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COMPATIBILITY OF CONFORMAL AND PROJECTIVE STRUCTURES ON MANIFOLDS

Summary

conformal str. $[g]$ from light propagation

$\nearrow$

Weyl (1921): Riem. geom. g of GRT

$\searrow$

projective str. $[\Gamma]$ from freely falling particles

Ehlers, Pirani, Schild (1972): how to reconstruct g from $[g]$ and $[\Gamma]$?
Necessary EPS condition: null geod's of $[g]$ must be autoparallels of $[\Gamma]$

Definition: $[g]$ and $[\Gamma]$ are compatible iff they both come from one (pseudo)Riemannian geometry

Vladimir S. Matveev (Sept. 2012) found simple tensorial equations equivalent to compatibility of $[g]$ and $[\Gamma]$; they can be used to find g explicitly.

My interest in this problem was initiated by Andrzej Krasiński, who asked me to write editor's comments on the paper (EPS)

J. Ehlers, F. A. E. Pirani and A. Schild, The geometry of free fall and light propagation

published originally in 1972 and reprinted as a 'Golden Oldie' in

Gen. Relativity Gravitation **44** 1581–1586 (2012).

Reading EPS led me to

H. Weyl, Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffasung. *Nachr. Gesellschaft Wiss. Göttingen, Math.-Phys. Kl.* 99–112 (1921).

Weyl points out that Riemannian geometry of GRT determines two (weaker) structures; he considers them so important that they appear in the title of the paper.

In this paper, Weyl introduces the tensor of projective curvature; the tensor of conformal curvature was defined by Weyl in a paper of 1918.

Formally, a **conformal structure** (geometry) on a n-dimensional manifold M is an equivalence class $\mathscr{C}$ of metric tensors with respect to the relation

$$g \sim g' \iff \text{there is a function } \varphi \text{ on } M \text{ such that } g' = g \exp 2\varphi.$$

If $g \in \mathscr{C}$, then g is said to generate $\mathscr{C}$ that can be denoted as $[g]$.

A conformal structure is trivial (flat) if it is generated by a g which, by a choice of coordinates, can be transformed to $g_{ij} =$const. (For $n > 3$ this is equivalent to the vanishing of the tensor of conformal curvature.)

Two symmetric linear connections $\Gamma = (\Gamma^i_{jk})$ and $\Gamma' = ({\Gamma'}^i_{jk})$ are said to be *projectively equivalent* if their geodesics differ only by parametrisation. To find the relation between Γ and Γ' consider the geodetic equation

$$\mathrm{d}u^i/\mathrm{d}t + \Gamma^i_{jk}u^j u^k = \lambda u^i, \quad i, j, k = 1, \ldots, n.$$

If every solution $u^i(t) = \mathrm{d}x^i(t)/\mathrm{d}t$ of the last equation is a solution of a similar equation with Γ and λ replaced by Γ' and λ', then

$$(\Gamma^i_{jk} - \Gamma'^i_{jk})u^j u^k = (\lambda - \lambda')u^i$$

for all vectors u. This is easily shown (by algebra) to be equivalent to the existence of a one-form ψ such that $\Gamma'^i_{jk} = \Gamma^i_{jk} + \delta^i_j\psi_k + \delta^i_k\psi_j$.

(In this form, this appears in Weyl 1921; but the result was first given in

T. Levi-Civita, Sulle trasformazioni delle equazioni dinamiche. *Ann. di Mat., serie 2a* **24** 255–300 (1896), a paper by Tullio Levi-Civita written when he was 23.)

Projective equivalence is clearly an equivalence relation on the set of all symmetric linear connections on M.

A **projective structure** is an equivalence class $\mathscr{P}$ with respect to the relation

$$\Gamma \sim \Gamma' \iff \exists \text{ 1-form } \psi \text{ so that } \Gamma'^i_{jk} = \Gamma^i_{jk} + \delta^i_j \psi_k + \delta^i_k \psi_j$$

If $\Gamma \in \mathscr{P}$, then Γ is said to generate $\mathscr{P} = [\Gamma]$.

A projective structure is trivial (flat) if there is $\Gamma \in \mathscr{P}$ such that, by a coordinate transformation, its coefficients can be reduced to 0. (For $n > 2$ this is equivalent to the vanishing of the projective curvature tensor.)

Every (pseudo)Riemannian metric g determines the Levi-Civita connection $\digamma(g)$ (read: dyegamma of g); in coordinates

$$\digamma^i_{jk}(g) = \tfrac{1}{2} g^{ip}(\partial_k g_{pj} + \partial_j g_{pk} - \partial_p g_{jk}),$$

so that g defines both $\mathscr{C}$ and $\mathscr{P}$ generated by $\digamma(g)$. Since

$$\digamma^i_{jk}(g \exp 2\varphi) = \digamma^i_{jk}(g) + \underline{\delta^i_j \partial_k \varphi + \delta^i_k \partial_j \varphi} - g^{ip} g_{jk} \partial_p \varphi, \tag{1}$$

if u is null, then $\digamma^i_{jk}(g \exp 2\varphi) u^j u^k - \digamma^i_{jk}(g) u^j u^k \| u^i$ so that null geodesics are well defined by a conformal geometry. Underlined terms in (1) are 'projective', but the last term is not: this observation Weyl used to show that if g and g' define the same $\mathscr{C}$ and $\mathscr{P}$, then $g' =$ const.g.

He did not, however, consider when a given pair $(\mathscr{C}, \mathscr{P})$ comes from a metric g.

As far as I know, EPS is the first paper where this problem was considered. Its authors give a necessary condition that the pair $(\mathscr{C}, \mathscr{P})$, for Lorentzian signature of $\mathscr{C}$, must satisfy: in order to come from a metric tensor:

(EPS condition) every null geodesic of $\mathscr{C}$ is a geodesic (autoparallel) of $\mathscr{P}$

The authors were aware that their condition was not sufficient.

It is clear that the statement: the pair $(\mathscr{C}, \mathscr{P})$ comes from a metric g is equivalent to the pair being *compatible* in the sense that

$$\boxed{\text{there is } g \text{ such that } g \in \mathscr{C} \text{ and } F(g) \in \mathscr{P}.}$$

This gives rise to the problem: find the necessary and sufficient conditions for the pair $(\mathscr{C}, \mathscr{P})$ to be compatible and, if it is, give a prescription how to determine the corresponding metric.

I described this to Paweł Nurowski. In September 2012, during the workshop *Interaction of geometry and representation theory* at the Erwin Schrödinger Institute in Vienna, Paweł described the problem to Vladimir Matveev, a Russian mathematician now established at the University of Jena. Vladimir then formulated and proved the theorem presented below and invited me to join him in writing a paper on this subject.

Tracy Thomas observed in

T. Y. Thomas, On the projective and equi-projective geometries of paths. *Proc. Natl. Acad. Sci. USA* 11, 199–203 (1925)

that, given two symmetric linear connections, it is easy to check whether they are projectively equivalent by computing the traceless quantity $\Pi(\Gamma)$, which is nowadays called the Thomas symbol,

$$\Pi^i_{jk}(\Gamma) = \Gamma^i_{jk} - \frac{1}{n+1}\delta^i_j \Gamma^p_{pk} - \frac{1}{n+1}\delta^i_k \Gamma^p_{pj}, \quad n = \dim M.$$

(Note: Π is a projection map and its kernel is the space of pure traces, such as $\delta^i_j \psi_k + \delta^i_k \psi_j$.) Two symmetric linear connections are projectively equivalent, if and only if, their Thomas symbols coincide,

$$\Pi(\Gamma) = \Pi(\Gamma') \iff \Gamma \text{ and } \Gamma' \text{ are projectively equivalent.}$$

Given $g \in \mathscr{C}$ and $\Gamma \in \mathscr{P}$, one obtains from the above

(2) $\mathscr{C}$ and $\mathscr{P}$ are compatible $\Longleftrightarrow \exists\, \varphi$ s. t. $\Pi(F(g \exp 2\varphi)) = \Pi(\Gamma)$.

Since the difference of two connection coefficients is a tensor, so is

$$T^i_{jk} \stackrel{\text{def}}{=} \Pi^i_{jk}(F(g) - \Gamma).$$

The components of this tensor depend on the components of the metric tensor and their first derivatives and on the components of the linear connection. Substituting (1) (formula for $F(g \exp 2\varphi)$) into (2), one infers that compatibility of $\mathscr{C}$ and $\mathscr{P}$ is equivalent to the existence of φ such that

(3) $$T^i_{jk} - g_{jk} g^{ip} \partial_p \varphi + \frac{1}{n+1} \delta^i_j \partial_k \varphi + \frac{1}{n+1} \delta^i_k \partial_j \varphi = 0.$$

Let

$$T^i \stackrel{\text{def}}{=} \frac{n+1}{(n+2)(n-1)} g^{jk} T^i_{jk} \quad \text{and} \quad T_i = g_{ij} T^j. \tag{4}$$

By contraction of (3) with g^{jk} one obtains

$$\partial_i \varphi = T_i \tag{5}$$

Substituting $\partial_i \varphi$ determined by (5) and (4) into (3), one obtains the following condition on g and Γ:

$$\boxed{T^i_{jk} - g_{jk} T^i + \frac{1}{n+1} \delta^i_j T_k + \frac{1}{n+1} \delta^i_k T_j = 0.} \tag{6}$$

Since the second partial derivatives of φ commute, from (5) one obtains

$$\boxed{\partial_j T_i - \partial_i T_j = 0.} \tag{7}$$

Left sides of (6) and (7) depend only on $\mathscr{C}$ and $\mathscr{P}$, not on the representatives $g \in \mathscr{C}$ and $\varGamma \in \mathscr{P}$: they measure the incompatibility between $\mathscr{C}$ and $\mathscr{P}$. **Problem:** what are consequences of compatibility on the relation between conformal and projective curvature tensors?

Theorem (Matveev). *The conditions* (6) *and* (7) *are necessary and sufficient for local compatibility of the conformal and projective structures, defined on* M *by* g *and* $\varGamma$*, respectively. If, moreover, the first cohomology group of* M *vanishes, then there is global compatibility.*

Proof is easy. $\square$

A simple application

Using the theorem one can confirm the existence of pairs $(\mathscr{C}, \mathscr{P})$ that are incompatible even though the EPS condition holds. Indeed, let g be a Lorentzian metric on an $3 \leqslant n$-dimensional manifold M, and $\mathscr{C} = [g]$.

Given a vector field (S^i) on M, one considers the projective structure $\mathscr{P} = [\Gamma]$ such that

$$\Gamma^i_{jk} = \digamma^i_{jk}(g) - S^i g_{jk}. \tag{8}$$

If u^i is a null vector, $g_{ij}u^iu^j = 0$, then $(\Gamma^i_{jk} - \digamma^i_{jk}(g))u^ju^k = 0$ so that a null geodesic with respect to $\mathscr{C}$ is also a geodesic with respect to $\mathscr{P}$ and the EPS condition is satisfied.

Computing now T^i_{jk} for Γ given by (8), one obtains $T^i = S^i$ and that the algebraic condition (6) is satisfied. Therefore, the pair $(\mathscr{C}, \mathscr{P})$ now under consideration is compatible if, and only if, the form $g_{ij}S^j\mathrm{d}x^i$ is closed.

Article

V. S. Matveev & A.T, A criterion for compatibility of conformal and projective structures,
Open access: *Commun. Math. Phys.* **329** *(2014) 821–825.*