wyklad_1a

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Notation : $\eta^{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$ Minkowski metric tensor

Levi-Civita tensor $\epsilon_{\mu\nu\alpha\beta} := \begin{cases} -1 & 0,1,2,3 \\ +1 & \text{for odd permutations of } 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$

$$\epsilon^{0123} = \eta^{0\mu}\eta^{1\nu}\eta^{2\alpha}\eta^{3\beta}\epsilon_{\mu\nu\alpha\beta} = 1(-1)^3\epsilon_{0123} = +1$$

$A_\mu B^\mu := \sum_{\mu=0,1,2,3} A_\mu B^\mu$

$\epsilon_{ijk} = \begin{cases} +1 & 1,2,3 \\ -1 & \text{for odd permutations} \\ 0 & \text{otherwise} \end{cases}$

$\vec{A} := (A^1, A^2, A^3)$

$\epsilon^{ijk} := \epsilon_{ijk}$

$\partial_\mu := \frac{\partial}{\partial x^\mu}$

$\Box := \partial_\mu \partial^\mu = \partial_0^2 - \underset{\frac{\partial^2}{\partial x^{i\,2}}}{\partial_i^2} = \partial_t^2 - \nabla^2$ – d'Alembertian

$x^\mu = (ct, x, y, z) = (ct, x^1, x^2, x^3)$

$$f\overleftrightarrow{\partial_\mu} g := f\partial_\mu g - (\partial_\mu f) g$$

$\not{A} := A_\mu \gamma^\mu$ for $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ – Dirac matrices

Fourier transform:

Fourier transform:

$$f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{f}(k), \qquad kx := k_\mu x^\mu$$

$$\tilde{f}(k) = \int d^4x \, e^{ikx} f(x)$$

$$f(\bar{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\bar{k}\bar{x}} \tilde{f}(\bar{k}) \qquad \bar{k}\cdot\bar{x} = k^i x^i$$

Dirac delta: $\int d^n x \, e^{ikx} = (2\pi)^n \delta^{(n)}(k)$

Natural units: $\hbar = c = 1 \Rightarrow$ the fine structure constant $\alpha = \frac{e^2}{4\pi(\hbar c)} \approx \frac{1}{137} = \frac{e^2}{4\pi}$

$c = 299\,792\,458 \, \frac{m}{s}$ one can measure velocity in units of c

velocity $v = \frac{1}{2}c \Rightarrow v = \frac{1}{2}$ ("c" is still there, we simply don't write it, we do know how to convert to SI units)

$c = 1 \Rightarrow [\text{length}] = [\text{time}]$

$[\text{velocity}] =$ dimensionless

$E = \hbar\nu \Rightarrow \boxed{[\text{energy}] = [\text{time}]^{-1} = [\text{length}]^{-1} = [\text{mass}]}$

$\hbar := \frac{h}{2\pi} = 6.582\,119\,569 \cdot 10^{-16} \, eV \cdot s$ $\qquad (mc^2)^2 = E^2 - (pc)^2$

typical length scale in particle physics: $1\,fm = 10^{-15}\,m$ (size of the proton)

2π

typical length scale in particle physics: $1\,\text{fm} = 10^{-15}\,\text{m}$ (size of the proton)
(one Fermi)

$$\boxed{\hbar c \cong 200\ \text{MeV fm}} \Rightarrow \text{in natural units} \quad 1\,\text{fm} = \frac{1}{200\ \text{MeV}}$$

useful to convert to ordinary units
e.g. the Compton radius

$$r_c = \frac{1}{m_e} \simeq \frac{200\ \text{MeV}\cdot\text{fm}}{0.5\ \text{MeV}} = 4\cdot 10^{-13}\,\text{m}$$

The Lorentz transformation

a linear transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ such that

$$x_\mu x^\mu = \eta_{\mu\nu} x^\mu x^\nu = t^2 - x^2 - y^2 - z^2 \quad \text{is invariant i.e.} = x'_\mu x'^\mu$$

$$x'_\mu x'^\mu = \eta_{\mu\nu} x'^\nu x'^\mu = \eta_{\mu\nu} \underbrace{\Lambda^\mu{}_\alpha x^\alpha}_{x'^\mu} \underbrace{\Lambda^\nu{}_\beta x^\beta}_{x'^\nu} = \underbrace{\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta = \eta_{\alpha\beta} x^\alpha x^\beta}_{\Downarrow}$$

$$\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$$

$$(\det \Lambda)^2 = 1 \quad \Leftarrow \quad \Lambda^T \eta \Lambda = \eta \quad \Leftarrow \quad \Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$$

Scalar fields

$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \phi(x) \to \phi'(x') = \phi(x)$

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$$\phi'(\Lambda x) = \phi(x)$$

Classical field theory

Classical mechanics

$q_i(t)$, $p_i(t)$

$i = 1, \dots, N$

$$[q_i(t), p_i(t)] = 0$$

first quantization $\longrightarrow$

Quantum mechanics

operators: $\hat{q}_i(t)$, $\hat{p}_i(t)$

$i = 1, \dots, N$

$$[\hat{q}_i(t), \hat{p}_i(t)] \neq 0$$

$\Big\downarrow$ $N \to \infty$

Classical field theory

$$q_i(t) \xrightarrow[N\to\infty]{} \phi(t, \bar{x}) \equiv \phi(x)$$

$i \leftrightarrow \bar{x}$

second quantization $\longrightarrow$

Quantum Field Theory (QFT)

$\hat{\phi}(t, \bar{x})$ — an operator

$\Downarrow$

particles as excitations of quantum fields e.g. Higgs bosons

History of field theory

Classical field theory (electrodynamics): • 1831 discovery of induction by Faraday
(1879 - first electric passenger train)

• 1861 Maxwell equations

QFT: • 1927-29 Dirac, Heisenberg, Pauli QED (quantum electrodynamics)

• 1937 Fermi, β- decay $n \to p e^+ \nu$

• 1936 Yukawa, nuclear forces

• 1958 Feynman, Gell-Mann

• 1967-68 Glashow, Salam, Weinberg

} the Standard Model of electroweak interactions

Classical mechanics in 1D

$$m\ddot{q} = -\frac{\partial V}{\partial q}, \quad q(t_0),\ \dot{q}(t_0) \Rightarrow \text{unique solution } q(t)$$

Lagrange function: $L(q,\dot{q}) = T - V = \frac{1}{2}m\dot{q}^2 - V(q)$

Action: $S[q(t)] = \int_{t_1}^{t_2} dt\, L(q(t), \dot{q}(t))$

Minimal action principle: $\boxed{\delta S = 0 \quad \text{with} \quad \underbrace{\delta q(t_1) = \delta q(t_2) = 0}_{\Updownarrow}}$

$\Downarrow$ motion

$q(t_1) = q_1, \quad q(t_2) = q_2$ (fixed ends)

$$\delta S = \delta \int_{t_1}^{t_2} L(q,\dot q)\, dt := \int_{t_1}^{t_2} dt \left\{ L(q+\delta q, \dot q + (\dot{\delta q})) - L(q,\dot q) \right\} = \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\frac{d}{dt}\delta q \right\} =$$

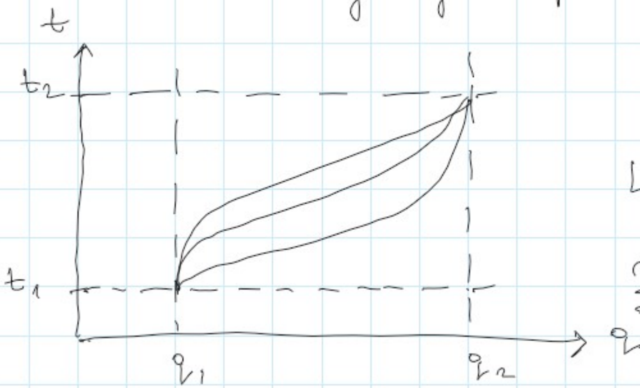
$$\boxed{\delta F[\phi] := F[\phi + \delta\phi] - F[\phi]}$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q}\delta q - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\delta q \right) + \int_{t_1}^{t_2} dt\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right) =$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right)\delta q + \underbrace{\frac{\partial L}{\partial \dot q}\delta q \Big|_{t_1}^{t_2}}_{=0} = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} \right] \delta q = 0$$

for any $\delta q(t)$
$\delta q(t_1) = \delta q(t_2)$

Euler - Lagrange equation: $\quad \dfrac{\partial L}{\partial q} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot q} = 0$



$$L = \frac{1}{2} m \dot q^2 - V(q)$$

$$\frac{\partial L}{\partial \dot q} = m \dot q \qquad \frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q} \quad \Rightarrow \quad \frac{d}{dt}(m\dot q) + \frac{\partial V}{\partial q} = 0$$

$$m \ddot q = -\frac{\partial V}{\partial q}$$

Hamilton formalism

Hamilton formalism

$$H(q,p) = p\cdot \dot q(p) - L(q,\dot q(p)) \quad \text{where} \quad p := \frac{\partial L}{\partial \dot q}$$

$$(q,\dot q) \to (q,p)$$

Hamilton equations: (first order equations)

$$\dot p = -\frac{\partial H}{\partial q}$$

$$\dot q = \frac{\partial H}{\partial p}$$

- Let's replace $L(q,\dot q)$ by a functional $L(\phi,\dot\phi)$
- Define variation : $\delta F[\phi] := F[\phi+\delta\phi] - F[\phi]$
- Define functional derivative $\frac{\delta F}{\delta \phi}$, such that $\delta F[\phi] = \int d^3x \, \frac{\delta F}{\delta \phi(x)} \, \delta\phi(x)$
- Action : $S = \int dt \, L(\phi,\dot\phi)$

$$\delta S = \int dt \, \delta L = \int dt \int_V d^3x \left[\frac{\delta L}{\delta \phi(x)} \delta\phi(x) + \frac{\delta L}{\delta \dot\phi(x)} \underbrace{\delta \dot\phi(x)}_{= \frac{d}{dt}\delta\phi(x)} \right]$$

assume $\delta\phi(t_1,x) = \delta\phi(t_2,x)\Big|_{x\in\partial V} = 0$

then from $\delta S = 0$ for any $\delta\phi(x)$ one gets the Euler-Lagrange eqs.

$$\frac{\delta L}{\delta \phi(x)} - \frac{d}{dt}\frac{\delta L}{\delta \dot\phi(x)} = 0$$

$$\frac{\delta L}{\delta \phi(x)} - \frac{d}{dt}\frac{\delta L}{\delta \dot\phi(x)} = 0$$

Assumptions:

1. We are going to consider theories for which there exists Lagrangian density $\mathcal{L}(\phi, \nabla\phi, \dot\phi)$, such that

$$L[\phi(t,\bar x), \bar\nabla\phi(t,\bar x), \dot\phi(t,\bar x)] = \int d^3x\, \mathcal{L}\left(\phi(t,\bar x), \bar\nabla\phi(t,\bar x), \dot\phi(t,\bar x)\right)$$

2. $\mathcal{L}$ depends quadratically on first derivatives only

3. For Lorentz invariance $\mathcal{L}$ depends on $\partial_\mu\phi$ only

4. Theory is local, i.e. $\mathcal{L}(t,\bar x)$ depends on fields in the same location.

For instance the following density would be disallowed

$$\mathcal{L}(t,\bar x) = \int k(t,\bar x; \bar y)\, \phi(t,\bar y)\, d^3y$$

5. L is a real function

$$\Downarrow$$

$$\boxed{S = \int_{t_1}^{t_2} dt \int d^3x\, \mathcal{L}\left(\phi(x), \partial_\mu\phi(x)\right)}$$

6. Operators that appear in $\mathcal{L}$ should have mass dimension ≤ 4

6. Operators that appear in $\mathcal{L}$ should have mass dimension ≤ 4 in order to ensure renormalizability

$$\delta L = \int_V d^3x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} \underbrace{\delta(\vec{\nabla}\phi)}_{= \vec{\nabla}(\delta\phi)} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta(\dot{\phi}) \right) =$$

$$= \int_V d^3x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \vec{\nabla}\left(\frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} \right) \delta\phi + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta(\dot{\phi}) \right] + \boxed{\int_{\partial V} d\vec{\sigma}\, \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} \delta\phi} = 0 \quad =$$

The Green's theorem: $\int_V d^3x\; \vec{F} \cdot \vec{\nabla} g = - \int_V d^3x \left(\vec{\nabla} \cdot \vec{F} \right) g + \int_{\partial V} \vec{F} \cdot g \, d\vec{\sigma}$

$$= \int_V d^3x \left[\underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \vec{\nabla}\left(\frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} \right) \right)}_{\frac{\delta L}{\delta \phi}} \delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}_{\frac{\delta L}{\delta \dot{\phi}}} \delta(\dot{\phi}) \right]$$

$$\boxed{\frac{\delta L}{\delta \phi(x)} - \frac{d}{dt} \frac{\delta L}{\delta \dot{\phi}} = 0}$$

Then the Euler-Lagrange equations look like

$$\frac{\partial \mathcal{L}}{\partial \phi} - \vec{\nabla} \frac{\partial \mathcal{L}}{\partial \vec{\nabla}\phi} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = 0}$$

For the Hamiltonian formalism we define the conjugate momenta

$$\pi(t,\bar{x}) = \frac{\delta L}{\delta \dot{\phi}(t,\bar{x})} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(t,\bar{x})}$$

Then $\quad H = \int d^3x \; \pi(t,\bar{x})\, \dot{\phi}(t,\bar{x}) - L(t)$

Classical mechanics $\longleftrightarrow$ Classical field theory

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad\qquad \pi(t,\bar{x}) = \frac{\delta L}{\delta \dot{\phi}(t,\bar{x})}$$

$$L(t) = L\big(q_i(t), \dot{q}_i(t)\big) \qquad\qquad L(t) = L\big[\phi(t,\bar{x}), \dot{\phi}(t,\bar{x})\big]$$

$$H(t) = \sum_j p_j\, \dot{q}_j(p) - L(q_i(t), \dot{q}_i(t)) \qquad\qquad H(t) = \int d^3x\, \pi(t,\bar{x})\, \dot{\phi}(t,\bar{x}) - L(t)$$

We assume that there exists the Hamiltonian density

$$H(t) = \int d^3x\, \mathcal{H}(t,\bar{x}) = \int d^3x \big[\pi(t,\bar{x})\, \dot{\phi}(t,\bar{x}) - \mathcal{L}(t,\bar{x})\big]$$

Hamilton equations:

$$\dot{\phi}(t,\bar{x}) = \frac{\delta H}{\delta \pi(t,\bar{x})} \qquad\qquad \dot{\pi}(t,\bar{x}) = -\frac{\delta H}{\delta \phi(t,\bar{x})}$$

Derivation

$$\delta H = \int d^3x \left(\delta\pi\, \dot{\phi} + \pi\, \delta\dot{\phi}\right) - \delta L$$

$$\delta H = \int d^3x \left(\delta\pi \dot{\phi} + \pi \delta\dot{\phi} \right) - \delta L$$

$$\delta L = \int d^3x \left(\frac{\delta L}{\delta \phi} \delta\phi + \underbrace{\frac{\delta L}{\delta \dot{\phi}}}_{\pi} \delta\dot{\phi} \right)$$

$$\frac{\delta L}{\delta \phi} = \dot{\pi} = \frac{d}{dt} \frac{\delta L}{\delta \dot{\phi}}$$

So $\delta H = \int d^3x \left(\delta\pi \dot{\phi} + \cancel{\pi \delta\dot{\phi}} - \dot{\pi}\delta\phi - \cancel{\pi \delta\dot{\phi}} \right) = \int d^3x \left(\frac{\delta H}{\delta \phi} \delta\phi + \frac{\delta H}{\delta \pi} \delta\pi \right)$

$$\frac{\delta H}{\delta \phi} = -\dot{\pi}, \qquad \frac{\delta H}{\delta \pi} = \dot{\phi}$$

$\square$