wyklad_1c

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Complex, U(1) symmetric, scalar field

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - \underbrace{V(|\varphi|)}_{= m^2 \varphi^* \varphi + \lambda(\varphi^* \varphi)^2}$$

E-L: $\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} = 0 \quad \Rightarrow \quad \begin{array}{l} (\Box + m^2)\varphi = 0 \\ (\Box + m^2)\varphi^* = 0 \end{array}$ for $\lambda = 0$

$\mathcal{L}$ is invariant under a global U(1) transformation:

$$\varphi(x) \to \varphi'(x) = e^{i\alpha}\varphi(x), \quad \alpha - \text{c-number}$$
$$\varphi^*(x) \to \varphi'^*(x) = e^{-i\alpha}\varphi^*(x)$$

Electrodynamics

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

E-L equations $\quad \underbrace{\frac{\partial \mathcal{L}}{\partial \phi_i}}_{0} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_i)} = 0 \qquad \phi_i \leftrightarrow A_\beta$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{4}\eta^{\mu\sigma}\eta^{\nu\lambda}\frac{\partial}{\partial(\partial_\alpha A_\beta)}\left(F_{\sigma\lambda}F_{\mu\nu}\right) = -\frac{1}{4}\eta^{\mu\sigma}\eta^{\nu\lambda}\frac{\partial}{\partial(\partial_\alpha A_\beta)}\left(\partial_\sigma A_\lambda - \partial_\lambda A_\sigma\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) =$$

$$= -\frac{1}{4}\eta^{\mu\sigma}\eta^{\nu\lambda}\left[\left(\delta^\alpha_\sigma \delta^\beta_\lambda - \delta^\alpha_\lambda \delta^\beta_\sigma\right)F_{\mu\nu} + F_{\sigma\lambda}\left(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu\right)\right] =$$

$$= -\frac{1}{4}\left[\left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha}\right)F_{\mu\nu} + F_{\sigma\lambda}\left(\eta^{\alpha\sigma}\eta^{\beta\lambda} - \eta^{\beta\sigma}\eta^{\alpha\lambda}\right)\right] =$$

$$= -\frac{1}{4}\left(F^{\alpha\beta} - F^{\beta\alpha} + F^{\alpha\beta} - F^{\beta\alpha}\right) = -F^{\alpha\beta}$$

E-L : $\partial_\alpha F^{\alpha\beta} = 0$ $\quad (*)$

$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow$ the Bianchi identity : $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} = 0 \quad (**)$

$\bar{E} = (E^1, E^2, E^3) \qquad \bar{B} = (B^1, B^2, B^3)$

$F^{12} = -B^3 \qquad F^{23} = -B^1 \qquad F^{31} = -B^2$

$F^{01} = -E^1 \qquad F^{02} = -E^2 \qquad F^{03} = -E^3$

$(*)$ $\quad \beta = 0 \Rightarrow \bar{\nabla}\cdot\bar{E} = 0$

$\quad \beta = i \Rightarrow \bar{\nabla}\times\bar{B} = \partial_t \bar{E}$

$(**)$

$\quad \alpha = 0 \Rightarrow \bar{\nabla}\cdot\bar{B} = 0$

$\quad \alpha = i \Rightarrow \bar{\nabla}\times\bar{E} = -\partial_t \bar{B}$

Gauge symmetry : $F_{\mu\nu}$ invariant under $A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x)$

(local symmetry: $\alpha = \alpha(x)$)

$\Downarrow$

$F_{\mu\nu}(x) \rightarrow F'_{\mu\nu}(x) = F_{\mu\nu}(x)$

$$\boxed{\text{Scalar elektrodynamics: } U(1) \text{ gauge theory}}$$

$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^*\varphi - \lambda(\varphi^*\varphi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$$\mathcal{L} = \underbrace{\partial_\mu \varphi^* \partial^\mu \varphi \underbrace{- m^2 \varphi^* \varphi - \lambda (\varphi^* \varphi)^2}_{-V(|\varphi|)}} \underbrace{- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

- globally symmetric (invariant) — invariant under local transformations

$$\varphi(x) \to e^{i\alpha} \varphi(x)$$

- vary under local transformations

$$\varphi(x) \to e^{i\alpha(x)} \varphi(x)$$

$$\Downarrow$$

$$\partial_\mu \varphi \to \partial_\mu (e^{i\alpha} \varphi) = i(\partial_\mu \alpha) e^{i\alpha} \varphi + e^{i\alpha} \partial_\mu \varphi = e^{i\alpha} (\partial_\mu + i \partial_\mu \alpha) \varphi$$

Question : How to construct a theory for φ and A_μ that would be locally symmetric?

hint : to modify $\partial_\mu \varphi$ so that $D_\mu \varphi = \partial_\mu \varphi - i X_\mu \varphi \to e^{i\alpha} D_\mu \varphi$

$$\partial_\mu \varphi - i X_\mu \varphi \longrightarrow \partial_\mu \varphi' - i X'_\mu \varphi' = e^{i\alpha} (\partial_\mu \varphi - i X_\mu \varphi)$$

$$\cancel{e^{i\alpha}} (\cancel{\partial_\mu \varphi} + i \partial_\mu \alpha \varphi) - i X'_\mu \cancel{e^{i\alpha}} \varphi = \cancel{e^{i\alpha}} (\cancel{\partial_\mu \varphi} - i X_\mu \varphi)$$

$$i (\partial_\mu \alpha) \cancel{\varphi} - i X'_\mu \cancel{\varphi} = - i X_\mu \cancel{\varphi}$$

$$\boxed{X'_\mu = X_\mu + \partial_\mu \alpha}$$ gauge transformation of electrodynamics!

$$\boxed{\mathcal{L} = (D_\mu \varphi)^* D^\mu \varphi - V(|\varphi|) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}$$

$$\mathcal{L} = (D_\mu\varphi)^* D^\mu\varphi - V(|\varphi|) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

$$D_\mu\varphi \equiv (\partial_\mu - iA_\mu)\varphi$$

symmetry : $\varphi \to e^{i\alpha(x)}\varphi$, $A_\mu \to A_\mu + \partial_\mu\alpha$

Rudiments of Lie group theory

group G is a set of elements $(a, b, c, \dots)$ with a multiplication law:

i) closure: $a, b \in G \Rightarrow ab \in G$

ii) associative: $a(bc) = (ab)c$

iii) identity: $ea = ae = a$ for any $a \in G$

iv) inverse: $aa^{-1} = a^{-1}a = e$ for any $a \in G$

if $ab = ba$ for all $a, b \in G$, the group is called Abelian

if the number of elements in G is finite, the group is a finite group

Given any two groups $G = \{g_1, g_2, \dots\}$ $H = \{h_1, h_2, \dots\}$ then, if g_is commute with h_is one can define a direct-product group $G \times H = \{g_i h_j\}$ with a multiplication law

$$g_k h_l \cdot g_m h_n = g_k \cdot g_m \cdot h_l \cdot h_n$$

eg. $SU(2) \times U(1)$

A Lie group is a group whose elements g depend in a continuous and differentiable way on a set of real parameters α_a, $a = 1 \dots n$

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A representation is a specific realization of the multiplication of the group elements by matrices

$$g \to D(g) \quad \text{such that if} \quad g \cdot h = k \quad \text{then} \quad D(g) \cdot D(h) = D(k)$$

convention : $\quad g(\alpha)\Big|_{\alpha=0} = e \quad \to \quad D(g(\alpha))\Big|_{\alpha=0} = \mathbb{1}$

$$D(d\alpha) = \mathbb{1} + i\, d\alpha_a T_a + \cdots$$

$$\Downarrow$$

$$\boxed{T_a \equiv -i \left.\frac{\partial D(\alpha)}{\partial \alpha_a}\right|_{\alpha=0}} \quad \text{– group generators}$$

it could be shown that a generic group element $g(\alpha)$ can always represented

$$\boxed{D[g(\alpha)] = e^{i\alpha_a X_a}}$$

$SU(n)$: group of $n\times n$ unitary matrices with $\det \omega = 1$

$$(\omega\omega^+ = \omega^+\omega = \mathbb{1})$$

$$\omega = e^{iH} \quad \text{for } H^+ = H \quad (\text{unitarity})$$

$$\det\left(e^{A}\right) = e^{\mathrm{Tr}A} \quad \Rightarrow \quad \mathrm{Tr}\, H = 0$$

Question: How many $n\times n$ traceless and Hermitian matrices do exist?

number of real parameters: $n\times n \rightarrow 2n^2 \xrightarrow{H=H^+} n^2 \xrightarrow{\mathrm{Tr}H=0} n^2-1$

$$H_{ij} = H_{ji}^* \qquad \frac{n^2-n}{2}\cdot 2 + n = n^2$$

$$H_{ji} = H_{ij}^*$$

$$\boxed{\omega = e^{i\sum_{a=1}^{n^2-1}\varepsilon_a T_a}}$$

ε_a – real parameters

T_a – generators

Example : $SU(2)$

$$\omega = e^{i\varepsilon_a \frac{\sigma_a}{2}} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ the Pauli matrices

$$\mathrm{Tr}\,\sigma_a = 0 \qquad T_a = \frac{\sigma_a}{2}$$

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k$$

$\mathrm{Tr}\, U_a = 0 \qquad T_a = \frac{\sigma_a}{2}$

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k$$

$$[T_i, T_j] = i\,\varepsilon_{ijk} T_k$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1} \quad \Rightarrow \quad \sigma_i^2 = \mathbb{1}$$

$$\sigma_i \sigma_j = \frac{1}{2}\{\sigma_i, \sigma_j\} + \frac{1}{2}[\sigma_i, \sigma_j] = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$$

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = a_i b_j \sigma_i \sigma_j = a_i b_j\left(\delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k\right) = \vec{a}\cdot\vec{b}\,\mathbb{1} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

Problem

Show that $e^{i\vec{a}\cdot\vec{\sigma}} = \mathbb{1}\cos a + i(\hat{n}\cdot\vec{\sigma})\sin a$ for $\vec{a} = a\hat{n}$, $|\hat{n}| = 1$

$$e^{i\vec{a}\cdot\vec{\sigma}} = e^{ia\hat{n}\cdot\vec{\sigma}} = \sum_{n=0}^{\infty} \frac{i^n (a\hat{n}\cdot\vec{\sigma})^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n (a\hat{n}\cdot\vec{\sigma})^{2n}}{(2n)!} + i\sum_{n=0}^{\infty}\frac{(-1)^n (a\hat{n}\cdot\vec{\sigma})^{2n+1}}{(2n+1)!} =$$

$$(\hat{n}\cdot\vec{\sigma})^{2n} = \left[(\hat{n}\cdot\vec{\sigma})^2\right]^n = [\mathbb{1}]^n = \mathbb{1}$$

$$(\hat{n}\cdot\vec{\sigma})^{2n+1} = \hat{n}\cdot\vec{\sigma}$$

$$= \mathbb{1}\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}}{(2n)!}}_{\cos a} + i\hat{n}\cdot\vec{\sigma}\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}}{(2n+1)!}}_{\sin a} = \mathbb{1}\cos a + i\hat{n}\cdot\hat{\sigma}\sin a \qquad \square$$

Yang-Mills theories : non-Abelian generalisations of electrodynamics

i) global

$$\varphi_i(x) \to \varphi_i'(x) = \omega_{ij}\,\varphi_j(x) \qquad \omega \in SU(N) \;:\; \det\omega = 1 \qquad \omega^+\omega = 1\!\!1$$

$$\varphi_i^*\varphi_i \to \varphi_i'^*\varphi_i' = \omega_{ij}^*\varphi_j^*\,\omega_{ik}\varphi_k = \varphi_j^+\varphi_k \underbrace{\omega_{ij}^*\omega_{ik}}_{(\omega^+\omega)_{jk} = \delta_{jk}} = \varphi_j^*\varphi_j$$

$$\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix}$$

$$\mathcal{L} = \partial_\mu\varphi^+\partial^\mu\varphi - m^2\varphi^+\varphi - \lambda\,(\varphi^+\varphi)^2$$

Question : what are the consequences of the symmetry? • equal masses
• equal couplings

ii) local

$$\varphi_i(x) \to \varphi_i'(x) = \omega_{ij}(x)\,\varphi_j(x) \quad \text{e.g.} \quad \omega(x) \in SU(N)$$

$$\partial_\mu\varphi \to \partial_\mu\varphi'(x) = \partial_\mu(\omega\varphi) = \underbrace{(\partial_\mu\omega)\varphi}_{\Downarrow\ \text{non-invariance}} + \omega(\partial_\mu\varphi) \qquad (*)$$

$$[\omega(\partial_\mu\varphi)]^+\omega(\partial^\mu\varphi) = \partial_\mu\varphi^+\underbrace{\omega^+\omega}_{=1\!\!1}\partial^\mu\varphi$$

Following ED we introduce a covariant derivative and require that it transforms the same way as φ

$$D_\mu\varphi \to (D_\mu\varphi)' = \omega(D_\mu\varphi)$$

and require that it transforms the same way as φ

$$D_\mu \varphi \to (D_\mu \varphi)' = \omega (D_\mu \varphi)$$

$$D_\mu \varphi = (\partial_\mu - i A_\mu)\varphi$$

$$\partial_\mu \varphi' - i A'_\mu \varphi' = \omega(\partial_\mu \varphi - i A_\mu \varphi)$$

$$(\partial_\mu \omega)\varphi + \omega\cancel{(\partial_\mu \varphi)} - i A'_\mu \omega \varphi = \omega \cancel{(\partial_\mu \varphi)} - i\omega A_\mu \varphi \qquad \text{for any } \varphi$$

$$\partial_\mu \omega - i A'_\mu \omega = -i\omega A_\mu \quad | \, \omega^+$$

$$i \, | \qquad (\partial_\mu \omega)\omega^+ - i A'_\mu = -i\omega A_\mu \omega^+$$

$$\boxed{A'_\mu = \omega A_\mu \omega^+ - i(\partial_\mu \omega)\,\omega^+ \overset{?}{=} \omega A_\mu \omega^+ + i\omega \partial_\mu \omega^+}$$

Question: How does it look like for $\omega \in U(1)$?

$$\omega = e^{i\alpha} \qquad A'_\mu = A_\mu - i\, i\, \partial_\mu \alpha = A_\mu + \partial_\mu \alpha \qquad \text{as it should.}$$

For Lie groups, in the vicinity of identity

$$\omega(x) \simeq \mathbb{1} + \varepsilon(x) \simeq \mathbb{1} + i\,\theta_a T_R^a$$

T_R^a – generators (algebra)

θ_a – parameters (real)

Let's consider an infinitesimal transformation, then

$$\omega^+ = \omega^{-1} = \mathbb{1} - \varepsilon(x)$$

For instance for $\omega \in SU(2)$

i) $\omega \partial_\mu \omega^+ = (1+\varepsilon)(-)\partial_\mu \varepsilon = -\partial_\mu \varepsilon + O(\varepsilon^2) \quad \Rightarrow \quad \omega \partial_\mu \omega^+ \in A\,SU(2)$

For instance for $\omega \in SU(2)$

i) $\omega \partial_\mu \omega^+ = (1+\varepsilon)(-)\partial_\mu \varepsilon = -\partial_\mu \varepsilon + O(\varepsilon^2) \Rightarrow \omega \partial_\mu \omega^+ \in ASU(2)$

ii) assume that $A_\mu \in ASU(2)$, then $\omega A_\mu \omega^+ = (1+\varepsilon) A_\mu (1-\varepsilon) =$

$$= A_\mu + [\varepsilon, A_\mu] + O(\varepsilon^2)$$

Problem

Construct kinetic term for non-Abelian gauge theories, an analog of $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ for E-D.

- for global transformations : $A_\mu \to A_\mu' = \omega A_\mu \omega^+$

$$\underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{} \to \omega (\partial_\mu A_\nu - \partial_\nu A_\mu) \omega^+$$

  - note that this is not invariant as in E-D
  - $(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$ could work

- however for local transformations $A_\mu \to A_\mu' = \omega A_\mu \omega^+ + i \omega \partial_\mu \omega^+$,

  $(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$ does not transfor simply

- we need new $F_{\mu\nu}$ such that $F_{\mu\nu} \Rightarrow F_{\mu\nu}' = \omega F_{\mu\nu} \omega^+$

$$\Downarrow$$

  then $Tr\{ F_{\mu\nu} F^{\mu\nu} \}$ is invariant

- let's try $\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]}$

$$\partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu(\omega A_\nu \omega^\dagger + i\omega\partial_\nu\omega^\dagger) - \partial_\nu(\omega A_\mu\omega^\dagger + i\omega\partial_\mu\omega^\dagger) =$$

$$= \partial_\mu\omega A_\nu\omega^\dagger + \omega\partial_\mu A_\nu\omega^\dagger + \omega A_\nu\partial_\mu\omega^\dagger + i\partial_\mu\omega\partial_\nu\omega^\dagger + \cancel{i\omega\partial_\mu\partial_\nu\omega^\dagger} +$$

$$- \partial_\nu\omega A_\mu\omega^\dagger - \omega\partial_\nu A_\mu\omega^\dagger - \omega A_\mu\partial_\nu\omega^\dagger - i\partial_\nu\omega\partial_\mu\omega^\dagger - \cancel{i\omega\partial_\nu\partial_\mu\omega^\dagger} =$$

$$\boxed{\omega(\partial_\mu A_\nu - \partial_\nu A_\mu)\omega^\dagger}$$

$$-i[A_\mu, A_\nu] = -i(A_\mu A_\nu - A_\nu A_\mu) \to -i(\omega A_\mu\omega^\dagger + i\omega\partial_\mu\omega^\dagger)(\omega A_\nu\omega^\dagger + i\omega\partial_\nu\omega^\dagger) +$$

$$+ i(\omega A_\nu\omega^\dagger + i\omega\partial_\nu\omega^\dagger)(\omega A_\mu\omega^\dagger + i\omega\partial_\mu\omega^\dagger)$$

$$= \boxed{-i\omega(A_\mu A_\nu - A_\nu A_\mu)\omega^\dagger} + \omega A_\mu\partial_\nu\omega^\dagger + \overbrace{\omega\partial_\mu\omega^\dagger\omega}^{-\partial_\mu\omega} A_\nu\omega^\dagger + i\overbrace{\omega\partial_\mu\omega^\dagger\omega}^{-\partial_\mu\omega}\partial_\nu\omega^\dagger +$$

$$- \omega A_\nu\partial_\mu\omega^\dagger - \underbrace{\omega\partial_\nu\omega^\dagger\omega}_{-\partial_\nu\omega} A_\mu\omega^\dagger - i\underbrace{\omega\partial_\nu\omega^\dagger\omega}_{-\partial_\nu\omega}\partial_\mu\omega^\dagger$$

$$0 = \partial_\mu(\omega^\dagger\omega) = \partial_\mu\omega^\dagger\omega + \omega^\dagger\partial_\mu\omega \quad\Rightarrow\quad \omega\partial_\mu\omega^\dagger\omega = -\partial_\mu\omega$$

$$\Downarrow$$

$$F_{\mu\nu} \to F'_{\mu\nu} = \omega(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu])\omega^\dagger = \omega F_{\mu\nu}\omega^\dagger$$

Alternative derivation

$$D_\mu\varphi = (\partial_\mu - iA_\mu)\varphi$$

$$D_\mu\varphi \to \omega D_\mu\varphi$$

Note that

$$[D_\mu, D_\nu] = [(\partial_\mu - iA_\mu), (\partial_\nu - iA_\nu)] = -i[\partial_\mu, A_\nu] - i[A_\mu, \partial_\nu] - [A_\mu, A_\nu] =$$

$$[\partial_\mu, A_\nu]\varphi = \partial_\mu(A_\nu\varphi) - A_\nu\partial_\mu\varphi = \partial_\mu A_\nu\varphi + \cancel{A_\nu\partial_\mu\varphi} - \cancel{A_\nu\partial_\mu\varphi} \qquad [\partial_\mu, A_\nu] = \partial_\mu A_\nu$$

$$[A_\mu, \partial_\nu]\varphi = A_\mu\partial_\nu\varphi - \partial_\nu(A_\mu\varphi) = \cancel{A_\mu\partial_\nu\varphi} - \partial_\nu A_\mu\varphi - \cancel{A_\mu\partial_\nu\varphi} \qquad [A_\mu, \partial_\nu] = -\partial_\nu A_\mu$$

$$[D_\mu, D_\nu] = -i\partial_\mu A_\nu - i(-\partial_\nu A_\mu) - [A_\mu, A_\nu] = -i\left(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\right) = -iF_{\mu\nu}$$

$$F_{\mu\nu} = i[D_\mu, D_\nu]$$

$$D_\mu\varphi \to D'_\mu\varphi' = D'_\mu\omega\varphi = \omega D_\mu\varphi \qquad \text{for any } \varphi$$

$$\Downarrow$$

$$D_\mu \to D'_\mu = \omega D_\mu \omega^+$$

$$\Downarrow$$

$$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \to \mathrm{Tr}(\omega F_{\mu\nu}\omega^+\omega F^{\mu\nu}\omega^+) = \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \quad \Leftarrow \quad F_{\mu\nu} \to i[D'_\mu, D'_\nu] = i\omega[D_\mu, D_\nu]\omega^+ = \omega F_{\mu\nu}\omega^+$$

$$\mathcal{L}_{kin} \propto \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$

$$\varphi \to \omega\varphi = e^{ig\theta^a T_a}\varphi$$

ω is a matrix in a given representation

T_a - group generators

$$D_\mu\varphi = (\partial_\mu - iA_\mu)\varphi \qquad \mathrm{Tr}(T_a, T_b) = \frac{1}{2}\delta_{ab} \text{ in fundamental representation}$$

$$\boxed{A_\mu = g\, A^a_\mu T_a}$$ g - gauge coupling (normalization of A^a_μ)

$$F_{\mu\nu} = g\left(\partial_\mu A^a_\nu T_a - \partial_\nu A^a_\mu T_a\right) - i g^2 A^a_\mu A^b_\nu \underbrace{[T_a, T_b]}_{i f_{abc} T_c} = g\underbrace{\left(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g A^a_\mu A^b_\nu f_{abc}\right)}_{\equiv F^c_{\mu\nu}} T_c$$

$$\boxed{F_{\mu\nu} = g F^a_{\mu\nu} T_a}$$

$$\mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right) = g^2 F^a_{\mu\nu} F^{b\,\mu\nu} \underbrace{\mathrm{Tr}\left(T_a T_b\right)}_{\frac{1}{2}\delta_{ab}} = \frac{g^2}{2} F^a_{\mu\nu} F^{a\,\mu\nu}$$

$$\Downarrow$$

$$\boxed{\mathcal{L}_{YM} = \frac{1}{2g^2}\,\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = \frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}}$$

Example : $SU(2)$

fundamental: $T_a = \frac{\tau_a}{2}$, $\left[\frac{\tau_a}{2}, \frac{\tau_b}{2}\right] = i\epsilon_{abc}\frac{\tau_c}{2}$, $\mathrm{Tr}\left(\frac{\tau_a}{2}, \frac{\tau_b}{2}\right) = \frac{1}{2}\delta_{ab}$

$$A_\mu = g A^a_\mu \frac{\tau_a}{2}$$

$$D_\mu = \partial_\mu - i A_\mu = \partial_\mu - i g A^a_\mu \frac{\tau_a}{2}$$

$$F_{\mu\nu} = \frac{\tau_a}{2} F^a_{\mu\nu} = \frac{\tau_c}{2}\left(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g\,\epsilon_{abc} A^a_\mu A^b_\nu\right)$$

The gauge invariant scalar-vector theory with scalar in the fundamental repr.

$$\boxed{\mathcal{L} = (D_\mu \varphi)^\dagger (D^\mu \varphi) + \frac{1}{2g^2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) \qquad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad D_\mu \varphi = \left(\partial_\mu - i g A^a_\mu \frac{\tau_a}{2}\right) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}}$$

Gauge (Yang-Mills) theories with fermions

Following Dirac, we are looking for linear (since the goal was to construct a relativistically covariant generalization of the Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$ which is linear in $\frac{\partial}{\partial t}$) in derivatives, such that its solutions satisfy the Klein-Gordon equation; $(\Box + m^2)\varphi = 0$.

We therefore postulate

$(i\gamma^\nu \partial_\nu + m) \cdot |$ $\qquad (i\gamma^\mu \partial_\mu - m)\psi = 0$

$\uparrow$ undefined coefficients

↑ undefined coefficients

$$\left(-\underbrace{\gamma^\nu\gamma^\mu}\partial_\nu\partial_\mu - mi\cancel{\gamma^\nu}\partial_\nu + mi\cancel{\gamma^\mu}\partial_\mu - m^2\right)\psi = 0$$

$$\hookrightarrow \frac{1}{2}\{\gamma^\nu,\gamma^\mu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]$$

$$\left(-\frac{1}{2}\{\gamma^\nu,\gamma^\mu\}\partial_\nu\partial_\mu - m^2\right)\psi = 0$$

In order to reproduce the Klein-Gordon equation we require $\{\gamma^\nu,\gamma^\mu\} = 2\eta^{\nu\mu}$

the Clifford algebra

1. γ^μ must be matrices $\Longleftarrow$

2. ψ must have several components

It is possible to prove that for a space-time of dimension D the dimension of matrices satisfying $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$ equals $2^{\left[\frac{D}{2}\right]}$,

$[x]$ denotes the integer part of x

e.g. $\left[\frac{4}{2}\right] = 2$, $\left[\frac{5}{2}\right] = 2$

$\Downarrow$

In 4-dimensional space time γ^μ are 4×4 matrices

$\gamma^0 \mid \quad i\gamma^0\partial_0\psi = (-i\gamma^i\partial_i + m)\psi \qquad \gamma^{0\,2} = \mathbb{1} \qquad \{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$

$\gamma^0 \cdot | \quad i\gamma^0\partial_0\psi = (-i\gamma^i\partial_i + m)\psi \qquad \gamma^{0\,2} = 1 \qquad \{\gamma^0,\gamma^0\} = 2\eta^{00}\mathbb{1}$

$$i\partial_0\psi = \Big(-i\underbrace{\gamma^0\gamma^i}_{\alpha_i}\partial_i + m\underbrace{\gamma^0}_{\beta}\Big)\psi \iff i\hbar\partial_t\psi = \hat{H}\psi \implies \begin{array}{l}\alpha_i^\dagger = \alpha_i \\ \beta^\dagger = \beta\end{array}$$

$$\boxed{\begin{array}{l}\gamma^{i\dagger} = -\gamma^i \\ \gamma^{0\dagger} = \gamma^0\end{array}} \Leftarrow \begin{array}{l}\gamma^{i\dagger}\gamma^{0\dagger} = \gamma^0\gamma^i \\ \gamma^{0\dagger} = \gamma^0\end{array}$$

The standard representation of Dirac matrices γ^μ

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \qquad \begin{array}{l}\sigma^\mu := (\mathbb{1}, \sigma^i) \\ \tilde{\sigma}^\mu := (\mathbb{1}, -\sigma^i)\end{array}$$

, where σ^i the Pauli matrices

$$\psi = \begin{pmatrix}\psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4\end{pmatrix} \qquad \boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}$$

the Dirac equation

$$S_D = \int d^4x\, \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \Rightarrow \delta S_D = 0 \text{ for } (i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\bar{\psi} = \psi^\dagger\gamma_0$$

- invariance under $\psi \to \omega\psi$ such that $\omega^\dagger\omega = \mathbb{1}$ and ω const.
- local invariance : $\partial_\mu \to D_\mu := \partial_\mu - iA_\mu$ covariant derivative

• local invariance :

$$\partial_\mu \to D_\mu := \partial_\mu - i A_\mu \qquad \text{covariant derivative}$$

Example : Dirac equation for SU(2) and fundamental (dim 2) representation

$$\left\{ i(\gamma^\mu)_{\alpha\beta} \underbrace{\left[\delta_{ij}\, \partial_\mu - i g \frac{(\tau^a)_{ij}}{2} A^a_\mu \right]}_{(D_\mu)_{ij}} - m\, \delta_{\alpha\beta}\, \delta_{ij} \right\} \psi_{j\beta}(x) = 0$$

Lorentz $\quad \mu = 0,1,2,3$

spinor $\quad \alpha,\beta = 1,2,3,4$

SU(2) $\quad i,j = 1,2$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{bmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{13} \\ \psi_{14} \end{pmatrix} \\ \begin{pmatrix} \psi_{21} \\ \psi_{22} \\ \psi_{23} \\ \psi_{24} \end{pmatrix} \end{bmatrix}$$