{\mu_1}^t)^2} \frac{|v_{\mu_2, \mu_1}|}{v_{\theta_1, \mu_1} v_{\theta_1, \theta_2}} (R_{\theta_1, \eta}^t \partial_\eta \rho_\eta \rho_{\mu_1}^t \rho_{\mu_1} f_{\mu_1} - \rho_{\theta_1}^t \rho_{\theta_1} f_{\theta_1} R_{\mu_1, \eta}^t \partial_\eta \rho_\eta), \end{aligned} \quad (1.18)$$

where we used the definition of inverse rotation matrix as $R^t R^{-t} = 1$. The last expression can be finally expressed as

$$\mathcal{I}_{\theta_1}^{\text{cross}} = 2\pi \int dk k^2 \hat{V}^2(k) \int d\sigma d\zeta \left[\rho_{\theta_1}^t (T_{\sigma, \zeta}^{\text{dr}})^2 \frac{|v_{\sigma, \zeta}|}{v_{\theta_1, \sigma} v_{\zeta, \theta_1}^2} - \frac{(\rho_\sigma^t)^2}{\rho_{\theta_1}^t} (T_{\zeta, \theta}^{\text{dr}})^2 \frac{|v_{\zeta, \theta_1}|}{v_{\sigma, \zeta} v_{\theta_1, \sigma}^2} \right] \mathcal{B}(\theta_1, \zeta, \sigma), \quad (1.19)$$

with the definition

$$\mathcal{B}(\theta, \zeta, \sigma) \equiv \frac{v_{\sigma, \zeta}}{\rho_\theta^t} R_{\theta, \eta}^t \partial_\eta \rho_\eta \rho_\sigma f_\sigma \rho_\zeta f_\zeta + \frac{v_{\zeta, \theta}}{\rho_\sigma^t} \rho_\theta f_\theta R_{\sigma, \eta}^t \partial_\eta \rho_\eta \rho_\zeta f_\zeta + \frac{v_{\theta, \sigma}}{\rho_\zeta^t} \rho_\theta f_\theta \rho_\sigma f_\sigma R_{\zeta, \eta}^t \partial_\eta \rho_\eta. \quad (1.20)$$

We now compute the self contributions to the scattering integral from Eq (1.18) and (1.12)

$$\begin{aligned} \mathcal{I}_{\theta_1}^{\text{self}} = & -2\pi \int (\delta_{\sigma, \nu_2} \delta_{\theta_1, \nu_3} + \delta_{\theta_1, \nu_2} \delta_{\sigma, \nu_3}) (\delta_{\nu_1, \zeta_1} \delta_{\nu_2, \xi_2} + \delta_{\nu_2, \zeta_1} \delta_{\nu_1, \xi_2}) \frac{T_{\sigma, \theta_1}^{\text{dr}} T_{\zeta_1, \xi_2}^{\text{dr}}}{\rho_{\theta_1}^t \rho_{\zeta_1}^t} \frac{1}{v_{\theta_1, \nu_1}} \frac{v_{\sigma, \theta_1} v_{\zeta_1, \xi_2}}{|v_{\nu_1, \nu_2}|} \frac{|v_{\nu_1, \nu_2}|}{|v_{\nu_2, \nu_3}| v_{\zeta_1, \nu_3}} \frac{v_{\nu_1, \nu_2}}{v_{\nu_2, \nu_3}} \times \\ & \times \rho_{\nu_1}^t \rho_{\xi_2} f_{\xi_2} (R_{\zeta_1, \gamma}^t \partial_\gamma \rho_\gamma \rho_{\nu_3}^t \rho_{\nu_3} f_{\nu_3} - \rho_{\zeta_1}^t \rho_{\zeta_1} f_{\zeta_1} R_{\nu_3, \gamma}^t \partial_\gamma \rho_\gamma) \int dk k^2 V_0^2 \hat{\varphi}(k) \hat{\varphi}\left(\frac{v_{\nu_1, \nu_2}}{v_{\nu_2, \nu_3}} k\right) + \\ & + 2\pi \int (\delta_{\sigma, \nu_1} \delta_{\eta, \nu_2} + \delta_{\eta, \nu_1} \delta_{\sigma, \nu_2}) (\delta_{\nu_1, \zeta_1} \delta_{\theta_1, \xi_2} + \delta_{\theta_1, \zeta_1} \delta_{\nu_1, \xi_2}) \frac{T_{\sigma, \eta}^{\text{dr}} T_{\zeta_1, \xi_2}^{\text{dr}}}{\rho_\eta^t \rho_{\zeta_1}^t} \frac{1}{v_{\theta_1, \eta}} \frac{v_{\sigma, \eta} v_{\zeta_1, \xi_2}}{|v_{\nu_1, \theta_1}|} \frac{|v_{\nu_1, \theta_1}|}{|v_{\nu_1, \nu_2}| v_{\zeta_1, \nu_2}} \frac{v_{\nu_1, \theta_1}}{v_{\nu_2, \nu_3}} \times \\ & \times \rho_\eta^t \rho_{\xi_2} f_{\xi_2} (R_{\zeta_1, \gamma}^t \partial_\gamma \rho_\gamma \rho_{\nu_2}^t \rho_{\nu_2} f_{\nu_2} - \rho_{\zeta_1}^t \rho_{\zeta_1} f_{\zeta_1} R_{\nu_2, \gamma}^t \partial_\gamma \rho_\gamma) \int dk k^2 V_0^2 \hat{\varphi}(k) \hat{\varphi}\left(\frac{v_{\nu_1, \theta_1}}{v_{\nu_1, \nu_2}} k\right). \end{aligned} \quad (1.21)$$

In order to write the former formula we used the Fourier transform property $FT[f(ax)](k) = (1/|a|)FT[f(x)](k/a)$. This expression is composed by two different types of contributions: proportional to $T_{\theta_1, \bullet}^{\text{dr}}, T_{\theta_1, \bullet}^{\text{dr}}$ or $T_{\theta_1, \bullet}^{\text{dr}}, T_{\bullet, \bullet}^{\text{dr}}$. More explicitly they can be written as

$$\mathcal{I}_{\theta_1}^{\text{self}} = 2\pi \int d\zeta d\sigma \int dk k^2 V_0^2 \hat{\varphi}(k) \rho_\sigma^t T_{\theta_1, \zeta}^{\text{dr}} \frac{v_{\zeta, \theta_1}}{v_{\theta, 1\sigma}} \left[\hat{\varphi}\left(\frac{v_{\zeta, \theta_1}}{v_{\theta_1, \sigma}} k\right) \frac{\rho_\zeta^t}{\rho_{\theta_1}^t} T_{\sigma, \theta_1}^{\text{dr}} \frac{1}{|v_{\theta_1, \sigma}| v_{\sigma, \zeta}} - 2\hat{\varphi}\left(\frac{v_{\zeta, \theta_1}}{v_{\sigma, \zeta}} k\right) T_{\zeta, \sigma}^{\text{dr}} \frac{1}{v_{\theta_1, \sigma} |v_{\zeta, \sigma}|} \right] \mathcal{B}(\theta_1, \zeta, \sigma). \quad (1.22)$$

This concludes the derivation of the generalised Landau equation.

2. EFFECTIVE DIFFUSION IN THE SMALL TIME LIMIT FOR TWO-POINT FUNCTIONS

In this section we show how, in the small time limit $t \rightarrow 0^+$, i.e., close to a GGE state, the $1/\xi$ correction to the convective spreading of two-point function is diffusive. We also show that the diffusion matrix is the one associated with the local equilibrium state around which we are linearising. For the sake of simplicity, we do this derivation in the small V_0 case, i.e. when the solution evolution equation for three-point function is given by (1.7), otherwise we

conjecture that the result is valid beyond this approximation. Hence, we consider the limit $t \rightarrow 0^+$ of the three-point function solution (1.9)

$$\begin{aligned} \lim_{t \rightarrow 0^+} g_{\theta_1, \theta_2, \theta_3}^{(3), \text{sym}}(0^+, z'; 0^+) &= \frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} \frac{1}{|v_{\gamma_1, \gamma_2}|} R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t M_{\mu_1, \mu_2}^\beta \partial_1 g_{\beta, \theta_3}^{(2)}(-z'; 0^+) \\ &+ \frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_1, \gamma_3}} [\text{sgn}(z') + \text{sgn}(v_{\gamma_1, \gamma_3})] R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_1, \mu_3}^\beta \partial_1 g_{\beta, \mu_2}^{(2)}\left(\frac{v_{\gamma_1, \gamma_2}}{v_{\gamma_1, \gamma_3}} z'; 0^+ - \frac{z'}{v_{\gamma_1, \gamma_3}}\right) \\ &+ \frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} R_{\theta_3, \gamma_3}^{-t} \frac{1}{v_{\gamma_2, \gamma_3}} [\text{sgn}(z') + \text{sgn}(v_{\gamma_2, \gamma_3})] R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t R_{\gamma_3, \mu_3}^t M_{\mu_2, \mu_3}^\beta \partial_1 g_{\beta, \mu_1}^{(2)}\left(\frac{v_{\gamma_2, \gamma_1}}{v_{\gamma_2, \gamma_3}} z'; 0^+ - \frac{z'}{v_{\gamma_2, \gamma_3}}\right). \end{aligned} \quad (2.1)$$

Here, we stress that, at $t = 0^+$, few point correlations in the system are not identically null, yet small. Crucially, imposing the positivity of time dependence in the argument of the two-point function in the last two lines, we observe that they are vanishing. More precisely, considering the second line, we have $z'/v_{\gamma_1, \gamma_3} < 0$. This imposes the identity $\text{sgn}(z') + \text{sgn}(v_{\gamma_1, \gamma_3}) = 0$. For analogous reasons, the third line is also vanishing. Hence, we find

$$\lim_{t \rightarrow 0^+} g_{\theta_1, \theta_2, \theta_3}^{(3), \text{sym}}(x_1^+, x_1^-, x_2; 0^+) = -\frac{1}{2} R_{\theta_1, \gamma_1}^{-t} R_{\theta_2, \gamma_2}^{-t} \frac{1}{|v_{\gamma_1, \gamma_2}|} R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t M_{\mu_1, \mu_2}^\beta \partial_{x_1} g_{\beta, \theta_3}^{(2)}(x_1, x_2; 0^+), \quad (2.2)$$

where we have expressed the derivative in terms of the space coordinate. Using this result in the equation for two-point function, we get the effective contribution

$$\begin{aligned} \lim_{t \rightarrow 0^+} \partial_{x_1} H_{\theta_1}^{\gamma, \eta} g_{\gamma, \eta, \theta_2}^{(3), \text{sym}}(x_1^+, x_1^-, x_2; t) &= -\frac{1}{2} \partial_{x_1} H_{\theta_1}^{\gamma, \eta} R_{\gamma, \gamma_1}^{-t} R_{\eta, \gamma_2}^{-t} \frac{1}{|v_{\gamma_1, \gamma_2}|} R_{\gamma_1, \mu_1}^t R_{\gamma_2, \mu_2}^t M_{\mu_1, \mu_2}^\beta \partial_{x_1} g_{\beta, \theta_2}^{(2)}(x_1, x_2; 0^+) \\ &= -\frac{1}{2} \partial_{x_1} \mathfrak{D}_{\theta_1}^\gamma \partial_{x_1} g_{\gamma, \theta_2}^{(2)}(x_1, x_2; 0^+) \end{aligned} \quad (2.3)$$

where the identity used in the second line has been proven in [141].

3. GBBGKY HIERARCHY FOR A SYSTEM OF MANY TUBES

In this section we extend the gBBGKY hierarchy introduced in the main text to the physically relevant case of a system composed by many one dimensional tubes interacting through long-range interactions. More precisely, we consider systems of N_t tubes evolving according to the global Hamiltonian

$$\hat{H} = \sum_{t=1}^{N_t} \left[\hat{H}_{a,t}^{\text{int}} + \frac{1}{2} \sum_{t'=1}^{N_t} \int dx \, dx' \, V_{(t,t')}(x - x') \hat{q}_0^{(t)}(x) \hat{q}_0^{(t')}(x') \right], \quad (3.1)$$

where $H_{a,t}$ is the unperturbed integrable Hamiltonian associated to the t -th tube, $\hat{q}_i^{(t)}$ its i -th charge density, and $V_{(t,t')}$ the interaction potential between t -th and t' -th tubes. In particular $V_{(t,t)}$ represent the intra tube interaction. Importantly, charge density operators associated to different tubes always commute

$$[\hat{q}_i^{(t)}, \hat{q}_j^{(t')}] = 0 \quad \text{for any } t' \neq t. \quad (3.2)$$

In this section we repeat the main steps for the derivation of the hierarchy in analogy with single tube case. The first step is to compute the quantum generalisation of the classical Liouville's equation

$$\partial_t \hat{q}_i^{(t)}(x) + \partial_x \hat{j}_i^{(t)}(x) = - \sum_{t'=1}^{N_t} \int dx' V'_{(t,t')}(x - x') \hat{j}_{i,0}^{(t)}(x) \hat{q}_0^{(t')}(x'), \quad (3.3)$$

where we have $i[\hat{H}_{a,t}^{\text{int}}, \hat{q}_i^{(t)}] = \partial_x \hat{j}_i^{(t)}$. It is already interesting to observe that the evolution equation is completely analogous to the single tube case. The evolution equation for the expectation value of the product of single charges produces the hierarchical structure. For the two function it reads

$$\partial_t \langle q_{i_1}^{(t_1)}(x_1) q_{i_2}^{(t_2)}(x_2) \rangle = - \left[\partial_{x_1} \langle j_{i_1}^{(t_1)}(x_1) q_{i_2}^{(t_2)}(x_2) \rangle - \sum_{t'=1}^{N_t} \int dy V'_{(t_1,t')}(x_1 - y) \langle j_{i_1,0}^{(t_1)}(x_1) q_{i_2}^{(t_2)}(x_2) q_0^{(t')}(y) \rangle \right]_{(1,2)}. \quad (3.4)$$

It is now important to specify that, because of the interactions between different tubes, the system will develop long-range correlations between them. Hence, in order to introduce them, we perform the average over the ensemble

$$\begin{aligned} \rho = \exp \Big(& - \sum_{t_1=1}^{N_t} \int dx \beta_{(t_1)}^i(x) q_i^{(t_1)}(x) - \sum_{\substack{t_1, t_2=1 \\ \text{if same tube}}}^{N_t} \int d\vec{x} \beta_{(t_1, t_2)}^{i, j}(x_1, x_2) \delta q_i^{(t_1)}(x_1) \delta q_j^{(t_2)}(x_2) \\ & - \sum_{\substack{t_1, t_2, t_3=1 \\ \text{if same tube}}}^{N_t} \int d\vec{x} \beta_{(t_1, t_2, t_3)}^{i, j, k}(x_1, x_2, x_3) \delta q_i^{(t_1)}(x_1) \delta q_j^{(t_2)}(x_2) \delta q_k^{(t_3)}(x_3) + \dots \Big). \end{aligned} \quad (3.5)$$

Here, $\beta_{(t_1, \dots, t_n)}^{i_1, \dots, i_n}$ is the Lagrange multiplier associated to the n -point function between tubes $t_1, \dots, t_n$. In addition, we observe that, since the tubes are spatially separated, the connected correlations between them do not exhibit any ultralocal component proportional to delta function, thus we have to exclude from the integral only intratube coinciding points. For example for two and three-point correlator we have

$$\langle q_{i_1}^{(t_1)}(x_1) q_{i_2}^{(t_2)}(x_2) \rangle^c = \delta_{t_1, t_2} \delta(x_1 - x_2) C_{i_1 i_2}^{(t_1)}(x_1) + g_{i_1 i_2}^{(t_1, t_2)}(x_1, x_2) \quad (3.6)$$

$$\langle q_{i_1}^{(t_1)} q_{i_2}^{(t_2)} q_{i_3}^{(t_3)} \rangle^c = \delta_{t_1, t_2} \delta_{t_1, t_3} \delta(x_1 - x_2) \delta(x_1 - x_3) C_{i_1 i_2 i_3}^{(t_1)} + [\delta_{t_1, t_2} \delta(x_1 - x_2) C_{i_1 i_2 j}^{(t_1)} C_{(t_1)}^{j k} g_{k, i_3}^{(t_1, t_3)}]_{(1, 2, 3)} + g_{i_1 i_2 i_3}^{(t_1, t_2, t_3)}. \quad (3.7)$$

Straightforwardly, if the correlations functions are computed in the same tube, this relations coincide with the ones already introduced in the main text. The state assumption (3.5), together with (3.3), (3.6) and (3.7) is enough to derive the full hierarchy of equations and express it in terms of quasiparticles occupations and correlations functions. In order to avoid confusions, we stress that in this context we changed notation: the connected regular few point functions now, instead of an upper index showing the number of points, have a string of indexes indicating the string of tubes connected by the function.

A. Example: Two tubes system

In this section we consider the simple case of two coupled one dimensional gases evolving through the Hamiltonian

$$\hat{H} = \hat{H}_{a,1}^{\text{int}} + \hat{H}_{a,2}^{\text{int}} + \int dx dx' V_{(1,2)}(x - x') \hat{q}_0^{(1)}(x) \hat{q}_0^{(2)}(x'). \quad (3.8)$$

We observe that, differently from the single tube case, here the potential doesn't have the symmetric factor 1/2 in front since the integration in x and x' is not double counting particles. Also, we stress that, in this specific example, the particles are not interacting within the same tube. The aim of this section is to write the first layers of the hierarchy and derive the associated Landau equation. The single charge density time evolution reads

$$\partial_t \hat{q}_i^{(1)}(x) + \partial_x \hat{j}_i^{(1)}(x) = - \int dx' V'_{(1,2)}(x - x') \hat{j}_{i,0}^{(1)}(x) \hat{q}_0^{(2)}(x'), \quad (3.9)$$

Taking the coarse grained ensemble average, for a homogeneous system, we find

$$\partial_t q_i^{(1)}(x) = - \int dx' V'_{(1,2)}(x - x') A_{i,0}^{(1),k} \langle q_k^{(1)}(x) q_0^{(2)}(x') \rangle^c. \quad (3.10)$$

Hence, the inter tube two-point correlation evolves according to

$$\begin{aligned} \partial_t \langle q_{i_1}^{(1)}(x_1) q_{i_2}^{(2)}(x_2) \rangle^c + \left[\partial_{x_1} \langle j_{i_1}^{(1)}(x_1) q_{i_2}^{(2)}(x_2) \rangle^c \right]_{(1,2)} = \\ = - \left[\int dy'_{(1,2)} (x_1 - y) j_{i_1,0}^{(1)}(x_1) \langle q_0^{(2)}(y) q_{i_2}^{(2)}(x_2) \rangle^c + \langle j_{i_1,0}^{(1)}(x_1) q_0^{(2)}(y) q_{i_2}^{(2)}(x_2) \rangle^c \right]_{(1,2)}. \end{aligned} \quad (3.11)$$

We here observe that the dynamics of this object depends on the intra tube correlation function, that evolves according to a similar equation that, for the sake of brevity, will not be written here. We now extract the singular part of the connected correlations and perform the space and time rescaling with ξ , finding

$$\partial_t \langle q_i^{(1)}(x) \rangle = - \int dx' V'_{(1,2)}(x - x') A_{i,0}^{(1),k} g_{k,0}^{(1,2)}(x, x'). \quad (3.12)$$

$$\begin{aligned}
& \partial_t g_{i_1, i_2}^{(1,2)}(x_1, x_2) + \left[\partial_{x_1} A_{i_1}^{(1),k} g_{k, i_2}^{(1,2)}(x_1, x_2) + \frac{1}{2} \partial_{x_1} H_{i_1}^{(1),jk} g_{jki_2}^{(1,1,2),\text{sym}}(x_1^+, x_1^-, x_2) \right]_{(1,2)} = \\
& = - \left[V'_{(1,2)}(x_1 - x_2) A_{i_1,0}^{(1),k} \langle q_k^{(1)} \rangle C_{0i_2}^{(2)} + A_{i_1,0}^{(1),k} g_{km}^{(1,2)}(x_1, x_2) C_{n,0,i_2}^{(2),mn} + \right. \\
& \left. - \int dy V'_{(1,2)}(x_1 - y) A_{i_1,0}^{(1),k} \langle q_k^{(1)} \rangle g_{0,i_2}^{(2,2)}(y, x_2) + A_{i_1,0}^{(1),k} g_{k,0,i_2}^{(1,2,2)}(x_1, y, x_2) \right]_{(1,2)}. \tag{3.13}
\end{aligned}$$

$$\begin{aligned}
& \partial_t g_{i_1 i_2 i_3}^{(1,1,2)}(x_1, x_2, x_3) + \partial_{x_1} A_{i_1}^{(1),k} g_{ki_2 i_3}^{(1,1,2)} + \partial_{x_2} A_{i_2}^{(1),k} g_{i_1 k i_3}^{(1,1,2)} + \partial_{x_3} A_{i_3}^{(2),k} g_{i_1 i_2 k}^{(1,1,2)} = \\
& = -\delta(x_1 - x_2) \left(A_{i_1}^{(1),k} C_{k,i_2,m}^{(1)} C^{(1),mn} - C_{i_1,i_2,k}^{(1)} C^{(1),km} A_m^{(1),n} + H_{i_1}^{(1),mn} C_{m,i_2}^{(1)} \right) \partial_{x_1} g_{n,i_3}^{(1,2)}(x_1, x_3) + \text{h.o.t.} \tag{3.14}
\end{aligned}$$

Here, we crucially observe that, differently from what found for the single tube case, the term proportional to Dirac delta function of RHS of Eq. (3.14) is not symmetric with respect to permutation of indexes, the consequence of this will be that absence of self contributions in the generalised Landau equation for two coupled tubes. We now proceed writing these equations in quasiparticle picture using the same machinery introduced in the main text, finding

$$\partial_t \rho_{\theta_1}^{(1)} = \int dx_2 d\theta_2 V'_{(1,2)}(x_1 - x_2) \partial_{\theta_1} g_{\theta_1, \theta_2}^{(1,2)}(x_1, x_2) \tag{3.15}$$

$$\begin{aligned}
& \partial_t g_{\theta_1, \theta_2}^{(1,2)}(x_1, x_2) + \left[\partial_{x_1} (A_{\theta_1}^{(1)\gamma} g_{\gamma, \theta_2}^{(1,2)}(x_1, x_2)) + \frac{1}{2} \partial_{x_1} (H_{\theta_1}^{(1),\gamma\gamma'} g_{\gamma, \gamma', \theta_2}^{(1,1,2),\text{sym}}(x_1^+, x_1^-, x_2)) \right]_{(1,2)} = \\
& = \left[V'_{(1,2)}(x_1 - x_2) \int d\theta_3 \partial_{\theta_1} \left(\rho_{\theta_1}^{(1)} C_{\theta_2 \theta_3}^{(2)} + C_{\theta_2, \theta_3}^{(1),\gamma} g_{\theta_1, \gamma}^{(1,2)}(x_1, x_2) \right) \right. \\
& \left. + \int dx_3 d\theta_3 V''_{(1,2)}(x_1 - x_3) \partial_{\theta_1} \left(\rho_{\theta_1}^{(1)} g_{\theta_2, \theta_3}^{(1,2)}(x_2, x_3) + g_{\theta_1, \theta_2, \theta_3}^{(1,2,2)}(x_1, x_2, x_3) \right) \right]_{(1,2)} \tag{3.16}
\end{aligned}$$

$$\begin{aligned}
& \partial_t g_{\theta_1, \theta_2, \theta_3}^{(1,1,2)}(x_1, x_2, x_3) + \partial_{x_1} A_{\theta_1}^{(1),\gamma} g_{\gamma \theta_2 \theta_3}^{(1,1,2)}(x_1, x_2, x_3) + \partial_{x_2} A_{\theta_2}^{(2),\gamma} g_{\theta_1 \gamma \theta_3}^{(1,1,2)}(x_1, x_2, x_3) + \partial_{x_3} A_{\theta_3}^{(3),\gamma} g_{\theta_1, \theta_2, \gamma}^{(1,1,2)}(x_1, x_2, x_3) = \\
& = \delta(x_1 - x_2) M_{\theta_1, \theta_2}^{\gamma} \partial_{x_1} g_{\gamma, \theta_3}^{(1,2)}(x_1, x_3) + \text{h.o.t.} \tag{3.17}
\end{aligned}$$

We now proceed by computing the Landau equation in this specific setup. The main difference from the single tube case is given by the time integration of three-point function, that here reads

$$g_{\theta_1, \theta_2, \theta_3}^{(1,1,2),\text{sym}}(0^+, z'; t) = -\frac{1}{2} R_{\theta_1, \gamma_1}^{(1),-t} R_{\theta_2, \gamma_2}^{(1),-t} \frac{1}{|v_{\gamma_1, \gamma_2}^{(1,1)}|} R_{\gamma_1, \mu_1}^{(1),t} R_{\gamma_2, \mu_2}^{(1),t} M_{\mu_1, \mu_2}^{(1),\beta} \partial_{z'} g_{\beta, \theta_3}^{(1,2),1}(-z'; t) \tag{3.18}$$

where we introduced $v_{\gamma_1 \gamma_2}^{(1,1)} \equiv v_{\gamma_1}^{(1)} - v_{\gamma_2}^{(1)}$. As compared to (1.9), this solution is simpler, since it only contains contributions local in time. The late time solution for $g^{(1,2),1}$ also reads

$$\lim_{t \rightarrow \infty} g_{\theta_1, \theta_2}^{(1,2),1}(z, t) = V'_{(1,2)}(z') R_{\theta_1, \zeta_1}^{(1),-t} R_{\theta_2, \zeta_2}^{(2),-t} \frac{1}{v_{\zeta_1, \zeta_2}^{(1,2)}} R_{\zeta_1, \eta_1}^{(1),t} R_{\zeta_2, \eta_2}^{(2),t} \left(\partial_{\eta_1} \rho_{\eta_1}^{(1)} C_{\gamma, \eta_2}^{(2)} - C_{\eta_1, \gamma}^{(1)} \partial_{\eta_2} \rho_{\eta_2}^{(2)} \right). \tag{3.19}$$

Now, the procedure to derive a scattering integral is exactly analogous to the one introduced in Sec.1. Defining the coupled kinetic equations as

$$\partial_t \rho_{\theta}^{(1)} = \partial_{\theta} \mathcal{I}^{(1)}[\rho^{(1)}, \rho^{(2)}]_{\theta} \quad , \quad \partial_t \rho_{\theta}^{(2)} = \partial_{\theta} \mathcal{I}^{(2)}[\rho^{(1)}, \rho^{(2)}]_{\theta}, \tag{3.20}$$

we write the scattering integral as

$$\begin{aligned}
& \mathcal{I}_{\theta_1}^{(1)} = 2\pi \int dk k^2 \hat{V}_{(1,2)}^2(z) \times \\
& \times \left[\int (\delta_{\theta_1, \mu_1} - \delta_{\theta_1, \mu_2}) (T_{\mu_2, \mu_1}^{(1)\text{dr}})^2 \frac{\rho_{\mu_2}^{(1)} f_{\mu_2}^{(1)} \rho_{\zeta}^{(2),\text{tot}}}{(\rho_{\mu_1}^{(1)\text{tot}})^2} \frac{|v_{\mu_2, \mu_1}^{(1,1)}|}{v_{\mu_1, \zeta}^{(1,2)} v_{\theta_1, \zeta}} (R_{\mu_1, \eta}^{(1),t} \partial_{\eta} \rho_{\eta}^{(1)} \rho_{\zeta}^{(2),\text{tot}} \rho_{\zeta}^{(2)} f_{\zeta}^{(2)} - \rho_{\mu_1}^{(2),\text{tot}} \rho_{\mu_1}^{(2)} f_{\mu_1}^{(2)} R_{\zeta, \eta}^{(2),t} \partial_{\eta} \rho_{\eta}^{(2)}) \right. \\
& \left. + \int (\delta_{\zeta, \mu_1} - \delta_{\zeta, \mu_2}) (T_{\mu_2, \mu_1}^{(2)\text{dr}})^2 \frac{\rho_{\mu_2}^{(2)} f_{\mu_2}^{(2)} \rho_{\zeta}^{(2),\text{tot}}}{(\rho_{\mu_1}^{(2),\text{tot}})^2} \frac{|v_{\mu_2, \mu_1}^{(2,2)}|}{v_{\theta_1, \mu_1}^{(1,2)} v_{\theta_1, \zeta}} (R_{\theta_1, \eta}^{(1),t} \partial_{\eta} \rho_{\eta}^{(1)} \rho_{\mu_1}^{(2),\text{tot}} \rho_{\mu_1}^{(2)} f_{\mu_1}^{(2)} - \rho_{\theta_1}^{(1),\text{tot}} \rho_{\theta_1}^{(1)} f_{\theta_1}^{(1)} R_{\mu_1, \eta}^{(2),t} \partial_{\eta} \rho_{\eta}^{(2)}) \right]. \tag{3.21}
\end{aligned}$$

Straightforwardly, the scattering integral $\mathcal{I}^{(2)}$ is equivalent up to exchange of tube indexes $1 \longleftrightarrow 2$. As already mentioned, we stress that in this case there is no cross terms proportional to $T^{(1)} \times T^{(2)}$ appear in the equation. This crucially represent the main qualitative difference between the two tubes late time dynamics, and the single tube case.

4. LINEARIZED GENERALISED LANDAU EQUATION

In this section we consider the generalised Landau equation

$$\partial_t \rho_\theta = \mathcal{I}_\theta[\rho] \quad (4.1)$$

close to stationary, thermal state. We thus write $\partial_t \delta \rho_\theta = \left(\frac{\delta \mathcal{I}}{\delta \rho} \right)_{\theta_\gamma} \delta \rho_\gamma$. Our goal is to change the degrees of freedom from $\delta \rho$ to perturbations of expectation values of conserved charges δq_n defined as $\delta q_n = \int d\theta h_n(\theta) \delta \rho(\theta)$.

Firstly, we note that in general it is more natural to work with perturbations of bare pseudoenergy $\delta \epsilon_0$, instead $\delta \rho$. We have the following relations (see SM of [77])

$$\delta \rho_\theta = -C_{\theta\gamma}(\delta \epsilon_0)_\gamma \quad (4.2)$$

and we also introduce

$$\left(\frac{\delta \mathcal{I}}{\delta \rho} \right)_{\theta_\gamma} \delta \rho_\gamma = \Gamma_{\theta\gamma}(\delta \epsilon_0)_\gamma. \quad (4.3)$$

In what follows we introduce hydrodynamic inner product defined as

$$(h|g) = h_\theta C_{\theta\theta'} g_{\theta'} \quad (4.4)$$

and expand

$$(\delta \epsilon_0)_\theta = \sum_m (h_m | \delta \epsilon_0) (h_m)_\theta \quad (4.5)$$

where h_m form an orthonormal set with respect to $(\cdot|\cdot)$, constructed with Gram-Schmidt algorithm performed on e.g. set of ultra-local charges $g_m(\theta) = \theta^m/m!$. We note that it follows from the formulas above that $\delta q_n = -(h_n | \delta \epsilon_0)$. We multiply $\square$ by h_n and integrate over $d\theta$ finally getting

$$\partial_t \delta q_n = \Gamma_{nm} \delta q_m, \quad \Gamma_{nm} = (h_n)_\theta \Gamma_{\theta\theta'} (h_m)_{\theta'}. \quad (4.6)$$

In what follows we determine the matrix Γ_{nm} . We start with expressions (67) and (68). As a first step, it is useful to rewrite $\mathcal{B}$ factor a bit using

$$R_{\theta\gamma}^t \partial_\gamma \rho_\gamma = -\rho_\theta f_\theta \partial_\theta \epsilon_\theta \quad (4.7)$$

where ϵ is the pseudoenergy of the state. This identity follows from standard TBA calculations and is derived in SM of [77]. It leads to

$$\mathcal{B}(\theta, \zeta, \sigma) = -\rho_\theta f_\theta \rho_\sigma f_\sigma \rho_\zeta f_\zeta \left[\frac{v_{\sigma,\zeta}}{\rho_\theta^t} \partial_\theta \epsilon_\theta + \frac{v_{\zeta,\theta}}{\rho_\sigma^t} \partial_\sigma \epsilon_\sigma + \frac{v_{\theta,\sigma}}{\rho_\zeta^t} \partial_\zeta \epsilon_\zeta \right]. \quad (4.8)$$

We note that for thermal state $\epsilon_\theta = \beta(\varepsilon_\theta - \mu)$. Moreover, $k'_\theta = 2\pi\rho_\theta^t$ and $v_\theta = \varepsilon'_\theta/k'_\theta$. Thus for thermal states

$$\mathcal{B}(\theta, \zeta, \sigma) = -2\pi\beta\rho_\theta f_\theta \rho_\sigma f_\sigma \rho_\zeta f_\zeta [v_{\sigma,\zeta} v_\theta + v_{\zeta,\theta} v_\sigma + v_{\theta,\sigma} v_\zeta] = 0 \quad (4.9)$$

as expected. We thus consider $\epsilon \rightarrow \epsilon_{\text{th}} + \delta \epsilon_0$ and linearize (to see why large Dressing appears in the expression below see SM of [77], where similar calculation was performed)

$$\begin{aligned} (\Gamma_0^{\text{cross}} \delta \epsilon_0)_\theta = & -2\pi \int dk k^2 \hat{V}^2(k) \int d\sigma d\zeta \left[\rho_\theta^t (T_{\sigma,\zeta}^{\text{dr}})^2 \frac{|v_{\sigma,\zeta}|}{v_{\theta,\sigma} v_{\zeta,\theta}^2} - \frac{(\rho_\sigma^t)^2}{\rho_\theta^t} (T_{\zeta,\theta}^{\text{dr}})^2 \frac{|v_{\zeta,\theta}|}{v_{\sigma,\zeta} v_{\theta,\sigma}^2} \right] \rho_\theta f_\theta \rho_\sigma f_\sigma \rho_\zeta f_\zeta \times \\ & \left[\frac{v_{\sigma,\zeta}}{\rho_\theta^t} \partial_\theta (\delta \epsilon_0)_\theta^{\text{Dr}} + \frac{v_{\zeta,\theta}}{\rho_\sigma^t} \partial_\sigma (\delta \epsilon_0)_\sigma^{\text{Dr}} + \frac{v_{\theta,\sigma}}{\rho_\zeta^t} \partial_\zeta (\delta \epsilon_0)_\zeta^{\text{Dr}} \right], \end{aligned} \quad (4.10)$$

where Dressing operation is defined as

$$f_\mu^{\text{Dr}} = f_\mu - \int d\mu F_{\mu,\lambda} n_\mu f'_\mu. \quad (4.11)$$

$$\begin{aligned} (\Gamma_0^{\text{self}} \delta\epsilon_0)_\theta = & - \int d\zeta d\sigma \int \frac{dk}{2\pi} k^2 \hat{V}(k) \rho_\sigma^t T_{\theta\zeta}^{\text{dr}} \frac{v_{\zeta\theta}}{v_{\theta\sigma}} \left[\hat{V}\left(\frac{v_{\zeta\theta}}{v_{\theta\sigma}} k\right) \frac{\rho_\zeta^t T_{\sigma\theta}^{\text{dr}}}{\rho_\theta^t} \frac{1}{|v_{\theta\sigma}| v_{\sigma\zeta}} - 2\hat{V}\left(\frac{v_{\zeta\theta}}{v_{\sigma\zeta}} k\right) T_{\zeta\sigma}^{\text{dr}} \frac{1}{v_{\theta\sigma} |v_{\zeta\sigma}|} \right] \times \\ & \rho_\theta f_\theta \rho_\sigma f_\sigma \rho_\zeta f_\zeta \left[\frac{v_{\sigma\zeta}}{\rho_\theta^t} \partial_\theta (\delta\epsilon_0)_\theta^{\text{Dr}} + \frac{v_{\zeta\theta}}{\rho_\sigma^t} \partial_\sigma (\delta\epsilon_0)_\sigma^{\text{Dr}} + \frac{v_{\theta\sigma}}{\rho_\zeta^t} \partial_\zeta (\delta\epsilon_0)_\zeta^{\text{Dr}} \right], \end{aligned} \quad (4.12)$$

To simplify these expressions we use that under integral we can replace

$$2 \frac{|v_{\sigma,\zeta}|}{v_{\theta,\sigma} v_{\zeta,\theta}^2} \rightarrow - \frac{|v_{\sigma,\zeta}| v_{\sigma,\zeta}}{v_{\theta,\sigma}^2 v_{\zeta,\theta}^2} \quad (4.13)$$

We get

$$\begin{aligned} \Gamma_{n,m}^{\text{cross}} = & \pi \int dk k^2 \hat{V}^2(k) \int d\theta d\sigma d\zeta (\rho_\theta^t)^2 (T_{\sigma,\zeta}^{\text{dr}})^2 \frac{|v_{\sigma,\zeta}|}{v_{\theta,\sigma}^2 v_{\zeta,\theta}^2} \rho_\theta f_\theta \rho_\sigma f_\sigma \rho_\zeta f_\zeta \times \\ & \left(\frac{v_{\sigma,\zeta}}{\rho_\theta^t} (h'_n)_\theta^{\text{dr}} + \frac{v_{\zeta,\theta}}{\rho_\sigma^t} (h'_n)_\sigma^{\text{dr}} + \frac{v_{\theta,\sigma}}{\rho_\zeta^t} (h'_n)_\zeta^{\text{dr}} \right) \left(\frac{v_{\sigma,\zeta}}{\rho_\theta^t} (h'_m)_\theta^{\text{dr}} + \frac{v_{\zeta,\theta}}{\rho_\sigma^t} (h'_m)_\sigma^{\text{dr}} + \frac{v_{\theta,\sigma}}{\rho_\zeta^t} (h'_m)_\zeta^{\text{dr}} \right), \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} \Gamma_{n,m}^{\text{self}} = & 2\pi \int dk k^2 \hat{V}(k) \int d\theta d\sigma d\zeta \rho_\zeta^t \rho_\sigma^t T_{\theta,\zeta}^{\text{dr}} T_{\theta,\sigma}^{\text{dr}} \hat{V}\left(\frac{v_{\theta,\zeta}}{v_{\sigma,\theta}} k\right) \frac{v_{\theta,\zeta}}{|v_{\sigma,\theta}| v_{\zeta,\sigma}^2 v_{\sigma,\theta}} \rho_\theta f_\theta \rho_\sigma f_\sigma \rho_\zeta f_\zeta \times \\ & \left(\frac{v_{\sigma,\zeta}}{\rho_\theta^t} (h'_n)_\theta^{\text{dr}} + \frac{v_{\zeta,\theta}}{\rho_\sigma^t} (h'_n)_\sigma^{\text{dr}} + \frac{v_{\theta,\sigma}}{\rho_\zeta^t} (h'_n)_\zeta^{\text{dr}} \right) \left(\frac{v_{\sigma,\zeta}}{\rho_\theta^t} (h'_m)_\theta^{\text{dr}} + \frac{v_{\zeta,\theta}}{\rho_\sigma^t} (h'_m)_\sigma^{\text{dr}} + \frac{v_{\theta,\sigma}}{\rho_\zeta^t} (h'_m)_\zeta^{\text{dr}} \right). \end{aligned} \quad (4.15)$$

5. SMALL-MOMENTUM LIMIT OF FERMI'S GOLDEN RULE COLLISION INTEGRAL

We start with the bare collision integral focusing on the $m = 3$ case

$$\mathcal{I}_0^{(3)}[\rho_p](\theta) = \int d\mathbf{p} d\mathbf{h} \delta(\theta, \mathbf{p}) B(\mathbf{p} \rightarrow \mathbf{h}) [\rho_h(\mathbf{p}) \rho_p(\mathbf{h}) - \rho_h(\mathbf{h}) \rho_p(\mathbf{p})], \quad (5.1)$$

where $\delta(\theta, \mathbf{p}) = \sum_{i=1}^3 \delta(\theta - p_i)$ and other quantities are defined as before. We introduce $\lambda_i = (p_i + h_i)/2$ and $q_i = k(p_i) - k(h_i)$. Assuming that $\alpha_i = p_i - h_i$ are small for all i we can approximate $q_i \approx k'(\lambda_i) \alpha_i$ leading to $p_i = \lambda_i + \frac{q_i}{2k'(\lambda_i)}$ and $h_i = \lambda_i - \frac{q_i}{2k'(\lambda_i)}$, from which follow the expansion of delta term

$$\delta(\theta, \mathbf{p}) = \delta(\theta, \lambda) + \sum_{i=1}^3 \frac{q_i}{2k'(\lambda_i)} \delta'(\theta - \lambda_i) + \mathcal{O}(q_i^2), \quad (5.2)$$

and difference of the density factors

$$\rho_h(\mathbf{p}) \rho_p(\mathbf{h}) - \rho_h(\mathbf{h}) \rho_p(\mathbf{p}) = \rho_h(\mathbf{h}) \rho_p(\mathbf{p}) \left(\sum_{i=1}^3 q_i \frac{\epsilon'(\lambda_i)}{k'(\lambda_i)} + \mathcal{O}(q_i^3) \right), \quad (5.3)$$

where we have introduced pseudoenergy $\epsilon(\lambda) = \log(\rho_h(\lambda)/\rho_p(\lambda))$. The conservation laws inside B factors become $\delta(q_1 + q_2 + q_3)$ for the momentum and $\delta(v_1 q_1 + v_2 q_2 + v_3 q_3)$ for the energy. From the particle-hole symmetry of form factors the first term in (5.2) vanishes. We change integration variables getting (93).

2.6 Article VI: *Burgers equation from non-thermal stationary states in nearly-integrable gases*

- P. Lisiak, M. Łebek, M. Panfil “*Burgers equation from non-thermal stationary states in nearly-integrable gases*”, [arXiv:2512.09726](https://arxiv.org/abs/2512.09726) (under review in Physical Review E)

The universality of NS hydrodynamics is closely tied to the universality stemming from thermodynamics through the local equilibrium assumption. It is then quite natural to ask: what kind of hydrodynamics can we expect, if the system does not relax locally to a maximum-entropy state? This question will be the main theme of the present article, in which we build upon the non-thermal stationary states discovered in coupled integrable models found in Article I.

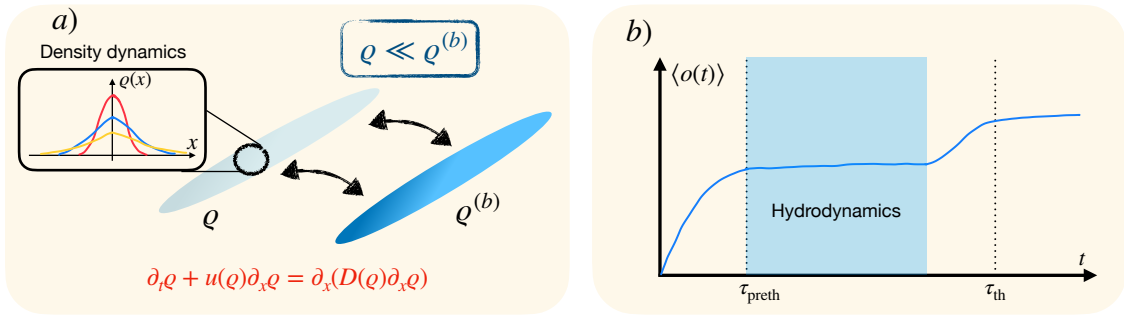


Figure 17: Panel *a*): the system studied. We consider the coupled integrable gases in the density imbalance limit. We derive hydrodynamic equation for density in system tube and in the most general case we find the nonlinear Burgers equation. Panel *b*): hydrodynamics is probed on top of a prethermal state in the temporal regime $\tau_{\text{preth}} \ll t_h \ll \tau_{\text{th}}$.

In comparison to the Article I, even on the level of relaxation in homogeneous systems, a more general perspective is taken. We consider two arbitrary gases, quantum or classical, weakly coupled with interactions that conserve the particle numbers, total momentum, and energy. We work in the imbalance limit, in which the density of the so-called system tube ρ is much smaller than the density of the bath tube $\rho^{(b)}$, see Fig. 17(a). In such a regime, the dynamics of the bath tube is effectively frozen. We seek the hydrodynamic description of the system tube on a timescales, on which the prethermal state is stationary, see Fig. 17(b).

The standard hydrodynamic equation that emerges in the system tube in the case of thermal bath is the diffusion equation. The only conserved charge in the system tube is the particle number, thus yielding a single hydrodynamic equation. In this work, it is shown that this result may be altered in the presence of non-thermal stationary states. We give an explicit condition under which a convective term $u(\rho)\partial_x \rho$ is brought into the diffusion equation. We also show that when the particles feature quantum statistics, $u(\rho)$ is not a constant. Thus, we arrive at the Burgers equation. Both the convective coefficient $u(\rho)$ and diffusion coefficient $D(\rho)$ are computed using a generalization of ChE method formulated for a single conserved charge.

To numerically test the predictions of the derived hydrodynamic equation, we proposed and simulated particle models with a stochastic two-body collision mechanism. For such models, we computed the transport coefficients analytically. We compared their molecular dynamics to the derived hydrodynamics, finding a very good agreement. Computations of transport coefficients were also made for the relaxation time approximation and for the coupled LL models studied in Article I.

Candidate's contribution: I led the analysis presented in Sec. 3 regarding athermal stationary states and the necessary conditions for their existence. I assisted in generalizing the ChE expansions to the case of a single conserved charge. I proposed the stochastic velocity-swap models and performed the analytical computations as well as the numerical simulations together with Paweł Lisiak. I contributed to writing the manuscript and I am the corresponding author.

Burgers equation from non-thermal stationary states in nearly-integrable gases

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When a gas of particles interacts with much a larger reservoir the density dynamics on large scales is typically governed by diffusion. We study this paradigmatic problem for weakly coupled integrable systems and show that this picture gets altered, when transport is investigated on top of long-lived non-thermal states. Remarkably, for states non-invariant under parity we find Burgers equation arising in the hydrodynamic limit. We explicitly compute the diffusion constant and nonlinear advective coefficient of the Burgers equation using a variant of the Chapman-Enskog theory. We find an excellent agreement between our theory and numerical simulations of a simplified model of stochastic two-body collisions. Our conclusions are based only on Galilean invariance, existence of a small system-bath coupling parameter and a small momentum exchange between the system and the bath particles during two-body scattering.

I. INTRODUCTION

The non-equilibrium dynamics of low-dimensional quantum many-body systems attracts a lot of attention due to recent advances in experimental, numerical and analytical understanding of fundamental phenomena such as thermalisation and transport [1–20]. In one dimension, one often encounters various instances of anomalous, or non-conventional hydrodynamics. This happens for instance in momentum-conserving fluids or anharmonic chains [21–24], Luttinger liquids [25, 26] and certain integrable spin chains [27–30]. Non-conventional transport occurs more broadly in integrable systems, due to the presence of infinite number of conservation laws. In the recent years, a special role in the field of 1D transport is indeed played by the family of such models [31, 32], for which there has been recently a lot of progress in understanding their equilibration and emergent hydrodynamics, dubbed generalized hydrodynamics (GHD) [33–35]. The developed techniques cover also situations in which the integrability is weakly broken [36–48].

In such case the resulting description is similar to a kinetic theory of dilute gases [49, 50]. Namely, the state of the system is described by a local distribution of quasiparticles which propagate with state-dependent effective velocities and occasionally scatter due to integrability breaking effects. Additionally, because of the presence of interactions in integrable theory there is a quasiparticle diffusion effect [51–55]. The scattering between the quasiparticles opens a possibility for a thermalisation channel which otherwise is absent. Another thermalisation mechanism is through introducing an external potential [56–59], but to simplify the problem we do not consider it here. The observation that integrability breaking modifies the dynamics of quasiparticles similarly as weak interactions alter the dynamics of free particles can be exploited by bringing tools from the kinetic theory. In particular, such approach

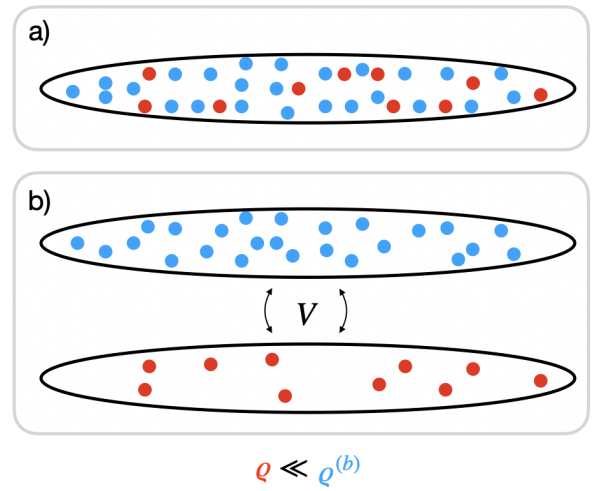


FIG. 1. We consider dynamics of particles with density q coupled to a much larger bath with density $q^{(b)} \gg q$. Both systems are described by integrable models and the long-range coupling breaks the integrability. One can think about two realizations of this scenario: mixed systems (a) and long-range interacting tubes (b). Particles of the observed system feature single conservation law and we consider the problem of hydrodynamics, which emerges on large space-time scales.

was used to study hydrodynamics of nearly-integrable quantum systems. An example of this idea is application of the Chapman-Enskog (ChE) theory [49] to understand the emergence of conventional hydrodynamics along with computation of transport coefficients in such systems [60, 61]. The latter was important to establish an existence of parametrically large space-time regime [60, 62–66], where the conventional hydrodynamics works before giving place for anomalous one [21–24]. This kinetic picture can be also lifted from the quasiparticles distribution to their many-body correlations leading to a generalization of the famous Born-Bogoliubov-Green-Kirkwood-Yvon (BBGKY) hierarchy [48]. The generalized BBGKY (gBBGKY) provides a new theoretical framework to understand the integrability-breaking-

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induced thermalization and was proven useful in description of long-range interacting hard rod gas numerics as well as experiments in 1D dipolar gases [67].

In this work we continue following that program and consider a canonical problem of diffusion of Galilean particles. The mechanism for diffusion is provided by a long-range coupling to a much larger bath, also assumed to be described by a one-dimensional integrable model. As shown in Fig. 1, one can naturally think about such scenario in the geometry of mixed systems, cf. Fig. 1(a) or in coupled 1D tubes, cf. Fig. 1(b). The particles of the smaller system, due to long-range interactions, exchange momentum and energy with the bath and therefore the only conserved quantity associated with them is their total number. On a phenomenological level, when the bath state is assumed to be thermal, the continuity equation for the particle density $\varrho(x, t)$, combined with the gradient expansion of the current, lead to the diffusion equation

$$\partial_t \varrho = \partial_x (\mathcal{D}(\varrho) \partial_x \varrho), \quad (1)$$

with the density-dependent diffusion coefficient $\mathcal{D}(\varrho)$. The standard form of the diffusion equation is obtained by linearising the theory with the diffusion coefficient attaining a constant value. In such case the diffusion coefficient can be also computed through the Green-Kubo formula from the velocity autocorrelation function.

The diffusion equation can be also obtained from microscopic considerations based on the kinetic theory and Boltzmann collision operator $\mathcal{I}[\rho_p, \rho_p^{(b)}]$ for system-bath interactions. Here $\rho_p(x, \lambda, t)$ is one-body density function (λ labels quasimomentum) and $\rho_p^{(b)}$ denotes (thermal) state of the bath. The density is $\varrho(x, t) = \int d\lambda \rho_p(x, \lambda, t)$ and under the assumption $\varrho \ll \varrho^{(b)}$ the dynamics of the bath is effectively frozen out. The diffusion coefficient can be computed with the ChE method, discussed later in the text, in Sec. IV. It is given by an integral equation formulated in terms of the collision operator $\mathcal{I}$.

Our results generalize in various way the diffusion equation (1). Firstly, the resulting diffusion coefficient incorporates in non-perturbative way the interactions of integrable theory, which extends the applicability of the standard kinetic theory methods to strongly correlated systems. Secondly, isolated integrable systems do not equilibrate to thermal states due to presence of an extensive number of conservation laws. While breaking the integrability through system-bath interactions ultimately thermalizes the system, if it is sufficiently weak it creates a time span in which the system might be in a prethermal quasi-stationary state [12, 45, 46, 48, 68]. This leads to a new physical situation in which the particles diffuse in a non-thermal environment. Finally, if the non-thermal state is furthermore not parity invariant there are new terms appearing in the diffusion equation. These together ultimately lead to the celebrated Burgers equation (studied recently in many contexts [69–71]) in

state \ statistics	classical	quantum
parity-symmetric	diff. eq. (1)	diff. eq. (1)
parity-asymmetric	diff. eq. (1)	Burgers eq. (2)

TABLE I. Hydrodynamic equations for the system for different particle statistics and properties of the stationary state.

the form

$$\partial_t \varrho + u(\varrho) \partial_x \varrho = \partial_x (\mathcal{D}(\varrho) \partial_x \varrho), \quad (2)$$

where $u(\varrho)$ is advective coefficient, which can be constant or not, depending on particles' statistics. For classical statistics, we find that $u(\varrho)$ is constant, and equivalent to a simple Galilean boost of the diffusion equation (1). The more interesting situation happens for quantum statistics, where $u(\varrho)$ is non-constant and we recover a genuine Burgers equation. This is summarized in Tab. I. Equation (2) for an almost homogeneous density profile, $\varrho \approx \varrho_0 + \delta \varrho$, after changing the reference frame to the one moving with $u(\varrho_0)$, simplifies to

$$\partial_t \delta \varrho + \delta \varrho \partial_x \delta \varrho = \nu \partial_x^2 \delta \varrho. \quad (3)$$

with time rescaled by $u'(\varrho_0)$ and $\nu = \mathcal{D}(\varrho_0)/u'(\varrho_0)$. This is a standard computation within a reductive perturbative method [72–74]. In the literature, it is equation (3) that is usually called Burgers equation. Here, for simplicity, we will refer to both equations (2) and (3) as Burgers equations.

We exemplify our approach by considering three different physical scenarios. First, we study a system of two coupled 1D Bose gases which mimics the experimental settings of cold atomic gases consisting of arrays of 1D systems with possibly tunable couplings [67]. It is known that the collision integral for such systems supports athermal stationary states [68]. This provides a concrete physical realisation for testing the ideas presented in this work. Next, we abstract from a concrete realization by considering a generic collision integral in the relaxation time approximation (RTA) with respect to athermal state and with particle number being the only conserved quantity. Within the RTA the ChE equation for the diffusion constant can be solved. This allows us to draw general conclusions for diffusion processes in such circumstances and show how Burgers equations arise for athermal and parity-breaking background states. Finally, we consider a simple stochastic model of classical or fermionic particles colliding with bath particles at constant rate. This serves as a minimal model in which all the phenomena seen in a more sophisticated situations are present, but without technical complications. This allows us to pinpoint the sufficient conditions required to see the Burgers equation. At the same time, this model

can be simulated efficiently and we successfully test the effective theory given by (2) against the molecular dynamics.

The paper is organized in the following way. In Sec. II we provide the necessary ingredients of the kinetic theory of nearly-integrable system, specifically we discuss in the details the Thermodynamic Bethe Ansatz (TBA), the GHD equation supplemented with the Boltzmann term and the Generalized Gibbs Ensemble (GGE) formulation of the quasiparticle distribution function. In Section III we discuss the collision integrals supporting athermal states. In Sec. IV we use the ChE method to derive from the GHD-Boltzmann equation the Burgers equation. In Sec. V, we extract the characteristic relaxation time from the collision integral for two coupled 1D gases which then serves as the input for the RTA. We compute the diffusion coefficient in the latter approximation and link it with thermodynamic properties of the gas also in athermal states. We also show how Burgers equation arises for states without parity invariance. In Sec. VI we demonstrate that our predictions match the microscopic simulations of the stochastic models. We relegated some more technical aspects of our work to the Appendices, to which we refer when needed.

II. PRELIMINARIES

We consider a system described by the following Hamiltonian

$$H = H_{\text{int}} + H_{\text{int}}^{(b)} + \alpha H_{\text{pert}}, \quad \alpha \ll 1, \quad (4)$$

where H_{int} and $H_{\text{int}}^{(b)}$ are some integrable Hamiltonians modelling system and bath, respectively and αH_{pert} is some small integrability-breaking perturbation Hamiltonian (modelling system-bath interaction). The system is assumed to be Galilean-invariant. We consider only the case when H_{pert} does not depend on position and time explicitly. The dynamics of nearly-integrable models, after the initial decoherence phase, is effectively described by kinetic theory [12, 38, 48, 55]. This kinetic theory is formulated in terms of the space-time dependent distribution function of quasiparticles $\rho_p(x, \lambda, t)$ with quasimomentum (rapidity) λ . To understand the basics of quasiparticle description we review first the TBA and GHD of integrable models.

A. Basics of TBA and GHD

Integrable systems are characterized by infinite number of local conserved charges Q_k fulfilling $[H_{\text{int}}, Q_k] = 0$. Their expectation values for a state described by quasiparticle distribution ρ_p are

$$\langle Q_k \rangle = \int d\lambda h_k(\lambda) \rho_p(\lambda). \quad (5)$$

For Galilean invariant models of particles with unit mass, as is assumed here, it is convenient to choose $h_k(\lambda) = \lambda^k/k!$ which implies that Q_k are ultra-local charges. Specifically, for total momentum $h_1(\lambda) = \lambda$ and for total energy $h_2(\lambda) = \lambda^2/2$. The quasiparticle picture is based upon the assumption of relaxation to GGE [75, 76] with the density matrix $\rho_{\text{GGE}} \propto \exp(-\sum_j \beta_j Q_j)$, where β_j are Lagrange multipliers.

We introduce now more ingredients of the TBA in similar way as presented in [77]. We start with free energy density $\mathfrak{f} = \frac{1}{2\pi} \int d\lambda F[\epsilon(\lambda)]$, where $\epsilon(\lambda)$ is so-called pseudoenergy and function F depends on particles' statistics (for fermionic particles $F[\epsilon(\lambda)] = -\log(1 + e^{-\epsilon(\lambda)})$). The pseudoenergy is determined from the following equation

$$\epsilon(\lambda) = \epsilon_0(\lambda) + \int d\lambda' \mathcal{T}(\lambda - \lambda') F[\epsilon(\lambda')], \quad (6)$$

with bare pseudoenergy given by $\epsilon_0(\lambda) = \sum_j \beta_j h_j(\lambda)$. Moreover, by $\mathcal{T}(\lambda - \lambda')$ we denoted model-dependent scattering kernel. The set $\{\beta_j\}_{j=0,1,\dots}$ specifies the state. From pseudoenergy we can compute statistical factor $f(\lambda) = (d^2 F/d\epsilon^2)/(dF/d\epsilon)$ and occupation function $n = dF/d\epsilon$ defined also as $n = \rho_p/\rho_t$, where ρ_t is the total density of states. It fulfills the following integral equation

$$\rho_t(\lambda) = \frac{1}{2\pi} + \int d\lambda' \mathcal{T}(\lambda - \lambda') \rho_p(\lambda'). \quad (7)$$

Next, we introduce dressing operations. The small dressing is defined through integral equation

$$g^{\text{dr}} = g + \mathcal{T}n \cdot g^{\text{dr}}. \quad (8)$$

The big Dressing is the result of the action of the operator $\mathbb{F}$, $g^{\text{Dr}} = \mathbb{F}^t g$ (t denotes transposition) with

$$\mathbb{F}(\lambda, \mu) = \delta(\lambda - \mu) + \partial_\lambda (n(\lambda) F(\lambda|\mu)). \quad (9)$$

Above we introduced backflow function $F(\lambda|\mu)$ [32], appearing naturally in studies of excitations in integrable systems. The two dressings are related

$$(g^{\text{Dr}})' = (g')^{\text{dr}}. \quad (10)$$

For future reference we introduce Dressed momentum $k(\lambda) = h_1^{\text{Dr}}(\lambda)$ and energy $\omega(\lambda) = h_2^{\text{Dr}}(\lambda)$. We also mention useful relation that $k'(\lambda) = 2\pi\rho_t(\lambda)$.

We move on and discuss ingredients of GHD [77]. The effective velocity v of quasiparticles obeys the integral equation

$$v(\lambda) = h_1(\lambda) + 2\pi \int d\lambda' \mathcal{T}(\lambda - \lambda') \rho_p(\lambda') (v(\lambda') - v(\lambda)). \quad (11)$$

Alternatively, the effective velocity can be written as $v(\lambda) = h_1^{\text{dr}}(\lambda)/h_0^{\text{dr}}(\lambda)$. For later use it will be convenient to introduce kernels of hydrodynamic matrices: flux Jacobian

$$\mathcal{A} = (\mathbf{1} - n\mathcal{T})^{-1}v(\mathbf{1} - n\mathcal{T}), \quad (12)$$

along with charge and current susceptibility matrix kernels

$$\mathcal{C} = (\mathbf{1} - n\mathcal{T})^{-1} \rho_p f (\mathbf{1} - \mathcal{T}n)^{-1}, \quad (13)$$

$$\mathcal{B} = (\mathbf{1} - n\mathcal{T})^{-1} v \rho_p f (\mathbf{1} - \mathcal{T}n)^{-1}. \quad (14)$$

The last ingredient is the diffusion operator, which we denote by $\mathfrak{D}_\rho = \mathfrak{D}_\rho(\lambda, \mu; x, t)$. It is state-dependent and expressible through objects already introduced. For specific form of $\mathfrak{D}$ and discussion of its properties we refer to Appendix D 1.

B. GHD-Boltzmann equation

We will assume here that the dynamics of integrable systems in the presence of integrability-breaking effects is given by GHD-Boltzmann equation [36, 38]

$$\partial_t \rho_p + \partial_x (v_\rho \rho_p) = \frac{1}{2} \partial_x (\mathfrak{D}_\rho \partial_x \rho_p) + \mathcal{I}[\rho_p]. \quad (15)$$

where $\mathcal{I}[\rho_p]$ is the collision operator responsible for interaction between the observed system and the bath. We indicated the dependence on state in effective velocity and diffusion operator by putting ρ in the lower index. Moreover, we assume that the bath system is much larger and in consequence, does not evolve in time. Equation (15) describes the system only and thus does not conserve momentum and energy. However, it does conserve particle number, which is reflected in collision integral

$$\int d\lambda h_0(\lambda) \mathcal{I}[\rho_p] = 0. \quad (16)$$

We also assume that $\mathcal{I}[\rho_p]$ has a family of stationary states ρ_p^{st} parametrised by their density ρ . We will give explicit examples of collision integrals with such properties in the following section.

III. ATHERMAL STATIONARY STATES

The aim of this section is to provide a construction of collision integrals that support athermal stationary states and to expose relevant physics behind them. We can focus here on the homogeneous systems. The generalization to inhomogeneous systems follows then from general arguments of separation of scales.

Athermal states relevant to the present context were first found in the setup of coupled Lieb-Liniger (LL) gases [68]. In this work, thanks to the recent development of gBBGKY [48], we present arguments which hold for arbitrary weakly coupled integrable systems. In gBBGKY for coupled integrable systems we assume that the system and bath interact through long-range density-density interaction given by

$$V(x) = \frac{V_0}{\xi} \varphi\left(\frac{x}{\xi}\right), \quad (17)$$

where ξ is the characteristic range of the potential, V_0 measures its strength and function φ gives the interaction profile. Importantly, the parameter ξ should be thought of as a large parameter in the theory in order to provide a mechanism for truncation of the gBBGKY hierarchy. Moreover, we will assume here that the bath does not evolve, which can be realized for instance by a large density imbalance between the two systems. The aforementioned kinetic framework predicts that the dynamics at sufficiently late times of the system is given by Boltzmann-like kinetic equation.

$$\partial_t \rho_p(\lambda) = \mathcal{I}_{2p}(\lambda) (1 + \mathcal{O}(\xi^{-1})), \quad (18)$$

where collision integral $\mathcal{I}_{2p}$ describes two-particle scattering processes, which will be discussed in the following. There are important corrections, written above as $\mathcal{O}(\xi^{-1})$, which involve collision operators responsible for three and more particle collisions. These processes are relevant on timescales longer than $\mathcal{I}_{2p}$ and eventually thermalize the system. In what follows, we neglect them focusing on timescales relevant for $\mathcal{I}_{2p}$. Let us also note that the same picture emerges from Fermi's Golden Rule (FGR) treatment of integrability breaking as developed in [36, 38]. There, the collision operator involve terms with more and more particle collision processes, with the two-particle scatterings dominating in the system-bath setup. The exact equivalence of between FGR and gBBGKY was established in [48].

To understand the simplest relevant physical scenario and minimal ingredients for athermal states let us abstract from the specific form of $\mathcal{I}_{2p}$ arising from a microscopic theory. We start broadly and consider general Master equation for two-particle scatterings between classical system and classical bath

$$\begin{aligned} \mathcal{I}[\rho_p](\lambda) = & \int d\lambda' d\lambda_b d\lambda'_b \mathcal{W}(\lambda, \lambda_b; \lambda', \lambda'_b) \times \\ & \delta(k)\delta(\omega) \left(\rho_p(\lambda) \rho_p^{(b)}(\lambda') - \rho_p(\lambda') \rho_p^{(b)}(\lambda) \right), \end{aligned} \quad (19)$$

where conservation laws for a scattering process are imposed by Dirac deltas with $k = k(\lambda) + k_b(\lambda_b) - k(\lambda') - k_b(\lambda'_b)$ and $\omega = \omega(\lambda) + \omega_b(\lambda_b) - \omega(\lambda') - \omega_b(\lambda'_b)$. In general, we assume that quasiparticle with quasimomentum λ carry momentum $k(\lambda)$ and energy $\omega(\lambda)$. In our notation k, k_b are momenta of system and bath particles, respectively, the same goes for energies ω, ω_b . The quantity $\mathcal{W}(\lambda, \lambda_b; \lambda', \lambda'_b)$ is naturally related to a rate of scattering $\lambda' + \lambda'_b \rightarrow \lambda + \lambda_b$.

Importantly, we assume that dispersion relations in both systems are not necessarily the same (meaning $\omega_b \neq \omega$ and $k_b \neq k$), which in the context of classical particles can be realized for instance by different masses, or different length of the rods (in the case of hard rod gas). Non-equal dispersions generically prohibit [68] the existence of athermal state and thermal states (given by Maxwell-Boltzmann distributions) are the only fixed points of (19).

The complementary situation of equal dispersion happens when both gases consist of identical free particles with $k(\lambda) = \lambda$ and $\omega(\lambda) = \lambda^2/2$. The momentum-energy conservation laws imply then that scattering amounts to permutation of velocities meaning that $\lambda = \lambda'_b$ and $\lambda_b = \lambda'$. We readily find the stationary state in such case which is

$$\rho_p^{\text{st}}(\lambda) = \frac{\varrho}{\varrho^{(b)}} \rho_p^{(b)}(\lambda). \quad (20)$$

Obviously, if $\rho_p^{(b)}$ is athermal then the state above is athermal as well. The case of two classical gases with equal dispersions is the simplest example of dynamics leading to non-thermal stationary state.

Remarkably, the conclusion about lack of non-thermal fixed points in the case of non-equal dispersions changes in the *grazing limit* of the equation (19), which amounts to restricting possible scattering processes to only these, where the momentum exchange is small. In that limit, the collision integral acquires the form of Landau-like kinetic equation [49, 78]

$$\begin{aligned} \mathcal{I}^{\text{gr}}[\rho_p](\lambda) = & -\partial_\lambda \int d\lambda_b \tilde{\mathcal{W}}(\lambda, \lambda_b) \delta(v(\lambda) - v_b(\lambda_b)) \times \\ & \rho_p(\lambda) \rho_p^{(b)}(\lambda_b) \left[\epsilon'(\lambda) - \frac{k'(\lambda)}{k'_b(\lambda_b)} \epsilon'_b(\lambda_b) \right], \end{aligned} \quad (21)$$

with $\tilde{\mathcal{W}}(\lambda, \lambda_b)$ directly related to collision rate $\mathcal{W}$. For the precise definition of grazing limit as well as expression for $\tilde{\mathcal{W}}$ see Appendix A.

The stationary state of (21) fulfils a set of two equations

$$v(\lambda) = v_b(\lambda_b), \quad \epsilon'(\lambda) = \frac{k'(\lambda)}{k'_b(\lambda_b)} \epsilon'_b(\lambda_b). \quad (22)$$

Let us emphasize that this stationary state appears even though two-body scattering *does not* lead to a simple momentum exchange. We also stress that the solution to the equation above is athermal in the case of athermal bath. In [68] such athermal states were found in the setup, where both the system and bath are described by the LL models.

In the generic interacting integrable models, the dispersion relation is a dynamic quantity, it depends on the state of the system which in turn reflects e.g. the interactions and the density. Therefore, two such systems that are weakly coupled with potential that allows only for small exchange of momentum naturally provide a physical realisation in which two body scatterings in (1+1)D lead to momentum redistribution. In this case we expect the stationary state described by (22).

Summing up, the existence of possible non-thermal states depends on dispersion relations in both systems and on grazing limit of collision operator, cf. Tab. II. Lastly, let us mention, that the found stationary states become thermal states in the case of thermal bath.

dispersions \ collisions	at small momenta (grazing limit)	all momenta
	same	different
same	atherm.	atherm.
different	atherm.	therm.

TABLE II. Conditions for possible existence of athermal states in dynamics driven by two-particle scattering processes. Athermal states are excluded only if the dispersions of the systems H_{int} , $H_{\text{int}}^{(b)}$ are different, and the grazing limit of small momentum exchange is absent.

In what follows we go beyond the case of classical particles described by (19) and present different treatments of weakly coupled integrable systems, quantum or classical. We start with the most concrete case of coupled LL models, which can be studied using FGR. This analysis is later generalized to arbitrary weakly coupled integrable models using the gBBGKY hierarchy. Lastly, we propose a simplified model of stochastic two-body collisions which offers particularly simple structure of collision integral, while capturing all the relevant features of the dynamics. It also has the advantage of efficient numerical simulation and analytical expressions for diffusion constants.

A. Fermi's Golden Rule

We start with the FGR treatment of nearly-integrable quantum models. Within this approximation, the collision integral involves matrix elements of the perturbing operator. In our case this originates from the integrability breaking term in the Hamiltonian which we assume couples to the local particles' density operator $q_0(x)$. However, we can be slightly more general and assume that it couples to density $q_j(x)$ of any of the conserved charges $Q_j = \int dx q_j(x)$.

The collision integral for two-particle processes takes then the following form

$$\mathcal{I}[\rho_p] = \mathbb{F} \mathcal{I}_0[\rho_p], \quad (23)$$

where the backflow operator (9) describes the response of the whole state on the scattering process described by the bare collision integral $\mathcal{I}_0[\rho_p]$. The latter is given by the usual form of the Master equation,

$$\mathcal{I}_0[\rho_p](\lambda) = \int d\mu [w(\lambda, \mu) \rho_p(\mu) \rho_h(\lambda) - (\lambda \leftrightarrow \mu)], \quad (24)$$

with the transition rate $w(\lambda, \mu)$ and density of holes defined as $\rho_h = \rho_t - \rho_p$. The transition rate is computed from the FGR and involves matrix elements of the perturbing operator $q_j(x)$. Typically, the transition rate obeys a detailed balance relation,

$$w(\lambda, \mu) = e^{-\beta(\omega(\lambda) - \omega(\mu))} w(\mu, \lambda), \quad (25)$$

and in such circumstances, the stationary state of (24) is thermal with inverse temperature β . Lack of the detailed balance relation open ways for athermal stationary states.

To illustrate a situation in which athermal stationary states exist we consider a two coupled 1D systems, see Fig. 1(b), with total Hamiltonian

$$H = H_1 + H_2 + V_{12}. \quad (26)$$

We assume that H_i are integrable, for example are given by the LL model [79, 80], and V_{12} breaks the integrability. In calculations, it is natural to treat system and bath on equal footing which is reflected in a slightly different notation than in Eq. (4). Experimentally relevant choice of V_{12} is the long-range interaction potential between two tubes that couples to local densities

$$V_{12} = \int dx_1 dx_2 V(x_1 - x_2) q_i(x_1) q_j(x_2). \quad (27)$$

In writing the perturbation term we allow for coupling between densities of different charges. The usual density-density interaction corresponds to $i = j = 0$.

In the homogeneous system, the quasiparticles density in the two tubes evolves according to the set of equations

$$\partial_t \rho_p^{(1)} = \mathcal{I}_{12}[\rho_p^{(1)}, \rho_p^{(2)}], \quad \partial_t \rho_p^{(2)} = \mathcal{I}_{21}[\rho_p^{(2)}, \rho_p^{(1)}], \quad (28)$$

with the corresponding transition rates

$$w_{12}(\lambda, \mu) = \tilde{V}^2(k) |F_i^{(1)}(\lambda, \mu)|^2 S_j^{(2)}(-k, -\omega), \quad (29)$$

$$w_{21}(\lambda, \mu) = \tilde{V}^2(k) |F_j^{(2)}(\lambda, \mu)|^2 S_i^{(1)}(-k, -\omega). \quad (30)$$

Here $F_i^{(1)}(\lambda, \mu)$ is the one particle-hole form factor of the operator q_i evaluated in the first tube. The rapidity of the particle and the hole are λ and μ respectively. With $k \equiv k(\lambda) - k(\mu)$ and $\omega \equiv \omega(\lambda) - \omega(\mu)$ we denote momentum and energy of the particle-hole pair. $\tilde{V}(k)$ is the Fourier transform of the interaction potential from Eq. (27). Finally, $S_j^{(2)}(k, \omega)$ is the dynamic structure factor in the second tube, namely Fourier transform of the connected two point function $\langle q_j^{(2)}(x, t) q_j^{(2)}(0) \rangle^c$ of density q_j . In principle, there are also contributions to the transition rates that involve form-factors with larger number of particle-hole pairs. As shown in [68] their contribution is subleading in the momentum transferred between the tubes and is important at larger time scales. Importantly, the higher particle-hole processes are responsible for ultimate thermalization of the system [38].

We consider now an imbalance limit in which density of particles in the second tube is much larger than in the first one. In the Boltzmann picture, the evolution proceeds as a succession of 2-body scatterings. Every such scattering process has much larger impact on a distribution in a first tube and instead, the much denser second tube evolves slowly. In the self-diffusion limit we assume that it does not evolve at all and distribution $\rho_p^{(2)}$

is fixed. The second tube acquires then a role of a bath. In such case the evolution in the first tube is given by equation (24) with the transition rate

$$w(\lambda, \mu) = \tilde{V}^2(k) |F_i^{(1)}(\lambda, \mu)|^2 S_j^{(2)}(-k, -\omega), \quad (31)$$

If the bath is in a thermal state then its correlation function obeys the detailed balance relation

$$S_j^{(2)}(k, \omega) = e^{-\beta\omega} S_j^{(2)}(-k, -\omega). \quad (32)$$

In the consequence the transition rates obey (25) since the form-factors are particle-hole symmetric and the stationary state is thermal. Otherwise, the stationary state of (24) is athermal.

Note also, that for illustration purposes the derivation of the collision integral was presented in the case of two coupled integrable models with density imbalance. However, the only information that we need of the bath is its dynamic response function $S_j^{(2)}$. Instead, for the description of the system, its integrability is crucial as it provides the quasiparticle picture.

This being said, if the bath is in fact integrable, then a form of a generalized detailed balance holds [81]

$$S_j^{(2)}(k, \omega) = e^{-k\mathcal{F}_2(\omega/k)} S_j^{(2)}(-k, -\omega), \quad (33)$$

where $\mathcal{F}_2(v) = \epsilon'_2(\lambda)/k'_2(\lambda)$ with λ fixed by $v_2(\lambda) = v$. This leads then to the stationary state equations

$$\frac{\epsilon'_1(\lambda)}{k'_1(\lambda)} = \mathcal{F}_2(v_1(\lambda)). \quad (34)$$

Upon substituting in this equation the expression for $\mathcal{F}_2$ we recover the stationary state equation as presented in [68] or in Eq. (22) above.

Following [68] let us discuss some specific cases. First, if we specialize to a thermal state, then $\epsilon'_2(\lambda) = \beta\omega'_2(\lambda)$ and $\mathcal{F}_2(v) = \beta v$. The stationary state of the system is then thermal with the same inverse temperature β . The second special case is when states and Hamiltonians in both tubes are the same. In this cases the collision integral (at the 1 particle-hole level) is exactly zero. However, such situation cannot occur in the self-diffusion problem since the densities, and therefore states, are different by construction. Finally, in the Tonks-Girardeau limit of the LL model the dressing is absent. Then the effective velocities are state independent and equal to $\hbar_1$. The stationarity condition simplifies to $\epsilon'_{1,0}(\lambda) = \epsilon'_{2,0}(\lambda)$. Crucially, in the Tonks-Girardeau gas the form-factors with larger number of particle-hole excitations vanish and the athermal states are stationary beyond the small momentum approximation, due to equal dispersions in both systems.

B. Generalized BBGKY hierarchy

In this section we take a more general perspective on the integrability breaking and employ recently developed

gBBGKY hierarchy for the dynamics of correlation functions [48]. We specialize to the system described by the Hamiltonian (26) and assume homogenous states. Importantly, the interaction potential is assumed to be long-ranged and given by the form (17). The full hierarchy is exposed in [48] and involves correlation functions of all orders. To understand the appearance of athermal states it is sufficient however to truncate the hierarchy at the level of two-particle (system-bath) connected correlation function $g_{\lambda_1, \lambda_2}(x_1, x_2)$. Such correlation is sufficient to capture two-body scattering physics discussed earlier. Obviously, there are more terms and equations in the hierarchy and they describe effects ultimately leading to thermalization. However, this will happen on a longer timescale and we neglect these effects in present discussion. On the level of hierarchy, this means that we have to deal with three dynamical equations (for the purpose of this section, we use the notation where index 1 refers to the system and 2 refers to the bath and quasimomentum dependence is indicated in lower index)

$$\begin{aligned} \partial_t \rho_{\lambda_1}^{(1)} &= \int dx_2 d\lambda_2 V'(x_1 - x_2) \partial_{\lambda_1} g_{\lambda_1, \lambda_2}(x_1, x_2), \\ \partial_t \rho_{\lambda_2}^{(2)} &= \int dx_1 d\lambda_1 V'(x_1 - x_2) \partial_{\lambda_2} g_{\lambda_1, \lambda_2}(x_1, x_2), \end{aligned} \quad (35)$$

$$\begin{aligned} &\partial_t g_{\lambda_1, \lambda_2}(x_1, x_2) + \\ &\mathcal{A}_{\lambda_1, \gamma}^{(1)} \partial_{x_1} g_{\gamma, \lambda_2}(x_1, x_2) + \mathcal{A}_{\lambda_2, \gamma}^{(2)} \partial_{x_2} g_{\lambda_1, \gamma}(x_1, x_2) = \\ &V'(x_1 - x_2) \int dx_3 \left(\partial_{\lambda_1} \rho_{\lambda_1}^{(1)} \mathcal{C}_{\lambda_2, \lambda_3}^{(2)} - \partial_{\lambda_2} \rho_{\lambda_2}^{(2)} \mathcal{C}_{\lambda_1, \lambda_3}^{(1)} \right). \end{aligned} \quad (36)$$

The hierarchy can be integrated in time, as in standard derivations of collision kernels from BBGKY hierarchy [48, 50], leading to coupled equations of the form of (28). In Appendix B we go through this calculation. The resulting collision integrals take the Landau form in the grazing limit. The explicit form is given in (B12). The resulting collision operator, upon specifying to LL model exactly agrees with FGR result. The athermal states featured by it are exactly of the form (22). The approach of gBBGKY is the most versatile as it captures both classical and quantum systems. In particular, it allowed for derivation of collision term for long-range interacting hard rods, successfully tested against exact molecular dynamics in [48]. In general, when specified to classical statistics the resulting collision integral exactly takes the form of (22) with

$$\tilde{W}(\lambda, \lambda_b) = 2\pi^2 \int dk |k| \tilde{V}^2(k) (k'_b(\lambda_b))^2 \quad (37)$$

where $\tilde{V}(k)$ denotes Fourier transform of interaction potential (17). For more details see Appendix B. Note also that both FGR and gBBGKY predict the scaling of collision integral as α^2 . The characteristic relaxation timescale is thus parametrically long in the inverse of the integrability breaking parameter $\tau_{\mathcal{I}} \sim \alpha^{-2}$.

C. Stochastic velocity swap models

The minimal version of two-particle scattering dynamics corresponds to putting $\mathcal{W}(\lambda, \lambda_b; \lambda', \lambda'_b) = \gamma = \text{const}$ in (19) meaning that all possible scatterings happen at the same rate. The collision integral simplifies then to

$$\mathcal{I} = \gamma \left(\varrho \rho_p^{(b)}(\lambda) - \varrho^{(b)} \rho_p(\lambda) \right). \quad (38)$$

This model can be understood as an adaptation of three-particle collision model studied in [65, 82] to our context. The proposed model can be efficiently simulated by imposing stochastic velocity-swapping collisions between ideal classical gases describing system and bath. Obviously, the stationary state of (38) is given by (20).

Due to classical statistics, the system displays standard diffusion equation (1) on large scales. As we would like to see Burgers equation (2) in molecular dynamics, we extend this model to fermions interacting with classical bath, again with a constant collision rate. Master equations for scattering processes with Pauli blocking is

$$\begin{aligned} \mathcal{I} &= \gamma \int d\lambda' \times \\ &\left[(1 - n(\lambda)) \rho_p^{(b)}(\lambda) \rho_p(\lambda') - (1 - n(\lambda')) \rho_p^{(b)}(\lambda') \rho_p(\lambda) \right]. \end{aligned} \quad (39)$$

The model features the following stationary state

$$\rho_p^{\text{st}}(\lambda) = \frac{1}{2\pi} \frac{\rho_p^{(b)}(\lambda)}{\rho_p^{(b)}(\lambda) + A}, \quad (40)$$

where constant A is fixed by normalization $\int d\lambda \rho_p^{\text{st}}(\lambda) = \varrho$. The algorithm which implements classical model as well as Pauli principle in scatterings is presented in Appendix C. The models (38) and (39) are introduced as benchmarks for our effective, hydrodynamic theory, which can be efficiently simulated. In Fig. 2 we present numerical simulations of classical version of the model in homogenous setting, confirming the existence of athermal stationary states as well as the correctness of kinetic equation (38). The characteristic time scale for relaxation behaves as $\tau_{\mathcal{I}} \sim \gamma^{-1}$. In the nearly-integrable regime it is thus again expected to be parametrically large.

IV. HYDRODYNAMIC EQUATION

The effective hydrodynamic equation is obtained using the ChE method known from the kinetic theory [49]. Before going into the details of the method, let us start with summarizing the most important results of this section.

The equation for density $\varrho(x, t)$ which we find on large scales in the most general case takes the form (2). Expressions for the advective coefficient $u(\varrho)$, related to the

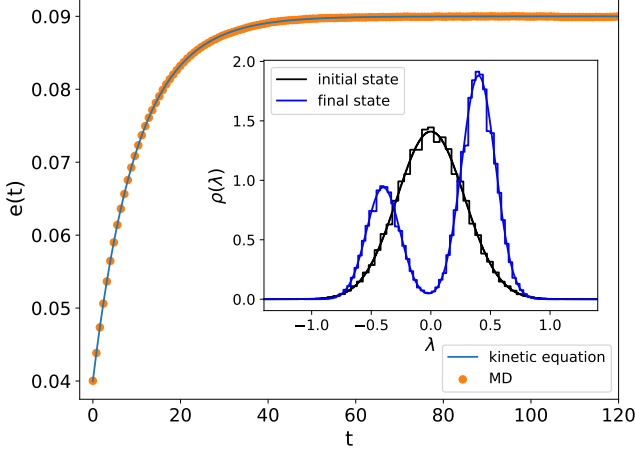


FIG. 2. Relaxation dynamics of kinetic energy per particle $e(t)$ in classical stochastic velocity swap model. The molecular simulation agrees very well with solution of kinetic equation (38). Inset: initial and final distributions of velocities. Solid lines are analytical distributions, in particular the microscopic final state agrees very well with analytical prediction (20). We consider homogenous system of $N = 1600$ particles with $\varrho = 1$, $\varrho^{(b)} = 100$ and with $\gamma = 0.001$. The initial system state is thermal with temperature $T = 0.08$, whereas the bath system state is asymmetric Bragg-split state given by $\rho_p^{(b)} \propto e^{-(\lambda+\lambda_B)^2/\sigma^2} + b e^{-(\lambda-\lambda_B)^2/\sigma^2}$ with $\sigma = 0.2$, $\lambda_B = 0.4$ and $b = 0.5$. The results were averaged over 1000 realizations.

local center-of-mass velocity of the system, and the diffusion coefficient $\mathcal{D}(\varrho)$ are

$$u(\varrho) = \int d\lambda h_1 \frac{\partial \rho_p^{\text{st}}}{\partial \varrho}, \quad \mathcal{D}(\varrho) = - \int d\lambda h_1 a(\lambda, \varrho). \quad (41)$$

Here $a(\lambda, \varrho)$ is a solution to the ChE equation detailed below. The parameter $u(\varrho)$ vanishes for parity-symmetric stationary states ρ_p^{st} , leading to diffusion equation (1). For parity-asymmetric state ρ_p^{st} , which cannot be brought to symmetric states by Galilean boost, there are two cases. For classical statistics of system particles, $u(\varrho) = \text{const}$ and it can be removed through boost. For quantum statistics, $u(\varrho)$ is density-dependent, leading to Burgers equation.

We move now to the ChE method. In papers [60, 61] similar computation is performed for nearly-integrable system with total number of particles, momentum and energy conserved. At the foundations of the ChE method lies the separation of scales between the microscopic collision integral relaxation time $\tau_{\mathcal{I}}$ and macroscopic relaxation due to $\partial_x (v_\rho \rho_p)$, $\partial_x (\mathfrak{D}_\rho \partial_x \rho_p)$ terms in the GHD equation. The separation of scales gives us a suitable small parameter in which various quantities can be expanded. To accommodate the GHD diffusion in the (perturbative) ChE method we must additionally assume separation between characteristic spatial scale of the GHD

diffusion and the characteristic spatial scale of ρ_p [61]. More precisely, we demand

$$\delta_{\mathcal{I}} := (\tau_{\mathcal{I}} \langle v \rangle) / \ell_h \ll 1, \quad \delta_{\mathfrak{D}} := \ell_{\text{mic}} / \ell_h \ll 1, \quad (42)$$

where ℓ_h is hydrodynamic length characterizing spatial variations of ρ_p and $\langle v \rangle$ is the characteristic velocity of the state ρ_p . By ℓ_{mic} we denoted characteristic spatial scale of the GHD diffusion process. The introduced small parameters $\delta_{\mathcal{I}}, \delta_{\mathfrak{D}}$ are called ChE parameters. In classical kinetic theory $\delta_{\mathcal{I}}$ is known as Knudsen number. For more discussion on the second ChE parameter we refer to [61]. Another important aspect of the ChE method is the assumption, that time enter ρ_p only implicitly through the time-dependence of conserved charge, i.e.,

$$\rho_p(\lambda, x, t) = \rho_p(\lambda, x, \varrho(x, t)). \quad (43)$$

Solution to GHD-Boltzmann equation satisfying this condition is called a normal solution.

A. Evaluation of ChE procedure

We start by integrating the GHD-Boltzmann equation over $d\lambda$. As a result, we find the continuity equation for density

$$\partial_t \varrho + \partial_x j_\rho = 0, \quad (44)$$

with the following particle current

$$j_\rho = \int d\lambda v_\rho \rho_p = \int d\lambda h_1 \rho_p, \quad (45)$$

where in the second equality we used that, for a Galilean invariant system, v_ρ satisfies integral equation (11), supplemented with a fact that $\mathcal{T}(\lambda)$ is symmetric [60]. In obtaining (44) we also used the Markovian property of diffusion matrix, discussed in Appendix D 1. Obviously, at this point (44) is not closed as the current depends on the whole distribution ρ_p . To make some progress we have to take advantage of smallness of ChE parameters.

To see how ChE parameters (42) appear in the GHD-Boltzmann equation (15) we introduce dimensionless variables [49, 60, 61] $\bar{t} = t/\tau_h$ and $\bar{x} = x/\ell_h$ where $\tau_h = \ell_h/\langle v \rangle$ is the characteristic hydrodynamic time scale. In the new variables equation (15) takes the form

$$\partial_{\bar{t}} \rho_p + \partial_{\bar{x}} (\bar{v}_\rho \rho_p) = \frac{\delta_{\mathfrak{D}}}{2} \partial_{\bar{x}} (\bar{\mathfrak{D}}_\rho \partial_{\bar{x}} \rho_p) + \frac{1}{\delta_{\mathcal{I}}} \bar{\mathcal{I}}[\rho_p], \quad (46)$$

where $\bar{v}_\rho = v_\rho/\langle v \rangle$, $\bar{\mathfrak{D}}_\rho = \mathfrak{D}_\rho/(\ell_{\text{mic}} \langle v \rangle)$ and $\bar{\mathcal{I}} = \tau_{\mathcal{I}} \mathcal{I}$ are respectively velocity, diffusion operator and collision integral in the natural units. In what follows we drop $\bar{\cdot}$ symbols from the notation and keep δ parameters. After truncating to a given order, we will restore the original units. Since we assumed ρ_p to have the normal form (43) we can use the particle conservation law (44) to write the equation above without explicit time dependence:

$$\frac{\partial \rho_p}{\partial \varrho} G + \partial_x (v_\rho \rho_p) = \frac{\delta_{\mathfrak{D}}}{2} \partial_x (\mathfrak{D}_\rho \partial_x \rho_p) + \frac{1}{\delta_{\mathcal{I}}} \mathcal{I}[\rho_p]. \quad (47)$$

In the equation above we introduced function $G(x, \varrho) = -\partial_x j_\rho$, such that the continuity equation (44) becomes

$$\partial_t \varrho = G(x, \varrho). \quad (48)$$

Solution of the ChE procedure will determine $G(x, \varrho)$ in terms of ϱ thus closing the equation.

Next we proceed with expansion in small ChE parameters (42). According to the usual scheme, we expand the density ρ_p and G in the two ChE parameters [49] and expand

$$\rho_p = \sum_{i,j=0}^{\infty} \rho_p^{(i,j)} \delta_{\mathcal{I}}^i \delta_{\mathcal{D}}^j, \quad G = \sum_{i,j=0}^{\infty} G^{(i,j)} \delta_{\mathcal{I}}^i \delta_{\mathcal{D}}^j, \quad (49)$$

with $G^{(i,j)} = -\int d\lambda h_1 \partial_x \rho_p^{(i,j)}$. For notational clarity we introduce symbol $\mathcal{O}(\delta^n)$ to mean all $\mathcal{O}(\delta_{\mathcal{I}}^i \delta_{\mathcal{D}}^j)$ contributions such that $i+j = n$, $i \geq -1$, $j \geq 0$. This notation becomes handy later.

In order to solve (47) perturbatively in ChE parameters we also have to expand $\mathcal{I}[\rho_p]$, v_{ρ_p} and $\mathfrak{D}_{\rho_p}$ in terms of ChE parameters. Proceeding in analogy to (49) we denote expansion coefficients as $\mathcal{I}^{(i,j)}$, $v^{(i,j)}$ and $\mathfrak{D}^{(i,j)}$. Since they are all functionals of ρ_p we need to relate each of them with $\rho_p^{(i,j)}$. This can be done by performing functional Taylor expansion of $\mathcal{I}^{(i,j)}$, $v^{(i,j)}$ and $\mathfrak{D}^{(i,j)}$ around $\rho_p^{(0,0)}$, since $\rho_p^{(0,0)}$ is the only factor of order unity in the expansion of ρ_p .

Finally, and contrary to other quantities, we do not expand the conserved density ϱ . Since the only quantity of order unity in expansion of ρ_p is $\rho_p^{(0,0)}$, for the theory to be consistent we require [49]

$$\varrho = \int d\lambda \rho_p^{(0,0)}, \quad 0 = \int d\lambda \rho_p^{(i,j)} \text{ for } i, j \neq 0. \quad (50)$$

These are the constitutive relations that uniquely determine $\rho_p^{(i,j)}$.

B. Euler scale

At the leading order $\mathcal{O}(\delta_{\mathcal{I}}^{-1})$, the GHD-Boltzmann equation (15) simplifies to

$$\mathcal{I}[\rho_p^{(0,0)}] = 0. \quad (51)$$

This condition fixes a state in each fluid cell to be stationary state of collision operator. From now on, we will denote the stationary state $\rho_p^{(0,0)}$ as ρ_p^{st} . In the present context, we assume that the stationary states are parametrized by their density. Therefore, equation (51) has a family of solutions $\rho_p^{\text{st}}(\lambda, \varrho)$ where $\varrho = \int d\lambda \rho_p^{\text{st}}$.

To find the Euler scale equations we recall that the density ϱ varies in space and the quasiparticle distribution is given by $\rho_p^{\text{st}}(\lambda, \varrho(x, t))$. We can evaluate now the leading contribution to G thus closing the continuity equation (48). We have

$$G^{(0,0)} = -\int d\lambda h_1 \partial_x \rho_p^{\text{st}} = -u \partial_x \varrho, \quad (52)$$

where we used the chain rule and introduced

$$u = \int d\lambda h_1 \frac{\partial \rho_p^{\text{st}}}{\partial \varrho}. \quad (53)$$

Changing variables to the original, dimensionful, ones we arrive at the Euler-scale evolution equation

$$\partial_t \varrho + u(\varrho) \partial_x \varrho = 0. \quad (54)$$

Let us now gain some intuition on the advective coefficient u . First of all, it can be shown that under the Galilean boost with velocity w , it transforms indeed as velocity: $u \rightarrow u + w$ (see Appendix D3). This renders the Euler-scale equation Galilean invariant as it should be.

Furthermore, for parity-symmetric states (this includes thermal states) $u = 0$. This follows from the observations that the derivative can be taken in front of the integral and that the first moment of a symmetric distribution is 0. A Galilean boost makes the first moment finite but density independent. Therefore, for standard collision integral with only thermal stationary term there is no dynamics at the Euler scale. A result known from the standard kinetic theory. However, for non-thermal stationary states u can be non-zero. This requires the stationary state to be asymmetric in λ and this asymmetry cannot be removable by a boost. Let us also note that even though it might be not evident at the first sight, equation (54) does have a structure of continuity equation for the particle density. This is seen most clearly when the term $u(\varrho) \partial_x \varrho$ is rewritten as

$$u(\varrho) \partial_x \varrho = \partial_x \left(\int d\lambda h_1 \rho_p^{\text{st}}(x, \lambda) \right) \equiv \partial_x j_{\varrho}^{\text{Eul}}. \quad (55)$$

C. Diffusive scale

Let us now turn to finding the first-order contribution to (47). There will be two contributions at order δ^0 from the collision integral

$$\frac{1}{\delta_{\mathcal{I}}} \mathcal{I}[\rho_p] = \Gamma^{\text{st}} \cdot \left[\rho_p^{(1,0)} + \frac{\delta_{\mathcal{D}}}{\delta_{\mathcal{I}}} \rho_p^{(0,1)} \right] + \mathcal{O}(\delta), \quad (56)$$

where we introduced linear operator

$$\Gamma^{\text{st}} = \frac{\delta \mathcal{I}}{\delta \rho_p} \Big|_{\rho_p^{\text{st}}}. \quad (57)$$

At the same time the GHD diffusion does not contribute at this order. This gives then two equations

$$\Gamma^{\text{st}} \cdot \rho_p^{(1,0)} = \frac{\partial \rho_p^{\text{st}}}{\partial \varrho} G^{(0,0)} + \frac{\partial (v^{\text{st}} \rho_p^{\text{st}})}{\partial x}, \quad (58)$$

$$\Gamma^{\text{st}} \cdot \rho_p^{(0,1)} = 0. \quad (59)$$

The second equation implies that $\rho^{(0,1)}$ is in the kernel of Γ^{st} . This implies that $\rho^{(0,1)} = 0$, due to constitutive relations. We are left with the equation for $\rho^{(1,0)}$. We substitute for $G^{(0,0)}$ and use the chain rule in the last term to obtain

$$\Gamma^{\text{st}} \cdot \rho^{(1,0)} = (\mathcal{A}^{\text{st}} - u) \cdot \frac{\partial \rho_p^{\text{st}}}{\partial \varrho} \frac{\partial \varrho}{\partial x}, \quad (60)$$

where we used that the hydrodynamic matrix $\mathcal{A} \equiv \mathcal{BC}^{-1}$ obeys $\delta(v\rho_p) = \mathcal{A} \cdot \delta\rho_p$ [83]. The integral equation (60) is simplified by the following ansatz

$$\rho_p^{(1,0)}(\lambda, \varrho(x)) = a(\lambda, \varrho(x)) \partial_x \varrho. \quad (61)$$

The resulting equation, free of explicit space-time dependence, is then

$$\Gamma^{\text{st}} \cdot a = (\mathcal{A}^{\text{st}} - u) \cdot \frac{\partial \rho_p^{\text{st}}}{\partial \varrho}. \quad (62)$$

The constitutive relation (50) additionally implies

$$\int d\lambda a(\lambda, \varrho) = 0. \quad (63)$$

The resulting contribution to the hydrodynamic equation is $G^{(1,0)} = \partial_x (\mathcal{D}(\varrho) \partial_x \varrho)$ where we defined the diffusion coefficient

$$\mathcal{D}(\varrho) = - \int d\lambda h_1(\lambda) a(\lambda, \varrho). \quad (64)$$

By combining the Euler scale equation with the contribution from the diffusive scale, we finally obtain the Burgers (2). The diffusion coefficient $\mathcal{D}(\varrho)$ is invariant under the Galilean transformations, see Appendix D 3. Note that there is no contribution from the diffusion present in the GHD as $G^{(0,1)} = 0$. We also mention that the diffusion coefficient in general scales as the characteristic timescale of $\mathcal{I}$, namely $\mathcal{D} \sim \tau_{\mathcal{I}}$. This concludes the derivation of the hydrodynamic equation for density.

With the characteristic scaling $\mathcal{D} \sim \tau_{\mathcal{I}}$ we can say something general about the typical regimes in the derived Burgers equation. For this purpose, one can define the so-called effective Péclet number, which is the ratio between the *non-linear* advective and the diffusive transport rates

$$\text{Pe} = \frac{\langle \Delta u \rangle l_h}{\langle \mathcal{D} \rangle}, \quad (65)$$

where $\langle \mathcal{D} \rangle$ is a typical value of the diffusion coefficient and Δu is a characteristic variation of advective coefficient. Even though the diffusion is large, our hydrodynamic equation is derived on length scales $l_h \gg \tau_{\mathcal{I}} \langle v \rangle$, cf. (42). Therefore, the relevant regime is the one of high Péclet numbers, where shock formation is dominant.

V. APPLICATIONS

In the following section we apply the ChE formalism to coupled Bose gases, stochastic velocity swap models and to the so-called relaxation time approximation of collision operator.

A. Relaxation time for two tubes system

We turn now to a more detailed picture of the collision integral which arises in a quasi-1D systems of two 1D gases that are coupled with a density-density interaction. In the density imbalance limit, as discussed in Section III, the collision integral $\mathcal{I}[\rho_p]$ in the minority tube, assumed to be tube 1, is given by the Master equation (24). The dynamic structure factor of the bath tube is given by [38]

$$S_0^{(2)}(k, \omega) = (2\pi)^2 \int d\lambda d\mu \rho_p^{(2)}(\lambda) \rho_h^{(2)}(\mu) \times |F_2(\lambda, \mu)|^2 \delta(k - k_2(\lambda, \mu)) \delta(\omega - \omega_2(\lambda, \mu)). \quad (66)$$

In the density imbalance limit evolution of the environmental tube is frozen i.e., $\partial_t \rho_p^{(2)} = \mathcal{I}_{21}[\rho_p^{(2)}, \rho_p^{(1)}] \approx 0$. Therefore we are free to choose $\rho_p^{(2)}$ to be any stationary state. We will denote the state in the environmental tube as $\rho_p^{(b)}$ and will refer to the particle density in the evolving tube $\rho_p^{(1)}$ simply as ρ_p . Let us assume that (homogenous) state of the system tube is close to a stationary state

$$\rho_p = \rho_p^{\text{st}} + \delta\rho_p, \quad (67)$$

where ρ_p^{st} satisfies $\mathcal{I}_{12}[\rho_p^{\text{st}}] = 0$. In such setup, evolution of the system tube can be approximated as

$$\frac{\partial (\delta\rho_p)}{\partial t} = \Gamma^{\text{st}} \cdot \delta\rho_p, \quad (68)$$

where

$$\Gamma^{\text{st}} = \left. \frac{\delta \mathcal{I}_{12}[\rho_p, \rho_p^{(b)}]}{\delta \rho_p} \right|_{\rho_p = \rho_p^{\text{st}}}. \quad (69)$$

Detailed computation of Γ^{st} is presented in the appendix E. Given that $\mathcal{I}_{12}$ has only one collision invariant, operator Γ^{st} has exactly one left null vector, corresponding to the conservation of the number of particles, see (16). Denoting eigenvalues as γ_i we have $0 = \gamma_0 > \gamma_1 \geq \gamma_2 \geq \dots$. Except for the non-decaying zeroth mode, each eigenmode of Γ^{st} will decay with the characteristic time $1/|\gamma_i|$. The characteristic relaxation time can be identified with a gap between zero mode and smallest non-zero eigenvalue of Γ^{st} [39].

$$\tau_{\text{RTA}} = |\gamma_1^{-1}|. \quad (70)$$

Numerical estimation of the relaxation time is discussed in the Appendix F. This gives a conservative thermalisation scale and observables that do not couple to the slowest decaying charge will thermalise at faster rate.

B. Relaxation Time Approximation

Given the complex nature of the collision integral, to be able to say more about the evolution without restricting to the specific system we need to make some approximation. One of the simplest approximation to apply is the RTA [39]. It amounts to the following approximation of collision integral

$$\mathcal{I}[\rho_p] \approx \tau_{\text{RTA}}^{-1} (\rho_p^{\text{st}}[\rho_p] - \rho_p), \quad (71)$$

where multiplication by τ_{RTA}^{-1} enforces relaxation at proper time scale. We only consider the case when the relaxation time is independent of ϱ . In the equation above ρ_p^{st} is a stationary state towards which ρ_p evolves. If the only non-trivially conserved charge is the total number of particle then ρ_p^{st} is related to ρ_p via $\int d\lambda \rho_p^{\text{st}} = \int d\lambda \rho_p$. This relation makes $\rho_p^{\text{st}}[\rho_p]$ highly non-linear functional of ρ_p [39]. Our ultimate goal in this section is to compute the diffusion coefficient (64) for the collision integral in the relaxation time approximation. First thing to do is to compute linearized collision integral (57):

$$\Gamma^{\text{st}} = \frac{1}{\tau_{\text{RTA}}} \left(\frac{\partial \rho_p^{\text{st}}}{\partial \varrho} h_0 - \mathbf{1} \right), \quad (72)$$

where $\mathbf{1}$ is the identity operator represented by the Dirac δ -function. Because of the constitutive relation (63), action of Γ on a is especially simple

$$\Gamma^{\text{st}} \cdot a = -\frac{1}{\tau_{\text{RTA}}} a, \quad (73)$$

since

$$\frac{\partial \rho_p^{\text{st}}(\lambda)}{\partial \varrho} \int d\mu h_0(\mu) a(\mu) = 0. \quad (74)$$

The integral equation is now simple to solve,

$$a = \tau_{\text{RTA}} (\mathcal{A}^{\text{st}} - u) \cdot \frac{\partial \rho_p^{\text{st}}}{\partial \varrho}. \quad (75)$$

By direct computations one can check that the constitutive relation (63) is fulfilled. The resulting diffusion coefficient is

$$\mathcal{D}(\varrho) = \tau_{\text{RTA}} \left(\int d\lambda h_1(\lambda) \mathcal{A}^{\text{st}} \cdot \frac{\partial \rho_p^{\text{st}}}{\partial \varrho} - u^2 \right). \quad (76)$$

This formula can be rewritten in terms of the momentum current

$$j_1 = \int d\lambda h_1(\lambda) v(\lambda) \rho_p^{\text{st}}(\lambda). \quad (77)$$

Computing the derivative of j_1 with respect to the density we find

$$\frac{\partial j_1}{\partial \varrho} = \int d\lambda h_1(\lambda) \frac{\partial(v(\lambda) \rho_p^{\text{st}})}{\partial \varrho} = \int d\lambda h_1(\lambda) \mathcal{A}^{\text{st}} \frac{\partial \rho_p^{\text{st}}}{\partial \varrho}. \quad (78)$$

Hence the diffusion coefficient is

$$\mathcal{D}_{\text{RTA}} = \tau_{\text{RTA}} \left(\frac{\partial j_1}{\partial \varrho} - u^2 \right). \quad (79)$$

In a thermal state $u = 0$ and the momentum current is equal to the thermodynamic pressure. The derivative of pressure is computed at fixed temperature which relates it to the isothermal compressibility κ_T such that

$$\mathcal{D}_{\text{RTA}}^{\text{th}} = \frac{\tau_{\text{RTA}}}{\varrho \kappa_T}, \quad \frac{1}{\kappa_T} = \varrho \left(\frac{\partial P}{\partial \varrho} \right)_T, \quad (80)$$

where P is the 1D pressure. The diffusion coefficient $\mathcal{D}_{\text{RTA}}^{\text{th}}$ is positive since $\kappa_T > 0$ reflecting the stability of thermal equilibrium.

In the upper part of Fig. 3 we present comparison of the diffusion coefficient computed in the RTA with the diffusion coefficient computed using the full collision integral for the two (Lieb–Liniger) tubes system [84]. For more detailed description of the numerical procedure we refer to the Appendix F. For the analyzed system we observe that RTA gives a good approximation of the diffusion coefficient. In the lower panel figure 3 we present numerically-computed ν for the bath tube in an asymmetric Bragg-split state, i.e., state of the form

$$\rho_p^{(b)}(\lambda) \propto \rho_p^{\text{th}}(\lambda + \lambda_B) + b \rho_p^{\text{th}}(\lambda - \lambda_B), \quad (81)$$

where ρ_p^{th} is a Gibbs thermal state, $b \in [0, 1]$ is an asymmetry parameter and λ_B fixes the peak separation. Numerically computing u' , we find that its value varies over several orders of magnitude, depending on the environmental tube's state. We find that u' attains the largest values in the regime of low temperatures, with λ_b being slightly smaller than $(\text{width of } \rho_p^{\text{th}})/2$, meaning that left and right peaks minimally overlap with each other.

Let us now present a qualitative way of deriving the diffusion coefficient for a classical thermal case. Suppose we have a viscous stationary fluid. A slowly moving particle experiences then a Stokes drag $F_d = -\zeta u$ where u is the particle's velocity and ζ is friction. Under such assumption the total force acting on the particle is $F = F_p + F_d$, where $F_p = -a \partial_x P$ is a force exerted by the pressure gradient on a particle of size (length) a . The net force F will cause the particle to accelerate towards the terminal velocity fixed by the condition $F = 0$. Thus, assuming the particle to reach the terminal velocity quickly, we get a relation between velocity of the particle and the pressure gradient

$$u = -\frac{da}{\zeta} \frac{\partial P}{\partial x}. \quad (82)$$

Conservation of the total number of particles leads to the continuity equation $\frac{\partial \varrho}{\partial t} + \partial_x(\varrho u) = 0$. Using the relation between u and the pressure gradient and assuming that each fluid cell is in a thermal state with fixed temperature and locally varying density we obtain

$$\frac{\partial \varrho}{\partial t} - \frac{\partial}{\partial x} \left(\frac{m}{\zeta} \frac{\partial P}{\partial \varrho} \frac{\partial \varrho}{\partial x} \right) = 0, \quad (83)$$

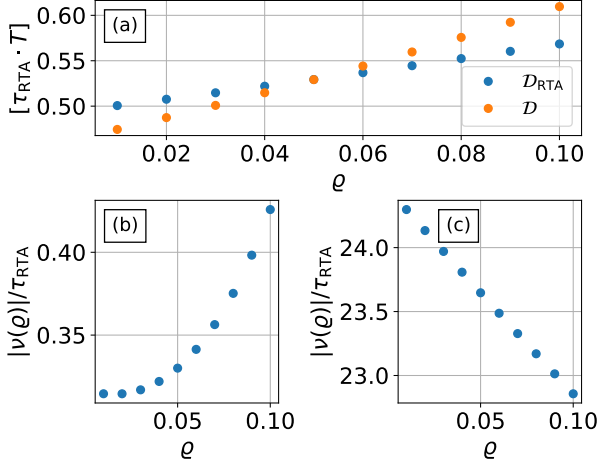


FIG. 3. Transport coefficients in the coupled LL models in the Tonks-Girardeau limit. Panel (a): comparison of diffusion coefficient stemming from RTA with non-approximated diffusion for the two-tubes system in a thermal state ($T = 4.0$). In the panels (b) and (c) we show the Burgers parameter $\nu = \mathcal{D}_{\text{RTA}}/u'$ for a system in asymmetric Bragg-split state (81). In (b) we present a system in a small temperature ($T = 1.0$), with asymmetry $b = 9$ with the peak separation $\lambda_B = 3.9$ chosen in a way so that they overlap over a small region in λ . For (c), we increased the temperature ($T = 10.0$), simultaneously increasing peak separation to $\lambda_B = 8.0$ with the same $b = 9$.

from which the diffusion coefficient can be read off. Comparing with the result of the ChE theory within the RTA we find the relation between the friction and the relaxation time given by $\tau_{\text{RTA}} = \frac{m}{\zeta}$. This provides a way to estimate the relaxation time from the kinematics of a single particle.

C. Stochastic velocity swap models

In this section we analytically compute diffusion coefficient $\mathcal{D}(\varrho)$ along with $u(\varrho)$ for the velocity swap model of classical and fermionic particles. We can treat both cases at the same time by introducing χ which is $\chi = 0$ for classical particles and $\chi = 1$ for fermions. We begin by introducing pseudoenergy functions

$$\epsilon = \log\left(\frac{1 - \chi n}{n}\right), \quad \epsilon^{(b)} = -\log(n^{(b)}). \quad (84)$$

First, let us find the stationary states. To this end we write the collision integrals (38) and (39) in a convenient form

$$\begin{aligned} \mathcal{I}[\rho_p](\lambda) = & \gamma \int d\lambda' (1 - \chi n(\lambda')) \rho_p^{(b)}(\lambda') \rho_p(\lambda) \\ & \times \left(e^{\epsilon(\lambda) - \epsilon(\lambda') - (\epsilon^{(b)}(\lambda) - \epsilon^{(b)}(\lambda'))} - 1 \right). \end{aligned} \quad (85)$$

The condition fixing the stationary state gives a simple relation between pseudo-energies

$$\epsilon^{\text{st}}(\lambda) = \epsilon^{(b)}(\lambda) + \log \tilde{A}, \quad (86)$$

where $\tilde{A}$ is some positive constant. From (84) we recover the stationary states (40) and (20). The constant $A = \frac{\tilde{A}}{2\pi}$ is fixed by the condition $\varrho = \int d\lambda \rho_p^{\text{st}}(\lambda)$. It follows that for a fixed state $\rho_p^{(b)}$, A is a function of ϱ only and so $\frac{\partial \rho_p^{\text{st}}(\lambda, \varrho)}{\partial \varrho} = \frac{\partial \rho_p^{\text{st}}(\lambda, A)}{\partial A} \frac{dA}{d\varrho}$. By a direct computation we get

$$\frac{\partial \rho_p^{\text{st}}(\lambda, A)}{\partial A} = -A^{-1} \rho_p^{\text{st}}(\lambda) (1 - \chi n^{\text{st}}(\lambda)). \quad (87)$$

Similarly $\frac{d\varrho}{dA} = -A^{-1} (h_0|h_0)_{\text{st}}$, where we defined

$$(\phi|\psi)_{\text{st}} := \int d\lambda \rho_p^{\text{st}}(\lambda) (1 - \chi n^{\text{st}}(\lambda)) \phi(\lambda) \psi(\lambda). \quad (88)$$

Therefore we get

$$\frac{\partial \rho_p^{\text{st}}(\lambda, \varrho)}{\partial \varrho} = \frac{\rho_p^{\text{st}}(\lambda) (1 - \chi n^{\text{st}}(\lambda))}{(h_0|h_0)_{\text{st}}}. \quad (89)$$

It follows that for the analyzed system u is given by

$$u(\varrho) = \frac{(h_1|h_0)_{\text{st}}}{(h_0|h_0)_{\text{st}}}. \quad (90)$$

In similar manner we find

$$u'(\varrho) = -2\chi \frac{(h_1 - u|n^{\text{st}})_{\text{st}}}{(h_0|h_0)_{\text{st}}^2}. \quad (91)$$

Now we turn to computing the diffusion coefficient. To do this we need to solve the linear integral equation (62). Before proceeding let us note that for the considered system $v^{\text{st}}(\lambda) = \lambda$ and we can identify $\mathcal{A} \equiv \lambda$. Thus the equation we need to solve is

$$\left. \frac{\delta \mathcal{I}}{\delta \rho_p} \right|_{\rho_p^{\text{st}}} \cdot a = \frac{(\lambda - u) \rho_p^{\text{st}}(\lambda) (1 - \chi n^{\text{st}}(\lambda))}{(h_0|h_0)_{\text{st}}}. \quad (92)$$

We proceed by writing

$$\begin{aligned} \left. \frac{\delta \mathcal{I}}{\delta \epsilon} \right|_{\rho_p^{\text{st}}} \cdot \delta \epsilon = & \gamma \int d\lambda' (1 - \chi n^{\text{st}}(\lambda')) \rho_p^{(b)}(\lambda') \rho_p(\lambda) \\ & \times [\delta \epsilon(\lambda) - \delta \epsilon(\lambda')], \end{aligned} \quad (93)$$

where we used that for a stationary state equation (86) is satisfied. From the definition (84) it follows that $\delta \rho_p = -\rho_p (1 - \chi n) \delta \epsilon$, which can be checked by a direct calculation. Introducing

$$b(\lambda) = -\gamma \frac{(h_0|h_0)_{\text{st}} a(\lambda)}{\rho_p^{\text{st}}(\lambda) (1 - \chi n^{\text{st}}(\lambda))} \quad (94)$$

the integral equation becomes

$$\int d\lambda' (1 - \chi n^{\text{st}}(\lambda')) \rho_p^{(b)}(\lambda') [b(\lambda) - b(\lambda')] = (\lambda - u) (1 - \chi n^{\text{st}}). \quad (95)$$

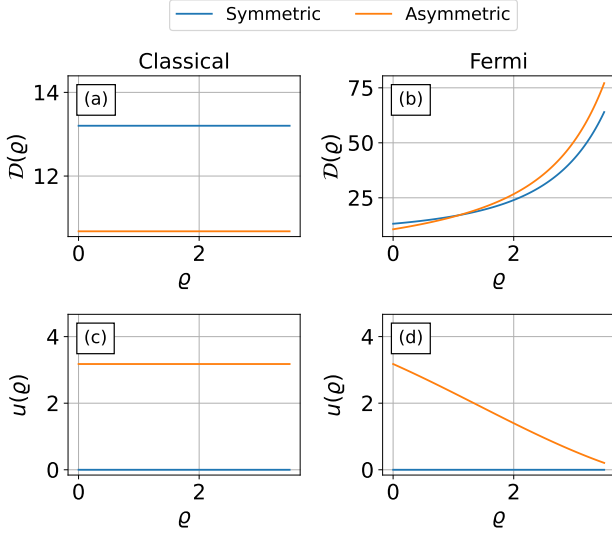


FIG. 4. Diffusion (panels (a) and (b)) and advective (panels (c),(d)) coefficients as a function of density ρ for the stochastic velocity swap models coupled to the classical bath. The bath is in the same state as in Fig. 5 and the collision rate $\gamma = 0.08$ is the same as well. Note that fermionic statistics make both diffusion and advective coefficients density-dependent. This has important consequences for the dynamics as visible in Fig. 5(d).

To solve the equation we differentiate both sides over λ

$$b'(\lambda) = \frac{1}{\left(h_0|\rho_p^{(b)} / \rho_p^{st}\right)_{st}} \left((\lambda - u) (1 - \chi n^{st}(\lambda))\right)', \quad (96)$$

and so

$$b(\lambda) = \frac{(\lambda - u) (1 - \chi n^{st}(\lambda)) - C}{\left(h_0|\rho_p^{(b)} / \rho_p^{st}\right)_{st}}, \quad (97)$$

with C being some constant.

To fix the constant we use the relation (63), which gives

$$b(\lambda) = \frac{(\lambda - u) (1 - \chi n^{st}(\lambda)) - \frac{(\lambda - u|1 - \chi n^{st})_{st}}{(h_0|h_0)_{st}}}{\left(h_0|\rho_p^{(b)} / \rho_p^{st}\right)_{st}}. \quad (98)$$

Let us now verify that the obtained solution indeed solves the integral equation. Plugging it into (95) we find that the equation has a solution iff

$$\int d\lambda' (1 - \chi n^{st}(\lambda')) \rho_p^{(b)}(\lambda') (\lambda' - u) (1 - \chi n^{st}(\lambda')) = 0. \quad (99)$$

To see why the equation above is always satisfied, note that for a stationary state

$$(1 - \chi n^{st}(\lambda')) \rho_p^{(b)}(\lambda') = A n^{st}(\lambda'), \quad (100)$$

which follows from the definition (84) and (86). Thus the solvability condition is equivalent to $(h_1 - u|h_0)_{st} = 0$,

which coincides with the definition of u , hence the solution we have found is the correct solution of (95). Having solved for $b(\lambda)$, we compute the diffusion coefficient using (64) which, after some algebraic manipulations, gives us

$$\mathcal{D}(\rho) = \frac{1}{\gamma} \frac{\left((h_1 - u)^2 |1 - \chi n^{st}\right)_{st}}{\left(h_0|h_0\right)_{st} \left(h_0|\rho_p^{(b)} / \rho_p^{st}\right)_{st}}. \quad (101)$$

For the classical particles $\chi = 0$ and the relation between the bath state and a stationary state is especially simple and given by (20). The convection velocity function $u(\rho)$ simplifies to

$$u = \frac{1}{\rho^{(b)}} \int d\lambda \lambda \rho_p^{(b)}(\lambda) = \text{const}, \quad (102)$$

and the diffusion coefficient is

$$\mathcal{D} = \frac{1}{\gamma \left(\rho^{(b)}\right)^2} \int d\lambda (\lambda - u)^2 \rho_p^{(b)}(\lambda) = \text{const}. \quad (103)$$

Hence, performing a boost to remove the convective term, the general Burgers-diffusion (2) equation reduces to the ordinary linear diffusion equation

$$\partial_t \rho = \mathcal{D} \partial_x^2 \rho. \quad (104)$$

For $\rho_p^{(b)}$ being a boosted thermal state with pseudo energy $\epsilon^{(b)} = \beta \mu^{(b)} + \kappa h_1(\lambda) + \beta h_2(\lambda)$, the diffusion coefficient reads

$$\mathcal{D} = \frac{T}{\gamma \rho^{(b)}}. \quad (105)$$

In Fig. 4 we present the transport coefficient computed for classical and fermionic particles for the $\rho^{(b)}$ being an asymmetric Bragg-split state.

VI. NUMERICAL SIMULATIONS

We consider now molecular dynamics of the stochastic velocity swap models to check the hydrodynamic theory for both classical and fermionic particles. Both cases can be simulated efficiently and on the level of kinetic equation description, they are given by (38) and (39).

In the simulation, particles are initialized such that the initial distribution of velocities is given by

$$\rho_p^{(b)} \propto e^{-(\lambda + \lambda_B)^2 / \sigma^2} + b e^{-(\lambda - \lambda_B)^2 / \sigma^2} \quad (106)$$

with $\sigma = 6$, $\lambda_B = 5.9$ and $b = 1$ for symmetric states and $b = 0.3$ for asymmetric states. The particles propagate freely and occasionally undergo two-body collisions with the bath. A collision amounts to velocity swap between system and a bath. When particles are classical and collisions happen at a constant rate the corresponding Master equation is precisely (38). Since the rate does

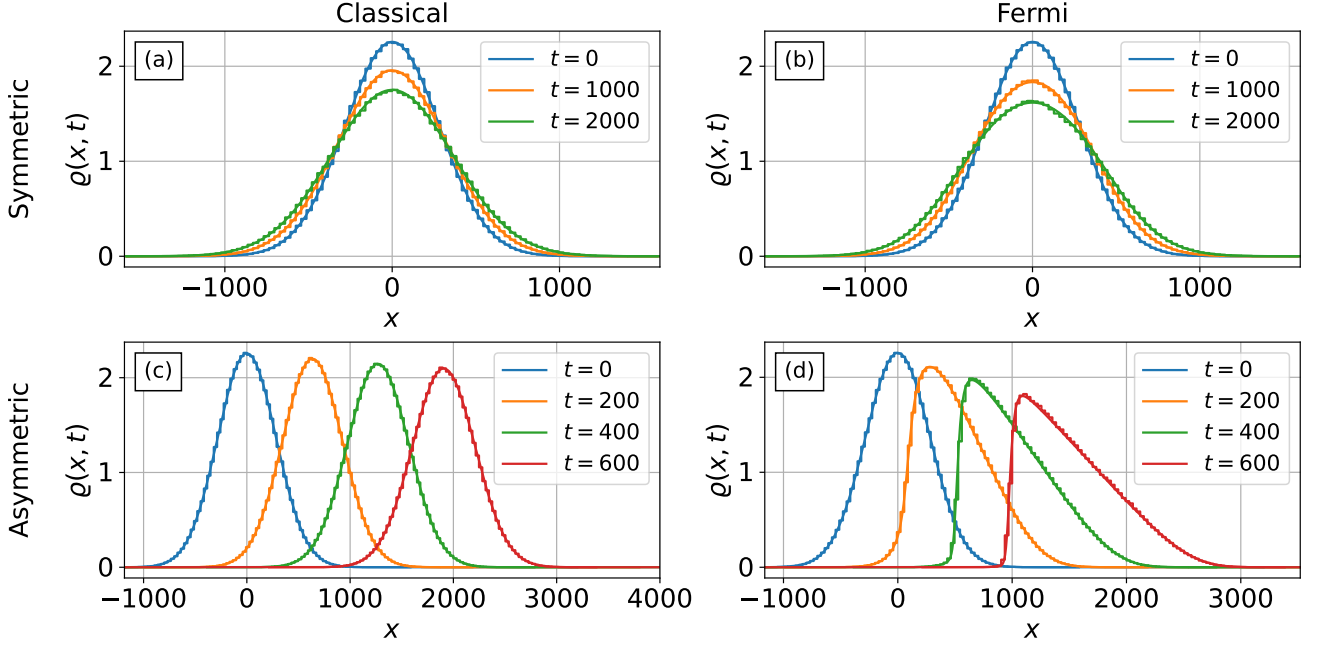


FIG. 5. Molecular dynamics simulations for velocity swap models ($\gamma = 0.08$, $\varrho^{(b)} = 50.0$, $N = 1600$) with initial states given by (106). We observe an excellent agreement between numerical simulations (histograms) and hydrodynamic equation (2) derived through ChE method (solid lines) for all cases. For symmetric states in bath system, cf. panels (a,b), both systems are described by diffusion equation (1). Upon introducing the asymmetry, the classical system is described by boosted diffusion equation, see panel (c). On the other hand, the fermionic model exhibits Burgers equation (2) dynamics, as visible in the dynamically generated asymmetry in the initially Gaussian shape, cf. panel (d). Results from microscopic simulations were averaged over 1000 realizations.

not depend on the state of the system, such collisions can be efficiently implemented with standard Gillespie algorithm [85]. The more complicated case is the simulation of fermionic kinetic equation given by (39). There, we partition the system into fluid cells and propose an algorithm, which effectively incorporates the Pauli principle into the collisions. Algorithms for both cases are discussed in Appendix C

The results are presented in Fig. 5. A very good agreement is found between molecular simulations and hydrodynamic equation (2). Depending on symmetry of the state (or lack thereof) and particle statistics, we find that numerical results excellently agree with diffusion equation (1) and with Burgers equation (2) for fermionic model with parity-asymmetric state.

Burgers equation in 1D is well-known for its breakdown of diffusive behavior in hydrodynamic two-point functions of the density field [86] close to homogenous equilibrium state. From this perspective, it is interesting to note that (finite) diffusion constant computed from kinetic theory is relevant for description of the numerics in our highly inhomogenous setting.

In the simulation, we used a Gaussian initial profile $\varrho = \varrho_h e^{-(x/l_h)^2}$, with $\varrho_h \approx 2.1$ and $l_h = 400$. For the values of ϱ attained during the simulation $\langle \Delta u \rangle \approx 2$ and

$\langle D \rangle \approx 20$. The corresponding Péclet number is

$$Pe \approx 40, \quad (107)$$

and signatures of shock formation are observed in the simulation, see Fig. 5(d). Note that due to $u'(\varrho) < 0$ (cf. Fig. 4(d)), a shock is formed at the trailing edge of the density profile.

VII. SUMMARY AND DISCUSSION

In this work, we have considered a canonical problem of diffusion in which a minority species undergoes an irreversible motion due to coupling to the majority species, whose dynamics is not monitored and which, as is customary to assume, is in a spatially homogenous state. Due to the conservation of particle number of the minority species, its density obeys a continuity equation. In the standard situation of classical particles this continuity equation leads to a (possibly non-linear) diffusion equation. In this work we have identified the conditions under which the density follows instead the Burgers equation.

The two sufficient conditions are: a) athermal and parity-asymmetric state of the majority species (bath) and b) quantum statistics of minority species (system).

We have demonstrated that these two conditions are fulfilled for effective dynamics of nearly-integrable coupled gases. In such scenario, the integrability naturally makes athermal states possible. We have provided a kinetic explanation for appearance of such athermal states through limitations of two-body scattering in 1D. In our approach, we assume a large density imbalance between the two gases allowing for a separation of timescales between the slow dynamics in the dense tube (bath) and fast dynamics in the dilute tube (system). Starting from the GHD-Boltzmann description, we have derived an effective hydrodynamic equation for the density of the system particles (minority species). We have shown that the resulting equations become a standard diffusion equation when the bath is in a thermal state. Instead, for athermal state of a bath we obtain Burgers equation. Our method provides also expressions for transport coefficients: advective coefficient $u(\varrho)$ and diffusion constant $\mathcal{D}(\varrho)$.

To unravel the essential ingredients leading to the Burgers equation we have considered also a simple stochastic model of two-particle scatterings between system and the bath. We observed an excellent agreement between microscopic numerical simulations and hydrodynamics derived from kinetic equation describing such random scattering events.

Lastly, let us discuss the features of the hydrodynamic density-density correlation function, which is a natural object appearing in the literature on 1D Burgers equation [86]. This is because under the presence of ballistic mode with a nonlinear current, the correlation function is expected to exhibit a breakdown of diffusive behavior at the largest scales [24]. Close to the integrability, we generically expect large diffusion constants, which push away the anomalous broadening to very large space-time scales. In this sense, we expect a scenario similar to the one analyzed in the context of Navier-Stokes equations [60]. We provide an analysis of the hydrodynamic correlation function along the lines of [60] in the Appendix G. We also note that one has to be careful when looking at asymptotically large scales in models with (late-time) thermalization channels. There are three-particle (and higher) collision processes, which provide a mechanism for thermalization at late times, assuming that the density of the bath is finite. This leads ultimately to the breakdown of Burgers equation. We plan to investigate the interesting problem of two-point functions using the models studied here in the future.

ACKNOWLEDGMENTS

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Appendix A: Grazing limit of the collision integral (19)

In this Appendix we formalize the grazing limit introduced in Sec. III. Our starting point is collision integral (19). We introduce new variables under the integral

$$\alpha = \lambda - \lambda', \quad \gamma = \frac{\lambda + \lambda'}{2} \quad (\text{A1})$$

and similarly we define α_b and γ_b . Note that the parameters α, α_b are nothing else but changes of momenta of system (bath) particles that take part in a collision process $\lambda' + \lambda'_b \rightarrow \lambda + \lambda_b$. The collision rate becomes

$$\mathcal{W}\left(\gamma + \frac{\alpha}{2}, \gamma_b + \frac{\alpha_b}{2}; \gamma - \frac{\alpha}{2}, \gamma_b - \frac{\alpha_b}{2}\right) \quad (\text{A2})$$

and the grazing limit is easiest to formulate when the expression above is viewed as a function of α and α_b . We essentially assume that it is localized in the region where both α, α_b are small as compared to typical range of variation of states ρ_p and $\rho_p^{(b)}$. This allows for performing expansions in α, α_b in other terms, that enter the integral. For convenience, we rewrite the collision integral into the form

$$\begin{aligned} \mathcal{I}[\rho_p](\mu) = & \int d\lambda d\lambda' d\lambda_b d\lambda'_b \delta(\mu - \lambda) \mathcal{W}(\lambda, \lambda_b; \lambda', \lambda'_b) \times \\ & \delta(k) \delta(\omega) \rho_p(\lambda') \rho_p^{(b)}(\lambda) \times \\ & (\exp[\epsilon(\lambda) + \epsilon_b(\lambda_b) - \epsilon(\lambda') - \epsilon_b(\lambda'_b)] - 1), \end{aligned} \quad (\text{A3})$$

For instance, the momentum conservation becomes

$$k'(\gamma)\alpha + k'_b(\gamma_b)\alpha_b = 0 \quad (\text{A4})$$

and similarly for energy and expression with pseudoenergies. This yields

$$\delta(k) \delta(\omega) = \frac{\delta(v(\gamma) - v_b(\gamma_b)) \delta\left(\alpha_b + \alpha \frac{k'(\gamma)}{k'_b(\gamma_b)}\right)}{|\alpha| k'(\gamma) k'_b(\gamma_b)}. \quad (\text{A5})$$

We also expand $\delta(\mu - \gamma - \frac{\alpha}{2}) = \delta(\mu - \gamma) - \partial_\mu \delta(\mu - \gamma) \frac{\alpha}{2} + \dots$. Moreover, we assume that the rate (A2) is symmetric function of α and thus the first term in the expansion of Dirac delta vanishes. We define

$$\begin{aligned} \tilde{\mathcal{W}}(\gamma, \gamma_b) = & \frac{1}{k'(\gamma) k'_b(\gamma_b)} \int d\alpha \frac{|\alpha|}{2} \times \\ & \mathcal{W}\left(\gamma + \frac{\alpha}{2}, \gamma_b - \alpha \frac{k'(\gamma)}{k'_b(\gamma_b)}; \gamma - \frac{\alpha}{2}, \gamma_b + \alpha \frac{k'(\gamma)}{k'_b(\gamma_b)}\right) \end{aligned} \quad (\text{A6})$$

and find that the collision integral indeed takes the form (21) upon truncating the Dirac delta expansion after the second term.

Appendix B: Derivation of collision operator from GBBGKY hierarchy

The following section is based heavily on formalism and notation introduced in [48], especially on the part

related to many-tube systems. Thus, for the purpose of clearer presentation in this section we change the notation slightly. The quasimomentum dependence will be denoted in lower index and we introduce rotation matrices $R = \mathbf{1} - \mathcal{T}n$. We also denote $\mathcal{A}_{\lambda_1, \lambda_2} = R_{\lambda_1, \gamma}^{-t} v_\gamma R_{\gamma, \lambda_2}^t$ and $\mathcal{C}_{\lambda_1, \lambda_2} = R_{\lambda_1, \gamma}^{-t} \rho_\gamma f_\gamma R_{\gamma, \lambda_2}^{-1}$. The upper indices (i) denote tube index. To system tube we will refer to with index (1), bath tube will be denoted with (2). Einstein summation convention will be assumed in integration over repeated quasimomenta. We start with the following set of equations:

$$\begin{aligned} \partial_t \rho_{\lambda_1}^{(1)} &= \int dx_2 d\lambda_2 V'(x_1 - x_2) \partial_{\lambda_1} g_{\lambda_1, \lambda_2}(x_1, x_2) \\ \partial_t \rho_{\lambda_2}^{(2)} &= \int dx_1 d\lambda_1 V'(x_1 - x_2) \partial_{\lambda_2} g_{\lambda_1, \lambda_2}(x_1, x_2) \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} &\partial_t g_{\lambda_1, \lambda_2}(x_1, x_2) + \\ &\mathcal{A}_{\lambda_1, \gamma}^{(1)} \partial_{x_1} g_{\gamma, \lambda_2}(x_1, x_2) + \mathcal{A}_{\lambda_2, \gamma}^{(2)} \partial_{x_2} g_{\lambda_1, \gamma}(x_1, x_2) = \\ &V'(x_1 - x_2) \int d\lambda_3 \left(\partial_{\lambda_1} \rho_{\lambda_1}^{(1)} \mathcal{C}_{\lambda_2, \lambda_3}^{(2)} - \partial_{\lambda_2} \rho_{\lambda_2}^{(2)} \mathcal{C}_{\lambda_1, \lambda_3}^{(1)} \right) \end{aligned} \quad (\text{B2})$$

We consider homogenous system with $z = x_1 - x_2$

$$\begin{aligned} \partial_t \rho_{\lambda_1}^{(1)} &= \int dz d\lambda_2 V'(z) \partial_{\lambda_1} g_{\lambda_1, \lambda_2}(z) \\ \partial_t \rho_{\lambda_2}^{(2)} &= \int dz d\lambda_1 V'(z) \partial_{\lambda_2} g_{\lambda_1, \lambda_2}(z) \end{aligned} \quad (\text{B3})$$

$$\partial_t g_{\lambda_1, \lambda_2} + \partial_z \left(\mathcal{A}_{\lambda_1, \gamma}^{(1)} g_{\gamma, \lambda_2} - \mathcal{A}_{\lambda_2, \gamma}^{(2)} g_{\lambda_1, \gamma} \right) = \mathcal{S}_{\lambda_1, \lambda_2} \quad (\text{B4})$$

with source term

$$\mathcal{S}_{\lambda_1, \lambda_2} = V'(z) \int d\lambda_3 \left(\partial_{\lambda_1} \rho_{\lambda_1}^{(1)} \mathcal{C}_{\lambda_2, \lambda_3}^{(2)} - \partial_{\lambda_2} \rho_{\lambda_2}^{(2)} \mathcal{C}_{\lambda_1, \lambda_3}^{(1)} \right) \quad (\text{B5})$$

To make progress with (B4) we rotate to normal modes of GHD as

$$g_{\lambda_1, \lambda_2} = (R^{(1)})_{\lambda_1, \gamma_1}^{-t} (R^{(2)})_{\lambda_2, \gamma_2}^{-t} \bar{g}_{\gamma_1, \gamma_2}, \quad (\text{B6})$$

which leads to (below we define $v_{\lambda_1, \lambda_2} := v_{\lambda_1}^{(1)} - v_{\lambda_2}^{(2)}$)

$$\partial_t \bar{g}_{\lambda_1, \lambda_2} + v_{\lambda_1, \lambda_2} \partial_z \bar{g}_{\lambda_1, \lambda_2} = \bar{\mathcal{S}}_{\lambda_1, \lambda_2}. \quad (\text{B7})$$

This equation is a linear equation with a source term, thus it can be solved easily. The Green's function is

$$\mathcal{G}_{\lambda_1, \lambda_2}(z, z'; t, t') = \frac{1}{2\pi} \int dk e^{ik(z+z'-v_{\lambda_1, \lambda_2} t)} \theta_H(t-t') \quad (\text{B8})$$

with θ_H denoting Heaviside function. The solution is given by

$$\bar{g}_{\lambda_1, \lambda_2}(z, t) = \int dz' dt' \mathcal{G}_{\lambda_1, \lambda_2}(z, z'; t, t') \bar{\mathcal{S}}_{\lambda_1, \lambda_2}(z', t'). \quad (\text{B9})$$

After rotating back we get

$$\begin{aligned} g_{\lambda_1, \lambda_2}[\rho](z, t) &= (R^{(1)})_{\lambda_1, \gamma_1}^{-t} (R^{(2)})_{\lambda_2, \gamma_2}^{-t} \int_0^t d\tau \int \frac{dk}{2\pi} \int dz' \times \\ &e^{ik(z+z'-v_{\gamma_1, \gamma_2} \tau)} (R^{(1)})_{\gamma_1, \eta_1}^t (R^{(2)})_{\gamma_2, \eta_2}^t \mathcal{S}_{\eta_1, \eta_2}[\rho](z', t-\tau). \end{aligned} \quad (\text{B10})$$

In the next step we plug this expression into (B3), and perform the standard approximations used in such derivations [78], related to separations of scales. In practice replace the time argument of $\mathcal{S}_{\eta_1, \eta_2}[\rho](z', t-\tau)$ with t and extend the integration range over τ to infinity. We also insert momentum representations of interaction potential $V(z) = \int dk e^{ikz} \tilde{V}(k)$ and use the property that $\int d\lambda R_{\lambda, \gamma}^{-t} = 2\pi \rho_\gamma^t$. After all these steps we find the collision integral

$$\partial_t \rho_{\lambda_1}^{(1)} = \mathcal{I}(\lambda_1), \quad \mathcal{I}(\lambda_1) = \partial_{\lambda_1} \left[(R^{(1)})_{\lambda_1, \eta}^{-t} \mathcal{I}_0(\eta) \right], \quad (\text{B11})$$

where

$$\mathcal{I}_0(\lambda_1) = (2\pi)^2 \int dk k^2 \tilde{V}^2(k) \int_0^\infty d\tau \int d\lambda_2 e^{-ikv_{\lambda_1, \lambda_2} \tau} \rho_{\lambda_2}^{t, (2)} (R^{(1)})_{\lambda_1, \eta_1}^t (R^{(2)})_{\lambda_2, \eta_2}^t \int d\eta_3 \left(\partial_{\eta_1} \rho_{\eta_1}^{(1)} \mathcal{C}_{\eta_2, \eta_3}^{(2)} - \partial_{\eta_2} \rho_{\eta_2}^{(2)} \mathcal{C}_{\eta_1, \eta_3}^{(1)} \right). \quad (\text{B12})$$

Where by $\rho_{\lambda_2}^{t, (2)}$ we denoted total density of states in tube 2. We proceed by using [78] $\int_0^\infty d\tau e^{i\tau x} = \pi \delta(x) + i\mathcal{P}(\frac{1}{x})$, where the second term is the principal value, which vanishes as the integrand is a symmetric function in k . In what follows we also use that [48] $\int d\lambda_2 \mathcal{C}_{\lambda_1, \lambda_2} = 2\pi R_{\lambda_1, \gamma}^{-t} \rho_\gamma f_\gamma \rho_\gamma^t$ and $R_{\lambda, \gamma}^t \partial_\gamma \rho_\gamma = -\rho_\lambda f_\lambda \partial_\lambda \epsilon_\lambda$. We get

$$\mathcal{I}(\lambda_1) = -2\pi^2 \int dk |k| \tilde{V}^2(k) \int d\lambda_2 \delta(v_{\lambda_1, \lambda_2}) (k'_2(\lambda_2))^2 \rho_{\lambda_1}^{(1)} f_{\lambda_1}^{(1)} \rho_{\lambda_2}^{(2)} f_{\lambda_2}^{(2)} \left(\epsilon'_{\lambda_1} - \frac{k'_1(\lambda_1)}{k'_2(\lambda_2)} \epsilon'_{\lambda_2} \right). \quad (\text{B13})$$

In the end we find collision integral which has essentially exactly the same stationary state (22) as the one found in the grazing limit of collision integral (19).

Appendix C: Molecular dynamics simulation algorithm

The classical system with collision integral given by (38) can be simulated fairly easy with a standard im-

plementation of Gillespie algorithm [85]. The fermionic

case (39) is more complicated due to Fermi exclusion principle, which we want to somehow model using dynamics of classical particles. We thus wish to construct a classical model for which the dynamics in phase-space would be described by kinetic equation (39). Our approach will turn out to be more general and cover both classical and fermionic case.

To account for spatial locality of the collision integral (95) we begin by dividing the system into sufficiently small cells of equal length and consider each cell as a separate system, while allowing the particles to flow between different cells. In the molecular dynamics simulation for the velocity-swap models we only scatter system particles with heat bath particles. For the classical case, we have no restriction on the momenta of the scattered particles, so number of such collisions in each cell per unit time (rate) is

$$R = \frac{\gamma \cdot N \cdot N_{\text{bath}}}{L/(\text{Number of cells})}, \quad (\text{C1})$$

where N and N_{bath} are respectively number of system particles and number of bath particles in the cell and L is total length of the system. In the fermionic case we need to account for the Pauli exclusion principle, which prohibits scattering if we already have a particle with the same (outgoing) momentum. Consequently, scattering rate needs to be modified for fermions:

$$R = \frac{\gamma \cdot N \cdot |\{v_{b,i} \notin V_{\text{system}} : i \in \{1, \dots, N_{\text{bath}}\}\}|}{L/(\text{Number of cells})}, \quad (\text{C2})$$

where by V_{system} we understand set of velocities of system particles in the cell and $v_{b,i}$ denotes velocity of i -th bath particle in the cell. For fermions we need to be careful, as dividing the system into cells we ran into a situation where the ballistic propagation of a particle can cause the exclusion principle to be violated. For the simulation presented in this paper, we resolve this problem by artificially scattering particles violating the exclusion principle with the particles from the bath. For the parameters used for the simulation presented in Fig. 5, the described situation is rare and we observed that no matter how this issue is resolved, density profiles do not change significantly.

In the simulations shown in Fig. 5 we performed the following variation of discrete step Monte Carlo algorithm [87] with a fixed time step, which we denote Δt :

1. Segregate all system particles into cells of equal size.
2. For each cell:
 - (a) Compute rate R .
 - (b) If we find exclusion principle to be violated (due to particle streaming), we scatter duplicates with the particles from the bath, so that there are no two particles with the same velocity in each cell.

- (c) Draw a random number from 0 to 1. If $R \cdot \Delta t$ is greater then the drawn number, go to next cell.
- (d) Otherwise, randomly choose system particle and bath particle in the cell and set the momentum of the system particle to the momentum of the bath particle.

3. Ballistically propagate system particles by a time Δt and go to the step 1.

Appendix D: Additional formulas and derivations

In this Appendix we present additional computations showing: Markovianity of the GHD diffusion operator, existence and uniqueness of the solution to the ChE equation and verify that parameter u transforms as a velocity while the diffusion coefficient is invariant under the Galilean transformations.

1. Diffusion operator and its Markovianity

We recall here the result of showing the GHD diffusion operator to be Markovian [52], namely

$$\int d\lambda \mathfrak{D}_\rho(\lambda, \lambda') = 0. \quad (\text{D1})$$

The GHD diffusion operator is [53, 88]

$$\mathfrak{D}_\rho = (\mathbf{1} - n\mathcal{T})^{-1} \rho_t^{-1} \tilde{\mathfrak{D}}_\rho \rho_t^{-1} (\mathbf{1} - n\mathcal{T}), \quad (\text{D2})$$

$$\tilde{\mathfrak{D}}_\rho(\lambda, \lambda') = \delta(\lambda - \lambda') w(\lambda) - W(\lambda, \lambda'), \quad (\text{D3})$$

with

$$W(\lambda, \lambda') = \rho_p(\lambda) f(\lambda) (\mathcal{T}^{\text{dr}}(\lambda, \lambda'))^2 |v_\rho(\lambda) - v_\rho(\lambda')|, \quad (\text{D4})$$

and $w(\lambda) = \int d\lambda' W(\lambda', \lambda)$. We note that the operator $(\mathbf{1} - n\mathcal{T})^{-1}$ implements dressing dr when acting to the left,

$$f^{\text{dr}} = f \cdot (\mathbf{1} - n\mathcal{T})^{-1}. \quad (\text{D5})$$

Relation (D1) follows from the fact that $2\pi\rho_{\text{tot}} = (h_0)^{\text{dr}}$. This gives

$$\frac{1}{2\pi} \int d\lambda \mathfrak{D}_\rho(\lambda, \lambda') = \int d\lambda \left(\tilde{\mathfrak{D}}_\rho \rho_t^{-1} (\mathbf{1} - n\mathcal{T}) \right) (\lambda, \lambda'), \quad (\text{D6})$$

and the latter integral vanishes because of the structure of $\tilde{\mathfrak{D}}_\rho(\lambda, \lambda')$.

2. Existence and uniqueness of the diffusion coefficient

The diffusion coefficient is expressed by the solution to the integral equation (62). We show here that the solution to this equation exists and is unique. This follows from the standard result in the theory of linear integral equations which we now recall following [49].

The equation in question is a linear inhomogeneous integral equation of the form

$$L \cdot a = b. \quad (\text{D7})$$

Let us assume that L is hermitian. The solution to such equation exists if b is orthogonal to all solutions ϕ_j of the homogeneous equation

$$L \cdot \phi_j = 0. \quad (\text{D8})$$

In the present context, there is only one solution, $\phi = \text{const}$, to the homogeneous equation $\Gamma^{\text{st}} \cdot \phi = 0$. It corresponds to the single conserved quantity, that is the particle number. Therefore, we need to check that

$$\mathbf{1} \cdot \mathcal{A}^{\text{st}} \cdot \frac{\partial \rho_p^{\text{st}}}{\partial \varrho} = u, \quad (\text{D9})$$

where we used that

$$\mathbf{1} \cdot \frac{\partial \rho_p^{\text{st}}}{\partial \varrho} = \frac{\partial \varrho}{\partial \varrho} = 1. \quad (\text{D10})$$

From the definitions of the hydrodynamic operator $\mathcal{A}$ and the effective velocity we have

$$\mathbf{1} \cdot \mathcal{A}^{\text{st}} = (h'_1)^{\text{dr}} v_\rho \cdot (\mathbf{1} - n\mathcal{T}) = h'_2 = h_1. \quad (\text{D11})$$

In the second equality we used that $v_\rho = (h'_2)^{\text{dr}} / (h'_1)^{\text{dr}}$ and that $(\mathbf{1} - n\mathcal{T})$ removes the dressing from the function that it acts upon. Recalling the definition (53) of u we confirm the sought after relation. This proves that if the operator Γ^{st} is hermitian then there is always a solution to the equation specifying the diffusion coefficient. The operators Γ^{st} obtained from the Fermi's golden rule or from RTA are hermitian which concludes the existence part of the proof.

The constitutive relation (63) guarantees the uniqueness of the solution by removing from the space of solutions cases that differ by the kernel of Γ^{st} .

3. Galilean transformations

The Galilean transformations describe the effect of the boost, by a constant velocity w , on the system. We denote with *bar* quantities referring to the system moving with velocity w with respect to the stationary one.

The quasiparticle distribution functions of the two systems obey $\bar{\rho}_p(\lambda) = \rho_p(\lambda - w)$. The parameter u transforms as velocity under the boost,

$$\bar{u} = u + w, \quad (\text{D12})$$

as can be directly verified from its definition (53) and the transformation rule for the quasiparticles distributions. In a similar fashion we find transformation rules for the effective velocity and the flux Jacobian transform

$$\begin{aligned} \bar{v}_\rho(\lambda) &= v_\rho(\lambda - w) + w, \\ \bar{\mathcal{A}}(\lambda, \lambda') &= \mathcal{A}(\lambda - w, \lambda' - w) + w. \end{aligned} \quad (\text{D13})$$

This implies that the solutions ChE equation (62) in the moving and in the stationary system are related by $\bar{a}(\lambda) = a(\lambda - w)$. The diffusion coefficient in the moving system is then

$$\bar{\mathcal{D}} = - \int d\lambda h_1(\lambda) \bar{a}(\lambda) = \mathcal{D} - \int d\lambda a(\lambda). \quad (\text{D14})$$

The remaining integral is zero due to the constitutive relation (63). This shows that the diffusion coefficient is invariant under the Galilean transformations.

Appendix E: Linearization of the collision integral

In this section we approach computation of the linearized collision integral operator Γ^{st} (69) for the two coupled Lieb–Liniger tubes. Our derivation resembles linearization performed in [61], although the context is a little different. We assume the inter-tube interaction potential (in the momentum space) to have a form

$$\tilde{V}(k) = \mathcal{V}(k/k_r), \quad \tilde{V}(k) = \tilde{V}(-k). \quad (\text{E1})$$

Constant k_r is a characteristic momentum exchanged between the tubes, which we assume to be small. It can also be regarded as an inverse of characteristic range of the potential. In the small momentum transfer limit, processes involving 1 particle-hole excitation are dominant and higher processes contribute at higher powers of k_r [68]. While it is possible to directly compute $\delta\mathcal{I}[\rho_p]/\delta\rho_p$, such path is not the easiest way to find the linear contribution to the collision integral. The task becomes much easier if instead of describing the perturbation with $\delta\rho_p$ we switch the description to pseudoenergy perturbation. In the language of pseudoenergy perturbation we define the linearized collision integral as

$$\tilde{\Gamma}^{\text{st}} = \mathbb{F}_1^{\text{st}} \cdot \frac{\delta\mathcal{I}_0}{\delta\epsilon_0} \Big|_{\rho_p^{\text{st}}}. \quad (\text{E2})$$

Using the identities [61]

$$\delta\rho_p = -\mathcal{C} \cdot \delta\epsilon_0, \quad \delta\epsilon = (\mathbf{1} - \mathcal{T}n)^{-1} \cdot \delta\epsilon_0 \quad (\text{E3})$$

and remembering that $\mathcal{I}_0[\rho_p^{\text{st}}] = 0$, we can relate $\tilde{\Gamma}^{\text{st}}$ to Γ^{st} (see (69)). The presented equalities follow directly from the definition of the pseudoenergy and the equation (6). For arbitrary functions f and g we have

$$\langle f | \tilde{\Gamma}^{\text{st}} g \rangle = \int d\lambda f^{\text{Dr, st}} \frac{\delta\mathcal{I}_0}{\delta\epsilon_0} \Big|_{\rho_p^{\text{st}}} \cdot g^{\text{st}}. \quad (\text{E4})$$

Writing the matrix element we used the identity $\int d\lambda f \cdot \mathbb{F} \cdot g = \int d\lambda f^{\text{Dr}} \cdot g$, which follows from the definition of the operator $\mathbb{F}$ [61]. Given a state in the non-evolving tube and a particle number in the evolving tube, stationary state in the evolving tube is fixed with the condition (34). As discussed in the main text, to linearize collision integral in the density imbalance limit we consider a perturbation $\rho_p = \rho_p^{\text{st}} + \delta\rho_p$, while keeping the state in the non-evolving tube $\rho_p^{(b)}$ fixed. Combining (29) with the dynamic structure factor (66) we write $\mathcal{I}_0$ in a form convenient for linearization

$$\mathcal{I}_0[\rho_p](p_1) = (2\pi)^2 \int dh_1 dp_2 dh_2 \times G(p_1, h_1, p_2, h_2) H(p_1, h_1, p_2, h_2),$$

where

$$\begin{aligned} G &= \tilde{V}^2(k(p_1, h_1)) \rho_p(h_1) \rho_h(p_1) \rho_p^{(b)}(p_2) \rho_h^{(b)}(h_2) \\ &\times |F(p_1, h_1)|^2 |F^{(b)}(p_2, h_2)|^2 \\ &\times \delta(k(p_1, h_1) + k^{(b)}(p_2, h_2)) \\ &\times \delta(\omega(p_1, h_1) + \omega^{(b)}(p_2, h_2)), \\ H &= \exp\left(\epsilon(p_1) - \epsilon(h_1) + \epsilon^{(b)}(p_2) - \epsilon^{(b)}(h_2)\right) - 1. \end{aligned}$$

We perturb ρ_p around stationary state and linearize. It follows that

$$\begin{aligned} \langle f | \tilde{\Gamma}^{\text{st}} g \rangle &= (2\pi)^2 \int dp_1 dh_1 dp_2 dh_2 f^{\text{Dr, st}}(p_1) \\ &\times G^{\text{st}}(p_1, h_1, p_2, h_2) (g^{\text{Dr, st}}(p_1) - g^{\text{Dr, st}}(h_1)), \end{aligned}$$

where we performed the trivial integral in λ . Next step is to change coordinates to the center of mass (CM) coordinates

$$\lambda_i = \frac{p_i + h_i}{2} \quad \text{and} \quad \mu_i = p_i - h_i, \quad (\text{E5})$$

with the Jacobian of change of variables being unity. Working in the small momentum limit, μ_i is small. In the CM variables Dressed momentum and energy are

$$\begin{aligned} k(p, h) &= k'(\lambda)\mu + \mathcal{O}(\mu^3), \\ \omega(p, h) &= \omega'(\lambda)\mu + \mathcal{O}(\mu^3). \end{aligned}$$

Also, in small momentum limit form factors simplify to $F(p, h) = k'(\lambda) + \mathcal{O}(\mu^2)$ [38]. The interaction potential term is expanded as

$$\tilde{V}^2(k(p) - k(h)) = \tilde{V}^2(k'(\lambda)\mu) + \mathcal{O}(\mu^3). \quad (\text{E6})$$

Expanding G^{st} and keeping terms up to order $\mathcal{O}(\mu^2)$

$$\begin{aligned} G^{\text{st}} &= \tilde{V}^2((k^{\text{st}})'(\lambda_1)\mu_1) (k^{\text{st}})'(\lambda_1) (k^{(b)})'(\lambda_2) \\ &\times \rho_p^{\text{st}}(\lambda_1) \rho_h^{\text{st}}(\lambda_1) \rho_p^{(b)}(\lambda_2) \rho_h^{(b)}(\lambda_2) \\ &\times \frac{1}{|\mu_1|} \delta\left(\mu_2 + \mu_1 \frac{(k^{\text{st}})'(\lambda_2)}{(k^{(b)})'(\lambda_1)}\right) \\ &\times \delta\left(v^{\text{st}}(\lambda_1) - v^{(b)}(\lambda_2)\right), \end{aligned}$$

where $v = \omega'/k'$ is the effective velocity (11). Changing variables under the Dirac delta we used that $k' = 2\pi\rho_t > 0$. Plugging expanded G^{st} into (E4) we find

$$\begin{aligned} \langle f | \tilde{\Gamma}^{\text{st}} g \rangle &= (2\pi)^2 \int d\lambda_1 d\mu_1 d\lambda_2 \tilde{V}^2((k^{\text{st}})'(\lambda_1)\mu_1) \\ &\times (k^{\text{st}})'(\lambda_1) (k^{(b)})'(\lambda_2) \rho_p^{\text{st}}(\lambda_1) \rho_h^{\text{st}}(\lambda_1) \rho_p^{(b)}(\lambda_2) \rho_h^{(b)}(\lambda_2) \\ &\times \frac{1}{|\mu_1|} \delta\left(v^{\text{st}}(\lambda_1) - v^{(b)}(\lambda_2)\right) \\ &\times \mu_1 (g^{\text{Dr, st}}(\lambda_1))' \left(f^{\text{Dr, st}}(\lambda_1) + \mu_1 (f')^{\text{dr, st}}(\lambda_1)\right). \end{aligned}$$

Writing the matrix element we took advantage of the relation between dressings (10). Proceeding we note that since $\tilde{A}$ is symmetric in momenta, the term $\sim f^{\text{Dr, st}}$ does not contribute to the matrix element. Changing variables $\mu_1 \rightarrow \mu_1/k'$, after some bookkeeping we arrive at

$$\begin{aligned} \langle f | \tilde{\Gamma}^{\text{st}} g \rangle &= (2\pi)^2 k_r^2 |\mathcal{V}^2| \\ &\times \int d\lambda \frac{\rho_p^{\text{st}} \rho_h^{\text{st}}}{(k^{\text{st}})'} \zeta^{(b)}(v^{\text{st}}) (f')^{\text{dr, st}} (g^{\text{Dr, st}})' + \mathcal{O}(k_r^3), \end{aligned}$$

where $|\mathcal{V}^2| = \int dz |z| \mathcal{V}^2(z)$ and

$$\zeta^{(b)}(v) = \frac{\rho_p^{(b)} \rho_h^{(b)} (k^{(b)})'}{|(v^{(b)})'|} \Big|_{(v^{(b)})^{-1}(v)}. \quad (\text{E7})$$

Finally we note that in the Tonks–Girardeau limit ($c \rightarrow \infty$) in both tubes, the matrix element takes particularly simple form

$$\langle f | \tilde{\Gamma}^{\text{st}} g \rangle = (2\pi)^2 k_r^2 |\mathcal{V}^2| \int d\lambda \rho_p^{\text{st}} \rho_h^{\text{st}} \rho_p^{(b)} \rho_h^{(b)} f' g', \quad (\text{E8})$$

since in this limit the dressing is absent.

Appendix F: Additional numerical results and verifications

In this section we describe numerical computations of the diffusion coefficient for the two density-imbalanced coupled Lieb–Liniger tubes. First, note that the only non-trivial step needed to compute diffusion coefficient is solving the integral equation (62). The equation is an example of a Fredholm integral equation of the first kind. A simple way to solve it is by choosing a basis and restricting it to a finite dimension. Using (E3) we rewrite (62) into equivalent form

$$\tilde{\Gamma}^{\text{st}} \cdot \alpha = (\mathcal{B}^{\text{st}} - u\mathcal{C}^{\text{st}}) \cdot \partial_\epsilon \epsilon_0, \quad (\text{F1})$$

where $\mathcal{C}^{\text{st}} \cdot \alpha = a$. Continuing, we write α in a polynomial basis, orthonormal with respect to hydrodynamic scalar product

$$(f|g)_{\text{st}} = \int d\lambda f(\lambda) (\mathcal{C}^{\text{st}} \cdot g)(\lambda). \quad (\text{F2})$$

The basis is constructed by performing Gram-Schmidt orthonormalization on the ultra-local charge basis [61]. We will denote elements of the new basis as $\{\mathbf{h}_i\}_{i \in \mathbb{N}}$. Note, that from the construction procedure follows $\mathbf{h}_0 \sim h_0$. Expanding $\alpha = \alpha_j \mathbf{h}_j$, multiplying from the left by h_i and integrating, we arrive at

$$\tilde{\Gamma}_{ij}^{\text{st}} \alpha_j = B_i^{\text{st}} - u C_i^{\text{st}}, \quad (\text{F3})$$

where Einstein's summation convention is assumed. Matrix elements are defined as

$$\begin{aligned} \tilde{\Gamma}_{ij}^{\text{st}} &= \langle \mathbf{h}_i | \tilde{\Gamma}^{\text{st}} \mathbf{h}_j \rangle, \\ B_i^{\text{st}} &= \int d\lambda v_{\rho_p^{\text{st}}} \rho_p^{\text{st}} f^{\text{st}} \mathbf{h}_i^{\text{dr, st}} (\partial_\rho \epsilon_0)^{\text{dr, st}}, \\ C_i^{\text{st}} &= \int d\lambda \rho_p^{\text{st}} f^{\text{st}} \mathbf{h}_i^{\text{dr, st}} (\partial_\rho \epsilon_0)^{\text{dr, st}}, \end{aligned} \quad (\text{F4})$$

Straightforward way to solve this set of equations is to truncate the basis to a finite dimension. Inverting the matrix $\tilde{\Gamma}_{ij}^{\text{st}}$ we encounter a complication. Because of the particle conservation law (16) we have $\tilde{\Gamma}_{0,j}^{\text{st}} = 0$. Thus the matrix is rank deficient. From computations performed in the section D 2 follows $B_0^{\text{st}} - u C_0^{\text{st}} = 0$. To be able to uniquely determine α_i , one needs to use the constitutive relation (63), which fixes $\alpha_0 = 0$, resolving the rank deficiency problem. Higher components of α are fixed by the matrix equation

$$\tilde{\Gamma}^{\text{st}} \cdot \vec{\alpha} = \vec{B}^{\text{st}} - u \vec{C}^{\text{st}}, \quad (\text{F5})$$

where $(\tilde{\Gamma}^{\text{st}})_{ij} = \tilde{\Gamma}_{i+1,j+1}^{\text{st}}$ is a full-rank matrix and $(\vec{B}^{\text{st}})_i = B_{i+1}^{\text{st}}$, $(\vec{C}^{\text{st}})_i = C_{i+1}^{\text{st}}$, $(\vec{\alpha})_i = \alpha_{i+1}$. We have verified that solution to (F5), upon increasing the basis size, converges to function solving the original integral equation.

We applied the presented method to the system of weakly coupled Lieb-Liniger tubes. Data presented in Fig. 3 were computed using the presented method.

We now turn to analyzing the function $\partial_\rho \epsilon_0$ for the system of two Lieb-Liniger tubes. Suppose the environmental tube is in a boosted thermal state. Pseudo energy is then $\epsilon^{(b)}(\lambda) = -\beta \mu^{(b)} + \kappa k^{(b)}(\lambda) + \beta \omega^{(b)}(\lambda)$, where $\beta = 1/T$ and κ is a potential related with the tube's velocity. In [68] it is shown that for the environmental tube in a boosted thermal state, the pseudoenergy of the stationary state in the coupled tube is $\epsilon^{\text{st}}(\lambda) = -\beta \mu + \kappa k^{\text{st}}(\lambda) + \beta \omega^{\text{st}}(\lambda)$. Consequently, the bare pseudo energy reads $\epsilon_0^{\text{st}}(\lambda) = \beta \mu + \kappa h_1(\lambda) + \beta h_2(\lambda)$. To compute the bare pseudoenergy we differentiated (6) and used the relation between dressing operations (10). We see that the only free parameter is the chemical potential μ , fixing number of particles in the evolving tube. Therefore, applying the chain rule, $\partial_\rho \epsilon_0^{\text{st}} = \frac{d\mu}{d\rho} \partial_\mu \epsilon_0^{\text{st}} = -\frac{d\mu}{d\rho} \beta$. The derivative $\frac{d\mu}{d\rho}$ can be further simplified

$$\frac{d\mu}{d\rho} = \int d\lambda \frac{\partial \rho_p^{\text{st}}}{\partial \mu} = - \int d\lambda C^{\text{st}} \cdot \frac{\partial \epsilon_0^{\text{st}}}{\partial \mu} = -\beta (h_0|h_0)_{\text{st}}, \quad (\text{F6})$$

where in the second equality we used (E3). Last equality follows from the definition of the hydrodynamic scalar product. Thus, for a system near a boosted thermal state

$$\partial_\rho \epsilon_0^{\text{st}} = \frac{1}{(h_0|h_0)_{\text{st}}}. \quad (\text{F7})$$

For an athermal stationary state in the bath tube and finite interaction strength, the factor $\partial_\rho \epsilon_0^{\text{st}}$ has much more complicated structure. To compute it we begin by writing $\epsilon^{\text{st}}(\lambda)$ as $(\epsilon^{\text{st}})'(\lambda) + \nu$, where ν is a constant parameter. For a fixed value of ν , using (34) we are able to find $\epsilon^{\text{st}}(\lambda)$ by solving the non-linear integral equation. Having solved equations for sufficiently many values of ν we are able to find $\varrho(\nu)$ and compute $\partial_\rho \epsilon_0^{\text{st}}(\lambda)$.

Finally, let us comment on the estimation of the relaxation time. As discussed in the main text it is customary to estimate relaxation time of Γ^{st} as a gap between zero and the smallest non-zero eigenvalue, see Fig. F. To numerically find the smallest non-zero eigenvalue of the operator Γ^{st} , algorithm similar to the one outlined above might be used. It is straightforward to verify that if γ is an eigenvalue of Γ^{st} , then it satisfies the equation

$$\tilde{\Gamma}_{ij}^{\text{st}} \alpha_j = -\gamma \alpha_i \quad (\text{F8})$$

for some vector α_i . To find the gap, one can for example perform the power iteration procedure on the $\tilde{\Gamma}$ matrix. We employed this method to find the approximation of the relaxation time presented in Fig. 3. The presented method has its weaknesses. Firstly, while we are able to verify that the found eigenvalue is indeed an eigenvalue of the operator Γ^{st} , by no means it allows us to verify that it indeed is the smallest eigenvalue of the integral operator. Lastly, we observed that the method fails to give reasonable approximation for larger temperatures. Detailed analysis showed that for some cases the eigenvector corresponding to the smallest eigenvalue is converging towards some function for small truncation dimensions. As we further increase the dimension, we observe that this convergence is lost before we find satisfactory approximation of the eigenvector, an effect similar to parameter overfitting.

Appendix G: Stochastic Burgers equation with a large diffusion constant

In this section, we consider the stochastic version of the derived Burgers equation, we follow [24, 60, 89]. To represent the degrees of freedom truncated in the Chapman-Enskog method, we add a noise term to the current, such that

$$j_\rho = j_\rho^{\text{Eul}} - \mathcal{D}(\varrho) \partial_x \varrho + \sigma \xi \quad (\text{G1})$$

with

$$\langle \xi(x, t) \xi(x', t') \rangle = \delta(x - x') \delta(t - t'), \quad \langle \xi(x, t) \rangle = 0. \quad (\text{G2})$$

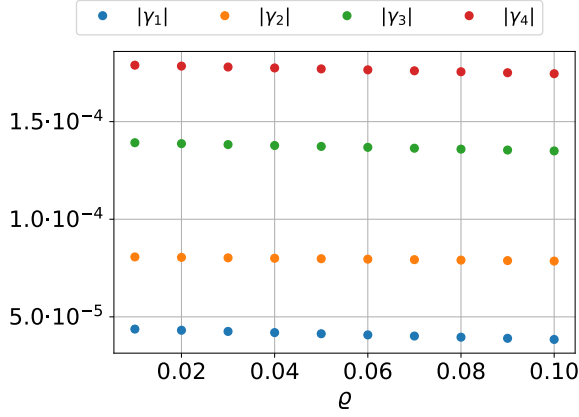


FIG. 6. The smallest non-zero eigenvalues of $\tilde{\Gamma}_{ij}^{\text{st}}$ as a function of ϱ for the system in the same state as in panel (a) in Fig. 3.

The strength of the noise is fixed by the value of equilibrium correlations as $\sigma^2 = 2\mathcal{C}_{00}\mathcal{D}_0$, where $\mathcal{C}_{00} = h_0\mathcal{C}h_0$. We consider states close to the homogeneous state with the density ϱ_0 and write $\varrho = \varrho_0 + \delta\varrho$. We keep quadratic term in the current expansion and get (below $u_0 \equiv u(\varrho_0)$ etc.)

$$\partial_t \delta\varrho + \partial_x(u_0 \delta\varrho + u'_0 \delta\varrho^2 - \mathcal{D}_0 \partial_x \delta\varrho + \sigma \xi) = 0. \quad (\text{G3})$$

We will be interested in the correlator $f(x, t) = \langle \delta\varrho(x, t) \delta\varrho(0, 0) \rangle$. The standard mode-coupling equation for the Fourier transform $\hat{f}(k, t)$ of such correlator in the NLFH reads [24]

$$\begin{aligned} \partial_t \hat{f}(k, t) = & -k^2 \left(\mathcal{D}_0 \hat{f}(k, t) + \right. \\ & \left. 2(u'_0)^2 \int_0^t ds \hat{f}(k, t-s) \int dq \hat{f}(q, s) \hat{f}(k-q, s) \right). \end{aligned} \quad (\text{G4})$$

The RHS of (G4) involves two terms. Upon neglecting the second one, we recover diffusive broadening of

the correlator. On the other, hand, when the first one is neglected, one finds integral equation leading to KPZ dynamical exponent $z = 3/2$ and a scaling function close to the scaling function of 1D KPZ equation [24]. To learn more about the equation, let us postulate the solution in the diffusive form

$$\hat{f}(k, t) = g(\mathcal{D}_0 k^2 t), \quad w \equiv \mathcal{D}_0 k^2 t. \quad (\text{G5})$$

Plugging this into (G4) we find

$$\begin{aligned} g'(w) = & - \left[g(w) + \frac{A}{w^{1/2}} \int_0^w d\mu \mu^{-1/2} g(w-\mu) \times \right. \\ & \left. \int dr r^{-1/2} g(r) g\left(\mu \left(1 - (r/\mu)^{1/2}\right)^2\right) \right], \end{aligned} \quad (\text{G6})$$

where

$$A = \frac{(u'_0)^2 t^{1/2}}{\mathcal{D}_0^{3/2}}. \quad (\text{G7})$$

Note that on time scales such that $A \ll 1$, the second term in (G6) can be neglected and the transport is diffusive with $g(w) = e^{-w}$, as expected. The crossover time between diffusive and anomalous transport can be determined by setting $A = 1$, we then get

$$t_c = \frac{\mathcal{D}_0^3}{(u'_0)^4}. \quad (\text{G8})$$

We crucially note that in the regime of small integrability breaking the diffusion constant is large and scales as follows $\mathcal{D}_0 \sim \alpha^{-2}$. On the other hand, u'_0 is a function of the state only and does not depend on α . This means, that the onset of anomalous transport takes place for times parametrically large in the inverse of integrability breaking

$$t_c \sim \alpha^{-6}, \quad (\text{G9})$$

and the transport is dominated by diffusion.

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- [1] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, *Rev. Mod. Phys.* **83**, 863 (2011).
 - [2] A. P. Luca D'Alessio, Yariv Kafri and M. Rigol, *Advances in Physics* **65**, 239 (2016).
 - [3] B. Bertini, F. Heidrich-Meisner, C. Karrasch, T. Prosen, R. Steinigeweg, and M. Žnidarič, *Reviews of Modern Physics* **93**, 025003 (2021).
 - [4] P. Calabrese, F. H. L. Essler, and G. Mussardo, *Journal of Statistical Mechanics: Theory and Experiment* **2016**, 064001 (2016).
 - [5] F. H. L. Essler and M. Fagotti, *Journal of Statistical Mechanics: Theory and Experiment* **6**, 064002 (2016).
 - [6] J.-S. Caux, *J. Stat. Mech.* **2016**, 064006 (2016).
 - [7] E. Ilievski, M. Medenjak, T. Prosen, and L. Zadnik, *Journal of Statistical Mechanics: Theory and Experiment* **2016**, 064008 (2016).
 - [8] R. Vasseur and J. E. Moore, *Journal of Statistical Mechanics: Theory and Experiment* **2016**, 064010 (2016).
 - [9] P. Calabrese, *SciPost Phys. Lect. Notes*, **20** (2020).
 - [10] A. Bastianello, J. De Nardis, and A. D. Luca, *Physical Review B* **102**, 161110 (2020).
 - [11] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, *Reviews of Modern Physics* **91**, 021001 (2019).
 - [12] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, *Physical Review X* **8**, 021030 (2018).
 - [13] N. Malvania, Y. Zhang, Y. Le, J. Dubail, M. Rigol, and D. S. Weiss, *Science* **373**, 1129 (2021).
 - [14] F. Møller, C. Li, I. Mazets, H.-P. Stimming, T. Zhou, Z. Zhu, X. Chen, and J. Schmiedmayer, *Physical Review*

- Letters **126**, 090602 (2021).
- [15] F. Cataldini, F. Møller, M. Tajik, J. Sabino, S.-C. Ji, I. Mazets, T. Schweigler, B. Rauer, and J. Schmiedmayer, *Physical Review X* **12**, 041032 (2022).
 - [16] P. N. Jepsen, J. Amato-Grill, I. Dimitrova, W. W. Ho, E. Demler, and W. Ketterle, *Nature* **588**, 403 (2020).
 - [17] A. Scheie, N. E. Sherman, M. Dupont, S. E. Nagler, M. B. Stone, G. E. Granroth, J. E. Moore, and D. A. Tennant, *Nature Physics* **17**, 726 (2021).
 - [18] D. Wei, A. Rubio-Abadal, B. Ye, F. Machado, J. Kemp, K. Srakaew, S. Hollerith, J. Rui, S. Gopalakrishnan, N. Y. Yao, I. Bloch, and J. Zeiher, *Science* **376**, 716 (2022).
 - [19] Y. Le, Y. Zhang, S. Gopalakrishnan, M. Rigol, and D. S. Weiss, *Nature* **618**, 494 (2023).
 - [20] T. Kinoshita, T. Wenger, and D. S. Weiss, *Nature* **440**, 900 (2006).
 - [21] D. Forster, D. R. Nelson, and M. J. Stephen, *Physical Review A* **16**, 732 (1977).
 - [22] O. Narayan and S. Ramaswamy, *Physical Review Letters* **89**, 200601 (2002).
 - [23] H. van Beijeren, *Physical Review Letters* **108**, 180601 (2012).
 - [24] H. Spohn, *Journal of Statistical Physics* **154**, 1191 (2014).
 - [25] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, 2003).
 - [26] V. B. Bulchandani, C. Karrasch, and J. E. Moore, *Proceedings of the National Academy of Sciences* **117**, 12713 (2020).
 - [27] M. Ljubotina, M. Žnidarič, and T. Prosen, *Nature Communications* **8**, 16117 (2017).
 - [28] S. Gopalakrishnan and R. Vasseur, *Physical Review Letters* **122**, 127202 (2019).
 - [29] J. D. Nardis, M. Medenjak, C. Karrasch, and E. Ilievski, *Physical Review Letters* **123**, 186601 (2019).
 - [30] V. B. Bulchandani, S. Gopalakrishnan, and E. Ilievski, *Journal of Statistical Mechanics: Theory and Experiment* **2021**, 084001 (2021).
 - [31] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, 1999).
 - [32] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, 1993).
 - [33] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, *Phys. Rev. X* **6**, 041065 (2016).
 - [34] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, *Phys. Rev. Lett.* **117**, 207201 (2016).
 - [35] B. Doyon, S. Gopalakrishnan, F. Møller, J. Schmiedmayer, and R. Vasseur, *Physical Review X* **15**, 010501 (2025).
 - [36] J. Durnin, M. Bhaseen, and B. Doyon, *Physical Review Letters* **127**, 130601 (2021).
 - [37] A. J. Friedman, S. Gopalakrishnan, and R. Vasseur, *Phys. Rev. B* **101**, 180302 (2020).
 - [38] M. Panfil, S. Gopalakrishnan, and R. M. Konik, *Physical Review Letters* **130**, 030401 (2023).
 - [39] J. Lopez-Piqueres, B. Ware, S. Gopalakrishnan, and R. Vasseur, *Physical Review B* **103**, L060302 (2021).
 - [40] J. Lopez-Piqueres, S. Gopalakrishnan, and R. Vasseur, *Journal of Physics A: Mathematical and Theoretical* **55**, 234005 (2022).
 - [41] J. Lopez-Piqueres and R. Vasseur, *Physical Review Letters* **130**, 247101 (2023).
 - [42] J.-S. Caux, B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, *SciPost Physics* **6**, 070 (2019).
 - [43] A. Bastianello, A. D. Luca, B. Doyon, and J. De Nardis, *Physical Review Letters* **125**, 240604 (2020).
 - [44] I. Bouchoule, B. Doyon, and J. Dubail, *SciPost Physics* **9**, 044 (2020).
 - [45] B. Bertini, F. H. Essler, S. Groha, and N. J. Robinson, *Physical Review Letters* **115**, 180601 (2015).
 - [46] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, *Physical Review B* **94**, 245117 (2016).
 - [47] A. Bastianello, A. D. Luca, and R. Vasseur, *Journal of Statistical Mechanics: Theory and Experiment* **2021**, 114003 (2021).
 - [48] L. Biagetti, M. Lebek, M. Panfil, and J. D. Nardis, Generalised bbgky hierarchy for near-integrable dynamics (2025), [arXiv:2408.00593](https://arxiv.org/abs/2408.00593).
 - [49] P. Résibois and M. De Leener, *Classical Kinetic Theory of Fluids* (John Wiley & Sons, 1977).
 - [50] J.-B. Fouvry, P.-H. Chavanis, and C. Pichon, *Physical Review E* **102**, 052110 (2020).
 - [51] J. D. Nardis, D. Bernard, and B. Doyon, *Physical Review Letters* **121**, 160603 (2018).
 - [52] J. De Nardis, D. Bernard, and B. Doyon, *SciPost Physics* **6**, 049 (2019).
 - [53] S. Gopalakrishnan, D. A. Huse, V. Khemani, and R. Vasseur, *Physical Review B* **98**, 220303 (2018).
 - [54] F. Hübner, L. Biagetti, J. D. Nardis, and B. Doyon, *Physical Review Letters* **134**, 187101 (2025).
 - [55] J. Durnin, A. D. Luca, J. De Nardis, and B. Doyon, *Journal of Physics A: Mathematical and Theoretical* **54**, 494001 (2021).
 - [56] X. Cao, V. B. Bulchandani, and J. E. Moore, *Physical Review Letters* **120**, 164101 (2018).
 - [57] A. Bastianello, A. D. Luca, B. Doyon, and J. D. Nardis, *Physical Review Letters* **125**, 240604 (2020).
 - [58] D. Bagchi, J. Kethepalli, V. B. Bulchandani, A. Dhar, D. A. Huse, M. Kulkarni, and A. Kundu, *Physical Review E* **108**, 064130 (2023).
 - [59] L. Biagetti, G. Cecile, and J. D. Nardis, *Physical Review Research* **6**, 023083 (2024).
 - [60] M. Lebek and M. Panfil, *SciPost Phys.* **18**, 186 (2025).
 - [61] M. Lebek and M. Panfil, *Phys. Rev. Lett.* **134**, 010405 (2025).
 - [62] S. Lepri, R. Livi, and A. Politi, *Physical Review Letters* **125**, 040604 (2020).
 - [63] S. Chen, J. Wang, G. Casati, and G. Benenti, *Physical Review E* **90**, 032134 (2014).
 - [64] H. Zhao and W. ge Wang, *Physical Review E* **97**, 010103 (2018).
 - [65] A. Miron, J. Cividini, A. Kundu, and D. Mukamel, *Physical Review E* **99**, 012124 (2019).
 - [66] A. Kundu, *SciPost Phys.* **15**, 038 (2023).
 - [67] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, *Phys. Rev. X* **8**, 021030 (2018).
 - [68] M. Lebek, M. Panfil, and R. M. Konik, *SciPost Phys.* **17**, 007 (2024).
 - [69] M. Kulkarni and A. G. Abanov, *Physical Review A* **86**, 033614 (2012).
 - [70] A. Abanov, T. Can, and S. Ganeshan, *SciPost Physics* **5**, 010 (2018).
 - [71] S. W. P. Kim, F. Hübner, J. P. Garrahan, and B. Doyon, (2025), [arXiv:2503.08788](https://arxiv.org/abs/2503.08788) [cond-mat.stat-mech].
 - [72] T. Taniuti and C.-C. Wei, *Journal of the*

- Physical Society of Japan **24**, 941 (1968), <https://doi.org/10.1143/JPSJ.24.941>.
- [73] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley Series in Pure and Applied Mathematics (Wiley, 1999).
 - [74] H. Leblond, *Journal of Physics B: Atomic, Molecular and Optical Physics* **41**, 043001 (2008).
 - [75] C. N. Yang and C. P. Yang, *Journal of Mathematical Physics* **10**, 1115 (1969).
 - [76] J. Mossel and J.-S. Caux, *Journal of Physics A: Mathematical and Theoretical* **45**, 255001 (2012).
 - [77] B. Doyon, *SciPost Phys. Lect. Notes*, 18 (2020).
 - [78] R. Balescu, *Statistical Dynamics* (PUBLISHED BY IMPERIAL COLLEGE PRESS AND DISTRIBUTED BY WORLD SCIENTIFIC PUBLISHING CO., 1997).
 - [79] E. H. Lieb and W. Liniger, *Physical Review* **130**, 1605 (1963).
 - [80] E. H. Lieb and W. Liniger, *Physical Review* **130**, 1605 (1963).
 - [81] J. D. Nardis and M. Panfil, *SciPost Phys.* **1**, 015 (2016).
 - [82] S. k. Ma, *Journal of Statistical Physics* **31**, 107 (1983).
 - [83] B. Doyon and H. Spohn, *SciPost Phys.* **3**, 039 (2017).
 - [84] The inter-tube interaction potential was chosen to be a Gaussian potential $V(x) = V_0 e^{-(x/x_r)^2} / (\sqrt{\pi} x_r)$ with $x_r = 2$. Constant V_0 can be expressed as $\tau_{\mathcal{I}}^{-1} = 2A_0^2 m / \hbar$ [68]. For the purpose of numerical computations we set $m = 1$. Characteristic relaxation time is extracted from spectrum of the linearized collision integral in the limit $\varrho \rightarrow 0$.
 - [85] D. T. Gillespie, *The Journal of Physical Chemistry* **81**, 2340 (1977).
 - [86] H. van Beijeren, R. Kutner, and H. Spohn, *Phys. Rev. Lett.* **54**, 2026 (1985).
 - [87] G. A. Bird, *Molecular Gas Dynamics And The Direct Simulation Of Gas Flows* (Oxford University Press Oxford, 1994).
 - [88] J. De Nardis and M. Panfil, *Journal of Statistical Mechanics: Theory and Experiment* **2018**, 033102 (2018).
 - [89] H. Spohn, Fluctuating hydrodynamics approach to equilibrium time correlations for anharmonic chains, in *Thermal Transport in Low Dimensions: From Statistical Physics to Nanoscale Heat Transfer*, edited by S. Lepri (Springer International Publishing, Cham, 2016) pp. 107–158.

3 Summary and outlook

In this dissertation the following research hypothesis was examined:

Dynamics of nearly-integrable gases can be studied using generalizations of frameworks developed by kinetic theory for weakly interacting models of particles.

In the course of this PhD project this hypothesis has proved to be correct in many ways and some of its implications were successfully tested against experimental data and *ab initio* numerics. Let us summarize the new features of 1D nearly-integrable gases, which were uncovered throughout the process:

- *A new mechanism for prethermalization:* Slow relaxation in 1D due to kinematic constraints in two-particle scattering was a well-known fact. It was not known, however, that the constrained nature of such processes may lead to prethermal states in coupled systems. It seems possible that such prethermal states escape the picture of long-lived GGE states, which is the standard picture of prethermalization in nearly-integrable systems.
- *Navier-Stokes equations from the GHD-Boltzmann equation:* The standard formalisms that are used to derive the NS equations can be adapted to the GHD-Boltzmann equation. In such equation, quasiparticles propagate with state-dependent effective velocities and feature additional diffusion effects as opposed to the standard Boltzmann equation. Together with the derivation of hydrodynamics, the transport coefficients: viscosity and thermal conductivity can be computed. The interactions encoded in the integrable Hamiltonian lead to finite viscosity as opposed to the case of nearly-ideal gas, for which it vanishes.
- *Three regimes of transport:* In nearly-integrable gases, anomalous transport and breakdown of diffusion happen on space-time scales that are parametrically large in the inverse of the integrability breaking parameter. This opens up a window for the diffusive behavior of hydrodynamic correlation functions. At shorter timescales, the effects of integrability breaking are not visible and system is described by the GHD. Thus, transport in nearly-integrable gases is characterized by three hydrodynamic regimes.
- *Generalization of BBGKY hierarchy:* A BBGKY-like framework can be derived to describe dynamics of correlations in nearly-integrable gases. The hierarchy can be efficiently truncated under the assumption of length scale separation between microscopic correlation length and the characteristic range of long-range potential. A collision integral can be derived from it. The predictions of the theory match experimental data for thermalization phenomena in 1D dipolar gases.
- *Burgers hydrodynamics in coupled gases:* For two coupled integrable gases in the system-bath limit, a Burgers equation can be found as an effective description of density dynamics in the system. It is driven by two mechanisms: quantum statistics of particles and athermal nature of bath state.

The topic of nearly-integrable gases is still full of open, interesting questions. Let us finish this dissertation with a discussion of possible extensions of this work.

Firstly, we covered mainly questions about non-equilibrium behavior of nearly-integrable gases. However, we did not address their equilibrium properties, such as thermal correlation functions. Here, a natural path forward would be to adapt the frameworks of Mayer expansions for thermodynamics and equilibrium hierarchies for correlation functions [3]. In particular, it would be interesting to obtain a generalization of the Born-Green-Yvon equation for the equilibrium two-point function of nearly-integrable gas. The pair distribution is a quantity directly accessible [149] to experiments with the LL model via Bragg spectroscopy and could potentially serve the role of yet another measure of integrability breaking in dipolar gases.

Secondly, it would be useful to study minimal models of a nearly-integrable gases that can be simulated efficiently. The simplest model of long-range interacting gas is perhaps the long-range interacting hard rod model, studied numerically in Article V. While the available computational resources were sufficient to study thermalization phenomena, the system sizes needed to study hydrodynamic crossovers seem to be out of reach. The remedy for that could be models of hard rods with stochastic collision processes, akin to the family of models introduced in Article VI. Even though they are not described by GBBGKY, as there is no long-range interaction, a version of the model with random three-particle scattering processes conserving momentum and energy is well-described by the GHD-Boltzmann equation. Numerical simulations of such models and comparisons with the derived NS equations are under way. Let us also note that such models are very flexible and one could in principle investigate variants with more exotic sets of conserved quantities. For instance, by imposing four-particle collisions, instead of three-particle collisions, it would be possible to study systems with four conserved charges.

Lastly, the effective hydrodynamics can also be derived through ChE methods in systems with different sets of charges. So far, we have covered the case of conserved particle number (for which we found the Burgers equation) and the case of conserved particle number, momentum and energy (for which we recovered the NS equations), but there are more possibilities. In particular, it would be interesting to check if it can be extended to systems that retain an infinite number of conserved quantities after integrability breaking such as hard rods with a backscattering mechanism [125].

References

- [1] J. R. Dorfman, H. van Beijeren, and T. R. Kirkpatrick, *Contemporary kinetic theory of matter* (Cambridge University Press, June 2021).
- [2] P. Résibois and M. De Leener, *Classical Kinetic Theory of Fluids* (John Wiley & Sons, 1977).
- [3] R. Balescu, *Equilibrium and nonequilibrium statistical mechanics* (John Wiley & Sons, 1975).
- [4] S. Chapman and T. Cowling, *The mathematical theory of non-uniform gases: an account of the kinetic theory of viscosity, thermal conduction and diffusion in gases*, Cambridge Mathematical Library (Cambridge University Press, 1990).
- [5] H. Grad, [Communications on Pure and Applied Mathematics](#) **2**, 331 (1949).
- [6] P. Résibois, [Journal of Statistical Physics](#) **2**, 21 (1970).
- [7] E. H. Lieb and W. Liniger, [Physical Review](#) **130**, 1605 (1963).
- [8] L. Tonks, [Physical Review](#) **50**, 955 (1936).
- [9] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, [Reviews of Modern Physics](#) **82**, 1225 (2010).
- [10] D. S. Petrov, D. M. Gangardt, and G. V. Shlyapnikov, [Journal de Physique IV \(Proceedings\)](#) **116**, 5 (2004).
- [11] M. Takahashi, *Thermodynamics of one-dimensional solvable models* (Cambridge University Press, Mar. 1999).
- [12] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum inverse scattering method and correlation functions* (Cambridge University Press, Aug. 1993).
- [13] B. Sutherland, *Beautiful models* (World Scientific, 2004).
- [14] M. Olshanii, [Physical Review Letters](#) **81**, 938 (1998).
- [15] E. H. Lieb, [Physical Review](#) **130**, 1616 (1963).

- [16] H. Bethe, *Zeitschrift fur Physik* **71**, 205 (1931).
- [17] M. Gaudin, *The Bethe wavefunction* (Cambridge University Press, Mar. 2014).
- [18] C. N. Yang, *Physical Review Letters* **19**, 1312 (1967).
- [19] M. Gaudin, *Physics Letters A* **24**, 55 (1967).
- [20] F. Mazzanti, G. E. Astrakharchik, J. Boronat, and J. Casulleras, *Physical Review Letters* **100**, 020401 (2008).
- [21] S. Kiedrzyński, E. Witkowska, and M. Panfil, *Physical Review B* **113**, 045401 (2026).
- [22] O. Gamayun and M. Panfil, *arXiv:2601.15108* (2026).
- [23] B. Doyon and H. Spohn, *Journal of Statistical Mechanics: Theory and Experiment* **2017**, 073210 (2017).
- [24] F. Hübner, L. Biagetti, J. De Nardis, and B. Doyon, *SciPost Physics Core* **9**, 010 (2026).
- [25] C. Boldrighini, R. L. Dobrushin, and Y. M. Sukhov, *Journal of Statistical Physics* **31**, 577 (1983).
- [26] C. Boldrighini and Y. Suhov, *Communications in Mathematical Physics* **189**, 577 (1997).
- [27] B. Doyon, *SciPost Physics Lecture Notes*, 18 (2020).
- [28] B. Doyon, J. Durnin, and T. Yoshimura, *SciPost Physics* **13**, 072 (2022).
- [29] Y. Jiang, *SciPost Physics* **12**, 191 (2022).
- [30] B. Doyon, F. Hübner, and T. Yoshimura, *Physical Review Letters* **132**, 251602 (2024).
- [31] B. Doyon, F. Hübner, and T. Yoshimura, *Annales Henri Poincaré*, 10.1007/s00023-026-01684-1 (2026).
- [32] P. Fendley, *Journal of Physics A: Mathematical and Theoretical* **52**, 335002 (2019).
- [33] M. Vanicat, L. Zadnik, and T. Prosen, *Physical Review Letters* **121**, 030606 (2018).
- [34] M. Ljubotina, L. Zadnik, and T. Prosen, *Physical Review Letters* **122**, 150605 (2019).
- [35] P. W. Claeys, J. Herzog-Arbeitman, and A. Lamacraft, *SciPost Physics* **12**, 007 (2022).
- [36] E. Vernier, H.-C. Yeh, L. Piroli, and A. Mitra, *Physical Review Letters* **133**, 050606 (2024).
- [37] F. Hübner, L. Biagetti, J. De Nardis, and B. Doyon, *Physical Review Letters* **134**, 187101 (2025).
- [38] M. V. Medvedyeva, F. H. L. Essler, and T. Prosen, *Physical Review Letters* **117**, 137202 (2016).
- [39] A. A. Ziolkowska and F. Essler, *SciPost Physics* **8**, 044 (2020).
- [40] L. Sá, P. Ribeiro, and T. Prosen, *Physical Review B* **103**, 115132 (2021).
- [41] P. Penc and F. H. L. Essler, *SciPost Physics* **20**, 058 (2026).
- [42] M. Mierzejewski, P. Prelovšek, and T. Prosen, *Physical Review Letters* **114**, 140601 (2015).
- [43] M. Mierzejewski, M. Kozarzewski, and P. Prelovšek, *Physical Review B* **97**, 064204 (2018).
- [44] I. Ulčakar and Z. Lenarčič, *Physical Review Letters* **132**, 230402 (2024).
- [45] J. Pawłowski, J. Herbrych, and M. Mierzejewski, *Physical Review B* **112**, 155130 (2025).
- [46] O. Shtanko, D. S. Wang, H. Zhang, N. Harle, A. Seif, R. Movassagh, and Z. Mineev, *Nature Communications* **16**, 2552 (2025).
- [47] V. Dunjko, V. Lorent, and M. Olshanii, *Physical Review Letters* **86**, 5413 (2001).
- [48] C. Menotti and S. Stringari, *Physical Review A* **66**, 043610 (2002).

- [49] K. V. Kheruntsyan, D. M. Gangardt, P. D. Drummond, and G. V. Shlyapnikov, [Physical Review A **71**, 053615 \(2005\)](#).
- [50] F. Riggio, Y. Brun, D. Karevski, A. Faribault, and J. Dubail, [Physical Review A **106**, 053309 \(2022\)](#).
- [51] J. K. Percus, [Journal of Statistical Physics **15**, 505 \(1976\)](#).
- [52] B. Doyon and T. Yoshimura, [SciPost Physics **2**, 014 \(2017\)](#).
- [53] A. Bastianello, A. D. Luca, B. Doyon, and J. De Nardis, [Physical Review Letters **125**, 240604 \(2020\)](#).
- [54] L. Biagetti, G. Cecile, and J. De Nardis, (2023).
- [55] X. Cao, V. B. Bulchandani, and J. E. Moore, [Physical Review Letters **120**, 164101 \(2018\)](#).
- [56] L. Biagetti, G. Cecile, and J. De Nardis, [Physical Review Research **6**, 023083 \(2024\)](#).
- [57] I. Bouchoule, B. Doyon, and J. Dubail, [SciPost Physics **9**, 044 \(2020\)](#).
- [58] F. Riggio, L. Rosso, D. Karevski, and J. Dubail, [Physical Review A **109**, 023311 \(2024\)](#).
- [59] A. Hutsalyuk and B. Pozsgay, [Physical Review E **103**, 042121 \(2021\)](#).
- [60] R. Pathria and P. Beale, *Statistical mechanics* (Elsevier, 2011).
- [61] J.-P. Hansen and I. McDonald, *Theory of simple liquids* (Elsevier, 2013).
- [62] R. Balescu, *Statistical dynamics* (Imperial College Press, 1997).
- [63] M. Kac, G. E. Uhlenbeck, and P. C. Hemmer, [Journal of Mathematical Physics **4**, 216 \(1963\)](#).
- [64] G. E. Uhlenbeck, P. C. Hemmer, and M. Kac, [Journal of Mathematical Physics **4**, 229 \(1963\)](#).
- [65] P. C. Hemmer, M. Kac, and G. E. Uhlenbeck, [Journal of Mathematical Physics **5**, 60 \(1964\)](#).
- [66] P. C. Hemmer, [Journal of Mathematical Physics **5**, 75 \(1964\)](#).
- [67] J. L. Lebowitz and O. Penrose, [Journal of Mathematical Physics **7**, 98 \(1966\)](#).
- [68] F. Deuretzbacher, J. C. Cremon, and S. M. Reimann, [Physical Review A **81**, 063616 \(2010\)](#).
- [69] F. Deuretzbacher, J. C. Cremon, and S. M. Reimann, [Physical Review A **87**, 039903 \(2013\)](#).
- [70] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, [Physical Review X **8**, 021030 \(2018\)](#).
- [71] L. Chomaz, I. Ferrier-Barbut, F. Ferlaino, B. Laburthe-Tolra, B. L. Lev, and T. Pfau, [Reports on Progress in Physics **86**, 026401 \(2023\)](#).
- [72] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, [Reports on Progress in Physics **72**, 126401 \(2009\)](#).
- [73] C. N. Yang and C. P. Yang, [Journal of Mathematical Physics **10**, 1115 \(1969\)](#).
- [74] J. Mossel and J.-S. Caux, [Journal of Physics A: Mathematical and Theoretical **45**, 255001 \(2012\)](#).
- [75] E. T. Jaynes, [Physical Review **106**, 620 \(1957\)](#).
- [76] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, [Physical Review Letters **98**, 050405 \(2007\)](#).
- [77] P. Calabrese, F. H. L. Essler, and M. Fagotti, [Physical Review Letters **106**, 227203 \(2011\)](#).

- [78] J.-S. Caux and F. H. L. Essler, [Physical Review Letters](#) **110**, 257203 (2013).
- [79] B. Wouters, J. De Nardis, M. Brockmann, D. Fioretto, M. Rigol, and J.-S. Caux, [Physical Review Letters](#) **113**, 117202 (2014).
- [80] J. De Nardis, B. Wouters, M. Brockmann, and J.-S. Caux, [Physical Review A](#) **89**, 033601 (2014).
- [81] J. Viti, J.-M. Stéphan, J. Dubail, and M. Haque, [EPL \(Europhysics Letters\)](#) **115**, 40011 (2016).
- [82] B. Dóra, M. Haque, and G. Zaránd, [Physical Review Letters](#) **106**, 156406 (2011).
- [83] T. Langen, S. Erne, R. Geiger, B. Rauer, T. Schweigler, M. Kuhnert, W. Rohringer, I. E. Mazets, T. Gasenzer, and J. Schmiedmayer, [Science](#) **348**, 207 (2015).
- [84] A. Bastianello, A. Tikan, F. Copie, S. Randoux, and P. Suret, [Physical Review A](#) **113**, 013514 (2026).
- [85] P. Calabrese and J. Cardy, [Journal of Statistical Mechanics: Theory and Experiment](#) **2007**, P10004 (2007).
- [86] J.-M. Stéphan and J. Dubail, [Journal of Statistical Mechanics: Theory and Experiment](#) **2011**, P08019 (2011).
- [87] P. Calabrese, F. H. L. Essler, and G. Mussardo, [Journal of Statistical Mechanics: Theory and Experiment](#) **2016**, 064001 (2016).
- [88] T. W. B. Kibble, [Journal of Physics A: Mathematical and General](#) **9**, 1387 (1976).
- [89] W. H. Zurek, [Nature](#) **317**, 505 (1985).
- [90] J. Dziarmaga, [Physical Review Letters](#) **95**, 245701 (2005).
- [91] A. Polkovnikov, [Physical Review B](#) **72**, 161201 (2005).
- [92] L. Cincio, J. Dziarmaga, M. M. Rams, and W. H. Zurek, [Physical Review A](#) **75**, 052321 (2007).
- [93] A. del Campo and W. H. Zurek, [International Journal of Modern Physics A](#) **29**, 1430018 (2014).
- [94] M. Schmitt, M. M. Rams, J. Dziarmaga, M. Heyl, and W. H. Zurek, [Science Advances](#) **8** (2022).
- [95] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, [Physical Review X](#) **6**, 041065 (2016).
- [96] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, [Physical Review Letters](#) **117**, 207201 (2016).
- [97] F. H. Essler, [Physica A: Statistical Mechanics and its Applications](#) **631**, 127572 (2023).
- [98] A. Bastianello, B. Bertini, B. Doyon, and R. Vasseur, [Journal of Statistical Mechanics: Theory and Experiment](#) **2022**, 014001 (2022).
- [99] B. Doyon, S. Gopalakrishnan, F. Møller, J. Schmiedmayer, and R. Vasseur, [Physical Review X](#) **15**, 010501 (2025).
- [100] I. Bouchoule and J. Dubail, [Journal of Statistical Mechanics: Theory and Experiment](#) **2022**, 014003 (2022).
- [101] N. Malvania, Y. Zhang, Y. Le, J. Dubail, M. Rigol, and D. S. Weiss, [Science](#) **373**, 1129 (2021).
- [102] A. Scheie, N. E. Sherman, M. Dupont, S. E. Nagler, M. B. Stone, G. E. Granroth, J. E. Moore, and D. A. Tennant, [Nature Physics](#) **17**, 726 (2021).
- [103] F. Møller, C. Li, I. Mazets, H.-P. Stimming, T. Zhou, Z. Zhu, X. Chen, and J. Schmiedmayer, [Physical Review Letters](#) **126**, 090602 (2021).

- [104] L. Dubois, G. Thémèze, F. Nogrette, J. Dubail, and I. Bouchoule, [Physical Review Letters **133**, 113402 \(2024\)](#).
- [105] L. Dubois, G. Thémèze, J. Dubail, and I. Bouchoule, [SciPost Physics **20**, 008 \(2026\)](#).
- [106] P. Schüttelkopf, M. Tajik, N. Bazhan, F. Cataldini, S.-C. Ji, J. Schmiedmayer, and F. Møller, [Science **391**, 290 \(2026\)](#).
- [107] L. García-Colín, R. Velasco, and F. Uribe, [Physics Reports **465**, 149 \(2008\)](#).
- [108] J. De Nardis, D. Bernard, and B. Doyon, [Physical Review Letters **121**, 160603 \(2018\)](#).
- [109] J. De Nardis, D. Bernard, and B. Doyon, [SciPost Physics **6**, 049 \(2019\)](#).
- [110] J. Durnin, M. J. Bhaseen, and B. Doyon, [Physical Review Letters **127**, 130601 \(2021\)](#).
- [111] B. Doyon, G. Perfetto, T. Sasamoto, and T. Yoshimura, [SciPost Physics **15**, 136 \(2023\)](#).
- [112] B. Doyon, G. Perfetto, T. Sasamoto, and T. Yoshimura, [Physical Review Letters **131**, 027101 \(2023\)](#).
- [113] M. Panfil, S. Gopalakrishnan, and R. M. Konik, [Physical Review Letters **130**, 030401 \(2023\)](#).
- [114] J. Pawłowski, M. Panfil, J. Herbrych, and M. Mierzejewski, [Physical Review B **109**, L161109 \(2024\)](#).
- [115] J. De Nardis and M. Panfil, [Journal of Statistical Mechanics: Theory and Experiment **2015**, P02019 \(2015\)](#).
- [116] J. De Nardis and M. Panfil, [SciPost Physics **1**, 015 \(2016\)](#).
- [117] J. De Nardis and M. Panfil, [Journal of Statistical Mechanics: Theory and Experiment **2018**, 033102 \(2018\)](#).
- [118] A. C. Cubero and M. Panfil, [Journal of High Energy Physics **2019**, 104 \(2019\)](#).
- [119] M. Panfil, [Journal of Statistical Mechanics: Theory and Experiment **2021**, 013108 \(2021\)](#).
- [120] M. Panfil and R. M. Konik, [Journal of High Energy Physics **2023**, 72 \(2023\)](#).
- [121] J.-B. Fouvry, B. Bar-Or, and P.-H. Chavanis, [Physical Review E **100**, 052142 \(2019\)](#).
- [122] J.-B. Fouvry, P.-H. Chavanis, and C. Pichon, [Physical Review E **102**, 052110 \(2020\)](#).
- [123] A. Bastianello, A. D. Luca, and R. Vasseur, [Journal of Statistical Mechanics: Theory and Experiment **2021**, 114003 \(2021\)](#).
- [124] J. Lopez-Piqueres, B. Ware, S. Gopalakrishnan, and R. Vasseur, [Physical Review B **103**, L060302 \(2021\)](#).
- [125] J. Lopez-Piqueres and R. Vasseur, [Physical Review Letters **130**, 247101 \(2023\)](#).
- [126] J.-S. Caux, B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, [SciPost Physics **6**, 070 \(2019\)](#).
- [127] A. J. Friedman, S. Gopalakrishnan, and R. Vasseur, [Physical Review B **101**, 180302 \(2020\)](#).
- [128] D. Forster, D. R. Nelson, and M. J. Stephen, [Physical Review A **16**, 732 \(1977\)](#).
- [129] O. Narayan and S. Ramaswamy, [Physical Review Letters **89**, 200601 \(2002\)](#).
- [130] H. van Beijeren, [Physical Review Letters **108**, 180601 \(2012\)](#).
- [131] B. J. Alder and T. E. Wainwright, [Physical Review A **1**, 18 \(1970\)](#).
- [132] H. Spohn, [Journal of Statistical Physics **154**, 1191 \(2014\)](#).
- [133] H. Spohn, “Fluctuating hydrodynamics approach to equilibrium time correlations for anharmonic chains”, in (2016), pp. 107–158.
- [134] M. Kardar, G. Parisi, and Y.-C. Zhang, [Physical Review Letters **56**, 889 \(1986\)](#).

- [135] M. Prähofer and H. Spohn, [Journal of Statistical Physics](#) **115**, 255 (2004).
- [136] S. Lepri, R. Livi, and A. Politi, [Physical Review Letters](#) **78**, 1896 (1997).
- [137] S. Lepri, R. Livi, and A. Politi, [Europhysics Letters](#) **43**, 271 (1998).
- [138] T. Prosen and D. K. Campbell, [Physical Review Letters](#) **84**, 2857 (2000).
- [139] S. G. Das, A. Dhar, K. Saito, C. B. Mendl, and H. Spohn, [Physical Review E](#) **90**, 012124 (2014).
- [140] C. B. Mendl and H. Spohn, [Physical Review E](#) **90**, 012147 (2014).
- [141] C. B. Mendl and H. Spohn, [Journal of Statistical Mechanics: Theory and Experiment](#) **2015**, P03007 (2015).
- [142] M. Kulkarni, D. A. Huse, and H. Spohn, [Physical Review A](#) **92**, 043612 (2015).
- [143] S. Lepri, R. Livi, and A. Politi, [Physical Review Letters](#) **125**, 040604 (2020).
- [144] S. Chen, J. Wang, G. Casati, and G. Benenti, [Physical Review E](#) **90**, 032134 (2014).
- [145] H. Zhao and W.-g. Wang, [Physical Review E](#) **97**, 010103 (2018).
- [146] W. Fu, Z. Wang, Y. Wang, Y. Zhang, and H. Zhao, [Physical Review Research](#) **8**, 013314 (2026).
- [147] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, [Journal of Physics B: Atomic, Molecular and Optical Physics](#) **51**, 112001 (2018).
- [148] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, [Physical Review B](#) **94**, 245117 (2016).
- [149] F. Meinert, M. Panfil, M. J. Mark, K. Lauber, J.-S. Caux, and H.-C. Nägerl, [Physical Review Letters](#) **115**, 085301 (2015).