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♠ Distance in general relativity

Along an arbitrary spacelike path P, the proper (physical) distance is given as the line integral

$$D_P = \int_P (-g_{\mu\nu} dx^\mu dx^\nu)^{1/2}$$

♠ Conformal time and comoving radius

Conformal time η

$$d\eta \equiv \frac{dt}{a(t)} \qquad \Rightarrow \qquad \eta(t) = \int_0^t \frac{dt'}{a(t')}$$

Then the FLRW metric

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right\}$$

could be rewritten in the “conformal” manner

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) \left\{ d\eta^2 - \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right) \right\}$$

Comoving radial coordinate χ (in contrast to the comoving radial coordinate r)

$$d\chi \equiv \pm \frac{dr}{(1-kr^2)^{1/2}} \qquad \Rightarrow \qquad \chi = \int_0^r \frac{dr'}{(1-kr'^{\,2})^{1/2}} = \begin{cases} \arcsin\, r \\ r \\ \mathrm{arcsinh}\, r \end{cases} \text{for} \quad k = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

Then

$$d\tau^2 = dt^2 - a^2(t)\left[d\chi^2 + S_k^2(\chi)\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]$$

where

$$S_k(\chi) = \begin{cases} \sin\, \chi \\ \chi \\ \sinh\, \chi \end{cases} \text{for} \quad k = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

♠ The recession velocity and the Hubble law

Discovery of expanding Universe

- Vesto Slipher, American astronomer performed the first measurements of radial velocities for galaxies, providing the empirical basis for the expansion of the universe. In 1912, he was the first to observe the shift of spectral lines of galaxies, making him the discoverer of galactic redshifts. In 1914, Slipher also made the first discovery of the rotation of spiral galaxies. By 1922 Slipher had collected radial velocities for 41 spiral nebulae (later identified as galaxies), almost all of which were redshifted. Unfortunately, he did not formally publish the full dataset in a journal; they became known to the community when they were published in an early textbook on general relativity (Eddington 1923) and in a paper by the astronomer Gustav Strömberg (1925). Interpreting the redshift as a non-relativistic Doppler effect Sliper was able to determine radial velocities of the observed nebulae.

  The "redshift" z definition:

  $$1 + z \equiv \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}}$$

  The Doppler (non-relativistic) effect:

  $$1 + \frac{v}{c} = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}}$$

The velocity-distance relation in GR

The proper (physical) distance (defined along the surface of constant time $dt = 0$) to an object located at the coordinate r at the moment t:

$$D(t) = \int_P (-g_{\mu\nu} dx^\mu dx^\nu)^{1/2} = a(t) \int_0^r \frac{dr'}{(1 - kr'^2)^{1/2}}$$

We calculate variation of the proper distance at a time t

$$\frac{d}{dt} D(t) = \dot{a}(t) \int_0^r \frac{dr'}{(1 - kr'^2)^{1/2}} + a(t) \frac{\dot{r}}{(1 - kr^2)^{1/2}} \tag{1}$$

The recession velocity is related (by definition) to the change of $D(t)$ caused by the evolution of $a(t)$ for a constant comoving coordinate r, therefore

$$v_{\mathrm{rec}}(t) = \dot{a}(t) \int_0^r \frac{dr'}{(1 - kr'^2)^{1/2}}$$

So we get the Hubble law as

$$v_{\mathrm{rec}}(t) = \dot{a}(t) \int_0^r \frac{dr'}{(1 - kr'^2)^{1/2}} = \left(\frac{\dot{a}(t)}{a(t)}\right) \left(a(t) \int_0^r \frac{dr'}{(1 - kr'^2)^{1/2}}\right) = H(t)D(t)$$

The second term in (1) describes the peculiar velocity related to a motion in the comoving frame with changes of the comoving coordinate r:

$$v_{\mathrm{pec}} = a(t) \frac{\dot{r}}{(1 - kr^2)^{1/2}}$$

Note that in the FLRW geometry for a photon emitted from receding galaxy we have $d\tau = 0$, so

$$d\tau^2 = dt^2 - a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right\}$$

$$dt = \pm a(t)\frac{dr}{(1-kr^2)^{1/2}}$$

therefore we conclude that $v_{\mathrm{pec}} = \pm 1$, so locally photons have velocity 1, as they should.

Assume that we observe light from a galaxy the recession velocity of which we want to determine. Then the velocity could be calculated (see class) as a function of the time at which we would like to know the velocity (i.e. t), coordinates of the emission (t_{em}, r) and detection $(t_{\mathrm{obs}}, 0)$ of photons from the observed object (galaxy):

$$v_{\mathrm{rec}}(t, t_{\mathrm{em}}, t_{\mathrm{obs}}) = \dot{a}(t)\int_0^r \frac{dr'}{(1-kr'^2)^{1/2}} = \dot{a}(t)\int_{t_{\mathrm{em}}}^{t_{\mathrm{obs}}} \frac{dt'}{a(t')}$$

and then assuming that the observation is performed now ($z = 0$) the velocity could be expressed (see class) as a function of the redshift ($z \leftrightarrow t_{\mathrm{em}}$) of the observed object

$$v_{\mathrm{rec}}(t) = \frac{\dot{a}(t)}{a_0}\int_0^z \frac{dz'}{H(z')} \qquad (2)$$

where $H(z')$ is a known function parametrized by Ω's of universe constituents

$$H(z') = H_o \left[\Omega^0_{\mathrm{rad}}(1+z')^4 + \Omega^0_m(1+z')^3 + \Omega^0_k(1+z')^2 + \Omega^0_\Lambda\right]^{1/2}$$

and z is referring to the moment of emission (t_{em}).
Comments:

- We observe an object with redshift z and we have to specify the time t at which we wish to know the recession velocity $v_{\mathrm{rec}}(t)$. If we choose $t = t_{\mathrm{em}}$ in (2) then we get

$$v_{\mathrm{rec}}(z) = \frac{\dot{a}(z)}{a_{\mathrm{obs}}} \int_0^z \frac{dz'}{H(z')}$$

  then $v_{\mathrm{rec}}(z)$ is the recession velocity at the moment photons were emitted.
- There exist superluminal galaxies receding with $v_{\mathrm{rec}}(z) > 1$, e.g. for $(\Omega^0_m, \Omega^0_\Lambda) = (0.3, 0.7)$ objects with $z > 1.46$ have $v_{\mathrm{rec}} > 1$.
- Special relativity relation (based on the Doppler effect) between v_{rec} and z is incorrect if applied for large distances (redshifts).

For farther reading see

- T. M. Davis and Ch. H. Lineweaver, "Superluminal Recession Velocities", AIP Conf. Proc. 555, 348 (2001)
- T. M. Davis, Ch. H. Lineweaver, (New South Wales U.), "Expanding confusion: common misconceptions of cosmological horizons and the superluminal expansion of the universe", e-Print: astro-ph/0310808
- G.F.R. Ellis and T. Rothman, "Lost horizons", Am. J. Phys. 61, pp. 883-893 (1993)

The velocity-redshift relation

- non-relativistic Doppler effect: $\beta \equiv \frac{v}{c} = z$
- relativistic Doppler effect: $\beta = \frac{(1+z)^2 - 1}{(1+z)^1 + 1}$
- GR: $v_{\rm rec}(z) = \frac{\dot{a}(z)}{a_{\rm obs}} \int_0^z \frac{dz'}{H(z')}$

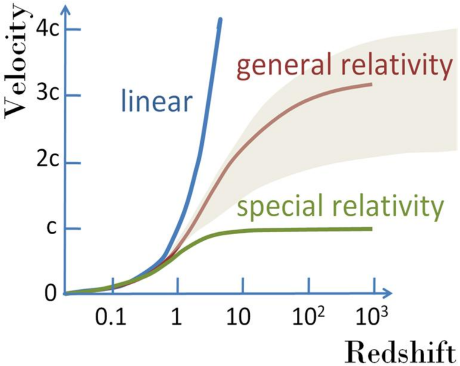


Figure 2: *Velocity-redshift relations, from T. M. Davis and Ch. H. Lineweaver, "Superluminal Recession Velocities", AIP Conf. Proc. 555, 348 (2001). The thick brown line corresponds to* $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$.

♠ The Hubble sphere and horizons

• The Hubble sphere

The Hubble sphere is defined as the surface that separates the region of the Universe beyond which the recession velocity exceeds the speed of light. So, we have the following condition for the proper (physical) distance $D_{Hs}(t)$ to the Hubble sphere:

$$v_{\rm rec} = 1 = \dot{a}(t)\int_0^{r_{Hs}(t)} \frac{dr'}{(1-kr'^2)^{1/2}} = \left[\frac{\dot{a}(t)}{a(t)}\right]\underbrace{\left[a(t)\int_0^{r_{Hs}(t)} \frac{dr'}{(1-kr'^2)^{1/2}}\right]}_{D_{Hs}(t)} = H(t)D_{Hs}(t)$$

where $r_{Hs}(t)$ are $D_{Hs}(t)$ are the coordinate and the distance to the sphere, respectively. So we get

$$D_{Hs}(t) = H^{-1}(t)$$

For a geometry such that $a(t) \propto t^\alpha$ (for RD $\alpha = 1/2$ for MD $\alpha = 2/3$) one gets $H(t) = \frac{\alpha}{t}$ so

$$D_{Hs}(t) = \frac{t}{\alpha}$$

• The past null cone

We are going to determine the worldline of photons that we observe now, i.e. at $t = t_0$ at $r_0 = 0$ (θ_0, φ_0 irrelevant). Since for the light signal $d\tau^2 = 0$, therefore

$$d\tau^2 = 0 = dt^2 - a^2(t)\frac{dr^2}{1-kr^2} \qquad \longrightarrow \qquad \frac{dt}{a(t)} = \pm\frac{dr}{\sqrt{1-kr^2}}$$

Consider light emitted at $t = t_e$ at some location $r = r_e$, then we integrate from emission to the observation (note that $dr < 0$)

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = -\int_{r_e}^{0} \frac{dr}{\sqrt{1-kr^2}} = \int_{0}^{r_e} \frac{dr}{\sqrt{1-kr^2}} \tag{3}$$

So the proper distance $D(t_e)$ at the emission time from the point of emission to the observer is given by

$$D(t_e) = a(t_e)\int_{0}^{r_e} \frac{dr}{\sqrt{1-kr^2}} = a(t_e)\int_{t_e}^{t_0} \frac{dt'}{a(t')}$$

Replacing t_e by t we obtain a worldline of photons which are reaching us today (at t_0), this is an analog of the past light cone in special relativity.

For cosmologies such that $a \propto t^\alpha$ one obtains

$$D(t) = \frac{t}{1-\alpha}\left[\left(\frac{t}{t_0}\right)^{\alpha-1} - 1\right]$$

Remarks:

- $D(t)$ is not a monotonic function of t, its maximum is at

$$t_{\max} = \frac{t_0}{\alpha^{\frac{1}{\alpha-1}}}$$

- $D(0) = D(t_0) = 0$

• The particle horizon

The fundamental question in cosmology: what fraction of the Universe is in causal contact?

More precisely:

For comoving observer with coordinate $r_0 = 0$ for what values of the emission coordinate r would a light signal emitted at $t = 0$ reach the observer at, or before, time t? The particle horizon is a surface of the region from which a light signal emitted at $t = 0$ may reach an observer at $r_0 = 0$ at time t. As for an ordinary horizon we could not see behind the "particle horizon".

- homogeneity $\longrightarrow$ we can choose $r_0 = 0$,
- for the light signal $d\tau^2 = 0$

$$d\tau^2 = dt^2 - a^2(t)\frac{dr^2}{1-kr^2} \qquad \longrightarrow \qquad \frac{dt'}{a(t')} = \pm\frac{dr}{\sqrt{1-kr^2}}$$

Emission at $(0, r_{ph})$, detection at $(t, 0)$, hence

$$\int_0^t \frac{dt'}{a(t')} = \int_0^{r_{ph}} \frac{dr}{\sqrt{1-kr^2}} \qquad \Rightarrow \qquad r_{ph} = r_{ph}(t) \tag{4}$$

Note that the signal is moving toward us, so $\frac{dr}{dt} < 0$.

The distance to the horizon at time t:

$$D_{ph}(t) = \int_0^{r_{ph}(t)} g_{rr}^{1/2}(t)dr = a(t)\int_0^{r_{ph}(t)} \frac{dr}{\sqrt{1-kr^2}}$$

Adopting (4) one gets

$$D_{ph}(t) = a(t)\int_0^t \frac{dt'}{a(t')}$$

For $a \propto t^\alpha$ one obtains at present

$$D_{ph}(t_0) = \frac{t_0}{1-\alpha}$$

Remarks:

- Note that for a static Universe, a = const., $D_{ph} = t_0$ (for $c = 1$).
- Note that $D_{ph}(t_0)$ is a distance <u>now</u> to a galaxy that emitted light at $t = 0$.
- As time flows $D_{ph}(t_0)$ grows as well, i.e. so far unseen regions of the Universe will become visible.
- For $0 < \alpha < 1$ we have $(1-\alpha)^{-1} > 1$ therefore the question emerges: How could light travel in time t_0 a distance greater than t_0 if its velocity is 1? $D_{ph}(t_0)$ is the present distance to the galaxy on the horizon which emitted light *we observe now*, in fact the galaxy was on top of us when the light was emitted!

- A galaxy on the horizon indeed might have traveled a distance greater than t_0, so *its velocity was greater than the speed of light.* Balloon analogy: two points separated by an arclength S and an angle θ, so $S = a\theta$, when the balloon inflates with a speed $\dot{a} = 1/2$ then $\dot{S} = 1/2 \times \theta$, so that for $\theta > 2$ rad they recede faster than light.
- The horizon is *not* the boundary that is moving with the speed of light (the Hubble sphere):

$$\frac{D_{ph}(t_0)}{D_{Hs}(t_0)} = \frac{\alpha}{1-\alpha}$$

  For $\alpha > 1/2$ the particle horizon is farther than the Hubble sphere.
- If $D_{ph}(t_0)$ is finite, then the particle horizon is a boundary between the part of the Universe that we have already seen and the remaining from where the light has not reached us yet.
- The finiteness of $D_{ph}(t_0)$ is determined by the behavior of $a(t)$ around $t = 0$, in the standard cosmology $D_{ph}(t)$ is finite since $\lim_{t\to 0}\,[t/a(t)] = 0$ ($a \to 0$ slower than t as $t \to 0$).

• The event horizon

The event horizon is a surface of the region from which a light signal emitted at (t_1, r_1) may reach an observer at $r_0 = 0$ if the observer waits long enough.

$$\int_0^{r_1} \frac{dr'}{(1 - kr'^2)^{1/2}} = \int_{t_1}^{t_{max}} \frac{dt'}{a(t')}$$

where t_{max} is the time-coordinate of the end of the universe, which would be infinite in the case of a universe that expands forever. The above allows to determine events (t_1, r_1) that are observable if we waited infinitely long (this is applicable for universes which expands forever). If the integral on the rhs diverges then the whole universe is observable if we wait long enough $(r_1 \to \infty)$. Then the distance at a given time t to the horizon reads

$$D_{eh}(t, t_1) = a(t) \int_0^{r_1} \frac{dr'}{(1 - kr'^2)^{1/2}} = a(t) \int_{t_1}^{t_{max}} \frac{dt'}{a(t')}$$

Usually the case with $t_1 = t$ is discussed, so in other words how far at the time t is the region beyond which we will never see signals emitted at the same time t, e.g. t could correspond to the present moment.

$$D_{eh}(t) = a(t) \int_t^{t_{max}} \frac{dt'}{a(t')}$$

For $a \propto t^\alpha$ one finds:

$$D_{eh}(t) = (1-\alpha)^{-1} \left(t^\alpha t_{\max}^{1-\alpha} - t \right)$$

♠ The Luminosity Distance

The total flux $\mathcal{F}$ (the energy *per time* per area measured by the detector) of light received by a telescope on Earth from an object of emitting power $\mathcal{L}$, called luminosity (energy produced *per time* by the source) can be calculated as follows. A "flash" of N_{emit} photons is emitted isotropically at the time $t = t_{\text{emit}}$ from a source located at the radial coordinate r. If there were no expansion then a telescope located at $r = 0$ would detect the total flux

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi[a(t_{\text{emit}})r]^2}$$

Note that $4\pi[a(t_{\text{emit}})r]^2$ is the area of the sphere containing photons emitted at $t = t_{\text{emit}}$.

The two-sphere analogy could be helpful to understand the presence of $[a(t_{\text{emit}})r]$.

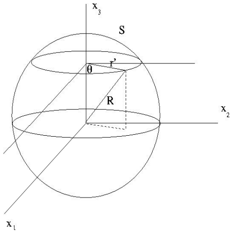


Figure 3: *The two sphere.*

However, because of the expansion of the sphere (the space time is expanding while the photon is traveling), at the detection time $t = t_{\text{obs}}$, the area of the spherical shell within which the photons travel has expanded to $4\pi[a(t_{\text{obs}})r]^2$, therefore the fraction should be corrected

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi[a(t_{\text{obs}})r]^2}$$

To compute properly the total flux, two other effects must be taken into account:

- Each emitted photon has its energy redshifted by the factor $\frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} = 1 + z$, so the observed photon energy is rescaled by the factor $\frac{1}{1+z}$.
- The observed flux is defined as energy *per time*, so that must be taken into account. If the time distance between photon flashes at the source is δt_{emit}, then the time distance between the detection of those flashes, δt_{obs}, will be increased according to the relation which we have obtained earlier:
$$\frac{\delta t_{\text{emit}}}{\delta t_{\text{obs}}} = \frac{a(t_{\text{emit}})}{a(t_{\text{obs}})} = \frac{1}{1+z}$$
So, the detected flux is suppressed by the factor $\frac{1}{1+z}$.

$$\Downarrow$$

The total flux observed now reads

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2} \qquad \text{for} \qquad d_L \equiv a(t_0)r(1+z)$$

where d_L is called the *luminosity distance* and the detection time is denoted by t_0. From now on the emission time will be denoted by t.

Note that r is unknown radial coordinate of the source. However, if the solution of the Friedmann equation is known then r could be related to the redshift z as follows. Let's recall the expansion of the scale factor around the present time:

$$a(t) = a_0 + a_0 \frac{\dot{a}}{a}_{|t=t_0} (t - t_0) - \frac{1}{2} a_0 \underbrace{\left[-\frac{\ddot{a}}{a}_{|t=t_0} \frac{1}{H_0^2} \right]}_{\equiv q_0} H_0^2 (t - t_0)^2 + \cdots$$

where q_0 is the *deceleration* parameter. We can eliminate the ratio $\frac{a(t)}{a(t_0)}$ using the relation $\frac{a(t)}{a(t_0)} = \frac{1}{1+z}$ (t_0 is the detection moment), so that

$$\frac{1}{1+z} = 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \cdots$$

Inverting we get

$$z = -H_0(t-t_0) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t-t_0)^2 + \cdots = H_0(t-t_0) \left[-1 + \left(1 + \frac{q_0}{2}\right) H_0 (t - t_0) + \cdots \right]$$

Therefore we can express the time difference $t_0 - t$ as a function of z:

$$t_0 - t = z H_0^{-1} \left[1 - \left(1 + \frac{q_0}{2}\right) z + \cdots \right]$$

Let's now recall the relation we have obtained for a massless wave traveling along a geodesic $d\tau^2 = 0$:

$$\int_t^{t_0} \frac{dt'}{a(t')} = \int_0^r \frac{dr'}{(1-kr'^2)^{1/2}} = \begin{cases} \sin^{-1} r = r + \frac{r^3}{6} + \cdots & k = +1 \\ r & k = 0 \\ \sinh^{-1} r = r - \frac{r^3}{6} + \cdots & k = -1 \end{cases} \tag{5}$$

Let's use the expansion

$$a(t) = a_0 + a_0 H_0 (t - t_0) - \frac{1}{2} a_0 q_0 H_0^2 (t - t_0)^2 + \cdots$$

on the lhs of (5) and keep only $\propto r$ terms on the rhs, then we get

$$a^{-1}(t_0) \left[(t_0 - t) + H_0 \frac{1}{2} (t_0 - t)^2 + \cdots \right] = r + \cdots$$

Substituting $t_0 - t = z H_0^{-1} \left[1 - \left(1 + \frac{q_0}{2} \right) z + \cdots \right]$ and keeping only terms $\mathcal{O}(z^2)$ we get

$$r = a_0^{-1} H_0^{-1} \left[z - \frac{1}{2} (1 + q_0) z^2 \right]$$

Now we are ready to use the above result in the expression for the luminosity distance $d_L = a(t_0)r(1+z)$

$$d_L = H_0^{-1}\left[z + \frac{1}{2}(1-q_0)z^2\right]$$

where we have kept only terms $\mathcal{O}(z^2)$. The above result yields a version of the Hubble law

$$H_0 d_L = z + \frac{1}{2}(1-q_0)z^2 + \cdots$$

Note that the above formula differs from the linear Hubble law for $q_0 \neq 1$, even though it was obtained for small z. Since q_0 depends on the cosmological model

$$q_0 = \frac{4\pi G}{3H_0^2}\sum_i(\rho_i^0 + 3p_i^0) = \frac{4\pi G}{3H_0^2}\sum_i(1+3w_i)\rho_i^0 = \frac{1}{2}\sum_i \Omega_i^0(1+3w_i)$$

therefore the measurement of $H_0 d_L$ offers a way to determine the fate of the Universe.

♠ The Angular Distance

Assume that there is an object of known diameter D located at the coordinate $r = r$, which emitted light at $t = t$, observed at $t = t_0$ at $r = 0$. From the FLRW metric we know that the angular diameter of the source, δ is given by

$$\delta = \frac{D}{a(t)r}$$

The angular distance d_A is defined as

$$d_A \equiv \frac{D}{\delta} = a(t)r$$

Since the luminosity distance is given by $d_L = a(t_0)r(1 + z)$ and we know the relation between the size of the scale factor at the corresponding redshift: $\frac{1}{1+z} = \frac{a(t)}{a(t_0)}$ therefore we can derive the relation between d_L and d_A:

$$d_A = \frac{d_L}{(1 + z)^2}$$

♠ Determination of Cosmological Parameters

Here we will discuss the determination of cosmological parameters such as H_0 and Ω_i^0 through a measurement of the luminosity distance d_L.
The luminosity distance d_L is defined through the total observed power

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2} \qquad \text{for} \qquad d_L \equiv a(t_0) r(1+z)$$

where $a(t_0)$, r and t are related by the equation of radial, null (light-like) geodesics for the FLRW metric ($d\theta = d\varphi = 0$):

$$d\tau = 0 \qquad \Rightarrow \qquad \frac{dr}{dt} = -\frac{(1-kr^2)^{1/2}}{a(t)}$$

Using the relation between the scale factor $a(t)$ and the redshift $1+z = \frac{a_0}{a(t)}$ we get

$$a_0 \frac{dr}{(1-kr^2)^{1/2}} = -(1+z)dt$$

The following relation (obtained earlier)

$$\frac{dt}{dz} = H_0^{-1}\frac{-1}{1+z}\,\frac{1}{[\Omega^0_{\mathrm{rad}}(1+z)^4+\Omega^0_m(1+z)^3+\Omega^0_k(1+z)^2+\Omega^0_\Lambda]^{1/2}}$$

could be adopted to swap dt and dz such that the integration could be performed

$$a_0\int_0^r \frac{dr'}{(1-kr'^2)^{1/2}} = H_0^{-1}\int_0^z \frac{dz'}{[\Omega^0_{\mathrm{rad}}(1+z')^4+\Omega^0_m(1+z')^3+\Omega^0_k(1+z')^2+\Omega^0_\Lambda]^{1/2}}$$

The lhs could be easily integrated

$$a_0\int_0^r \frac{dr'}{(1-kr'^2)^{1/2}} = a_0\begin{cases}\sin^{-1} r & k=+1\\ r & k=0\\ \sinh^{-1} r & k=-1\end{cases}$$

Thus we are able to express r as a function of z, this is exactly what is needed to find the luminosity distance as a function of z, that way we get e.g. for $k=+1$

$$r(z) = \sin\left\{(a_0H_0)^{-1}\int_o^z \frac{dz'}{[\Omega^0_{\mathrm{rad}}(1+z')^4+\Omega^0_m(1+z')^3+\Omega^0_k(1+z')^2+\Omega^0_\Lambda]^{1/2}}\right\}$$

Using the definition of $\Omega_k^0 = \frac{-k}{(a_0 H_0)^2}$ we will get rid of $a_0 H_0$ obtaining

- $k = +1$

$$\begin{aligned} d_L &= a(t_0)(1+z)r(z) = \frac{a_0 H_0}{H_0}(1+z)r(z) = H_0^{-1}(1+z)\left(|\Omega_k^0|\right)^{-1/2} \times \\ & \sin\left\{\left(|\Omega_k^0|\right)^{1/2}\int_o^z \frac{dz'}{[\Omega_{\rm rad}^0(1+z')^4 + \Omega_m^0(1+z')^3 + \Omega_k^0(1+z')^2 + \Omega_\Lambda^0]^{1/2}}\right\} \\ \Omega_k^0 &= 1 - \Omega_{\rm rad}^0 - \Omega_m^0 - \Omega_\Lambda < 0 \end{aligned}$$

- $k = 0$

$$\begin{aligned} d_L &= a(t_0)(1+z)r(z) = \frac{a_0 H_0}{H_0}(1+z)r(z) = \\ & H_0^{-1}(1+z)\int_o^z \frac{dz'}{[\Omega_{\rm rad}^0(1+z')^4 + \Omega_m^0(1+z')^3 + \Omega_k^0(1+z')^2 + \Omega_\Lambda^0]^{1/2}} \\ \Omega_k^0 &= 0 \end{aligned}$$

- $k = -1$

$$
\begin{aligned}
d_L &= a(t_0)(1+z)r(z) = \frac{a_0 H_0}{H_0}(1+z)r(z) = H_0^{-1}(1+z)\left(|\Omega_k^0|\right)^{-1/2} \times \\
&\quad \sinh\left\{\left(|\Omega_k^0|\right)^{1/2} \int_o^z \frac{dz'}{[\Omega_{\mathrm{rad}}^0(1+z')^4 + \Omega_m^0(1+z')^3 + \Omega_k^0(1+z')^2 + \Omega_\Lambda^0]^{1/2}}\right\} \\
\Omega_k^0 &= 1 - \Omega_{\mathrm{rad}}^0 - \Omega_m^0 - \Omega_\Lambda > 0
\end{aligned}
$$

So, a measurement of d_L provides a constraint on H_0 and Ω_{rad}^0, Ω_m^0 and Ω_Λ^0.

♠ The General Form of the Redshift Dependence of Particle Horizon

As we have shown the distance to the particle horizon is given by

$$D_{ph}(t) = a(t) \int_0^t \frac{dt'}{a(t')}$$

Our goal is to find the distance D_{ph} as a function of z (earlier we obtained $d_L = d_L(z)$ for small and large z), therefore it is convenient to change variables from t' to z'. For that we can adopt the relation obtained earlier

$$\frac{dt'}{dz'} = H_0^{-1} \frac{-1}{1+z'} \frac{1}{[\Omega^0_{\rm rad}(1+z')^4 + \Omega^0_m(1+z')^3 + \Omega^0_k(1+z')^2 + \Omega^0_\Lambda]^{1/2}}$$

Then

$$D_{ph}(t) = a(t) \int_0^t \frac{dt'}{a(t')} = a(t) \int_\infty^z a_0^{-1} \frac{a_0}{a(t')} \frac{dt'}{dz'} dz'$$

Inserting $\frac{dt'}{dz'}$ and adopting $\frac{a_0}{a(t')} = 1 + z'$ we obtain

$$D_{ph}(z) = \frac{a(t)}{a_0} \int_\infty^z (1+z') H_0^{-1} \frac{-1}{1+z'} \frac{dz'}{[\Omega^0_{\rm rad}(1+z')^4 + \Omega^0_m(1+z')^3 + \Omega^0_k(1+z')^2 + \Omega^0_\Lambda]^{1/2}}$$

Using $1 + z = \frac{a_0}{a(t)}$ we have

$$D_{ph}(z) = \frac{1}{H_0(1+z)} \int_z^\infty \frac{dz'}{[\Omega^0_{\rm rad}(1+z')^4 + \Omega^0_m(1+z')^3 + \Omega^0_k(1+z')^2 + \Omega^0_\Lambda]^{1/2}}$$

If we allow for an extra component of the Universe with the equation of state $p = w_x \rho$ then the above result is modified such that the horizon distance reads

$$D_{ph}(z) = \frac{1}{H_0(1+z)} \times$$

$$\int_z^{\infty} \frac{dz'}{[\Omega^0_{\rm rad}(1+z')^4 + \Omega^0_m(1+z')^3 + \Omega^0_k(1+z')^2 + \Omega^0_\Lambda + \Omega_x(1+z')^{3(1+w_x)}]^{1/2}}$$

Comment:

- It is important to realize that various powers of $(1+z)$ present above (or just on the rhs of the Friedmann equation, where they come from) originate from different dependence of energy densities on a (e.g. $\propto a^{-3}$ for matter, $\propto a^{-4}$ for radiation). The dependence on a was derived from the first law of thermodynamics *separately* for each kind of Universe constituents while the first law of thermodynamics applies for the *total* energy density and pressure. In general (before decoupling) non-relativistic matter interacts with radiation and the precise picture is more involved. So, strictly speaking what we are doing applies for the period when the radiation and the non-relativistic matter do not interact.

Parallax-based methods:
The most important direct distance measurements come from the parallax. The Earth's motion around the sun causes small shifts in stellar positions. These shifts are angles in a right triangle, with 1 AU making the short side of the triangle and the distance to the star being the long side. One pc is the distance of a star whose parallax is one arc second.

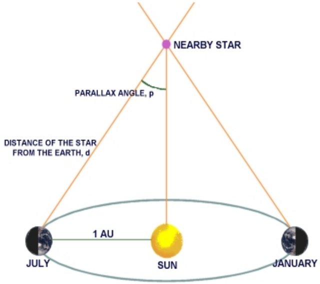


Figure 4: *The parallax.*

A standard candle is a class of astrophysical objects, such as supernovae or variable stars, which have known luminosity due to some characteristic quality possessed by the entire class of objects.

- Cepheids
  Cepheid is a variable star that has a fairly tight correlation between its period of variability and intrinsic brightness. Because of this correlation (discovered and stated by Henrietta Swan Leavitt in 1908 and given precise mathematical form by her in 1912), a Cepheid can be used as a "standard candle" to determine the distance to its host cluster or galaxy.

    - The variation in luminosity is caused by a cycle of ionization of helium in the star’s atmosphere, followed by expansion and deionization. While ionized, the atmosphere is more opaque to light.
    - The luminosity of cepheid stars range from 10^3 to 10^4 times that of the Sun. A three-day period Cepheid has a luminosity of about 800 times that of the Sun. A thirty-day period Cepheid is 10^4 times as bright as the Sun. The scale has been calibrated using nearby Cepheid stars, for which the distance was already known (a source of some uncertainties). This high luminosity, and the precision with which their distance can be estimated, makes Cepheid stars the ideal standard candle to measure the distance of clusters and external galaxies.

- First let's define apparent magnitude m of a celestial body as a measure of its brightness as seen on Earth:

$$m = -2.5 \log_{10} \mathcal{F} + \text{const.}$$

where $\mathcal{F}$ is the total flux (energy/area/time) observed on Earth while "const." is a constant to be determined by the requirement that the star Vega has apparent magnitude $m = 0$. Then the period-luminosity relationship for Type I Cepheids could be written as follows:

$$M = -2.81 \log_{10}(P) - (1.43 \pm 0.1)$$

where M is the absolute magnitude (an apparent magnitude of the object if it was be at 10 pc distance from the observer) and P is the period measured in days. The above relation was obtained by Henrietta Leavitt. She was working at the Harvard College Observatory, studying photographic plates of the Large (LMC) and Small (SMC) Magellanic Clouds, compiled a list of 1777 periodic variables. Eventually she classified 47 of these in the two clouds as Cepheid variables and noticed that those with longer periods were brighter than the shorter-period ones. She correctly concluded that as the stars were in the same distant clouds they were all at much the same relative distance from us.

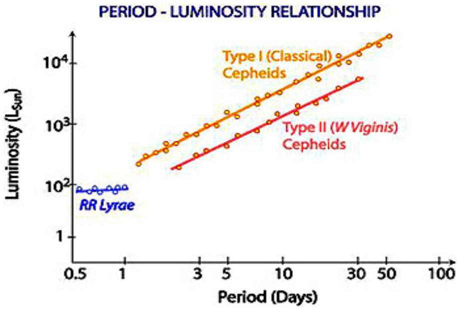
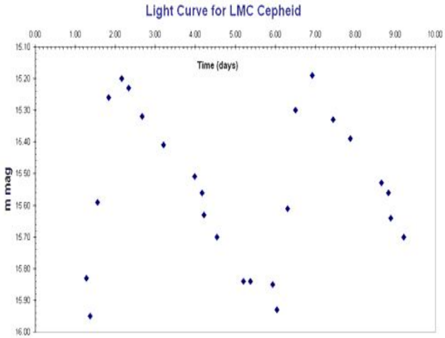


Figure 5: *Period-luminosity relationship for Cepheids and RR Lyrae stars.*

Any difference in apparent magnitude was therefore related to a difference in absolute magnitude. When she plotted her results for the two clouds she noted that they formed distinct relationships between brightness and period.

Let us now see how this relationship can be used to determine the distance to a Cepheid. For this procedure we will assume that we are dealing with a Type I, Classical Cepheid but the same method applies for W Virginis and RR Lyrae-type stars.

1. Photometric observations, by the naked-eye estimates, photographic plates, or photoelectric CCD images provide the apparent magnitude values for the Cepheid.
2. Plotting apparent magnitude values from observations at different times results in a light curve such as that below for a Cepheid in the LMC.



3. From the light curve and the photometric data, two values can be determined; the average apparent magnitude, m, of the star and its period in days. In the example above the Cepheid has a mean apparent magnitude of 15.56 and a period of 4.76 days.
4. Knowing the period of the Cepheid we can now determine its mean absolute magnitude, M, by adopting the relation found by Henrietta Leavitt
$$M = -2.81 \log_{10}(P) - (1.43 \pm 0.1)$$
Alternatively one can put a Cepheid on the period-luminosity plot as shown in (7). The one shown below is based on Cepheids within the Milky Way. The vertical axis shows absolute magnitude whilst period is displayed as a log value on the horizontal axes.

5. Once both apparent magnitude, m, and absolute magnitude, M are known we can simply substitute them into the distance-modulus formula and rewrite it to find a value for d_L the luminosity distance to the Cepheid.

$$M = -\frac{5}{2}\log_{10}\mathcal{F}_{10} + \text{const.} \qquad \text{and} \qquad m = -\frac{5}{2}\log_{10}\mathcal{F} + \text{const.}$$

where $\mathcal{F}_{10}$ is the total power observed at the distance of 10 pc (according to the definition of M). Since $\mathcal{F} \propto d_L^{-2}$ we obtain

$$5\log_{10}\left(\frac{d_L}{\text{Mpc}}\right) = m - M - 25, \qquad (6)$$

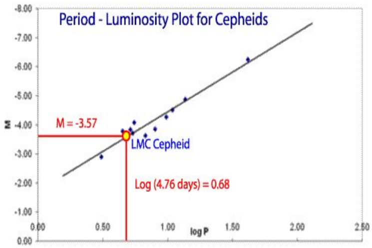


Figure 7: *The log of 4.76 days = 0.68. When this is plotted a value of about -3.6 results for absolute magnitude.*

- Type Ia Supernovas
  A supernova (plural: supernovae or supernovas) is a stellar explosion that creates an extremely luminous object. A supernova causes a burst of radiation that may briefly outshine its entire host galaxy before fading from view over several weeks or months. During this short interval, a supernova can radiate as much energy as the Sun could emit over its life span. The explosion expels much or all of a star's material at a velocity of up to a tenth the speed of light, driving a shock wave into the surrounding interstellar medium.
  Type Ia Supernova could be formed as follows. If a carbon-oxygen white dwarf accreted enough matter to reach the Chandrasekhar limit (the maximum non-rotating mass which can be supported against gravitational collapse) of about 1.38 solar masses, (note that this is for white dwarfs, not for any stars) it would no longer be able to support the bulk of its plasma and would begin to collapse. Increasing temperature and density inside the core triggers carbon fusion. Within a few seconds, a substantial fraction of the matter in the white dwarf undergoes nuclear fusion, releasing enough energy ($1 - 2 \times 10^{44}$ J) to unbind the star in a supernova explosion.

An outwardly expanding shock wave is generated, with matter reaching velocities of the order of $5,000 - 20,000$ km/s, or roughly 3% of the speed of light. There is also a significant increase in luminosity, reaching an absolute magnitude of -19.3 (or 5 billion times brighter than the Sun), with little variation.

One model for the formation of a Type Ia explosion involves the merger of two white dwarf stars, with the combined mass momentarily exceeding the Chandrasekhar limit. A white dwarf could also accrete matter from other types of companions (if the orbit is sufficiently close). For the list of supernovae see http://www.cfa.harvard.edu/iau/lists/Supernovae.html.

Supernovae are very rare, one per few hundred years per galaxy, however since there are many galaxies we can observe many supernovae "simultaneously".

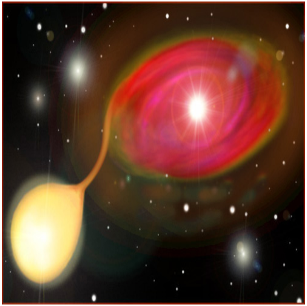


Figure 8: *A binary system before the explosion.*

The supernova explosions always release roughly the same amount of energy, and studies of relatively nearby type Ia supernovae have shown that they reach almost the same peak brightness in every case. Therefore it can be used as standard candle to determine their true distance. The absolute magnitude for the Type Ia supernovae has been calibrated to be $M = -19.33 \pm 0.25$, therefore a measurement of the apparent luminosity m allows us to determine the luminosity distance d_L according to (6).

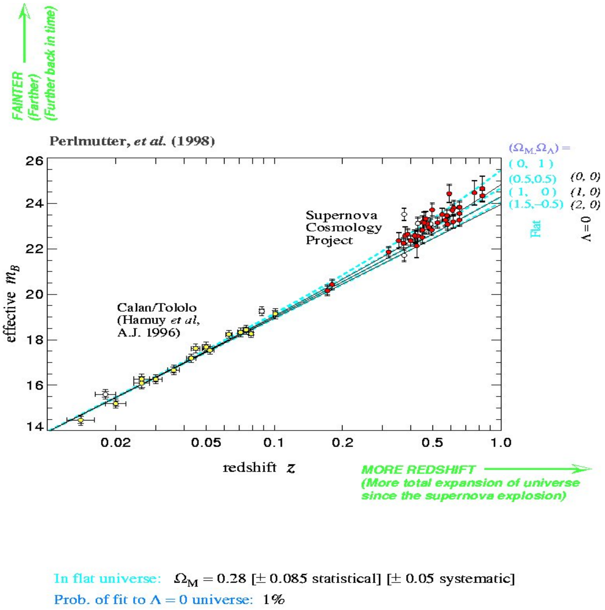


Figure 9: *Hubble diagram with 42 high-redshift supernovae (log redshift scale), from SCP.*

The data (from the *Supernova Cosmology Project* shown in fig. 9 favour a flat (k = 0) Universe (CMB) with a positive cosmological constant, $\Omega_\Lambda = 0.75 \pm 0.08$. The current data set of high-redshift Type Ia supernovas is not sufficient to break the degeneracy of the density terms, see (10). The results can be approximated by the linear combination $0.8\Omega_m - 0.6\Omega_\Lambda \simeq -0.2 \pm 0.1$.

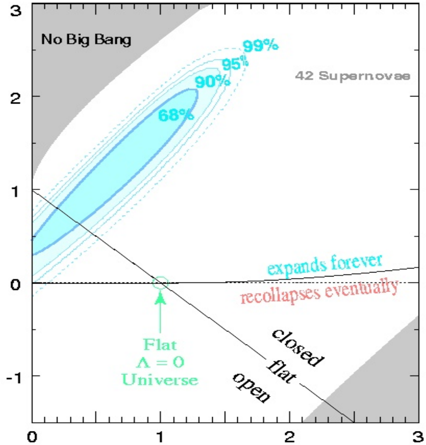


Figure 10: *Confidence region in Ω_Λ vs. Ω_m plane, from SCP.*

The geometry of the Universe is determined by $\Omega = \Omega_m + \Omega_\Lambda$ ($\Omega_k = 1 - \Omega$, for $\Omega_k \equiv -k/(H_0^2 a_0^2)$):

- $\Omega > 1 \quad \Rightarrow \quad k = +1$ closed Universe
- $\Omega < 1 \quad \Rightarrow \quad k = -1$ open Universe
- $\Omega = 1 \quad \Rightarrow \quad k = 0$ flat Universe

$\Omega_\Lambda = 1 - \Omega_m$ separates regions of closed ($k = +1$) and open ($k = -1$) Universes.

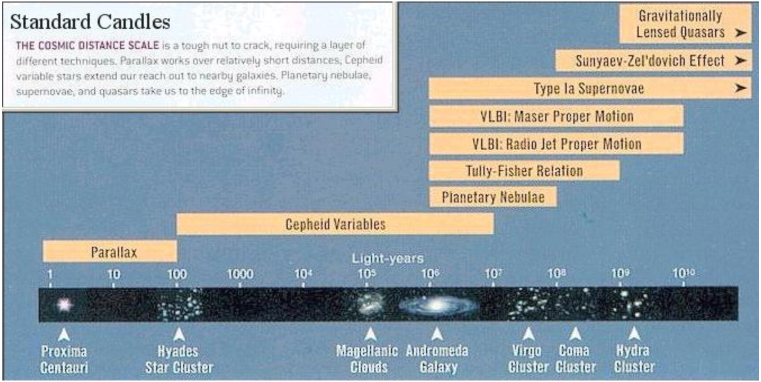


Figure 11: *The cosmic distance ladder*

$$
\begin{aligned}
\mathrm{m} &= 5.08 \times 10^{15}\ \mathrm{GeV}^{-1} \\
\mathrm{s} &= 1.51 \times 10^{24}\ \mathrm{GeV}^{-1} \\
\mathrm{pc} &= 3.09 \times 10^{16}\ \mathrm{m} = 1.57 \times 10^{32}\ \mathrm{GeV}^{-1}
\end{aligned}
$$

Then

$$\begin{aligned} H_0 &= h \times 10^2 \text{ km s}^{-1} \text{ Mpc}^{-1} = h \times 2.14 \times 10^{-42} \text{ GeV} \\ G &= 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} = 6.89 \times 10^{-39} \text{ GeV}^{-2} \end{aligned}$$

So

$$\rho^0_{\mathrm{crit}} = \frac{3H_0^2}{8\pi G} = h^2 \times 7.94 \times 10^{-47} \text{ GeV}^4$$

Comments:

- An explanation of *smallness* of the cosmological constant is one of the most outstanding problems of modern theoretical physics. In units with $\hbar = c = 1$, the energy density for $\Omega^0_\Lambda = \rho^0_\Lambda/\rho^0_{\rm crit} \simeq 1$ is $\rho_\Lambda \simeq 10^{-46}$ GeV^4. Since the origin of Λ seems to be gravitational, therefore the natural size of ρ_Λ should be a 4th power of the Planck mass, $\sim \mathcal{O}(M^4_{Pl})$, $M_{Pl} = 1.2 \cdot 10^{19}$ GeV, that gives $\rho_\Lambda \simeq 10^{76}$ GeV^4, while the observed value is smaller by 122 orders of magnitude! Theoretically, it is much easier to explain that a quantity is zero, then to show that it is so small, unfortunately the data require $\Omega_\Lambda \simeq 1$.
- There are some problems concerning the distance determination using standard candles. The principal one is calibration, determining exactly what the absolute magnitude of the candle is. This includes defining the class well enough that members can be recognized, and finding enough members with well-known distances that their true absolute magnitude can be determined with enough accuracy. The second lies in recognizing members of the class, and not mistakenly using the standard candle calibration upon an object which does not belong to the class. At extreme distances, which is where one most wishes to use a distance indicator, this recognition problem can be quite serious.

Non-Homogeneous Universe; the Lamaître-Tolman Cosmological Model

Consider spherically symmetric dust universe with radial inhomogeneities observed from the origin ($x^i = 0$). The line element takes the following form

$$d\tau^2 = dt^2 - X^2(r,t)dr^2 - a^2(r,t)(d\theta^2 + \sin^2\theta d\phi^2)$$

The FLRW metric is a limiting case of the Lamaître-Tolman (LT):

$$X(r,t) \to \frac{a(t)}{(1-kr^2)^{1/2}}, \qquad a(r,t) \to a(t)r$$

The energy-momentum tensor in that case reads

$$T_{\alpha\beta}(r,t) = \rho_m(r,t)U_\alpha U_\beta$$

for U_α being perfect fluid 4-velocity, so $U_0 = 1$ and $U_i = 0$ in the comoving frame.

The Einstein equations lead to the following set of differential equations:

$$\begin{aligned}
-2\frac{a''}{aX^2} + 2\frac{a'X'}{aX^3} + 2\frac{\dot{X}\dot{a}}{Xa} + \frac{1}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 - \left(\frac{a'}{aX}\right)^2 &= 8\pi G\rho_m \\
\dot{a}' &= a'\frac{\dot{X}}{X} \\
2\frac{\ddot{a}}{a} + \frac{1}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 - \left(\frac{a'}{aX}\right)^2 &= 0 \\
-\frac{a''}{aX^2} + \frac{\ddot{a}}{a} + \frac{\dot{X}\dot{a}}{Xa} + \frac{a'X'}{aX^3} + \frac{\ddot{X}}{X} &= 0
\end{aligned} \tag{7}$$

where $a' \equiv \partial a/\partial r$ and $\dot{a} \equiv \partial a/\partial t$. Only three of the above four equations are independent.

Eq.7 could be easily solved by

$$X(r,t) = C(r)a'(r,t)$$

The function $C(r)$ (to be determined by boundary conditions) could be written as follows:

$$C(r) \equiv \frac{1}{[1 - k(r)]^{1/2}},$$

where $k(r) < 1$.

Then the LT metric could rewritten as

$$d\tau^2 = dt^2 - \frac{[a'(r,t)]^2}{1-k(r)}dr^2 - a^2(r,t)(d\theta^2 + \sin^2\theta d\phi^2)$$

(The FLRW case could be obtained for $k(r) \to kr^2$ and $a(r,t) \to a(t)r$.) Then the two independent Einstein equations read

$$\frac{\dot{a}^2 + k(r)}{a^2} + \frac{2\dot{a}\dot{a}' + k'(r)}{aa'} = 8\pi G\rho_m \tag{8}$$

$$\dot{a}^2 + 2a\ddot{a} + k(r) = 0 \tag{9}$$

It is easy to verify (apply $\partial/\partial t$) that the first integral of (9) is

$$a\dot{a}^2 = F(r) - ak(r)$$

for $F(r)$ to be determined by boundary conditions. Then we get the generalized Friedmann equation for the *local* Hubble parameter $H(r,t) \equiv \dot{a}(r,t)/a(r,t)$:

$$H^2(r,t) + \frac{k(r)}{a^2} = \frac{F(r)}{a^3} \tag{10}$$

Instead of $F(r)$ and $k(r)$ one can define $\Omega_m^0(r)$ and $\Omega_k^0(r)$

$$\begin{aligned} F(r) &= H_0^2(r)\Omega_m^0(r)a_0^3(r) \\ k(r) &= -H_0^2(r)\Omega_k^0(r)a_0^2(r) \end{aligned}$$

where

$$\Omega_m^0(r) \equiv \frac{\rho_m(r,t_0)}{\rho_{\rm crit}(r,t_0)}, \quad \Omega_k^0(r) \equiv \frac{\rho_k(r,t_0)}{\rho_{\rm crit}(r,t_0)}, \quad H_0(r) \equiv H(r,t_0) \;\; \text{and} \;\; a_0(r) \equiv a(r,t_0)$$

Then the generalized Friedmann equation (10) reads

$$H^2(r,t) = H_0^2(r)\left[\Omega_k^0(r)\left(\frac{a_0(r)}{a(r,t)}\right)^2 + \Omega_m^0(r)\left(\frac{a_0(r)}{a(r,t)}\right)^3\right]$$

That should be compared with the FLRW Friedmann equation in the presence of the cosmological constant

$$H^2(t) = H_0^2\left[\Omega_k^0\left(\frac{a_0}{a(t)}\right)^2 + \Omega_m^0\left(\frac{a_0}{a(t)}\right)^3 + \Omega_\Lambda\right]$$

Comments

- The "observed" acceleration of the Universe is not a direct measurement, but a consequence of interpretation of the supernova data within the standard (FLRW) cosmology. Within FLRW Ω_Λ is a possible explanation of the observed maximal luminosity of supernovae (the observed luminosity is lower than one expected in FLRW model with $\Omega_\Lambda = 0$). Therefore in the concordance model we found $\Omega_\Lambda \simeq 0.7$ and $\Omega_m \simeq 0.3$. *Non-zero Ω_Λ and the standard Friedmann's equations imply $\ddot{a} > 0$*:

$$q_0 = \frac{4\pi G}{3H_0^2}\sum_i(\rho_i^0 + 3p_i^0) = \frac{4\pi G}{3H_0^2}\sum_i(1 + 3w_i)\rho_i^0 = \frac{1}{2}\sum_i \Omega_i^0(1 + 3w_i)$$

  So, the conclusion that $\ddot{a} > 0$ and $\Omega_\Lambda \neq 0$ are consequences of *the assumed FLRW geometry.*
- When light travels from a supernova toward us it "feels" $H(r,t)$ on its way. That is seen through the expression for luminosity distance d_L. It turns out (see e.g. H. Iguchi, T. Nakamura and K. i. Nakao, "Is dark energy the only solution to the apparent acceleration of the present universe?", Prog. Theor. Phys. **108**, 809 (2002) [arXiv:astro-ph/0112419]) that the extra freedom that appears within the LT geometry (i.e. $H(r,t)$) allows to fit the supernova data without invoking the cosmological constant.