Homework problems # 2a

(deadline: 13.12.21)

11. (1 pt) Show for non-Abelian gauge symmetry group that the covariant derivative of fermionic field transforms the same way as the field itself.

12. (4 pts) Consider $SO(3)$ Yang-Mills theory coupled to a real scalar field ϕ transforming according to the 3-dimensional representation of $SO(3)$. Assume non-zero vacuum expectation value of the scalar field. Find mass matrix squared for scalar and vector fields. Find the remaining symmetry.

13. (3 pts) For quantum real field theory derive the commutation relations for the creation and annihilation operators $a_p^\dagger$ and a_q from the equal-time commutation relations between the field operator $\hat{\phi}(t, \vec{x})$ and the corresponding canonical momentum $\hat{\pi}(t, \vec{x})$:

$$[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \quad \text{and} \quad [\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})] = [\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{y})] = 0\,.$$

14. (4 pts) The $U(1)$-symmetric complex scalar field ϕ theory is described by the Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi\,.$$

Hamiltonian of this theory reads

$$\hat{H} = \int d^3x : \hat{\pi}^\dagger \hat{\pi} + \vec{\nabla}\hat{\phi}^\dagger \cdot \vec{\nabla}\hat{\phi} + m^2 \hat{\phi}^\dagger \hat{\phi} :,$$

where $\hat{\pi}$ is the canonical momentum corresponding to $\hat{\phi}$.

(a) Assuming that the state $|\alpha\rangle$ is an eigenstate of the Hamiltonian, $\hat{H}|\alpha\rangle = E_\alpha|\alpha\rangle$, check if the state $|\beta\rangle = a_p^\dagger|\alpha\rangle$ is also its eigenstate.

(b) Assuming that the state $|\alpha\rangle$ is an eigenstate of the $U(1)$ charge operator,

$$\hat{Q} = i \int d^3x : \phi^\dagger \overset{\leftrightarrow}{\partial^0} \phi :$$

i.e. $\hat{Q}|\alpha\rangle = q_\alpha|\alpha\rangle$, check if the state $|\beta\rangle = \phi^\dagger|\alpha\rangle$ is also its eigenstate.

(c) Show that

$$[\hat{Q}, \hat{\phi}(x)] = -\hat{\phi}(x) \qquad \text{and} \qquad [\hat{Q}, \hat{\phi}^\dagger(x)] = \hat{\phi}^\dagger(x)\,.$$

15. (1 pt) Complex scalar field $\phi(x)$ can be replaced by two real fields

$$\phi(x) = \frac{1}{\sqrt{2}} \left[\phi_r(x) + i\phi_i(x)\right] .$$

Show that the creation operators for the complex field, $a_p^\dagger$, $b_p^\dagger$, defined by the decomposition

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \left(a_p e^{-ipx} + b_p^\dagger e^{ipx}\right)$$

are related to the creation operators $a_{\alpha,\,p}^\dagger$ $(\alpha = r, i)$ for the real fields $\phi_r(x)$ and $\phi_i(x)$ by

$$a_p^\dagger = \frac{1}{\sqrt{2}} \left(a_{r,\,p}^\dagger - i a_{i,\,p}^\dagger\right) \qquad b_p^\dagger = \frac{1}{\sqrt{2}} \left(a_{r,\,p}^\dagger + i a_{i,\,p}^\dagger\right)$$

16. (3 pts) Propagators $\Delta^\pm$ are defined as follows:

$$\Delta^\pm(x-y) \equiv [\hat{\phi}^\pm(x), \hat{\phi}^\mp(y)]$$

The Pauli-Jordan function $\Delta(x-y)$ is defined as

$$\Delta(x-y) \equiv \Delta^+(x-y) + \Delta^-(x-y) .$$

- Show that $\Delta(x-y)$ satisfies the Klein-Gordon equation

$$(\Box_x + m^2)\Delta(x-y) = 0 .$$

- Find the explicit form of $\Delta(x)$.
- Show that $\Delta(x) = 0$ dla $x^2 < 0$.

17. (2 pts) Show that the Hamiltonian for a complex scalar field

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \left(a_p e^{-ipx} + b_p^\dagger e^{ipx}\right)$$

reads

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} E_p \left(a_p^\dagger a_p + b_p^\dagger b_p\right) .$$