Cosmological Models

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The de Sitter Model

The Einstein's Field Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

don't allow for static solutions when applied for homogeneous and isotropic Universe. The idea of a static universe or "Einstein's universe" is one which demands that space is not expanding nor contracting but rather is dynamically stable. Einstein proposed such a model by adding a cosmological constant Λ to his equations of general relativity to counteract the dynamical effects of gravity which in a universe of matter would cause the universe to collapse. This motivation evaporated after the discovery by Edwin Hubble that the universe is not static, but expanding; in particular, Hubble discovered a relationship between redshift and distance, which forms the basis for the modern expansion paradigm.

Even after Hubble's observations, Fritz Zwicky proposed that a static universe could still be viable if there was an alternative explanation of redshift due to a mechanism that would cause light to lose energy as it traveled through space, a concept that would come to be known as "tired light". However, subsequent cosmological observations have shown that this model has not been a viable alternative either.

With the cosmological constant the Einstein's equations read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

The Λ term could be written as a part of the energy-momentum tensor:

$$T^{\Lambda}_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} = \rho_{\Lambda} g_{\mu\nu} \qquad \text{for} \qquad \rho_{\Lambda} \equiv \frac{\Lambda}{8\pi G}$$

Then the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G \left(T_{\mu\nu} + T^{\Lambda}_{\mu\nu} \right)$$

Recall that in the rest frame of an element of perfect fluid $T_{\mu\nu}$ has the form:

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

while $T^{\Lambda}_{\mu\nu}$ in a freely falling system reads

$$T^{\Lambda}_{\mu\nu} = \begin{pmatrix} \rho_{\Lambda} & & & \\ & -\rho_{\Lambda} & & \\ & & -\rho_{\Lambda} & \\ & & & -\rho_{\Lambda} \end{pmatrix}$$

So, $\rho = \rho_\Lambda$ and $p = -\rho_\Lambda$ (*negative pressure!*).
We assume an empty space (no matter, so $T_{\mu\nu} = 0$), but $\Lambda \neq 0$ and solve the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu}$$

The cosmological principle implies that

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right\}$$

Using the results we obtained for the FLRW metric we have

$$R_{tt} = 3\frac{\ddot{a}}{a} \qquad R_{it} = R_{ti} = 0 \qquad R_{ij} = -\tilde{g}_{ij}(a\ddot{a} + 2\dot{a}^2 + 2k)$$

The Einstein's equations could be written as

$$R_{\mu\nu} = -8\pi G S_{\mu\nu}$$

where

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda = (\rho + p) U_\mu U_\nu - \frac{1}{2}(\rho - p) g_{\mu\nu}$$

for $T^{\mu\nu} = -pg^{\mu\nu} + (p + \rho) U^\mu U^\nu$ ($T^\alpha{}_\alpha = \rho - 3p$). So for $U^t = 1$ and $U^i = 0$ we have

$$S_{tt} = \frac{1}{2}(\rho + 3p) = -\rho_\Lambda \qquad S_{ti} = S_{it} = 0 \qquad S_{ij} = \frac{1}{2}(\rho - p) a^2 \tilde{g}_{ij} = a^2 \rho_\Lambda \tilde{g}_{ij}$$

One gets the Friedmann's equations for this case

- $(0,0)$ component:

$$3\frac{\ddot{a}}{a} = -8\pi G(-\rho_\Lambda) = \Lambda$$

- (i,j) component:

$$-(a\ddot{a} + 2\dot{a}^2 + 2k)\tilde{g}_{ij} = -8\pi G\rho_\Lambda a^2\tilde{g}_{ij} = -a^2\Lambda\tilde{g}_{ij}$$

In other terms

$$3\frac{\ddot{a}}{a} = \Lambda \qquad \text{and} \qquad \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = \Lambda$$

Eliminating $\ddot{a}$ we get

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} = \Lambda$$

For $k = 0$ and $\Lambda > 0$ we get

$$H^2(t) \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} \quad \Rightarrow \quad a(t) = a(t_0)e^{H\cdot(t-t_0)} \quad \text{for} \quad H^2 = \frac{\Lambda}{3} = \text{const.}$$

This is the exponential *inflation*: exponential growth of the scale factor (de Sitter universe):

It is easy to show (see class) using the Friedmann equations that the sufficient conditions for the exponential inflation are:

$$k = 0 \qquad \text{and} \qquad p = -\rho$$

or

$$k \neq 0 \qquad \text{and} \qquad p = -\frac{\rho}{3} - \frac{H^2}{4\pi G}\,, \tag{1}$$

with constant H. Note that ρ and p do not need to be constant.

The easiest way to derive the above equation of state is to subtract the Friedmann equations

$$\begin{aligned} 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} &= -8\pi G p \\ \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} &= \frac{8\pi G}{3}\rho \end{aligned}$$

eliminating that way $(\dot{a}/a)^2$ and k/a^2. Then we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p).$$

Comments:

- $p \simeq -\rho$ could be easily arranged within a scalar field theory.
- For small a the term $\propto k/a^2$ dominates. However if a is growing then for a large enough, the curvature term $\frac{k}{a^2} \sim \Lambda$. Since the Universe is expanding Λ will dominate and the expansion will be exponential.
- Non-relativistic equation of motion for a material point of mass m that corresponds to the Friedmann equation with cosmological constant

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8}{3}\pi G\rho + \frac{\Lambda}{3}$$

  reads (see class)

$$m\ddot{r} = -G\frac{Mm}{r^2} + \frac{\Lambda}{3}mr\,,$$

  therefore positive Λ implies presence of a force proportional to r, the acceleration increases with r! The universe is growing exponentially.

The Cosmological Principle implies

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right\}$$

The Friedmann equations read

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \qquad \text{for} \qquad \rho \equiv \sum_i \rho_i = \rho_m + \rho_{\mathrm{rad}} + \rho_\Lambda \tag{2}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G p \qquad \text{for} \qquad p \equiv \sum_i p_i = p_{\mathrm{rad}} + p_\Lambda \tag{3}$$

where the sum runs over all contributions to the energy density and pressure. The conservation of the energy-momentum tensor ($T^{\mu\nu}{}_{;\nu} = 0$) implies

$$\dot{p}a^3 = \frac{d}{dt}\left[a^3(\rho + p)\right] \qquad \Rightarrow \qquad \frac{d}{dt}\left(\rho a^3\right) = -p\frac{d}{dt}a^3 \tag{4}$$

Hereafter we will assume the following equation of state

$$p = w\rho,$$

with a constant w.

- For the non-relativistic matter (see class): $\rho \simeq nm + \frac{3}{2}p$, so if $p \ll nm$ then $w = 0$,
- For ultra-relativistic matter (e.g. photons): $p = \frac{1}{3}\rho$, so $w = \frac{1}{3}$.
- For the cosmological constant: $p = -\rho$, so $w = -1$.

Let's solve (4) for $p = w\rho$:

$$\frac{d}{dt}\left(\rho a^3\right) = \dot{\rho} a^3 + \rho 3a^2 \dot{a} = -p\frac{d}{dt}a^3 = -w\rho 3a^2 \dot{a}$$

$$\Downarrow$$

$$\frac{\dot{\rho}}{\rho} = -3(w+1)\frac{\dot{a}}{a} \qquad \Rightarrow \qquad \rho(t) = \rho^0 \left[\frac{a(t)}{a_0}\right]^{-3(w+1)} \propto \left[a(t)\right]^{-3(w+1)} \tag{5}$$

- Matter dominated Universe ($w = 0$), so called dust: $\rho \propto a^{-3}$
- Radiation dominated Universe ($w = \frac{1}{3}$): $\rho \propto a^{-4}$
  We have shown that for photons emitted at t_0 and detected at t (*attention notation changed*) the following relation holds

$$\lambda(t) = \lambda(t_0)\frac{a(t)}{a(t_0)} \propto a(t)$$

  Since $\nu\lambda = c = 1$, so we have

$$\nu(t) = \nu(t_0)\frac{a(t_0)}{a(t)} \propto a^{-1}(t)$$

  Therefore the photon energy $E = h\nu$ suffers from another extra suppression because of the expansion, so

$$\rho \propto a^{-3}a^{-1} = a^{-4}$$

- Cosmological constant dominated Universe: ($w = -1$): ρ = const.

Comment: If the Universe is composed of several components then the result $\rho_i(t) \propto a(t)^{-3(w_i+1)}$ is valid only if interactions between the components could be neglected. Note that (4) holds for *total* energy density and *total* pressure. In other words for multi-component Universe we sum contributions to the total energy density (or pressure) *assuming* that they scale with a as if they were the only components.

Now we can try to solve the Friedmann equation (2) assuming $p = w\rho$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

First let's neglect the curvature k, then we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho^0 a_0^{3(w+1)} a^{-3(w+1)}$$

$$\Downarrow$$

$$\dot{a} \propto a^{-\frac{1}{2}(3w+1)} \tag{6}$$

We can look for a power-like solution $a(t) \propto t^\alpha$, then substituting into (6) we can determine α:

$$t^{\alpha-1} = t^{-\frac{\alpha}{2}(3w+1)}$$

hence $\alpha = \frac{2}{3(w+1)}$. Therefore

$$a(t) \propto t^{\frac{2}{3(w+1)}} = \begin{cases} t^{2/3} & \text{matter} \\ t^{1/2} & \text{radiation} \\ e^{Ht} & \text{cosmological constant} \end{cases}$$

Let's verify if $a(t) \propto t^\alpha$ is also a solution for $k \neq 0$. Using (5) in the Friedmann equation one finds:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}$$

Then, inserting

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^\alpha$$

results in the following condition

$$\left(\frac{\alpha}{t_0}\right)^2 + \frac{k}{a_0^2}\left(\frac{t_0}{t}\right)^{2(\alpha-1)} = \frac{8\pi G}{3}\rho_0 \left(\frac{t_0}{t}\right)^{3\alpha(1+w)-2}$$

So if $k \neq 0$ then $a(t) \propto t^\alpha$ may satisfy the Friedmann equation only if $\alpha = 1$ and $w = -1/3$.

Note that the solution $a(t) \propto t$ is the same as the one obtained in 1932 (long before FRW) by Milne for an empty universe ($\rho = 0$, $\Lambda = 0$), it requires a negative curvature $k = -1$. However in the case considered here if there exists a matter with $w = -1/3$ then k could be $0, \pm 1$.

Subtracting (2) and (3) we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{7}$$

Since presently $p \sim 0$ and $\rho > 0$ therefore we would be tempted to conclude that the Universe is decelerating at present. This is not consistent with observations which suggest that $\ddot{a} > 0$. If we add Λ then we get an extra contribution to the rhs: $(\rho + 3p) = \rho_\Lambda + 3(-\rho_\Lambda) = -2\rho_\Lambda < 0$, so that $\ddot{a} > 0$ in agreement with observations.

Let's expand $a(t)$ around the present time $t = t_0$:

$$a(t) = a_0 + a_0 \left.\frac{\dot{a}}{a}\right|_{t=t_0}(t - t_0) - \frac{1}{2}a_0 \underbrace{\left[-\left.\frac{\ddot{a}}{a}\right|_{t=t_0}\frac{1}{H_0^2}\right]}_{\equiv q_0} H_0^2(t - t_0)^2 + \cdots$$

where q_0 is the *deceleration* parameter. Now let's expand the Hubble parameter $H(t)$

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} = H_0\left[1 - (q_0 + 1)H_0(t - t_0) + \cdots\right]$$

In general $H(t)$ is not constant, but a time dependent function.

Questions to test alertness of students:

1. What is the condition to obtain $H(t) = \text{const.} + \mathcal{O}[H_0^2(t - t_0)^2]$?
2. What kind of matter in the Universe leads to $H(t) = \text{const.}$ at any time?

Answers:

1. In the next to the leading order: $q_0 = -1$.
2.

$$H(t) = \frac{\dot{a}}{a} = \text{const.} \qquad \Rightarrow \qquad a(t) \propto e^{Ht}$$

- the cosmological constant, $p = -\rho$, for $k = 0$, or
- *something* that satisfies $p = -\frac{\rho}{3} - \frac{H^2}{4\pi G}$ for $k \neq 0$.

Define the critical density of the Universe as

$$\rho_{\text{crit}} \equiv \frac{3H^2}{8\pi G}$$

Note that ρ_{crit} is a function of time, at present

$$\rho^0_{\text{crit}} = 1.9 \times 10^{-32} h^2 \frac{\text{kg}}{\text{cm}^3} \qquad \text{for} \qquad h \equiv \frac{H_0}{100 \text{ km s}^{-1} \text{ Mpc}^{-1}} \qquad \text{for} \qquad 0.6 \lesssim h \lesssim 0.8$$

Then we can rewrite the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$$

as follows

$$\frac{k}{a^2 H^2} = \Omega - 1 \qquad \text{for} \qquad \Omega \equiv \frac{\rho}{\rho_{\text{crit}}}$$

$$\Downarrow$$

The geometry of the Universe is determined by Ω

- $\Omega > 1 \quad \Rightarrow \quad k = +1$ closed Universe
- $\Omega < 1 \quad \Rightarrow \quad k = -1$ open Universe
- $\Omega = 1 \quad \Rightarrow \quad k = 0$ flat Universe

Let's calculate the deceleration parameter q_0 assuming $p_i = w_i \rho_i$ and using (7)

$$q_0 = \frac{4\pi G}{3H_0^2}\sum_i (\rho_i^0 + 3p_i^0) = \frac{4\pi G}{3H_0^2}\sum_i (1 + 3w_i)\rho_i^0 = \frac{1}{2}\sum_i \Omega_i^0 (1 + 3w_i)$$

for

$$\Omega_i^0 \equiv \frac{\rho_i^0}{\rho_{\text{crit}}^0} = \frac{\rho_i^0}{\frac{3H_0^2}{8\pi G}}$$

Since $w = -1$ for Λ, therefore in the presence of the cosmological constant q_0 may be negative.

Consider energy density composed of matter, radiation and the cosmological constant, then we can rewrite the Friedmann equation at the present time as

$$H_0^2 + \frac{k}{a_0^2} = \frac{8\pi G}{3}(\rho_m^0 + \rho_{\rm rad}^0 + \rho_\Lambda)$$

Dividing by H_0^2 and adopting the definition $\rho_{\rm crit}^0 \equiv \frac{3H_0^2}{8\pi G}$ we obtain

$$1 = -\frac{k}{H_0^2 a_0^2} + \Omega_m^0 + \Omega_{\rm rad}^0 + \Omega_\Lambda^0$$

Introducing $\Omega_k^0 \equiv -\frac{k}{H_0^2 a_0^2}$ we have

$$1 = \Omega_k^0 + \Omega_m^0 + \Omega_{\rm rad}^0 + \Omega_\Lambda^0$$

Let's return to the Friedmann equation

$$H^2 = -\frac{k}{a^2} + \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad} + \rho_\Lambda) \qquad (8)$$

The matter and radiation densities scale as

$$\rho_m = \rho_m^0 \left(\frac{a_0}{a}\right)^3 \qquad \text{and} \qquad \rho_{\rm rad} = \rho_{\rm rad}^0 \left(\frac{a_0}{a}\right)^4$$

while ρ_Λ remains constant.

The fractional energy densities are defined as follows

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\rm crit}} \qquad \text{for} \qquad \rho_{\rm crit} \equiv \frac{3H^2}{8\pi G}$$

while at the present time

$$\Omega_i^0 \equiv \frac{\rho_i^0}{\rho_{\rm crit}^0} \qquad \text{for} \qquad \rho_{\rm crit}^0 \equiv \frac{3H_0^2}{8\pi G}.$$

So the densities could be rewritten as

$$\begin{aligned} \rho_{\rm rad} &= \rho_{\rm rad}^0 \left(\frac{a_0}{a}\right)^4 = \frac{3}{8\pi G} H_0^2 \Omega_{\rm rad}^0 \left(\frac{a_0}{a}\right)^4 \\ \rho_m &= \rho_m^0 \left(\frac{a_0}{a}\right)^3 = \frac{3}{8\pi G} H_0^2 \Omega_m^0 \left(\frac{a_0}{a}\right)^3 \\ \rho_\Lambda &= \rho_\Lambda^0 = \frac{3}{8\pi G} H_0^2 \Omega_\Lambda^0 \end{aligned}$$

The curvature terms will be written as

$$-\frac{k}{a^2} = \underbrace{-\frac{k}{a_0^2 H_0^2}}_{\Omega_k^0} H_0^2 \left(\frac{a_0}{a}\right)^2 = \Omega_k^0 H_0^2 \left(\frac{a_0}{a}\right)^2$$

Now, using the relation $1 + z = \frac{a_0}{a}$, we are ready to express the densities corresponding to a given scale factor as functions of the red-shift:

$$
\begin{aligned}
\rho_{\rm rad} &= \frac{3}{8\pi G} H_0^2 \Omega^0_{\rm rad} \left(\frac{a_0}{a}\right)^4 = \frac{3}{8\pi G} H_0^2 \Omega^0_{\rm rad} (1+z)^4 \\
\rho_m &= \frac{3}{8\pi G} H_0^2 \Omega^0_m \left(\frac{a_0}{a}\right)^3 = \frac{3}{8\pi G} H_0^2 \Omega^0_m (1+z)^3 \\
-\frac{k}{a^2} &= \Omega^0_k H_0^2 \left(\frac{a_0}{a}\right)^2 = H_0^2 \Omega^0_k (1+z)^2 \\
\rho_\Lambda &= \frac{3}{8\pi G} H_0^2 \Omega^0_\Lambda
\end{aligned}
$$

Let's insert the above formulas into the Friedmann equation (8):

$$
H^2 = H_0^2 \left[\Omega^0_{\rm rad}(1+z)^4 + \Omega^0_m (1+z)^3 + \Omega^0_k (1+z)^2 + \Omega^0_\Lambda\right]
$$

So, we have shown how to determine the expansion rate at a given epoch ($H(z)$) knowing its present value and present energy densities.

Matter or Radiation dominated Universe

The Friedmann equation could be integrated to provide the age of the Universe. Two periods must be separately considered (the possibility of the existence of Λ will not be considered at this moment): radiation domination (early Universe) when $\rho = \rho_{\rm rad} = \rho^0(a_0/a)^4$ and matter domination (present Universe) when $\rho = \rho_m = \rho^0(a_0/a)^3$:

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad} + \rho_\Lambda) \leftarrow \rho_m = \rho^0\left(\frac{a_0}{a}\right)^3, \ \rho_{\rm rad} = \rho_\Lambda = 0 \ \ (\text{MD})$$

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad} + \rho_\Lambda) \leftarrow \rho_{\rm rad} = \rho^0\left(\frac{a_0}{a}\right)^4, \ \rho_m = \rho_\Lambda = 0 \ \ (\text{RD})$$

$$\left(\frac{\dot a}{a_0}\right)^2 + \frac{k}{a_0^2} = \frac{8\pi G}{3}\rho^0\frac{a_0}{a} \qquad (\text{MD}) \qquad (9)$$

$$\left(\frac{\dot a}{a_0}\right)^2 + \frac{k}{a_0^2} = \frac{8\pi G}{3}\rho^0\left(\frac{a_0}{a}\right)^2 \qquad (\text{RD}) \qquad (10)$$

The age t of the Universe of a size $a(t)$ is defined as $t \equiv \int_{t(0)}^{t(a)} dt'$

Changing integration variables to $a' = a'(t')$ we can write

$$t = \int_0^{a(t)} \frac{da'}{\dot{a}'}$$

Using the relation

$$\frac{k}{a_0^2 H_0^2} = \Omega_m^0 + \Omega_{\rm rad}^0 + \Omega_\Lambda^0 - 1$$

to eliminate $\frac{k}{a_0^2}$, defining $x \equiv \frac{a}{a_0}$ and using $\Omega^0 \equiv \rho^0/\rho_{\rm crit} = \rho^0 8\pi G/(3H_0^2)$ one can rewrite (9) for $\Omega_{\rm rad}^0 = \Omega_\Lambda^0 = 0$ and $\Omega_m^0 = \Omega^0$

$$\left(\frac{\dot{a}}{a_0}\right)^2 = -H_0^2(\Omega^0 - 1) + \underbrace{\frac{8\pi G}{3}\rho^0}_{H_0^2\Omega^0} x^{-1} = H_0^2\left(\Omega^0 x^{-1} + 1 - \Omega^0\right)$$

So,

$$\dot{a}'(x) = a_0 H_0 \left(\Omega^0 x^{-1} + 1 - \Omega^0\right)^{1/2} \qquad \text{and} \qquad da' = a_0 dx$$

Then expressing the scale factor a in terms of the redshift z ($1 + z = \frac{a_0}{a} = \frac{1}{x}$) one gets the age for MD as

$$t^{(MD)} = \int_0^{a(t)} \frac{da'}{\dot{a}'} = \int_0^{(1+z)^{-1}} \frac{a_0 dx}{\dot{a}'(x)}$$

$$t^{(MD)} = H_0^{-1}\int_0^{(1+z)^{-1}} \frac{dx}{[1-\Omega^0+\Omega^0 x^{-1}]^{1/2}} \quad \text{(MD)} \tag{11}$$

$$t^{(RD)} = H_0^{-1}\int_0^{(1+z)^{-1}} \frac{dx}{[1-\Omega^0+\Omega^0 x^{-2}]^{1/2}} \quad \text{(RD)} \tag{12}$$

Comments:

- For $a \lesssim l_{Pl}$ (the Planck length $l_{Pl} \equiv \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.6\times 10^{-35}$ m) our knowledge of the Universe is uncertain. However if we assume that $a(t) = a_0\left(\frac{t}{t_0}\right)^{\alpha}$ $(0<\alpha<1)$ then this first period of the expansion (from $a=0$ till $a = l_{Pl}$) contributes a tiny piece to the total age

$$\int_0^{l_{Pl}} \frac{da'}{\dot{a}'} = \frac{t_0}{\alpha a_0^{1/\alpha}}\int_0^{l_{Pl}} \frac{da'}{a'^{(\alpha-1)/\alpha}} = t_0\left(\frac{l_{Pl}}{a_0}\right)^{1/\alpha}$$

  Since $1/\alpha > 1$ this contribution could be neglected.
- Note that if I wanted to include more Universe components in the same spirit as in (11-12) then I would be allowed just to add various Ω's only if interaction between them could be neglected. Otherwise the scaling of $\rho = \rho(a)$ is more complicated. For instance the interaction between matter and radiation, could be neglected after neutral atoms were created (the recombination), so that photons stopped interacting with matter.

First the (MD) Universe. The present age could be obtained substituting $z = 0$ in (11), then integrating for $\Omega^0 > 1$ we obtain

$$t_0^{(MD)} = H_0^{-1} \frac{\Omega^0}{2(\Omega^0 - 1)^{3/2}} \left[\cos^{-1}\left(2\Omega^{0\,-1} - 1\right) - \frac{2}{\Omega^0}\left(\Omega^0 - 1\right)^{1/2}\right] \tag{13}$$

and for $\Omega^0 < 1$

$$t_0^{(MD)} = H_0^{-1} \frac{\Omega^0}{2(1 - \Omega^0)^{3/2}} \left[\frac{2}{\Omega^0}\left(1 - \Omega^0\right)^{1/2} - \cosh^{-1}\left(2\Omega^{0\,-1} - 1\right)\right] \tag{14}$$

For $\Omega^0 = 1$ we have $t_0^{(MD)} = \frac{2}{3} H_0^{-1}$.

Note that $t_0 = t_0(\Omega^0)$ is a decreasing function of Ω^0.
Expanding (13-14) around $\Omega^0 = 1$ we obtain

$$t_0^{(MD)} = \frac{2}{3} H_0^{-1} \left[1 - \frac{1}{5}(\Omega^0 - 1) + \cdots \right]$$

The present age of the (MD) Universe could be easily estimated assuming $\Omega^0 \simeq 1$

$$t_0^{(MD)} = \frac{2}{3} H_0^{-1} = 6.5 \times 10^9 h^{-1} \text{ yr} \qquad \text{for} \qquad 0.6 \lesssim h \lesssim 0.8$$

for $H_0 = 100$ km s^{-1} Mpc$^{-1} h$. The above estimates assumes that the Universe was MD from the very beginning till today.
For the (RD) Universe we obtain at $z = 0$

$$t_0^{(RD)} = H_0^{-1} \frac{1}{\Omega^{0\,1/2} + 1} = \frac{1}{2} H_0^{-1} \left[1 - \frac{1}{4}\left(\Omega^0 - 1\right) + \cdots \right]$$

Matter domination leads to larger age: $t_0^{(MD)}(\Omega^0) > t_0^{(RD)}(\Omega^0)$.

Universe made of Matter and Cosmological Constant

Let's now discuss the age of the Universe for a model that is flat ($k = 0$) but that contains both matter and $\Lambda > 0$, so

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda)$$

Following the same steps as above for the (MD) case and using the fact that $\Omega_m^0 + \Omega_\Lambda^0 = 1$ (it follows from $k = 0$) we find (see class) that

$$t_0^{(\Lambda)} = H_0^{-1} \int_0^1 \frac{dx}{(\Omega_m^0 x^{-1} + \Omega_\Lambda^0 x^2)^{1/2}} = \frac{2}{3} H_0^{-1} \frac{1}{\Omega_\Lambda^{0\,1/2}} \ln \left[\frac{1 + \Omega_\Lambda^{0\,1/2}}{(1 - \Omega_\Lambda^0)^{1/2}} \right]$$

Comments:

- For $\Omega_\Lambda^0 \gtrsim \frac{3}{4}$, $t_0^{(\Lambda)} \gtrsim H_0^{-1}$, unlike $t_0^{(MD)}$ and $t_0^{(RD)}$.
- $t_0^{(\Lambda)} = t_0^{(\Lambda)}(\Omega_\Lambda)$ is an increasing function of Ω_Λ,

$$\lim_{\Omega_\Lambda^0 \to 0} t_0^{(\Lambda)} = \frac{2}{3} H_0^{-1} \qquad \text{and} \qquad \lim_{\Omega_\Lambda^0 \to 1} t_0^{(\Lambda)} = \infty$$

The expansion rate at a given epoch (z) as a function of its present value and present energy densities:

$$H^2 = H_0^2 \left[\Omega^0_{\rm rad}(1+z)^4 + \Omega^0_m(1+z)^3 + \Omega^0_k(1+z)^2 + \Omega^0_\Lambda\right] \quad (15)$$

Using the above form of the Friedmann equation we will derive a general formula that allows to determine the age of the Universe at a given redshift (the "lookback time"). The Hubble parameter could be written as

$$H = \frac{d}{dt}\ln\left(\frac{a(t)}{a_0}\right) = \frac{d}{dt}\ln\left(\frac{1}{1+z}\right) = \frac{-1}{1+z}\frac{dz}{dt}$$

Then using (15) we get

$$\frac{dt}{dz} = H_0^{-1}\frac{-1}{1+z}\,\frac{1}{[\Omega^0_{\rm rad}(1+z)^4 + \Omega^0_m(1+z)^3 + \Omega^0_k(1+z)^2 + \Omega^0_\Lambda]^{1/2}}$$

Integrating we obtain

$$t_0 - t = H_0^{-1}\int_0^z \frac{dz'}{(1+z')[\Omega^0_{\rm rad}(1+z')^4 + \Omega^0_m(1+z')^3 + \Omega^0_k(1+z')^2 + \Omega^0_\Lambda]^{1/2}} \quad (16)$$

Note that Ω_i^0 are not independent as they satisfy

$$1 = \Omega_k^0 + \Omega_m^0 + \Omega_{\mathrm{rad}}^0 + \Omega_\Lambda^0$$

Choosing $t = 0$ and $z = \infty$ in (16) we have the present age of the Universe. Note that (as we have anticipated) the scale of the lookback time is set by H_0^{-1}, which is called the Hubble time.

Future of the Universe

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad} + \rho_\Lambda) \tag{17}$$

$$\rho_m \propto a^{-3} \qquad \rho_{\rm rad} \propto a^{-4} \qquad \rho_\Lambda = \text{const.}$$

In general (17) is difficult to solve. However at present $\rho_{\rm rad} = 2 \times 10^5$ eV m^{-3} for the CMB while $\rho_{\rm baryon} = 10^9$ eV m^{-3}, so we can assume that $\rho_m \gg \rho_{\rm rad}$.

♠ No cosmological constant: $\rho_\Lambda = 0$

Therefore we have (neglecting ρ_Λ temporarily)

$$\rho = \rho^0 \frac{a_0^3}{a^3} \qquad \Rightarrow \qquad \dot{a}^2 + k = \frac{8\pi G}{3}\rho^0 a_0^2 \frac{a_0}{a} \tag{18}$$

- Suppose for a moment that $\rho^0 = 0$, then $k < 0$ is required to have real-valued solutions for a (so called Milne model). The solution is (in general $\pm|k|^{1/2}t + \text{const.}$)

$$a_{\text{Milne}}(t) = |k|^{1/2}t = t$$

- For $\rho^0 > 0$ and $k = 0$ the solution for initial conditions such that $a(0) = 0$ (for the general solution see class) reads

$$a(t) \propto t^{2/3}$$

- For $\rho^0 > 0$ and $k = -1$ we observe that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} > 0 \tag{19}$$

Since $\rho = \rho^0 \frac{a_0^3}{a^3}$ we find that

$$\dot{a}^2 + k = \frac{8\pi G}{3} \rho^0 a_0^2 \frac{a_0}{a}$$

Therefore one can see that if a is large enough (and $\dot{a} > 0$ at some moment), then matter term becomes sub-dominant and the Universe turns out to expand forever a'la Milne: $a(t) \propto t$.

- Now let's consider the dust ($\rho^0 > 0$) and $k = +1$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho^0 \frac{a_0^3}{a^3} - \frac{k}{a^2}$$

  Then we observe that in this case there exists $a = a_{\text{crit}}$ such that $\dot{a}(t) = 0$ for $a = a_{\text{crit}}$:

$$\frac{8\pi G}{3} \rho^0 \left(\frac{a_0}{a_{\text{crit}}}\right)^3 = \frac{k}{a_{\text{crit}}^2} \qquad \Rightarrow \qquad a_{\text{crit}} = \frac{8\pi G \rho^0 a_0^3}{3}$$

  Since we know that

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G}{3} \rho_m$$

  therefore in our case (no Λ and $p = 0$) we have $\ddot{a} < 0$. At $a = a_{\text{crit}}$ the expansion stops, since the Universe is decelerating ($\ddot{a} < 0$), the contraction starts (note that the Universe can not stay at a_{crit}) and after some time ends as a "Big Crunch".

♠ Cosmological constant: $\rho_\Lambda \neq 0$

Let's now consider $\Lambda \neq 0$ (still neglecting $\rho_{\rm rad}$), then we should solve

$$\dot{a}^2 = \frac{8\pi G}{3}\rho^0 \frac{a_0^3}{a} - k + \frac{\Lambda a^2}{3} \tag{20}$$

Comments:

- From (20) we can see that even if Λ was negligible for small a (at the beginning of the expansion) it will eventually dominate over all other forms of matter (including curvature).
- If $\Lambda < 0$ and $k = 0, \pm 1$ then (20) tells us that $a(t)$ cannot be arbitrarily large since $\dot{a}(t)$ must be real. So, the maximal size of the scale factor is determined by the solution of

$$\frac{8\pi G}{3}\rho^0 \frac{a_0^3}{a} = k + \frac{|\Lambda| a^2}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i(\rho_i + 3p_i) = -\frac{4\pi G}{3}\left(-2\rho_\Lambda + \rho^0 \frac{a_0^3}{a^3}\right) = -\frac{4\pi G}{3}\left(\frac{|\Lambda|}{4\pi G} + \rho^0 \frac{a_0^3}{a^3}\right) < 0$$

  The Universe is decelerating, so again we have an oscillating Universe (regardless of the value of k).

- If $\Lambda > 0$ and $k = 0$ or $k = -1$ we have

$$\dot{a}^2 = \frac{8\pi G}{3}\rho^0 \frac{a_0^3}{a} + |k| + \frac{\Lambda a^2}{3} > 0$$

  So, the Universe is expanding forever (as it was expanding at the beginning: $a(t) \propto t^{1/2}$) and after some time the cosmological constant starts dominating and the Universe enters a period of exponential expansion (de Sitter model).
- For $\Lambda > 0$ and $k = +1$ the picture is more complicated. It is possible to find $\Lambda = \Lambda_E = (4\pi G\rho^0 a_0^3)^{-2}$ such that $\dot{a}(t) = \ddot{a}(t) = 0$ for some $a = a_E = 4\pi G\rho^0 a_0^3$, so that $\Lambda_E \cdot a_E^2 = 1$ (see class). This is a static Universe, the existence of this solution (not consistent with the present data) motivated Einstein to introduce Λ.
  - For $\Lambda = \Lambda_E$ the Universe is static (although unstable, see class).
  - For $\Lambda > \Lambda_E$ the repulsion from Λ (Why is Λ repulsive?) dominates and the Universe expands forever.
  - For $\Lambda < \Lambda_E$ there is a range of a: $a_a \leq a \leq a_b$ such that

$$\dot{a}^2 = \frac{8\pi G}{3}\rho^0 \frac{a_0^3}{a} - k + \frac{\Lambda a^2}{3} \leq 0$$

    that is forbidden (see class). So, for $0 < \Lambda < \Lambda_E$ and $k = +1$:
    - If the initial position is such that $a(0) < a_a$ then the Universe is oscillating between $a = 0$ and $a = a_a$, or
    - If the initial position is such that $a(0) > a_b$ then it always expands.