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TWO APPROACHES TO SPINOR FIELDS ON MANIFOLDS

What are they?

1. Weyl, Wigner and Fock (1929): refer spinor fields on curved manifolds to orthonormal frames (Weyl: *Achsenkreuze*; later: *Vierbeine*), use constant Dirac matrices; (much) later formalized by mathematicians as spin structures (principal bundles)

2. Tetrode (1928), Schrödinger (1932), Infeld and van der Waerden (1933), Bergmann (1957): assume Dirac matrices to be point-dependent and covariantly constant (vector bundles)

Plan of talk

Spinors according to the Ancient Greeks
Words of the masters and the controversies
Two approaches to vector and spinor fields
Double valuedness of spinor fields on $\mathbb{S}_2$ and the Schrödinger solution
Dirac operator on hypersurfaces
Example: the sphere $\mathbb{S}_m \subset \mathbb{R}^{m+1}$

SPINORS ACCORDING TO THE ANCIENT GREEKS

Spinors are implicit in solution of the Pythagorean equation

$$x^2 + y^2 = z^2$$

which is equivalent to

$$\text{(G)} \qquad \begin{pmatrix} z+x & y \\ y & z-x \end{pmatrix} = 2 \begin{pmatrix} p \\ q \end{pmatrix} \begin{pmatrix} p & q \end{pmatrix}$$

and gives (**Euclid**) $x = p^2 - q^2,\ y = 2pq,\ z = p^2 + q^2$.
In $\mathbb{R}^3$ with g of signature $(2,1)$: null vector = (spinor)^2.

Lessons from Euclid

1. **Generalize to higher dim**

totally null multivector of max dim = (pure spinor)^2.

(Veblen, Cartan; pure essential in dim $\geqslant 7$)

2. **Spin groups**

Multiply (G) on the left by real unimodular matrix A, on the right by its transpose, take det to get

$$1 \to \mathbb{Z}_2 \to \mathrm{SL}_2(\mathbb{R}) = \mathrm{Spin}^0_{2,1} \to \mathrm{SO}^0_{2,1} \to 1$$

NB. If $A \in \mathrm{GL}_2(\mathbb{R})$, then vectors transform by rotations and dilations, but not under general linear transformations.

3. **Idea of Clifford algebras**

Multiply (G) by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the left, take square

$$\begin{pmatrix} y & z-x \\ -z-x & -y \end{pmatrix}^2 = (x^2+y^2-z^2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

"Quadratic form linearized"

Replace z by iz to get complex spinors, Pauli matrices, $\mathrm{Spin}_3 = \mathrm{SU}_2$, etc.

4. Non-trivial topology involved

Rotation by α in (p,q) plane:

$$p' = p\cos\alpha + q\sin\alpha, \quad q' = -p\sin\alpha + q\cos\alpha$$

induces rotation by 2α in (x,y) plane:

$$x' = x\cos 2\alpha + y\sin 2\alpha, \quad y' = -x\sin 2\alpha + y\cos 2\alpha$$

Take $0 \leqslant \alpha \leqslant \pi$ to conclude: Spinors change sign when rotated by 2π

WORDS OF THE MASTERS

The relativity theory is based on nothing but the idea of invariance and develops from it the conception of tensors as a matter of necessity; and it is rather disconcerting to find that apparently something has slipped through the net, so that physical quantities exist,which it would be, to say the least, very artificial and inconvenient to express as tensors.

C. G. Darwin, *Proc. Roy. Soc. London* **A118** (1928) 654.

The orthogonal transformations are the automorphisms of Euclidean vector space. Only with the spinors do we strike that level in the theory of its representations on which Euclid himself, flourishing ruler and compass, so deftly moves in the realm of geometric figures.

H. Weyl, *The Classical Groups*, Princeton U.P., 1946

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THE CONTROVERSY

E. Schrödinger mildly criticized the approach to spinor fields based on Weyl's *Achsenkreuze*:

Bei diesem Verfahren ist es ein bißchen schwer, zu erkennen, ob die Einsteinsche Idee des Fernparallelismus, auf die teilweise direkt Bezug genommen wird, wirklich hereinspielt oder ob man davon unabhängig ist.

E. Schrödinger, Diracsches Elektron im Schwerefeld I, *S. B. preuss. Akad. Wiss.*, 1932, 105–128.

NB. Remarkable paper; contains the first derivation of formula for the square of the Dirac operator. Criticized by Cartan.

Quotations from É. Cartan, *The Theory of Spinors*, English translation from the French published in 1938.

Footnote on p. 150:

Certain physicists regard spinors as entities which are, in a sense, unaffected by the rotations which classical geometric entities (vectors, etc.) can undergo, and of which the components in a given reference system are susceptible of undergoing linear transformations which are in a sense autonomous. See for example L. Infeld and B. L. van der Waerden, "Die Wellengleichungen des Elektrons in der allgemeinen Relativitätstheorie", *S. B. preuss. Akad. Wiss.*, 1933, 380.

Note that Cartan dismisses van der Waerden as a 'certain physicist'

From p. 151:

Theorem. With the geometric sense we have given to the word "spinor" it is impossible to introduce fields of spinors into the classical Riemannian technique; that is, having chosen an arbitrary system of co-ordinates x^i for the space, it is impossible to represent a spinor by any finite number N whatsoever, of components u_α such that the u_α have covariant derivatives of the form

$$u_{\alpha,i} = \partial u_\alpha/\partial x^i + \Lambda^\beta_{\alpha i} u_\beta$$

where the $\Lambda^\beta_{\alpha i}$ are determinate functions of x^h.

Cartan defines spinor fields by reference to orthonormal frames and emphasizes that mere curvilinear coordinates are not enough

TWO APPROACHES TO VECTOR AND SPINOR FIELDS

Recall two ways of describing **vector fields** on m-dim manifold:

1. start from principal bundle P of linear frames

$$\begin{tikzcd}
\mathrm{GL}_m(\mathbb{R}) \arrow[r] & P \arrow[r, "X"] \arrow[d] & \mathbb{R}^m \\
& M &
\end{tikzcd}$$

vector field X is equivariant map: $X(ea) = a^{-1}X(e)$, $a \in \mathrm{GL}_m(\mathbb{R})$, $e \in P$. If $s : M \to P$ is a section, then $X \circ s : M \to \mathbb{R}^m$ gives the components of X with respect to the field of frames s.

2. tangent bundle

$$\begin{tikzcd}
TM \arrow[d, shift right, "\pi"'] \\
M \arrow[u, shift right, "X'"']
\end{tikzcd}$$

vector field X' is section of the tangent bundle

Connection between the two:

1⇒ **2** by forming associated bundle:

$$(P \times \mathbb{R}^m)/\,\mathrm{GL}_m(\mathbb{R}) = TM$$

1⇐ **2** define

$$P = \{e : \mathbb{R}^m \to T_xM \text{ linear isom.} \mid x \in M\}$$

Consider analogous definitions for **spinors.** For simplicity of notation assume (M, g) to be proper Riemannian, orientable manifold. Recall that given a quadratic space (V, h), one has the Clifford algebra $\mathrm{Cl}(V, h)$ and the spin group $\mathrm{Spin}(V, h) \subset \mathrm{Cl}(V, h)$; if $V = \mathbb{R}^m$ and h definite, then write Cl_m, Spin_m, SO_m, etc. A representation $\gamma : \mathrm{Cl}_m \to \mathrm{End}\, S$ defines by restriction a representation

$$\gamma : \mathrm{Spin}_m \to \mathrm{GL}(S)$$

in the vector space S of spinors

1. start from principal spin bundle Q, a *reduction* of the bundle of *orthonormal* frames P on M,

$$\begin{tikzcd}
\mathrm{Spin}_m \arrow[r] \arrow[d] & Q \arrow[r, "\varPsi"] \arrow[d] & S \\
\mathrm{SO}_m \arrow[r] & P \arrow[d] & \\
 & M &
\end{tikzcd}$$

spinor field $\varPsi$ of type γ is equivariant map:

$$\varPsi(qa) = \gamma(a^{-1})\varPsi(q), \quad a \in \mathrm{Spin}_m \,.$$

The Levi-Civita connection lifts from P to Q and defines Dirac operator $\gamma^\mu \nabla_\mu$ acting on spinor fields $\varPsi : Q \to S$. No teleparallelism required.

2. To define bundle of spinors $\varSigma \to M$, analogous to $TM \to M$, consider the Clifford bundle

$$\mathrm{Cl}(g) = \textstyle\bigcup_{x\in M} \mathrm{Cl}(T_xM, g_x) \to M$$

and a map of bundles

$$\gamma : \mathrm{Cl}(g) \to \mathrm{End}\,\varSigma$$

such that for $x \in M$, the restriction

$$\gamma_x : \mathrm{Cl}(T_xM, g_x) \to \mathrm{End}\,\varSigma_x$$

is a representation of the Clifford algebra $\mathrm{Cl}(T_xM, g_x)$ in $\varSigma_x$. Sections $\varPsi : M \to \varSigma$ are spinor fields.

NB. Schrödinger assumed γ_x to be the Dirac repr., whereas Bergmann considered 2-component Weyl spinors and split $\varSigma = \varSigma_+ \oplus \varSigma_-$ so that $\gamma = \begin{pmatrix} 0 & \tau \\ \sigma & 0 \end{pmatrix}$.

If $X : M \to TM \to \mathrm{Cl}(g)$ is a vector field, then $\gamma(X) : M \to \mathrm{End}\,\varSigma$ is a field of matrices acting on spinor fields. Covariant differentiation is introduced by requiring

$$\nabla_X(\gamma(Y)\varPsi) = \gamma(\nabla^{\mathrm{LC}}_X Y)\varPsi + \gamma(Y)\nabla_X\varPsi$$

1⇒**2** is by forming the associated bundle

Going from **2** to **1** is somewhat subtle, especially in the non-orientable, odd-dim case; see Th. Friedrich and A. Trautman, *Ann. Global Anal. Geom.* **18** (2000) 221–240.

DOUBLE VALUEDNESS OF SPINOR FIELDS ON $\mathbb{S}_2$ AND THE SCHRÖDINGER TRICK

As example, construct spin structure on $M = \mathbb{S}_2$; note first $\mathrm{Spin}_2 = \mathrm{U}_1$. The spin bundle is

$$Q = \mathrm{Spin}_3 = \mathrm{SU}_2 = \mathbb{S}_3 = \{q = (q_1, q_2) \in \mathbb{C}^2 : |q_1|^2 + |q_2|^2 = 1\},$$

the action of Spin_2 on Q is $(q, a) \mapsto qa = (q_1a, q_2a)$, $a \in \mathrm{U}_1$, and the bundle of orth. frames $P = \mathrm{SO}_3 = \mathrm{SU}_2/\mathbb{Z}_2$ so that

$$Q \to P \ \text{ is } \ (q_1, q_2) \mapsto \pm(q_1, q_2) \quad \text{and}$$

$$Q \to \mathbb{S}_2 \text{ is } (q_1, q_2) \mapsto (2q_1\bar{q}_2 = x + \mathrm{i}y,\ |q_1|^2 - |q_2|^2 = z)$$

Consider Euler angles (θ, φ, χ) on $Q = \mathbb{S}_3$; they define local coords so that

$$q_1 = e^{\frac{1}{2}\mathrm{i}(\chi+\varphi)} \cos \tfrac{1}{2}\theta, \quad q_2 = e^{\frac{1}{2}\mathrm{i}(\chi-\varphi)} \sin \tfrac{1}{2}\theta$$

and $Q \to \mathbb{S}_2$ is $(q_1, q_2) \mapsto (e^{\mathrm{i}\varphi} \sin\theta,\ \cos\theta)$.

The 'usual' field of orth. frames $(\mathrm{d}\theta, \sin\theta\, \mathrm{d}\varphi)$ is def on the 2-sphere $\mathbb{S}_2^\bullet$ with the two poles ($\theta = 0$ and π) removed; it corresponds to the section $\chi = 0$, i.e. to

$$(\theta, \varphi) \mapsto \pm(e^{\frac{1}{2}\mathrm{i}\varphi} \cos \tfrac{1}{2}\theta,\ e^{-\frac{1}{2}\mathrm{i}\varphi} \sin \tfrac{1}{2}\theta)$$

Note map well defined at $\varphi = 0 \equiv 2\pi$ because of $\pm$.

But it does not lift to a map from $\mathbb{S}_2^\bullet$ to $\mathbb{S}_3$: for this, a meridian, such as $\varphi = 0$, has to be removed. If this is not done, then spinor fields on $\mathbb{S}_2$ become double-valued. Indeed, the Dirac repr. $\gamma : \mathrm{Spin}_2 = \mathrm{U}_1 \to \mathrm{GL}_2(\mathbb{C})$ is faithful so that $\gamma(-1) = -I$; a Dirac spinor field $\Psi : Q = \mathbb{S}_3 \to S = \mathbb{C}^2$ satisfies $\Psi(qa) = \gamma(a^{-1})\Psi(q)$ so that $\Psi(-q) = -\Psi(q)$.
Take section $s : \mathbb{S}_2^\bullet \setminus \{\varphi = 0\} \to \mathbb{S}_3$

$$s(\theta, \varphi) = (e^{\frac{1}{2}\mathrm{i}\varphi} \cos \tfrac{1}{2}\theta,\ e^{-\frac{1}{2}\mathrm{i}\varphi} \sin \tfrac{1}{2}\theta)$$

then $\lim_{\varphi\to 0} s = -\lim_{\varphi\to 2\pi} s$. Look at the 'usual' description of spinor field: $\Psi \circ s$.

This and similar problems were noticed, considered at length, and solved by the masters:

E. Schrödinger, Eigenschwingungen des sphärischen Raums, *Comment. Pontificia Acad. Scientiarum* **2** (1938) 321–364.

E. Schrödinger, Die Mehrdeutigkeit der Wellenfunktion, *Helv. Phys. Acta* **32** (1938) 49–55.

W. Pauli, Über ein Kriterium für Ein- oder Zweiwertigkeit der Eigenfunktionen in der Wellenmechanik, *Helv. Phys. Acta* **12** (1939) 147–168.

Note: double valuedness does not arise if one starts from a field of orth. frames on $\mathbb{S}_2$ coming from the stereographic projection; there is then the section
$s' : \mathbb{S}_2 \setminus \{\theta = 0\} \to \mathbb{S}_3$, $\quad s'(\theta, \varphi) = (e^{\mathrm{i}\varphi} \cos \frac{1}{2}\theta, \sin \frac{1}{2}\theta)$ and $\Psi \circ s'$ is single-valued. This approach taken in theory of spin spherical harmonics.

Schrödinger's solution of the problem:

note that the Dirac representation of Spin_{2n} in 2^n-dim space of spinors extends to the Pauli representation of Spin_{2n+1} in the *same* space ("add γ_5"); in particular

$$\begin{tikzcd}
\mathrm{Spin}_2 \arrow[r, "\gamma"] \arrow[d, "\mathrm{inj}"'] & \mathrm{GL}_2(\mathbb{C}) \arrow[d, equal] \\
\mathrm{Spin}_3 \arrow[r, "\sigma"'] & \mathrm{GL}_2(\mathbb{C})
\end{tikzcd}$$

Consider the spinor field (equivariant map) $\Psi : \mathrm{Spin}_3 \to \mathbb{C}^2$. Form $\Phi(q) = \sigma(q)\Psi(q)$, then for $a \in \mathrm{Spin}_2$ one has $\Phi(qa) = \Phi(q)$ so that Φ descends to a globally defined, single-valued map $\psi : \mathbb{S}_2 \to \mathbb{C}^2$. This shows: bundle Σ of spinors on $\mathbb{S}_2$ is trivial. But the tangent bundle is not.

Generalize: **Theorem.** Every hypersurface M immersed in $\mathbb{R}^{m+1}$ has a (s)pin structure and the bundle of Dirac (m even) or Pauli (m odd) spinors, associated with it, is trivial.

If m even and M orientable, then there is the bundle of Weyl (chiral) spinors over M; it need not be trivial.

The Dirac operator, originally defined to act on the equivariant map $\Psi : Q \to S$, for a hypersurface M descends to act on the corresponding field $\psi : M \to S$.

DIRAC OPERATOR ON HYPERSURFACES

Assume M to be an even-dimensional orientable hypersurface in $\mathbb{R}^{m+1}$ and let n^μ, $\mu = 1, \ldots, m+1$, be a field of unit normals on it. Let γ_μ be the Dirac matrices satisfying

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2\delta_{\mu\nu} I$$

and put $N = \gamma_\mu n^\mu$,

$$\sigma_{\mu\nu} = \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \ \text{ and } \ L_{\mu\nu} = n_\nu \partial_\mu - n_\mu \partial_\nu .$$

The vector fields $L_{\mu\nu}$ are tangent to M. One derives the following formula for the Dirac operator acting on $\psi : M \to S$:

$$D = N(\sigma^{\mu\nu} L_{\mu\nu} - \tfrac{1}{2} \operatorname{div} n)$$

and

$$DN + ND = 0$$

so that N plays the role of γ_5 in SRT. (Analogy with Polyakov's hedgehog solution.)

There is more if there is a foliation of $\mathbb{R}^{m+1}$ by hypersurfaces; introducing $\frac{\partial}{\partial r} = n^\mu \partial_\mu$ one has the decomposition

$$\gamma^\mu \partial_\mu = D + N(\tfrac{\partial}{\partial r} + \tfrac{1}{2} \operatorname{div} n)$$

EXAMPLE: THE SPHERE $\mathbb{S}_m \subset \mathbb{R}^{m+1}$

Consider $\mathbb{R}^{m+1}$ foliated by spheres $r =$ const., then $n^\mu = x^\mu / r$ and $\operatorname{div} n = m/r$; for $m = 3$ the vector with components $rL_{23}, rL_{31}, rL_{12}$ is the angular momentum operator.

Recall formula for the Laplacian:

$$\Delta_{\mathbb{R}^{m+1}} = r^{-2} \Delta_{\mathbb{S}_m} + r^{-m} \tfrac{\partial}{\partial r} r^m \tfrac{\partial}{\partial r}$$

Evaluating on a harmonic polynomial homogeneous of degree l gives the eigenvalues $-l(l+m-1)$ of $\Delta_{\mathbb{S}_m}$.

An analogous formula for the Dirac operator is

$$\gamma^\mu \partial_\mu = r^{-1} D_{\mathbb{S}_m} + N r^{-m/2} \tfrac{\partial}{\partial r} r^{m/2}$$

Take now for $\phi : \mathbb{R}^{m+1} \to S$ a harmonic, homogeneous of degree $l+1$, S-valued polynomial, then $\psi = \gamma^\mu \partial_\mu \phi$ is of degree l and satisfies $\gamma^\mu \partial_\mu \psi = 0$, therefore

$$D_{\mathbb{S}_m} \psi = -(l + \tfrac{1}{2} m)\psi \ \text{ and } \ D_{\mathbb{S}_m} N\psi = (l + \tfrac{1}{2} m) N\psi$$

so that the spectrum of the Dirac operator on $\mathbb{S}_m$ is the set $\pm(l + \frac{1}{2} m)$, $l = 0, 1, \ldots$.

The author's paper "The Dirac operator on hypersurfaces" *Acta Phys. Polon. B* **26**, 1283–1310 (1995) contains further information and references.