Cosmology Course
problems for the written exam
February 4th 2019

1. Show, using the following form of the energy-momentum tensor

$$T^{\mu\nu}(x) = g^{-1/2}\sum_n \frac{p_n^\mu p_n^\nu}{E_n}\delta^3(\vec{x}-\vec{x}_n)$$

that for a gas of non-relativistic particles of mass m, the energy density (ρ), pressure (p) and the number density (n) are related by $\rho = nm + \frac{3}{2}p$.

2. Show, using the following form of the energy-momentum tensor

$$T^{\mu\nu} = g^{-1/2}\sum_n \frac{p_n^\mu p_n^\nu}{E_n}\delta^3(\vec{x}-\vec{x}_n)$$

that for a gas of relativistic particles, the energy density (ρ) and pressure (p) are related by $\rho = 3p$.

3. The fundamental equations for cosmology are

   - The Friedmann equation
   $$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2$$
   - The acceleration equation:
   $$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = -8\pi G p$$
   - The energy-momentum conservation (the first law of thermodynamics):
   $$\dot{p}R^3 = \frac{d}{dt}\left[R^3(\rho+p)\right]$$

   Show that only two of them are independent.

4. Show that if the energy density is dominated by positive cosmological constant and Universe is spatially flat ($k=0$) then the expansion is exponential.

5. Find the necessary and sufficient conditions for the exponential inflation in terms of the equation of state, consider both $k=0$ and $k\neq 0$.

6. Assuming a single component Universe, find the condition which must be satisfied by the equation of state in order to ensure positive acceleration of the Universe.

7. Find the present Universe age t_0 in terms of $\Omega^0_{\rm rad}$ and H_0 assuming radiation domination for $k=0$ and $k\neq 0$. Plot t_0H_0 as a function of $\Omega^0_{\rm rad}$.

8. Find the present Universe age t_0 in terms of $\Omega^0_{\rm m}$ and H_0 assuming matter domination for $k=0$ and $k\neq 0$. Hint:

$$\int_0^1 \frac{dx}{(a+bx^{-1})^{1/2}}\Bigg|_{a+b=1} = \begin{cases} \frac{1}{1-b} + \frac{b}{2(b-1)^{3/2}}\arccos(\frac{2}{b}-1) & \text{for} \quad b>1 \\ \frac{1}{1-b} - \frac{b}{2(1-b)^{3/2}}\text{arcosh}(\frac{2}{b}-1) & \text{for} \quad 0<b<1 \end{cases}$$

Plot t_0H_0 as a function of Ω^0_m.

9. Find the present Universe age t_0 in terms of Ω_Λ^0 and H_0 assuming that it is flat ($k = 0$) and it contains both matter and cosmological constant.
Hint:
$$\int_0^1 \frac{dx}{(ax^{-1} + bx^2)^{1/2}}\bigg|_{a+b=1} = \frac{1}{3b^{1/2}} \ln\left[\frac{1 + b^{1/2}}{1 - b^{1/2}}\right]$$
Plot $t_0 H_0$ as a function of Ω_Λ^0.

10. Assuming $\Lambda > 0$ and $k = 1$ construct the static Universe ($R(t) = R_E$) containing cosmological constant Λ and non-relativistic matter, i.e. determine Λ_E and R_E, such that $\dot{R}(t) = 0$. Find a relation between Λ_E and R_E. Describe the evolution if
   - $\Lambda < \Lambda_E$
   - $\Lambda > \Lambda_E$

11. Assuming $\Lambda > 0$ and $k = 1$ construct the static Universe ($R(t) = R_E$) containing cosmological constant Λ and non-relativistic matter, i.e. determine Λ_E and R_E, such that $\dot{R}(t) = 0$ and verify its stability.

12. Find the proper (physical) distance D to the Hubble sphere and the past null light cone as a function of time for geometries such that $R(t) \propto t^\alpha$. Plot both in (D, t) plane.

13. Find a relation between time t and redshift z for mater dominated and radiation dominated flat universes.

14. Find the coordinate system in which the LFRW metric has the following form
$$ds^2 = dt^2 - R^2(t)\left[d\chi^2 + I_k^2(\chi)(d\theta^2 + \sin^2\theta d\varphi^2)\right],$$
determine $I_k^2(\chi)$.

15. Determine the boundary (i.e. find the appropriate condition for $\Omega_\Lambda^0 = \Omega_\Lambda^0(\Omega_m^0)$) between regions "expands forever" and "recollapses eventually" in figure 1.

16. Using results obtained for the static Einstein Universe ($k = 1$ and $\Lambda > 0$):
$$R_E = \frac{3}{2}b, \qquad \Lambda_E = \left(\frac{2}{3b}\right)^2, \qquad \text{for} \qquad b \equiv \frac{1}{3}8\pi G\rho_0 R_0^3$$
where ρ_0 is the matter energy density corresponding to the scale factor R_0, determine the boundary of the region "No Big Bang" in figure 1.

17. Show that in the radiation dominated Universe the deceleration parameter q_0 equals Ω_{rad}^0.

18. Evaluate the deceleration parameter as a function of the redshift, $q = q(z)$, neglect radiation but include a "matter" satisfying equation of state $p_X = w_X \rho_X$.

19. Assuming numbers accepted by the concordance model $\Omega_m^0 = 0.3$ and $\Omega_\Lambda^0 = 0.7$ find the redshift at which the observed presently acceleration began.

20. In terms of present CMB temperature T_0 and Ω_m^0, determine the red-shift, cosmic time and temperature for which radiation and matter contributions to the energy density were equal.

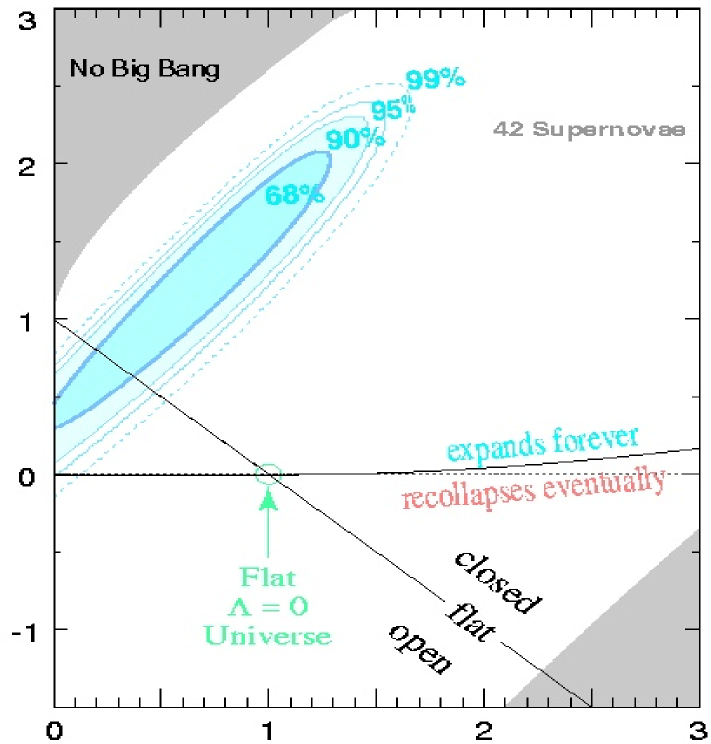

Figure 1: Allowed regions for $(\Omega_m^0, \Omega_\Lambda^0)$ space, from SCP.

21. Calculate the fraction of matter-dominated universe that is visible, as a function of the redshift for $k = +1$ (for $k < 1$ the problem does not make sense), by the fraction one means $d_H(z)/(2\pi R(z))$, where $2\pi R(z)$ is the circumference of the universe.

22. Find the visible fraction of the dust universe at the moment of maximal expansion for $k = +1$.

23. Assuming that the graviton-SM scattering cross-section scales as

$$\langle \sigma_{\rm grav} v \rangle \sim \frac{T^2}{M_{Pl}^4}$$

find relation between the present graviton background temperature and the present temperature of CMB. Estimate present graviton contribution to the energy density. Hint: graviton is massless.

24. Assuming present CMB temperature $T_0 = 2.73$ K fill the table 1.

Hint:
Conversion factors: 1 K $= 4.4$ cm^{-1} $= 8.6 \cdot 10^{-14}$ GeV $= 1.5 \cdot 10^{-37}$ g, 1 Mpc $= 1.6 \cdot 10^{38}$ GeV^{-1}, G $= 6.7 \cdot 10^{-39}$ GeV^{-2} and $H_0 = h\, 2.1 \cdot 10^{-42}$ GeV.

25. Assuming relativistic dark matter χ and $g_{\star S}(x) = $ const. in the considered range of x, find a solution of the Boltzmann equation

$$\frac{x}{Y_{\chi\,EQ}} \frac{dY_\chi(x)}{dx} = -\frac{\Gamma(x)}{H(x)} \left[\left(\frac{Y_\chi(x)}{Y_{\chi\,EQ}} \right)^2 - 1 \right]$$

	γ	ν
$g_\star =$		
$g_{\star S} =$		
$\rho =$		
$n =$		
$s =$		
$\Omega^0 h^2 =$		

Table 1: CMB and ν-background parameters.

26. Assuming the existence of hypothetical heavy stable neutrino with $m \gg 1$ MeV, and

$$\langle \sigma v \rangle = G_F^2 m^2,$$

derive the Lee-Weinberg bound on its mass.

27. Derive the Saha equation.

28. Find how many horizon volumes at the time of recombination $t_{\rm rec}$ has expanded to fill the presently observed Universe, i.e. calculate the ratio of $V_0(t_{\rm rec})$ (the volume of the presently observed Universe at the recombination) and $V_{\rm rec}(t_{\rm rec})$ (the horizon volume at the recombination) for an expansion of Universe dominated by matter or by radiation.