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The Freeze-Out, the Boltzmann Transport Equation and the Dark Matter

"Freeze-out" examples:

- 1 sec - few minutes after the Big Bang: synthesis of light elements (nucleosynthesis)
- ~ 1 sec after the Big Bang: neutrino decoupling
- $\sim 10^5$ years after the Big Bang: decoupling of photons from the matter (recombination)

Consider a particle χ that could be a candidate for dark matter. Assume that χ are stable and their number can change only through annihilations to some SM particles X (it could be quark, lepton, Higgs boson etc.):

$$\bar{\chi}\chi \longleftrightarrow \bar{X}X$$

In addition we assume that:

- the above process can take place in both directions, if the speed in both directions is the same, we call it chemical equilibrium, then $\mu_\chi + \mu_{\bar{\chi}} = \mu_X + \mu_{\bar{X}}$,
- X and $\bar{X}$ have thermal (equilibrium) distributions (they usually have other interactions, e.g. electromagnetic, so the assumption is often satisfied) with $\mu_X, \mu_{\bar{X}} \approx 0$, (show that for charged particles $\mu_X = -\mu_{\bar{X}}$ so $n_X - n_{\bar{X}} \propto \mu_X$),
- $g_X = g_{\bar{X}}$, so $f_X = f_{\bar{X}}$ if $\mu_X, \mu_{\bar{X}} \approx 0$ (assumed for all SM particles),
- T invariance holds, so $\mathcal{M}_{\bar{\chi}\chi \to \bar{X}X} = \mathcal{M}_{\bar{X}X \to \bar{\chi}\chi}$,
- symmetric dark matter (χ): $g_\chi = g_{\bar{\chi}}$, $\mu_\chi = \mu_{\bar{\chi}}$ (so $f_\chi = f_{\bar{\chi}}$ and $n_\chi = n_{\bar{\chi}}$ always, not only in equilibrium),
- the Bose-Einstein (for bosons) and the Fermi-Dirac (for fermions) distribution functions could be approximated by the Maxwell-Boltzmann distribution functions:

$$f(\vec{p}, T) = \frac{1}{e^{[E(\vec{p})-\mu]/T} \pm 1} \simeq e^{-[E(\vec{p})-\mu]/T}.$$

- scattering processes of the DM with the thermal bath enforce kinetic (thermal) equilibrium (also after decoupling and out of chemical equilibrium), so that phase-space distribution functions for particles involved in the scattering satisfy, see Dodelson, 2003, sec. 3.1:

$$f_\chi(E,T) = e^{(-E+\mu_\chi)/T} = e^{\mu_\chi/T} f_\chi^{EQ}(E,T), \tag{1}$$

  where $f_\chi^{EQ}(E,T)$ is the thermal Maxwell-Boltzmann equilibrium distribution function for zero chemical potential. In this form of $f_\chi(E,T)$ the whole uncertainty in the determination of $f_\chi(E,T)$ (also after decoupling and out of chemical equilibrium) is encoded in the function $\mu_\chi = \mu_\chi(t)$.
  Since

$$n_\chi(T) = g_\chi e^{\mu_\chi/T} \int \frac{d^3 p_\chi}{(2\pi)^3} f_\chi^{EQ}(E,T) = e^{\mu_\chi/T} n_\chi^{EQ}(T),$$

  therefore (1) could be written as

$$f_\chi(E,T) = \frac{n_\chi(T)}{n_\chi^{EQ}(T)} \cdot f_\chi^{EQ}(E,T),$$

Our goal is to determine the evolution of the number density $n_\chi = n_\chi(t)$. When it happens that $n_\chi > n_\chi^{EQ}$ then the reaction would go faster to the right, so $\bar{\chi}\chi$ pairs will annihilate faster than they are created. The depletion rate should be proportional to $\sigma(\bar{\chi}\chi \to \bar{X}X)|\vec{v}|n_\chi^2$ (quadratic in density, as it should be proportional to the product of n_χ and $n_{\bar{\chi}}$, while these are equal). At the same time $\bar{\chi}\chi$ are also produced in the process $\bar{X}X \to \bar{\chi}\chi$ with a rate proportional to

$$f_X f_{\bar{X}} = e^{-(E_X+E_{\bar{X}})/T} = e^{-(E_\chi+E_{\bar{\chi}})/T} = f_\chi^{EQ} f_{\bar{\chi}}^{EQ},$$

where X and $\bar{X}$ were assumed to be in equilibrium with $\mu_X = \mu_{\bar{X}} = 0$. So we get

$$\boxed{\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle\sigma(\bar{\chi}\chi \to \bar{X}X)|\vec{v}|\rangle[n_\chi^2 - (n_\chi^{EQ})^2]}$$

where the lhs comes from $\frac{1}{a^3}\frac{d}{dt}(n_\chi a^3)$. The term $3H$ takes care of the dilution that comes from the Hubble expansion. The expression $\langle\sigma(\bar{\chi}\chi \to \bar{X}X)|\vec{v}|\rangle$ denotes a thermal average of the cross-section times velocity:

$$\langle\sigma(\bar{\chi}\chi \to \bar{X}X)|\vec{v}|\rangle \equiv \left(n_\chi^{EQ}\right)^{-2}(g_\chi)^2(g_X)^2\cdot$$
$$\int d\Phi_\chi d\Phi_{\bar{\chi}} d\Phi_X d\Phi_{\bar{X}} (2\pi)^4\delta^4(p_\chi + p_{\bar{\chi}} - p_X - p_{\bar{X}})|\mathcal{M}|^2 e^{-(E_\chi+E_{\bar{\chi}})/T}$$

In general after summing over all possible final states (all annihilation channels) one gets the Boltzamann equation

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle\sigma_A|\vec{v}|\rangle[n_\chi^2 - (n_\chi^{EQ})^2] \tag{2}$$

where σ_A is the total (inclusive) annihilation cross-section.
In order to scale out the effect of the Universe expansion let's define a new variable $Y_\chi \equiv n_\chi/s$ where s is the total entropy density and hence sa^3 is constant (this is an approximation) as the entropy in the comoving volume a^3, therefore

$$\dot{Y}_\chi \equiv \frac{d}{dt}\left(\frac{n_\chi}{s}\right) = \frac{\dot{n}_\chi}{s} - n_\chi\frac{\dot{s}}{s^2} = \frac{1}{s}(\dot{n}_\chi - n_\chi\frac{\dot{s}}{s})$$

Since sa^3 = const. therefore

$$\frac{d}{dt}(sa^3) = \dot{s}a^3 + 3a^2\dot{a}s = 0 \qquad \Rightarrow \qquad 3\frac{\dot{a}}{a}s = -\dot{s} \qquad \Rightarrow \qquad \frac{\dot{s}}{s} = -3H$$

Hence

$$\dot{Y}_\chi = \frac{1}{s}\left(\dot{n}_\chi - n_\chi\frac{\dot{s}}{s}\right) = \frac{1}{s}\left(\dot{n}_\chi + 3Hn_\chi\right)$$

Therefore the Boltzmann equation could be written as

$$
\begin{aligned}
s\dot{Y}_\chi &= -\langle\sigma_A|\vec{v}|\rangle[n_\chi^2 - (n_\chi^{EQ})^2] = -\langle\sigma_A|\vec{v}|\rangle \left[\left(\frac{n_\chi}{n_\chi^{EQ}}\right)^2 - 1\right](n_\chi^{EQ})^2 = \\
&-\langle\sigma_A|\vec{v}|\rangle \left[\left(\frac{Y_\chi}{Y_{\chi\,EQ}}\right)^2 - 1\right] Y_{\chi\,EQ}^2 s^2
\end{aligned}
$$

Hence

$$
\frac{\dot{Y}_\chi}{Y_{\chi EQ}} = -\langle\sigma_A|\vec{v}|\rangle \left[\left(\frac{Y_\chi}{Y_{\chi\,EQ}}\right)^2 - 1\right] Y_{\chi\,EQ}\, s = -n_{\chi\,EQ}\langle\sigma_A|\vec{v}|\rangle \left[\left(\frac{Y_\chi}{Y_{\chi\,EQ}}\right)^2 - 1\right]
$$

Defining the interaction rate $\Gamma \equiv n_{\chi EQ}\langle\sigma_A|\vec{v}|\rangle$ we can write the Boltzmann equation in the following form

$$
\frac{\dot{Y}_\chi}{Y_{\chi EQ}} = -\Gamma \left[\left(\frac{Y_\chi}{Y_{\chi\,EQ}}\right)^2 - 1\right]
$$

Recall the relation between temperature and time obtained for the domination of radiation

$$
t = 0.30\frac{M_{\rm Pl}}{T^2 g_\star^{1/2}} \tag{3}
$$

Let's define $x \equiv \frac{m}{T}$ and rewrite dt in terms of dx in order to change variables in the Boltzmann equation

$$dt = -0.30 \frac{M_{\rm Pl}}{g_\star^{1/2}} \frac{2}{T^3} dT = 2 \cdot 0.30 \frac{M_{\rm Pl}}{g_\star^{1/2} m^2} x dx$$

Note that (3) follows from the Friedmann equation with the energy density replaced by $\rho = \frac{\pi^2}{30} g_\star T^4$:

$$H = 1.66 \frac{g_\star^{1/2} T^2}{M_{\rm Pl}}$$

Hence we can write dt as

$$dt = 2 \cdot 0.30 \frac{M_{\rm Pl}}{g_\star^{1/2} m^2} x dx = \frac{1}{1.66} \frac{M_{\rm Pl}}{g_\star^{1/2}} \left(\frac{x}{m}\right)^2 \frac{dx}{x} = \left[\frac{1}{1.66} \frac{M_{\rm Pl}}{g_\star^{1/2}} \frac{1}{T^2}\right] \frac{dx}{x} = \frac{1}{H} \frac{dx}{x}$$

Therefore the equation (2) could be written as

$$\frac{x}{Y_{\chi\,EQ}} \frac{dY_\chi}{dx} = -\frac{\Gamma}{H} \left[\left(\frac{Y_\chi}{Y_{\chi\,EQ}}\right)^2 - 1\right] \tag{4}$$

In the non-relativistic ($x \equiv \frac{m}{T} \gg 3$) and ultra-relativistic ($x \ll 3$) cases Y_{EQ} has the following limiting forms

$$Y_{\chi\,EQ} = \begin{cases} \frac{45}{2\pi^4} \left(\frac{\pi}{8}\right)^{1/2} \frac{g_\chi}{g_{\star\,S}} x_\chi^{3/2} e^{-x_\chi} = 0.145 \cdot \frac{g_\chi}{g_{\star\,S}} x_\chi^{3/2} e^{-x_\chi} & \text{for} \quad x_\chi \gg 3\ (\text{non-rel}) \\ b_\chi \frac{\zeta(3) 45}{2\pi^4} \frac{g_\chi}{g_{\star\,S}} = 0.278 \cdot b_\chi \frac{g_\chi}{g_{\star\,S}} & \text{for} \quad x_\chi \ll 3\ (\text{rel}) \end{cases}$$

where $b_\chi = 1$ or $\frac{3}{4}$ for bosons and fermions, respectively.

Comments:

- The destruction rate of $\bar{\chi}\chi$ per comoving volume is proportional to the annihilation rate Γ.
- The destruction rate is balanced by inverse processes when $n_\chi = n_{\chi\, EQ}$ as expected.
- The creation (the inverse) process is suppressed for $T \ll m$ ($Y_{\chi\, EQ} \ll 1$), since only a small portion of $\bar{X}X$ pairs can have an energy sufficient to create $\bar{\chi}\chi$ pairs.
- The change of χ number density is controlled by $\frac{\Gamma}{H}$ as we have argued before. If $\frac{\Gamma}{H} \ll 1$ then, since $\Delta Y_\chi / Y_\chi \propto \Gamma/H$, we obtain for the relative change of Y_χ: $\Delta Y_\chi / Y_\chi \propto \Gamma/H \ll 1$, so the annihilations "freeze-out" while the number χ's "freezes in".
- $\Gamma = n_{EQ}\langle \sigma v \rangle$, so in the
    - relativistic regime $\Gamma \sim T^3 \cdot T^k/\Lambda^{k+2} \sim T$, while
    - in the non-relativistic regime $\Gamma \sim (mT)^{3/2} e^{-m/T} \cdot T^k/\Lambda^{k+2}$

  In both cases Γ decreases as T decreases, so usually eventually the interaction rate becomes too small to maintain the equilibrium, roughly at $\Gamma \simeq H$ (for $x \equiv x_f \simeq 25$ for cold dark matter), thus for $x \lesssim x_f$ we expect $Y(x) \simeq Y_{EQ}(x)$ while for $x \gtrsim x_f$ the abundance "freezes in": $Y(x \gtrsim x_f) \simeq Y_{EQ}(x_f)$.

♠ Hot relics: $x_f \lesssim 3$

We assume that the freeze-out occurs when the species are still relativistic and that Y_{EQ} does not change with time (or temperature). Note that $Y_{\chi\,EQ}(x) \propto \frac{g_\chi}{g_{\star\,S}(x)}$, it turns out that the proper choice of x in $g_{\star\,S}(x)$ is $x = x_f$. We are going to integrate the Boltzmann equation from $x = x_f$ till $x \to \infty$.

Then the Boltzmann equation

$$\frac{x}{Y_{\chi\,EQ}}\frac{dY_\chi}{dx} = -\frac{\Gamma}{H}\left[\left(\frac{Y_\chi}{Y_{\chi\,EQ}}\right)^2 - 1\right]$$

has a fixed point at $x \to \infty$ such that:

$$\left[\left(\frac{Y_\chi}{Y_{\chi\,EQ}}\right)^2 - 1\right] = 0$$

Note that

$$\frac{dY_\chi}{dx} < 0 \quad \text{for} \quad Y\chi > Y_{\chi\,EQ} \quad \text{and} \quad \frac{dY_\chi}{dx} > 0 \quad \text{for} \quad Y\chi < Y_{\chi\,EQ}$$

If $g_{\star\,S}(x)$ is constant then

- The asymptotic value of $Y_\infty \equiv \lim_{x\to\infty} Y_\chi(x)$ is not sensitive to the initial value of $Y_\chi(x_f)$, it is just $Y_{\chi\,EQ}$.
- For $x < x_f$ the solution $Y_\chi(x)$ follows the equilibrium yield $Y_{\chi\,EQ}(x)$.

It could be shown (see class) that if it is assumed that $g_{\star\,S}(x) = \text{const.}$, then the Boltzmann equation is satisfied by

$$Y_\chi(x) = Y_{\chi\,EQ}(x_f)\tanh(\alpha x)$$

where $\alpha \equiv M_{\mathrm{Pl}}/m \gg 1$.

At large x we get

$$Y_\chi(x) \to Y_\infty = Y_{\chi\,EQ}(x_f) = 0.278 \cdot b_\chi \frac{g_\chi}{g_{\star\,S}(x_f)} \qquad \text{for} \qquad x_f \lesssim 3$$

So, in the range where $g_{\star\,S}(x_f) = \text{const.}$ the resulting asymptotic (now) abundance is independent of the freeze-out temperature. Hence the present number density reads

$$n_{\chi\,0} = s_0 Y_\infty = 2906 \;\; Y_\infty \text{ cm}^{-3} = 807 \;\; b_\chi \cdot \frac{g_\chi}{g_{\star\,S}(x_f)} \text{ cm}^{-3}$$

where $s_0 = 2906 \text{ cm}^{-3}$ was used (see class). Today the energy density of a particle which was relativistic at the freeze-out and is non-relativistic now is saturated by its mass:

$$\rho_{\chi\,0} \simeq n_{\chi\,0} m = 2.91 \cdot 10^3 \; Y_\infty \left(\frac{m}{1 \text{ eV}}\right) \text{ eV cm}^{-3}$$

That leads to

$$\Omega^0_\chi = \frac{8\pi G}{3H_0^2}\rho_{\chi\,0} = h^{-2}\,7.8\cdot 10^{-2} b_\chi \cdot \frac{g_\chi}{g_{\star\,S}(x_f)}\left(\frac{m}{1\text{ eV}}\right)$$

Let's consider a contribution to Ω that comes from neutrinos for which $b_\nu = 3/4$ and $g_\nu = 2$. As we already know neutrinos decouple at $T \simeq 1$ MeV, the total entropy is conserved so we can calculate the entropy just above 1 MeV where the relativistic species are γ, $e^\pm$ and $(\nu, \bar{\nu})$:

$$g_{\star\,S}(x_f) = 2 + \frac{7}{8}(4 + 3 * 2) = 10\frac{3}{4}$$

Hence we obtain for $\nu\bar{\nu}$ pairs

$$\Omega^0_{\nu\bar{\nu}} h^2 = 0,011\frac{\sum_i m_{\nu_i}}{1\text{ eV}} \qquad \Rightarrow \qquad \sum_i m_{\nu_i} = (\Omega^0_{\nu\bar{\nu}} h^2)\cdot 91.9\text{ eV}$$

If we require that neutrinos do not overclose the Universe, so $\Omega_{\nu\bar{\nu}} h^2 < 1$ then we get the celebrated cosmological bound on the mass of *stable* neutrinos of single chirality

$$\sum_i m_{\nu_i} \ < \ 91.9\text{ eV}$$

♠ Cold relics: $x_f \gtrsim 3$

Let's consider the case where the freeze-out occurs when the species is non-relativistic ($x_f \gtrsim 3$), then while T is decreasing, $Y_{\chi\,EQ}(x)$ is decreasing exponentially:

$$Y_{\chi\,EQ}(x) = \frac{45}{2\pi^4}\left(\frac{\pi}{8}\right)^{1/2}\frac{g_\chi}{g_{\star\,S}}x_\chi^{3/2}e^{-x_\chi} = 0.145\cdot\frac{g_\chi}{g_{\star\,S}}x_\chi^{3/2}e^{-x_\chi}$$

There is no fixed point in this case. Assume that the following parameterization could be adopted

$$\langle\sigma_A|v|\rangle = \sigma_0\left(\frac{T}{m}\right)^n = \sigma_0 x^{-n} \qquad \text{for} \qquad x \gtrsim 3 \qquad \text{and} \qquad n \geq 0$$

Then the Boltzmann equation

$$\frac{x}{Y_{\chi\,EQ}}\frac{dY_\chi}{dx} = -\frac{\Gamma}{H}\left[\left(\frac{Y_\chi}{Y_{\chi\,EQ}}\right)^2 - 1\right]$$

could be written as (note that $\Gamma \equiv n_{\chi\,EQ}\langle\sigma_A|v|\rangle$)

$$\frac{dY_\chi}{dx} = -\frac{\Gamma}{xH}\frac{Y_{\chi\,EQ}^2}{Y_{\chi\,EQ}}\left[\left(\frac{Y_\chi}{Y_{\chi\,EQ}}\right)^2 - 1\right] = -\frac{\langle\sigma_A|v|\rangle\, s}{xH}\left[Y_\chi^2 - Y_{\chi\,EQ}^2\right]$$

where we have used the fact that $n_{\chi\,EQ} = sY_{\chi\,EQ}$.

Next let's recall that if the non-relativistic contribution to the energy and entropy densities could be neglected, then

$$s = \frac{2\pi^2}{45} g_{\star\, s} T^3 \qquad \text{and} \qquad H = \left(\frac{8\pi G}{3}\right)^{1/2} g_\star^{1/2} T^2$$

what could be written as

$$s = \frac{2\pi^2}{45} g_{\star\, s} \frac{m^3}{x^3} \qquad \text{and} \qquad H = \left(\frac{8\pi G}{3}\right)^{1/2} g_\star^{1/2} \frac{m^2}{x^2}$$

Let me rewrite the coefficient $\frac{\langle \sigma_A |v| \rangle \ s}{xH}$ as a function of x

$$\frac{\langle \sigma_A |v| \rangle \ s}{xH} = \frac{\sigma_0 x^{-n} \frac{2\pi^2}{45} g_{\star\, s} \frac{m^3}{x^3}}{x \left(\frac{8\pi G}{3}\right)^{1/2} g_\star^{1/2} \frac{m^2}{x^2}} = \underbrace{m \frac{2\pi^2}{45} \left(\frac{3}{8\pi G}\right)^{1/2} \frac{g_{\star\, s}}{g_\star^{1/2}} \sigma_0}_{\lambda} \, x^{-(n+2)} \equiv \lambda x^{-(n+2)}$$

So, the Boltzmann equation in this case reads

$$\frac{dY_\chi}{dx} = -\lambda x^{-(n+2)}\left[Y_\chi^2 - Y_{\chi\, EQ}^2\right]$$

where

$$\begin{aligned}\lambda &= m\frac{2\pi^2}{45}\left(\frac{3}{8\pi G}\right)^{1/2}\frac{g_{\star\, S}}{g_\star^{1/2}}\sigma_0 = 0.264\ M_{\rm Pl}\ m\sigma_0\frac{g_{\star\, S}}{g_\star^{1/2}}\\ Y_{\chi\, EQ} &= 0.145\cdot\frac{g_\chi}{g_{\star\, S}}x^{3/2}e^{-x}\end{aligned}$$

Let's define a departure form equilibrium

$$\Delta \equiv Y_\chi - Y_{EQ}$$

then the Boltzmann equation for Δ reads

$$\Delta' = -Y'_{EQ} - \lambda x^{-(n+2)}\Delta(2Y_{EQ} + \Delta)$$

We assume that at early times ($1 < x \ll x_f$), Y_χ follows closely Y_{EQ} so both Δ and Δ' are small, so setting $\Delta' = 0$ one gets approximately

$$\Delta \approx -\lambda^{-1}x^{n+2}\frac{Y'_{EQ}}{2Y_{EQ} + \Delta} \approx \frac{x^{n+2}}{2\lambda}, \tag{5}$$

where in the last equality Δ was neglected in the denominator and $1/x$ was dropped (note that $1 < x$)

$$\frac{Y'_{EQ}}{2Y_{EQ} + \Delta} \approx \frac{Y'_{EQ}}{2Y_{EQ}} = \frac{1}{2}\left(\frac{3}{2}x^{-1} - 1\right) \approx -\frac{1}{2}. \tag{6}$$

At late times ($x \gg x_f$) Δ is large, so $\Delta \approx Y \gg Y_{EQ}$, so the terms containing Y_{EQ} and Y'_{EQ} could be dropped, so the Boltzmann equation reads

$$\Delta' \approx -\lambda x^{-n-2}\Delta^2$$

Then integrating we get

$$\int_{x_f}^{\infty} \frac{\Delta'}{\Delta^2} dx \approx -\lambda \int_{x_f}^{\infty} \frac{dx}{x^{n+2}}$$

So then

$$\int_{\Delta_f}^{\Delta_\infty} \frac{d\Delta}{\Delta^2} \approx -\frac{\lambda}{(n+1)x_f^{n+1}}$$

where $\Delta_\infty \equiv \lim_{x\to\infty} \Delta(x)$ and $\Delta_f \equiv \Delta(x_f)$. Finally we obtain

$$\frac{1}{\Delta_\infty} = \frac{1}{\Delta_f} + \frac{\lambda}{(n+1)x_f^{n+1}}$$

Defining the freeze-out criterion by $\Delta(x_f) = cY_{EQ}(x_f)$ (with $c \sim \mathcal{O}(1)$) we get from the early time solution (5) and (6)

$$\begin{aligned} \Delta_f &\approx -\frac{1}{\lambda}x_f^{n+2}\frac{Y'_{EQ}(x_f)}{2Y_{EQ}(x_f)+\Delta_f} \\ &\approx -\frac{1}{\lambda}x_f^{n+2}\frac{2}{(2+c)}\frac{Y'_{EQ}(x_f)}{2Y_{EQ}(x_f)} \approx -\frac{1}{\lambda}x_f^{n+2}\frac{2}{(2+c)}\frac{-1}{2} = \frac{x_f^{n+2}}{\lambda(2+c)} \end{aligned}$$

therefore

$$\frac{1}{\Delta_\infty} \approx \frac{\lambda(2+c)}{x_f^{n+2}} + \frac{\lambda}{(n+1)x_f^{n+1}}$$

Since by assumption $x_f \gtrsim 3$ and $n \geq 0$ we may try to neglect the first term above.

Then, since $\Delta_\infty \approx Y_\infty$ (late time) we obtain

$$Y_\infty \approx \frac{(n+1)x_f^{n+1}}{\lambda} = \frac{(n+1)x_f^{n+1} g_\star^{1/2}}{0.26 M_{\mathrm{Pl}} m \sigma_0 g_{\star S}} \propto \frac{1}{m\sigma_0}$$

The above allows to determine the asymptotic (present) number density. However one still needs to determine the freeze-out temperature x_f. The explicit form of the freeze-out condition $\Delta(x_f) = cY_{EQ}(x_f)$ is the following

$$\Delta(x_f) = \frac{x_f^{n+2}}{\lambda(2+c)} = cY_{EQ}(x_f) = ca x_f^{3/2} e^{-x_f}$$

for $a \equiv 0.145(g/g_{\star S})$. Choosing $c(c+2) = n+1$ provides the best approximation to the exact solution, so we get

$$x_f \approx \ln\left[\frac{(n+1)\lambda a}{x_f^{n+\frac{1}{2}}}\right] \tag{7}$$

Let's adopt for notation

$$A_n \equiv (n+1)\lambda a$$

Then keeping two first terms one can write down the solution of (7) as follows

$$x_f = \ln A_n - (n + \frac{1}{2}) \ln x_f \approx \ln A_n - (n + \frac{1}{2}) \ln(\ln A_n) + \cdots$$

Class: Illustrate the dependence of x_f on λa assuming $\sigma_0 \approx G_F^2 m_Z^2$ for $G_F = 1.16 \cdot 10^{-5}$ GeV^{-2} for $n = 0, 1, 2, 3, 4$, assume $A_n = (n + 1) \cdot 10^p$ and vary $p \in [10, 16]$.

Having x_f and Y_∞ determined one can calculate the present number and mass densities:

$$n_{\psi\,0} = s_0 Y_\infty = 2906\, Y_\infty \text{ cm}^{-3} = 1.1 \cdot 10^4 \frac{(n+1) x_f^{n+1}}{(g_\star s / g_\star^{1/2}) M_{\rm Pl} m \sigma_0} \text{ cm}^{-3}$$

$$\boxed{\Omega_\psi h^2 = 1.1 \cdot 10^9 \frac{(n+1) x_f^{n+1} \text{ GeV}^{-1}}{(g_\star s / g_\star^{1/2}) M_{\rm Pl} \sigma_0}}$$

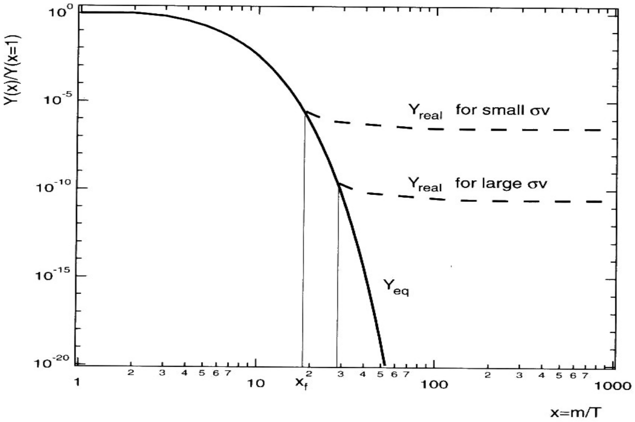

Fig. 9.1. The freeze-out of a massive particle. At a certain value $x_f = m_\chi/T_f$ the number density Y (normalized to the entropy density s, and in the figure arbitrarily normalized to the value at $x = 1$) leaves the equilibrium abundance curve Y_{eq} (the solid line) and gives an actual abundance $Y_{\rm real}$ shown by the dashed lines. As can be seen, a higher annihilation rate σv means a smaller relic abundance, since the actual curve tracks the equilibrium curve to smaller temperatures. For weakly interacting massive particles, x_f is of the order of 20. Adapted from [26].

(An illustration for the freeze-out for cold relic is taken from Bergström & Goobar.)

An example of cold relics is a hypothetical heavy Dirac stable neutrino with $m \gg 1$ MeV. The large mass implies that such a neutrino would decouple as non-relativistic, though not necessarily at the same temperature $T \sim 1$ MeV as ordinary light neutrinos. The annihilation through the Z boson exchange leads to various final states $\bar{\nu}_i \nu_i$, $\bar{l} l$, $\bar{q}_i q_i$ etc.. Then for $T \lesssim m \lesssim M_Z$ (assuming, as verified below, that the neutrino is non-relativistic)

$$\sigma_0 \sim G_F^2 m^2 \qquad \text{with} \qquad n = 0$$

Taking $g = 2$ and $g_\star \simeq 60$ one gets (class/homework perhaps)

$$x_f \simeq 16.6 + 3 \ln\left(\frac{m}{1\text{ GeV}}\right) \qquad \text{and} \qquad Y_\infty \simeq 5.1 \cdot 10^{-9} \left(\frac{1\text{ GeV}}{m}\right)^3 \left[1 + \frac{3}{16.6} \ln\left(\frac{m}{1\text{ GeV}}\right)\right]$$

then

$$\Omega_{\bar{\nu}\nu} h^2 \simeq 1.5 \cdot 2 \left(\frac{1\text{ GeV}}{m}\right)^2 \left[1 + \frac{3}{16.6} \ln\left(\frac{m}{1\text{ GeV}}\right)\right],$$

where it has been taken into account that $\Omega_{\bar{\nu}\nu} = 2\Omega_\nu$ because of identical abundance of neutrinos and anti-neutrinos.

Note that the freeze-out takes place at

$$T_F \simeq \frac{m}{15} \simeq 60\text{ MeV} \left(\frac{m}{1\text{ GeV}}\right)$$

Requiring $\Omega_{\bar{\nu}\nu} h^2 \lesssim 1$ we get the famous Lee-Weinberg bound

$$m \gtrsim 2\text{ GeV}$$

♠ The Baryon number of the Universe

The net baryon number density is $n_B \equiv n_b - n_{\bar{b}}$ where n_b and $n_{\bar{b}}$ are the baryon and anti-baryon number densities, respectively. From $\rho_c = 1.05\ h^2$ eV cm^{-3} and $m_N \simeq 940$ MeV (the nucleon mass) we get

$$\Omega_B = \frac{m_N n_B}{\rho_c} = \frac{m_N n_B}{\frac{3H_0^2}{8\pi G}} \qquad \Rightarrow \qquad n_B = \Omega_B h^2 \cdot 1.11 \cdot 10^{-6}\ \mathrm{cm}^{-3}$$

Since $s \propto a^{-3}$ $(s = S/a^3)$, $B \equiv n_B/s \propto n_B a^3 =$ const. is the net baryon number of the Universe ($V = a^3$). In the absence of baryon number violating interactions, it is conserved.

It is useful to relate s and photon number density n_γ:

$$\left.\begin{array}{l} s = \frac{2\pi^2}{45} g_{\star\, s} T^3 \\ n_\gamma = \frac{\zeta(3)}{\pi^2} g_\gamma T^3 \end{array}\right\} \longrightarrow \frac{s}{n_\gamma} = \frac{2\pi^4}{45\zeta(3)} \frac{g_{\star\, s}}{g_\gamma} \longrightarrow s = \frac{\pi^4}{45\zeta(3)} g_{\star\, s} n_\gamma \simeq 1.80 g_{\star\, s} n_\gamma$$

If $g_{\star\, s} =$ const. then one can use s and n_γ interchangeable, for instance since e^+e^- annihilation till today $s \simeq 7.04 \cdot n_\gamma$.

For the Universe baryon number we get

$$B = \left.\frac{n_B}{s}\right|_{\text{today}} = \frac{\Omega_B h^2 \cdot 1.11 \cdot 10^{-6}\ \text{cm}^{-3}}{2970\ \text{cm}^{-3}} \simeq 3.74 \cdot 10^{-10} \Omega_B h^2$$

where we have used the fact (see class) that $s = \frac{2\pi^2}{45} g_{\star\, s} T^3|_{\text{today}} \simeq 2970\ \text{cm}^{-3}$ ($g_{\star\, s}|_{\text{today}} = 3.91$).
Since the epoch of $e^{\pm}$ annihilation, s and n_γ were related by $s \simeq 7.04 \cdot n_\gamma$, so we get for the η parameter

$$\eta \equiv \left.\frac{n_B}{n_\gamma}\right|_{\text{today}} = \left.\frac{n_B}{s}\frac{s}{n_\gamma}\right|_{\text{today}} \simeq 7.04 \cdot B \simeq 2.63 \cdot 10^{-9} \cdot \Omega_B h^2$$

♠ The nuclear statistical equilibrium

Now we are in position to discuss consequences of the nuclear statistical equilibrium:

- kinetic (thermal, local) equilibrium for non-relativistic species

$$\Downarrow$$

$$n_A = g_A \left(\frac{m_A T}{2\pi} \right)^{3/2} e^{(\mu_A - m_A)/T}$$

- chemical equilibrium ${}^A Z \leftrightarrow Zp + (A - Z)n$ (the same speed)

$$\Downarrow$$

$$\mu_A = Z\mu_p + (A - Z)\mu_n$$

Let's find the number density for the nuclear species ${}^{A}Z$. First we calculate $e^{\mu_A/T}$ using the above relation and

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{(\mu_i - m_i)/T} \qquad \text{for} \qquad i = n, p.$$

So

$$\begin{aligned} e^{\mu_A/T} &= e^{(Z\mu_p + (A-Z)\mu_n)/T} = \left(e^{\mu_p/T}\right)^Z \left(e^{\mu_n/T}\right)^{A-Z} = \\ &= \left(\frac{n_p}{g_p}\right)^Z \left(\frac{2\pi}{m_p T}\right)^{3Z/2} \left(e^{m_p/T}\right)^Z \cdot \left(\frac{n_n}{g_n}\right)^{A-Z} \left(\frac{2\pi}{m_n T}\right)^{3(A-Z)/2} \left(e^{m_n/T}\right)^{A-Z} \\ &= n_p^Z n_n^{A-Z} 2^{-A} \left(\frac{2\pi}{m_N T}\right)^{3A/2} e^{[Zm_p + (A-Z)m_n]/T} \end{aligned}$$

where $m_N \simeq m_p \simeq m_n$ and $g_p = g_n = 2$.

nucleus	${}^A_Z X$	B_A	g_A
deuteron	${}^2_1 H$	2.22 MeV	3
triton	${}^3_1 H$	8.48 MeV	2
helium-3	${}^3_2 He$	7.72 MeV	2
helium-4	${}^4_2 He$	28.3 MeV	1
carbon-12	${}^{12}_6 C$	92.2 MeV	1

Table 1: The binding energies of some light nuclei.

Using the expression for the binding energy

$$B_A \equiv Zm_p + (A - Z)m_n - m_A$$

we get

$$\begin{aligned} n_A &= g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} e^{-m_A/T} \cdot n_p^Z n_n^{A-Z} 2^{-A} \left(\frac{2\pi}{m_N T}\right)^{3A/2} e^{[Zm_p+(A-Z)m_n]/T} \\ &= g_A A^{3/2} 2^{-A} \left(\frac{2\pi}{m_N T}\right)^{3(A-1)/2} n_p^Z n_n^{A-Z} e^{B_A/T} \end{aligned}$$

For all species $n_i \propto a^{-3}$ therefore it is useful to factor out and cancel the change related exclusively to the expansion. The following variable (mass fraction) proves to be convenient:

$$X_A \equiv \frac{An_A}{n_N} \qquad \text{with} \qquad \sum_A X_A = 1$$

where $n_N \equiv n_n + n_p + \sum_i (An_A)_i$ is the total nucleon density (that is also equal to the total baryon density n_B). Let's first find An_A in terms of T , X_p and X_n:

$$An_A = g_A A^{5/2} 2^{-A} \left(\frac{2\pi}{m_N T}\right)^{3(A-1)/2} \underbrace{n_p^Z n_n^{A-Z}}_{X_p^Z X_n^{A-Z} n_B^A} e^{B_A/T}$$

From the definition of η we obtain

$$n_N = n_B = \eta n_\gamma = \eta\zeta(3)\frac{2}{\pi^2}T^3 \tag{8}$$

Hence

$$X_A = g_A A^{5/2} 2^{-A} \left(\frac{2\pi}{m_N T}\right)^{3(A-1)/2} X_p^Z X_n^{A-Z} n_B^{A-1} e^{B_A/T}$$

Then inserting n_N from (8) we have

$$X_A = g_A \zeta(3)^{A-1} 2^{-(3A+5)/2} \pi^{(1-A)/2} A^{5/2} \eta^{A-1} X_p^Z X_n^{A-Z} \left(\frac{T}{m_N}\right)^{3(A-1)/2} e^{B_A/T}$$

♠ $p \longleftrightarrow n$ transitions : $T \gg 1$ MeV ($t \ll 1$ s)

The following reactions are responsible for the balance between protons and neutrons:

$$
\begin{aligned}
n &\longleftrightarrow p + e^- + \bar{\nu}_e \\
n + \nu_e &\longleftrightarrow p + e^- \\
n + e^+ &\longleftrightarrow p + \bar{\nu}_e
\end{aligned}
$$

For chemical equilibrium one obtains

$$\mu_n + \mu_{\nu_e} = \mu_p + \mu_e$$

Then we can calculate n_n/n_p which is of fundamental importance for the formation of light nuclei

$$\frac{n}{p} \equiv \frac{n_n}{n_p} = \frac{g_n \left(\frac{m_n T}{2\pi}\right)^{3/2} e^{(\mu_n - m_n)/T}}{g_p \left(\frac{m_p T}{2\pi}\right)^{3/2} e^{(\mu_p - m_p)/T}} = e^{-(m_n - m_p)/T - (\mu_p - \mu_n)/T} = e^{-Q/T + (\mu_e - \mu_{\nu_e})/T} \tag{9}$$

where $Q \equiv m_n - m_p = 1.293$ MeV.

In order to estimate the relevance of the chemical potential term lets find a net fermion number for a given fermionic species. Assume that there are rapid transitions of the form: $f\bar{f} \longleftrightarrow \gamma + \gamma$ (here we will consider temperatures 100 MeV $> T >$ 1 MeV, so the transition $e^+e^- \longrightarrow \gamma + \gamma$ takes place). Then $\mu_f + \mu_{\bar{f}} = 2\mu_\gamma$, since $\mu_\gamma = 0$ we get $\mu_f = -\mu_{\bar{f}}$. In general we have

$$n_f(T) = \frac{g_f}{2\pi^2}\int_{m_f}^{\infty} \frac{(E^2 - m_f^2)^{1/2}}{\exp\left[(E-\mu_f)/T\right] + 1} EdE$$

Hence assuming $g_f = g_{\bar{f}}$ we get for the net fermionic number density (in fact this applies to any additive $U(1)$ quantum number) corresponding to the species f

$$\begin{aligned} n_f - n_{\bar{f}} &= \frac{g_f}{2\pi^2}\int_{m_f}^{\infty} dEE(E^2 - m_f^2)^{1/2}\left[\frac{1}{\exp\left[(E-\mu_f)/T\right]+1} - \frac{1}{\exp\left[(E+\mu_f)/T\right]+1}\right] \\ &= \begin{cases} \frac{g_f T^3}{6\pi^2}\left[\pi^2\left(\frac{\mu_f}{T}\right) + \left(\frac{\mu_f}{T}\right)^3\right] & \text{for} \quad T \gg m_f \\ 2g_f\left(\frac{m_f T}{2\pi}\right)^{3/2}\sinh\left(\frac{\mu_f}{T}\right)\exp\left(-\frac{m_f}{T}\right) & \text{for} \quad T \ll m_f \end{cases} \end{aligned} \tag{10}$$

Let's focus on the relativistic case $T \gg m_f$ and introduce the notation $\Delta_{n_f} \equiv n_f - n_{\bar{f}}$, then

$$\Delta_{n_f} = \frac{g_f T^3}{6} \left(\frac{\mu_f}{T}\right) \left[1 + \frac{1}{\pi^2} \left(\frac{\mu_f}{T}\right)^2\right]$$

Since

$$s = \frac{2\pi^2}{45} g_{\star S} T^3$$

therefore we have

$$\frac{\Delta_{n_f}}{s} = \frac{g_f 45}{g_{\star S} 12\pi^2} \left(\frac{\mu_f}{T}\right) \left[1 + \frac{1}{\pi^2} \left(\frac{\mu_f}{T}\right)^2\right]$$

Let's specify now to $f = e$. If, in addition we assume that $\mu_e/T \ll 1$ (later we will see that this is indeed the case) then we obtain

$$\frac{\mu_e}{T} \sim \frac{g_{\star S} 12\pi^2}{g_f 45} \frac{\Delta_{n_e}}{s} \sim 1.4 \cdot 10 \, \frac{\Delta_{n_e}}{s} \tag{11}$$

for $g_{\star S} = 10.75$ and $g_e = 2$.

From electric neutrality of the Universe we have

$$\frac{\mu_e}{T} \simeq \frac{\Delta_{n_e}}{s} = \frac{\Delta_{n_p}}{s} \tag{12}$$

where only contributions for e^- and p to the total charge of the Universe was taken into account (heavier leptons and baryons are negligible at the temperature of interest): $(\Delta_{n_p} - \Delta_{n_e})/s = 0$. The baryon number of the Universe is given by

$$B = \frac{n_B}{s} = \frac{\Delta_{n_p} + \Delta_{n_n}}{s} \simeq 3.74 \cdot 10^{-10} \Omega_B h^2$$

Therefore (assuming $\Delta_{n_p} \sim \Delta_{n_n}$)

$$\frac{\Delta_{n_p}}{s} \sim 10^{-10}\ \Omega_B h^2$$

and we get from (12)

$$\frac{\mu_e}{T} \sim 10^{-9}\ \Omega_B h^2$$

Note that the above result allows to skip the electron chemical potential contribution to n/p as a consequence of experimental data.

On the other hand, to estimate the contribution from the neutrino chemical potential we *assume* that lepton numbers

$$L_i \equiv \frac{\Delta_{n_i} + \Delta_{\nu_i}}{s}$$

are small (as the baryon number B does), then we have from (11) (assuming no cancellations)

$$\frac{\mu_{\nu_e}}{T} \ll 1$$

so that we can approximate (9) by

$$\left.\frac{n}{p}\right|_{EQ} \equiv \frac{n_n}{n_p} = e^{-Q/T+(\mu_e-\mu_{\nu_e})/T} \simeq e^{-Q/T}$$

Therefore if $T \gg Q = m_n - m_p = 1.293$ MeV then the number of protons and neutrons are very much the same. As we know for certain temperature the interaction between p and n are expected to be too slow to maintain equilibrium between them. For the interaction rate for $n + \nu_e \longleftrightarrow p + e^-$ one gets (see e.g. Kolb & Turner for details)

$$\Gamma = \begin{cases} \frac{1}{\tau_n}\left(\frac{T}{m_e}\right)^3 \exp\left(-\frac{Q}{T}\right) & \text{for} \quad T \ll Q, m_e \\ \frac{7\pi}{60}(1+3g_A^2)G_F^2 T^5 \simeq G_F^2 T^5 & \text{for} \quad T \gg Q, m_e \end{cases}$$

where $\tau_n = 885.7 \pm 0.8$ s is the neutron life time and $g_A \simeq 1.26$ is the axial vector coupling of the nucleon.

Recall that

$$H = \left[\frac{8\pi G}{3}\rho_{\rm tot}(T)\right]^{1/2} = \left[\frac{8\pi G}{3}\frac{\pi^2}{30}g_\star T^4\right]^{1/2} = 1.66\,\frac{g_\star^{1/2}T^2}{M_{\rm Pl}} \simeq 5.4\,\frac{T^2}{M_{\rm Pl}}$$

where $g_\star = 2 + \frac{7}{8}(3\cdot 2 + 4) = 10\frac{3}{4}$ was adopted. For $T \gtrsim Q, m_e$ one gets

$$\frac{\Gamma}{H} \sim \left(\frac{T}{0.8\ \text{MeV}}\right)^3$$

So for $T \gtrsim 0.8$ MeV one expects the ratio n/p to have its equilibrium value

$$\frac{1}{5} \lesssim \left.\frac{n}{p}\right|_{EQ} \simeq e^{-Q/T} \lesssim 1$$

what implies $X_p \simeq X_n$.

The strategy that we apply to estimate abundances of light elements is to assume that the thermal evolution is equilibrium like. Then one can determine the freeze-out temperature and assume that the abundance at this temperature is the same as at present asymptotic temperature. More precisely in order to predict abundances of light elements one has to solve (as functions of T) the following set of equations

$$\begin{aligned} \frac{X_n}{X_p} &= \exp\left(-\frac{Q}{T}\right) \\ X_A &= g_A \zeta(3)^{A-1} 2^{-(3A+5)/2} \pi^{(1-A)/2} A^{5/2} \eta^{A-1} X_p^Z X_n^{A-Z} \left(\frac{T}{m_N}\right)^{3(A-1)/2} e^{B_A/T} \\ 1 &= X_p + X_n + X_2 + X_3 + X_4 + X_{12} \end{aligned}$$

for $A = {}^2H, {}^3He, {}^4He$ and ${}^{12}C$ (in the simplest case).

Questions: show suppression factors in the formula for X_A, when (for what temperature) the abundance of A might be substantial?

$\star$ $t \sim 10^{-2}$ s ($T \sim 10$ MeV)

- The energy density dominated by the radiation, relativistic degrees of freedom: $e^{\pm}$, γ, 3 neutrino species, $g_\star = 10\frac{3}{4}$.
- Weak reaction rates are large: $\Gamma \gg H$, so $n/p = (n/p)_{EQ} \simeq 1$.
- $T_\nu = T$
- $X_n \simeq X_p \simeq 0.5$, $X_2 - X_{12} \sim 10^{-12} - 10^{-126}$

$\star\ t \sim 1\ \mathrm{s}\ (T \sim 1\ \mathrm{MeV})$

- Neutrinos decoupled just before this epoch.
- At $T \simeq m_e/3 \simeq 0.2$ MeV $e^{\pm}$ pairs annihilate heating photons relative to neutrinos by the factor $(11/4)^{1/3}$.
- Weak interactions that interconvert neutrons and protons freeze-out (so $\Gamma \lesssim H$), then

$$\left(\frac{n}{p}\right)_{\text{freeze-out}} \simeq e^{-Q/T} \simeq \frac{1}{6}$$

  and

$$X_p \simeq \frac{6}{7}, \qquad X_n \simeq \frac{1}{7} \qquad \text{and} \qquad X_2 - X_{12} \sim 10^{-12} - 10^{-108}$$

The ratio $\frac{n}{p}$ starts slowly decreasing below $1/6$ after the freeze-out because of occasional free neutron decays.

$\star\ t \sim 1 - 2\,\mathrm{min}\ (T \sim 0.3 - 0.1\ \mathrm{MeV})$

- At that time $g_\star$ decreases to its present value 3.36.
- The ratio $\frac{n}{p}$ has decreased (as a consequence of decays with $\tau_n = 885.7 \pm 0.8$ s) from $\sim \frac{1}{6}$ to $\sim \frac{1}{7}$ (its equilibrium value would be $\frac{1}{74}$ for $T = 0.3$ MeV). *Before having time to decay*, most neutrons ends up in helium nuclei through one of the chains:

$$\begin{aligned} p + n &\longrightarrow {}^2H + \gamma \\ {}^2H + {}^2H &\longrightarrow {}^3He + n \\ {}^3He + {}^2H &\longrightarrow {}^4He + p \end{aligned}$$

or

$$\begin{aligned} p + n &\longrightarrow {}^2H + \gamma \\ {}^2H + {}^2H &\longrightarrow {}^3H + p \\ {}^3H + {}^2H &\longrightarrow {}^4He + n \end{aligned}$$

The ratio of the rate for $p + n \longrightarrow {}^2H + \gamma$ to the expansion rate

$$\frac{\Gamma_{pn}}{H} \simeq 2 \cdot 10^3 \left(\frac{T}{0.1\ \mathrm{MeV}}\right)^5 \frac{n_p}{n_p + n_n} \Omega_B h^2$$

turns out to be large for $T \gg 0.1$ MeV. For $T \gtrsim 0.1$ MeV the photodisintegration $p + n \longrightarrow {}^2H + \gamma$ is very efficient and not much helium can be produced. However for $T \lesssim 0.1$ MeV ${}^2H$ abundance rises to $\sim 10^{-5} - 10^{-3}$, which leads to rapid ${}^2H + {}^2H$ fusion, that uses most of the available neutrons, so that the estimate of the helium abundance is

$$X_4 \simeq \frac{4 n_{^4He}}{n_N} = \frac{4(n_n/2)}{n_n + n_p} = \frac{2\frac{n}{p}}{1 + \frac{n}{p}}$$

where for the ratio n/p one should adopt $\sim 1/7$ which leads to

$$X_4 \simeq \frac{1}{4}$$

$X_4 \simeq 0.25$ agrees with observations of helium abundance in stars and gas clouds. Note that the depletion of $\frac{n}{p}$ (due to neutron decays) from $\sim 1/6$ to $\sim 1/7$ is essential to fit the data (for $\frac{n}{p} = \frac{1}{6}$ one gets $X_4 \simeq \frac{2}{7} \simeq 0.29$).

- For other species X_A are still very small.
- Note that the time from the weak interaction freeze-out till formation of ^4He is approximately $t \sim 200$ s (roughly the Universe age at $T \sim 0.1$ MeV) which is of the order of the neutron life time $\tau_n = 885.7 \pm 0.8$ s, this is a very spectacular coincidence since:
  - If the time was longer more neutrons would decay and the formation of the observed helium abundance would not be possible.
  - If the Universe cooled faster (so the time was shorter) fewer neutrons would have time to decay before being saved into its stable existence inside helium nuclei, so that helium abundance would have increased.

  Since $H^2 \propto \rho_{\text{tot}}$ and ρ_{tot} is dominated by relativistic species therefore an addition of extra (besides 3 present in the SM) neutrinos would speed up the expansion increasing the helium abundance beyond the observed value. From that (see class) one can obtain the limit $N_\nu \lesssim 4$ (confirmed later by the LEP measurement of the number of light ($\lesssim m_Z/2$) neutrinos $N_\nu = 3$.
- The nucleosynthesis restricts the value of the baryon to photon ratio: $\eta = (5-6) \cdot 10^{-10}$

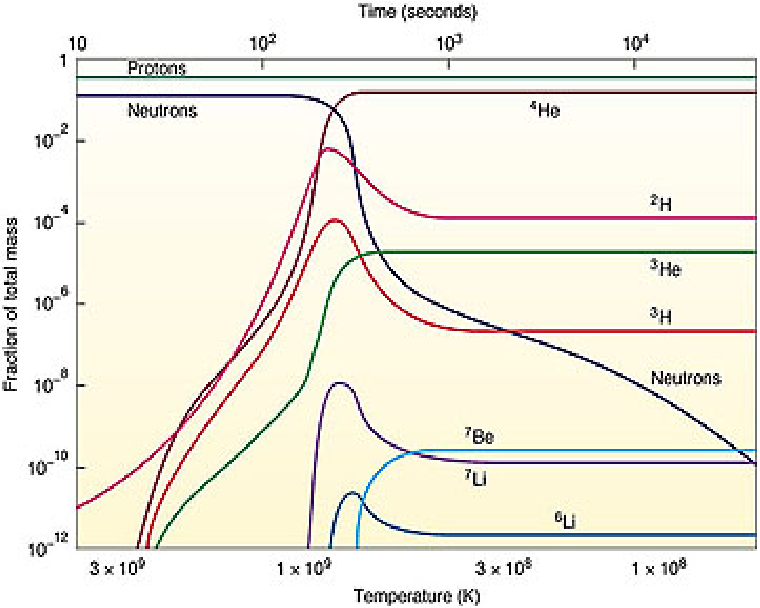

Figure 1: *Mass fractions relative to hydrogen (from Astronomica.org.),* $1\ K = 8.6 \cdot 10^{-4}\ eV$.

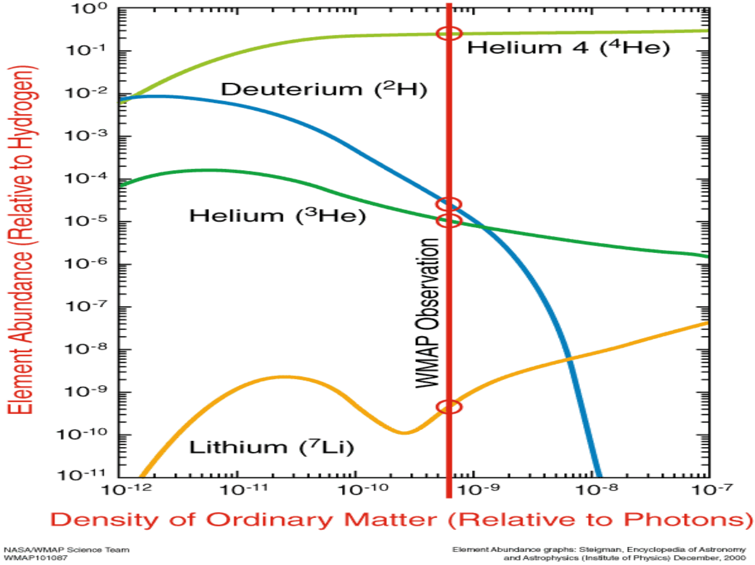

Figure 2: *The light element abundance predictions from BBN theory plotted against the baryon-to-photon ratio. From top to bottom are the mass fraction of* ^4He *and the relative mole fractions* D/H, $^3He/H$ *and* $^7Li/H$. *From https://map.gsfc.nasa.gov/universe/bb_tests_ele.html*

Recombination

Now we are going to discuss what happened at the temperature far below $T \sim 0.3 - 0.1$ MeV (when the nucleosynthesis take place). Here we focus on $T \sim 1$ eV, we assume $n_{e^+} = 0$, $n_{\bar{p}} = 0$ and $n_e = n_p$ (as the Universe is electrically neutral). The electrons and photons are still in thermal equilibrium, the Thomson scattering $\gamma + e^- \longrightarrow \gamma + e^-$ is responsible for maintaining the equilibrium. In the limit $E_\gamma \ll m_e$ the cross-section and the interaction rate could be estimated as

$$\langle \sigma_T v \rangle \simeq \frac{\alpha^2}{m_e^2} \qquad \Rightarrow \qquad \Gamma_\gamma \simeq n_e \langle \sigma_T v \rangle$$

It is easy to see that for $T \sim 1 - 10$ eV the condition $\Gamma_\gamma > H$ is no longer satisfied so that photons and electrons decouple. However there appears a new difficulty while calculating n_e, namely electrons may disappear by combining with protons (so forming hydrogen atoms), thus we should consider the reaction $p + e^- \longrightarrow H + \gamma$ that would be responsible for the electron number density, hence (since photons have $\mu_\gamma = 0$)

$$\mu_p + \mu_e = \mu_H$$

in equilibrium.

Let's introduce the total baryon number (for simplicity we neglect here the baryon number carried by ${}^4He$, so protons may be either free or bound in hydrogen)

$$n_B = n_p + n_H$$

Here we are interested in $T \lesssim 10$ eV (note the hydrogen binding energy in the ground state is $B_1 = 13.6$ eV) therefore e^-, p and H are non-relativistic, hence

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right) \qquad \text{for} \qquad i = e, p, H$$

Using $\mu_p + \mu_e = \mu_H$ and $m_H = m_e + m_p - B$ (definition of the binding energy) we get

$$\begin{aligned} n_H &= g_H \left(\frac{m_H T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_H - m_H}{T}\right) = \\ &= \frac{g_H}{g_e g_p} g_e g_p \left(\frac{m_H T}{2\pi}\right)^{3/2} \exp\left(\frac{(\mu_e + \mu_p) - (m_e + m_p - B)}{T}\right) = \\ &= \frac{g_H}{g_e g_p} \left[g_e \exp\left(\frac{\mu_e - m_e}{T}\right) \left(\frac{m_e T}{2\pi}\right)^{3/2}\right] \left[g_p \exp\left(\frac{\mu_p - m_p}{T}\right) \left(\frac{m_p T}{2\pi}\right)^{3/2}\right] \times \\ & \quad \exp\left(\frac{B}{T}\right) \left[\frac{(2\pi)^2}{m_e m_p T^2}\right]^{3/2} \left(\frac{m_H T}{2\pi}\right)^{3/2} = \frac{g_H}{g_e g_p} n_e\, n_p \exp\left(\frac{B}{T}\right) \left(\frac{2\pi m_H}{m_e m_p T}\right)^{3/2} \end{aligned}$$

Define the ionization fraction as

$$X_e \equiv \frac{n_p}{n_B} = \frac{n_p}{n_p + n_H}$$

Then we can express n_H in terms of X_e as a function of T

$$n_H = \frac{1 - X_e}{X_e} n_p = \frac{g_H}{g_e g_p} n_e\, n_p \exp\left(\frac{B}{T}\right) \left(\frac{2\pi m_H}{m_e m_p T}\right)^{3/2}$$

Hence, since $n_e = n_p$ and $m_H \simeq m_p$ we get

$$\frac{1 - X_e}{X_e} = \frac{g_H}{g_e g_p} n_p \exp\left(\frac{B}{T}\right) \left(\frac{2\pi}{m_e T}\right)^{3/2}$$

Expressing n_p through the baryon to photon ratio $\eta = n_B/n_\gamma$ and X_e we obtain

$$\frac{1 - X_e}{X_e} = \frac{g_H}{g_e g_p} [X_e n_B] \exp\left(\frac{B}{T}\right) \left(\frac{2\pi}{m_e T}\right)^{3/2}$$

Since

$$n_B = \eta n_\gamma = \eta \frac{\zeta(3)}{\pi^2} g_\gamma T^3$$

we finally get (adopting $g_H = 4$, $g_\gamma = g_e = g_p = 2$) the so-called Saha equation for the fractional ionization *at equilibrium*:

$$\frac{1 - X_e}{X_e^2} = 4 \left(\frac{2}{\pi}\right)^{1/2} \zeta(3)\eta \left(\frac{T}{m_e}\right)^{3/2} \exp\left(\frac{B}{T}\right)$$

As we already know the nucleosynthesis restricts η: $\eta = (5 - 6) \cdot 10^{-10}$ (through the relation $\eta = 2.7 \cdot 10^{-8} \Omega_B h^2$ it corresponds to $\Omega_B h^2 \sim 0.02$). Therefore the Saha equation could be solved for $X_e = X_e(T)$, or equivalently as $X_e = X_e(z)$ using $T = 2.73\,(1 + z)$ K.

The Fig.3 (from Kolb & Turner) shows X_e as a function of the redshift z.

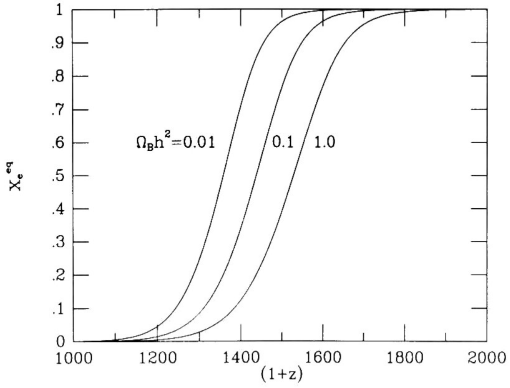

Fig. 3.9: The equilibrium ionization fraction as a function of $(1 + z)$.

Figure 3: *The ionization fraction (from Kolb & Turner).*

The ionization decreases below 10% for $z \sim 1200 - 1300$, so at that $z(= z_{\text{rec}})$ electrons begins to be captured by protons forming neutral hydrogen (the recombination). The corresponding temperature and time are

$$\begin{aligned} T_{\text{rec}} &= T_0(1 + z_{\text{rec}}) \sim 2.7 \cdot 1300 \text{ K} = 3500 \text{ K} \sim 0.3 \text{ eV} \\ t_{\text{rec}} &= \frac{2}{3} H_0^{-1} \Omega_m^{0\,-1/2} (1 + z_{\text{rec}})^{-3/2} \sim \frac{1.4 \cdot 10^5}{(\Omega_m^0)^{1/2} h} \text{ yr} \end{aligned} \qquad (13)$$

where we have assumed that the Universe was matter dominated (see Kolb&Turner) so $t \simeq \frac{2}{3}(1 + z)^{-3/2} H_0^{-1} \Omega_m^{0\,-1/2}$. For radiation domination $1.4 \cdot 10^5$ would be replaced by $2.9 \cdot 10^3$, the exact value (radiation and matter) is $2.7 \cdot 10^5$.

Comments:

- Note that naively one could expect the recombination to happen at $T \simeq B = 13$ eV, that is not the case because of the long tail of energies larger than T, there are so many photons relative to baryons ($\eta = n_b/n_\gamma = 2.7 \cdot 10^{-8}\Omega_B h^2$) that the reionization easily may happen even for $T < 13$ eV.
- So far we have considered the case of equilibrium so $p + e^- \longrightarrow H + \gamma$ with the rate faster than the expansion rate. It turns out that this is indeed the case for $z \gtrsim 1100$. After that the equilibrium can not be maintained and the ionization fraction is frozen at its value for $z \sim 1100$.
- It could be shown that for $z \simeq 1050$ the mean free path of photons is comparable with the radius of observable Universe, so the region of $z \sim 1100$ is sometimes referred to as the surface of last scattering of the cosmic microwave background.

To determine the freeze-out temperature of the ionization fraction more precisely we have to consider the Boltzmann equation for $p + e^- \longrightarrow H + \gamma$. In a close analogy with the case considered before we obtain

$$\dot{n}_e + 3Hn_e = -\langle\sigma_{\rm rec}|\vec{v}|\rangle \left[n_e^2 - (n_e^{EQ})^2\right] \tag{14}$$

where for the thermally averaged cross-section one can get

$$\langle\sigma_{\rm rec}|\vec{v}|\rangle = 4.7 \cdot 10^{-24} \left(\frac{1\ \text{eV}}{T}\right)^{1/2}\ \text{cm}^2$$

Solving the equation (14) numerically one finds

$$T_f \sim 0.25\ \text{eV}$$

and hence the remaining ionization fraction (see class perhaps)

$$X_e(\infty) \sim 2.7 \cdot 10^{-5} \frac{\Omega_m^0}{\Omega_B h} \sim 1.4 \cdot 10^{-3}$$

which means that only one proton per 10^3 baryons is free!
Comments:

- At the moment of recombination photons temperature was $T = T_f \sim 0.25$ eV to be compared with the present CMB temperature $T_{\rm CMB} = 2.35 \cdot 10^{-4}$ eV. The difference (ratio) is due to the redshift.

Brief thermal history of the Universe

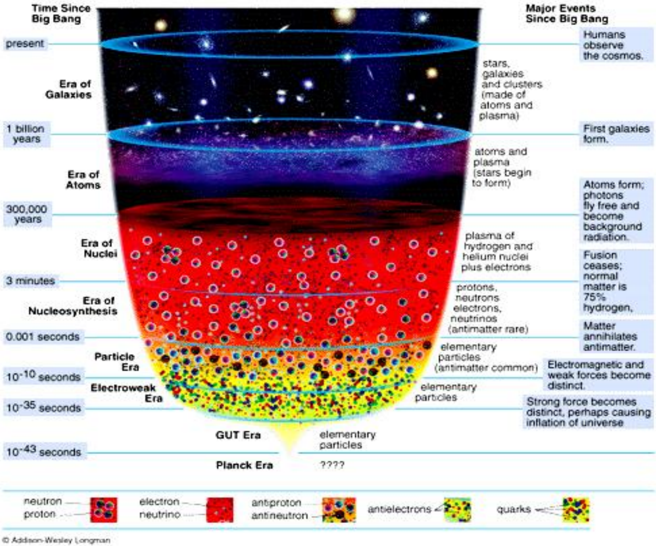

Figure 4: *History of the Universe. Form physics.lakeheadu.ca/.../2330/Cosmology/.*

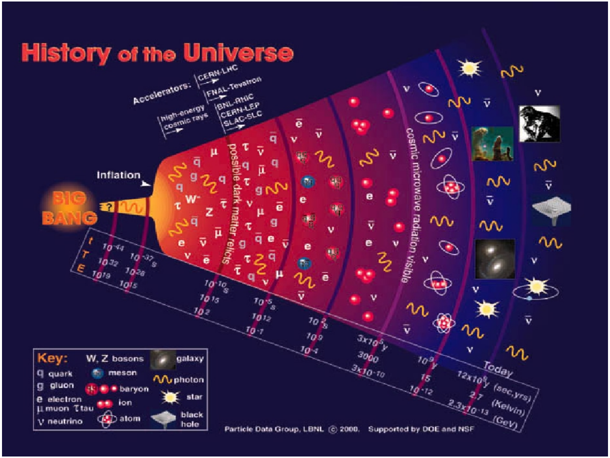

Figure 5: *History of the Universe. Form conferences.fnal.gov.*

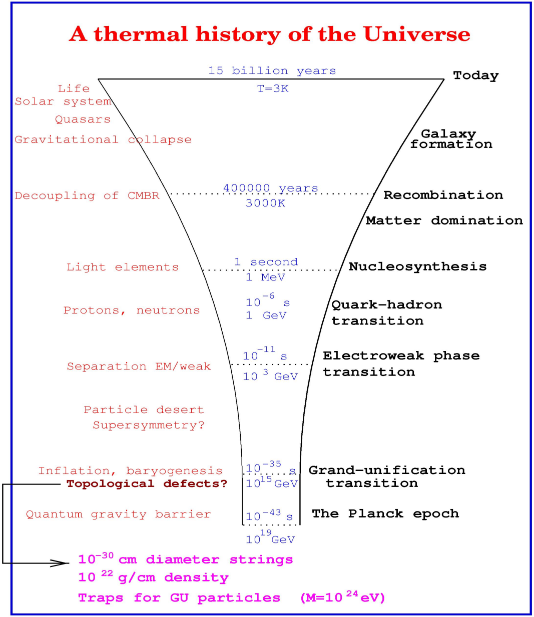

Figure 6: *History of the Universe. Form lpnhe-auger.in2p3.fr/slides/vulg/.*