Past light cone shape and refocusing in cosmology, A Response to Michael Rauch's "Comments on 'Lost Horizons' " [Am. J. Phys. 63, 87 (1995)]

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We thank Dr. Rauch for his thoughtful comments on our paper "Lost Horizons".[1] Although we do not agree with his main conclusion, the question he raises is an interesting one, and serves to highlight a point we did not get quite right in Sec. VI of our own discussion.

At issue is whether the reconvergence of light in the universe, which is manifested in the "onion" shape of the past light cones in our Figs. 4 (Ref. 1), is due to the gravitational refocusing of light caused by matter in the universe, or is a kinematical effect. In our paper we stated that the shape of the past null cone is an example of gravitational lensing. Rauch points out that in a universe without matter the same effect still occurs, therefore it cannot be due to gravitational lensing. He then argues that this must be a kinematical effect, i.e., one due to the expansion of the universe and to the coordinate systems involved. To make this clearer, consider our Eq. (7), in the special case with only the spatial-curvature term present on the right

$$\left(\frac{\dot{R}}{R}\right)^2 = -\frac{k}{R^2}. \tag{1}$$

(Here, we have set the speed of light $=1$.) This cosmological model is known as the "Milne Universe" and contains no matter, only curvature. Necessarily we have $k=-1$, an "open" universe, and one can immediately integrate Eq. (1) to find $R \propto t$. Plotting the past null cone gives qualitatively the same result as we got in our paper [Eq. (17)] for the Einstein–de Sitter universe. Rauch then argues that since this universe is empty, the reconvergence of light rays cannot be caused by gravitational lensing.

Rauch's argument that this is a kinematic effect can be summarized as follows: In comoving coordinates (coordinates attached to galaxies) the velocity of a galaxy is proportional to its distance (Hubble's law). The Hubble velocity, v_H can be arbitrarily large. In comoving coordinates, a photon emitted from a distant galaxy approaches us monotonically, thus entering regions of ever smaller v_H. As our Figs. 4 show, however, in proper distance coordinates, photons emitted at the beginning of the universe initially move away from us, reach a maximum distance, then move toward us. The velocity of photons in proper-distance coordinates is

$$v_p = v_H - c$$

[from our Eq. (25) or Rauch Eq. (1)]. Clearly, v_p will reverse sign when the photons enter a region where v_H drops below c. At this point the photons appear to be coming toward us in proper distance coordinates. However, the turnaround point, given by $v_H = c$ is also by definition the speed-of-light sphere (SLS). Therefore, the fact that the SLS intersects the null cone at its maximum is not a coincidence. We are grateful to Rauch for pointing this out. That none of this is apparently due to the matter content of the universe (for it is true in the Milne universe as well as the Einstein–de Sitter universe) leads Rauch to conclude that the effect is kinematic, rather than dynamic.

It is here we disagree, but the disagreement is partly due to an oversight in our own paper. To summarize what follows, there is actually no focusing of light in the Milne universe, in the usually accepted definition of the term. The usual definition of refocusing is that the *angular diameter* of a source reaches a maximum at some time (or redshift) in the past and then reconverges. Although we showed that the *radius* of the past null cone has a maximum [Eq. (17)] at some redshift, we tacitly assumed without proof that the *angular diameter* had a maximum at the same redshift. In other words, we assumed the usual Euclidean relationship between radius and circumference. Now this was in fact true in the case we explicity considered (the Einstein–de Sitter universe), but it does not hold more generally in the curved space–times of relatively; the maximum radius need not occur at the same redshift as the maximum angular diameter.

Indeed, for the Milne universe, there is no maximum angular diameter at all! This comes about because one can by appropriate choice of coordinates transform the Milne metric into an expanding Minkowski metric (see Refs. 2 and 3). The Milne universe is therefore in fact flat space–time. One can calculate the convergence of null rays in these coordinates, and the result is clearly the same as in flat space–times: they never reconverge.

In more detail:

In Ref. 1, we gave the correct formula for the "shape" of the light cone

$$l(t_e) = R(t_e)u_e, \quad u_e \equiv \int_{t_e}^{t_0} c\, dt/R(t). \tag{2}$$

Here, $l(t_e)$ refers to the radius of the past null cone. However, Eq. (2) is not the whole story. We neglected to give the formula for the *area* of the past light cone at a given time. (Or what is equivalent, the null cone's circumference; the two are effectively the same, and are worked out in the usual expression for the observed angular diameter in cosmology—what is often called the "observer area distance."[4]) In fact, the circumference of the light cone at time t_e is

$$C(t_e) = 2\pi R(t_e) f(u_e), \tag{3}$$

where

$$f(x) = \sin x, x, \sinh x \text{ for } k = +1, 0, -1. \tag{4}$$

The area is $4\pi R^2(t_e) f^2(u_e)$, essentially the square of the quantity (3). Now, it is immediately clear that the maximum of Eqs. (2) and (3) need not coincide except when $k=0$; then $f(u_e) = u_e$ and Eq. (3) becomes the same as Eq. (2) (up to a constant). When $k=-1$, the refocusing occurs *after* the light cone has reached its maximum size (if refocusing occurs at all).

We plead guilty to not having checked this, and also confess to being surprised at the result, although in retrospect it is obvious, since the Euclidean relationship between radius and circumference is not generally true in relativity. However, in the Einstein–de Sitter case, which concerned us in Ref. 1, $k=0$ and the maxima/minima *do* coincide, as stated there. This is a coincidence!

The case of the Milne universe is best examined through the "null Raychaudhuri equation," from which Eq. (3) above follows. [See Eq. (4.35) of Ref. 5, with the shear and vorticity set to zero.] This reads

$$\frac{d\hat{\theta}}{ds}=-R_{ab}k^a k^b-(1/2)\hat{\theta}^2, \tag{5}$$

where $\hat{\theta}$ is the expansion of the null rays. It is this equation which leads one to say that the matter causes refocusing of the light cone R_{ab} is determined directly from the matter stress tensor T_{ab} through the Einstein field equations; [see Eqs. (7) and (16) of Ref. 1].

However, as already indicated, in the Milne universe both the Ricci and Weyl tensors are zero, that is, $R_{ab}=0=C_{abcd}$, and so this is just flat space–time in expanding coordinates. Then Eq. (5) shows there is no refocusing of light, in the sense that there is no minimum angular diameter. The Milne universe must give the usual flat-space relationship between radius and area of the null cone (in usual flat-space coordinates).

We see that bending of light is not a kinematic effect [Eq. (5) relates angular sizes to the matter present, through the Einstein field equations]. It does not occur in the Milne universe. The maximum size of the light cone as defined by the nonmonotonic behavior of $l(t_e)$ is a kinematic effect in the sense that because the Milne universe is empty, there is no preferred timelike direction. By choosing different t=const surfaces one can therefore calculate different spatial distances for the size of the light cone.

In less specialized language, there is no matter in the Milne universe to define a preferred rest frame. Arbitary Lorentz boosts will lead to arbitrary length measurements. In the Einstein–de Sitter universe, the presence of matter defines a preferred rest frame. The worldlines of particles at rest with respect to this matter are orthogonal to the spatial (constant-time) surfaces. Only when we choose the expanding worldlines of test particles in the Milne universe to be orthogonal to the constant-time surfaces, as in the Einstein–de Sitter case, does the Milne universe give the same "maximum radius feature" that we found for Einstein–de Sitter. But this will not be true for all choices of constant-time surfaces.

However, in nonempty universes, the worldlines of matter are unique and uniquely define a set of t=const surfaces orthogonal to these worldlines. Therefore, if we insist on, (1) nonempty universes (thus excluding Milne and de Sitter), and (2) measuring spatial distances in the (unique) surfaces orthogonal to the matter worldlines, we can claim that not only is gravitational lensing a dynamical effect, but so is the maximum size of the past light cone; for the size of the light cone, as defined in these unique surfaces, is determined by the dynamics of the matter in the universe.

Once again, we are grateful to Dr. Rauch for his comments: We hope this response clarifies the matter.

[1]G. F. R. Ellis and T. Rothman, "Lost Horizons," Am. J. Phys. **61** (10), 883–893 (1993).
[2]C. W. Misner, K. S. Thorn, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 743.
[3]G. F. R. Ellis and R. M. Williams, *Flat and Curved Space–Times* (Oxford University, London, 1988).
[4]G. F. R. Ellis, in *General Relativity and Cosmology*, edited by R. K. Sachs, XLVII Enrico Fermi Summer School (Academic, London, 1971), pp. 152–159.
[5]S. W. Hawking and G. F. R. Ellis, *The Large-Scale Structure of Space–Time* (Cambridge University, Cambridge, 1973).

Erratum: "The World Around Us, E. Leonard Jossem's acceptance speech for the 1994 Oersted Medal presented by the American Association of Physics Teachers, 6 January 1994" [Am. J. Phys. 62 (7), 589–595 (1994)]

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Reference 18, page 595, is incomplete and should have noted that the reference to Sellery's book applies to the woodcut only. The title "The Lecturer: Sinaistic Reducer" is due to Professor Gerald R. Holton and appears in his essay "What is Conveyed by Demonstrations" in the book *Physics Demonstration Experiments*, edited by H. F. Meiners, Vol. I, pages 61–82 (The Ronald Press, New York, 1970). The author thanks Professor Holton for calling this to his attention and regrets the inadvertent omission of this information.