Noether Theorem

$x^\mu \to x'^\mu = x^\mu + \epsilon^a A^\mu_a(x)$ $\quad |\epsilon^a| \ll 1$, $\epsilon^a \equiv \epsilon^a(x)$

$\phi_i(x) \to \phi'_i(x') = \phi_i(x) + \epsilon^a F_{i,a}(\phi, \partial\phi)$

Assume that the action is invariant under a global transformation ($\epsilon^a = \text{const}$).
In general variation of the action reads

$$S[\phi'] = S[\phi] + \int_M d^4x \left[\epsilon^a K_a(\phi,\partial\phi) - (\partial_\mu \epsilon^a) j^\mu_a(\phi,\partial\phi) + O(\partial\partial\epsilon, \epsilon^2)\right]$$

The invariance under global transformation implys that $K_a(\phi, \partial\phi) = 0$ for any a.

Now let's calculate $\delta S \equiv S[\phi'] - S[\phi]$ for a generic $\epsilon^a(x)$, so

$$\delta S \equiv S[\phi'] - S[\phi] \underset{\text{by parts}}{=} \int_M d^4x\, \epsilon^a (\partial_\mu j^\mu_a) + \underbrace{\int_{\partial M} d\sigma_\mu\, \epsilon^a j^\mu_a}_{\to 0} + \underbrace{O(\partial\partial\epsilon, \epsilon^2)}_{\text{I'll be dropping these terms}}$$

$j^\mu_a = j^\mu_a(\phi, \partial\phi)$

$$\delta S = \int_M d^4x\, \epsilon^a (\partial_\mu j^\mu_a) \quad \text{for any } \epsilon^a$$

Now we assume that ϕ entering $j^\mu_a = j^\mu_a(\phi,\partial\phi)$ satisfies the E-L equation, i.e. $\delta S = 0$ for any variation of fields, so

$$\delta S = 0 \Rightarrow \boxed{\partial_\mu j^\mu_a(\phi, \partial\phi) = 0}$$

We are going to find an explicit form of j^μ_a in terms of $\mathcal{L}$ and the symmetry properties.

$x^\mu \to x'^\mu = x^\mu + \epsilon^a A^\mu_a(x)$

$\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \epsilon^a F_{i,a}(\phi, \partial\phi)$

We assume symmetry (invariance) under the above infinitesimal transformation, i.e.
for $\epsilon^a = \text{const}$ $\quad \delta S \equiv S(\phi') - S(\phi) = 0$

where $S(\phi) = \int_M d^4x\, \mathcal{L}[\phi(x), \partial_\mu \phi(x)]$, $S(\phi') = \int_{M'} d^4x'\, \mathcal{L}[\phi'(x'), \partial'_\mu \phi'(x')]$

Let's calculate δS for $\epsilon^a = \epsilon^a(x)$

$$S(\phi') = \int_M d^4x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left[\phi'[x'(x)], \partial'_\mu \phi'[x'(x)]\right]$$

change of variables: $x'^\mu = x'^\mu(x) = x^\mu + \epsilon^a A^\mu_a(x)$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \underbrace{(\partial_\nu \epsilon^a) A^\mu_a + \epsilon^a \partial_\nu A^\mu_a}_{k^\mu{}_\nu} \Rightarrow \left|\frac{\partial x'}{\partial x}\right| = \begin{vmatrix} (1+k^0{}_0) & k^0{}_1 & k^0{}_2 & k^0{}_3 \\ k^1{}_0 & (1+k^1{}_1) & k^1{}_2 & k^1{}_3 \\ k^2{}_0 & k^2{}_1 & (1+k^2{}_2) & k^2{}_3 \\ k^3{}_0 & k^3{}_1 & k^3{}_2 & (1+k^3{}_3) \end{vmatrix} = 1 + k^\mu{}_\mu + O(k^2)$$

$k \ll 1$

$$\left|\frac{\partial x'^\mu}{\partial x^\nu}\right| = 1 + (\partial_\mu \epsilon^a) A^\mu_a + \epsilon^a (\partial_\mu A^\mu_a)$$

$$\delta S = \int_M d^4x \left[1 + (\partial_\mu \epsilon^a) A^\mu_a + \epsilon^a(\partial_\mu A^\mu_a)\right] \mathcal{L}\Big[\phi_i(x) + \epsilon^a F_{i,a}, \underbrace{\frac{\partial}{\partial x'^\mu}\left(\phi_i(x) + \epsilon^a(x) F_{i,a}(x)\right)}\Big] - \int_M d^4x\, \mathcal{L}[\phi_i(x), \partial_\mu \phi_i(x)]$$

$$\frac{\partial}{\partial x^\nu}\left(\phi_i + \epsilon^a F_{i,a}\right) \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}} = \left[\partial_\nu \phi_i + \partial_\nu(\epsilon^a F_{i,a})\right]\left[\delta^\nu_\mu - \partial_\mu(\epsilon^a A^\nu_a)\right] = \partial_\mu \phi_i - (\partial_\nu \phi_i)\partial_\mu(\epsilon^a A^\nu_a) + \partial_\mu(\epsilon^a F_{i,a}) + \cdots$$

$$\hookrightarrow \delta^\nu_\mu - \frac{\partial}{\partial x'^\mu}(\epsilon^a A^\nu_a) \simeq \delta^\nu_\mu - \frac{\partial}{\partial x^\mu}(\epsilon^a A^\nu_a) + O(\partial\epsilon, \epsilon^2)$$

$$\delta S = \int_M d^4x \left\{ \left[(\partial_\mu \epsilon^a) A^\mu_a + \epsilon^a(\partial_\mu A^\mu_a)\right] \mathcal{L}(\phi_i, \partial_\mu \phi_i) + \frac{\partial \mathcal{L}}{\partial \phi_i} \epsilon^a F_{i,a} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \left[-(\partial_\nu \phi_i)\partial_\mu(\epsilon^a A^\nu_a) + \partial_\mu(\epsilon^a F_{i,a})\right] \right\} =$$

$$= \int_M d^4x \left\{ \epsilon^a \left[(\partial_\mu A^\mu_a)\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi_i} F_{i,a} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \left[-(\partial_\nu \phi_i)(\partial_\mu A^\nu_a) + \partial_\mu F_{i,a}\right]\right] + (\partial_\mu \epsilon^a)\left[A^\mu_a \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \left(-(\partial_\nu \phi_i) A^\nu_a + F_{i,a}\right)\right] \right\}$$

The coefficient of ϵ^a vanishes as a consequence of the global symmetry ($K_a(\phi, \partial\phi) = 0$).

$\delta S = +\int_M d^4x\, \epsilon^a \partial_\mu j^\mu_a$ + boundary terms, for $\boxed{j^\mu_a = -A^\mu_a \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\left[F_{i,a} - (\partial_\nu \phi_i) A^\nu_a\right]}$

$$\delta S = +\int_M d^4x\, \epsilon^a \partial_\mu j^\mu_a \;+\; \text{boundary terms} \quad , \quad \text{for} \quad \boxed{j^\mu_a = -A^\mu_a \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\left[F_{i,a} - (\partial_\nu \phi_i) A^\nu_a\right]}$$

Question : why and when $\delta S = 0$?

Therefore we conclude that $\boxed{\partial_\mu j^\mu_a(\phi, \partial\phi) = 0}$ for any a. $\square$

- For internal symmetries $A^\mu_a = 0$, so

$$j^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} F_{i,a}(\phi, \partial\phi)$$

- For linear internal symmetries $F_{i,a} = i(T_a)_i{}^j \phi_j$

- Conserved charges:

$$Q_a \equiv \int d^3x\, j^0_a$$

$$\frac{d}{dt} Q_a = \int_V d^3x\, \partial_0 j^0_a \underset{\partial_\mu j^\mu_a = 0}{=} -\int_V d^3x\, \partial_i j^i_a = +\int_{\partial V} dS_i\, j^i_a = 0$$

The energy-momentum tensor $\Theta^{\mu\nu}$

We will consider only theories that are invariant under space-time translations :

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu = x^\mu + \epsilon^\nu \delta^\mu_\nu$$

so $a \to \nu$

$A^\mu_a \to \delta^\mu_\nu$

$F_{i,a} \to 0$

then for any ν we have a conserved current $j^\mu_{(\nu)} \equiv \Theta^\mu{}_\nu$

$$j^\mu_a = -A^\mu_a \mathcal{L} - \frac{\partial \mathcal{L}}{\cdots}\left[F_{i,a} - (\partial_\nu \phi_i) A^\nu_a\right]$$

$$j^\mu_a = -A^\mu_a \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\left[F_{i,a} - (\partial_\nu \phi_i) A^\nu_a\right]$$

$$\Theta^{\mu\nu} = \eta^{\nu\rho}\Theta^\mu{}_\rho = \eta^{\nu\rho}\left[-\delta^\mu_\rho \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\delta^\alpha_\rho \partial_\alpha \phi_i\right]$$

$$\boxed{\Theta^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial^\nu \phi_i}$$

- The conserved charges:

$$P^\nu \equiv \int d^3x\, \Theta^{0\nu}$$

- $\Theta^{\mu\nu}$ is called the canonical energy-momentum tensor, in general $\Theta^{\mu\nu} \neq \Theta^{\nu\mu}$,

it is possible to "improve" $\Theta^{\mu\nu}$ as follows

$$T^{\mu\nu} \equiv \Theta^{\mu\nu} + \partial_\rho A^{\rho\mu\nu}$$

1) if $A^{\rho\mu\nu} = -A^{\mu\rho\nu}$ then $T^{\mu\nu}$ is also conserved:

$$\partial_\mu T^{\mu\nu} = \underbrace{\partial_\mu \Theta^{\mu\nu}}_{\shortparallel \atop 0} + \underbrace{\partial_\mu \partial_\rho A^{\rho\mu\nu}}_{\shortparallel \atop 0} = 0$$

2) the momentum remains the same:

$$\int_V d^3x\, T^{0\nu} = \int_V d^3x\, \Theta^{0\nu} + \int_V d^3x\, \underbrace{\partial_\rho A^{\rho 0 \nu}}_{= \partial_i A^{i0\nu}} = \int_V d^3x\, \Theta^{0\nu} + \underbrace{\int_{\partial V} d\sigma_i\, A^{i0\nu}}_{\to 0}$$

3) $A^{\rho\mu\nu}$ could be chosen so that

$$T^{\mu\nu} = T^{\nu\mu}$$

Problem

Show by explicite calculation that $\Theta^{\mu\nu}$ is indeed

Problem

Show, by explicit calculation that $\Theta^{\mu\nu}$ is indeed conserved if E-L eqs. are satisfied.

E-L : $\dfrac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\alpha \dfrac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i)} = 0$

$$\Theta^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial^\nu \phi_i$$

$$\partial_\mu \Theta^{\mu\nu} = -\eta^{\mu\nu}\left[\cancel{\frac{\partial \mathcal{L}}{\partial \phi_i}\partial_\mu \phi_i} + \cancel{\frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)}\partial_\mu(\partial_\rho \phi_i)}\right] + \underbrace{\partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right]}_{= \frac{\partial \mathcal{L}}{\partial \phi_i}}\partial^\nu \phi_i + \cancel{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\mu \partial^\nu \phi_i}$$

$\square$

Real scalar field ϕ

$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$ with $V(\phi) = \frac{1}{2}m^2\phi^2 + \lambda\phi^4 + \text{const}$ symmetry : Z_2 : $\phi \to -\phi$

Problem : write $\mathcal{L}$ without Z_2.

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi) \quad \text{with} \quad V(\phi) = m_3^3\phi + \frac{1}{2}m_2^2\phi^2 + m_1\phi^3 + \lambda\phi^4 + \text{const}$$

Show that e.g. the linear term, $m_3^3\phi$, could be eliminated by an appropriate field reparametrization

$\varphi^4 + 4\varphi^3 m + 6\varphi^2 m^2 + 4\varphi m^3 + m^4$

$$\varphi^4 + 4\varphi^3 m + \overbrace{6\varphi^2 m^2 + 4\varphi m^3 + m^4}$$

Introduce $\varphi := \phi - m$, then

$$V(\varphi) = m_3^3(\varphi + m) + \frac{1}{2} m_2^2 \underbrace{(\varphi+m)^2}_{\varphi^2 + 2\varphi m + m^2} + m_1 \underbrace{(\varphi+m)^3}_{\varphi^3 + 3\varphi^2 m + 3\varphi m^2 + m^3} + \lambda(\varphi+m)^4 =$$

$$\begin{aligned}
= \; & m_3^3 m + \frac{1}{2} m_2^2 m^2 + m_1 m^3 + \lambda m^4 \; + \\
& \varphi \left(m_3^3 + m_2^2 m + 3 m_1 m^2 + 4\lambda m^3 \right) + \\
& \varphi^2 \left(\frac{1}{2} m_2^2 + 3 m_1 m + 6\lambda m^2 \right) + \\
& \varphi^3 \left(m_1 + \lambda m \right) + \\
& \varphi^4 (\lambda)
\end{aligned}$$

Adjust m so that $\left(m_3^3 + m_2^2 m + 3 m_1 m^2 + 4\lambda m^3 \right) = 0$, then

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) \quad \text{with} \quad V(\varphi) = \frac{1}{2} m_2'^2 \varphi^2 + m_1' \varphi^3 + \lambda \varphi^4 + \text{const.}$$

E-L $\quad \dfrac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \underbrace{\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}} = 0 \quad$ for a free field $\lambda = 0$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\Downarrow \quad (m^2 + \overbrace{\partial^\mu \partial_\mu}^{\Box}) \phi = 0 \quad \text{the Klein-Gordon equation}$$

$$\frac{\partial}{\partial(\partial_\mu \phi)} \left(\frac{1}{2} \partial_\nu \phi \partial^\nu \phi \right) = \frac{1}{2} \frac{\partial}{\partial(\partial_\mu \phi)} \left(\eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right) = \frac{\eta^{\alpha\beta}}{2} \left(\delta^\mu_\alpha \partial_\beta \phi + \partial_\alpha \phi \delta^\mu_\beta \right) =$$

$$= \frac{1}{2} \left(\partial^\mu \phi + \partial^\mu \phi \right) = \partial^\mu \phi$$

plane wave solution: e^{ipx}

$$(m^2+\Box)e^{ipx} = (m^2-p^2)e^{ipx} = 0 \quad \Rightarrow \quad p^2=m^2$$

$\phi(x) = \phi^*(x)$ $\Rightarrow$ the most general solution: $\displaystyle\int \frac{d^3p}{(2\pi)^3 \underbrace{(2E_p)^{1/2}}_{\text{conventional normalization of } a_p}}\left(a_p e^{-ipx} + a_p^* e^{-ipx}\right)\Big|_{E_p=p^0}$

$E_p = +(m^2+\bar{p}^2)^{1/2}$

Momentum conjugate to ϕ:

$$\Pi_\phi \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi = \dot{\phi}$$

The Hamiltonian density

$$\mathcal{H} = \Pi_\phi\,\partial_0\phi - \mathcal{L} = (\Pi_\phi)^2 - \left[\frac{1}{2}\partial_0\phi\,\partial^0\phi + \frac{1}{2}\partial_i\phi\,\partial^i\phi - \frac{1}{2}m^2\phi^2\right] =$$

$$= \frac{1}{2}(\Pi_\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

The energy-momentum tensor

$$\Theta^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi = -\eta^{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - V(\phi)\right) + \partial^\mu\phi\,\partial^\nu\phi$$

$$\Theta^{00} = -\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi + V + \partial^0\phi\,\partial^0\phi = \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$$

$\boxed{\text{Complex scalar field}}$

Complex scalar field

standard scenario : $\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - \underbrace{\left[m^2 \phi^* \phi + \lambda (\phi^* \phi)^2\right]}_{V(|\phi|)}$ symmetry : $\phi(x) \to e^{i\alpha} \phi(x)$

α – "c-number" (global symmetry)

Question : Is $\mathcal{L}$ the most general Lagrangian consistent with $\phi \to e^{i\alpha}\phi$?

$$\Delta\mathcal{L} = \phi \Box \phi^* = \phi^* \partial^\mu \partial_\mu \phi = -\partial^\mu \phi^* \partial_\mu \phi + \underbrace{\partial^\mu (\phi^* \partial_\mu \phi)}$$

total derivative : could be dropped

$$\Delta\mathcal{L} = \Box(\phi^* \phi) = \partial^\mu\left[\partial_\mu (\phi^* \phi)\right]$$

$$\Delta\mathcal{L} = (\phi^* \phi) \partial_\mu \phi \partial^\mu \phi^* \qquad \text{dim } 6 \Rightarrow \text{non-renormalizability}$$

Problem

write most general mass terms for a complex scalar field ϕ.

$$V(\phi) = m^2 \phi^* \phi + \mu^2 (\phi^2 + \phi^{*2}) = \qquad \text{assuming } \mu^2 \in \mathbb{R}$$

$$= \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) + \mu^2 \, 2 \,\mathrm{Re}(\phi^2) = \qquad \phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$= \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) + \cancel{2}\mu^2 \frac{1}{\cancel{2}} (\phi_1^2 - \phi_2^2) = \frac{1}{2}(\phi_1, \phi_2) \begin{pmatrix} m^2 + 2\mu^2 & 0 \\ 0 & m^2 - 2\mu^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Problem

Find the energy-momentum tensor of a complex scalar field (U(1)-symmetric)

$$\Theta^{\mu\nu} = -\eta^{\mu\nu}\mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial^\nu \phi_i \qquad \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(|\,|)$$

$$\phi_1 = \phi \quad , \quad \phi_2 = \phi^* \quad , \quad \mathcal{L} = \partial_\mu \phi_1 \partial^\mu \phi_2 \quad V$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} = \partial^\mu \phi_2 \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \partial^\mu \phi_1$$

$$\Theta^{\mu\nu} = -\eta^{\mu\nu}\left(\partial_\alpha \phi^* \partial^\alpha \phi - V\right) + \partial^\mu \phi_2 \partial^\nu \phi_1 + \partial^\mu \phi_1 \partial \,|_2$$

$$\partial^\mu \phi^* \partial^\nu \phi + \partial^\mu \phi \quad \phi^*$$

$$\Theta^{00} = -\left(\partial_0 \phi^* \partial^0 \phi + \partial_i \phi^* \partial^i \phi - V\right) + \partial^0 \phi^* \partial^0 \phi \cdot 2 = \partial^0 \phi^* \partial^0 \phi + \vec{\nabla} \phi^* \vec{\nabla} \phi + V(|\phi|)$$

Problem

Derive the E-L equations for a) $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$ and b) $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(|\phi|)$

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0$$

a) $\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial V}{\partial \phi} \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \qquad$ E-L : $-\frac{\partial V}{\partial \phi} - \partial_\mu \partial^\mu \phi = 0 \qquad \Box \phi = -\frac{\partial V}{\partial \phi}$

b) ϕ and ϕ^* as independent components

b) ϕ and ϕ^* as independent components

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = -\frac{\partial V}{\partial \phi^*} \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi \quad \Longrightarrow \quad \Box \phi = -\frac{\partial V}{\partial \phi^*}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial V}{\partial \phi} \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^* \quad \Longrightarrow \quad \Box \phi^* = -\frac{\partial V}{\partial \phi}$$

Problem

Find the Noether current corresponding to the U(1) symmetry.

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(|\phi|)$$

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \begin{pmatrix} \phi \\ \phi^* \end{pmatrix} \qquad \mathcal{L} \text{ invariant under } \begin{array}{l} \phi \to e^{i\alpha}\phi \\ \phi^* \to e^{-i\alpha}\phi^* \end{array}$$

$$x^\mu \to x'^\mu = x^\mu + \epsilon^a A^\mu_a(x)$$

$$\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \epsilon^a F_{i,a}(\phi, \partial\phi)$$

$$\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} \to \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} \simeq \left[\mathbb{1} + i\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}$$

$$\underbrace{\epsilon^a F_{i,a}}_{} = \alpha \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}}_{}$$

$a = 1$ $\qquad F_1 = i\varphi^1 \qquad F_2 = -i\varphi^2$

$$\mathcal{L} = \partial_\mu \varphi^1 \partial^\mu \varphi^2 - V(\varphi^{1\,2} + \varphi^{2\,2})$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^1)} = \partial^\mu \varphi^2 \ , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^2)} = \partial^\mu \varphi^1$$

$$j^\mu_a = -\underbrace{A^\mu_a \mathcal{L}}_{=0} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\left[F_{i,a} - \underbrace{(\partial_\nu \phi_i) A^\nu_a}_{=0} \right]$$

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^i)} F_i = -\big(\underbrace{\partial^\mu \varphi^2 \, i\varphi^1}_{i=1} + \underbrace{\partial^\mu \varphi^1 (-i) \varphi^2}_{i=2} \big) = -\left(\partial^\mu \phi^* \, i \phi + \partial^\mu \phi (-i) \phi^* \right) =$$

$$= i\left(\partial^\mu \phi \, \phi^* - \partial^\mu \phi^* \phi \right) = i \phi^* \overleftrightarrow{\partial^\mu} \phi$$

Problem

Consider the theory of complex scalar field with Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(|\phi|)$$

and introduce two real fields ϕ_1 and ϕ_2 defined as

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

write the Lagrangian in terms of $\phi_{1,2}$, write the symmetry transformation $\phi \to e^{i\alpha}\phi$ in terms of $\phi_{1,2}$, determine $F_{i,a}$ and construct the current. Show that it agrees with the one we have found.

Solution:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - V\left(\frac{1}{2}(\phi_1^2+\phi_2^2)\right)$$

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \to e^{i\alpha}\phi = \frac{1}{\sqrt{2}}(\cos\alpha + i\sin\alpha)(\phi_1 + i\phi_2) = \frac{1}{\sqrt{2}}\left[(\cos\alpha\,\phi_1 - \sin\alpha\,\phi_2) + i(\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2)\right]$$

$$\phi_1 \to \cos\alpha\,\phi_1 - \sin\alpha\,\phi_2$$
$$\phi_2 \to \sin\alpha\,\phi_1 + \cos\alpha\,\phi_2$$

$$\Rightarrow \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} =$$

$$= \begin{pmatrix}1 & -\alpha\\ \alpha & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} + O(\alpha^2)$$

$$x^\mu \to x'^\mu = x^\mu + \epsilon^a A^\mu_a(x)$$

$$\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \epsilon^a F_{i,a}(\phi, \partial\phi)$$

$$\epsilon\begin{pmatrix}F_1\\ F_2\end{pmatrix} = \alpha\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

$$\epsilon = \alpha \qquad F_1 = -\phi_2$$
$$F_2 = \phi_1$$

$$\boxed{j^\mu_a = -A^\mu_a\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\left[F_{i,a} - (\partial_\nu\phi_i)A^\nu_a\right]}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi_1 \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)} = \partial^\mu\phi_2$$

$$\to\ j^\mu = -\left(\partial^\mu\phi_1(-\phi_2) + \partial^\mu\phi_2(\phi_1)\right) = (\partial^\mu\phi_1)\phi_2 - (\partial^\mu\phi_2)\phi_1 = i\left(\partial^\mu\phi\,\phi^* - \partial^\mu\phi^*\phi\right) = i\phi^*\overleftrightarrow{\partial^\mu}\phi$$

$$\mathrm{Re}\left[i\,\partial^\mu\frac{(\phi_1 + i\phi_2)}{\sqrt{2}}\frac{(\phi_1 - i\phi_2)}{\sqrt{2}}\right] = \frac{1}{2}\left(i\,\partial^\mu\phi_1(-i)\phi_2 + i\partial^\mu i\phi_2\phi_1\right) =$$

$$= (\partial^\mu\phi_1)\phi_2$$

$$= \left(\partial^\mu \phi_1\right)\phi_2$$

Problem

Show that the Noether current for $U(1)$ is conserved if E-L equations are satisfied.

Solution: $$j^\mu = -i\left(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right)$$

$$\partial_\mu j^\mu = -i\left(\cancel{\partial_\mu\phi\,\partial^\mu\phi^*} + \phi\,\underbrace{\Box\phi^*}_{-\frac{\partial V}{\partial\phi}} - \cancel{\partial_\mu\phi^*\,\partial^\mu\phi} - \phi^*\underbrace{\Box\phi}_{-\frac{\partial V}{\partial\phi^*}}\right) = -i\left(\phi\,\frac{\partial V}{\partial\phi} - \phi^*\frac{\partial V}{\partial\phi^*}\right) = -i\,\frac{\partial V}{\partial|\phi|^2}\left(|\phi|^2 - |\phi|^2\right) = 0$$

$$V = V(\phi^*\phi)$$

$$\underbrace{\frac{\partial V}{\partial\phi} = \frac{\partial V}{\partial|\phi|^2}\,\phi^* \qquad \frac{\partial V}{\partial\phi^*} = \frac{\partial V}{\partial|\phi|^2}\,\phi}$$

Problem

Show that the Noether current is conserved in a general case.

$$j^\mu_a = -A^\mu_a\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\left[F_a - (\partial_\nu\phi)A^\nu_a\right], \qquad \frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} = 0$$

$$\partial_\mu j^\mu_a = -\partial_\mu A^\mu_a\mathcal{L} - A^\mu_a\left[\cancel{\frac{\partial\mathcal{L}}{\partial\phi}\partial_\mu\phi} + \cancel{\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\partial_\nu\phi}\right] - \underbrace{\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}}_{-\frac{\partial\mathcal{L}}{\partial\phi}}\left[F_a - \cancel{(\partial_\nu\phi)A^\nu_a}\right] - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\left[\partial_\mu F_a - \cancel{(\partial_\mu\partial_\nu\phi)A^\nu_a} - (\partial_\nu\phi)\partial_\mu A^\nu_a\right]$$

$$= -\partial_\mu A^\nu_a\left[\mathcal{L}\,\delta^\mu_\nu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi\right] - \frac{\partial\mathcal{L}}{\partial\phi}F_a - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu F_a = 0 \qquad \Box$$

$$\delta S = \int_M d^4x \left\{ \left[(\partial_\mu \epsilon^a) A^\mu_a + \epsilon^a (\partial_\mu A^\mu_a) \right] \mathcal{L}(\phi_i, \partial_\mu \phi_i) + \frac{\partial \mathcal{L}}{\partial \phi_i} \epsilon^a F_{i,a} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \left[-(\partial_\nu \phi_i) \partial_\mu (\epsilon^a A^\nu_a) + \partial_\mu (\epsilon^a F_{i,a}) \right] \right. =$$

$$= \int_M d^4x \left\{ \epsilon^a \underbrace{\boxed{\left[(\partial_\mu A^\mu_a) \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi_i} F_{i,a} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \left[-(\partial_\nu \phi_i)(\partial_\mu A^\nu_a) + \partial_\mu F_{i,a} \right] \right]}}_{= K_a(\phi, \partial\phi)} + \right.$$

$$\left. + (\partial_\mu \epsilon^a) \left[A^\mu_a \mathcal{L} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \left(-(\partial_\nu \phi_i) A^\nu_a + F_{i,a} \right) \right] \right\}$$

The coefficient of ϵ^a vanishes as a consequence of the global symmetry ($\boxed{K_a(\phi, \partial\phi) = 0}$).

$$\delta S = + \int_M d^4x \, \epsilon^a \partial_\mu j^\mu_a + \text{boundary terms}, \quad \text{for} \quad \boxed{j^\mu_a = - A^\mu_a \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \left[F_{i,a} - (\partial_\nu \phi_i) A^\nu_a \right]}$$