LECTURE 3

Examples of atomic BEC
Quasi-particles in solid
Light-matter coupling
Exciton-polaritons

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EXPERIMENTAL PROCEDURE

1. Atoms are released from a source. $N = 10^{10}$ atoms.

2. Atoms are trapped in a magneto-optical trap. (MOT)

3. Doppler cooling. $N = 10^9$ atoms. $T = 100$ mikroK.

4. Atoms are trapped in a magnetic trap.

5. Evaporative cooling. $N = 10^7$ atoms. $T = 100$ nanoK.
**HISTORY**

Experimental results

2D velocity distributions

- Produced in vapor of $^{87}$Rb atoms
- Fraction of condensed atoms first appear near $T = 170$ nK & $n = 2.5 \cdot 10^{12}$ cm$^{-3}$
- Could be preserved for more than 15 seconds
- BEC on top of broad thermal velocity
- Fraction of atoms that were in this low-velocity peak increases abruptly
- Non-thermal, anisotropic velocity distribution expected of minimum-energy quantum state of magnetic trap

*Velocity distribution of gas of Rb atoms.*


E. Cornell et al., JILA, 1995

![Bose-Einstein Condensation of Rb 87](http://www.colorado.edu/physics/2000/bec/images/evap2.gif)
Expanding cloud cooled to just above the transition point; middle: just after the condensate appeared; right: after further evaporative cooling has left an almost pure condensate.

The "sharp peak" is the Bose-Einstein condensate, characterized by its slow expansion observed after 6 msec time of flight.

The width of the images is 1.0 mm. The total number of atoms at the phase transition is about 700,000, the temperature at the transition point is 2 microkelvin.

W. Ketterle et al. MIT, 1995

http://cua.mit.edu/ketterle_group/Popular_papers/Ultralow_temperatures.htm
Bose-Einstein Condensation of Strontium

Simon Stellmer, Meng Khoon Tey, Bo Huang, Rudolf Grimm, and Florian Schreck
Phys. Rev. Lett. 103, 200401 – Published 9 November 2009

Absorption images and integrated density profiles showing the BEC phase transition for different times of the evaporative cooling ramp.

The images are along the vertical direction, 25 ms after release from the trap. The solid line represents a fit with a bimodal distribution, while the dashed line shows the Gaussian-shaped thermal part, from which the given temperature values are derived.
Bose-Einstein Condensation of Strontium

Simon Stellmer, Meng Khoon Tey, Bo Huang, Rudolf Grimm, and Florian Schreck
Phys. Rev. Lett. 103, 200401 – Published 9 November 2009

Inversion of the aspect ratio during the expansion of a pure BEC. The images field of view 250 μm×250 μm. The first image is an in situ image recorded at the time of release. The further images are taken 5, 10, 15, and 20 ms after release.
ABSORPTION IMAGING

- Switching off trap ⇒ condensate falling down (gravity) and ballistically expands
- Illuminating atoms with nearly resonant laser beam and imaging shadow cast on charge-coupled device camera (CCD-camera)
- Cloud heats up by absorbing photons (about one recoil energy per photon)
- Single destructive image
- Provides reliable density distributions of which properties of condensates and thermal clouds can be inferred

Stefan Kienzle
Technische Universitat Munchen
**ABSORPTION IMAGING**

- 2D probe absorption images after 6 ms time of flight. Width of images is 870 μm.
- Velocity distribution of cloud just above transition point.
- Shows difference between isotropic thermal distribution and elliptical core attributed to expansion of dense condensate.
- Almost pure condensate (after further evaporative cooling).

Stefan Kienzle
Technische Universitat Munchen
ORDER PARAMETER

\[ \psi(\vec{r},t) = \sqrt{n(\vec{r},t)} \ e^{i\theta(\vec{r},t)} \]

Phase coherence!!!

**spatial and temporal:**

- Coherence in time
- Long-range order = spatial coherence

during the phase transition

**T=0:**
Pure Bose condensate
"Giant matter wave"

interference !!
INTERFERENCE BETWEEN TWO BEC

Evidence for coherence of BEC’s

Cut atom trap in half (double-well potential) by focusing far-off-resonant laser light into center of magnetic trap

Cool atoms in these two halves to form two independent condensates

Quickly turn off laser and magnetic fields, allowing atoms to fall and expand freely

Both condensates start to overlap and interfere with each other
**Interference between two atomic BEC**


! Interference of a matter wave !
Quasi-particles in solid
Light-matter coupling
<table>
<thead>
<tr>
<th>Particle</th>
<th>Composed of</th>
<th>In</th>
<th>Coherence seen in</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooper pair</td>
<td>$e^-, e^-$</td>
<td>metals</td>
<td>superconductivity</td>
</tr>
<tr>
<td>Cooper pair</td>
<td>$h^+, h^+$</td>
<td>copper oxides</td>
<td>high-$T_c$, superconductivity</td>
</tr>
<tr>
<td>exciton</td>
<td>$e^-, h^+$</td>
<td>semiconductors</td>
<td>luminescence and drag-free transport in Cu$_2$O</td>
</tr>
<tr>
<td>biexciton</td>
<td>$2(e^-, h^+)$</td>
<td>semiconductors</td>
<td>luminescence and optical phase coherence in CuCl</td>
</tr>
<tr>
<td>positronium</td>
<td>$e^-, e^+$</td>
<td>crystal vacancies</td>
<td>(proposed)</td>
</tr>
<tr>
<td>hydrogen</td>
<td>$e^-, p^+$</td>
<td>magnetic traps</td>
<td>(in progress)</td>
</tr>
<tr>
<td>$^4$He</td>
<td>$^4$He$_{2^+}, 2e^-$</td>
<td>He-II</td>
<td>superfluidity</td>
</tr>
<tr>
<td>$^3$He pairs</td>
<td>$2(3^2$He$_{2^+}, 2e^-)$</td>
<td>$^3$He-A,B phases</td>
<td>superfluidity</td>
</tr>
<tr>
<td>cesium</td>
<td>$^{133}$Cs$^{55+}, 55e^-$</td>
<td>laser traps</td>
<td>(in progress)</td>
</tr>
<tr>
<td>interacting bosons</td>
<td>$nn$ or $pp$</td>
<td>nuclei</td>
<td>excitations</td>
</tr>
<tr>
<td>nucleonic pairing</td>
<td>$nn$ or $pp$</td>
<td>nuclei neutron stars</td>
<td>moments of inertia, superfluidity and pulsar glitches</td>
</tr>
<tr>
<td>chiral condensates</td>
<td>$\langle\bar{q}q\rangle$</td>
<td>vacuum</td>
<td>elementary particle structure</td>
</tr>
<tr>
<td>meson condensates</td>
<td>pion condensate $= (ud)$, etc.</td>
<td>neutron star matter</td>
<td>neutron stars, supernovae (proposed)</td>
</tr>
<tr>
<td></td>
<td>kaon condensate $= (us)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Higgs boson</td>
<td>$\langle\bar{t}t\rangle$ condensate</td>
<td>vacuum</td>
<td>elementary particle masses</td>
</tr>
</tbody>
</table>
The probability of filling the quantum state of the energy $E$ 
$E_F$ – chemical potential

**Fermions:** \[ f_0 = \frac{1}{e^{\frac{E-E_F}{k_BT}} + 1} \]

**Bosons:** \[ f_0 = \frac{1}{e^{\frac{E-E_F}{k_BT}} - 1} \]

**Boltzmann distribution:** \[ f_0 = \frac{1}{e^{\frac{E-E_F}{k_BT}} \pm 1} \approx e^{\frac{E-E_F}{k_BT}} \]

Electrons
Holes
Trions (charged excitons)
Polaritons
Phonons
Magnons
Excitons, biexcitons
Plazmons

Anyons – np. composite fermions \[ |\psi_1 \psi_2\rangle = e^{i\theta} |\psi_2 \psi_1\rangle \]
Slave fermions (chargon, holon, spinon) = fermion+bozon in spin-charge separation

\[ E_F = \frac{\partial F}{\partial n_i} \]
\[ F = U - TS \]
Few remarks up to now

Facts:
• crystals have highly ordered microscopic structure
• in consequence, energy bands are formed
• properties (electrical, optical, ...) are determined by electrons distributed over the bands
• electrons are fermions and obey Fermi-Dirac distribution
Optical properties of semiconductors

Optical absorption spectra are governed by the density of electronic states in the valence and conduction bands.

\[ g(E) = \frac{\partial n}{\partial E} \]

- \( n \) - number of quantum states per unit area

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Number Density Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-D (bulk)</td>
<td>[ g(E) = \frac{1}{2\pi^2} \left( \frac{2m^*}{\hbar^2} \right)^{3/2} \sqrt{E_g - E} ]</td>
</tr>
<tr>
<td>2-D (slab)</td>
<td>[ g(E) = \frac{m^*}{\pi\hbar^2} \sigma(E_g - E) ]</td>
</tr>
<tr>
<td>1-D (wire)</td>
<td>[ g(E) = \frac{m^<em>}{\pi\hbar} \sqrt{\frac{m^</em>}{m}} ]</td>
</tr>
<tr>
<td>0-D (dot)</td>
<td>[ g(E) = 2\delta(E_g - E) ]</td>
</tr>
</tbody>
</table>

\( \sigma(E_g - E) \) is a constant representing the optical absorption coefficient.
Optical properties of semiconductors

Excitons in bulk

Absorption spectra in semiconductors (at low temperature) exhibit sharp peaks below the edge of the inter-band absorption.


Manifestation of resonant light - matter coupling in semiconductor.
An exciton is a bound state of an electron and an imaginary particle called an electron hole in an insulator or semiconductor.

- The overall charge for this quasiparticle is zero.
- It carries no electric current.
- ! It is a composite BOSON !
Excitons - 3D

Consider an electron-hole pair bound by the coulomb interactions:

\[-\frac{\hbar^2}{2\mu} \nabla^2 f(r) - \frac{\varepsilon^2}{4\pi\varepsilon_0 r} f(r) = Ef(r)\]

- dielectric constant of a crystal

- effective mass: \(\mu = m_e m_h / (m_e + m_h)\)

- e-h distance: \(r = \sqrt{x^2 + y^2 + z^2}\)

Equation is analogous to Schrodinger equation for a hydrogen atom with the following renormalisations:

- \(m_0 \rightarrow \mu, \quad \varepsilon^2 \rightarrow \varepsilon^2 / \varepsilon\)
Excitons - 3D

Consider an electron-hole pair bound by the coulomb interactions:

\[ -\frac{\hbar^2}{2\mu} \nabla^2 f(r) - \frac{e^2}{4\pi \varepsilon \varepsilon_0 r} f(r) = E f(r) \]

Dielectric constant of a crystal

Bohr radius:

\[ a_B = \frac{4\pi \hbar^2 \varepsilon \varepsilon_0}{\mu e^2} \]

Binding energy of a ground state:

\[ E_B = \frac{\mu e^4}{(4\pi)^2 2\hbar^2 \varepsilon \varepsilon_0^2} = \frac{\hbar^2}{2\mu a_B^2} \]

Wavefunction of the 1s state:

\[ f_{1s} = \frac{1}{\sqrt{\pi a_B^3}} e^{-r/a_B} \]
<table>
<thead>
<tr>
<th>Semiconductor crystal</th>
<th>$E_g$ (eV)</th>
<th>$m_e/m_0$</th>
<th>$E_B$ (eV)</th>
<th>$a_B$ (Å)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PbTe*</td>
<td>0.17</td>
<td>0.024/0.26</td>
<td>0.01</td>
<td>17 000</td>
</tr>
<tr>
<td>InSb</td>
<td>0.237</td>
<td>0.014</td>
<td>0.5</td>
<td>860</td>
</tr>
<tr>
<td>Cd$<em>{0.3}$Hg$</em>{0.7}$Te</td>
<td>0.257</td>
<td>0.022</td>
<td>0.7</td>
<td>640**</td>
</tr>
<tr>
<td>Ge</td>
<td>0.89</td>
<td>0.038</td>
<td>1.4</td>
<td>360</td>
</tr>
<tr>
<td>GaAs</td>
<td>1.519</td>
<td>0.066</td>
<td>4.1</td>
<td>150</td>
</tr>
<tr>
<td>InP</td>
<td>1.423</td>
<td>0.078</td>
<td>5.0</td>
<td>140</td>
</tr>
<tr>
<td>CdTe</td>
<td>1.606</td>
<td>0.089</td>
<td>10.6</td>
<td>80</td>
</tr>
<tr>
<td>ZnSe</td>
<td>2.82</td>
<td>0.13</td>
<td>20.4</td>
<td>60</td>
</tr>
<tr>
<td>GaN***</td>
<td>3.51</td>
<td>0.13</td>
<td>22.7</td>
<td>40</td>
</tr>
<tr>
<td>Cu$_2$O</td>
<td>2.172</td>
<td>0.96</td>
<td>97.2</td>
<td>38****</td>
</tr>
<tr>
<td>SnO$_2$</td>
<td>3.596</td>
<td>0.33</td>
<td>32.3</td>
<td>86****</td>
</tr>
</tbody>
</table>

Table 4.2 Strongly anisotropic conduction and valence bands, direct transitions far from the centre of the Brillouin zone.
* Strongly anisotropic conduction and valence bands, direct transitions far from the centre of the Brillouin zone.
** In the presence of a magnetic field of 5 T.
*** An exciton in hexagonal GaN.
**** The ground-state corresponds to an optically forbidden transition, data given for $n = 2$ state.
Excitons - 3D

\[ E_n = -\left(\frac{m^*}{m_0}\right) \frac{1}{\varepsilon^2} \frac{1}{\text{Ry}} \frac{1}{n^2} \]

**Giant Rydberg excitons in the copper oxide Cu\textsubscript{2}O**

Excitons - 2D (in quantum well)

The Schrödinger equation for an exciton in a quantum well (QW) reads:

\[
\left( -\frac{\hbar^2}{2m_e} \nabla_e^2 - \frac{\hbar^2}{2m_h} \nabla_h^2 + V_e(z_e) + V_h(z_h) - \frac{e^2}{4\pi \varepsilon \varepsilon_0 |r_e - r_h|} \right) \Psi = E \Psi
\]

Solutions are again similar to 2D hydrogen atom:

Bohr radius:

\[ a_{2D}^B = \frac{a_B}{2} \]

Binding energy of a ground state:

\[ E_{2D}^B = 4E_B \]

Wavefunction of the 1s state:

\[ f_{1S}(\rho) = \sqrt{\frac{2}{\pi}} \frac{1}{a_{2D}^B} \exp(-\rho/a_{2D}^B) \]

Energies of the excited states:

\[ E_n = -\frac{Ry^*}{(n - \frac{1}{2})^2} \]
Excitons - 2D (in quantum well)
Optical properties of semiconductors

Excitons in quantum well

Linear absorption spectrum of the Ge multiple quantum well structure and a schematic sketch of the electronic structure.

N. S. Köster et al., New Journal of Physics, Volume 15, July 2013
Exciton family

- **Exciton**
- **Biexciton**
- **Trion - exciton positively charged**
- **Trion - exciton negatively charged**

Exciton bound on donor or acceptor
Exciton condensation
particle mass significance

\[ T \sim \frac{(2\pi \hbar)^2}{2m k_B} n^{2/3} \]

Light mass implies Bose-Einstein effects at higher temperature!

<table>
<thead>
<tr>
<th></th>
<th>atoms</th>
<th>EXCITONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>Rb: (10^4 m_e)</td>
<td>(10^{-2} m_e)</td>
</tr>
<tr>
<td>( T_C )</td>
<td>(10^{-7} K)</td>
<td>(\sim 4 K) possible</td>
</tr>
<tr>
<td>( n )</td>
<td>(10^{14}/cm^3)</td>
<td>limit: (10^{17}/cm^3 ) or (10^{11}/cm^2)</td>
</tr>
<tr>
<td>lifetime</td>
<td>(\infty)</td>
<td>typically (\sim) 100 ns up to 1 ms in specially designed samples</td>
</tr>
</tbody>
</table>
Exciton condensation
particle mass significance

\[ T \sim \frac{(2\pi \hbar)^2}{2mk_B} n^{2/3} \]

Light mass implies Bose-Einstein effects at higher temperature!

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<thead>
<tr>
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<th>atoms</th>
<th>EXCITONS</th>
<th>EXCITON POLARITONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>Rb: (10^4m_e)</td>
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</tr>
<tr>
<td>( T_C )</td>
<td>(10^{-7})K</td>
<td>(\sim 4) K possible</td>
<td>RT possible</td>
</tr>
</tbody>
</table>
| \( n \) | \(10^{14}/\text{cm}^3\) | limit:
\(10^{17}/\text{cm}^3\) or \(10^{11}/\text{cm}^2\) | \(<10^{11}/\text{cm}^2\) |
| lifetime | \(\infty\) | typically \(\sim 100\) ns up to 1 ms in specially designed samples | 10 ps |

EXCITON POLARITON = EXCITON + PHOTON
SHORT HISTORY OF LIGHT-MATTER COUPLING IN SEMICONDUCTORS

1951 - Huang: interaction between the electromagnetic field with the crystal lattice excitation
(Maxwell equations + classical lattice vibrations)

1956 - Fano & 1958 Hopfield

interaction between the electromagnetic field with excitons (quantum approach)

polaritons - coupled modes of electromagnetic waves and any excitation propagating in the material with complex dielectric function

phonon polaritons
exciton polaritons
magnon polaritons
plasmon polaritons
<table>
<thead>
<tr>
<th>Light-matter interaction</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>spontaneous emission</strong></td>
</tr>
<tr>
<td>emission to the space of infinite number of modes</td>
</tr>
<tr>
<td><strong>weak coupling</strong></td>
</tr>
<tr>
<td>emissions in well-defined mode, but the space is open (dissipation, decoherence, cavity losses) reabsorption process is not possible</td>
</tr>
<tr>
<td><strong>strong coupling</strong></td>
</tr>
<tr>
<td>emission to well defined mode with strong reabsorption process</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>----------------------</td>
</tr>
<tr>
<td><strong>perturbation theory with Fermi Golden rule</strong></td>
</tr>
<tr>
<td><strong>normal modes propagating through the crystal</strong></td>
</tr>
<tr>
<td><strong>Purcell effect</strong></td>
</tr>
<tr>
<td><strong>eigen states</strong></td>
</tr>
<tr>
<td>$\tau_C$ - exciton lifetime</td>
</tr>
<tr>
<td>$\tau_X$ - photon lifetime</td>
</tr>
</tbody>
</table>
Light-matter interaction

two-levels in an external field

- $|1\rangle$ and $|2\rangle$ form an orthonormal basis for the system \(i.e. \langle i|j\rangle = \delta_{ij}\) for \(i, j = 1, 2\);
- Photon frequency: \(\omega = \omega_0 + \Delta\);
- Detuning: \(\Delta\);
- Resonant frequency: \(\omega_0\).

The bare Hamiltonian:

\[
H_0 = \hbar \begin{pmatrix}
0 & 0 \\
0 & \omega_0
\end{pmatrix}
\]

\[
\hat{H}_0|1\rangle = 0|1\rangle,
\]

\[
\hat{H}_0|2\rangle = \hbar \omega_0|2\rangle
\]
Light-matter interaction
two-levels in an external field

\[
\psi = \begin{pmatrix} \langle 1 | \psi \rangle \\ \langle 2 | \psi \rangle \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}
\]

Any two-level quantum state can be expressed as \( |\psi\rangle = c_1 |1\rangle + c_2 |2\rangle \), where \( c_1 \) and \( c_2 \) are complex state amplitudes and \( |c_1|^2 + |c_2|^2 = 1 \). Such a state can be represented by a two-component vector;

The probability of finding the system in state \( |i\rangle \) is \( |\langle i | \psi \rangle|^2 = |c_i|^2 \), (for \( i = 1, 2 \)).
Light-matter interaction

two-levels in an external field

What about the driving field? What does it do? It induces a dipole (electric or magnetic) moment between the states $|1\rangle$ and $|2\rangle$. The electromagnetic field interacts with this dipole, resulting in an oscillatory perturbation. This perturbation is represented by the operator:

$$H_{int} = \hbar \begin{pmatrix} 0 & \Omega \cos(\omega t) \\ \Omega^* \cos(\omega t) & 0 \end{pmatrix}$$

the Rabi frequency $\Omega = \mathcal{E} \mu / \hbar$
Light-matter interaction
two-levels in an external field

\[ H_0 = \hbar \begin{pmatrix} 0 & 0 \\ 0 & \omega_0 \end{pmatrix} \]

\[ H_{int} = \hbar \begin{pmatrix} 0 & \Omega \cos(\omega t) \\ \Omega^* \cos(\omega t) & 0 \end{pmatrix} \]

\[ \psi(x, t) = c_1(t)\phi_1(x) + c_2(t)\phi_2(x) \]

The time dependent coefficients satisfy the Schrödinger equation in matrix form

\[ i\hbar \frac{d}{dt} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} \]
Light-matter interaction
two-levels in an external field

\[ H_0 = \hbar \begin{pmatrix} 0 & 0 \\ 0 & \omega_0 \end{pmatrix} \]

\[ H_{int} = \hbar \begin{pmatrix} 0 & \Omega \cos(\omega t) \\ \Omega^* \cos(\omega t) & 0 \end{pmatrix} \]

\[ \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = E \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \]

\[ \begin{vmatrix} H_{11} - E & H_{12} \\ H_{21} & H_{22} - E \end{vmatrix} = 0. \]
Light-matter interaction

two-levels in an external field

\[ E_- = \frac{H_{11} + H_{22}}{2} - \sqrt{\left( \frac{H_{22} - H_{11}}{2} \right)^2 + |H_{12}|^2} \]

\[ E_+ = \frac{H_{11} + H_{22}}{2} + \sqrt{\left( \frac{H_{22} - H_{11}}{2} \right)^2 + |H_{12}|^2} \]
Light-matter interaction

two-level interaction

any parameter
(temperature, position, magnetic field, etc.)
Light-matter interaction

two-level interaction

anti-crossing

Energy

any parameter

(temperature, position, magnetic field, etc.)
Light-matter interaction
two-levels in an external field

\[ |c_1(t)|^2 = \frac{\Omega^2}{\Omega^2_R} \sin^2 \left( \frac{\Omega_R t}{2} \right), \]
\[ |c_2(t)|^2 = \frac{\Delta^2}{\Omega^2_R} + \frac{\Omega^2_R}{\Omega^2_R} \cos^2 \left( \frac{\Omega_R t}{2} \right), \]
\[ \Omega^2_R \equiv \Omega^2 + \Delta^2. \]

This means that the probabilities to be in state \(|1\rangle\) or \(|2\rangle\) oscillate with the frequency \(\Omega_R\) defined above, the total Rabi frequency. From this result, it is clear that states \(|1\rangle\) and \(|2\rangle\) are no longer stationary states of the system. It is remarkable that the dynamic behaviour of the system is governed (at this point) by only two parameters. These parameters are the coupling strength \(\Omega\) (proportional to the electromagnetic field strength) and the detuning \(\Delta\) (how far the field is away from resonance).
Light-matter interaction
two-levels in an external field

\[ |1\rangle \quad \Omega = 2\pi, \Delta = 0 \]

\[ |2\rangle \quad \Omega = 4\pi, \Delta = 0 \]

\[ |1\rangle \quad \Omega = 2\pi, \Delta = 2\pi \]

\[ |2\rangle \quad \Omega = 2\pi, \Delta = 4\pi \]