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OPTICAL GEOMETRY*

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ABSTRACT

The geometry of classical physics is Lorentzian; but weaker geometries are often more appropriate : null geodesics and electromagnetic fields, for example, are well known to be objects of conformal geometry. To deal with a single null congruence, or with the radiative electromagnetic fields associated with it, even less is needed : "flag geometry" for the first, "optical geometry", with which this paper is chiefly concerned, for the second. We establish a natural one-to-one correspondence between optical geometries, considered locally, and three-dimensional Cauchy-Riemann structures. A number of Lorentzian geometries are shown to be equivalent from the optical point of view. For example the Gödel universe, the Taub-NUT metric and Hauser's twisting null solution have an optical geometry isomorphic to the one underlying the Robinson congruence in Minkowski space. We present general results on the problem of "lifting" a CR structure to a Lorentz manifold and, in particular, to Minkowski space ; and we exhibit the relevance of the deviation form to this problem.

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* Presented by A.Trautman

1. INTRODUCTION

Much of the work done by theoretical physicists consists in using perturbation methods to find approximate solutions suitable for the description of the phenomena under consideration. It is often important to know, however, not only whether the perturbation series converges, but also what is the form of the exact solution itself. Thus, for example, in Einstein's theory of general relativity, a rather rudimentary approximate description of the gravitational field of the Sun suffices to account for effects such as the deflection of light and the motion of planetary perihelia. The striking properties of black holes are apparent only when the appropriate exact solutions of Einstein's equations are considered. The development of gauge theories of the Yang-Mills type has led physicists and mathematicians to study self-dual connections on non-trivial principal bundles over compact, Riemannian four-manifolds. In this case, approximate and perturbation expansion methods are also of little use.

Exact solutions of classical differential equations of mathematical physics - such as those associated with the names of Maxwell, Einstein, Dirac, Yang and Mills - are often found by imposing symmetry restrictions on the solutions. To be more precise, if one considers, say, the Maxwell equation over a Riemannian or Lorentzian manifold admitting a Lie group G of isometries, then, given a Lie subgroup H of G, one can demand that the solution be invariant with respect to the action of H. In many cases, when H is sufficiently large, all such solutions can be found in closed form. There is another method of singling out and finding special solutions, adapted to the study of classical fields on Lorentz manifolds. The method, which may be traced back to Harry Bateman (1910)[1], makes use of the existence, on such manifolds, of null (light-like, optical) vectors and two-forms. Initially, the application of the method had been restricted to electromagnetism, but its real strength appeared in connection with work on relativistic gravitation. Recently, it has become clear what is the geometry underlying the method and how it is related to the notion of Cauchy-Riemann structures (Wells[2], Chern and Moser[3], Burns and Shnider[4], Penrose[5]). In this article, we present a self-contained account of the basis of this "optical geometry".

The paper is organized as follows: a section summarizing our notation and terminology is followed by one containing examples and heuristic considerations that motivate this research. The next section describes a rather weak "flag geometry", adapted to account for the geodetic property of a congruence of null curves. There then comes the main body of the paper devoted to optical geometry proper and its relation to CR structures and Lorentzian geometry. We give a novel derivation of the Sachs equation describing the propagation of complex expansion and shear. A final section is devoted to a brief history of the subject.

2. NOTATION AND PRELIMINARIES

In this paper, most of the time, we adhere to the standard terminology and notation of differential geometry and its applications to mathematical physics; see, for example, Abraham and Marsden[6]. The following paragraphs contain a summary of our conventions (see also Trautman[7]).

The Grassmann algebra of an n-dimensional real vector space V is denoted by

$$\Lambda V^* = \bigoplus_{l=0}^{n} \Lambda^l V^*$$

where $\Lambda^0 V^* = \mathbb{R}$ and $\Lambda^1 V^* = V^*$ is the dual of V. If s is a linear map from V^* to $\Lambda^{l+1} V^*$ (l = -1,0,...,n-1), then i(s) is the *graded derivation* of ΛV^* of degree l, defined by

$$i(s) : \Lambda V^* \to \Lambda V^* \quad \text{is linear,}$$

$$i(s)\alpha = s(\alpha) \quad \text{for any } \alpha \in V^*,$$

$$i(s)(\beta \wedge \gamma) = (i(s)\beta) \wedge \gamma + (-1)^{kl} \beta \wedge i(s)\gamma$$

for any $\beta \in \Lambda^k V^*$ and $\gamma \in \Lambda V^*$. If l = -1, i.e. if $s \in V$, then we write $s \lrcorner \gamma$ instead of $i(s)\gamma$. If $u \in V$ and $\xi \in V^*$, then the tensor $u \otimes \xi$ is identified with the linear map $V^* \to V^*$ given by $(u \otimes \xi)\alpha = \alpha(u)\xi$ and we have

$$i(u \otimes \xi)\gamma = \xi \wedge (u \lrcorner \gamma).$$

The graded bracket

$$[i(s), i(t)] = i(s) \circ i(t) - (-1)^{lm} i(t) \circ i(s)$$

of the derivations $i(s)$ and $i(t)$, of degrees l and m, respectively, is a graded derivation of degree $l + m$.

Sometimes it is convenient to express tensors in terms of their components with respect to a basis (e_μ) in V and the dual basis (e^μ) in V^*,

$$e_\mu \lrcorner e^\nu = \delta^\nu_\mu \;;\; \mu, \nu = 1, \ldots, n.$$

A *metric tensor* on V is defined as a map $g: V \times V \to \mathbb{R}$ which is bilinear, symmetric and non-singular. For any $u \in V$, we denote by $g(u)$ the one-form such that $v \lrcorner g(u) = g(u,v)$ for any $v \in V$; in other words, we use the letter g to denote also the isomorphism $V \to V^*$ induced by the metric tensor. A metric g on a four-dimensional vector space is said to be *Lorentzian* if its signature is (1,3), i.e. if there is a basis (e^μ) in the complexified space $\mathbb{C} \otimes V^*$ such that

$$g = e^0 \otimes e^3 + e^3 \otimes e^0 - e^1 \otimes e^2 - e^2 \otimes e^1 \,, \tag{2.1}$$

where

$$e^2 = \overline{e^1} \quad \text{and} \quad e^0, e^3 \text{ are real.}$$

Following the tradition of classical differential geometry, we shall omit the symbol of the tensor product in all formulae for g ; instead of (2.1) we write

$$g = 2e^0 e^3 - 2e^1 e^2 \tag{2.2}$$

or

$$g = g_{\mu\nu} e^\mu e^\nu \,, \text{ where } \mu, \nu = 0, \ldots, 3.$$

A heavy dot denotes the *contraction* of covariant tensors - and, in particular, forms - defined according to the pattern

$$(S\cdot T)_{\mu\nu\rho} = S_{\mu\nu\alpha} g^{\alpha\beta} T_{\beta\rho}$$

where $g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\nu}$ and

$$S = S_{\mu\nu\rho} e^{\mu} \otimes e^{\nu} \otimes e^{\rho}, \text{ etc.}$$

If L is a vector subspace of V, then

$$L^{0} = \{\alpha \in V^{*} \ : \ \text{if} \ \ u \in L \ \ \text{then} \ \ u \lrcorner \alpha = 0\}$$

is a vector subspace of V^{*}. If K is another subspace of V and $K \subset L$, then $L^{0} \subset K^{0}$ and the vector spaces $(L/K)^{*}$ and K^{0}/L^{0} are isomorphic to one another in a natural manner.

If V has an orientation and a metric tensor g, then one can define the *Hodge dual*

$$*(g) : \Lambda V^{*} \rightarrow \Lambda V^{*}$$

in the usual way ; and

$$u \lrcorner *(g)\alpha = *(g)(\alpha \wedge g(u)) \tag{2.3}$$

for any $u \in V$ and $\alpha \in \Lambda V^{*}$. When g is fixed, then one usually writes $*\alpha$ instead of $*(g)\alpha$; but we shall occasionally need the more elaborate notation to account for the dependence of the dual on the metric.

All manifolds and maps are of class C^{∞} or real analytic. The tangent and cotangent bundles of a manifold M are denoted by TM and $T^{*}M$, respectively. If $\varphi : M \rightarrow N$ is a differentiable map and g is covariant tensor field on N, then $\varphi^{*}g$ denotes its *pull-back* to M. A vector field k on M generates a *flow* $(\varphi_{t})_{t\in\mathbb{R}}$ on M, i.e. a local, one-parameter group of local transformations of M. The *Lie derivative* of the tensor field g with respect to k is defined by

$$\mathcal{L}_k g = \frac{d}{dt} \varphi_t^* g \Big|_{t=0}.$$

If α is a p-form field on M, then

$$\mathcal{L}_k \alpha = k \lrcorner d\alpha + d(k \lrcorner \alpha), \tag{2.4}$$

where d denotes the *exterior derivative* and the contraction is defined "pointwise", $(k \lrcorner \alpha)(x) = k(x) \lrcorner \alpha(x)$ for any $x \in M$. There are similar pointwise extensions of the algebraic operations defined on vector spaces to TM and the associated fibre bundles over M ; we use them without further explanation. We often omit the word "field" and speak of a metric tensor or a form on M when we mean a metric tensor field or a field of forms.

A *Lorentz space* is a four-dimensional manifold M with a metric tensor g such that g restricted to any tangent space to M is Lorentzian. For example, the Minkowski space $\mathbb{R}^4$ with coordinates (u,x,y,r) and

$$g = 2\, du\, dr - dx^2 - dy^2 \tag{2.5}$$

is a Lorentz space. The Levi-Civita connection ∇ on a Riemannian or Lorentz space may be computed from the *Christoffel formula*

$$\nabla_X Y = \frac{1}{2} g^{-1}((\mathcal{L}_X g)(Y) + (\mathcal{L}_Y g)(X) - d(g(X,Y))) + \frac{1}{2} [X,Y], \tag{2.6}$$

where X and Y are vector fields, $[X,Y] = \mathcal{L}_X Y$ is their bracket, $\nabla_X Y$ denotes the *covariant derivative* of Y in the direction of X and the Lie derivative $\mathcal{L}_X g$ is interpreted as a vector bundle map $TM \to T^*M$ so that $(\mathcal{L}_X g)(Y)$ is a one-form on M. Since $\nabla g = 0$ one has also

$$\nabla_X g(Y) = \frac{1}{2} X \lrcorner dg(Y) + \frac{1}{2} (\mathcal{L}_Y g)(X). \tag{2.7}$$

It is sometimes convenient to use the formula

$$\mathcal{L}_X\alpha = \nabla_X\alpha + i(\nabla X)\alpha \tag{2.8}$$

valid for any p-form α on M.

3. HEURISTIC CONSIDERATIONS AND EXAMPLES

Optical geometry evolved from the study of simple, null electromagnetic fields and its extension to gravitation and other classical, relativistic fields. Before presenting formal definitions and results, we recall a few known facts and examples, with the intention of "setting the stage" and justifying the abstractions we make.

Consider an electromagnetic field described in Minkowski space-time with Cartesian coordinates (t,x,y,z) by the vectors $E = (E_x, E_y, E_z)$ and $B = (B_x, B_y, B_z)$. Introducing the two-form

$$F = dt\wedge(E_x dx + E_y dy + E_z dz) - B_x dy\wedge dz - B_y dz\wedge dx - B_z dx\wedge dy$$

and its dual

$$*F = dt\wedge(B_x dx + B_y dy + B_z dz) + E_x dy\wedge dz + E_y dz\wedge dx + E_z dx\wedge dy$$

we can write *Maxwell's equations* in a region free of charges as

$$dF = 0 \quad \text{and} \quad d{*F} = 0. \tag{3.1}$$

These equations can be used, in the same form, in any oriented Lorentz four-manifold with metric tensor g ; the Hodge dual is then understood to be taken with respect to g. A solution of (3.1) will be referred to as a Maxwell field.

The ratio

$$v = E \times B/\tfrac{1}{2}(E^2 + B^2)$$

of the Poynting vector to the energy density is a vector characterizing the velocity of propagation of the field ;

its magnitude is never greater than 1 ; it is equal to 1 if, and only if, the electromagnetic field is *null*, i.e. if the vectors E and B are orthogonal to each other and of equal length. Let

$$F = E + \sqrt{-1}\, B \quad \text{and} \quad f = \begin{pmatrix} -F_x + \sqrt{-1}\, F_y & F_z \\ F_z & F_x + \sqrt{-1}\, F_y \end{pmatrix} .$$

The property of being null is algebraic ; for this reason let us restrict our attention to a point of space-time and assume that $F \neq 0$ is the two-form of the electromagnetic field at that point. The following conditions are then equivalent :

(i) the electromagnetic field is null ;

(ii) there is a vector $k \neq 0$ such that $k \lrcorner F = 0$ and $k \lrcorner *F = 0$;

(iii) there is a one-form $\varkappa \neq 0$ such that $\varkappa \wedge *F = 0$ and $\varkappa \wedge F = 0$;

(iv) there is a vector $k \neq 0$ such that

$$k \lrcorner F = 0 \quad \text{and} \quad g(k) \wedge F = 0 ; \tag{3.2}$$

(v) the two invariants of the field vanish,

$$F \wedge F = 0 \quad \text{and} \quad F \wedge *F = 0 ; \tag{3.3}$$

(vi) the complex vector F is null, $F^2 = 0$;

(vii) the matrix f is of rank 1.

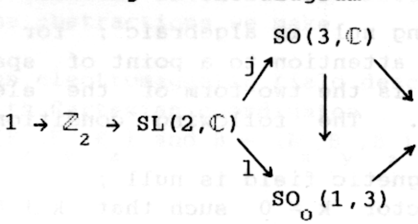
It is an easy matter to check the equivalence of these conditions; for example, if the field is null, then the vector and the one-form referred to in (ii), (iii) and (iv) may be taken, respectively, as

$$k = \partial/\partial t + v_x \partial/\partial x + v_y \partial/\partial y + v_z \partial/\partial z$$

and

$$\varkappa = dt - v_x dx - v_y dy - v_z dz.$$

Since $F \neq 0$, it follows from (iv) that k is null (orthogonal to itself). The property of the symmetric matrix f listed under (vii) is the basis of the spinor interpretation of null two-forms : there is a complex vector (spinor) $\varphi = (\varphi_1, \varphi_2)$ such that $F_z = \varphi_1\varphi_2$, $-F_x + \sqrt{-1}\,F_y = \varphi_1^2$ and $F_x + \sqrt{-1}\,F_y = \varphi_2^2$; the spinor is determined by F up to a sign. The last remark shows that the set of all non-zero null two-forms can be given the structure of a four-manifold diffeomorphic to $\mathbb{RP}_3 \times \mathbb{R}$. A unimodular automorphism $\varphi \mapsto U\varphi$ induces a complex orthogonal transformation j(U) of the vector F associated with φ and also a Lorentz transformation l(U) of the two-form F. In a standard manner one derives from these considerations the exact sequences of group homomorphisms occuring in the diagram



(3.4)

where the vertical arrow is an isomorphism of the group of complex rotations in $\mathbb{C}^3$ onto the connected component of the identity of the Lorentz group.

Maxwell's and Yang-Mills' equations on a four-manifold are *conformally invariant*. This well-known fact follows from the following simple lemma : Let M be a four-manifold and g and g' be two metric tensors on M. Then $*(g)F = *(g')F$ for any two-form F on M if, and only if, there exists a function h on M such that $g' = hg$.

Let now M be the open submanifold of $\mathbb{R}^4$, with coordinates (u,x,y,r), such that both of the metric tensors (2.5) and

$$g' = (1-2m/r)du^2 + 2du\,dr - (r/(1+\tfrac{1}{4}(x^2+y^2)))^2(dx^2+dy^2)$$

are well defined on M. Clearly, g is the Minkowski metric tensor and g' is the Schwarzschild solution of mass m. Let A be a smooth complex function on M and $F = \mathrm{Re}\,\Phi$, where

$$\Phi = A\, du \wedge (dx + \sqrt{-1}\, dy) \qquad (3.5)$$

One easily checks that

$$*(g)F = *(g')F \qquad (3.6)$$

so that F is a null Maxwell field in both geometries if, and only if, A depends on u and $x + \sqrt{-1}\, y$ only. The metric tensors g and g' are not conformally related to each other. The equality (3.6) fails to hold for two-forms F other than those represented by (3.5). Incidentally, the interpretation of the solution F is different in the two geometries : in Minkowski space-time, F represents a plane-fronted electromagnetic wave whereas in the other case F is a field with a spherical wave-front propagating over the Schwarzschild background.

As another heuristic example, consider again the Minkowski metric tensor (2.5) and introduce new coordinates (u',x',y',r') by putting

$$u=u'+r'(x'^2+y'^2),\ x + \sqrt{-1}y = 2(r'+\sqrt{-1})(x'+ \sqrt{-1}y'),\ r=r'.$$

Transforming (2.5) and dropping the primes one obtains

$$g = 2\varkappa dr - 2(r^2 + 1)(dx^2 + dy^2)$$

where

$$\varkappa = du + 2(xdy - ydx).$$

The two-form $\Phi = A\varkappa \wedge (dx + \sqrt{-1}\, dy)$ is self-dual and null; the real form $F = \operatorname{Re} \Phi$ is a Maxwell field if, and only if, the complex function A satisfies $dA \wedge \varkappa \wedge (dx + \sqrt{-1}dy) = 0$. This is equivalent to $\partial A/\partial r = 0$ and (Trautman[55])

$$\frac{\partial A}{\partial x} + \sqrt{-1}\, \frac{\partial A}{\partial y} - 2\sqrt{-1}\, (x + \sqrt{-1}\, y)\, \frac{\partial A}{\partial u} = 0,$$

the homogeneous form of the celebrated *Hans Lewy*[8] *equation* which gave impetus to the development of the modern theory of CR structures (Wells[2,9]).

Optical geometry has been developed to put into a perspective observations such as those given above. Roughly speaking, it is the weakest geometry sufficient to write Maxwell's equations for null electromagnetic fields associated with a line subbundle of the tangent bundle of a four-manifold. The line bundle in question is spanned by the vectors k appearing in the characterizations of null electromagnetic fields given under (ii) and (iv). To support null and nowhere vanishing Maxwell fields, the geometry should be invariant with respect to the flow generated by a section of the line bundle ; if it is, then the quotient geometry is that of a CR space. Its Levi form is proportional to what physicists call the curl or twist of the congruence (foliation) defined by the line bundle.

4. FLAG GEOMETRY

It is convenient to introduce optical geometry in two steps : first, we define a "flag geometry" sufficient to account for the null geodetic property of a congruence of curves. It is known that conformally related Lorentz metrics have the same null geodesics (light rays). Restricting one's attention to a particular congruence of such curves, one can subject the metric to tranformations more general than conformal without altering the null geodetic property.

A *flag geometry* on an n-dimesional manifold M, $n\geq 3$, is a pair (K,L) of vector bundles such that $K \subset L \subset TM$ and the fibre dimensions of K and L are 1 and n-1, respectively. Dually, a flag geometry can be defined by the pair (K^{0}, L^{0}), where $L^{0} \subset K^{0} \subset T^{*}M$. If k and $\varkappa$ are any sections of $K \to M$ and $L^{0} \to M$, respectively, then

$$k \,\lrcorner\, \varkappa = 0. \tag{4.1}$$

A *metric tensor* g on M is said to be *adapted* to the flag geometry if

$$g(k) \wedge \varkappa = 0$$

for any two such sections. If the metric tensor is considered as an isomorphism $g\,:\ TM \to T^{*}M$ of vector bundles, then the property of being adapted is expressed by $g(K) = L^o$. Since g is symmetric the latter condition is equivalent to $g(L) = K^o$. The bundle L may be thought of as the bundle orthogonal to K with respect to each adapted metric tensor g ; the elements of K are orthogonal to themselves, i.e. null with respect to g. The notion of "being adapted" to the flag geometry can be extended to tensors of various types. For example, a tensor field of valence (1,1), i.e. a vector bundle map $s : TM \to TM$, is said to be adapted if

$$s(K) \subset K \tag{4.2a}$$

and

$$s(L) \subset L. \tag{4.2b}$$

Such a tensor field defines a linear endomorphism of the quotient bundle

$$\hat{s}: L/K \to L/K$$

and $\hat{s} = 0$ if, and only if,

$$s = k \otimes \xi + u \otimes \varkappa$$

for some one-form ξ and vector u.

We say that a p-form F on M is *adapted* to (K,L) if

$$k \lrcorner F = 0 \quad \text{and} \quad \varkappa \wedge F = 0 \tag{4.3}$$

for any sections k and $\varkappa$ of $K \to M$ and $L^o \to M$, respectively. It is clear that if a two-form is adapted, then it is null with respect to any adapted metric ; given a flag geometry and an adapted metric g, there are two-forms that are null with respect to g, but not adapted to the flags. It follows from (2.3) that if both the metric tensor g and a p-form F are adapted, then so is $*(g)F$.

Let $(\varphi_t(k))_{t\in\mathbb{R}}$ be the flow generated by a section k of $K \to M$. The vector bundle L is invariant with respect to the flow if, and only if,

$$\mathcal{L}_k \varkappa \wedge \varkappa = 0. \tag{4.4}$$

Clearly, if it is invariant with respect to $(\varphi_t(k))$, then it is also invariant with respect to $(\varphi_t(fk))$, where f is any function on M. It is meaningful, therefore, to define L as being invariant with respect to K if condition (4.4) holds for any sections k and $\varkappa$ of $K \to M$ and $L^o \to M$, respectively. We have proved elsewhere (Robinson and Trautman[10,11])

PROPOSITION 1. *The following properties of a flag geometry are equivalent :*

(i) *L is invariant with respect to K ;*

(ii) *the three-form $\varkappa \wedge d\varkappa$ is adapted to (K,L);*

(iii) *the lines of the flow $(\varphi_t(k))$ define a congruence of null geodesics with respect to any metric tensor adapted to (K,L);*

(iv) *if* F *is any adapted form on* M, *then* $\mathcal{L}_k F$ *is also adapted;*

(v) *if* F *is an adapted (n-2)-form on* M, *then* $\varkappa \wedge dF=0$;

(vi) *if* g *is an adapted metric, then the tensor* $s = g^{-1} \circ \mathcal{L}_k g$ *is adapted.*

A flag geometry which has any - and therefore all - of properties (i)-(vi) is said to be *geodetic*. If the bundle is *integrable*, i.e. such that $\varkappa \wedge d\varkappa = 0$, then the flag geometry is geodetic. An integrable bundle L defines on M a foliation of co-dimension one ; in this case the congruence is said to be hypersurface-orthogonal. In the non-integrable case, physicists say that the congruence is *twisting*.

We assume throughout this paper that the foliation of M defined by K is regular in the sense that the quotient set $N = M/K$ has a natural structure of an (n-1)-manifold such that the canonical projection

$$\pi : M \to N$$

is a submersion. If the flag geometry (K,L) on M is geodetic, then L projects to a vector bundle $H \subset TN$ of co-dimension one on N. If λ is a section of $H^o \to N$, then $\varkappa = \pi^* \lambda$ is a section of $L^o \to M$ and

$$\mathcal{L}_k \varkappa = 0.$$

Clearly, the integrability of L is equivalent to that of H.

5. OPTICAL GEOMETRY

A flag geometry is not sufficient to write the second Maxwell equation (3.1) for adapted two-forms : the dual $*(g)F$ depends on the choice of g. Assume that M is a four-dimensional oriented manifold with a flag geometry (K,L) and denote by A the corresponding set of adapted Lorentz metric tensors on M. If F is a nowhere vanishing adapted two-form on M, then

$$gRg' \quad \text{iff} \quad *(g)F = *(g')F \tag{5.1}$$

defines an *equivalence relation* R in A. The relation R does not depend on the choice of the nowhere vanishing adapted two-form F : indeed, since any other such two-form F' can be written as

$$F' = aF + b*(g)F,$$

for some functions a and b, and

$$*(g)*(g) = -\mathrm{id} \quad \text{on two-forms,} \tag{5.2}$$

we see that $*(g)F' = *(g')F'$ is equivalent to $*(g)F = *(g')F$.

One easily checks that two adapted Lorentz metrics g and g' are in relation R if, and only if, there is a function f and a one-form ξ such that

$$g' = fg + 2\varkappa\xi \tag{5.3}$$

where $\varkappa$ is a nowhere vanishing section of $L^0 \rightarrow M$. (Our considerations do not, in fact, depend on the existence of nowhere vanishing and globally defined objects such as $\varkappa$ and F above ; all relevant definitions and propositions can be "localized" by simple rephrasing).

Definition 1. *An optical geometry on a four-dimensional oriented manifold M consists of*
(a) *a flag geometry (K,L) and*
(b) *an equivalence class B, with respect to R, of adapted Lorentz metric tensors on M.*

Condition (b) occuring in the definition can be replaced by an equivalent one. Let F and $\varkappa$ be nowhere vanishing adapted two- and one-forms, respectively. We have

$$F = \varkappa \wedge \alpha$$

where α is a section of $K^{\circ} \to M$, which is defined by F and $\varkappa$ only up to addition of multiples of $\varkappa$. In other words, F and $\varkappa$ define a section $[\alpha]$ of the quotient bundle $K^{\circ}/L^{\circ} \to M$ and it is clear that the bundle of adapted two-forms is isomorphic to the bundle $(K^{\circ}/L^{\circ}) \otimes L^{\circ} \to M$. Let $g \in B$ and put

$$*(g)F = \varkappa \wedge \beta.$$

The linear map

$$J : L/K \to L/K \tag{5.4}$$

given by

$$\langle J_p(l \bmod K_p),\ [\alpha]_p\rangle = \langle l \bmod K_p, [\beta]_p\rangle$$

where K_p is the fibre of K over $p \in M$, $l \in L_p$, etc., defines, by virtue of (5.2),

(b') *a complex structure in the real plane bundle* $L/K \to M$.

Conversely, given such a complex structure J, one defines B by declaring that $g \in A$ belongs to B if, and only if, g induces on L/K the same conformal geometry as J.

Noting that the bundle L is orthogonal to K with respect to each $g \in B$ we arrive at the equivalent

Definition 1'. *An optical geometry on a four-dimensional oriented manifold* M *consists of a line bundle* $K \subset TM$ *and a set* B *of Lorentz metric tensors on* M *such that the elements of* K *are null with respect to each* $g \in B$ *and the following holds: if* $g \in B$ *and* k *is a nowhere vanishing section of* K, *then* $g' \in B$ *if, and only if, there is a function* f *and a vector field* u *on* M *such that*

$$g' = fg + 2g(k)g(u) \tag{5.5}$$

With this definition in mind, we shall often refer to an optical geometry given by the pair (g,k), where g is a Lorentz metric tensor and k a nowhere vanishing vector field which is null with respect to g. Given such a pair, K is defined as spanned by k and B is generated from g and k according to (5.5).

If F is a two-form adapted to (K,L) and $g \in B$, then the complex two-form

$$\Phi = F - \sqrt{-1} * (g) F$$

is also adapted and

$$*(g)\, \Phi = \sqrt{-1}\, \Phi \quad , \quad *(g) \bar{\Phi} = - \sqrt{-1}\, \bar{\Phi} \tag{5.6}$$

With a slight abuse of the language, we say that Φ and $\bar{\Phi}$ are adapted, *self-* and *antiself-dual* two-forms, respectively. If F nowhere vanishes and Φ' is another adapted self-dual two-form, then there is a complex function χ on M such that

$$\Phi' = \Phi \exp \sqrt{-1}\, \chi \tag{5.7}$$

and any adapted two-form is a linear combination of Φ and $\bar{\Phi}$.

Let $s : TM \to TM$ be a tensor adapted to (K,L); if F is an adapted two-form, the so is $i(s)F$. Therefore, there exist complex functions ρ and σ on M such that

$$i(s)\Phi = \rho\Phi + \sigma\bar{\Phi} \,. \tag{5.8}$$

When Φ is replaced by (5.7), the function ρ is left invariant, but the phase of σ changes by $2 \operatorname{Re} \chi$. The absolute value of σ is an invariant called the *shear* of s.

Writing

$$i(s) = i(s)_{+} + i(s)_{-}$$

where

$$i(s)_{\pm} = \frac{1}{2}\left(i(s) \mp *(g) i(s) *(g)\right)$$

one obtains

$$*(g) i(s)_{\pm} = \pm\, i(s)_{\pm} *(g) \quad \text{on two-forms}$$

so that

$$i_+(s)\Phi = \rho\Phi \quad \text{and} \quad i_-(s)\Phi = \sigma\bar{\Phi}$$

If s is symmetric with respect to g, i.e., $s^* = g \circ s \circ g^{-1}$, then (Trautman[12])

$$*(g)i(s) + i(s)*(g) = (\mathrm{Tr}\ s)*(g)$$

so that

$$i_-(s) = i(s - \frac{1}{4}(\mathrm{Tr}\ s)\mathrm{id}) \quad \text{on two-forms.}$$

If s is symmetric and adapted, then

$$i(s)\Phi = 0 \quad \text{iff} \quad s = k \otimes g(u) + u \otimes g(k) - \frac{1}{2} g(k,u)\mathrm{id}$$

for some vector field u. The tensor s is said to be *fully adapted* to (K,L) if it satisfies (4.2a) and

$$s(TM) \subset L.$$

This implies $i(s)\varkappa = 0$ and

$$i(s^2)F = i(s)^2 F$$

for any adapted two-form F. Moreover, if s is symmetric and fully adapted, then its shear vanishes if, and only if,

$$s - \frac{1}{2}(\mathrm{Tr}\ s)\mathrm{id} = k \otimes g(u) + u \otimes g(k) \tag{5.9}$$

Let M_1 and M_2 be two oriented manifolds with optical geometries defined by (K_i, B_i), $i = 1,2$. An orientation preserving diffeomorphism $\varphi : M_1 \to M_2$ is said to be an *optical isomorphism* if

$$\varphi_* K_1 = K_2 \quad \text{and} \quad \varphi^* B_2 = B_1 .$$

A diffeomorphism $\varphi : M \to M$ is an optical automorphism if, for any section k of $K \to M$, the vector field $\varphi_* k$ is parallel to k and $g' = \varphi^* g$ is of the form (5.5) for any $g \in B$. A flow (φ_t) generated by a vector field X on M consists of optical automorphisms if $[X,k]$ is parallel to k and $\mathcal{L}_X g$ is of the form given by the right-hand side of (5.5).

If the optical geometries on M_1 and M_2 are defined by the pairs (g_1,k_1) and (g_2,k_2), respectively, and φ is a conformal transformation, $\varphi^* g_2 = f g_1$, mapping k_1 into a vector parallel to k_2 , then φ is an optical isomorphism. Such a φ is called a *trivial* optical isomorphism.

PROPOSITION 2. *The flow (φ_t) generated by a section k of $K \to M$ consists of optical automorphisms if, and only if, the tensor*

$$s = g^{-1} \circ \mathcal{L}_k g \,, \quad \textit{where} \quad g \in B \tag{5.10}$$

is adapted and its shear vanishes.

Indeed, we note first that the properties stated in Proposition 2 are invariant by replacement of k by another section of the same bundle and do not depend on the choice of g in B. Secondly, if k generates a flow of optical automorphisms, then the tensor (5.10) is of the form

$$s = f \,\mathrm{id} + k \otimes g(u) + u \otimes g(k).$$

It is adapted and $i(s)\Phi = 2f\Phi$, so that its shear vanishes. Conversely, if s is adapted and its shear vanishes, then, since $g \circ s$ is symmetric, $i(s)\Phi = 2f\Phi$, where f is a real function. Therefore, $i(s - f\,\mathrm{id})\Phi = 0$ and $\mathcal{L}_k g = g \circ s$ is of the form (5.5).

If the flow (φ_t) generated by k consists of optical automorphisms, then the associated congruence is null geodetic (Prop. 1., (iii) and (vi)) and the shear of s vanishes : one says that the optical geometry - and the associated congruence - is shear-free. An equivalent characterization, which explains the term "shear-free", is as follows : the flow (φ_t) consists of optical automorphisms if, and only if, the associated congruence is null geodetic and the conformal geometry in the fibres of L/K is preserved by the flow. The importance of shear-free optical geometries results from

PROPOSITION 3. *If an optical geometry admits an adapted nowhere vanishing Maxwell field, then it is shear-free.*

Indeed, let F be such a field on M with an optical geometry defined by the pair (g,k). Since F is adapted,

$$k \lrcorner F = 0 \text{ and } k \lrcorner *(g)F = 0.$$

Maxwell's equations

$$dF = 0 \quad \text{and} \quad d*(g)F = 0 \tag{5.11}$$

then imply

$$\mathcal{L}_k F = 0 \quad \text{and} \quad \mathcal{L}_k *(g)F = 0 \tag{5.12}$$

so that both F and $*(g)F$ are invariant with respect to the flow (φ_t) generated by k,

$$*(g)F = \varphi_t^*(*(g)F) = *(\varphi_t^* g)\varphi_t^* F = *(\varphi_t^* g)F.$$

By comparing this with (5.1) we see that the flow (φ_t) consists of optical automorphisms. The converse to Proposition 3 - which is true under suitable regularity hypotheses - is presented in the next section. The following Proposition is a simple consequence of our definitions :

PROPOSITION 4. *Optical isomorphisms transform adapted Maxwell fields into fields of the same kind.*

The first example in Section 3 contains the description of an optical isomorphism transforming plane-fronted waves in Minkowski space into spherically-fronted waves on a Schwarzchild background. Another example of a non-trivial isomorphism of optical geometries is obtained as follows. Let $t = (t^\mu)$, $\mu = 0,1,2,3$ be coordinates in Minkowski space $\mathbb{R}^4$ with the metric tensor

$$g_{\mu\nu} dt^\mu dt^\nu = (dt^0)^2 - (dt^1)^2 - (dt^2)^2 - (dt^3)^2. \tag{5.13}$$

Consider a time-like wordline $\tau : \mathbb{R} \to \mathbb{R}^4$ parametrized so that

$$g_{\mu\nu}\dot{\tau}^{\mu}\dot{\tau}^{\nu} = 1 \text{ , where } \dot{\tau}^{\mu}(s) = d\tau^{\mu}(s)/ds.$$

Assume τ to be such that, for any $t \in \mathbb{R}^4$ the wordline meets the past lightcone of t, i.e. there is a function $u: \mathbb{R}^4 \to \mathbb{R}$ such that

$$g_{\mu\nu}(t^{\mu} - \tau^{\mu}(u(t)))(t^{\nu} - \tau^{\nu}(u(t))) = 0 \quad \text{and} \quad t^0 \geq \tau^0(u(t))$$

for any $t \in \mathbb{R}^4$. Introduce a local coordinate chart (u,x,y,r) in $\mathbb{R}^4$ by putting

$$t^{\mu} = \tau^{\mu}(u) + rl^{\mu}(x,y)/p(u,x,y),$$

where (l^{μ}) is the null vector with components

$$(1 + \frac{1}{4}(x^2 + y^2),\ x,\ y,\ 1 - \frac{1}{4}(x^2 + y^2)),$$

$$r = g_{\mu\nu}\dot{\tau}^{\mu}(t^{\nu} - \tau^{\nu}) \quad \text{and} \quad p = g_{\mu\nu}\dot{\tau}^{\mu}l^{\nu}.$$

The domain of the new chart is the complement M' in $\mathbb{R}^4$ of the two-dimensional submanifold with boundary

$$S' = \{t \in \mathbb{R}^4 | t^{\mu} = \tau^{\mu}(u) + rn^{\mu},\ -\infty < u < \infty,\ r \geq 0\},$$

where (n^{μ}) is the null vector with components $(1,0,0,-1)$. In this chart, the metric tensor (5.13) is

$$g' = (1 - 2p^{-1}\dot{p}r)du^2 + 2\,du\,dr - r^2p^{-2}(dx^2 + dy^2) \qquad (5.14)$$

where $\dot{p} = \partial p/\partial u$. Let $S = \{t \in \mathbb{R}^4 \mid t^1 = t^2 = 0$ and $t^0 + t^3 \geq 0$ or $t^0 + t^3 \leq 0\}$ take another copy of Minkowski space, put $M = \mathbb{R}^4 \backslash S$ and introduce a coordinate system (u,x,y,r) on M by $u = t^0 - t^3$, $2r = t^0 + t^3$, $x = t^1$, $y = t^2$ so that the metric tensor g is of the form (3.4). Consider now optical geometries in M and M' defined by the pairs (g,k) and (g',k'), where g and g' are given by (3,4) and (5.14), respectively, and $k = k' = \partial/\partial r$. The map h from M' to M which reduces to the identity when expressed in coordinates (u,x,y,r) is an optical isomorphism. If F is a plane-fronted wave described in Section 3, then h^*F is a spherically-fronted wave emanating from a point source whose (accelerated) motion is given by the wordline τ .

Proposition 3 and 4 can be extended to Yang-Mills fields (Trautman[13]).

6. CR SPACES ASSOCIATED WITH SHEAR-FREE OPTICAL GEOMETRIES

The complex structure in the fibres of the bundle $L/K \to M$ is invariant under the action of the flow generated by any section of the line bundle K underlying an optical geometry without shear. Therefore, the complex structure descends to the plane bundle $H \subset TN$ associated with the quotient manifold N. By definition, such a complex structure makes the three-dimensional manifold N into a *Cauchy-Riemann space*. To simplify the notation, we use the same letter J to denote the complex structure on H and on L/K. Let λ be a section of $H^0 \to N$ (cf. Sec.4) and let X be a nowhere vanishing section of $H \to N$. The vector field J(X) is also a section of $H \to N$; if V is a vector field on N such that $V \lrcorner \lambda = 1$, then the triple (V,X,J(X)) of vector fields spans at each point the tangent space of N at that point. It is convenient to introduce the complex vector field $Z = X - \sqrt{-1}\, J(X)$ and its complex conjugate $\bar{Z}$. Let μ be a complex one-form on N such that the triple $(\lambda,\mu,\bar{\mu})$ constitutes a basis dual to $(V,Z,\bar{Z})$ in the sense that one has also $Z \lrcorner \mu = 1$, $Z \lrcorner \bar{\mu} = 0$ and $V \lrcorner \mu = 0$. The forms λ and μ are defined by the structure of the CR space up to replacements by

$$\lambda' = \alpha\lambda \tag{6.1a}$$

and

$$\mu' = \beta\mu + \gamma\lambda, \tag{6.1b}$$

where α is a real function and both β and γ are complex functions on N ; both α and β are nowhere zero.

Given a CR space N, one can construct an associated shear-free optical geometry as follows. Take $M = N \times \mathbb{R}$, let $\pi : M \to N$ be the projection on the first factor and denote by r the standard coordinate on $\mathbb{R}$. To simplify the notation, we omit π^* when considering pull-backs of forms from N to M. The optical geometry on M defined by the pair (g,k), where

$$g = 2\lambda dr - 2\mu\bar{\mu} \quad \text{and} \quad k = \partial/\partial r ,$$

is invariant with respect to transformations (6.1). It is shear-free since $\mathcal{L}_k g = 0$. Therefore, as far as local properties are concerned, there is a one-to-one, natural correspondence between CR spaces and optical geometries without shear.

The differential of any complex function f on N can be represented as

$$df = f_o\lambda + f_1\mu + f_2\bar{\mu} \tag{6.2}$$

where

$$f_o = V \lrcorner\, df, \quad f_1 = Z \lrcorner\, df, \quad f_2 = \bar{Z} \lrcorner\, df.$$

The *tangential Cauchy-Riemann equation*

$$f_2 = 0, \tag{6.3}$$

which is equivalent to

$$df \wedge \lambda \wedge \mu = 0, \tag{6.4}$$

is of the type considered by Hans Lewy ; the assumption of C^∞ smoothness on N, H and J is not sufficient to guarantee the existence of non-trivial solutions to (6.3) (Jacobowitz and Trèves[14]). If the CR space is real-analytic, then such solutions can be found. Assume now that there do exist two solutions w and z of (6.3) such that the map

$$(w,z) : N \to \mathbb{C}^2 \tag{6.5}$$

is an immersion, i.e. that its tangent map is of rank 3. The image of N by (6.5) is a hypersurface in $\mathbb{C}^2$; let

$$G(w,z,\bar{w},\bar{z}) = 0 \qquad (6.6)$$

be its equation. Here G is a real function vanishing identically when its arguments are replaced by the solutions w and z of (6.3); moreover, G may be chosen so that its gradient is nowhere zero,

$$|G_w|^2 + |G_z|^2 > 0$$

where $G_w = \partial G/\partial w$, etc. The one-form

$$\sqrt{-1}\,(G_w dw + G_z dz) \qquad (6.7)$$

is real, nowhere zero and annihilated by Z and $\bar{Z}$. It is, therefore, proportional to λ; without changing the CR structure we may now assume λ to be given by (6.7),

$$Z = G_w \partial/\partial z - G_z\, \partial/\partial w,$$

and

$$\mu = (G_{\bar{w}} dz - G_{\bar{z}} dw)/(|G_z|^2 + |G_w|^2).$$

If the *"Levi form"* (Levi[15])

$$2a = [Z,\bar{Z}] \lrcorner\ \lambda/\sqrt{-1}$$

$$= G_w G_{\bar{w}} G_{z\bar{z}} + G_z G_{\bar{z}} G_{w\bar{w}} - G_z G_{\bar{w}} G_{\bar{z}w} - G_{\bar{z}} G_w G_{z\bar{w}}$$

is nowhere zero, then the real vector field $\sqrt{-1}\,[Z,\bar{Z}]$ is at no point linearly dependent on Z and $\bar{Z}$. At any point of N at least one of the partial derivatives G_w and G_z is different from zero. Restricting our attention to a sufficiently small neighbourhood of a point where $G_w \neq 0$, we can replace Z and μ by

$$Z' = G_w^{-1} Z$$

and

$$\mu' = dz,$$

respectively, without changing the CR structure. The triple $(V, Z', \bar{Z}')$, where

$$V = |G_w|^2 \; [Z', \bar{Z}']/2\sqrt{-1}\; a,$$

constitutes at any point of N a linear basis dual to $(\lambda, \mu', \bar{\mu}')$.

The immersion (6.5) provides a (local) embedding of the CR space into $\mathbb{C}^2$; one says that the *CR structure* of N *is realized on a hypersurface* in $\mathbb{C}^2$. Roger Penrose[5] has pointed out that there may be non-realizable CR structures of interest in physics.

Let (ω, ζ) be a (possibly local) biholomorphic transformation of $\mathbb{C}^2$ into itself. The functions

$$w' = \omega(w,z) \quad \text{and} \quad z' = \zeta(w,z)$$

are also solutions of (6.4) and

$$(w', z') : N \to \mathbb{C}^2$$

is another realization of N in $\mathbb{C}^2$; the equation of the corresponding hypersurface is $G'(w,z,\bar{w},\bar{z}) = 0$, where

$$G'(w',z',\bar{w}',\bar{z}') = G(w,z,\bar{w},\bar{z}).$$

Elie Cartan[16] solved the local equivalence problem for such hypersurfaces : he found a set of differential invariants associated with hypersurfaces in $\mathbb{C}^2$ such that the pointwise equality of the corresponding invariants for two hypersurfaces is a necessary and sufficient condition for the existence of a biholomorphic map transforming one hypersurface into another. Cartan's method was simplified and generalized to hypersurfaces in $\mathbb{C}^n$ by Chern and Moser[3] and Tanaka[17].

If the CR space is realizable in $\mathbb{C}^2$ and A denotes an arbitrary analytic function of two complex variables, then the complex two-form

$$\Phi = A(w,z)dw \wedge dz, \tag{6.8}$$

is closed; its pull-back to M is self-dual and adapted to the optical geometry defined by (6.2) with λ given by (6.7) and $\mu = dz$. This proves

PROPOSITION 5. *If the CR space associated with a shear-free optical geometry on M is realizable in $\mathbb{C}^2$, then M admits an adapted, non-zero Maxwell field.*

If one is given a solution z of (6.3) such that

$$\lambda \wedge dz \wedge d\bar{z} \neq 0$$

then one can take

$$\mu = dz$$

and find a real function u on N such that the triple (u, Re z, Im z) is a system of (local) coordinates, i.e.,

$$du \wedge dz \wedge d\bar{z} = 0.$$

Putting

$$q = \bar{Z} \lrcorner\, du = u_{\bar{z}}$$

one obtains

$$Z = \partial/\partial z + \bar{q}\, \partial/\partial u.$$

If w is another solution of (6.3) such that (6.5) is an immersion then $dw \wedge dz \wedge d\bar{z} \neq 0$, therefore $w_{\bar{o}} \neq 0$ and one can express u in function of w, z and $\bar{z}$,

$$u = U(w,z,\bar{z}).$$

Since $w_{\bar{z}} = z_{\bar{z}} = 0$ and $\bar{z}_{\bar{z}} = 1$, we have

$$q = \partial U/\partial \bar{z} \tag{6.9}$$

and the equation (6.6) of the hypersurface is obtained by taking

$$\sqrt{-1}\, G(w,z,\bar{w},\bar{z}) = U(w,z,\bar{z}) - \bar{U}(\bar{w},\bar{z},z) \tag{6.10}$$

so that

$$\sqrt{-1}\ (G_w dw + G_z dz) = du - q\, dz - \bar{q}\, d\bar{z}. \tag{6.11}$$

If the bundle H is integrable, then there exists a real function u on N and a choice of the integrating factor α such that $\lambda = du$. One of the solutions of (6.3) is $w=u$ and the equation (6.6) of the embedded hypersurface may be taken as $w - \bar{w} = 0$. The field (6.8) coincides with (3.5) and is a solution of Maxwell's equations if A depends analytically on z; the dependence on the real variable u may be merely smooth; waves associated with an integrable flag geometry can be encoded with information (Trautman[18]). Optical geometries with integrable L have been used in general relativity in connection with research on gravitational waves (Robinson and Trautman[19], Kramer et al.[20]). From now on we consider exclusively optical geometries without shear, and the associated CR spaces, such that $\lambda \wedge d\lambda$ nowhere vanishes.

The second example of Section 2 corresponds to a structure considered already by Henri Poincare[21]: since, in this case $\lambda = du + \sqrt{-1}\ (zd\bar{z} - \bar{z}dz)$, a solution of (6.4) is provided by $w = u + \sqrt{-1}\ z\bar{z}$ and the equation (6.6) is that of a "*hyperquadric*",

$$w - \bar{w} - 2\sqrt{-1}\ z\bar{z} = 0. \tag{6.12}$$

The fractional linear map

$$w' = (w - \sqrt{-1})/(w + \sqrt{-1}), \quad z' = 2z/(w + \sqrt{-1})$$

transforms (6.12) into the equation of S_3,

$$w'\bar{w}' + z'\bar{z}' = 1. \tag{6.13}$$

Introducing the Euler angles (ψ, ϕ, θ) on S_3 by

$$w' = \exp\tfrac{1}{2}\sqrt{-1}\ (\psi - \phi)\ \sin\tfrac{\theta}{2},\ z' = \exp\tfrac{1}{2}\sqrt{-1}(\psi + \phi)\cos\tfrac{\theta}{2}$$

one easily finds that the forms characterizing the CR structure associated with (6.12) can be chosen as

$$\lambda = 2(d\psi + \cos\theta\, d\phi) \quad \text{and} \quad \mu = d\theta - \sqrt{-1}\,\sin\theta\, d\phi.$$

With this notation, the Taub-NUT metric tensor can be written (Misner[22])

$$g = l^2[2\lambda(dr + \tfrac{1}{2}c\lambda) - (r^2+1)\mu\bar{\mu}\,], \text{where } c = 1 - 2(mr+1)/(r^2+1).$$

The pair (g,k), where $k = \partial/\partial r$, defines an optical geometry without shear ; its quotient CR structure is equivalent to that of the hyperquadric.

The Cartan invariants characterizing CR spaces are fairly complicated ; the simplest among them is a (relative) invariant I that vanishes if, and only if, the CR space is locally equivalent to the hyperquadric. To give the explicit formula for I Cartan normalizes the forms λ and μ so as to have

$$d\mu = 0 \quad \text{and} \quad d\lambda = \sqrt{-1}\,\mu \wedge \bar{\mu} + \lambda \wedge (\bar{b}\mu + b\bar{\mu})$$

for some complex function b. Such a normalization is always possible if $\lambda \wedge d\lambda \neq 0$ and then

$$I = b_{122} - 3bb_{12} - b_1 b_2 + 2b^2 b_1 + 2\sqrt{-1}\, b_{02} - 4\sqrt{-1}\, bb_0 . \qquad (6.14)$$

An *automorphism* of an optical geometry without shear permutes the curves of the underlying congruence of null geodesics and preserves the complex structure in the fibres of the bundle *L*/*K*. Therefore, it descends to an automorphism of the quotient CR space and may be characterized in terms of its structure as follows : if λ and μ are forms defining the CR space N, then a diffeomorphism $\varphi : N \to N$ is an automorphism if, and only if, the pull-backs $\lambda' = \varphi^*\lambda$ and $\mu' = \varphi^*\mu$ are related to λ and μ by a transformation (6.1). From the work of B.Segrè[23] it follows that the automorphism group of CR space with non-integrable *H* is a Lie group. E. Cartan[16] has classified all homogeneous CR spaces. Locally, each such space is equivalent to either the hyperquadric or a hypersurface in $\mathbb{C}^2$ admitting a three-dimensional group of automorphisms of Bianchi type IV, VI, VII, VIII or IX (Taub[24]). The hyperquadric admits an eight-dimensional group of CR automorphisms, locally isomorphic to SU(2,1).

7. OPTICAL GEOMETRY AND LORENTZ MANIFOLDS

A Lorentz metric g on an oriented four-manifold M, together with a line bundle $K \subset TM$ of null directions, defines a structure which is richer than optical geometry. All optical notions can be expressed in terms of the data derived from the pair (g,K), but there is more, as we now proceed to show. From now on we assume that the pair (g,K) is given, L is the orthogonal bundle to K and the optical geometry is defined in terms of the metric induced by g in the fibres of L/K. All covariant derivatives are taken with respect to the Levi-Civita connection associated with g and the Hodge dual $*$ is evaluated with the help of g.

Let Φ be a non-zero self-dual two-form, adapted to (K,L). Clearly, the contraction $\Phi \cdot \Phi$ vanishes; the real symmetric tensor $\Phi \cdot \bar{\Phi}$ is a section of the bundle $N \to M$ of squares of non-zero elements of L^o. This section does not change if Φ is multiplied by a point-dependent phase factor. In other words, there is a circle bundle

$$U(1) \to F \to N$$

where F is the bundle of non-zero, self-dual adapted two-forms. Locally, any section of $F \to M$ defines two (opposite) sections of $L^o \to M$. We choose one of them, call it $\varkappa : M \to L^o$, and say that it is associated with $\Phi : M \to F$ so that

$$\Phi \cdot \bar{\Phi} = \varkappa \otimes \varkappa \tag{7.1}$$

This being so, let X be a vector field on M; the two-form $\varkappa \wedge \nabla_X \varkappa$ is adapted and depends linearly on X; there thus exists a complex one-form δ such that

$$\varkappa \wedge \nabla_X \varkappa + \bar{\delta}(X)\Phi + \delta(X)\bar{\Phi} = 0, \tag{7.2}$$

where $\varkappa$ is associated with Φ. If Φ is replaced by $(\exp\sqrt{-1}\chi)\Phi$ where χ is a real function, then the "*deviation*" form (Plebański and Robinson[25], Robinson[26]) δ is replaced by $(\exp\sqrt{-1}\chi)\delta$. By contracting both sides of (7.2) with Φ and using (7.1) one obtains

$$i(\nabla k)\Phi = \varkappa \wedge \delta, \tag{7.3}$$

where ∇k is the covariant derivative of the vector field $k = g^{-1}(\varkappa)$. With the above notation in mind, we formulate

PROPOSITION 6. *For any four-manifold* M *with a Lorentz metric* g, *a line bundle* K *of null directions, and the associated optical geometry, the following conditions are equivalent:*

(i) *the underlying flag geometry is geodetic;*
(ii) *the tensor* ∇k *is adapted to it;*
(iii) *the tensor* ∇k *is fully adapted;*
(iv) $\delta(k) = 0$;
(v) *there exist complex functions* ρ *and* σ *on* M *such that*

$$\varkappa \wedge \delta = \rho\Phi + \sigma\bar{\Phi}. \tag{7.4}$$

Indeed, by Proposition 1, condition (i) is equivalent to the property of $g^{-1} \circ \mathcal{L}_k g$ being adapted and this implies (ii) and conversely ; since k is null, the tensor ∇k is adapted if, and only if, it is fully adapted; by virtue of (7.2), the geodetic property is equivalent to (iv); finally, if $s = \nabla k$ is adapted, then one can define ρ and σ by (5.8) and use (7.3) to prove (7.4); conversely, (7.4) implies (iv).

Consider now a geometry (g, *K*) which is geodetic in the sense that it satisfies the conditions of Proposition 6. There then exists (locally) a nowhere zero section k of the bundle such that

$$\nabla_k k = 0 \tag{7.5}$$

Since covariant differentiation commutes with the Hodge dual, there also exist on M (local) sections Φ of F such that

$$\nabla_k \Phi = 0. \tag{7.6}$$

Let

$$\Psi = \begin{pmatrix} \Phi \\ \overline{\Phi} \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \rho & \sigma \\ \overline{\sigma} & \overline{\rho} \end{pmatrix}$$

then equation (5.8) for $s = \nabla k$ implies the matrix equation

$$i(\nabla k)\Psi = \Sigma\Psi. \tag{7.7}$$

Denoting $\nabla_k \Sigma$ by Σ', computing the covariant derivative of both sides of (7.7) in the direction of k and using (7.6), one obtains

$$i(\nabla_k \nabla k)\Psi = \Sigma'\Psi. \tag{7.8}$$

On the other hand, from the definition of the curvature tensor R in terms of ∇ and vector fields X,Y,Z,

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})\, Z = R(X,Y)Z$$

and taking into account (7.5), one derives

$$(\nabla_k \nabla k)(X) = R(k,X)k - (\nabla k)^2 (X).$$

From the symmetries of the curvature tensor it follows that the tensor S, defined by $S(X) = R(k,X)k$, is fully adapted. Since ∇k is also fully adapted, $i((\nabla k)^2)\Psi = i(\nabla k)^2\Psi$ and (7.8) implies the Sachs equation (Sachs[27]),

$$\Sigma' + \Sigma^2 + P = 0, \tag{7.9}$$

where the matrix P is defined by

$$i(S)\Psi = P\Psi.$$

If the optical geometry is shear-free, then $\sigma = 0$ and the matrix Σ is diagonal. The Sachs equation implies that the shear of S also vanishes. Since S is symmetric and fully adapted, equation (5.9) easily leads to

PROPOSITION 7. *If the pair* (g,K) *defines on* M *an optical geometry without shear, then the symmetric traceless tensor* E,

$$E(X) = C(k,X)k,$$

where C *is the Weyl (conformal curvature) tensor of* g*, is fully adapted and shear-free; the complex expansion* ρ *of* ∇k*, subject to (7.5) satisfies the propagation equation*

$$\rho' + \rho^2 = \tfrac{1}{2}\,\mathrm{Tr}\,S. \tag{7.10}$$

In the notation of tensor calculus, in terms of local coordinates x^μ, one puts $\varkappa = k_\mu dx^\mu$, $k = k^\mu e_\mu$, $e_\mu \lrcorner\, dx^\nu = \delta^\nu_\mu$, $R(e_\rho, e_\sigma)e_\nu = R^\mu{}_{\nu\rho\sigma} e_\mu$ and similarly for the Weyl tensor. One can then write

$$\mathrm{Tr}\,S = \mathrm{Ric}\,(k,k),$$

where Ric is the Ricci tensor, $\mathrm{Ric}(X,Y) = \langle R(X,e_\mu)Y, dx^\mu\rangle$.

The property of E being fully adapted and shear-free reads

$$k_{[\lambda}C_{\mu]\nu\rho[\sigma}k_{\tau]}k^\nu k^\rho = 0 \tag{7.11}$$

and is referred to by saying that K is a bundle of principal null directions of the Weyl tensor (Penrose[28]).

The tensor D,

$$D(X,Y) = C(X,Y)k,$$

defines a graded derivation i(D) of degree 1 and

$$[i(D), i(k)] = i(E). \tag{7.12}$$

If the optical geometry is shear-free and Φ is a section of F, then $i(E)\Phi = 0$ and (7.12) implies that $i(D)\Phi$ is an adapted three-form. There thus exists a complex function A such that

$$*i(D)\Phi = A\lambda \tag{7.13}$$

It can be shown (Robinson[26], Eqs (8.32) and(9.23)) that

$$4\sqrt{-1}\, *(\delta \wedge d\delta) = A\delta + \Phi \cdot \mathrm{Ric} \cdot \delta. \tag{7.14}$$

Moreover, the following conditions are equivalent :

(i) $A = 0$;

(ii) the tensor field E is a section of the bundle $K \otimes L^{o} \to M$;

(iii) the bundle K consists of multiple principal null directions of the Weyl tensor,

$$C^{\mu}{}_{\nu\rho[\sigma}k_{\tau]}k^{\nu}k^{\rho} = 0,$$

which is then said to be algebraically special or degenerate.

If

$$\mathrm{Ric}(L) \subset L^{o} \tag{7.15}$$

then $A = 0$ by the generalized Goldberg-Sachs theorem (Robinson and Schild[29]) and

$$\Phi \cdot \mathrm{Ric} \cdot \delta = 0$$

so that (7.14) implies the integrability of δ,

$$\delta \wedge d\delta = 0. \tag{7.16}$$

8. LIFTINGS OF CR SPACES TO LORENTZ MANIFOLDS

We say that the pair (g,K), where g is a Lorentz metric on M and $K \to M$ is a bundle of null directions is a lifting of a CR space N to M if the optical geometry defined by (g,K) is shear-free and the quotient M/K is a

CR space equivalent to N. We restict our attention to realizable CR spaces.

Let λ and μ be one-forms on N giving its CR structure; we use the same letters to denote their pull-backs to M and define the functions a and b on N by requiring

$$d\mu = 0 \quad \text{and} \quad d\lambda = 2\sqrt{-1}\, a\mu \wedge \bar{\mu} + \lambda \wedge (\bar{b}\mu + b\bar{\mu}) \tag{8.1}$$

Since M is locally diffeomorphic to $\mathbb{R} \times N$, we can choose the vector field k to be $\partial/\partial r$, where r is a coordinate on $\mathbb{R}$. Any Lorentz metric which, together with k, defines a lifting of N to M is of the form

$$g = 2\lambda(dr + \nu) - 2P^2 \mu\bar{\mu}, \tag{8.2}$$

where P is a nowhere vanishing function and

$$\nu = \frac{1}{2} c\lambda + \bar{f}\mu + f\bar{\mu} \tag{8.3}$$

is a one-form on M. The form (8.2) of the metric is invariant under the replacements (6.1), combined with appropriate transformations of P and ν. In many cases the process of lifting singles out a simple relation between the forms μ and δ. In particular, if M is partially Ricci-flat in the sense that (7.15) is satisfied, then δ is proportional to a gradient lying in the plane spanned by λ and μ. We may, therefore, specify the direction of μ by requiring that it be parallel to δ. Alternatively, having fixed the direction of μ in advance, we can restrict the lifting to satisfy

$$\delta \wedge \mu = 0. \tag{8.4}$$

To do this we remark that the two-form $\Phi = P\lambda \wedge \mu$ is self-dual and adapted to (g,k), where g is given by (8.2) and $k = \partial/\partial r$. Moreover, $\Phi \cdot \bar{\Phi} = \lambda \otimes \lambda$ and the deviation form δ can be read off from (7.2) with $\varkappa = \lambda = g(k)$, after the covariant derivative has been evaluated with the help of (2.7),

$$\delta = \rho P\mu - \frac{1}{2} P^{-1}(b + f')\lambda \tag{8.5}$$

where

$$\rho = P^{-1}P' - \sqrt{-1}\ aP^{-2} \tag{8.6}$$

and prime denotes differentiation with respect to r. We can, therefore, satisfy the condition (8.4) by putting

$$f' = -b. \tag{8.7}$$

Since, in general,

$$\delta + P\nabla_K\mu = 0, \tag{8.8}$$

the restriction (8.4) may be interpreted as consisting in choosing the local frame of reference along the lines of the null congruence so as to have the least possible rotation and to eliminate the "centrifugal forces" which would have made appearance had f not been constrained by (8.7) to be linear in r.

Any CR space admits many inequivalent liftings; we have already noted that the hyperquadric lifts to Minkowski space and to the Taub-NUT geometry. Transforming the metric of the Gödel universe to the form

$$g = x^{-2}[2(xdu - dy)(xdv - dy) - dx^2 - dy^2]$$

one sees that both $(g,\partial/\partial u)$ and $(g,\partial/\partial v)$ define an optical geometry corresponding to the hyperquadric, as recently noticed by L.K. Koch[30]. Hauser's metric (Hauser[31], Ernst and Hauser[32]) is yet another lifting of the same CR geometry, corresponding to the same choice of μ as in the standard lifting to Minkowski space.

If the components of the Ricci tensor in the direction of K vanish,

$$\mathrm{Tr}\ S = 0, \tag{8.9}$$

then, after a suitable adjustment of the coordinate r, solutions of (7.10) and (8.6) can be represented as

$$\rho = 1/(r + \sqrt{-1}\ ap^2) \tag{8.10}$$

and

$$P^{-2} = p^2\rho\bar{\rho} \quad \text{or} \quad P^2 = p^{-2}r^2 + p^2a^2, \tag{8.11}$$

where p is a function on N.

It seems reasonable to ask what further conditions can be imposed on the Riemann tensor of M over N. Given a CR space, one can ask whether it lifts to Minkowski space, a Ricci-flat space, an Einstein space, etc. These are difficult questions. From the twistor description of shear-free congruences of null geodesics (Penrose[33]) it is clear that very few CR spaces lift to Minkowski space (Penrose[5]). It is not known to us whether there are any CR spaces which have a lifting to a Ricci-flat Lorentz manifold without having any liftings to Minkowski space. The most thoroughly studied solutions of Einstein's equations admitting a twisting shear-free congruence of null geodesics are optically isomorphic to the geometry of either the Robinson (Penrose[33]) or the Kerr[34] congruence, but there are many solutions with an underlying CR structure different from those two (Robinson and Robinson[35]).

If the Ricci tensor is restricted to satisfy (7.15) - and therefore also (8.9) - then the metric (8.2) can be subject to (8.7), (8.10) and (8.11). The remaining information contained in (7.15) leads to

$$f = -b\rho^{-1} + \sqrt{-1}\ (ap^2)_2 \tag{8.12}$$

and

$$c = K - 2Hr - (m\rho + \overline{m\rho}) \tag{8.13}$$

where

$$K = p(p_{12} + p_{21}) - 2p_1p_2 - p^2(b_1 + \bar{b}_2), \tag{8.14}$$

$$H = p^{-1}p_0, \tag{8.15}$$

and m is a complex function on N whose imaginary part is determined by p and the CR geometry as follows :

introduce a real function U on N such that

$$U_o = p \;; \tag{8.16}$$

then

$$\mathrm{Im}\,(m + p^3 U_{1122}) = 0. \tag{8.17}$$

The conditions for the remaining components of the Ricci tensor to vanish have been discussed extensively elsewhere (Kramer et al.[20]). Here we shall confine ourselves to the special case when M is flat.

9. LIFTINGS TO MINKOWSKI SPACE

According to Robinson, Robinson and Zund[36] if $m = 0$ and

$$U_{22} = 0, \tag{9.1}$$

then the metric defined in the preceding Section is flat; conversely if the metric is flat, then the first of these equations holds and the second can be satisfied by a suitable choice of $\mu = dz$ and U. After this specialization, we still have at our disposal the fractional linear transformations of z, which correspond to Lorentz tranformations in Minkowski space and induce a suitable change of U; there are also "gauge transformations" of U induced by the translations.

Since the tangential CR equation (6.3) has only two functionally independent solutions on N, it follows from (9.1) that the three functions U_o, $U - \bar{z}U_2$ and z are functionally dependent. Introducing Cartesian coordinates, as in (Robinson, Robinson and Zund[36]), we recognize this observation to imply the Kerr theorem (Kerr and Schild[37], Penrose and Rindler[38]).

It follows from (8.16) that U cannot be a function of z and $\bar{z}$ only : consequently, the functions U_2 and $U-\bar{z}U_2$ cannot both depend only on z. We may, therefore, take one of them to be w and express the other as a function of w and z. Consider first the case when U_2 is not a function of z only; put $u = U$, $w = U_2$ and

$$U - \bar{z}U_2 = h(w,z), \quad \text{i.e.} \quad u = w\bar{z} + h(w,z). \tag{9.2}$$

Using (6.2) to evaluate du, we obtain

$$\lambda = p^{-1}(du - \bar{w}dz - wd\bar{z}). \tag{9.3}$$

If U_2 is a function of z only, say $U_2 = l(z)$, then we put

$$w = U - \bar{z}U_2, \quad \text{i.e.} \quad u = w + \bar{z}l(z), \tag{9.4}$$

and obtain

$$\lambda = p^{-1}(du - \bar{l}(\bar{z})dz - l(z)d\bar{z}) \tag{9.5}$$

The second case is, in fact, a special case of the first, as may be seen by making in (9.4) and (9.5) the replacements

$u \to u/z\bar{z}$, $z \to 1/z$, $w \to w/z$ and identifying $zf(1/z)$ with $h(w,z)$.

The most general expression for the differential form λ which, together with $\mu = dz$ defines a CR structure liftable to Minkowski space, is given by (9.3) with w determined by (9.2) as a function of the coordinates u, Re z, and Im z. The factor p is disposable; we can use it, for example, to impose the Cartan normalization, $a = 1/2$, or, in very special cases, to obtain $b = 0$. From the expression for λ, one obtains the lifted metric by means of Eqs (8.2,3,10-16) with $m = 0$. Incidentally, the special case when $h(w,z)$ is linear in w is used to provide the "Minkowski background" for the construction of a fairly large class of Ricci-flat Lorentz metrics (Robinson and Robinson[35]).

10. A LITTLE OF HISTORY AND CONCLUDING REMARKS

Shortly after E. Cunnigham[39] (1910) had established the conformal invariance of Maxwell's equations, H. Bateman[1] observed that null electromagnetic fields admit a larger group of automorphisms, consisting of what we now call optical transformations. He also developed general methods of constructing such null fields. In a short note published in 1922, E. Cartan[40] reported the existence of four privileged null ("optical", as he called them) directions at any point of a Lorentz space where the tensor of conformal curvature does not vanish. He also mentioned that, in the case of the Schwarzschild solution, these directions degenerate to two pairs of coinciding null lines; see (Robinson and Robinson[41])for further remarks on that paper. Cartan's observations went unnoticed for about 50 years. In the meantime, A.Z. Petrov[42] (1954) developed an algebraic classification of the Weyl tensor and F.A.E. Pirani[43] (1957) pointed out its physical relevance. Using spinors, R. Penrose[28] sharpened the Petrov classification and gave a simple description of the four principal null directions. This and subsequent work by Penrose (Penrose and Rindler[38]) played a fundametal role in the development of the subject. Another significant discovery was that of the shear-free property of congruences of null geodesics associated with null Maxwell fields (Robinson[44])and of the relation between the existence of such congruences and the properties of the Weyl tensor : the Goldberg-Sachs[45] theorem (1962) and its generalization (Robinson and Schild[29]). R.K. Sachs[27] established an optical interpretation of the scalars associated with null congruences and derived their propagation equations. During the years 1958-1967 L. Bel, M. Cahen, R. Debever, J. Ehlers, A. Lichnerowicz, E.T. Newman, A. Schild and several other scientists made important contributions to the study of algebraically degenerate Weyl tensors, the associated Lorentz spaces and their relation to gravitational waves and radiation (Kramer et al.[20]).

The shear-free condition turned out to be a restriction on the Lorentzian metric tensor well-suited for the study of solutions of Einstein's equations. On the one hand, the restriction is strong enough to reduce the equations to a manageable form; on the other, it is sufficiently weak to allow for metrics and gravitational fields of interest to physics. The integrable case is easy and was studied first: the corresponding solutions include gravitational waves with plane and spherical fronts as well as the Schwarzschild metric. R.P. Kerr[34)] had discovered the metric that bears his name looking for solutions admitting twisting shear-free congruences of null geodesics; its significance for the description of rotating black holes was understood later.

The study of twisting congruences was initiated by performing complex transformations of coordinates in Minkowski space, such as (3.7), and of the associated null Maxwell fields; cf. the work by I. Robinson reported by A. Trautman[46)]. It influenced Penrose[33,47)] (1967) in the early stages of his work on twistors; he coined the expression "Robinson congruence" to denote the one described at the end of Section 3. The projective twistor space is $\mathbb{C}P_3$ with the quadric Q defined, in terms of homogeneous coordinates $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \setminus \{0\}$, by

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = 0. \tag{10.1}$$

The quadric is a five-dimensional CR manifold and its points are in a bijective correspondence with null lines in compactified Minkowski space. According to the twistor form of the Kerr theorem any analytic shear-free congruence of null geodesics in compactified Minkowski space corresponds to the intersection N of Q with a complex surface of equation $h(z_1, z_2, z_3, z_4) = 0$, where h is a holomorphic and homogeneous function of its arguments (Penrose and Rindler[38)] (1986)). If h is linear, then by means of a transformation belonging to SU(2,2) and thus preserving (10.1), it can be reduced to $h(z_1, \ldots, z_4) = z_1$ or z_4, say; and $N = S_3$ is then given by $|z_1|^2 + |z_2|^2 = |z_3|^2$; this is the case of the Robinson congruence. The submanifold N of Q is a three-dimensional CR space; however, as Penrose points out, the freedom in defining N involves one complex holomorphic function of two variables whereas a general, realizable CR space may be defined by an analytic function of three variables. In other words, *most CR spaces do not* *lift to Minkowski space.* Penrose[5] extends the construction of the five-dimensional CR manifold Q to arbitrary Lorentz spaces. His construction depends, in an essential manner, on the choice of a space-like hypersurface in the Lorentz space-time. Our approach is more restricted: it is limited to Lorentz manifolds with a shear-free congruence of null geodesics; however, *the construction of our CR space N is natural in the sense that it does not require the introduction of any extraneous elements.*

The relation between shear-free congruences of null geodesics and CR geometry has been in the air for a long time. It is already apparent in the occurence of the Cauchy-Riemann operator in the process of solving Einstein's equations in the twist-free case (Robinson and Trautman[19] (1962)). P.Sommers[48,49] and J. Tafel[50] pointed out the appearence of the tangential CR operator in connection with twisting congruences. In particular, Tafel observed that the proof of the Robinson theorem requires finding a non-trivial solution to Eq. (6.4) and, therefore, for the theorem to be valid, it is not enough to assume that the underlying geometry is of class C^{∞}. The Cauchy-Riemann aspect of the geometry of light rays was implicit in early work on twistors; explicitly, it seems to have been mentioned for the first time by Penrose at the Helsinki Congress of Mathematicians (1978)(Penrose[51]). C.L. Fefferman[52] defined a natural conformal Lorentz geometry on a circle bundle over a CR space realized as the boundary of a pseudoconvex domain in $\mathbb{C}^2$ (see also Lewandowski[53]) and G.A.J. Sparling[54] studied its relation to twistor theory.

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