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AUTOREFERAT

Contents

1. Name

Piotr Sułkowski

2. Diplomas and scientific degrees

1. **Scientific degree: PhD in Physical Sciences**
Institution: Faculty of Physics, University of Warsaw
Year: 2007 (with distinction)
Title of PhD thesis: "Calabi-Yau crystals in topological string theory"
Advisors:
- Prof. Robbert Dijkgraaf (University of Amsterdam)
- Dr hab. Jacek Pawełczyk, prof. UW

2. **MSc diploma**
Institution: Faculty of Physics, University of Warsaw
Year: 2002 (with distinction)

3. **"Postgraduate course" diploma**
Institution: Durham University (United Kingdom)
Year: 2001 (with distinction)

3. Information about employment in scientific institutions

1. **University of Warsaw**, Faculty of Physics
Adjunct (assistant professor), from 2011

2. **University of Amsterdam (UvA)**, Institute for Theoretical Physics
Postdoctoral position, 2012-2013

3. **California Institute of Technology**, Division of Physics, Mathematics and Astronomy
Postdoctoral position, 2009-2012
Visiting faculty, 2012-2013

4. **Harvard University**, Department of Physics
Postdoctoral position, 2009

5. **University of Bonn**, Physics Institute
Postdoctoral position, 2007-2009

6. **A. Sołtan Institute for Nuclear Studies**
Adjunct (assistant professor), 2007-2011

4. Scientific achievement, in the sense of article 16, paragraph 2 of the *Act on academic degrees and academic title and degrees and title in art* (Dz. U. nr 65, poz. 595 ze zm.)

A) Title of the scientific achievement

Quantization and deformations of Riemann surfaces
in quantum field theories and string theory

B) Publications included in the scientific achievement

H1. Hiroyuki Fuji, Sergei Gukov, Marko Stosic, Piotr Sułkowski,
"*3d analogs of Argyres-Douglas theories and knot homologies*",
JHEP 1301 (2013) 175, pages 1-37, arXiv: 1209.1416 [hep-th].

H2. Hiroyuki Fuji, Sergei Gukov, Piotr Sułkowski,
"*Super-A-polynomial for knots and BPS states*",
Nucl. Phys. B867 (2013) 506-546, arXiv: 1205.1515 [hep-th].

H3. Hiroyuki Fuji, Sergei Gukov, Piotr Sułkowski, (with an appendix by Hidetoshi Awata)
"*Volume Conjecture: Refined and Categorified*",
Adv. Theor. Math. Phys. 6 (2012) 1669-1777, arXiv: 1203.2182 [hep-th].

H4. Sergei Gukov, Piotr Sułkowski,
"***A-polynomial, B-model, and Quantization",***
JHEP 1202 (2012) 070, pages 1-56, arXiv: 1108.0002 [hep-th].

H5. Piotr Sułkowski,
"***BPS states, crystals and matrices",***
Advances in High Energy Physics (2011) 357016, pages 1-52, arXiv: 1106.4873 [hep-th].

H6. Piotr Sułkowski,
"***Refined matrix models from BPS counting",***
Phys. Rev. D83 (2011) 085021, pages 1-12, arXiv: 1012.3228 [hep-th].

H7. Piotr Sułkowski,
"***Wall-crossing, open BPS counting and matrix models",***
JHEP 1103 (2011) 089, pages 1-19, arXiv: 1011.5269 [hep-th].

H8. Hirosi Ooguri, Piotr Sułkowski, Masahito Yamazaki,
"***Wall Crossing As Seen by Matrix Models",***
Commun. Math. Phys. 307 (2011) 429-462, arXiv: 1005.1293 [hep-th].

H9. Piotr Sułkowski,
"***Matrix models for ß-ensembles from Nekrasov partition functions*",**
JHEP 1004 (2010) 063, pages 1-28, arXiv: 0912.5476 [hep-th].

H10. Piotr Sułkowski,
"***Wall-crossing, free fermions and crystal melting*",**
Commun. Math. Phys. 301 (2011) 517-562, arXiv: 0910.5485 [hep-th].

H11. Piotr Sułkowski,
"***Matrix models for 2* theories*",**
Phys. Rev. D80 (2009) 086006, pages 1-9, arXiv: 0904.3064 [hep-th].

H12. Albrecht Klemm, Piotr Sułkowski,
"Seiberg-Witten theory and matrix models",
Nucl. Phys. B819 (2009) 400-430, arXiv: 0810.4944 [hep-th].

H13. Robbert Dijkgraaf, Lotte Hollands, Piotr Sułkowski,
"***Quantum Curves and D-modules",***
JHEP 0911 (2009) 047, pages 1-58, arXiv: 0810.4157 [hep-th].

H14. Robbert Dijkgraaf, Lotte Hollands, Piotr Sułkowski, Cumrun Vafa,
"Supersymmetric Gauge Theories, Intersecting Branes and Free Fermions",
JHEP 0802 (2008) 106, pages 1-56, arXiv: 0709.4446 [hep-th].

C) Description of the scientific aim of the above publications and the results

Contents

I Introduction

Physics is intimately related to mathematics. In particular, mathematics is often regarded as a language of physics. Relations to mathematics are well understood in particular in classical physics, in which many problems – e.g. in mechanics or theory of gravity – can be expressed at the mathematical level of rigor, using definitions, axioms, theorems and proofs.

The situation in broadly understood quantum physics is different. Even though it relies very much on advanced mathematical formalism, apart from a few simple models its rigorous foundations are not known. In particular we do not know rigorous foundations of the Standard Model of particles and interactions, the theory which describes our world – at least in some parameter range – to very high precision. Independently of mathematical formulation, and despite impressive agreement of Standard Model predictions with experiments, there are many reasons to believe that this is not the most fundamental theory of our Universe. Therefore many theoretical physicists put much effort in constructing more general theories of high energy physics, such as supersymmetric quantum field theories or string theory. One should stress, that most important motivations in such constructions are of phenomenological nature.

Nonetheless, it turns that analysis of such theories immediately leads to new, fascinating mathematical problems. Such problems arise in particular in simplified and exactly solvable models of supersymmetric theories and string theory. We stress that exact solvability in this case does not necessarily mean, that we understand fundamental mathematical formulation of a given theory. In exactly solvable theories we can compute certain quantities analytically – e.g. quantum amplitudes or correlation functions – even though underlying mathematical structures may remain mysterious. Nonetheless, it is believed that understanding of such structures in simplified theories will result in understanding of more complicated theories, in which various simplifications are absent, while newly found mathematical structures will provide a new language to describe Nature on the quantum level and in high energies. Moreover, mathematical structures found in this way are often very interesting in themselves, and become an inspiration which leads to important purely mathematical discoveries. Examples of such structures, related to or arising from problems in high energy physics, include moduli spaces, topological invariants of low dimensional manifolds (e.g. Donaldson, Seiberg-Witten, or Chern-Simons invariants), Calabi-Yau spaces, mirror symmetry and Gromov-Witten invariants, *etc.* All these issues are related to supersymmetric field theories or string theory, or the so-called topological versions of these theories. Let us also stress, that Donaldson, Jones, Kontsevich and Witten received Fields medals – the highest prize in mathematics – for various works related to these problems. This confirms the statement that physics is often a source of important ideas in mathematics. At the same time, in recent years these achievements also find other applications, and even become an inspiration in disciplines related to new technologies – for example topological theories and Chern-Simons invariants are important in description of the fractional quantum Hall effect and in quantum information theory. Undoubtedly understanding of mathematical structures behind exact solutions in quantum field theory and string theory is an important task; we are convinced, that many such structures will be important in description of the world around us.

In this work[1] we analyze exactly solvable theories related to high energy physics – in particular 3-dimensional and 4-dimensional quantum field theories with extended supersymmetry, string theory, and Chern-Simons theory – and new mathematical structures which arise in such theories. The main aim of this work is:

- to derive and analyze exact results, i.e. various quantum amplitudes and correlation functions, in the above mentioned supersymmetric quantum field theories, string theory, and Chern-Simons theory,
- to present that – despite completely different formulation of all these theories – important information about them is encoded in intriguing mathematical objects, which we will call as "quantum Riemann surfaces" or "quantum algebraic curves",
- to introduce a quantization formalism which associates "quantum Riemann surfaces" to "classical" Riemann surfaces, and to present how deformations of such surfaces encode various physical aspects of the above mentioned theories.

Because Riemann surfaces can be presented in terms of complex algebraic equations, and in view of various conventions in literature, in this work we often use notions of a "Riemann surface" and an "algebraic curve" interchangeably. Until recently a few unrelated examples of

[1]By "work" we mean a series of publications [H1-H14] listed above (in pages 3-4), which constitute a scientific achievement relevant for the habilitation degree.

Riemann surfaces had been known, which play important roles in various exactly solvable physical theories. These examples provide an important motivation for this work. One example of such surfaces are Seiberg-Witten curves, which arise in the solution of the so-called Seiberg-Witten theories, i.e. 4-dimensional supersymmetric theories of $\mathcal{N} = 2$ type [1][2]. The second example of such surfaces are the so-called A-polynomials, which are knot invariants taking form of algebraic curves [3]. As is well known, knot theory is intimately related to quantum field theory of Chern-Simons type [4], and in particular A-polynomials have a natural interpretation in Chern-Simons theory with non-compact gauge group $SL(2, \mathbb{C})$ [5]. Moreover a "quantum" generalization of A-polynomial is known, i.e. an operator expression $\widehat{A}$ which imposes recursion relations for the so-called colored Jones polynomials. Yet another example of (classical) Riemann surfaces are the so-called spectral curves in random matrix theory, which describe eigenvalue distribution in matrix models [6]. In some cases such matrix models are interpreted as simplified models of quantum field theory; in particular their solutions in the limit of large size of matrices take form of a topological expansion, found by 't Hooft in the context of strong interactions [7]. Finally, an interesting class of Riemann surfaces, the so-called "mirror curves", arises in the context of mirror symmetry in description of toric Calabi-Yau manifolds [9]. Because mirror symmetry is closely related to topological string theory on Calabi-Yau spaces, mirror curves also play an important role in string theory.

All the above mentioned examples of Riemann surfaces, which provided inspiration for this work, were found independently, in analysis of seemingly unrelated theoretical models. They also have rather different character – e.g. Seiberg-Witten curves are given in terms of complex equations with coefficients depending of physical parameters, and are considered as classical objects; on the other hand A-polynomials, as well as their quantum counterparts $\widehat{A}$, are given by equations with fixed, integer coefficients. In this work I showed that for all these, and many other surfaces, quantum generalizations can be formulated (which are called "quantum Riemann surfaces"), and I found a universal quantization procedure related to the formalism of the so-called "topological recursion", which enables to determine the form of such quantum surfaces. I also characterized how various parameters arising in physical theories are represented by deformations of classical and quantum Riemann surfaces.

It should be stressed, that apart from a formulation of a universal quantization procedure and a description of deformations of Riemann surfaces, another very important aspect of this work is a discovery of many new examples of classical and quantum Riemann surfaces in various physical theories. In particular in this work:

- I found a new class of surfaces describing the so-called "wall-crossing" phenomena [H5-H8],
- I discovered deformed Seiberg-Witten curves and mirror curves, which encode information about refinement [H5,H6,H9],
- I found a new interpretation of Seiberg-Witten curves and mirror curves, as spectral curves of a new class of matrix models [H9,H11,H12],
- I found a new interpretation of Seiberg-Witten curves and mirror curves, related to a system of intersecting D4-brans and D6-branes [H13,H14],

[2]In this summary, references [1], [2], [3], *etc.* refer to publications listed in pages 27-28, while references denoted [H1], [H2], [H3], *etc.* refer to publications listed in pages 3-4.

- I found new interpretations of Riemann surfaces in various other physical theories (e.g. in the so-called $c = 1$ model, minimal models of conformal field theory, A-polynomial) as spectral curves [H4],
- I discovered a very interesting class of classical and quantum curves, which have been named as "super-A-polynomials" [H1,H2,H3]; these are highly nontrivial generalizations of A-polynomials, which describe homological knot invariants on one hand, and 3-dimensional $\mathcal{N} = 2$ supersymmetric quantum field theories on the other hand; a discovery of super-A-polynomials is one of the most important aspects of this work.

The plan of this summary is as follows. In section II a notion of a "quantum Riemann surface" is introduced. In section III a quantization formalism based on matrix models and the topological recursion is presented, which enables to determine quantum Riemann surfaces in various exactly solvable physical theories. In section IV relations between various parameters of physical theories and deformations of associated Riemann surfaces are discussed. In section V all these ideas are discussed in explicit examples, and various classical and quantum Riemann surfaces (related to quantum field theories and string theory) which I discovered are presented.

II Classical and quantum Riemann surfaces

Before presenting new examples of Riemann surfaces analyzed in this work, in this section we explain what we mean by quantization of a Riemann surface. The idea of quantum Riemann surfaces is presented in papers [H4,H13,H14], and their various deformations – described in detail in section IV – are considered in all publications [H1-H14]. W should stress, that the ideas presented in this section are very general, and they have some particular realization in each physical theory considered in what follows; such realizations are discussed in section V.

II.A Classical Riemann surfaces

From mathematical viewpoint, a (classical) Riemann surfaces is defined as a complex one-dimensional manifold (or, as a real manifold, it is an oriented two-dimensional surface with a conformal structure). In general one can consider Riemann surfaces with or without boundaries, and possibly with some number of punctures. Riemann surfaces are characterized by a genus g, i.e. (informally) the number of "holes". More precisely, the genus can be defined recursively, so that it is equal $g = 0$ for a sphere, while cutting two disks in a surfaces of genus g (to get a surfaces whose boundary is homeomorphic to two circles), and connecting them by a cylinder, gives a surface of genus $g + 1$.

A number of Riemann surfaces considered in this work, associated to some physical theories, have no boundaries or punctures. However, in our analysis surfaces with boundaries also appear (e.g. in string theory they are related to D-branes), as well as surfaces with punctures (they are important in the formalism of the topological recursion). We also recall, that Riemann surfaces can be presented as complex algebraic curves, i.e. one-dimensional complex manifolds defined by a system of n polynomial equations in $(n + 1)$ complex variables. For all Riemann surfaces associated to physical theories described in this work we are able to find a representation as planar curves, i.e. for which $n = 1$; in other words, such curves are defined by a single polynomial equation

$$A(x, y) = \sum_{i,j} a_{ij} x^i y^j = 0, \tag{1}$$

for two complex variables x and y. In general coefficients in this equation a_{ij} depend on physical parameters in a given theory. It is also important that – depending on a theory being considered – variables x or y belong to $\mathbb{C}$ or $\mathbb{C}^*$. In the latter case we often represent them as $x = e^u$ and $y = e^v$, for $u, v \in \mathbb{C}$. As will be explained below, the quantization and deformation rules which I found are universal and do not depend on whether x, y are $\mathbb{C}$ or $\mathbb{C}^*$ variables.

II.B Quantum Riemann surfaces

One of the main aims of this work is to associate to a Riemann surface (1) a new object, which we will call a "quantum Riemann surface", or a "quantum curve". This idea is presented in detail in publications [H4,H13,H14]. From the viewpoint of quantization, in physical theories considered in this work, two-dimensional complex space $\mathbb{C} \times \mathbb{C}$ (or respectively $\mathbb{C}^* \times \mathbb{C}^*$) with (x, y) coordinates has a natural interpretation as a phase space and is equipped in a canonical holomorphic symplectic form, given respectively (for $\mathbb{C}$ or $\mathbb{C}^*$ coordinates) as

$$\omega = \frac{i}{\hbar} dx \wedge dy, \qquad\qquad \omega = \frac{i}{\hbar} \frac{dx}{x} \wedge \frac{dy}{y}, \tag{2}$$

where $\hbar$ is a parameter which plays an important physical role in each theory we consider. In classical theory the coordinates $x, y \in \mathbb{C}$ satisfy the Poisson bracket, which upon quantization is replaced by the commutation relation

$$[\hat{x}, \hat{y}] = \hbar, \tag{3}$$

where classical coordinates are promoted to operators $\hat{x}, \hat{y}$. For coordinates $x = e^u, y = e^v \in \mathbb{C}^*$ the commutation relation takes form $[\hat{u}, \hat{v}] = \hbar$, so that in consequence $\hat{x} = e^{\hat{u}}, \hat{y} = e^{\hat{v}}$ satisfy the Weyl relation

$$\hat{y}\hat{x} = q\hat{x}\hat{y}, \tag{4}$$

where the parameter

$$q = e^{\hbar} \tag{5}$$

plays an important role in subsequent considerations. Therefore upon quantization the algebra of functions on the phase space is deformed and becomes the non-commutative algebra of operators. In particular, the polynomial $A(x, y)$ defining the surface (1) is promoted to the operator

$$\widehat{A}(\widehat{x}, \widehat{y}) \;=\; \widehat{A}_0 + \hbar \widehat{A}_1 + \hbar^2 \widehat{A}_2 + \ldots\,, \tag{6}$$

where $A = A(x, y) \equiv \widehat{A}_0$. To present the above expression in a unique way, we have to make a choice of polarization and ordering of operators. In most of our analysis we choose the following representation

$$\hat{x} = x \qquad , \qquad \hat{y} = \hbar \frac{\partial}{\partial x}\,, \tag{7}$$

for $\mathbb{C}$ coordinates (and respectively $\hat{v} = \hbar \partial_u = \hbar x \partial_x$ for $x = e^u, y = e^v \in \mathbb{C}^*$ coordinates), and choose the ordering such that $\hat{y}$ arise to the right of $\hat{x}$.

To sum up, in case of $\mathbb{C}$ or $\mathbb{C}^*$ variables, a classical polynomial $A(x, y)$ which defines a Riemann surface (1) is promoted to the operator expression $\widehat{A}(\hat{x}, \hat{y})$. This is the object which we call the "quantum Riemann surface". At the same time, in the above mentioned two-dimensional complex phase space, the curve $A(x, y) = 0$ is (automatically) a lagrangian submanifold with respect to the holomorphic symplectic form ω. In the quantization process, one associates to the lagrangian submanifold (1) a quantum state characterized by a wave-function $Z(x)$. The main characteristic of quantum Riemann surfaces which we analyze is the fact, that they provide a differential (or difference, for $\mathbb{C}^*$ variables) equation for this wave-function

$$\widehat{A}(\hat{x},\hat{y})Z(x) = 0. \tag{8}$$

An important aspect of quantum Riemann surfaces described above is also the fact, that they are related to quantum amplitudes in physical theories. In theories which we analyze two types of such amplitudes arise, which – in the language of string theory – can be interpreted as open string amplitudes $S_k(x)$ (or more generally "open" amplitudes $W^g_n(x_1,\ldots,x_n)$), and closed string amplitudes F_g, also referred to as free energies. Open amplitudes depend on various parameters of a theory (in string theory interpretation related to D-branes) denoted here schematically by x, while closed string amplitudes do not depend on such parameters; moreover, both types of amplitudes may depend on other quantities characterizing a given theory (which we temporarily suppress in the notation). The above amplitudes arise in the expansion of the partition function Z of a given theory in the parameter $\hbar$, which – depending on a given theory – may also have some particular physical interpretation. Irrespective of such an interpretation, appropriately understood partition functions related to open amplitudes have the following expansion in the limit of small values of $\hbar$

$$Z(x) = \exp\Big(\frac{1}{\hbar}S_0(x) + \sum_{k=1}^{\infty} S_k(x)\hbar^{k-1}\Big), \tag{9}$$

while partition functions related to closed amplitudes have an expansion

$$Z = \exp\Big(\frac{1}{\hbar^2}F_0 + \sum_{g=1}^{\infty} F_g\hbar^{2g-2}\Big). \tag{10}$$

In both cases above we explicitly wrote the leading terms in the expansions $S_0(x)$ i F_0, because of their important role and a direct relation to a Riemann surface arising in a given theory – i.e. the main object of our interest. Namely, the leading term in the expansion of open amplitudes S_0 turns out to be given by an integral along an open path on such Riemann surface

$$S_0(x) = \int^x \lambda, \tag{11}$$

where the canonical one-form λ, in $\mathbb{C}$ or $\mathbb{C}^*$ variables, respectively takes form

$$\lambda = y dx, \qquad\qquad \lambda = \log y \frac{dx}{x}. \tag{12}$$

The above relation also implies, that from the knowledge of $S_0(x)$ one can determine explicitly the form of such a Riemann surface. Namely, determining y from the above relation for $\mathbb{C}$ and $\mathbb{C}^*$ variables we find respectively

$$y = \frac{\partial S_0(x)}{\partial x}, \qquad\qquad y = e^{x\frac{\partial S_0(x)}{\partial x}}. \tag{13}$$

This condition can be accompanied by some additional equations, which ensure that $S_0(x)$ is a saddle point of $Z(x)$. After elimination of additional variables from such saddle equations, the relation (13) can be transformed into the form of the classical curve $A(x,y) = 0$. Quantization of this curve leads to the quantum curve (8), and at the same time determines higher corrections $S_k(x)$ in (9).

In case of closed amplitudes, the leading term F_0 in the expansion of the partition function Z is related to the integral of the one-form λ along canonical cycles A_i and B_i on the surface

$$t_i = \oint_{A_i} \lambda, \qquad \frac{\partial F_0}{\partial t_i} = \oint_{B_i} \lambda, \tag{14}$$

where t_i parameters play important role in each physical theory that we consider (e.g. this is how moduli in Seiberg-Witten theory arise, or eigenvalue distribution in matrix models).

The first subleading terms in the expansion of (9) and (10) represent one-loop quantum corrections in associated physical theories, and also have well-defined geometrical meaning [H4,H13,H14]. In particular F_1 computes the chiral determinant of the Dirac operator $\overline{\partial}$ on the Riemann surface

$$F_1 = \log\det\overline{\partial}, \tag{15}$$

which is manifested e.g. in a relation between the partition function Z and free fermion theory on the Riemann surface [H5,H6,H7,H8,H10,H13,H14]. On the other hand the subleading term in the open amplitude is related to a certain torsion $T(x)$ associated to the surface [H4]

$$S_1(x) = -\frac{1}{2}\log T(x). \tag{16}$$

Higher terms in the expansion of (9) and (10), i.e. all the remaining $S_k(x)$ and F_g, represent higher quantum corrections (in the corresponding physical theories), and they also have very intricate and interesting structure [H1-H14].

To sum up, the quantum Riemann surface $\widehat{A}(\hat{x},\hat{y})$ is an operator which annihilates the wave-function $Z(x)$ (expressed in terms of open amplitudes) of a state associated to a classical lagrangian submanifold $A(x,y)=0$; partition functions representing closed amplitudes F_g are also associated in a definite way to the Riemann surface $A(x,y)=0$. It should be stressed, that partition functions Z and $Z(x)$ in physical theories are usually defined via path integrals, or (in string theory interpretation) may arise as generating functions of BPS states in some D-brane configurations. Showing that these independently defined partition functions indeed share the same properties as (9) and (10) is a nontrivial task; the fact that the representation of these partition functions related to a (quantum) Riemann surface exists is itself fascinating. In publications [H1-H14] I determined and analyzed relations between the above two types of partition functions for various physical theories and associated Riemann surfaces; these relations are discussed in the following chapters of this summary.

So far we presented partition functions Z and $Z(x)$, as well as the operator (6) representing the quantum Riemann surface, in terms of an expansion in $\hbar$. However, for surfaces embedded in $\mathbb{C}^*\times\mathbb{C}^*$, it is often possible to find another expansion in parameter $q=e^{\hbar}$. Even though a transformation between two such expansions is nontrivial, both of them are directly – and also nontrivially – related to a quantum Riemann surface. Above we discussed that in case of $\hbar$ expansion this relation takes form (8), where the operator $\widehat{A}$ is given as an expansion (6), and in the classical limit $\hbar\to 0$ it reduces to the classical Riemann surface

$$\widehat{A}(\widehat{x},\widehat{y}) \;=\; \widehat{A}_0 + \hbar\widehat{A}_1 + \hbar^2\widehat{A}_2 + \ldots \qquad \underset{\hbar\to 0}{\longrightarrow} \qquad A(x,y)\equiv A_0. \tag{17}$$

On the other hand, in case of expansion in q, the operator $\widehat{A}$ takes form

$$\widehat{A}(\hat{x},\hat{y}) = \sum_{i,j} a_{ij}(q)\hat{x}^i\hat{y}^j \qquad \underset{q\to 1}{\longrightarrow} \qquad A(x,y) = \sum_{i,j} a_{ij}(1)x^i y^j, \tag{18}$$

and on the right hand side we also explicitly presented the classical limit. The fact that such a limit reproduces the original surface $A(x,y)=0$, i.e. that the coefficients $a_{ij}(1)$ reproduce a_{ij} in the equation (1), is in general very nontrivial, and it is discussed many times in publications [H1-H14].

At this stage we can also present some of the results found in [H4]. Firstly, in this paper I derived an infinite hierarchy of differential equations which relates the form of the operator (6) with the expansion (9). This hierarchy takes form

$$\sum_{r=0}^{n} \mathfrak{D}_r A_{n-r} \; = \; 0\,, \tag{19}$$

where A_{n-r} is a symbol which uniquely determines (by the replacement $y \mapsto \hat{y}$, with $\hat{y}$ ordered to the right of $\hat{x}$) corrections $\widehat{A}_{n-r}$ arising in (6), while operators $\mathfrak{D}_r$ are determined by the generating function

$$\sum_{r=0}^{\infty} \hbar^r \mathfrak{D}_r \; = \; \exp\left(\sum_{n=1}^{\infty} \hbar^n \mathfrak{d}_n\right), \qquad \mathfrak{d}_n \; = \; \sum_{r=1}^{n+1} \frac{\partial_x^r S_{n+1-r}}{r!} (\partial_y)^r. \tag{20}$$

In [H4] I showed that this hierarchy takes the same form for both $\mathbb{C}$ oraz $\mathbb{C}^*$ variables; in the latter case one only has to replace ∂_x by $x\partial_x$ and ∂_y by $y\partial_y$ in the expression for $\mathfrak{d}_n$. These equations allow to determine uniquely the quantum Riemann surface from the expansion (9), or to determine $Z(x)$ from the knowledge of (6).

II.C Uniqueness of quantization

The relation between quantum objects $\widehat{A}$ and $Z(x)$ is given in a natural way by the hierarchy of equations (19). However in [H4] I discovered much more surprising and nontrivial relation between the classical surface $A(x,y)=0$ and the quantum surface $\widehat{A}(\hat{x},\hat{y})$. Namely, in [H4] I showed, that the quantum Riemann surfaces in physical theories that we consider are *determined* by the classical surfaces, and I also described a unique quantization procedure which determines $\widehat{A}$ based on the form of $A(x,y)$, as well as some particular universal (independent of the form of $A(x,y)$) quantization rules. In other words, this provides a unique correspondence

$$A(x,y) \quad \rightsquigarrow \quad \widehat{A}(\hat{x},\hat{y}) \tag{21}$$

(the inverse of this highly nontrivial relation is of course the classical limit (17), (18)). We stress that this is a very surprising result – it is known, that in broadly understood mathematical physics many quantization schemes were proposed, which in general may lead to inequivalent results. Nonetheless, the quantization procedure which I found in [H4] works universally for many exactly solvable theories, which are related to supersymmetry or string theory. This quantization formalism is related to the so-called topological recursion, which is described in the next section.

One more highly nontrivial result found in [H4] is as follows. Expansions (6) in parameter $\hbar$ of the operator $\widehat{A}$ representing a quantum Riemann surface have infinite number of terms $\widehat{A}_n$. In [H4] I showed that for a large class of examples such expansions are equivalent to *finite* expansions of the same operator $\widehat{A}$ in parameter q (as in (18)). This means in particular, that the knowledge of a *finite* number of terms in the full expansion (6), or a *finite* number of amplitudes S_k in (9), allows to determine *exactly* the form of $\widehat{A}$ – in particular as a finite expression in q. Moreover, it is discussed in [H4] that for a large class of theories the full quantum expansion is determined just by the first subleading term in the expansion (6) or (9) (i.e. $\widehat{A}_1$ lub S_1) – so that the quantum curve (with nontrivial dependence on q) in this case is fully determined by the classical curve (i.e. its $q \to 1$ limit) and the subleading correction, which does not even require using the universal quantization rules associated to the topological recursion described in the next chapter (however, these results are of course consistent with those quantization rules).

II.D Mathematical and physical aspects of quantization

In [H13,H14] I also discussed a formalism of D-modules and its relations to theories of free fermions on Riemann surfaces; in particular I reinterpreted partition functions of the form (9) and (10) as partition functions for such fermionic theories. I also showed that various physical theories considered by us are related by a chain of dualities to a system of D4-branes and D6-branes in type IIA superstring theory, intersecting along a Riemann surface; in this case free fermions arise as massless states of open strings stretched between D4-branes and D6-branes. Such dualities relate a system of D4-branes and D6-branes with topological string theory, 4-dimensional $\mathcal{N}=2$ supersymmetric field theories, or a theory of M5-branes in 11-dimensional M-theory, *etc.*; therefore understanding the system of intersecting D4- and D6-branes is important for all these theories. In a system of intersecting D-branes the parameter $\hbar$ introduced above represents nontrivial string-theoretic B-field, which is responsible for the non-commutative character of coordinates $\hat{x}$ and $\hat{y}$. The quantum Riemann surface can therefore be interpreted in the language of non-commutative geometry, with the non-commutativity parameter determined by the value of the B-field, analogously as in Seiberg-Witten theory [10]. It should also be stressed, that the free fermion formalism plays an important role in publications [H5,H6,H7,H8,H10].

In publications [H1-H4] I also analyzed the so-called "quantizability conditions" which are necessary for the existence of a quantum Riemann surface, and the relation of these conditions to a branch of mathematics known as K-theory. For $Z(x)$, interpreted as the open wave-function, to be well defined and annihilated by the quantum curve $\widehat{A}$ according to (8), in particular the leading term $S_0(x)$ must be uniquely determined and independent on an integration cycle in (11). This condition implies, that the following equalities (for $\mathbb{C}^*$ variables) must hold

$$\begin{aligned} \oint_\gamma \Big(\log|x| d(\arg y) - \log|y| d(\arg x) \Big) &= 0\,, \\ \frac{1}{4\pi^2}\oint_\gamma \Big(\log|x| d\log|y| + (\arg y) d(\arg x) \Big) &\in \mathbb{Q}, \end{aligned} \tag{22}$$

for all cycles γ along the curve $A(x,y)=0$. In [H4], after analysis of the above conditions, more general quantizability criterion was introduced and formulated in K-theory terms; this criterion states that a Riemann surface $A(x,y)=0$ is quantizable if and only if the so-called K-theoretic symbol $\{x,y\} \in K_2(\mathbb{C})$ represents a certain torsion element in K-theory. As described in [H4], this criterion also implies the following relatively simple condition which is necessary for quantizability of a curve $A(x,y)=0$:

$$\widehat{A}(\hat{x},\hat{y}) \ exists \quad \Rightarrow \quad \begin{pmatrix} face\ polynomials\ of\ a\ Newton\ polygon \\ for\ A(x,y)=0\ curve\ are\ tempered \end{pmatrix} \tag{23}$$

This condition should be understood as follows. By definition a polynomial is tempered, if all its roots are (complex) roots of unity. To ensure that a quantum counterpart of a curve

$A(x, y) = 0$ exists, one should consider its Newton polygon (i.e. a set of points (i, j) for which $a_{i,j} \neq 0$ in expression (1)). For each face of this polygon, represented by a set of coefficients $a_{ij} \equiv a_k$, a face polynomial $w(z) = \sum_k a_k z^k$ is constructed. The above condition is satisfied, if roots of all such polynomials $w(z)$ (for each face of the Newton polygon) are roots of unity.

The general condition (22), and in particular the necessary condition (23), can be satisfied and $\widehat{A}(\hat{x}, \hat{y})$ may exists only for some particular curves, for which coefficients a_{ij} satisfy the above relations. These conditions are particularly hard to satisfy when coefficients a_{ij} depend on some physical parameters (e.g. moduli representing expectation values of scalar fields, or the refinement parameter t). Nonetheless I showed that for Riemann surfaces considered in this work (in particular for super-A-polynomials and analogs of Seiberg-Witten curves for $\mathcal{N} = 2$, 3-dimensional theories, as well as mirror and other curves), these conditions are satisfied in a surprising way. This is a nontrivial result, which confirms that theories considered by us have very nontrivial structure. Quantizability of some particular Riemann surfaces considered in [H1-H3] is described in more detail in section V.

III Matrix models, topological recursion, and quantization

In publications [H1-H14] I formulated the quantization formalism for Riemann surfaces in various physical theories. This formalism is related to matrix models and the so-called topological recursion. After a brief summary of these topics in next two subsections, this quantization formalism will be presented in subsection III.C.

III.A Matrix models

Matrix models, or random matrices, can be regarded as simplified models of quantum field theory, for which a path integral is represented as an integral over a certain ensemble of matrices M (e.g. hermitian matrices, unitary ones, *etc.*) of size $N \times N$

$$Z = \int \mathcal{D}M \exp\Big(-\frac{1}{\hbar}\mathrm{Tr}V(M)\Big), \tag{24}$$

where the function $V(M)$ is called a potential of a model. In the above expression one can always integrate out off-diagonal elements of M; in a hermitian model, irrespective of the form of the potential, a contribution from the off-diagonal terms takes form of the Vandermonde determinant $\prod_{i<j}(\lambda_i - \lambda_j)^2$, and the above integral reduces to an integral over eigenvalues λ_i of matrices M

$$Z = \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j}(\lambda_i - \lambda_j)^2 \exp\Big(-\frac{1}{\hbar}\sum_{k=1}^{N} V(\lambda_k)\Big). \tag{25}$$

It is very interesting to analyze such models in the limit of large N. As 't Hooft showed, in the limit presently referred to by his name, and introducing the so-called 't Hooft parameter

$$T = \hbar N = const, \tag{26}$$

the partition function (24) has the so-called topological expansion and takes form analogous to (10)

$$Z = \exp\Big(\sum_{g=0}^{\infty} F_g \hbar^{2g-2}\Big). \tag{27}$$

Feynman diagrams arising from such an expansion take form of fatgraphs (or so-called ribbon graphs) [8], so that contributions to terms F_g arise from fatgraphs which can be drawn on a two-dimensional surface of genus g.

In matrix models various correlation functions also play an important role. One important class of such functions are multi-resolvents, i.e. expectation values (computed under the measure (24), i.e. with the weight $e^{-\frac{1}{\hbar}\mathrm{Tr}V(M)}$) of products of the following traces

$$\Big\langle \mathrm{Tr}\Big(\frac{1}{x_1-M}\Big)\cdots\mathrm{Tr}\Big(\frac{1}{x_n-M}\Big)\Big\rangle_{\mathrm{conn}} = \sum_{g=0}^{\infty} \hbar^{2g-2+n}\frac{W_n^g(x_1,\ldots,x_n)}{dx_1\cdots dx_n}. \tag{28}$$

The lower index $\langle\ldots\rangle_{\mathrm{conn}}$ denotes here contributions from connected diagrams, and on the right side of the above equality the above expectation value is expanded in a series in powers of $\hbar$, which defines symmetric multi-differentials $W_n^g = W_n^g(x_1,\ldots,x_n)$ (a factor in the denominator on the right side of the equation denotes contraction with a symmetric n-form $dx_1\cdots dx_n$, which by definition is also a factor in W_n^g). Amplitudes W_n^g are associated to certain auxiliary surfaces of genus g and n punctures.

Another important quantity is an expectation value of the determinant $Z(x)=\langle\det(x-M)\rangle$, whose asymptotic expansion in $\hbar$ has a form analogous to (9)

$$Z(x) = \Big\langle \det(x-M)\Big\rangle = \exp\Big(\sum_{k=0}^{\infty} S_k(x)\hbar^{k-1}\Big). \tag{29}$$

It is not hard to show [H4] that coefficients $S_k(x)$ in the above equation can be expressed in terms of multi-differentials W_n^g defined in (28)

$$S_k(x) = \sum_{2g-1+n=k}\frac{1}{n!}\underbrace{\int^x\cdots\int^x}_{n\ \text{times}} W_n^g(x_1',\ldots,x_n'). \tag{30}$$

The key object associated to a given matrix model is the so-called spectral curve. This is an algebraic curve which encodes equilibrium distribution of eigenvalues in the limit $N\to\infty$. Schematically this curve arises as follows. In the model defined by (25) eigenvalues, on one hand, approach minima of the potential $V(\lambda_i)$. On the other hand, the Vandermonde determinant in (25), after writing it the exponential form, imposes repelling interactions between eigenvalues λ_i – which therefore behave like fermions and obey the Pauli exclusion principle. It can be shown, that in the limit $N\to\infty$ a distribution of eigenvalues becomes continuous and takes form of several cuts localized along minima of the potential $V(x)$, in the complex x-plane. This x-plane with cuts defines a Riemann surface, known as the spectral curve of a given matrix model. More precisely, the spectral curve is a solution to the Riemann-Hilbert problem for the leading term of the one-point resolvent $W_1^0(x)$ (defined by (28) for $g=0$ and $n=1$). Namely, along the cuts in the x-plane, the resolvent is related to the potential $V(x)$ of a matrix model via

$$W_1^0(x-i\epsilon)+W_1^0(x+i\epsilon) = \frac{V'(x)}{T}, \tag{31}$$

and its discontinuity along these cuts defines density of eigenvalues

$$\rho(x) = \frac{1}{2\pi i}\big(W_1^0(x-i\epsilon)-W_1^0(x+i\epsilon)\big). \tag{32}$$

Above relations, together with a condition at infinity $\lim_{x\to\infty}W_1^0(x)=\frac{1}{x}$, determine the form of $W_1^0(x)$, which is often encoded in a new variable

$$y = \frac{T}{2}\big(W_1^0(x-i\epsilon)-W_1^0(x+i\epsilon)\big). \tag{33}$$

For a broad class of matrix models (in particular with a polynomial potential $V(x)$), such variable y turns out to be related to x via a polynomial equation of the form $A(x,y)=0$, which defines a (complex) algebraic curve. This curve (or a pair of variables x,y) can be parametrized by a complex parameter q, so that the polynomial equation is automatically satisfied

$$A(x,y)=0 \qquad \leftrightsquigarrow \qquad \begin{cases} x=x(q) \\ y=y(q) \end{cases} \tag{34}$$

The above curve is known as the spectral curve of a matrix model.

III.B Topological recursion

In random matrix theory the key role is played by free energies F_g which appear in (27) and amplitudes W^g_n in (28). Amplitudes W^g_n are associated to auxiliary surfaces of genus g and with n punctures; these amplitudes also satisfy an infinite set of equations, which – regarding (24) as a model of quantum field theory – are interpreted as Ward identities. In [11] a general solution of this system of equations was found. This solution takes form of recursion relations, which determine amplitudes W^g_n with higher indices (g,n) (in appropriate ordering) from the knowledge of amplitudes with lower indices. These relations are referred to as the topological recursion. The initial condition for this recursion is the spectral curve (34) presented in a parametric form $x=x(q), y=y(q)$. Important ingredient of these recursion equations is computation of residues along branch points of the spectral curve q^*_i, which are defined via $dx(q)|_{q=q^*_i}=0$; moreover in a neighborhood of each such branch point a conjugate point $\bar{q}$ is defined, such that $x(\bar{q})=x(q)$. With this notation, the topological recursion takes form

$$\begin{aligned} W^g_{n+1}(p,\vec{p}_N) \;\; &= \;\; \sum_{q^*_i} \mathrm{Res}_{q\to q^*_i} K(q,p)\Big(W^{g-1}_{n+2}(q,\bar{q},\vec{p}_N)+ \\ &\qquad + \sum_{m=0}^{g}\sum_{J\subset N} W^m_{|J|+1}(q,\vec{p}_J)W^{g-m}_{n-|J|+1}(\bar{q},\vec{p}_{N/J})\Big), \end{aligned} \tag{35}$$

where $\vec{p}_N$ denotes a set of n punctures with indices in a set N, the sum over J runs over all subsets of N, and $K(q,p)$ is appropriately defined recursion kernel [12], [H4]. These relations have a graphical representation shown in figure 1. Moreover, the knowledge of the one-point amplitudes W^g_1 allows to determine free energies for $g\geq 2$

$$F_g = \frac{1}{2g-2}\sum_{q^*_i}\mathrm{Res}_{q\to q^*_i} S_0(q) W^g_1(q)\,, \tag{36}$$

where $S_0(q)=\int^q \lambda$ is the primitive of (12). Moreover, the terms F_0 and F_1 are defined independently of the recursion relations, as discussed in detail in [12].

While recursion relations (35) determine an important class of amplitudes and free energies in arbitrary matrix model from the knowledge of its spectral curve, in [12] much more powerful statement was formulated: it was shown that such recursion relations can be considered for an *arbitrary* algebraic curve, not necessarily associated to a matrix model. From this perspective, recursion relations associate to such (arbitrary) curve an infinite family of multi-differentials W^g_n, and quantities S_k related to them via (30), as well as an infinite number of free energies F_g. Generating functions of these quantities in a natural way take form of amplitudes (27) and (29). It was shown in [12] that F_g are symplectic invariants of a given algebraic curve and they share many interesting properties. The recursion (35) is often called the Eynard-Orantin (the authors of [12]) recursion.

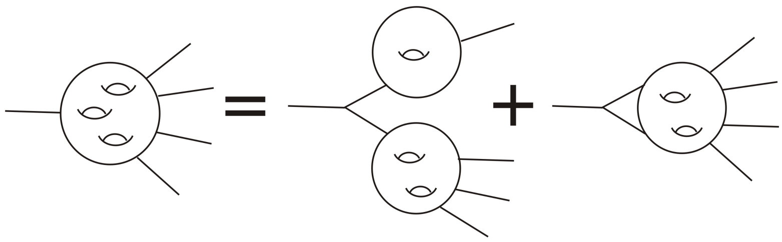

Figure 1: *Schematic representation of the topological recursion (35). The amplitudes* W^g_n *are represented by auxiliary surfaces of genus* g *and* n *punctures.*

III.C Main hypothesis: quantization of Riemann surfaces

At this stage we can finally present the main hypothesis of this work (already briefly mentioned in section II.C). This hypothesis states, that *Riemann surfaces in various exactly solvable supersymmetric field theories and string theory can be quantized using the universal formalism of the topological recursion.* I introduced and presented this formalism most generally in [H4], and many of its aspects and explicit examples (summarized in section V) are presented in all publications [H1-H14]. The quantization procedure in this formalism is as follows:

- classical Riemann surfaces $A(x,y)=0$ arising in exactly solvable physical theories should be identified with algebraic curves that provide the initial condition for the topological recursion (i.e. they play analogous role to spectral curves in matrix models (34)),
- using the topological recursion for a given surface $A(x,y)=0$ one can determine multi-differentials W^g_n using (35), and subsequently $S_k(x)$ defined via (30),
- an infinite hierarchy of equations (19) uniquely associates to amplitudes $S_k(x)$ corrections $\widehat{A}_k$ to the quantum curve (6); therefore, to the original Riemann surface $A(x,y)=0$ in a given physical theory, in this way one associates the quantum curve

$$A(x,y) \quad \rightsquigarrow \quad \widehat{A}(\hat{x},\hat{y}), \tag{37}$$

  in agreement with the earlier statement (21),
- independently of this quantum curve, the above identifications imply, that closed amplitudes (partition functions) Z and open amplitudes $Z(x)$ arising in physical exactly solvable theories can be identified with generating functions of amplitudes F_g and S_k, which are associated to the curve $A(x,y)=0$ via the topological recursion; in other words, physical partition functions Z and $Z(x)$ have properties analogous to partition functions (27) and (29) in matrix models; showing that such identifications indeed hold is a non-trivial and important result, which I showed for various theories in [H1-H14].

It should be stressed once more, that in publications [H1-H14] I showed that the above formalism applies in a universal way to a very large class of seemingly unrelated physical theories, such as: supersymmetric quantum firled theories in 3 and 4 dimensions, topological string theories on toric Calabi-Yau manifolds and the associated "wall-crossing" phenomena, Chern-Simons theory, $c=1$ model, *etc.* The universal character of this quantization procedure can be explained from several perspectives. Firstly, in each of the above mentioned (and considered in [H1-H14]) theories, an important role is played by some Riemann surface, as well as free fermions defined on this surface. As we explained in section III, matrix models and topological recursions are also intimately related to such representations (in particular eigenvalues in matrix models have fermionic character). Moreover, in [H13-H14] various dualities were discussed between the above mentioned theories, and a system of D4-branes and D6-branes intersecting along a Riemann surface; massless states of strings stretched between these D-branes also give rise to free fermions defined on this surface. It is natural to identify free fermions in physical theories with free fermions in a system of intersecting D4- and D6-branes, as well as with fermions in matrix models; and in consequence to identify quantization of physical theories with quantization via the topological recursion in matrix models. Moreover, physical theories which we consider are also related to the physics of M5-branes in M-theory, as presented in [H1,H2,H3,H13,H14]. So far the theory of M5-branes is not well understood, and its proper formulation is one of the most important challenges in high energy physics. Nonetheless, the fact that physical theories which we consider are directly related (via various dualities) to M5-branes suggests, that they have a universal underlying description. As follows from the results in publications [H1-H14], quantum Riemann surfaces and the quantization formalism presented above constitute an essential ingredient of such a description.

IV Deformations of Riemann surfaces

In the above chapters a universal quantization formalism of Riemann surfaces was presented. We described how to associate a quantum curve to a classical curve (37), i.e. how to determine corrections in $\hbar$ to the classical curve (6), and how the quantum curve depends on $q = e^{\hbar}$ (18). Nonetheless, Riemann surfaces which arise in physical theories often depend on many other parameters, which can be regarded as deformations. Two most important classes of such deformations include the so-called refinement (usually denoted by a parameter t, which is related to the interpretation in terms of BPS states), and deformations associated to moduli (whose set we temporarily denote by a single parameter Q), which represent various geometric properties of manifolds on which a given theory is considered. In [H1-H3] I showed that both these classes of parameters are independent of the quantization parameter $\hbar$, meaning that their change does not affect the form of commutation relations (3) or (4). Dependence on such deformation parameters also does not affect the quantization formalism presented in section III.C. Such deformation parameters can be encoded in coefficients $a_{ij} = a_{ij}(Q,t)$ of a classical Riemann surface (1), or coefficients of quantum curves $a_{ij}(q) = a_{ij}(q;Q,t)$, so that there exists a well defined limit $\lim_{q\to 1} a_{ij}(q;Q,t) = a_{ij}(Q,t)$. At the same time, dependence on deformation parameters appears in open (9) and closed (10) partition functions, $Z(x)$ and Z, associated to such Riemann surfaces.

In physical theories which we consider (in in [H1-H14]), partition functions $Z(x)$ and Z can be interpreted as generating functions, or indices, of BPS states on Calabi-Yau manifolds. Indeed, supersymmetric field theories in 3 and 4 dimensions, in particular Seiberg-Witten theories, are effective theories, which arise in the so-called field theory limit of type IIA superstring theory. Such superstring theory is formulated in 10 dimensions, whose 4-dimensional subspace is identified with spacetime, and 6 dimensions are compactified on a Calabi-Yau manifold. Properties of an effective field theory in 4 spacetime dimensions depend crucially on properties of this Calabi-Yau space, and in particular its Kähler moduli Q (this symbol denotes here the whole set of such moduli).

An example of a Calabi-Yau manifold considered several times in [H1-H14] is the so-called conifold, i.e. a line bundle of the form

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1. \tag{38}$$

This is a simple, toric Calabi-Yau manifold, which is a deformation of the singularity $xy = zw$ (for complex x, y, z, w), in which a singular point is replaced by a 2-cycle $\mathbb{P}^1$. This manifold has one Kähler parameter $Q = e^{-T}$, where T represents a size of $\mathbb{P}^1$. In this work the conifold was analyzed from many perspectives: it is related to a string-theoretic interpretation of knot theory [H1-H3], it provides one of basic examples of mirror symmetry [H4,H5,H8,H12], it is a model example of "wall-crossing" phenomena [H5-H8,H10], it is related to 5-dimensiona $U(1)$ Seiberg-Witten theory [H9,H11-H14]. As will be described in section V, in [H1-H14] I also derived various results for more complicated toric Calabi-Yau manifolds.

The so-called open and closed BPS states provide an important class of observables in effective 4-dimensional theories. These states arise from (open or closed) D2-branes (and its bound states with D0-branes) wrapped on 2-dimensional cycles of Calabi-Yau manifolds (e.g. on $\mathbb{P}^1$ in a conifold). These states belong to a Hilbert space $\mathcal{H}$, and – depending on a given theory and interpretation – closed and open partition functions Z and $Z(x)$ for such an effective theory have an interpretation of a trace over this Hilbert space

$$Z,\, Z(x) \quad \simeq \quad \mathrm{Tr}_{\mathcal{H}} Q^{\gamma} q^P t^F, \qquad \gamma \in H_2(\text{Calabi-Yau}), \tag{39}$$

where P and F denote (described in more detail below) generators of rotations in 4-dimensional spacetime. In the effective 4-dimensional theory the parameter q is identified with the quantum parameter (5). Dependence of the above trace on parameters t (or, equivalently, β) and Q is described below; its immediate consequence is also a dependece of (an associated to a given theory) Riemann surface on the same deformation parameters. In section V we will discuss many explicit examples of such deformations, which arise in theories analyzed in [H1-H14].

IV.A BPS states and the refinement

The above mentioned physical interpretation of the refinement, related to the counting of BPS states in string theory, generalizes many other mathematical and physical constructions, such as the so-called Ω background in Seiberg-Witten theories, β-deformations of matrix models, or homological knot invariants, which all have been considered in [H1-H14]. In this section we describe these constructions in more detail.

BPS states. Apart from dependence on Calabi-Yau moduli Q, the trace (39) in general may depend on parameters q and t, which are related to spins of BPS states in low-dimensional effective theory. In case of closed BPS states and partition functions Z, interpretation of these parameters is as follows. In 4-dimensional theory, in Euclidean metric, spatial rotation group $SO(4) = SU(2)_P \times SU(2)_F$ has two generators P and F, which represent "left" and "right" spins, while q (identified with (5)) and t are generating parameters for enumeration of these spins. It should be stressed, that for compact Calabi-Yau manifolds a dependence on t in (39) cannot arise, because in this case it is related to nontrivial complex deformations of these manifolds and the above trace is not well defined. However non-compact, toric manifolds – with conifold (38) being a simple example – provide a very interesting class of models which do not posses nontrivial complex deformations. In this case the trace (39) is well defined, and the dependence on t is usually quite interesting. Many results in [H1-H14] are derived for toric, non-compact Calabi-Yau manifolds, and include interesting dependence on t.

In case of open BPS states and open partition functions $Z(x)$, in string theory configuration one should take into account an additional D4-brane, wrapped on a lagrangian (3-dimensional) submanifold of a Calabi-Yau space. Open D2-branes, also wrapped on cycles of the Calabi-Yau, may end on such a D4-brane. Moreover, apart from three dimensions of D4-brane wrapped on a lagrangian submanifold, its remaining two dimensions extend along two-dimensional subspace of the 4-dimensional spacetime. In this case the trace (39) is computed for the effective theory in this two-dimensional part of the D4-brane, for which $SO(2) = U(1)$ is the Euclidean rotation group with generator P. Moreover in this case F is the generator of rotations in the remaining, two normal directions of 4-dimensional spacetime. Similarly as for closed BPS states, nontrivial dependence on t arises for non-compact, toric Calabi-Yau manifolds. Moreover, in such a system one should take into account an additional parameter, which represents a location of the D4-brane inside the Calabi-Yau space; this parameter should be identified with x, which is an argument of the open partition function $Z(x)$ (formally, in case of open D2-branes, the homology group in (39) can be replaced by a relative homology $H_2(\text{Calabi-Yau}, L)$, where L denotes a lagrangian submanifold on which D4-brane is wrapped; generating parameters in this case are represented by a pair (Q, x)).

The above interpretation can be extended to 11-dimensional M-theory. In this case D2-branes and the D4-brane are generalized respectively to M2-branes and the M5-brane. The M5-brane is still wrapped on a lagrangian submanifold inside Calabi-Yau space, while its remaining 3-dimensional part constitutes a spacetime for an effective, 3-dimensional $\mathcal{N} = 2$ supersymmetric theory, dual to Chern-Simons theory [H1-H3].

Nekrasov partition functions and Ω background. Independently of the above interpretation, in many cases partition functions Z and $Z(x)$ have another representation, related to path integrals in quantum field theories. In particular such an interpretation is important for 4-dimensional and 5-dimensional Seiberg-Witten theories, and the corresponding partition functions (presently known as Nekrasov partition functions) were found by Nekrasov [13], who considered such theories in a deformed spacetime, in the so-called Ω-background [13, 14]. This background is characterized by two parameters ϵ_1 and ϵ_2, which are related to q and t so that

$$q = e^{(\epsilon_1-\epsilon_2)/2}, \qquad t = e^{-(\epsilon_1+\epsilon_2)/2}, \tag{40}$$

and the special case $\epsilon_1 = -\epsilon_2 = \hbar$ corresponds to the unrefined theory. Parameters ϵ_1 and ϵ_2 are often presented as

$$\epsilon_1 = \sqrt{\beta}\hbar, \qquad \epsilon_2 = -\frac{\hbar}{\sqrt{\beta}}, \tag{41}$$

so that $\beta = -\frac{\epsilon_1}{\epsilon_2}$. In this convention, the unrefined case corresponds to $\beta = 1$.

β-deformation of matrix models. The parameter β also appears in the so-called β-deformed matrix models, in which the Vandermonde determinant in (25) appears in power β

$$Z = \int \prod_{i=1}^{N} d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^{2\beta} \exp\Big(-\frac{1}{\hbar} \sum_{k=1}^{N} V(\lambda_k)\Big). \tag{42}$$

In the unrefined case $\beta = 1$ this model reduces to a hermitian matrix model, while for $\beta = 2$ and $\beta = 1/2$ we get matrix integrals over orthogonal and symplectic matrices. Even though there is no immediate interpretation of Z as an integral over some ensemble of matrices for general β, it is still very interesting to analyze dependence on β in (42). In particular a formalism of the topological recursion [12] can be generalized to the case with arbitrary β.

Matrix models with β deformation play an important role in the so-called AGT conjecture [15]. This very interesting conjecture – presently proved in several cases – states that Nekrasov partition functions reproduce certain amplitudes in the Liouville theory. In [16] Dijkgraaf and Vafa proposed that the relation between Nekrasov partition functions and the Liouville theory can be explained after relating both of them to β-deformed matrix models, and I found explicit realizations of this statement in [H9] (as will be described in more detail in section V.B below).

Homological knot theories. A version of the refinement (i.e. a deformation in parameter t) is a very important aspect of modern knot theory, and it plays an important role in its relations to quantum field theory. In this work I considered such deformations in publications [H1-H3]. Let us recall first the basic relation between quantum field theory and knot theory. As shown by Witten in [4], expectation values of Wilson loops $e^{\oint_K A}$, computed along a knot K in 3-dimensional Chern-Simons theory with $SU(2)$ gauge group and in the fundamental representation $R = \square$, reproduce a famous knot invariant known as the Jones polynomial $J(q)$ for the knot K. In case of $SU(N)$ gauge group and its arbitrary representation R, expectation values of Wilson loops reproduce the so-called colored HOMFLY-PT polynomials $P_R(a,q)$, in which the entire dependence on N arises only through $Q = q^N \equiv a$. Jones and HOMFLY-PT polynomials are well known, classical knot invariants. However in recent years new, the so-called homological knot theories have been introduced, in which the invariants take form of chain complexes of graded vector spaces. Generalization of knot polynomials into such complexes is an example of the so-called categorification. The first example of categorification in knot theory, in which Jones polynomial is lifted to doubly graded homology theories, was found by Khovanov [17]. At present the most general homological knot theories are known to categorify the colored HOMFLY-PT polynomial, and the corresponding homological spaces $\mathcal{H}^R_{ijk}$ are triply graded. One can assign a new polynomial invariant of knots to such spaces, which is known as superpolynomial [18]. The superpolynomial takes form of the Poincare characteristic for $\mathcal{H}^R_{ijk}$

$$P_R(a,q,t) = \sum_{i,j,k} a^i q^j t^k \mathrm{dim}\mathcal{H}^R_{ijk}. \tag{43}$$

For $t = -1$ the above sum reduces to the Euler characteristic, and in this case the superpolynomial reduces to the colored HOMFLY-PT polynomial, $P_R(a,q,-1) = P_R(a,q)$.

It turns out that the above idea of categorification also has an interpretation in string theory, which involves open BPS states described earlier. In this interpretation the homological spaces $\mathcal{H}^R_{ijk}$ are identified with spaces $\mathcal{H}$ of open BPS of D2-branes ending on D4-branes in the conifold geometry. With such an identification, the trace (39) is identified with the superpolynomial (43)

$$Z(x) = P_{S^n}(a,q,t), \tag{44}$$

where we made a choice of symmetric representations $R = S^n$, and $x = q^n$. From this perspective the physical parameter q, related to the generator P of $U(1)$ rotations on a 2-dimensional part of the D4-brane, is identified with the basic parameter q in knot polynomials. Moreover, the physical parameter t, related to the generator F of rotations in the plane normal to D4-brane, is identified with the parameter which appears in the Poincare characteristic (43) for homological knot invariants.

IV.B Deformations related to moduli of Calabi-Yau spaces

Apart from the quantum parameter $\hbar$ (or $q = e^\hbar$), and the refinement parameter t (or, equivalently, the deformation parameter β), Riemann surfaces may also depend on parameters denoted schematically by Q in the trace (39). As we mentioned above, if a given theory is related to type IIA superstring theory, these parameters represent Kähler moduli of Calabi-Yau manifolds. Nonetheless, independently of relations to string theory, parameters Q may also have other interpretations. In Seiberg-Witten theories, and in particular in Nekrasov partition functions, these parameters label moduli of scalar fields [H11,H12]. On the other hand, in 3-dimensional Chern-Simons theories with gauge group $SU(N)$, a single parameter $Q = q^N$ is identified as a variable of knot polynomials (e.g. of HOMFLY-PT polynomial, or the superpolynomial (43)). String-theoretic formulation of knot theory also involves the conifold (38) as a Calabi-Yau space, and the parameter $a \equiv Q$ is identified with the Kähler parameter of the 2-cycle $\mathbb{P}^1$ of the conifold. All these relations play a crucial role in theories considered in this work and discussed in section V.

V Classical and quantum Riemann surfaces in exactly solvable models

In earlier chapters we presented the idea and the formalism of quantization of Riemann surfaces, and discussed the interpretation of deformations of such surfaces in physical theories. In this chapter we discuss explicit realization of these ideas in various models analyzed in [H1-H14], in particular presenting newly found classes of classical and quantum Riemann surfaces.

V.A Riemann surfaces for Seiberg-Witten theories

Seiberg-Witten theory is one of the most important exactly solvable models in high energy theoretical physics [1, 2]. This is $\mathcal{N} = 2$ supersymmetric quantum field theory, in which one can analyze such phenomena as confinement and chiral symmetry breaking as a consequence of condensation of magnetic monopoles, Montonen-Olive electric-magnetic duality, *etc.* Important role in the solution of this theory is played by a Riemann surface known as the Seiberg-Witten curve [1]. By analysis of cycle integrals in an equation analogous to (14), Seiberg and Witten managed to determine leading terms in the expansion of the prepotential F_0 for this theory; the prepotential is an important quantity that captures a lot of information about phenomena mentioned above. By considering a generalization of Seiberg-Witten theory in the Ω-background, Nekrasov derived an exact representation of the partition function Z of such theories, and proved, that in the special case $\epsilon_1 = -\epsilon_2 = \hbar$ this partition function takes the same form as in (10), where F_g play role of gravitational corrections to the field theory [13]. Nekrasov partition functions are expressed in terms of infinite sums over Young diagrams, which represent contributions from instanton configurations [13, 14].

In publications [H11-H14] I analyzed Seiberg-Witten theories in the unrefined case $\epsilon_1 = -\epsilon_2 = \hbar$; these theories are also related to some special cases of results which I found in [H8]. In all those papers I showed that the Seiberg-Witten curves can be identified with spectral curves of matrix models which I found explicitly, and that all amplitudes F_g found by Nekrasov have the same representation as in matrix models (25); in particular, this means that these F_g can be found from the topological recursion (35). The most important results which I found in this context are as follows:

- in [H14] I showed that Seiberg-Witten theory is related by a chain of dualities to a system of D4-branes intersecting D6-branes along a Riemann surface, which takes form of the

Seiberg-Witten curve; massless states of strings stretched between D4- and D6-branes are fermionic degrees of freedom defined on this curve, and the presence of the B-field results in the non-commutative character of this curve and turns it into a quantum curve,

- in [H13], taking advantage of a fermionic representation of closed and open Nekrasov partition functions, I found quantum versions $\widehat{A}(\hat{x}, \hat{y})$ of Seiberg-Witten curves for theories with $SU(N)$ gauge group; in particular I showed that the hierarchies of differential equations associated to such quantum surfaces are related to Meijer-G functions,

- in [H11,H12] I found an explicit form of matrix models, whose partition functions reproduce Nekrasov partition functions, for many different theories: in dimensions 4 and 5 and 6, with supersymmetry of type $\mathcal{N} = 2$ or $\mathcal{N} = 2^*$, in presence of Chern-Simons terms, *etc.*; I also showed that Seiberg-Witten curves can be identified with spectral curves of those matrix models, and in consequence free energies F_g in the expansion of Nekrasov functions arise from the topological recursion (35) and (36) for such models,

- in [H8] I found a matrix model for 5-dimensional Seiberg-Witten theory with $U(1)$ gauge group and I showed, that the spectral curve of this model reproduces the correct Seiberg-Witten curve; this result is a particular case of more general analysis of wall-crossing phenomena in [H8], which are summarized in section V.D.

V.B β-deformed matrix models for Seiberg-Witten theory

Seiberg-Witten theories and Nekrasov functions for general Ω-background, with generic values of ϵ_1 and ϵ_2, are much more complicated than in the case of $\epsilon_1 = -\epsilon_2 = \hbar$. In [H9] I found an explicit representation of such Nekrasov partition functions (with generic ϵ_1, ϵ_2) as β-deformed matrix models of a form (42). This implies, that expansions of such generic Nekrasov partition functions can be found from the β-deformed topological recursion. General arguments for the existence of such β-deformed matrix models for general Ω-background were given by Dijkgraaf and Vafa [16] in their interpretation of the AGT conjecture [15]; thus the results which I found in [H9] can be regarded as a particular realization of such an interpretation.

Independently of the above results, in [H5,H6] I discovered yet another matrix model representation of Nekrasov partition functions for 5-dimensional Seiberg-Witten theories, with generic values of ϵ_1 and ϵ_2. This model arises as a special limit of a more general model, which describes wall-crossing phenomena, as summarized in section V.D. In this case the matrix model is of usual, hermitian type (25), while the dependence on β appears in the matrix model potential $V(M)$ (instead of the Vandermonde determinant). Thus in this case the spectral curve is β-dependent, and free energies F_g satisfy ordinary topological recursions (35) and (36).

V.C Mirror curves

Mirror symmetry is a very important branch of both high energy physics, as well as mathematics [9]. A consequence of mirror symmetry is the fact, that (complex) 3-dimensional Calabi-Yau manifolds exist in pairs, for which two different theories, the so-called A-model and B-model, are defined. Two manifolds in each such pair are different – for example, their Hodge diamonds are related by mirror reflection (hence the name "mirror") – however the A-model and B-model theories are equivalent. In particular, for the A-model theory on one Calabi-Yau manifold Gromov-Witten invariants can be assembled into a generating function Z of the form

(10), which is the same as the partition function in the B-model theory on the mirror manifold (after taking into account the mirror map, which relates Kähler parameters of one manifold to complex parameters of its mirror). The relation of (complex) 3-dimensional Calabi-Yau manifolds to physics is also a consequence of the fact, that in string theory they arise as potential compactifications of 10-dimensional space into 4-dimensional spacetime. In particular one can consider special, 6-(real)-dimensional string theories, known as topological string theories, which are defined precisely for such Calabi-Yau three-folds. Partition functions of such theories are exactly computable and they reproduce Gromov-Witten invariants.

Toric, non-compact spaces constitute an important class of 3-dimensional Calabi-Yau manifolds for A-model theories. Their mirror manifolds for the B-model are defined by the equation

$$zw = A(x, y), \tag{45}$$

where $z, w \in \mathbb{C}$ and $x, y \in \mathbb{C}^*$. Therefore these manifolds are determined by a Riemann surface $A(x, y) = 0$, which is called the "mirror curve". In particular, the mirror curve for the conifold (38) takes form $A(x, y) = 1 + x + y + Qxy$, and in the limit $Q = 0$ it reduces to the mirror curve for $\mathbb{C}^3$ geometry $A(x, y) = 1 + x + y$.

It turns out, that 5-dimensional Seiberg-Witten theories are intimately related to toric Calabi-Yau manifolds – for each such theory one can find a toric three-fold, whose partition function reproduces (after appropriate identification of parameters) the generating function of Gromov-Witten invariants (or, equivalently, topological string amplitudes) [19]. I considered such toric manifolds, related to 5-dimensional Seiberg-Witten theories, in [H9,H11,H12]. In those papers I found matrix models, whose partition functions reproduce topological string amplitudes, and I showed that mirror curves of those toric manifolds can be identified with spectral curves of such matrix models.

Moreover, as mentioned in section IV, toric manifolds arise also in a certain limit of wall-crossing phenomena for BPS states of D-branes. In [H5-H8] I discovered matrix models (to be summarized in the next section) which describe such BPS states, and in particular I showed that in appropriate limit their spectral curves reproduce relevant mirror curves.

Finally, in [H4], I found quantum curves $\widehat{A}(\hat{x}, \hat{y})$ for a broad class of mirror curves, for toric three-folds without compact 4-cycles. I also showed that the exact form of such quantum curves is determined simply by the first correction $\widehat{A}_1$ in (6) – this is an example of the phenomenon described in section II.C. Moreover, in [H4] I also showed, that Gromov-Witten invariants and topological string amplitudes for this class of Calabi-Yau manifolds can be determined by the topological recursions (35) with boundary conditions provided by mirror curves.

V.D "Wall-crossing" phenomena

In section IV we discussed generating functions of BPS states in 4-dimensional field theory, which arise from D2-branes wrapped on 2-cycles of Calabi-Yau manifolds. One can consider also a related system, with an additional D6-brane wrapped on entire Calabi-Yau space. In this case, in an effective theory the so-called closed BPS states appear, which are bound states of a D6-brane with D2-branes and n D0-branes. Such BPS states belong to a Hilbert space $\mathcal{H}$, and their generating functions can be computed as a trace

$$Z = \mathrm{Tr}_{\mathcal{H}} t^P q^\alpha Q^\gamma = \sum_{j,\alpha,\gamma} \Omega(j, n, \gamma) t^j q^n Q^\gamma, \qquad \gamma \in H_2(\text{Calabi-Yau}), \tag{46}$$

where α is an operator counting the number of D0-branes. In the above expression we have already taken into account a more general case of deformed BPS states, which are weighted by an additional parameter t, and P is one generator of the rotation group in 4 dimensions; in our conventions, the unrefined case corresponds to $t = -1$. $\Omega(j, n, \gamma) \in \mathbb{Z}$ denotes the number of BPS states of a given spin, which form a bound state with n D0-branes and γ D2-branes. From general properties of BPS states one can conclude, that the numbers $\Omega(j, n, \gamma)$ should be invariant under deformations of physical fields in the effective theory, or moduli of Calabi-Yau space. This is indeed the case, however – as showed e.g. in [20] – only locally. It turns out that the entire moduli space of Calabi-Yau is divided into chambers, and for each chamber $\Omega(j, n, \gamma)$ is constant. However discrete jumps of $\Omega(j, n, \gamma)$ may take place when walls (of the so-called marginal stability) between chambers are crossed. This phenomenon is referred to as the "wall-crossing". A choice of a chamber is referred to as a choice of stability conditions. Changes of $\Omega(j, n, \gamma)$ are not arbitrary and satisfy certain relations, known as "wall-crossing formulas". These formulas, for a broad class of Calabi-Yau manifolds (not necessarily toric), were derived in [20], and their most general form has been formulated mathematically in [21].

I analyzed the wall-crossing phenomena in publications [H5-H8] (the unrefined case $t = -1$ in [H7,H8], a general value of t in [H6], and in [H5] I summarized all these results). In those papers I found a description of wall-crossing phenomena in terms of matrix models and derived explicit form of such models, I found Riemann surfaces describing these phenomena, and I showed that generating functions (46) are associated to such surfaces via the topological recursion. It should also be stressed, that the free fermions formalism which I introduced in [H10] is an important ingredient in derivation of such matrix models. In [H10] I also provided an interpretation of free fermion amplitudes, in the context of wall-crossing, in terms of crystal models. The formalism of free fermions introduced in [H10] is essential in the analysis of all theories considered in [H5-H8].

In [H8] I showed that for a given Calabi-Yau space, its Kähler parameters, together with parameters which determine chambers in the moduli space, constitute a larger Calabi-Yau manifold, which contains two copies of the original manifold. The B-model theories for these larger spaces are defined via their mirror curves, which I determined as spectral curves of matrix models whose partition functions reproduce (46). Moreover, in the matrix model which I determined in [H8] two copies of the original manifold are glued to each other by the 't Hooft parameter (26). In the limit of an infinite 't Hooft parameter, the two copies of the original manifold become independent, and one of them describes the original Calabi-Yau manifold, and the other one describes stability conditions.

In the space of stability conditions there is one particular chamber, whose choice corresponds to an "ordinary" topological string theory on the original Calabi-Yau manifold. In [H8] I showed, that this chamber is characterized by infinite values of Kähler parameters of the second copy of the original manifold (which encodes stability conditions). Therefore, together with the limit of an infinite 't Hooft parameter, in this chamber the total Calabi-Yau space reduces to the original Calabi-Yau manifold, and the wall-crossing theory reduces to the topological string theory. In particular, matrix models describing wall-crossing phenomena reduce then to matrix models for topological string theories and mirror curves. This is this limit which was mentioned in the earlier sections, in the context of Seiberg-Witten theories with $U(1)$ gauge group.

An interesting class of wall-crossing phenomena analyzed in [H8] is related to the conifold geometry (38). The conifold is characterized by a single Kähler parameter, which determines the size of $\mathbb{P}^1$, as well as a single parameter which determins stability conditions. In [H8] I showed, that together with the 't Hooft parameter, these three parameters define a Calabi-Yau manifold known as the "closed topological vertex". In [H8] I found a matrix model which describes this manifold and I showed, that its spectral curve reproduces the appropriate mirror curve. Moreover, in the limit of an infinite stability condition and an infinite 't Hooft parameter, the matrix model for the "closed topological vertex" reduces to the matrix model for the conifold.

In [H7] I found an interesting interpretation of matrix models discovered in [H8]. Namely, I showed that integrands of such models can be identified with a counterpart of generating functions (46) for open BPS states. Open BPS states arise when an additional D4-brane, wrapping a lagrangian submanifold of the Calabi-Yau space, is considered in a system of D0-D2-D6 branes. Open D2-branes can end on such D4-brane. Therefore generating functions of such open BPS states are analogous to $Z(x)$ in the trace (39), however at present we consider an additional D6-brane wrapping entire Calabi-Yau manifold. Relations of such open BPS states to integrands of matrix models were also illustrated in many examples in [H7].

While in [H7,H8] I analyzed the unrefined case $t = -1$, in [H6] I derived matrix model representations of (46) for refined theories, for generic values of t. In particular I showed that such models are still hermitian (25), and the dependence on t appears only in the potential $V(M)$. Therefore spectral curves which I determined for these models are also refined and depend nontrivially on t. Moreover I also considerd the above mentioned limit of infinite value of stability conditions, which leads to ordinary topological string theories. In this limit the spectral curves which I found reduce to refined mirror curves (and for $t = -1$ they reproduce well known mirror curves). Deriving the form of refined mirror curves in [H6] is an important result, which provides new information about the so-called refined topological string theory, which is still not fully understood.

V.E Riemann surfaces for conformal field theories

Riemann surfaces appear in a description of various 2-dimensional conformal field theories, such as minimal models or the so-called $c = 1$ model. I analyzed such theories and associated Riemann surfaces in [H4]. In particular I found quantum versions of such surfaces $\widehat{A}(\hat{x}, \hat{y})$, and showed that amplitudes in such conformal field theories satisfy the topological recursion (35) with boundary conditions provided by these curves.

V.F Super-A-polynomials

One of the most interesting results of the present work is a discovery of a new class of classical and quantum Riemann surfaces, which I called "super-A-polynomials". I found and analyzed these objects in publications [H1-H3]. Super-A-polynomials are related to two classes of physical theories: refined Chern-Simons theories, and 3-dimensional $\mathcal{N} = 2$ supersymmetric field theories; at the same time these surfaces constitute a new and very interesting class of knot invariants. The super-A-polynomial characterizes properties and can be associated to each knot. In [H1-H3] I determined super-A-polynomials for many knots, in particular for infinite families of torus and twist knots.

Classical super-A-polynomials, denoted $A(x, y; a, t)$, and quantum super-A-polynomials, denoted $\widehat{A}(\hat{x}, \hat{y}; a, q, t)$, depend on all types of deformations discussed in chapter IV. Their refinement (t-deformation) is related to the Poincare characteristic of homological knot invariants (43). Their dependence on a, in string theory interpretation, encodes dependence on the Kähler parameter $a \equiv Q$ of the conifold (38), as described in section IV.B.

Quantum super-A-polynomials are difference operators which annihilate open partition functions $Z(x)$, i.e. $\widehat{A}(\hat{x},\hat{y};a,q,t)Z(x)=0$, which have various interpretations. In the knot theory context, $Z(x)$ are identified with superpolynomials (43), which for $t=-1$ reduce to HOMFLY-PT polynomials. At the same time $Z(x)$ can be regarded as partition functions in refined Chern-Simons theory. In string theory interpretation, $Z(x)$ are identified with generating functions of BPS states on a D4-brane wrapped on a lagrangian submanifold of the conifold (39). $Z(x)$ also have an interpretation of partition functions of 3-dimensional, $\mathcal{N}=2$ supersymmetric theories, which are related to the refined Chern-Simons theory by the so-called 3d-3d duality.

Super-A-polynomials are highly non-trivial generalizations of the so-called A-polynomials, which are well known Riemann surfaces associated (in knot theory) to knots. For ordinary classical and quantum A-polynomials, coefficients a_{ij} in (1) are integers. For super-A-polynomials these coefficients become interesting functions of parameters a and t, whose form is related in a non-trivial way to quantization conditions (22) and (23). According to (23), $\widehat{A}(\hat{x},\hat{y};a,q,t)$ exists if face polynomials for $A(x,y;a,t)$ are tempered. One might suspect that this condition imposes severe constraints on a and t, possibly allowing only some finite set of their values. As I showed in [H1-H3], this problem is solved in a very surprising way – it turns out, that conditions (23) imply, that both a and t should be roots of unity, i.e. of the form $e^{i\phi}$ for $\phi\in\mathbb{Q}$. Therefore their set is dense. Moreover, this result is consistent with the relation $a=q^N$ in Chern-Simons theory; if q is a root of unity – which often follows from other assumptions – then a should also be a root of unity, in agreement with the statement above.

Finally it should be stressed, that A-polynomials are related to the quantization via the topological recursion, in agreement with (37). In particular, asymptotic expansion of colored Jones polynomials turns out to take form of an open partition function (9), for which S_k are determined via the topological recursion. This hypothesis was formulated and confirmed in detailed calculations (which, however, included some discrepances) in [22]. These discrepancies were further explained in [23]. On the other hand, in [H4] I explained how quantization via the topological recursion can be used to determine the form of quantum A-polynomial from the knowledge of the classical A-polynomial, in agreement with the main result of this work (37).

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5. Description of other scientific achievements

A) Grants and research projects

1. Project type: **ERC Starting Grant** (awarded budget: 1 345 080 EUR)
Project title: **Quantum fields and knot homologies**
Place and starting year of realization: **Faculty of Physics, University of Warsaw (2013)**
Funding agency: **European Research Council (ERC)**
Role in the project: **Principal Investigator**

2. Project type: **Iuventus Plus**
Project title: **Matrix models, effective geometries, and Schroedinger equations**
Place and starting year of realization: **Faculty of Physics, University of Warsaw (2013)**
Funding agency: **Ministry of Science and Higher Education (MNiSW)**
Role in the project: **Principal Investigator**

3. Project type: **Interdisciplinary project „Skills/Inter"**
Project title: **On topology, interacting RNA, and quantum physics**
Place and starting year of realization: **Faculty of Physics, University of Warsaw (2013)**
Funding agency: **Foundation for Polish Science**
Role in the project: **Principal Investigator**

4. Project type: **Grant Homing-Plus**
Project title: **Topological recursion and exact results in quantum theories**
Place and starting year of realization **Faculty of Physics, University of Warsaw**, in collaboration with **California Institute of Technology (2012)**
Funding agency: **Foundation for Polish Science**
Role in the project: **Principal Investigator**

5. Project type: **Marie-Curie Research Grant (International Outgoing Fellowship)**
Project title: **Black Holes, BPS states, and topological string theory**
Place and period of realization: **California Institute of Technology** and **University of Amsterdam (2009-2013)**
Funding agency: **European Commission (under the 7th Framework Programme)**
Role in the project: **Principal Investigator**

6. Project type: **Humboldt Research Fellowship**
Project title: **Physical aspects of the topological string theory**
Place and period of realization: **University of Bonn, Germany (2007-2009)**
Funding agency **Humboldt Foundation (Germany)**
Role in the project: **Principal Investigator**

7. Project type: **Promotor grant ("Grant promotorski"), number N202 004 31/0060**
Project title: **Calabi-Yau crystals in topological string theory**
Place and period of realization: **Faculty of Physics, University of Warsaw (2006-2007)**
Funding agency: **Ministry of Science and Higher Education (MNiSW)**
Role in the project: **main investigator**

B) Awards

1. Name and year of the award: **Award/stipend for outstanding young scientists**
Year of award: **2011**
Awarding/funding institution: **Ministry of Science and Higher Education (MNiSW)**
To whom the award is given: **„for outstanding young scientists"**

2. Name and year of the award: **Fulbright Fellowship, Senior Advanced Research Grant**
Year of award: **2009**
Awarding/funding institution **Fulbright Commission**
To whom the award is given: **research grant** – I had to decline to accept this grant, because the same year I was awarded Marie-Curie grant, and simultaneous realization of both these grants was not possible for formal reasons

3. Name and year of the award: **Research stay at Harvard University**
Year of award: **2009**
Awarding/funding institution: **Foundation for Polish Science**
To whom the award is given: **„for outstanding young researchers"**

4. Name and year of the award: **Award/stipend for young scientists "Start"**
Year of award: **2009**
Awarding/funding institution: **Foundation for Polish Science**
To whom the award is given: **„for outstanding young researchers"**

5. Name and year of the award: **Maria Bardadin-Otwinowska award**
Year of award: **2003**
Awarding/funding institution: **Faculty of Physics, University of Warsaw**
To whom the award is given: **for original MSc. thesis**

C) Bibliometric data

Total Impact Factor: **138.354**

Total number of citations according to Web of Science data base (however, this data base does not find all citations to my publications in high energy physics): **256**

Total number of citations, including citations in high energy physics from „INSPIRE-HEP" data base (the official data base in high energy physics), and citations in biophysics from Web of Science data base: **570**

Hirsch index (including citations in high energy physics from „INSPIRE-HEP" data base, and citations in biophysics from Web of Science data base): **17**

The number of all publications: **32**
The number of articles published in journals from Journal Citation Report (JCR) data base: **28**
The number of publications as a „corresponding author": **23**
The number of publications as a single author: **9**

D) A list and description of other publications

Publications in high energy physics and mathematical physics

1. Jorgen Andersen, Leonid Chekhov, Robert Penner, Christian Reidys, Piotr Sułkowski,
"*Topological recursion for chord diagrams, RNA complexes, and cells in moduli spaces*",
Nucl. Phys. B866 (2013) 414-443, arXiv: 1205.0658 [hep-th].

2. Vincent Bouchard, Andrei Catuneau, Oliver Marchal, Piotr Sułkowski,
"*The remodeling conjecture and the Faber-Pandharipande formula*",
Lett. Math. Phys. 103, 1 (2013) 59-77, arXiv: 1108.2689 [math.AG].

3. Vincent Bouchard, Piotr Sułkowski,
"*Topological recursion and mirror curves*",
Adv. Theor. Math. Phys. 16, 5 (2012) 1443-1483, arXiv: 1105.2052 [hep-th].

4. Piotr Sułkowski,
"*Deformed boson-fermion correspondence, Q-bosons, and topological strings on the conifold*",
JHEP 0810 (2008) 104, pages 1-16, arXiv: 0808.2327 [hep-th].

5. Robbert Dijkgraaf, Piotr Sułkowski,
***"Instantons on ALE spaces and orbifold partitions*",**
JHEP 0803 (2008) 013, pages 1-23, arXiv: 0712.1427 [hep-th].

6. Piotr Sułkowski,
***"Crystal Model for the Closed Topological Vertex Geometry*",**
JHEP 0612 (2006) 030, pages 1-20, arXiv: hep-th/0606055.

7. Nicholas Halmagyi, Annamaria Sinkovics, Piotr Sułkowski,
"Knot invariants and Calabi-Yau crystals",
JHEP 0601 (2006) 040, pages 1-31, arXiv: hep-th/0506230.

8. Piotr Sułkowski,
"On Tachyon Potential in Boundary String Field Theory and Problems with Boundary Fermions",
Acta Phys. Polon. B34 (2003) 4167-4183.

Publications in biophysics

9. Jorgen Andersen, Leonid Chekhov, Robert Penner, Christian Reidys, Piotr Sułkowski,
"*Enumeration of RNA complexes via random matrix theory*",
Biochemical Society Transactions 41 (2013) 652-655, arXiv: 1303.1326.

10. Joanna I. Sułkowska, Piotr Sułkowski, Piotr Szymczak, Marek Cieplak,
"Untying knots in proteins",
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12. Joanna I. Sułkowska, Piotr Sułkowski, Jose N. Onuchic,
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14. Joanna I. Sułkowska, Piotr Sułkowski, Piotr Szymczak, Marek Cieplak,
"Tightening of knots in proteins",
Phys. Rev. Lett. 100 (2008) 058106, pages 1-4, arXiv: 0706.2380 [q-bio].

<u>Monographs, conference proceedings, and other publications</u>

15. Sergei Gukov, Piotr Sułkowski,
"*A-polynomial, B-model, and Quantization",*
Springer, Lecture Notes in Mathematics, pages 1-60, accepted for publication (to appear in 2013).

16. Hiroyuki Fuji, Piotr Sułkowski,
"Super-A-polynomial",
Proceedings of String-Math conference, Bonn (2012), arXiv: 1303.3709 [math.AG].

17. Motohico Mulase, Piotr Sułkowski,
"Spectral curves and the Schroedinger equations for the Eynard-Orantin recursion",
arXiv: 1210.3006 [math-ph].

18. Joanna I. Sułkowska, Piotr Sułkowski, Piotr Szymczak, Marek Cieplak,
"Stretching the knotted protein YibK and its unknotted constructs",
Proceedings of the conference on "Knots and soft-matter physics", Kyoto University, Japan, 2009.

Description of achievements presented in the above publications

My other achievements and scientific results are presented in publications [1-18] listed above (in pages 31-32). Several of those publications ([1-8] and [15-17]), whose results we summarize first, are devoted to problems in mathematical physics, high energy physics, and string theory. Other publications ([9-14] and [18]), which we summarize afterwards, present the results of my research in a different branch of physics, namely biophysics.

Achievements in mathematical physics, high energy physics, and string theory

The results which I found, related to mathematical physics, high energy physics, and string theory, belong to several separate groups of topics presented below.

Publication [8] is concerned with properties of tachyons in string theory, and relates to problems which I analyzed during my master (MSc) studies. In this paper I determined the potential for tachyonic degrees of freedom in the system of D1-brane and anti-D1-brane in type I superstring theory, and analyzed the process of tachyon condensation.

Publications [6,7] are devoted to the so-called crystal models, which arise in the context of topological string theory on toric Calabi-Yau manifolds. In [7] I described models which I found for various configurations of D-branes on such toric manifolds, and I showed their relations to the theory of the topological vertex, as well as to knot invariants which are related to topological amplitudes for such D-branes. On the other hand, in [6] I discovered a very interesting crystal model for the manifold known as the "closed topological vertex" (which is a certain deformation of $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity). Crystal models presented in [6,7] are 3-dimensional, as they are related to Calabi-Yau manifolds of (complex) dimension 3 (i.e. real dimension 6). Moreover, in [5] I discovered 2-dimensional crystal models, which describe supersymmetric Yang-Mills quantum field theories on (real) 4-dimensional ALE ("Asymptotically Locally Euclidean") spaces. These theories can also be derived from superstring theory with D-branes wrapped on the torus geometry. Publications [5,6,7] are related to my research during PhD studies, and their results are also presented in my PhD thesis. It should be stressed, that some of those results were found in collaboration with Prof. Robbert Dijkgraaf (*"Distinguished Professor"* at the University of Amsterdam, currently also a director of the *Institute for Advanced Study* (IAS) in Princeton). The collaboration with Prof. Dijkgraaf and the University of Amsterdam was established from my own initiative, and it had crucial (and very fruitful) impact on my PhD studies and the scientific development. Prof. Robbert Dijkgraaf, together with Prof. Jacek Pawełczyk (University of Warsaw), became coadvisors of my PhD studies.

In the work [4] I discovered connections and proved the equivalence of the following three models: a certain deformation of the fermion-boson correspondence in two dimensions, the so-called Q-bosons, and topological string theory in the conifold geometry.

Papers [2,3] are devoted to an important hypothesis formulated by Marino and the Bouchard-Klemm-Marino-Pasquetti collaboration (*"Remodeling the B-model"*, Commun. Math. Phys. 287 (2009) 117-178), which is also known as the "remodeling conjecture". This hypothesis states, that Gromov-Witten invariants for toric Calabi-Yau manifolds satisfy a system of topological recursions. This conjecture was analyzed and proved in several particular cases, however it had not been analyzed before whether it is consistent with the behavior of constant (i.e. independent of Kähler moduli) contributions to Gromov-Witten invariants. In [3] I generalized this conjecture to such cases, and showed by explicit computations to a certain high order in the topological expansion, for several Calabi-Yau manifolds, that constant contributions indeed obey such generalized conjecture. Moreover, in [2] I found an analytic proof (to all orders in the topological expansion) of this conjecture for $\mathbb{C}^3$ manifold.

In [1] I discovered a new matrix model, which describes chord diagrams, and – independently – moduli spaces for Riemann surfaces with boundaries. In particular free energies of this model reproduce the numbers of various classes of chord diagrams, as well as the numbers of cells in the cell decomposition of Riemann moduli spaces. In [1] I derived many explicit examples of generating functions of such numbers, using the tools of random matrix theory and the formalism of the topological recursion.

In [17] I conducted an analysis of two combinatorial models, related respectively to the generalized Catalan number (and, at the same time, to $c = 1$ model), as well as the generalized Hurwitz numbers. In this paper I found the Schroedinger equation which is satisfied by the generating functions of these generalized numbers, and I proved that it holds to all orders in the quantum expansion.

The paper [16] presents my discovery of super-A-polynomials, and at the same time this paper is a summary of my talk at the String-Math conference in Bonn in 2012.

The paper [15], accepted as *Lecture Notes in Mathematics* in *Springer*, presents an approach to quantization based on the B-model theory. At the same time it is a summary of my talk at the conference *Mirror symmetry and tropical geometry* in Cetraro (Italy) in 2011.

Achievements related to biophysics

Independently of my research in mathematical physics and high energy physics, my scientific activities are also related to biophysics. My achievements in this context are devoted to two sets of topics: the topological properties of proteins, and the classification of RNA structures.

My papers [10-14] and [18] are devoted to the analysis of topological properties of proteins. I analyzed these properties in various contexts, including both stretching and folding of proteins. The main question in the context of proteins' topology is whether their backbone can be knotted in its native conformation. Until recently (i.e. around the end of 20'th century) it was believed, that the process of creation of knotted proteins would be too complicated and such proteins cannot exist. However, extensive searches of the PDB data base (i.e. Protein Data Bank, which contains crystallographic data about native configurations of proteins) revealed, that many such proteins exist – currently several hundreds of such knotted structures have been identified. In my studies I analyzed various properties of such proteins, based most of all on the results of computer simulations.

Even though knots in proteins are directly related to the branch of mathematics known as knot theory, it should be stressed, that knots in proteins arise on open chains. From mathematical viewpoint such object are not properly defined – in mathematical knot theory only knots on closed chains can be considered. Nonetheless, as long as the termini of a protein are located significantly outside its knotted structure, it is usually possible to connect them in a unique way and turn the protein backbone into a closed curve, for which a type of a knot can be uniquely determined. Knots in proteins, which I analyzed in papers listed above, are considered in such sense, after appropriate closing of the protein backbone.

The results which I found, related to the topological properties of proteins, are as follows. In [14] I showed, that the process of stretching of knotted proteins is significantly different that stretching of knotted, uniform polymers, and it consists of a series of jumps – so that locations of such jumps are correlated with sharp turns in the protein backbone. In [13], by analyzing two similar proteins, one knotted and another one unknotted (as well as a theoretical unknotted construction, obtained by removing a knot from a knotted protein upon a small change in geometry of its backbone), I showed, that the presence of a knot increases stability and may affect functions of proteins. These results are also presented in publication [18]. In [12] I theoretically predicted a folding pathway which leads to knotted native configurations of YibK and YbeA proteins, and I presented via computer simulations that such a pathway may indeed arise. In [11] I considered the so-called slipknot configurations; these are topologically trivial configurations, which however contain some nontrivial loop (which can be tightened). In this paper I showed how tightening or untying of the slipknot's loop depends on stretching conditions; I also presented that such stretching is a two-state process, in which a tightened slipknot may arise as an intermediate state. Finally, in [10] I analyzed how knotted proteins can be untied via stretching in various directions and by choosing different amino acids as stretching points. It should also be stressed, that in recent years analysis of topology of proteins has become an independent subfield at the border of biophysics, biochemistry, mathematics, and related fields, which involves both theoretical and experimental aspects – nowadays more and more papers devoted to these topics are published, special conferences are organized, *etc.* I believe that my research has had some impact on the development of this discipline.

The second group of biophysics problems which I analyzed concerns the classification of interacting RNA chains. Analysis of RNA structures is a vast topic, in which it is very useful to introduce various classifications of such structures. The number of RNA chains, as well the number of hydrogen bonds (e.g. associated to Watson-Crick pairs) they contain, are two obvious parameters one should consider in any classification. In addition, to each such configuration one can uniquely associate one more parameter, i.e. the genus of a surface associated to this configuration after representing it as a chord diagram. It turns out, that chord diagrams in this problem are the same, as diagrams which I considered in [1]. In order to enumerate configurations of RNA with a given number of chains, hydrogen bonds, and of a given genus, one can use the matrix model and generating functions which I determined in [1]. These results, reformulated and interpreted in the context of RNA, I analyzed and presented in publication [9].

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