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Remarks on the history of the notion of Lie differentiation[1]

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1. The derivative $X(f)$ of a function f, defined on a smooth manifold, in the direction of the vector field X and the bracket of two vector fields, introduced by Sophus Lie himself, are the first examples of what is now called the *Lie derivative*. Another early example comes from the Killing equation. David Hilbert [1], in his derivation of the Einstein equations, used the expression

$$X^\rho \partial_\rho g^{\mu\nu} - g^{\mu\rho} \partial_\rho X^\nu - g^{\rho\nu} \partial_\rho X^\mu$$

and stated that it is a tensor field for every tensor field g and vector field X. Around 1920, Élie Cartan defined a natural differential operator $\mathcal{L}(X)$ acting on fields of exterior forms. He noted that it commutes with the exterior derivative d and gave, in equation (5) on p. 84 in [2], the formula[3]

$$\mathcal{L}(X) = d \circ i(X) + i(X) \circ d, \tag{1}$$

where $i(X)$ is the contraction with X.

2. Władysław Ślebodziński, in his article of 1931 [5], wrote an explicit formula for the Lie derivative (without using that name) in the direction of X of a tensor field

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[3] In this note, I transcribe all equations from the form given by their authors to the notation in current usage. All manifolds and maps among them are assumed to be smooth. Good references for my notation and terminology are [3] and [4].

A of arbitrary valence. He gave also an equation equivalent to

$$\mathcal{L}(X)(A \otimes B) = (\mathcal{L}(X)A) \otimes B + A \otimes \mathcal{L}(X)B$$

and noted that $\mathcal{L}(X)$ commutes with contractions over pairs of tensorial indices. He then applied his results to Hamilton's canonical equations of motion. For a function $H(p, q)$, $p = (p_\mu)$, $q = (q^\mu)$, $\mu = 1, \dots, n$, Ślebodziński defined the vector field

$$X_H = \frac{\partial H}{\partial p_\mu} \frac{\partial}{\partial q^\mu} - \frac{\partial H}{\partial q^\mu} \frac{\partial}{\partial p_\mu},$$

introduced the symplectic form $A = dq^\mu \wedge dp_\mu$, the Poisson bivector $B = \partial/\partial q^\mu \wedge \partial/\partial p_\mu$ and showed that $\mathcal{L}(X_H)A = 0$ and $\mathcal{L}(X_H)B = 0$. This allowed him to generalize some results of Théophile de Donder in the theory of invariants [6].

The priority of Ślebodziński in defining the Lie derivative in the general case was recognized by David van Dantzig who wrote, in footnote on p. 536 of [7], *Der Operator* [the Lie derivative] *wurde zum ersten Mal von W. Ślebodziński eingeführt*. It was van Dantzig who introduced, in the same paper, the name *Liesche Ableitung*. Also Jan Arnoldus Schouten, in footnote 1 on p. 102 of [8], lists the 1931 paper by Ślebodziński as the first reference for the notion of Lie differentiation. Van Dantzig complemented the approach of Ślebodziński by pointing out that the Lie derivative can be defined as the difference between the value of a geometric object A at a point and the value of that object at the same point obtained by an infinitesimal 'dragging along' a vector field. In contemporary notation this is expressed by the formula

$$\mathcal{L}(X)A = \frac{d}{dt} \varphi_t^* A_{|t=0}, \tag{2}$$

where $\varphi_t^* A$ is the pull-back of A by the flow $(\varphi_t, t \in \mathbb{R})$ generated by X. In view of the equation

$$\frac{d}{dt} \varphi_t^* A = \varphi_t^* \mathcal{L}(X)A,$$

the vanishing of $\mathcal{L}(X)A$ is equivalent to the invariance of A with respect to the flow generated by X; see, e.g., §24 in [9].

3. For quite some time, physicists had been using Lie derivatives, without reference to the work of mathematicians. Léon Rosenfeld [10] introduced what he called a 'local variation' $\delta^* A$ of a geometric object A induced by an infinitesimal transformation of coordinates generated by X. He noted that δ^* commutes with differentiation. It is easily seen that his $\delta^* A$ is $-\mathcal{L}(X)A$; see, e.g., [11]. Assuming that A is a tensor of type determined by a representation ρ of $\mathrm{GL}(4, \mathbb{R})$ in the vector space $\mathbb{R}^N$ and denoting by $\rho_\mu^\nu \in \operatorname{End} \mathbb{R}^N$ the matrices of the corresponding representation of the Lie algebra of $\mathrm{GL}(4, \mathbb{R})$, one can deduce from Rosenfeld's equations the following formula for the Lie derivative

$$\mathcal{L}(X)A = X^\mu \partial_\mu A - \partial_\nu X^\mu \rho_\mu^\nu A.$$

In particular, assuming that L is a Lagrange function depending on the components of A and on their first derivatives and such that $\int Ld^4x$ is an invariant, Rosenfeld showed that

$$\mathcal{L}(X)L = \partial_\mu(LX^\mu)$$

and used the formula

$$\mathcal{L}(X)L = \frac{\partial L}{\partial A}\mathcal{L}(X)A + \frac{\partial L}{\partial(\partial_\mu A)}\partial_\mu(\mathcal{L}(X)A)$$

to derive a set of identities of the Noether type, and the conservation laws of energy-momentum and of angular momentum. One of the main results of that paper was the symmetrization of the canonical energy-momentum tensor t achieved by adding to it an expression linear in the derivatives of the spin tensor s.

Incidentally, it is remarkable that this symmetrization, derived independently also by F. J. Belinfante, is a natural consequence of the Einstein–Cartan theory of gravitation. In that theory, based on a metric tensor g and a linear connection $\omega^\mu{}_\nu = \Gamma^\mu_{\nu\rho}dx^\rho$ which is metric, but may have torsion, there are field equations relating curvature and torsion to t and s, respectively; see [12] and the references given there. If these Sciama–Kibble field equations are satisfied and X is a vector field generating a symmetry of space-time so that

$$\mathcal{L}(X)g = 0 \quad \text{and} \quad \mathcal{L}(X)\omega = 0$$

then, denoting by t_μ and $s_{\mu\nu}$ the 3-forms (densities) of energy-momentum and spin, and the covariant derivative with respect to the transposed connection $\tilde{\omega}^\mu_\nu = \Gamma^\mu_{\rho\nu}dx^\rho$ by $\widetilde{\nabla}$, one has the conservation law $dj = 0$, where

$$j = X^\mu t_\mu + \tfrac{1}{2}\widetilde{\nabla}^\nu X^\mu s_{\mu\nu}.$$

In the limit of special relativity, if X generates a translation, then j reduces to the corresponding component of the density of energy-momentum; for X generating a Lorentz transformation, one obtains a component of the density of total angular momentum.

4. The Lie derivative defines a homomorphism of the Lie algebra $\mathcal{V}(M)$ of all vector fields on an n-dimensional manifold M into the Lie algebra of derivations of the algebra of all tensor fields on M,

$$\mathcal{L}([X,Y]) = [\mathcal{L}(X), \mathcal{L}(Y)].$$

The *Cartan algebra* $\mathcal{C}(M) = \bigoplus_{p=0}^n \mathcal{C}^p(M)$ of all exterior forms on M is $\mathbb{Z}$-graded by the degree p of the forms. A derivation D of degree $q \in \mathbb{Z}$ maps linearly $\mathcal{C}^p(M)$ to $\mathcal{C}^{p+q}(M)$ and satisfies the graded Leibniz rule,

$$D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^{pq}\alpha \wedge D\beta \quad \text{for every } \alpha \in \mathcal{C}^p(M).$$

Derivations of odd degree are often called antiderivations. The vector space $\mathrm{Der}\,\mathcal{C}(M)$ of all derivations of $\mathcal{C}(M)$ is a *super Lie algebra* with respect to the bracket

$$[D, D'] = D \circ D' - (-1)^{\deg D \deg D'} D' \circ D. \tag{3}$$

The degree of $[D, D']$ is the sum of the degrees of D and D' and there holds a super Jacobi identity; see [13] for an early review of super Lie algebras, written for physicists. In particular, d is a derivation of degree $+1$ and, if $X \in \mathcal{V}(M)$, then $\mathcal{L}(X)$ and $i(X)$ are derivations of degrees 0 and -1, respectively. The Cartan formula (1) represents $\mathcal{L}(X)$ as a bracket, as defined in (3), of d and $i(X)$.

The contraction $i(X)$ generalizes to fields of vector-valued exterior forms. Let $X \in \mathcal{V}(M)$, $\xi \in \mathcal{C}^p(M)$, $p = 0, \ldots, n$, and $Y = X \otimes \xi$, then Y is a vector-valued p-form and $i(Y)$ is a derivation of the Cartan algebra, of degree $p - 1$, defined by

$$i(Y)\alpha = \xi \wedge i(X)\alpha, \quad \alpha \in \mathcal{C}(M).$$

By linearity one extends $i(Y)$ to arbitrary vector-valued p-forms. The bracket $[d, i(Y)]$ is now a derivation of degree p; by the super Jacobi identity its bracket with d is zero and every derivation (super) commuting with d is of this form. If Y and Z are vector-valued forms of degrees p and q, respectively, then the bracket $[[d, i(Y)], [d, i(Z)]]$ super commutes with d and, therefore, there exists a vector-valued $(p + q)$-form $[Y, Z]$ such that

$$[d, i([Y, Z])] = [[d, i(Y)], [d, i(Z)]]. \tag{4}$$

The *Fröhlicher–Nijenhuis* [14] bracket $[Y, Z]$, defined by (4), generalizes the Lie bracket of vector fields; it is super anticommutative,

$$[Z, Y] = -(-1)^{pq}[Y, Z],$$

and makes the vector space of all vector-valued forms into a super Lie algebra. For example, an almost complex structure J on an even-dimensional manifold is a vector-valued 1-form and $[J, J]$ is its Nijenhuis torsion.

5. A convenient framework to generalize the definition (2) of Lie derivatives is provided by *natural bundles*. A natural bundle is a functor F from the category of manifolds to that of bundles such that $\pi_M : F(M) \to M$ is a bundle and if $\varphi : M \to N$ is a diffeomorphism, then $F(\varphi) : F(M) \to F(N)$ is an isomorphism of bundles covering φ. If A is a section of $\pi_N : F(N) \to N$, i.e. a field on N of geometric objects of type F, then $\varphi^* A = F(\varphi^{-1}) \circ A \circ \varphi$ is its pull-back by φ to M. All tensor bundles are natural, but spinor bundles are not. The vertical bundle $VF(M)$ is the subbundle of the tangent bundle $TF(M)$ consisting of all vertical vectors, i.e. vectors that are annihilated by $T\pi_M$. Let $(\varphi_t, t \in \mathbb{R})$ be the flow generated by $X \in \mathcal{V}(M)$ and let A be a section of π_M. The curve $t \mapsto (\varphi_t^* A)(x)$ is

vertical for every $x \in M$ and the Lie derivative $\mathcal{L}(X)A$ is now defined as the section of the vector bundle $VF(M) \to M$ such that $(\mathcal{L}(X)A)(x)$ is the vector tangent to $t \mapsto (\varphi_t^* A)(x)$ at $t = 0$. The monograph by Kolář, Michor and Slovák [15] contains a full account of this approach and, in Ch. XI, an even more general definition of Lie differentiation.

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