From AdS amplitudes to dS cosmology

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based mostly on [2207.02872] and [2309.xxxxx]
work with **Kostas Skenderis** and **Paul McFadden**

Correlators in Cortona, September 21, 2023

From AdS to dS (1/3)

- dS/CFT correspondence was proposed more than 20 years ago [Strominger (2001)] (with earlier work in [Hull (1998)] [Witten (2001)]).
- The status has remained controversial.
- Different formulations/versions have appeared through the years
  - Wavefunction of the universe [Maldacena (2002)] ...
  - Domain-wall/cosmology correspondence [Skenderis, Townsend (2006)], [AB, McFadden, Skenderis (2009-2013)]
  - Cosmological bootstrap [Arkani-Hamed etal (2018)]
  - ...
- There is a useful and working version of dS/CFT perturbatively in $1/N$.
- It is unclear whether such dualities exist non-perturbatively in $1/N$.
- There is no known embedding in/derivation from string theory.

From AdS to dS (2/3)

CFT data

$$\langle \mathcal{O}(\boldsymbol{k}_1)\dots\mathcal{O}(\boldsymbol{k}_n)\rangle$$

Depends on $N, \lambda, \boldsymbol{k}_i$

holography (CFT data ↔ AdS data)

AdS data

$$\mathrm{d}s^2_{AdS} = \frac{{L_{AdS}}^2}{z^2}\left[\mathrm{d}z^2 + \mathrm{d}{\boldsymbol{x}_{AdS}}^2\right]$$

$$Z[\varphi^{AdS}_{(0)}]$$

Depends on $\ell_P^{(AdS)}, L_{AdS}, \boldsymbol{k}_i$

analytic continuation (AdS data → dS data)

dS data

$$\mathrm{d}s^2_{dS} = \frac{{L_{dS}}^2}{(-\tau)^2}\left[-\mathrm{d}\tau^2 + \mathrm{d}{\boldsymbol{x}_{dS}}^2\right]$$

$$\Psi[\varphi^{dS}_{(0)}]$$

Depends on $\ell_P^{(dS)}, L_{dS}, \bar{\boldsymbol{k}}_i$

$\boldsymbol{\tau \to 0^-}$ (dS data → Cosmology)

Cosmology

$$\langle \varphi_{(0)}(\bar{\boldsymbol{k}}_1)\dots\varphi_{(0)}(\bar{\boldsymbol{k}}_1)\rangle$$

Depends on $\bar{N}, \bar{\lambda}, \bar{\boldsymbol{k}}_i$

Outline

Goals:

1. Derive continuation formulas valid for renormalized correlators.
2. Investigate the effect of renormalization on both AdS and dS data.

Outline:

1. Continuation from AdS to dS:
   - The continuation formulas.
2. Need for regularization and renormalization:
   - Dimensional regularization.
   - Renormalization in AdS and dS.
   - Continuation formulas for renormalized correlators.
3. Some implications:
   - Weight-shifting operators.
   - Tools we developed.

From AdS to dS (3/3)

Metrics:

$$\mathrm{d}s^2_{AdS} = \frac{{L_{AdS}}^2}{z^2}\left[\mathrm{d}z^2 + \mathrm{d}{\boldsymbol{x}_{AdS}}^2\right] \qquad \mathrm{d}s^2_{dS} = \frac{{L_{dS}}^2}{\tau^2}\left[-\mathrm{d}\tau^2 + \mathrm{d}{\boldsymbol{x}_{dS}}^2\right]$$

Actions:

$$S_{AdS} = (\ell_P^{(AdS)})^{1-d}\int \mathrm{d}^{d+1}x\sqrt{g_{AdS}}\times \left[\frac{1}{2}(\partial\varphi_{AdS})^2 + \frac{1}{2}{m_{AdS}}^2{\varphi_{AdS}}^2 + (\ell_P^{(AdS)})^{-2}V_{int}^{AdS}(\varphi_{AdS})\right]$$

$$S_{dS} = -(\ell_P^{(dS)})^{1-d}\int \mathrm{d}^{d+1}x\sqrt{-g_{dS}}\times \left[\frac{1}{2}(\partial\varphi_{dS})^2 + \frac{1}{2}{m_{dS}}^2{\varphi_{dS}}^2 + (\ell_P^{(dS)})^{-2}V_{int}^{dS}(\varphi_{dS})\right]$$

States:

Regularity:

$\varphi_{AdS} \sim e^{-kz}$ as $z \to \infty$

Bunch-Davies vacuum $|0\rangle$

$\varphi_{dS} \sim e^{\mathrm{i}k\tau}$ as $\tau \to \infty$

Correlators:

Euclidean (Schwinger)

In-in correlators

Analytic continuation in Planck units

We keep

$$\varphi_{AdS} = \varphi_{dS}, \qquad V_{int}^{AdS} = V_{int}^{dS}, \qquad {m_{AdS}}^2 = -{m_{dS}}^2$$

Analytic continuation in Planck units

In Planck units $\ell_P^{(AdS)} = \ell_P^{(dS)} = 1$. Then we continue

$$L_{AdS} = \mathrm{i} L_{dS}, \qquad z = -\mathrm{i}\tau, \qquad q_{dS} = q_{AdS}$$

- In particular $\varphi_{(0)}^{AdS} = (-\mathrm{i})^{d-\Delta} \varphi_{(0)}^{dS}$.
- This is the continuation used in [Maldacena (2002)].

Analytic continuation in AdS units

Analytic continuation in AdS units

In AdS units $L_{AdS} = L_{dS} = 1$. Then

$$q_{AdS} = \mathrm{i} q_{dS}, \qquad \ell_P^{(AdS)} = -\mathrm{i}\ell_P^{(dS)}.$$

- In particular $\varphi_{(0)}^{AdS} = \varphi_{(0)}^{dS}$.
- This is the continuation used in [McFadden, Skenderis (2009)].
- This continuation can be expressed purely in terms of the CFT data: ${q_{AdS}}^2 = -{q_{dS}}^2$ and ${N_{AdS}}^2 = -{N_{dS}}^2$ (when the gauge group in $SU(N)$).
- We will use it here and argue this is the form of the AdS/dS dictionary best suited for renormalized correlators.

Set-up

- Fourier transform in the boundary direction: $\boldsymbol{x} \mapsto \boldsymbol{k}$ or $\boldsymbol{q}$ or $\boldsymbol{p}$.
- Consider scalar fields φ_i with generic 3- and 4-point interactions.
- Masses are parameterized as

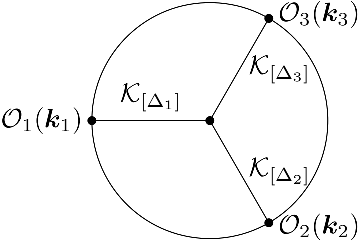
$$m^2_{AdS} = -m^2_{dS} = \Delta(\Delta - d), \qquad \frac{d}{2} < \Delta \leq d.$$

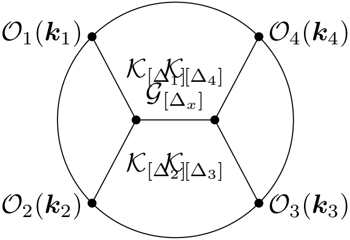
- By $\mathcal{K}_{[\Delta]}$ and $\mathcal{G}_{[\Delta]}$ we denote the associated propagators.
- We are mostly interested in $d = 3$ with $\Delta = 2$ or 3, *i.e.*, conformally coupled or massless scalars.
- We use notation

$$\langle \mathcal{O}(\boldsymbol{k}_1) \ldots \mathcal{O}(\boldsymbol{k}_n) \rangle = (2\pi)^d \delta \left(\sum_{i=1}^{n} \boldsymbol{k}_i \right) \langle\!\langle \mathcal{O}(\boldsymbol{k}_1) \ldots \mathcal{O}(\boldsymbol{k}_n) \rangle\!\rangle.$$

- Momenta lengths (magnitudes), $k_j = |\boldsymbol{k}_j|$, $s = |\boldsymbol{k}_1 + \boldsymbol{k}_2|$.

AdS amplitudes (1/2)





- The amplitudes are

$$i_{[\Delta_1\Delta_2\Delta_3]}(k_1,k_2,k_3) = \int_0^\infty \mathrm{d}z\, z^{-d-1}\, \mathcal{K}_{[\Delta_1]}(z,k_1)\,\mathcal{K}_{[\Delta_2]}(z,k_2)\,\mathcal{K}_{[\Delta_3]}(z,k_3),$$

$$\begin{aligned} &i_{[\Delta_1\Delta_2;\Delta_3\Delta_4 x\Delta_x]}(k_1,k_2,k_3,k_4,s) \\ &= \int_0^\infty \mathrm{d}z\, z^{-d-1}\, \mathcal{K}_{[\Delta_1]}(z,k_1)\,\mathcal{K}_{[\Delta_2]}(z,k_2)\times \\ &\quad \times \int_0^\infty \mathrm{d}\zeta\, \zeta^{-d-1}\, \mathcal{G}_{[\Delta_x]}(z,s;\zeta)\,\mathcal{K}_{[\Delta_3]}(\zeta,k_3)\,\mathcal{K}_{[\Delta_4]}(\zeta,k_4). \end{aligned}$$

AdS amplitudes (2/2)

- One would need an action

$$S^{\text{asym}} = \frac{1}{2}(\ell_P^{(AdS)})^{-d+1}\int \mathrm{d}^d x\sqrt{g}\sum_{j=1,2,3}\left[\partial_\mu\varphi_j\partial^\mu\varphi_j + \frac{1}{2}m^2_{\Delta_j}\varphi_j^2\right] - (\ell_P^{(AdS)})^{-d-1}\lambda\int \mathrm{d}^d x\sqrt{g}\varphi_1\varphi_2\varphi_3$$

- From action to diagrams:

$$\langle\langle \mathcal{O}_{[\Delta_1]}(\boldsymbol{k}_1)\mathcal{O}_{[\Delta_2]}(\boldsymbol{k}_2)\mathcal{O}_{[\Delta_3]}(\boldsymbol{k}_3)\rangle\rangle = (\ell_P^{(AdS)})^{-(d-1)-2V}\lambda^V \times \times i_{[\Delta_1\Delta_2\Delta_3]}(k_1,k_2,k_3) + O(\lambda^2).$$

with $V = 1$.

- There exists an AdS action turning a single scalar (no gauge symmetries) AdS amplitude into the full correlator.
- We can work digram by diagram. Also in the context of renormalization.

dS amplitudes (1/3)

- Use Schwinger-Keldysh formalism to implement the in-in calculations,

$$\langle\varphi(\tau,\boldsymbol{x}_1)\dots\varphi(\tau,\boldsymbol{x}_n)\rangle = \int \mathcal{D}\varphi_+\mathcal{D}\varphi_- \left(\prod_{i=1}^{n}\varphi_+(\tau,\boldsymbol{x}_i)\right)\times$$
$$\times \exp\Big(iS_+[\varphi_+] - iS_-[\varphi_-]\Big),$$

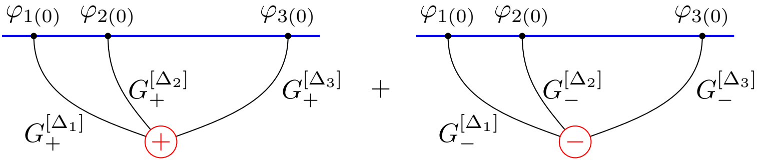
  where both fields $\varphi_\pm$ coincide at late times.
- The boundary (cosmological) field is

$$\varphi_{(0)}(\boldsymbol{x}) = \lim_{\tau_0\to 0}\left[(-\tau_0)^{\Delta-d}\varphi(\tau,\boldsymbol{x})\right].$$

- Apply correct integration contours to make sure we use the Bunch-Davies vacuum.

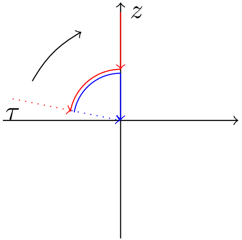
dS amplitudes (2/3)

$$\langle\!\langle \varphi_{1(0)}(\boldsymbol{k}_1)\varphi_{2(0)}(\boldsymbol{k}_2)\varphi_{3(0)}(\boldsymbol{k}_3)\rangle\!\rangle = (\ell_P^{(dS)})^{-(d-1)-2V}\,\lambda\, ds_{[\Delta_1\Delta_2\Delta_3]} + O(\lambda^2).$$



$$ds_{[\Delta_1\Delta_2\Delta_3]} = 2\,\mathrm{Re}\left[-\mathrm{i}\int_{-\infty(1-\mathrm{i}\epsilon)}^{0} \frac{\mathrm{d}\tau}{(-\tau)^{d+1}} G_+^{[\Delta_1]}(q_1,\tau)G_+^{[\Delta_2]}(q_2,\tau)G_+^{[\Delta_3]}(q_3,\tau)\right]$$

dS amplitudes (3/3)



Analytic continuations:

- $$\int_{-\infty(1-i\epsilon)}^{0} \frac{d\tau}{(-\tau)^{d+1}} = e^{\frac{i\pi d}{2}} \int_0^\infty \frac{dz}{z^{d+1}},$$
- $$G_{\pm}^{[\Delta]}(q, \pm iz) = e^{\pm\frac{i\pi}{2}(\Delta - d)} ds_{[\Delta\Delta]}(q)\, \mathcal{K}_{[\Delta]}(q,z).$$

$$ds_{[\Delta_1\Delta_2\Delta_3]} = 2\sin\left[\frac{\pi}{2}(\Delta_t - 2d)\right] \prod_{j=1}^{3} ds_{[\Delta_j\Delta_j]}(q_j) \times i_{[\Delta_1\Delta_2\Delta_3]}.$$

- where $\Delta_t = \Delta_1 + \Delta_2 + \Delta_3$.
- Since $\mathrm{Im}(iq)^D = \sin(\pi D/2) q^D$ and $\Delta_t - 2d$ is the total dimension of $i_{[\Delta_1\Delta_2\Delta_3]}$ we can write

$$\sin\left[\frac{\pi}{2}(\Delta_t - 2d)\right] i_{[\Delta_1\Delta_2\Delta_3]}(q_1, q_2, q_3) = \mathrm{Im}\, i_{[\Delta_1\Delta_2\Delta_3]}(iq_1, iq_2, iq_3).$$

Continuation formulas

$$ds_{[\Delta\Delta]}(q) = -\frac{1}{2\,\mathrm{Im}\,i_{[\Delta\Delta]}(\mathrm{i}q)},$$

$$ds_{[\Delta_1\Delta_2\Delta_3]}(q_1, q_2, q_3) = -\frac{1}{4}\frac{\mathrm{Im}\,i_{[\Delta_1\Delta_2\Delta_3]}(\mathrm{i}q_1, \mathrm{i}q_2, \mathrm{i}q_3)}{\prod_{j=1}^{3}\mathrm{Im}\,i_{[\Delta_j\Delta_j]}(\mathrm{i}q_j)},$$

$$ds_{[\Delta_1\Delta_2\Delta_3\Delta_4]}(q_i) = -\frac{1}{8}\frac{\mathrm{Im}\,i_{[\Delta_1\Delta_2\Delta_3\Delta_4]}(\mathrm{i}q_i)}{\prod_{j=1}^{4}\mathrm{Im}\,i_{[\Delta_j\Delta_j]}(\mathrm{i}q_j)},$$

$$ds_{[\Delta_1\Delta_2;\Delta_3\Delta_4 x\Delta_x]}(q_i, s) = \frac{1}{8}\prod_{j=1}^{4}\frac{1}{\mathrm{Im}\,i_{[\Delta_j\Delta_j]}(\mathrm{i}q_j)}\Bigg[\mathrm{Im}\,i_{[\Delta_1\Delta_2;\Delta_3\Delta_4 x\Delta_x]}(\mathrm{i}q_i, \mathrm{i}s) - \frac{\mathrm{Im}\,i_{[\Delta_1\Delta_2\Delta_x]}(\mathrm{i}q_1, \mathrm{i}q_2, \mathrm{i}s)\,\mathrm{Im}\,i_{[\Delta_x\Delta_3\Delta_4]}(\mathrm{i}s, \mathrm{i}q_3, \mathrm{i}q_4)}{\mathrm{Im}\,i_{[\Delta_x\Delta_x]}(\mathrm{i}s)}\Bigg].$$

- Various forms of these exist in the literature, [Maldacena (2002)] [McFadden, Skenderis (2010-11)] [AB, McFadden, Skenderis (2011-13)] [Pimentel, Maldacena (2011)] [Hartle, Hawking, Hertog (2012)] [Anninos, Denef, Harlow (2012)] [Anninos, Hartman, Strominger (2012)] [Mata, Raju,Trivedi (2012)] [Kundu, Shukla,Trivedi (2014)] [Arkani-Hamed, Maldacena (2015)] [Sleight, Toronna (2018-2022)] [Arkani-Hamed, Baumann, Lee, Pimentel (2018)] [Baumann et al (2019-21)] [Pajer et al (2021-23)] [Melville et al (2020)] [Di Petro, Gorbenko, Komatsu (2021)] [Raju et al (2023)] [Wang, Pimentel, Achucarro (2023)]

Example

- We are interested in $d = 3$ and scalars with $\Delta = 2, 3$, *i.e.*, conformally coupled and massless scalars
- Propagators simplify to elementary functions.
- In $d = 3$ we have $\mathcal{K}_{[2]}(z, \boldsymbol{k}) = ze^{-kz}$ and thus

$$i_{[222]} = \int_0^\infty \frac{e^{-k_t z}}{z} \mathrm{d}z = \infty$$

  where $k_t = k_1 + k_2 + k_3$.
- On the dS side $\Delta_t - 2d = 0$ and thus the sine vanishes. Does the amplitude vanish?
- Must regulate and renormalize.

Divergences in AdS amplitudes

- We are mostly interested in $d = 3$ with $\Delta = 2$ or 3, *i.e.*, conformally coupled or massless scalars.

3-point amplitude	$\hat{i}_{[\Delta_1 \Delta_2 \Delta_3]}$
$[222]$	1
$[322]$	1
$[332]$	1
$[333]$	1

External operators	Contact	$\Delta_x = 2$	$\Delta_x = 3$
$[22; 22x\Delta_x]$	0	0	0
$[32; 22x\Delta_x]$	1	2	1
$[33; 22x\Delta_x]$	1	1	2
$[32; 32x\Delta_x]$	1	2	1
$[33; 32x\Delta_x]$	1	2	2
$[33; 33x\Delta_x]$	1	1	2

Divergences in derivative amplitudes

$\Delta_x =$	C	2	3
$[22; 22x\Delta_x]$	0	0	0
$[22; 32x\Delta_x]$	1	2	1
$[22; 33x\Delta_x]$	1	1	2
$[32; 22x\Delta_x]$	0	1	0
$[32; 32x\Delta_x]$	0	1	0
$[32; 33x\Delta_x]$	1	1	2
$[33; 22x\Delta_x]$	0	0	0
$[33; 32x\Delta_x]$	1	2	1
$[33; 33x\Delta_x]$	0	0	1

$\Delta_x =$	C	2	3
$[22; 22x\Delta_x]$	0	0	0
$[32; 22x\Delta_x]$	0	1	0
$[33; 22x\Delta_x]$	0	0	0
$[32; 32x\Delta_x]$	0	0	0
$[33; 32x\Delta_x]$	0	1	0
$[33; 33x\Delta_x]$	0	0	0

$\Delta_x =$	2	3
$[22; 22x\Delta_x]$	0	0
$[32; 22x\Delta_x]$	1	0
$[33; 22x\Delta_x]$	0	0
$[32; 32x\Delta_x]$	0	0
$[33; 32x\Delta_x]$	1	1
$[33; 33x\Delta_x]$	0	0

Divergences in dS amplitudes

	dS	AdS
$[222]$	0	1
$[322]$	1	1
$[332]$	0	1
$[333]$	1	1

	de Sitter			Anti-de Sitter		
$\Delta_x =$	C	2	3	C	2	3
$[22; 22x\Delta_x]$	0	0	2	0	0	0
$[32; 22x\Delta_x]$	0	1	1	1	2	1
$[33; 22x\Delta_x]$	1	1	2	1	1	2
$[32; 32x\Delta_x]$	1	2	1	1	2	1
$[33; 32x\Delta_x]$	0	1	1	1	2	2
$[33; 33x\Delta_x]$	1	1	2	1	1	2

Dimensional regularization

- We use dimensional regularization

$$d \longmapsto \hat{d} = d + 2u\epsilon, \qquad \Delta_j \longmapsto \hat{\Delta}_j + (u + v_j)\epsilon,$$

where ϵ is the regulator and u, v fixed parameters.

- Things simplify considerably in the beta scheme: $u = 1$ and $v_j = 0$ since

$$\hat{\beta}_j = \hat{\Delta}_j - \frac{\hat{d}}{2} = \Delta_j - \frac{d}{2} = \beta_j.$$

- Regulated amplitude

$$\hat{i}_{[222]} = \int_0^\infty \frac{e^{-k_t z}}{z^{1-\epsilon}} \mathrm{d}z = k_t^{-\epsilon}\Gamma(\epsilon)$$
$$= \frac{1}{\epsilon} - \log k_t - \gamma_E + O(\epsilon).$$

- There is no cut-off.

Renormalization (1/2)

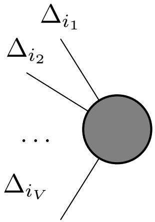
- Renormalize by adding boundary counterterms built up with the sources $\varphi_{(0)i}$ and operators $\mathcal{O}_i$.
- In our example

$$S_{\text{ct}} = -\lambda\Gamma(\epsilon)\mathfrak{a}\int \mathrm{d}^{\hat{d}}x\,\sqrt{\gamma}\varphi_{1(0)}\varphi_{2(0)}\varphi_{3(0)}\mu^{-\epsilon},$$

  where
  - $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon)$ is the required divergence.
  - $\mathfrak{a} = 1 + \epsilon\,\mathfrak{a}^{(1)} + \epsilon^2\,\mathfrak{a}^{(2)} + O(\epsilon^3)$ keeps scheme-dependence.
  - μ is the renormalization scale, due to the shift in dimensions.
  - $\varphi_{j(0)}$ is the source for $\mathcal{O}_{[\Delta_j]}$.
- In total

$$\begin{aligned} i^{\text{ren}}_{[222]} &= \lim_{\epsilon\to 0}\left[\hat{i}_{[222]} - \Gamma(\epsilon)\mathfrak{a}\mu^{-\epsilon}\right] \\ &= -\log\left(\frac{k_t}{\mu}\right) - \mathfrak{a}^{(1)}. \end{aligned}$$

Renormalization (2/2)



- Condition for divergences at each subdiagram:

$$\left\{ \begin{array}{c} d-\Delta_{i_1} \\ \Delta_{i_1} \end{array} \right\} + \ldots + \left\{ \begin{array}{c} d-\Delta_{i_V} \\ \Delta_{i_V} \end{array} \right\} + 2r = d$$

  for $r \in \{0, 1, 2, \ldots\}$.
- The condition is of type n if the bottom row is chosen n times.
- Divergences accumulate from each subdiagram.

- The corresponding counterterm:

$$S_{ct} \sim \int \mathrm{d}^{\hat{d}}x \, \sqrt{\gamma} \, \mu^{(2-V)\epsilon} \, \partial^{2r} \left\{ \begin{array}{c} \varphi_{i_1}(0) \\ \mathcal{O}_{i_1} \end{array} \right\} \times \ldots \times \left\{ \begin{array}{c} \varphi_{i_V}(0) \\ \mathcal{O}_{i_V} \end{array} \right\}$$

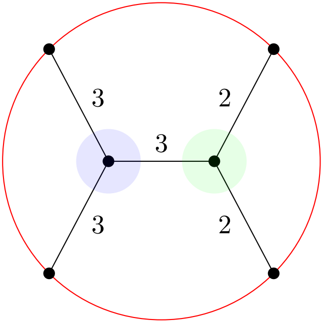
Example 1: $\hat{i}_{[222]}$

- All Δ_j are such that $\frac{d}{2} < \Delta_j < d$: only type-0 divergences appear.
- Type-0 condition: the total dimension $D = \Delta_t - (n-1)d$ satisfies $D = 2r$, $r \in \{0, 1, 2, \ldots\}$
- The counterterm generates anomaly

$$S_{ct} \sim \int \mathrm{d}^{\hat{d}}x \sqrt{\gamma}\, \mu^{(2-V)\epsilon}\, \partial^{2r} \varphi_{1(0)} \cdots \varphi_{i_n(0)}$$

- Btw: the sine $\sin(\pi D/2)$ in the continuation formulas vanishes when $D = 2n$ for integral n.
- Almost always the vanishing sine implies divergence of the amplitude.

Example 2: $\hat{i}_{[33;22x3]}$



The counterterms

$$\mathfrak{c}_1 \mu^{-\epsilon} \int \varphi_{[3]} \varphi_{[2]} \mathcal{O}_{[2]}$$

$$\mathfrak{c}_2 \mu^{-\epsilon} \int \varphi_{[3]}^2 \mathcal{O}_{[3]}$$

$$\mathfrak{c}_3 \mu^{-2\epsilon} \int \varphi_{[3]}^2 \varphi_{[2]} \mathcal{O}_{[2]}$$

- If $\frac{d}{2} < \Delta_j \leq d$ only type-0 and type-1 conditions can be satisfied.
- Source renormalization

$$\varphi_{[2]} \mapsto \varphi_{[2]} \left[1 + \mathfrak{c}_1 \mu^{-\epsilon} \varphi_{[3]} + \mathfrak{c}_3 \mu^{-2\epsilon} \varphi_{[3]}^2 + O(\varphi_{[3]}^3) \right],$$

$$\varphi_{[3]} \mapsto \varphi_{[3]} \left[1 + \mathfrak{c}_2 \mu^{-\epsilon} \varphi_{[3]} + O(\varphi_{[3]}^2) \right].$$

- Induces beta functions.

Renormalization in dS

- Renormalize dS amplitudes by introducing counterterm living at $\tau = 0$.
- In the Schwinger-Keldysh formalism we renormalize both actions $S_\pm$,

$$S_\pm[\varphi_\pm] \longmapsto S_\pm[\varphi_\pm] + S_{\rm ct}[\varphi_{(0)}, J_\pm]$$

- Counterterms with no J cancel.
- No anomalies in de Sitter.
- Only source renormalization: only beta functions.

$$S^{dS}_{\rm ct}[\varphi_{(0)}, (J_+ - J_-); \epsilon, \mu, \mathfrak{a}^{dS}_j] = \int d^d x (J_+ - J_-) f\left(\varphi_{(0)}; \epsilon, \mu, \mathfrak{a}^{dS}_j\right)$$

Continuation formulas

- When the dust settles the continuation formulas hold except that the map $A_{ij} : \mathfrak{a}_i^{AdS} \mapsto \mathfrak{a}_j^{dS}$ may be non-trivial.

$$ds^{\text{ren}}_{[\Delta_1\Delta_2\Delta_3]}(q_i;\mu,\mathfrak{a}_i) = -\frac{1}{4}\frac{\operatorname{Im} i^{\text{ren}}_{[\Delta_1\Delta_2\Delta_3]}(\mathrm{i}q_i;\mu,A(\mathfrak{a}_i))}{\prod_{j=1}^3 \operatorname{Im} i^{\text{ren}}_{[\Delta_j\Delta_j]}(\mathrm{i}q_j)},$$

$$ds^{\text{ren}}_{[\Delta_1\Delta_2\Delta_3\Delta_4]}(q_i;\mu,\mathfrak{a}_i) = -\frac{1}{8}\frac{\operatorname{Im} i^{\text{ren}}_{[\Delta_1\Delta_2\Delta_3\Delta_4]}(\mathrm{i}q_i;\mu,A(\mathfrak{a}_i))}{\prod_{j=1}^4 \operatorname{Im} i^{\text{ren}}_{[\Delta_j\Delta_j]}(\mathrm{i}q_j)},$$

$$ds^{\text{ren}}_{[\Delta_1\Delta_2;\Delta_3\Delta_4 x\Delta_x]}(q_i,s;\mu,\mathfrak{a}_i) = \frac{1}{8}\prod_{j=1}^{4}\frac{1}{\operatorname{Im} i^{\text{ren}}_{[\Delta_j\Delta_j]}(\mathrm{i}q_j)}\Big[\operatorname{Im} i^{\text{ren}}_{[\Delta_1\Delta_2;\Delta_3\Delta_4 x\Delta_x]}(\mathrm{i}q_i,\mathrm{i}s;\mu,A(\mathfrak{a}_i)) - \frac{\operatorname{Im} i^{\text{ren}}_{[\Delta_1\Delta_2\Delta_x]}(\mathrm{i}q_1,\mathrm{i}q_2,\mathrm{i}s;\mu,A(\mathfrak{a}_i))\operatorname{Im} i^{\text{ren}}_{[\Delta_x\Delta_3\Delta_4]}(\mathrm{i}s,\mathrm{i}q_3,\mathrm{i}q_4;\mu,A(\mathfrak{a}_i))}{\operatorname{Im} i^{\text{ren}}_{[\Delta_x\Delta_x]}(\mathrm{i}s)}\Big].$$

Example 1: $i^{\text{ren}}_{[222]}$

- We found

$$i^{\text{ren}}_{[222]} = -\log\left(\frac{k_t}{\mu}\right) - \mathfrak{a}^{(1)}.$$

- Use

$$\log\left(\frac{q}{\mu}\right) \longmapsto \log\left(\frac{\text{i}q}{\mu}\right) = \log\left(\frac{q}{\mu}\right) + \frac{\text{i}\pi}{2}.$$

- We get

$$\begin{aligned} ds^{\text{ren}}_{[222]} &= -\frac{1}{4}\frac{\text{Im}\, i^{\text{ren}}_{[222]}(\text{i}q_1, \text{i}q_2, \text{i}q_3)}{\prod_{j=1}^{3} \text{Im}\, i^{\text{ren}}_{[22]}(\text{i}q_j)} \\ &= -\frac{\pi}{8 q_1 q_2 q_3}. \end{aligned}$$

Example 2: $i^{\text{ren}}_{[33;22x3]}$

In[1]:= `dS4ptX[3, {3, 3, 2, 2, 3}]`

Out[1]=

$$
\begin{aligned}
&\frac{1}{8\,k_1^3\,k_2^3\,k_3\,k_4}\Bigg(\frac{\mathrm{Log}\left[\frac{s+k_1+k_2}{s+k_3+k_4}\right](s^3+k_1^3+k_2^3)}{3\,s^2}+\frac{1}{24}\pi^2(k_3+k_4)+\frac{1}{12}\mathrm{Log}\left[\frac{s+k_1+k_2}{s+k_3+k_4}\right]^2(k_3+k_4)-\frac{1}{6}\mathrm{Log}\left[\frac{s+k_3+k_4}{\mu}\right]^2(k_3+k_4)-\\
&\frac{1}{6}\Bigg(\frac{1}{2}\mathrm{Log}\left[\frac{s+k_1+k_2}{k_1+k_2+k_3+k_4}\right]^2-\frac{1}{2}\mathrm{Log}\left[\frac{s+k_3+k_4}{k_1+k_2+k_3+k_4}\right]^2-\mathrm{PolyLog}\left[2,\frac{-s+k_1+k_2}{k_1+k_2+k_3+k_4}\right]+\mathrm{PolyLog}\left[2,\frac{-s+k_3+k_4}{k_1+k_2+k_3+k_4}\right]+\\
&\frac{\left(-\frac{\pi^2}{6}+\mathrm{Log}\left[\frac{s+k_1+k_2}{k_1+k_2+k_3+k_4}\right]\mathrm{Log}\left[\frac{s+k_3+k_4}{k_1+k_2+k_3+k_4}\right]+\mathrm{PolyLog}\left[2,\frac{-s+k_1+k_2}{k_1+k_2+k_3+k_4}\right]+\mathrm{PolyLog}\left[2,\frac{-s+k_3+k_4}{k_1+k_2+k_3+k_4}\right]\right)(k_1^3+k_2^3)}{s^3}\Bigg)(k_3+k_4)-\\
&\frac{\mathrm{Log}\left[\frac{k_1+k_2+k_3+k_4}{s+k_3+k_4}\right](k_1^2-k_1k_2+k_2^2)(k_1+k_2+k_3+k_4)}{3\,s^2}+\frac{1}{3}\left(-\frac{7s}{3}-k_1-k_2-\frac{k_1^2-k_1k_2+k_2^2}{s}-\frac{43}{18}(k_3+k_4)\right)+\\
&\frac{1}{3}\mathrm{Log}\left[\frac{s+k_3+k_4}{\mu}\right]\left(s+(k_3+k_4)\left(\frac{4}{3}-\alpha[333][1]\right)\right)+\frac{1}{3}s\,(1+\alpha[333][1])+\frac{1}{3\,s^3}\left(-s+(k_3+k_4)\left(-1+\mathrm{Log}\left[\frac{s+k_3+k_4}{\mu}\right]+\alpha[322][1]\right)\right)\\
&\left(4\,s\,k_1k_2-(s+k_1+k_2)(s\,k_1+s\,k_2+k_1k_2)+(s^3+k_1^3+k_2^3)\left(-\frac{4}{3}+\mathrm{Log}\left[\frac{s+k_1+k_2}{\mu}\right]+\alpha[333][1]\right)\right)+\\
&\frac{1}{3}(k_3+k_4)\left(1+\alpha[333][1]+\alpha[333][2]-\frac{1}{2}\alpha[33223][2]\right)\Bigg)
\end{aligned}
$$

Our repository

Repository of AdS amplitudes

You can find all regulated and renormalized 2-, 3-, and 4-point amplitudes for $d = 3$ and $\Delta = 2$ or 3 in the HandbooK Mathematica package.

- The package is attached to the arXiv paper at https://arxiv.org/abs/2207.02872.
- The package provides all regulated and renormalized 2-, 3- and 4-point amplitudes for $d = 3$ and $\Delta = 2, 3$.
- Regulated amplitudes are evaluated in an arbitrary (u, v)-scheme.

Raising/lowering operators (1/2)

- Explicit expressions for amplitudes give us opportunity to test implicit results.
- Raising/lowering operators $\mathcal{W}_{12}^{\sigma_1\sigma_2}$ were introduced in [Karateev, Kravchuk, Simmons-Duffin (2017)] [Arkani-Hamed, Maldacena (2018)] [Baumann et al (2019)],

$$\Delta_{1,2} \longmapsto \Delta_{1,2} + \sigma_{1,2}, \qquad \sigma_{1,2} = \pm 1.$$

- For example, $\mathcal{W}_{12}^{++} i_{[22,22x3]} \sim i_{[33,22x3]}$? Impossible!
- The lowering operator is $\mathcal{W}_{12}^{--} = \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{k}_1^{\mu}} + \frac{\partial}{\partial \boldsymbol{k}_2^{\mu}}\right)^2$.
- The raising operator uses inversion,

$$\mathcal{S}_i(f) = k_i^{-2\Delta_i + d} f, \qquad \mathcal{W}_{12}^{++} = \mathcal{S}_1^{-1}\mathcal{S}_2^{-1}\mathcal{W}_{12}^{--}\mathcal{S}_1\mathcal{S}_2.$$

Raising/lowering operators (2/2)

Resolution

Combinations of exchange and contact diagrams,

$$\mathcal{W}_{12}^{\sigma_1\sigma_2}\,\hat{i}_{[\Delta_1,\Delta_2,\Delta_3,\Delta_4\,x\,\Delta_x]} = \mathcal{N}_{exch.}^{\sigma_1\sigma_2}\,\hat{i}_{[\Delta_1+\sigma_1,\Delta_2+\sigma_2,\Delta_3,\Delta_4\,x\,\Delta_x]} + \mathcal{N}_{cont.}^{\sigma_1\sigma_2}\,\hat{i}_{[\Delta_1+\sigma_1,\Delta_2+\sigma_2,\Delta_3,\Delta_4]}.$$

- Sometimes $\mathcal{W}_{12}^{\sigma_1\sigma_2}$ can yield an amplitude associated with a derivative vertex in the action, such as $\partial_\mu\varphi_1\partial^\mu\varphi_2\,\varphi_3$. This requires a special condition to be satisfied.
- Action of $\mathcal{W}_{12}^{\sigma_1\sigma_2}$ on renormalized correlators can yield additional, local contributions, *e.g.*,

$$\mathcal{W}_{12}^{++}\,i^{\text{ren}}_{[22,22x2]} = -i^{\text{ren}}_{[33,22x2]} - \frac{1}{2}i^{\text{ren}}_{[3322]} + \frac{k_3+k_4}{8}\left(3+2\mathfrak{a}^{(1)}_{[3322]} - 2\mathfrak{a}^{(1)}_{[33,22x2]}\right)$$

- For $\mathcal{W}_{12}^{++}\hat{i}_{[22,22x3]}$ we have $\mathcal{N}_{exch.}^{\sigma_1\sigma_2} = -\frac{1}{2}(-3+\epsilon)\epsilon$. One cannot obtain $\hat{i}_{[33,22x3]}$ from $\hat{i}_{[22,22x3]}$ at all.

Summary

- We present the detailed renormalization procedure for 2-, 3-, and 4-point dS and AdS amplitudes.
- This includes most of the amplitudes involving conformally coupled and massless scalars.
- Our continuation formulas hold for renormalized amplitudes (up to scheme-dependence).
- Be very careful when using raising/lowering operators: they mix exchange and contact amplitudes.
- Continuation formulas are not just the shadow transform.
- You don't have to renormalize every time: use renormalized amplitudes.
- Use our package HANDBOOK from [2207.02872] for 2-, 3- and 4-point amplitudes.