

Special relativity

Kinematics

$$m_N \frac{d^2 \bar{x}_N}{dt^2} = G \sum_H \frac{m_N m_H (\bar{x}_H - \bar{x}_N)}{|\bar{x}_H - \bar{x}_N|^3}$$

invariant under transformations from the Galileo group (GG)

$$\bar{x} \rightarrow \bar{x}' = R \bar{x} + \bar{v} t + \bar{a}$$

$$t \rightarrow t' = t + \tau$$

The Principle of Galilean Relativity: the laws of motion must be invariant under transformations from the Galileo group

The Maxwell's equations:

$$\nabla \cdot \bar{E} = \rho$$

$$\nabla \times \bar{B} = \frac{\partial \bar{E}}{\partial t} + \bar{J}$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = - \frac{\partial \bar{B}}{\partial t}$$

are NOT INVARIANT under transformations from the GG

but the M's equations are correct



↙ Einstein's special relativity

The Principle of Special Relativity: the laws of motion must be invariant under Lorentz transformations

$$x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\beta} + a^{\alpha}$$

where

$$\Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} \eta_{\alpha\beta} = \eta_{\gamma\delta}$$

$$\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$$

$$\alpha, \beta, \dots = 0, 1, 2, 3, \quad i, j, k, \dots = 1, 2, 3$$

System of units: $\hbar = c = 1$

$$[c] = \frac{L}{T} \quad [E] = E \cdot T = M \left(\frac{L}{T} \right)^2 T = M \frac{L^2}{T}$$

↑
energy ($E = \hbar \omega$)

$$[L] = [M^{-1}]$$

$$[x^0] = [x^i] = L$$

$$x'^{\alpha} = \underbrace{\Lambda^{\alpha}_{\beta} x^{\beta}}_{\text{summation over } \beta} + e^{\alpha} = \Lambda^{\alpha}_0 x^0 + \Lambda^{\alpha}_1 x^1 + \Lambda^{\alpha}_2 x^2 + \Lambda^{\alpha}_3 x^3 + e^{\alpha}$$

$$dT^2 = dt^2 - d\vec{x}^2 = \gamma_{\alpha\beta} dx^{\alpha} dx^{\beta} \quad \text{— the "proper time"}$$

in another inertial frame $dx'^{\alpha} = \Lambda^{\alpha}_{\beta} dx^{\beta}$

$$\begin{aligned} d\tau'^2 &= \gamma_{\alpha\beta} dx'^{\alpha} dx'^{\beta} = \\ &= \gamma_{\alpha\beta} \Lambda^{\alpha}_{\gamma} dx^{\gamma} \Lambda^{\beta}_{\delta} dx^{\delta} = \underbrace{\gamma_{\alpha\beta} \Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta}}_{\gamma_{\gamma\delta}} dx^{\gamma} dx^{\delta} = d\tau^2 \end{aligned}$$

↓
 $d\tau$ is invariant

Michelson - Morley exp. ↙

$$d\tau = 0 \quad \text{for light} \quad \frac{d\vec{x}}{dt} = 1$$

↓

The same in all inertial frames

$$x^\alpha \rightarrow x'^\alpha = \Lambda^\alpha_\beta x^\beta + a^\alpha$$

- the inhomogeneous Lorentz group
(the Poincaré group)

$a^\alpha = 0$ - the homogeneous Lorentz group

$$\Lambda^0_0 \geq 1, \text{ Det } \Lambda = +1 \quad \text{proper} \begin{cases} \text{homogeneous} \\ \text{inhomogeneous} \end{cases}$$

$$\begin{aligned} \Lambda^0_0 < -1, \text{ Det } \Lambda = +1 & \quad (\text{time inversion}) \\ \Lambda^0_0 \geq 1, \text{ Det } \Lambda = -1 & \quad (\text{space inversion}) \end{aligned} \quad \left. \vphantom{\begin{aligned} \Lambda^0_0 < -1, \text{ Det } \Lambda = +1 \\ \Lambda^0_0 \geq 1, \text{ Det } \Lambda = -1 \end{aligned}} \right\} \text{improper} \quad \left. \vphantom{\begin{aligned} \Lambda^0_0 < -1, \text{ Det } \Lambda = +1 \\ \Lambda^0_0 \geq 1, \text{ Det } \Lambda = -1 \end{aligned}} \right\} \text{violated by Nature}$$

$$\Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta} \quad \Rightarrow \quad (\Lambda^0_0)^2 - (\Lambda^i_0)^2 = 1$$

$\gamma = \delta = 0$

$$(\Lambda^0_0)^2 = 1 + \sum_{i=1,2,3} (\Lambda^i_0)^2 \geq 1$$

$$\Lambda^T \eta \Lambda = \eta \quad \Rightarrow \quad (\text{Det } \Lambda)^2 = 1$$

Rotations: $\Lambda^i_j = R_{ij} \quad \Lambda^i_0 = \Lambda^0_i = 0 \quad \Lambda^0_0 = 1$

with $\text{Det } R = 1 \quad R^T R = 11$

Booster: $\begin{aligned} O &: x^\mu = (x^0, 0, 0, 0) \quad \text{at rest} \\ O' &: x'^\mu = (x'^0, x'^1, x'^2, x'^3) \quad \text{moving} \end{aligned}$

$$dx'^\alpha = \Lambda^\alpha_\beta dx^\beta \quad \left. \vphantom{dx'^\alpha = \Lambda^\alpha_\beta dx^\beta} \right\} \begin{aligned} dt' &= \Lambda^0_0 dt \\ dx'^i &= \Lambda^i_0 dt \end{aligned}$$

$$\left. \begin{aligned} \frac{\Lambda^i_0}{\Lambda^0_0} &= v_i \quad \frac{dx'^i}{dt'} \\ \Lambda^i_0 &= v_i \Lambda^0_0 \end{aligned} \right\}$$

$$1 = (\Lambda^0_0)^2 - \sum_i (\Lambda^i_0)^2$$

$$\Lambda^\alpha_\gamma \Lambda^\beta_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta} \quad \Rightarrow \quad \Lambda^0_0 = \gamma \quad \Lambda^i_0 = \gamma v_i$$

$\gamma = 1/\sqrt{1-v^2}$

Λ^i_j are not uniquely determined because of possible rotation, a convenient choice is

$$\Lambda^i_j = \delta^i_j + v_i v_j \frac{\gamma - 1}{v^2}$$

$$\Lambda^0_j = \gamma v_j$$

$$\Lambda^0_0 = \gamma$$

Dynamics

Let us "define" the relativistic force f^α acting on a particle with coordinates $x^\alpha(\tau)$

$$f^\alpha = m \frac{d^2 x^\alpha}{d\tau^2}$$

where f^α is related to the Newtonian force by using that:

A) if the particle is momentarily at rest then $d\tau = dt$ and $f^\alpha = F^\alpha$ with $F^0 = 0$ and F^i being Cartesian components of the nonrelativistic force $\vec{F}$

B) Under the general Lorentz transformation

$$dx'^\alpha = \Lambda^\alpha_\beta dx^\beta \quad \text{while}$$

$$d\tau' = d\tau \quad \text{is invariant}$$

$\downarrow$

$$f'^\alpha = \Lambda^\alpha_\beta f^\beta \quad (\text{a four-vector})$$

Suppose that our particle has velocity $\vec{v}$ at some moment to and introduce a new coordinate system x'^α

$$x^\alpha = \Lambda^\alpha_\beta(\vec{v}) x'^\beta$$

where $\Lambda(\vec{v})$ is the boost that carries a particle from rest to velocity $\vec{v} \Rightarrow$ our particle is at rest in x'^α

↓

the force four-vector is x^α at to
is equal to the nonrelativistic force F^α , so

$$f^\alpha = \Lambda^\alpha_\beta(\vec{v}) F^\beta$$

since $F^0 = 0$

$$f^i = \Lambda^i_j F^j = \left(\delta_{ij} + v_i v_j \frac{\gamma-1}{v^2} \right) F^j = F^i + (\gamma-1) v_i \frac{\vec{v} \cdot \vec{F}}{v^2}$$

$$f^0 = \Lambda^0_j F^j = \gamma \vec{v} \cdot \vec{F}$$

$$f^\alpha = m \frac{d^2 x^\alpha}{d\tau^2} \Rightarrow x^\alpha = x^\alpha(\tau)$$

initial conditions: $-1 = \gamma_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$

Energy and momentum

$$f^\alpha = m \frac{d^2 x^\alpha}{d\tau^2} \Rightarrow p^\alpha \equiv m \frac{dx^\alpha}{d\tau}$$

$$\downarrow$$

$$f^\alpha = \frac{dp^\alpha}{d\tau}$$

$$d\tau = (dt^2 - d\vec{x}^2)^{1/2} =$$

$$= (1 - \vec{v}^2)^{1/2} dt$$

$$\vec{v} \equiv \frac{d\vec{x}}{dt}$$

$$p^i = m \frac{dx^i}{d\tau} = m (1 - \vec{v}^2)^{-1/2} \frac{dx^i}{dt} = m \gamma v^i$$

$$p^0 = m \frac{dx^0}{d\tau} = m \frac{dt}{d\tau} = m \gamma$$

p^α is a four-vector

$$p'^\alpha = \Lambda^\alpha_\beta p^\beta$$

p^α are conserved (see Noether's theorem)

↓

p'^α is also conserved

$$\left(\Delta \sum_n p'^\alpha_n = \Lambda^\alpha_\beta \Delta \sum_n p^\beta_n \right)$$

$$\rightarrow p^0 \equiv E = (\vec{p}^2 + m^2)^{1/2}$$

limiting $\vec{v}$

Vectors and Tensors

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four-vector : $v^\alpha \rightarrow v'^\alpha = \Lambda^\alpha_\beta v^\beta$ e.g. $dx^\alpha, f^\alpha, p^\alpha$
 contravariant
 for $x^\alpha \rightarrow x'^\alpha = \Lambda^\alpha_\beta x^\beta$

covariant four-vector : $U_\alpha \rightarrow U'_\alpha = \Lambda_\alpha^\beta U_\beta$

where $\Lambda_\alpha^\beta \equiv \eta_{\alpha\delta} \eta^{\delta\gamma} \Lambda^\gamma_\sigma$ and $\eta^{\beta\delta} = \eta_{\beta\delta}$

$$\eta^{\beta\delta} \eta_{\alpha\delta} = \delta^\beta_\alpha = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

Note that Λ_α^β is the inverse of Λ^α_β

$$\Lambda_\alpha^\gamma \Lambda^\alpha_\beta = \eta_{\alpha\delta} \eta^{\delta\gamma} \Lambda^\alpha_\sigma \Lambda^\sigma_\beta = \eta^{\gamma\sigma} \eta_{\sigma\beta} = \delta^\gamma_\beta$$

$U_\alpha v^\alpha$ is invariant :

$$U'_\alpha v'^\alpha = \Lambda_\alpha^\beta U_\beta \Lambda^\alpha_\gamma v^\gamma = U_\beta v^\beta$$

$v_\alpha \equiv \eta_{\alpha\beta} v^\beta$ is indeed covariant :

$$v_\alpha \rightarrow v'_\alpha = \eta_{\alpha\beta} v'^\beta = \eta_{\alpha\beta} \Lambda^\beta_\gamma v^\gamma = \eta_{\alpha\beta} \Lambda^\beta_\gamma \eta^{\gamma\delta} v_\delta = \Lambda_\alpha^\delta v_\delta$$

$$\frac{\partial}{\partial x^\alpha} \rightarrow \frac{\partial}{\partial x'^\alpha} = \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial}{\partial x^\beta} = \Lambda_\alpha^\beta \frac{\partial}{\partial x^\beta} \Rightarrow \frac{\partial}{\partial x^\alpha} - \text{covariant four-vector}$$

$$\Lambda_\alpha^\gamma \left| \begin{aligned} x'^\alpha &= \Lambda^\alpha_\beta x^\beta \\ x^\gamma &= \Lambda_\alpha^\gamma x'^\alpha \\ x^\beta &= \Lambda_\alpha^\beta x'^\alpha \Rightarrow \frac{\partial x^\beta}{\partial x'^\alpha} = \Lambda_\alpha^\beta \end{aligned} \right. \Rightarrow \Lambda_\alpha^\gamma x'^\alpha = \Lambda_\alpha^\gamma \Lambda^\alpha_\beta x^\beta = x^\gamma$$

$\frac{\partial}{\partial x^\alpha}$ - covariant, $\frac{\partial}{\partial x_\alpha}$ - contravariant

$\square^2 \equiv \eta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} = \frac{\partial^2}{\partial x^0^2} - \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i} = \frac{\partial^2}{\partial t^2} - \nabla^2$
 Lorentz invariant

Tensors

$T^\alpha_{\beta} \Rightarrow T'^\alpha_{\beta} = \Lambda^\alpha_\delta \Lambda^\epsilon_\beta \Lambda^\delta_\epsilon T^\epsilon_\epsilon$

$T^\alpha_\beta \equiv a P^\alpha_\beta + b S^\alpha_\beta$ - tensor if P^α_β and S^α_β tensors

• direct product: $T^\alpha_\beta{}^\gamma = A^\alpha_\beta B^\gamma$ - tensor if ...

• contraction: $T^{\alpha\delta} = T^\alpha_\beta{}^{\delta\beta}$

$T^{\alpha\gamma} \Rightarrow T'^{\alpha\delta} = T^{\alpha\gamma}{}^{\delta\beta} = \Lambda^\alpha_\delta \Lambda^\epsilon_\beta \Lambda^\gamma_\epsilon \Lambda^\delta_\epsilon T^{\epsilon\epsilon}{}^{\delta\epsilon} = \Lambda^\alpha_\delta \Lambda^\gamma_\epsilon T^{\epsilon\epsilon}{}^{\delta\epsilon}$
 $T^{\alpha\gamma}$ rank 2 tensor

• differentiation: $T^\alpha{}_{\beta\gamma} = \frac{\partial}{\partial x^\alpha} T^{\beta\gamma}$ - tensor as a product of tensors

• the Minkowski tensor:

definition of the Lorentz trans.: $\eta_{\alpha\beta} = \Lambda^\gamma_\alpha \Lambda^\delta_\beta \eta_{\gamma\delta} \Rightarrow \eta_{\alpha\beta}$ tensor covariant

$\eta^{\alpha\beta} = \Lambda^\gamma_\alpha \Lambda^\delta_\beta \eta^{\gamma\delta}$ contravariant tensor
 $\eta_{\alpha\beta} \Rightarrow \eta'_{\alpha\beta} = \eta_{\alpha\beta} = \Lambda^\gamma_\alpha \Lambda^\delta_\beta \eta_{\gamma\delta}$
 see the next page

Let us show that $\gamma_{\alpha\beta}$ is a covariant ^{second rank tensor} ~~four vector~~ which is invariant under the Lorentz transformation

$$\gamma_{\alpha\beta} \rightarrow \gamma'_{\alpha\beta} = \Lambda_{\alpha}^{\gamma} \Lambda_{\beta}^{\delta} \gamma_{\gamma\delta}$$

from the definition of the LT we have

$$\gamma_{\alpha\beta} \Lambda^{\alpha}_{\gamma} \Lambda^{\beta}_{\delta} = \gamma_{\gamma\delta} \quad \uparrow$$

$$\gamma_{\alpha\beta} \Lambda^{\alpha\gamma} \Lambda^{\beta}_{\delta} = \delta^{\gamma}_{\delta} \quad | \quad (\Lambda^{-1})^{\delta}_{\tau}$$

$$\gamma_{\alpha\beta} \Lambda^{\alpha\gamma} \underbrace{\Lambda^{\beta}_{\delta} (\Lambda^{-1})^{\delta}_{\tau}}_{\delta^{\beta}_{\tau}} = (\Lambda^{-1})^{\gamma}_{\tau}$$

$$\Lambda^{\gamma}_{\tau} = \gamma_{\alpha\tau} \Lambda^{\alpha\gamma} = (\Lambda^{-1})^{\gamma}_{\tau} \quad | \quad \Lambda^{\alpha}_{\gamma}$$

$$\Lambda^{\alpha}_{\gamma} \Lambda^{\gamma}_{\tau} = \Lambda^{\alpha}_{\gamma} (\Lambda^{-1})^{\gamma}_{\tau} = \delta^{\alpha}_{\tau} \quad \downarrow$$

$$\Lambda^{\alpha}_{\gamma} \Lambda^{\gamma}_{\tau} = \gamma_{\alpha\tau}$$

$$\gamma_{\gamma\delta} \Lambda^{\delta}_{\alpha} \Lambda^{\gamma}_{\tau} = \gamma_{\alpha\tau}$$

therefore

$$\gamma_{\alpha\beta} \rightarrow \gamma'_{\alpha\beta} = \Lambda_{\alpha}^{\gamma} \Lambda_{\beta}^{\delta} \gamma_{\gamma\delta} = \gamma_{\alpha\beta}$$

• the Levi-Civita tensor

$$\varepsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ even permutation of } 0123 \\ -1 & \text{odd} \\ 0 & \text{otherwise} \end{cases}$$

note that

$$\Lambda^\alpha_\varepsilon \Lambda^\beta_\gamma \Lambda^\gamma_\kappa \Lambda^\delta_\lambda \varepsilon^{\varepsilon\zeta\kappa\lambda} \propto \varepsilon^{\alpha\beta\gamma\delta}$$

as LHS must be odd under any permutation of indices $\alpha\beta\gamma\delta$

Currents and Densities

particles with position $x_u(t)$ and charge e_u

electric current : $\vec{J}(\vec{x}, t) = \sum_u e_u \delta^3(\vec{x} - \vec{x}_u(t)) \frac{d\vec{x}_u(t)}{dt}$

charge density : $\rho(\vec{x}, t) = \sum_u e_u \delta^3(\vec{x} - \vec{x}_u(t))$

for $\int d^3x \delta^3(\vec{x} - \vec{y}) f(\vec{x}) = f(\vec{y})$

let us make a "four-vector" prototype, such that

$$J^0 \equiv \rho$$

$$J^\mu(x) = \sum_u e_u \delta^3(\vec{x} - \vec{x}_u(t)) \frac{dx_u^\mu(t)}{dt} \quad \text{for } x_u^0(t) = t$$

let us show that $J^\mu(x)$ is indeed a four-vector

$$\begin{aligned} J^\mu(x) &= \int dt' \sum_u e_u \delta^4(x - x_u(t')) \frac{dx_u^\mu(t')}{dt'} = \left(t' \rightarrow \tau \right) = \\ &= \int d\tau \sum_u e_u \underbrace{\delta^4(x - x_u(\tau))}_{\text{invariant since } \text{Det } \Lambda = 1} \underbrace{\frac{dx_u^\mu(\tau)}{d\tau}}_{\text{four-vector}} \end{aligned}$$

invariant

$\Downarrow$

$J^\mu(x)$ is a four-vector

$\delta^{(4)}(x-x_0)$ is Lorentz invariant

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$$\beta \mid x^\beta - x_0^\beta = 0$$

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta$$

$$x'^\alpha - x_0'^\alpha = 0$$

$$\delta[f(x)] = \sum_u \frac{f(x-x_u)}{\left| \frac{df}{dx} \right|_{x=x_u}}$$

$$\delta^{(4)}(x^\beta - x_0^\beta) = \delta^{(4)}(x'^\alpha - x_0'^\alpha)$$

$$\int d^4x f(x) \delta^4(x-x_0) = \int d^4x' \underbrace{|\det \tilde{\Lambda}|}_1 f[x(x')] \delta^{(4)}(x'-x_0') = f[x(x_0')] = f(x_0)$$

↑
change of variables

$$\int d^4x \delta^4(x-x_0) f(x) = f(x_0)$$

For invariance we require

$$\int d^4x \delta^4(x'-x_0') f(x) = f(x_0)$$

where

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta$$

$$x_0'^\alpha = \Lambda^\alpha_\beta x_0^\beta$$

$$\int d^4x' \det(\tilde{\Lambda}) \delta^4(x'-x_0') f(\tilde{\Lambda}^{-1}x') = \det(\tilde{\Lambda}) f(\underbrace{\tilde{\Lambda}^{-1}x_0'}_{x_0})$$

change

of integration
variables

$$x' = \Lambda x$$

$$= \det \tilde{\Lambda}^{-1} f(x_0) = f(x_0)$$

" 1

□

Homework $\rightarrow \frac{\partial}{\partial x^\alpha} J^\alpha(x) = 0 \Rightarrow \frac{\partial}{\partial x^0} J^0 + \vec{\nabla} \cdot \vec{J} = 0$

Now we can show that $Q \equiv \int d^3x J^0(x)$ is time independent: $\frac{dQ}{dt} = 0$

$$\frac{dQ}{dt} = \int d^3x \frac{\partial}{\partial x^0} J^0(x) = - \int d^3x \underbrace{\vec{\nabla} \cdot \vec{J}(x)}_{\frac{\partial}{\partial x^i} J^i(x)} = 0$$

↑ provided all particles are localized at some finite distance $x_\mu(t)$

Electrodynamics

the Maxwell's equations:

(ME)

$\vec{E} = (E_1, E_2, E_3)$

$\vec{B} = (B_1, B_2, B_3)$

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \rho \\ \vec{\nabla} \times \vec{B} &= \frac{\partial \vec{E}}{\partial t} + \vec{J} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= - \frac{\partial \vec{B}}{\partial t} \end{aligned}$$

$\left. \begin{aligned} &(*) \\ &(**) \end{aligned} \right\}$

to write the ME in a covariant fashion let us define $F^{\alpha\beta}$ such that

$$\begin{aligned} F^{12} &= B_3 & F^{23} &= B_1 & F^{31} &= B_2 \\ F^{01} &= E_1 & F^{02} &= E_2 & F^{03} &= E_3 \end{aligned}$$

$F^{\alpha\beta} = -F^{\beta\alpha}$

then $(*)$ reads

$$\frac{\partial}{\partial x^\alpha} F^{\alpha\beta} = -J^\beta$$

$\beta = 0$:

$$\begin{aligned} \frac{\partial}{\partial x^\alpha} F^{\alpha 0} &= -J^0 = -\rho \\ &= \frac{\partial}{\partial x^i} F^{i0} = - \frac{\partial}{\partial x^i} F^{0i} = - \frac{\partial}{\partial x^i} E_i \end{aligned}$$

o.k.

$\beta = i$:

$$\begin{aligned} \frac{\partial}{\partial x^\alpha} F^{\alpha i} &= -J^i \\ &= \frac{\partial}{\partial x^0} F^{0i} + \frac{\partial}{\partial x^j} F^{ji} = \frac{\partial}{\partial t} E_i + \underbrace{\epsilon_{jik} \frac{\partial}{\partial x^j} B_k}_{-(\vec{\nabla} \times \vec{B})_i} \end{aligned}$$

o.k.

$\epsilon_{abk} \epsilon_{ijk} = \delta_{aj} \delta_{bk} - \delta_{ak} \delta_{bj}$

$B_i \equiv \frac{1}{2} \epsilon_{ijk} F^{jk} \Rightarrow F^{ji} = \epsilon_{jik} B_k$

1. Show that $\frac{\partial}{\partial t} J^L(x) = 0$

$$\begin{aligned} \nabla \cdot \vec{J}(x, t) &= \sum_n e_n \frac{\partial}{\partial x^i} \delta^3(\vec{x} - \vec{x}_n(t)) \frac{dx_n^i(t)}{dt} = \\ &= - \sum_n e_n \frac{\partial}{\partial x_n^i} \delta^3(\vec{x} - \vec{x}_n(t)) \frac{dx_n^i(t)}{dt} = \\ &= - \frac{\partial}{\partial t} \sum_n e_n \delta^3(\vec{x} - \vec{x}_n(t)) = - \frac{\partial E(\vec{x}, t)}{\partial t} \end{aligned}$$

□

2. Show that $\gamma_{\alpha\beta}$ is a tensor, i.e.

$$\gamma_{\alpha\beta} \rightarrow \gamma'_{\alpha\beta} = \Lambda_{\alpha}^{\gamma} \Lambda_{\beta}^{\delta} \gamma_{\gamma\delta}$$

and prove that $\gamma'_{\alpha\beta} = \gamma_{\alpha\beta}$

(**) could be written as

$$\epsilon^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^\beta} F_{\gamma\delta} = 0 \quad (F_{\gamma\delta} = \eta_{\gamma\alpha} \eta_{\delta\beta} F^{\alpha\beta})$$

check : $\alpha = 0$

$$\epsilon^{ijk} \frac{\partial}{\partial x^i} F_{jk} = \epsilon^{ijk} \frac{\partial}{\partial x^i} \epsilon_{jkl} B_l =$$

$$= - \underbrace{\epsilon^{ijk} \epsilon_{jkl}}_{2\delta^i_l} \frac{\partial}{\partial x^i} B_l = -2 \frac{\partial}{\partial x^i} B_i = 0$$

$$\nabla \cdot \vec{B} = 0 \quad \text{o.k.}$$

$\alpha = i$

$$\begin{aligned} \epsilon^{i\beta\gamma\delta} \frac{\partial}{\partial x^\beta} F_{\gamma\delta} &= \epsilon^{i0kL} \frac{\partial}{\partial x^0} F_{kL} + \epsilon^{ij\gamma\delta} \frac{\partial}{\partial x^j} F_{\gamma\delta} \\ &= \underbrace{-\epsilon^{iKL} \frac{\partial}{\partial x^0} \epsilon_{KLm} B_m}_{-2 \frac{\partial B_i}{\partial t}} + \underbrace{\epsilon^{ijok} \frac{\partial}{\partial x^j} F_{ok} + \epsilon^{ijk0} \frac{\partial}{\partial x^j} F_{k0}}_{-\epsilon^{ijk} \frac{\partial}{\partial x^j} (\vec{E}_k + \vec{E}_k)} = -2(\nabla \times \vec{E})_i \end{aligned}$$

o.k.

Since $\frac{\partial}{\partial x^\alpha} F^{\alpha\beta} = -j^\beta$ and

$\frac{\partial}{\partial x^\alpha}$ and j^β are vectors $\Rightarrow F^{\alpha\beta}$ is a tensor
if $F^{\alpha\beta}$ has a component with $\frac{\partial}{\partial x^\alpha} F^{\alpha\beta} = 0$ then we can't tell it transforms solely in terms of j^β

The electromagnetic force on a charged particle

$$\left. \begin{aligned} \text{(homework)} \quad f^\alpha &= e \eta_{\beta\gamma} F^{\alpha\beta} \frac{dx^\gamma}{dt} \\ \frac{dp^\alpha}{dt} &= f^\alpha \end{aligned} \right\} \Rightarrow \frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v} \times \vec{B})$$

An alternative for $\epsilon^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^\beta} F_{\gamma\delta} = 0$:

$$\frac{\partial}{\partial x^\alpha} F_{\beta\gamma} + \frac{\partial}{\partial x^\beta} F_{\gamma\alpha} + \frac{\partial}{\partial x^\gamma} F_{\alpha\beta} = 0$$

$\epsilon^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^\beta} F_{\gamma\delta} = 0 \Rightarrow$ there exists a four-vector potential

The Energy - Momentum tensor

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In analogy to the four-vector current construction

$$\left\{ \begin{aligned} T^{\alpha 0}(t, \vec{x}) &\equiv \sum_n p_n^\alpha(t) \delta^3(\vec{x} - \vec{x}_n(t)) && \text{(density of } p^\alpha) \\ &\quad \downarrow \text{four-momentum} \\ T^{\alpha i}(t, \vec{x}) &\equiv \sum_n p_n^\alpha(t) \frac{dx_n^i(t)}{dt} \delta^3(\vec{x} - \vec{x}_n(t)) && \text{(current of } p^\alpha) \\ \rightarrow T^{\alpha\beta}(x) &= \sum_n p_n^\alpha(t) \frac{dx_n^\beta(t)}{dt} \delta^3(\vec{x} - \vec{x}_n(t)) && \text{where } x_n^0(t) \equiv t \end{aligned} \right.$$

from the definition of the four-momentum we have

$$p_n^\beta = E_n \frac{dx_n^\beta}{dt}$$

$$\Rightarrow T^{\alpha\beta}(x) = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\vec{x} - \vec{x}_n(t)) \Rightarrow T^{\alpha\beta} = T^{\beta\alpha}$$

to see that $T^{\alpha\beta}$ is indeed a tensor let's write it as

$$T^{\alpha\beta}(x) = \sum_n \int d\tau p_n^\alpha \frac{dx_n^\beta}{d\tau} \delta^4(x - x_n(\tau))$$

Four-momentum (total : including the EM field)

time independent :

$$P_{\text{total}}^\alpha = \int d^3x T^{\alpha 0}(t, \vec{x})$$

as a consequence of the tensor conservation

$$\left(\text{homework } \partial_\alpha T^{\alpha\beta}_{\text{total}} = 0 \right)$$

The energy-momentum conservation

$$T^{\alpha\beta}(\bar{x}, t) = \sum_n p_n^\alpha(t) \frac{dx_n^\beta(t)}{dt} \delta^3(\bar{x} - \bar{x}_n(t))$$

$$\begin{aligned} \frac{\partial}{\partial x^i} T^{\alpha i}(\bar{x}, t) &= - \sum_n p_n^\alpha(t) \frac{dx_n^i(t)}{dt} \frac{\partial}{\partial x_n^i} \delta^3(\bar{x} - \bar{x}_n(t)) = \\ &= - \sum_n p_n^\alpha(t) \frac{\partial}{\partial t} \delta^3(\bar{x} - \bar{x}_n(t)) = \\ &= - \frac{\partial}{\partial t} T^{\alpha 0}(\bar{x}, t) + \sum_n \frac{dp_n^\alpha(t)}{dt} \delta^3(\bar{x} - \bar{x}_n(t)) \end{aligned}$$

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = G^\alpha$$

└ density of force

$$\frac{dp^\alpha}{dt} = f^\alpha \quad \Rightarrow \quad \frac{dp^\alpha}{dt} \frac{dt}{dt} = f^\alpha$$

$$G^\alpha = \sum_n \frac{dt}{dt} f_n^\alpha(t) \delta^3(\bar{x} - \bar{x}_n(t))$$

For free particles $f_n^\alpha = 0$, then

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = 0$$

Consider a gas of charged particles with charges e_n
then

$$f_n^\alpha = e_n F^\alpha{}_\gamma(x) \frac{dx_n^\gamma}{dt}$$

$$\begin{aligned} \frac{\partial}{\partial x^\beta} T^{\alpha\beta} &= \sum_n \frac{dt}{dt} e_n F^\alpha{}_\gamma(x) \frac{dx_n^\gamma}{dt} \delta^3(\bar{x} - \bar{x}_n(t)) = \\ &= \sum_n e_n F^\alpha{}_\gamma(x) \frac{dx_n^\gamma}{dt} \delta^3(\bar{x} - \bar{x}_n(t)) = F^\alpha{}_\gamma(x) J^\gamma(x) \end{aligned}$$

Relativistic Hydrodynamics

A perfect fluid is defined as having at each point a velocity $\vec{v}$, such that an observer moving with this velocity sees the fluid around him as isotropic (no viscosity!)

suppose we are in a frame $(\sim)$ of reference in which fluid is at rest at some particular position and time

↓ from the definition of the perfect fluid

$$\tilde{T}^{ij} = \rho \delta_{ij} \quad \tilde{T}^{i0} = 0 \quad \tilde{T}^{00} = \rho$$

$\uparrow$ pressure $\uparrow$ proper energy density

go to a frame at rest in the lab and suppose that the fluid in this frame is moving (at the given space-time point) with velocity $\vec{v}$

$$x^\alpha = \Lambda^\alpha_\beta(v) \tilde{x}^\beta$$

$$\Lambda^0_0 = \gamma \quad \Lambda^i_0 = \gamma v_i \quad \gamma = (1 - v^2)^{-1/2}$$

$$\Lambda^i_j = \delta_{ij} + v_i v_j \frac{\gamma - 1}{v^2} \quad \Lambda^0_j = \gamma v_j$$

calculate $T^{\alpha\beta} = \Lambda^\alpha_\gamma(v) \Lambda^\beta_\delta(v) \tilde{T}^{\gamma\delta}$

$$T^{ij} = p \delta_{ij} + (p+s) \frac{v_i v_j}{1-v^2}$$

$$T^{i0} = (p+s) \frac{v_i}{1-v^2} \quad T^{00} = \frac{s + p v^2}{1-v^2}$$

that could be written as

$$T^{\alpha\beta} = -p \eta^{\alpha\beta} + (p+s) U^\alpha U^\beta$$

$$\bar{U} = \frac{d\bar{x}}{d\tau} = \gamma \bar{v}$$

four velocity vector

$$U^0 = \frac{dt}{d\tau} = \gamma \quad \text{with} \quad U_\mu U^\mu = +1$$

$\Downarrow$

$T^{\alpha\beta}$ is indeed a tensor
(at a given point)

$n \equiv$ particle number density for a conserved quantity in the comoving frame:

$$\tilde{N}^0 := n$$

$$\tilde{N}^i := 0$$

particle current

in other frames

$$N^0 = \Lambda^0_\beta(\bar{v}) \tilde{N}^\beta = \gamma n$$

$$N^i = \Lambda^i_\beta(\bar{v}) \tilde{N}^\beta = \gamma v_i n, \quad \text{for}$$

$$\Lambda^0_0 = \gamma$$

$$\Lambda^i_j = \delta_{ij} + v_i v_j \frac{\gamma-1}{v^2}$$

$$\Lambda^i_0 = \gamma v_i$$

$$\Lambda^0_j = \gamma v_j$$

$$\gamma = (1-v^2)^{-1/2}$$

$$\Downarrow$$

$$N^\alpha = n U^\alpha$$

the velocity four-vector

the energy momentum conservation:

11e

$$0 = \partial_\beta T^{\alpha\beta} = \partial_\beta [p \gamma^{\alpha\beta} + (p+s) u^\alpha u^\beta] \quad N = \int n^0 d^3x$$

conservation of the particle current, the continuity equation

$$0 = \partial_\alpha N^\alpha = \partial_\alpha (n u^\alpha) = \partial_0 (n \gamma) + \vec{\nabla} \cdot (n \gamma \vec{v})$$

$$\vec{\nabla} = \partial_1, \partial_2, \partial_3$$

$\lambda = i$

$$0 = +\partial_i p + \underbrace{\partial_t [(p+s) u^i u^0]}_{\partial_t T^{i0}} + \vec{\nabla} [(p+s) u^i \bar{v}] =$$

$$u^i = v^i u^0$$

$$= +\partial_i p + \partial_t [(p+s) \gamma^2 v^i u^{02}] + \partial_j [(p+s) u^0 v^i u^0 v^j] =$$

$\lambda = 0$

$$0 = \partial_t p + \partial_t [(p+s) u^{02}] + \partial_j [(p+s) u^0 u^0 v^j]$$

$$= +\partial_i p + \underbrace{\left[v^i \partial_t [(p+s) u^{02}] + v^i \partial_j [(p+s) u^{02} v^j] \right]}_{+v^i \partial_t p} + (p+s) u^{02} \partial_t v^i + (p+s) u^{02} v^j \partial_j v^i =$$

$$= +\partial_i p + v^i \partial_t p + (p+s) \gamma^2 [\partial_t v^i + (\vec{v} \cdot \vec{\nabla}) v^i]$$

$$\partial_t \bar{v} + (\vec{v} \cdot \vec{\nabla}) \bar{v} = - \frac{1-\gamma^2}{p+s} (\bar{\nabla} p + \bar{v} \partial_t p)$$

the Euler equation

$s \rightarrow p+s$, usually
 $p \ll s$

non-relativistic form: $\partial_t \bar{v} + (\vec{v} \cdot \vec{\nabla}) \bar{v} = - \frac{\bar{\nabla} p}{p}$

The scalar equation reads:

11 f

$$\partial_\beta [-p \eta^{\alpha\beta} + (p+s) u^\alpha u^\beta] = 0 \quad | \cdot u_\alpha$$

$$-(\partial^\alpha p) u_\alpha + \partial_\beta [(p+s) u^\alpha u^\beta] u_\alpha = 0$$

$$0 = \partial_\beta (\underbrace{u_\alpha u^\alpha}_{+1}) = 2 u_\alpha \partial_\beta u^\alpha$$

$$= -(\partial^\beta p) u_\beta + \partial_\beta [(p+s) u^\beta] \underbrace{u^\alpha u_\alpha}_{+1} + (p+s) u^\beta (\underbrace{\partial_\beta u^\alpha}_{=0}) u_\alpha =$$

$$= -u^\beta (\partial_\beta p) + \partial_\beta [(p+s) u^\beta] =$$

from the conservation of the particle current $\partial_\beta (n u^\beta) = 0$

$$= u^\beta [\cancel{\partial_\beta p} + \partial_\beta (p+s)] + (p+s) \underbrace{\partial_\beta u^\beta}_{=0} =$$

$$\partial_\beta (n u^\beta) = u^\beta \partial_\beta n + n \partial_\beta u^\beta \stackrel{=0}{=} \bar{n}^{-1} u^\beta \partial_\beta n$$

$$= u^\beta \left[\partial_\beta s - (p+s) \underbrace{\bar{n}^{-1} \partial_\beta n}_{-n \partial_\beta \bar{n}^{-1}} \right] = n u^\beta \left[\bar{n}^{-1} \partial_\beta s + (p+s) \partial_\beta \bar{n}^{-1} \right] =$$

$$= n u^\beta \left[p \partial_\beta \left(\frac{1}{n} \right) + \partial_\beta \left(\frac{s}{n} \right) \right] = 0$$

The second law of thermodynamics tells us

$$p d\left(\frac{1}{n}\right) + d\left(\frac{s}{n}\right) = T dS$$

the entropy per particle

$n = \frac{N}{V}$, $\frac{1}{n} = \frac{V}{N}$
 volume per particle (fluid)

$$\Rightarrow 0 = u^\beta \partial_\beta S \propto \left[\frac{\partial S}{\partial t} + (\vec{v} \cdot \vec{\nabla}) S \right] = 0$$

Consider a fluid composed of structureless point particles, then

$$T^{\alpha\beta} = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\vec{x} - \vec{x}_n)$$

In the comoving frame

$$T^{\alpha\beta} = \text{diag}(\rho, p, p, p)$$

so

$$p = \frac{1}{3} \sum_{i=1}^3 T^{ii} = \frac{1}{3} \sum_n \frac{\bar{p}_n^2}{E_n} \delta^3(\vec{x} - \vec{x}_n)$$

$$\rho = T^{00} = \sum_n E_n \delta^3(\vec{x} - \vec{x}_n)$$

For the particle number density we have

$$n = \sum_n \delta^3(\vec{x} - \vec{x}_n)$$

As $\bar{p}_n^2 = E_n^2 - m_n^2$, we have

$$p = \frac{1}{3} \sum_n \frac{1}{E_n} (E_n^2 - m_n^2) \delta^3(\vec{x} - \vec{x}_n) = \frac{1}{3} \rho - \frac{1}{3} \sum_n \frac{m_n^2}{E_n} \delta^3(\vec{x} - \vec{x}_n)$$

$$\downarrow$$

$$0 \leq p \leq \frac{\rho}{3}$$

For a non-relativistic gas $E_n \approx m_n + \frac{\bar{p}_n^2}{2m_n} + \dots$
 then for $m_n = m$ we get

$$\begin{aligned} \rho &\approx m \cdot n + \sum_n \frac{\bar{p}_n^2}{2m} \delta^3(\vec{x} - \vec{x}_n) = m \cdot n + \\ &+ \sum_n \frac{\bar{p}_n^2}{2} \frac{1}{E_n - \frac{\bar{p}_n^2}{2m} + \dots} \delta^3(\vec{x} - \vec{x}_n) = m \cdot n + \frac{3}{2} p + \dots \\ &\quad \rho = m \cdot n + \frac{3}{2} p \end{aligned}$$

For a high-relativistic gas

$$E_u = |\vec{p}_u| \gg m, \quad \text{keep } 11h$$

$$g \approx \sum_u |\vec{p}_u| \delta^3(\vec{x} - \vec{x}_u) \approx \sum_u \frac{|\vec{p}_u|^2}{E_u} \delta^3(\vec{x} - \vec{x}_u) = 3p$$

$$\frac{|\vec{p}_u|^2}{|\vec{p}_u|} = \frac{|\vec{p}_u|^2}{E_u + \dots}$$

$$g = 3p$$

The variational principle

classical mechanics :

$$S[x(t)] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \frac{d\bar{x}}{dt} \frac{d\bar{x}}{dt} - V(x) \right] =$$

$$= \int_{t_1}^{t_2} dt L(\bar{x}, \dot{\bar{x}})$$

classical trajectories $\Leftrightarrow$ extremum of $\underbrace{S[x(t)]}_{\hat{q}}$

$$\delta S[x] \equiv S[x^i + \delta x^i] - S[x^i] = \delta S[x] = 0$$

$$= - \int_{t_1}^{t_2} dt \delta x^i \left[m \frac{d^2 x^i}{dt^2} + \frac{\partial V(x)}{\partial x^i} \right] + m \left(\delta x^i \frac{dx^i}{dt} \right) \Big|_{t_1}^{t_2} + o(\delta x^2)$$

$$= 0 \quad \text{for any } \delta x^i(t) \text{ such that } \delta x^i(t_1) = \delta x^i(t_2) = 0$$

$$\Downarrow$$

$$m \frac{d^2 x^i}{dt^2} + \frac{\partial V}{\partial x^i} = 0$$

classical field theory :

$$S[q] = \int_0^T dt \left(\sum_n \frac{1}{2} m_n \dot{q}_n^2 - V(q_1, \dots, q_N) \right)$$

let us consider $V(\dots) = \sum_{a,b,c} \frac{1}{2} k_{ab} q_a q_b + \frac{1}{2} g_{abc} q_a q_b q_c + \dots$

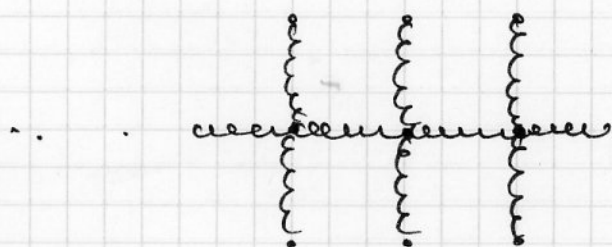
$$\Downarrow$$

$$m_a \ddot{q}_a = - \left(\sum_b k_{ab} q_b + \dots \right)$$

$\uparrow$
 e_a harmonic oscillator
 $\leftarrow$ constant force

$\uparrow$
expansion
around
certain
equilibrium

- let us keep only k_{ab} (oscillator) and consider a
lattice in the
limit $l \rightarrow 0$



only nearest neighbors interact

- we can replace the label "a" by a position vector $\vec{x}$

so $q(t, \vec{x})$ instead of $q_a(t)$

tradition $\rightarrow \varphi(t, \vec{x}) \leftarrow$ this is called a field

$$\sum_c \frac{1}{2} m_c \dot{q}_c^2 \rightarrow \int d^3x \frac{1}{2} \sigma \left(\frac{\partial \varphi}{\partial t} \right)^2 \quad \left(\sum_c \rightarrow \frac{1}{L^3} \int d^3x \right. \\ \left. \frac{m_c}{L^3} \rightarrow \sigma \right) \\ m_c = m$$

$$V = \sum_{ab} \frac{1}{2} k_{ab} q_a q_b + \dots$$

keep only nearest neighbors

$$2 q_a q_b = (q_a - q_b)^2 - q_a^2 - q_b^2$$

$$\sum_{ab} \frac{1}{2} k_{ab} q_a q_b$$

$\downarrow$

single sum

$\downarrow$

$$\frac{1}{L^3} d^3x$$

$$S[q] \rightarrow S[\varphi] \equiv \int_0^T dt \int d^3x \mathcal{L}(\varphi) = \int_0^T dt \int d^3x \frac{1}{2} \left\{ \sigma \left(\frac{\partial \varphi}{\partial t} \right)^2 - \sigma (\nabla \varphi)^2 + \right. \\ \left. - \tau \varphi^2 - g \varphi^3 + \dots \right\}$$

the Lorentz invariance requires

that $\sigma = g$

In general we will consider scalar field theories described by:

$$S[\varphi] = \int d^4x \left[\frac{1}{2} \underbrace{(\partial \varphi)^2}_{\partial_\mu \varphi \partial^\mu \varphi} - \frac{1}{2} m^2 \varphi^2 - \frac{g}{3!} \varphi^3 - \frac{\lambda}{4!} \varphi^4 + \dots \right]$$

$$\rightarrow \left(\frac{\partial}{\partial x^\mu} \varphi \right) \left(\frac{\partial}{\partial x_\mu} \varphi \right)$$

Symmetry:

Lorentz invariance + of most two powers of $\frac{\partial}{\partial t}$



$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi)$$

polynomial if

expansion is allowed

$$\delta S[\varphi] = 0 \Rightarrow \text{Euler-Lagrange equations} \\ (\text{equations of motion})$$

$$\begin{aligned} \delta S[\varphi] &= \int_M d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi(x) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta (\partial_\mu \varphi(x)) \right] = \\ &= \int_M d^4x \delta \varphi(x) \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \right) \right] + \underbrace{\int_{\partial M} d^3x n_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \delta \varphi(x)}_{\partial M} \end{aligned}$$

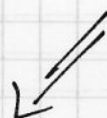
$$\delta [\partial_\mu \varphi(x)] = \partial_\mu \delta \varphi(x)$$

we will assume

either $\delta \varphi(x) = 0$ for $x \in \partial M$

or $\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} \rightarrow 0$ if $\partial M \rightarrow \infty$

(matter localized at finite distance)



$$\boxed{\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} = 0}$$

For $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4$

symmetry
($\varphi \rightarrow -\varphi$)
imposed

E-L equation is $\partial_\mu \partial^\mu \varphi + m^2 \varphi + \frac{\lambda}{6} \varphi^3 = 0$

$$(\square + m^2) \varphi + \frac{\lambda}{6} \varphi^3 = 0$$

interaction term

if 0 then the Klein-Gordon equation

Nature: pions, kaons, the Higgs boson

Complex scalar fields

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi - V(\varphi^* \varphi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} = 0 \quad \Rightarrow \quad \begin{aligned} \partial^\mu \partial_\mu \varphi + m^2 \varphi &= 0 && \leftarrow \delta \varphi^* \\ \partial^\mu \partial_\mu \varphi^* + m^2 \varphi^* &= 0 && \leftarrow \delta \varphi \end{aligned}$$

$\mathcal{L}$ is invariant under a global transformation

$$\varphi(x) \rightarrow \varphi'(x) = e^{i\alpha} \varphi(x)$$

$$\varphi^*(x) \rightarrow \varphi'^*(x) = e^{-i\alpha} \varphi^*(x)$$

Electrodynamics

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

E-L equations: $\partial_\mu F^{\mu\nu} = 0$

$$\downarrow$$
$$\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

the Bianchi identity

gauge invariance:

$F_{\mu\nu}$ is invariant under

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x)$$

$$\downarrow$$
$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = F_{\mu\nu}$$

Scalar electrodynamics

$$\mathcal{L} = \underbrace{\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi}_{\text{vary under local transformations}} - \underbrace{\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{invariant under local transformations}}$$

vary under local transformations

$$\varphi(x) \rightarrow e^{i\alpha(x)} \varphi(x)$$

invariant under local transformations

$$\hookrightarrow \partial^\mu \varphi \Rightarrow (\partial^\mu - ieA^\mu) \varphi \equiv D^\mu \varphi$$

then $\varphi(x) \rightarrow \varphi'(x) = e^{i\alpha(x)} \varphi$

$$D^\mu \varphi(x) \rightarrow (D^\mu \varphi(x))' = e^{i\alpha(x)} D^\mu \varphi(x)$$

Yang-Mills theory (non-Abelian generalization of electrodynamics)

- global non-Abelian symmetries

$$\varphi_i(x) \rightarrow \varphi'_i(x) = \omega_{ij} \varphi_j(x)$$

$$\omega \in \text{SU}(N)$$

$$\varphi_i^* \varphi_i \rightarrow \varphi_j^* \underbrace{\omega_{ij}^* \omega_{ik}}_{(\omega^\dagger)_{jk} = \delta_{jk}} \varphi_k = \varphi_j^* \varphi_j$$

$$\det \omega = 1 \quad \omega^\dagger \omega = 11$$

$$\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix}$$

$$\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi - \lambda (\varphi^\dagger \varphi)^2$$

- let us generalize from global to local non-Abelian symmetries:

$$\varphi_i(x) \rightarrow \varphi'_i(x) = \omega_{ij}(x) \varphi_j(x)$$

$$\text{example } \omega(x) \in \text{SU}(2)$$

$$\text{but } \partial^\mu \varphi'_i(x) = \underbrace{(\partial^\mu \omega_{ij}) \varphi_j(x)}_{\text{the problem}} + \omega_{ij}(x) \partial^\mu \varphi_j(x) \quad (*)$$

following E-D let us introduce a covariant derivative such that

$$(D^\mu \varphi)' = \omega D^\mu \varphi$$

from (*) it is clear that it can be achieved if

$$D^\mu \varphi = (\partial^\mu + \underbrace{A^\mu}_{\text{matrix acting on } \varphi}) \varphi$$

$$(D^\mu \varphi)' = \partial^\mu \varphi' + A'^\mu \varphi' = (\partial^\mu \omega) \varphi + \omega \partial^\mu \varphi + A'^\mu \omega \varphi = \omega \partial^\mu \varphi + \omega A^\mu \varphi$$

$$A'^\mu = -(\partial^\mu \omega) \omega^\dagger + \omega A^\mu \omega^\dagger \text{ for any } \varphi$$

$$A^\mu(x) \rightarrow A'^\mu(x) = \omega A^\mu \omega^\dagger + \omega \partial^\mu \omega^\dagger \leftarrow \text{non-Abelian}$$

$$A_\mu \rightarrow A'_\mu = \omega A_\mu \omega^{-1} + \omega \partial_\mu \omega^{-1}$$

let us consider an infinitesimal transformation

$$\omega(x) = 1 + \epsilon(x)$$

$$\omega^{-1}(x) = 1 - \epsilon(x)$$

$\epsilon \in$ Lie algebra of $SU(2)$

anti-Hermitian 2×2 traceless matrix

$$\left\{ \begin{array}{l} \text{i) } \omega \partial_\mu \omega^{-1} = \\ \quad = (1 + \epsilon) \partial_\mu \epsilon = -\partial_\mu \epsilon + \mathcal{O}(\epsilon^2) \end{array} \right. \quad \text{so } \omega \partial_\mu \omega^{-1} \in \mathfrak{su}(2)$$

ii) assume that

$$A_\mu \in \mathfrak{su}(2), \text{ then } \omega A_\mu \omega^{-1} = (1 + \epsilon) A_\mu (1 - \epsilon) =$$

$$= A_\mu + [\epsilon, A_\mu] + \mathcal{O}(\epsilon^2) \in \mathfrak{su}(2)$$

- now we should (may want) to introduce a kinetic term for the vector field A_μ in analogy to $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ in E-D

- for global transformations we have

$$A_\mu \rightarrow A'_\mu = \omega A_\mu \omega^{-1}$$

- we expect that $F_{\mu\nu}$ should contain $\partial_\mu A_\nu - \partial_\nu A_\mu$
global transformations
↓

$$\omega (\partial_\mu A_\nu - \partial_\nu A_\mu) \omega^{-1}$$

so now $F_{\mu\nu}$ is not going to be invariant or in ED

- let us try

$$F_{\mu\nu}(x) \rightarrow F'_{\mu\nu}(x) = \omega(x) F_{\mu\nu}(x) \omega^{-1}(x)$$

local transformation

↓

$\text{Tr}(F_{\mu\nu} F^{\mu\nu})$ is invariant

- the right form of $F_{\mu\nu}$ is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad \mathcal{L} = +\frac{1}{2g^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu})$$

- for $SU(2)$

$$A_\mu(x) = -ig \frac{\tau^a}{2} A_\mu^a(x)$$

↓

$$F_{\mu\nu}(x) = -ig \frac{\tau^a}{2} F_{\mu\nu}^a(x) =$$

↓

$$F_{\mu\nu}(x) = -ig \frac{\tau^a}{2} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g \epsilon^{abc} A_\mu^b A_\nu^c \right)$$

$$\mathcal{L} = +\frac{1}{2g^2} \text{Tr}(F^2) =$$

$$\left[\frac{\tau^a}{2}, \frac{\tau^b}{2} \right] = i \epsilon^{abc} \frac{\tau^c}{2}$$

$$F_{\mu\nu}^a(x)$$

$$= \frac{1}{2g^2} (ig)^2 F_{\mu\nu}^a F^{\mu\nu b} \underbrace{\text{Tr} \left(\frac{\tau^a}{2} \frac{\tau^b}{2} \right)}_{\frac{1}{2} \delta^{ab}}$$

$$= -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}$$

A gauge invariant scalar - vector theory is described by the following Lagrangian

$$\mathcal{L} = (D_\mu \psi)^\dagger (D^\mu \psi) - V(\psi^\dagger \psi) + \frac{1}{2g^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu})$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$D_\mu \psi \equiv \left(\partial_\mu - ig \frac{T^a}{2} A_\mu^a \right) \psi$$

$$F_{\mu\nu} \equiv -ig \frac{T^a}{2} F_{\mu\nu}^a$$

Fermions

Following Dirac, we are looking for a ~~first order~~ differential equation linear in derivatives, such that its solutions satisfy the Klein-Gordon equation

$$(\Box^2 + m^2)\psi = 0$$

since the goal was to construct a relativistic covariant generalization of the Schrödinger equation $i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$ which is linear in $\frac{\partial}{\partial t}$

$$(-p^2 + m^2)\psi = 0$$

$$(\gamma^\mu \partial_\mu + m) \psi = 0 \quad \leftarrow \text{postulate}$$

↑ undefined coefficients

$$(\gamma^\mu \partial_\mu + m)(\gamma^\nu \partial_\nu - m) \psi = \left[\frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \partial_\mu \partial_\nu - m^2 \right] \psi = 0$$

in order to have the K-G equation reproduced we require

$$\{ \gamma^\mu, \gamma^\nu \} = 2 \eta^{\mu\nu} \quad \leftarrow \text{the commutation relations for the Clifford algebra}$$

1. γ^μ - matrices

2. ψ must have several components



One can prove that for a space-time of dimension D the minimal dimension of matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ equals $2^{\lfloor \frac{D}{2} \rfloor}$, where $\lfloor x \rfloor$ denotes the integer part of x
 $(\lfloor \frac{4}{2} \rfloor = 2, \lfloor \frac{5}{2} \rfloor = 2)$

$\Downarrow$
 γ^μ are 4×4 matrices in $D=4$ dim

Hermiticity of the Hamiltonian requires $\gamma_0^\dagger = \gamma_0$, $\gamma_i^\dagger = \beta \gamma_i \beta^{-1}$ defined such that $\gamma^0 = \beta$, $\gamma^i = \beta \alpha_i$

$$\Downarrow \quad \gamma^{0\dagger} = \gamma^0 \quad \Downarrow \quad \gamma^{i\dagger} = -\gamma^i$$

'standard representation' of Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \quad \begin{matrix} \sigma^\mu = (1, \sigma^i) \\ \tilde{\sigma}^\mu = (1, -\sigma^i) \end{matrix}$$

$\uparrow$ Pauli matrices

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

$$\delta S = 0 \Rightarrow S = \int d^4x \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi \Rightarrow (\gamma^\mu \partial_\mu - m) \psi = 0$$

where $\bar{\psi} \equiv \psi^\dagger \gamma_0$

— invariance under $\psi(x) \rightarrow U \psi(x)$ where that $U^\dagger U = 1$ global

— to generalize for local transformations one needs to replace ∂_μ by $\partial_\mu + A_\mu$, then $(i \gamma^\mu \partial_\mu - m) \psi = 0$ in the Dirac eq.

for $SU(2)$

$$\left\{ i (\gamma^\mu)_{\alpha\beta} \left[\delta_{ij} \partial_\mu - i \frac{g}{2} (\tau^a)_{ij} A_\mu^a \right] - m \delta_{\alpha\beta} \delta_{ij} \right\} \psi_{\beta j}(x) = 0$$

β, j Lorentz index $\in SU(2)$

The Noether's theorem

$$\delta S[\varphi] = 0 \quad \Rightarrow \quad \text{E-L equations} \quad \frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} = 0$$

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

Let us consider global transformations of fields only
(no transformation of space-time coordinates)

$$\phi^i(x) \rightarrow \phi'^i(x) = (\delta^{ij} + \epsilon^a t_a^{ij}) \phi^j \quad (*)$$

$\uparrow$ infinitesimal parameters
(independent of x^μ)

Assume that the Lagrangian density is invariant under $(*)$:

$$\delta \mathcal{L} = \mathcal{L}(\phi + \delta \phi, \partial_\mu \phi + \delta \partial_\mu \phi) - \mathcal{L}(\phi, \partial_\mu \phi) = 0$$

$$\delta \phi = \epsilon^a t_a^{ij} \phi^j \quad \delta \partial_\mu \phi = \epsilon^a t_a^{ij} \partial_\mu \phi^j \quad (= \partial_\mu \delta \phi)$$

$$\rightarrow \frac{\partial \mathcal{L}}{\partial \phi^i} \epsilon^a t_a^{ij} \phi^j + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} \epsilon^a t_a^{ij} \partial_\mu \phi^j = 0$$

all ϵ^a are independent, so

$$\frac{\partial \mathcal{L}}{\partial \phi^i} t_a^{ij} \phi^j + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} t_a^{ij} \partial_\mu \phi^j = 0$$

Only E-L equation we get

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} t_a^{ij} \phi^j \right] = 0 \quad \Rightarrow \quad \partial_\mu j^{a\mu} = 0$$

$$\text{for } j^{a\mu} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^i)} t_a^{ij} \phi^j$$

(only for fields that satisfy the E-L equation)

$$Q \equiv \int d^3x j^{a0} \text{ is time independent} \quad \frac{dQ}{dt} = 0$$

$$Q = \int d^3x \partial_0 \phi^i \partial_1 \phi^i \quad \partial_5 \phi^i \rightarrow 0$$

Example: $\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - V(\varphi^* \varphi)$

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let us consider φ and φ^* as independent fields:

$$\Phi^i = (\varphi, \varphi^*)^T \quad i=1,2$$

$\mathcal{L}$ is invariant under $\varphi \rightarrow \varphi' = e^{i\alpha} \varphi$
 $\varphi^* \rightarrow \varphi'^* = e^{-i\alpha} \varphi^*$

for infinitesimal transformation $\varphi' = (1+i\alpha)\varphi$
 $\varphi'^* = (1-i\alpha)\varphi^*$

$$\Phi = \begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix} \rightarrow \left[1 + i \underbrace{\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}}_{\alpha \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}} \right] \begin{pmatrix} \varphi \\ \varphi^* \end{pmatrix}$$

$$\alpha \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$E' = \alpha \quad t_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$e=1$$

$\downarrow$

$$j^\mu = \partial^\mu \varphi^* i \varphi - \partial^\mu \varphi i \varphi^* = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} t_e^{ij} \Phi_j$$

$$= i[(\partial^\mu \varphi^*) \varphi - (\partial^\mu \varphi) \varphi^*]$$

it is easy to see that

$$\partial_\mu j^\mu = 0 \quad \text{for fields satisfying the Klein-Gordon eq.}$$

$$(\square + m^2) \varphi = 0$$

Now let us consider space-time translations

$$\Phi^i(x^\mu) \rightarrow \Phi'^i(x^\mu) = \Phi^i(x^\mu + \epsilon^\mu) = \Phi^i(x^\mu) + \partial_\mu \Phi^i(x^\mu) \epsilon^\mu + \dots$$

let us assume that the Lagrangian doesn't depend explicitly on x , then

$$\mathcal{L}(\Phi^i, \partial_\mu \Phi^i) = \mathcal{L}(x^\mu + \epsilon^\mu)$$

$$\mathcal{L}(\Phi, \partial_\mu \Phi) + \underbrace{\frac{\partial \mathcal{L}}{\partial \Phi^i}}_{\downarrow} \partial_\mu \Phi^i \epsilon^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi^i)} \partial_\nu \partial_\mu \Phi^i \epsilon^\mu = \mathcal{L}(x) + \underbrace{\partial_\mu \mathcal{L} \epsilon^\mu}_{\partial_\nu \delta^\nu_\mu \mathcal{L} \epsilon^\mu}$$

$$\partial_\nu \delta^\nu_\mu \mathcal{L} \epsilon^\mu$$

$$\partial_\nu \delta^\nu_\mu \mathcal{L} \epsilon^\mu$$

$$\partial_\nu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi^i)} \partial_\mu \phi^i - \delta^\nu_\mu \mathcal{L} \right] \varepsilon^\mu = 0$$

ε^μ independent parameter, so

$$\partial_\nu T^\nu_\mu = 0$$

for $T^\nu_\mu \overset{\text{flower}}{=} \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi^i)} \partial_\mu \phi^i - \delta^\nu_\mu \mathcal{L}$

$$E = \int d^3x T^{00} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^i} \dot{\phi}^i - \mathcal{L} \right)$$

$$p^i = \int d^3x T^{0i} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^i} \partial^i \phi^i \right)$$

Comment: - $\bar{j}^\mu_a \equiv j^\mu_a + \partial^\nu f^\mu_{\nu a}$ $f^\mu_{\nu a} = -f^\mu_{a\nu}$

$$\Downarrow$$

$$\partial^\mu \bar{j}^\mu_a = 0$$

$$- \bar{T}^\nu_\mu = T^\nu_\mu + \partial_\lambda \Omega^{\nu\lambda}_\mu$$

$$\Omega^{\nu\lambda}_\mu = -\Omega^{\lambda\nu}_\mu \rightarrow \partial_\nu \bar{T}^\nu_\mu = 0$$

- $T^{\mu\nu} \neq T^{\nu\mu}$ but one can construct symmetric tensor $\bar{T}^{\mu\nu}$ which is also conserved

$$\partial_\mu j^\mu = 0$$

$$\bar{j}^\mu = j^\mu + \partial_\sigma f^{\sigma\mu}$$

$$f^{\sigma\mu} = -f^{\mu\sigma}$$

$$\bar{Q} := \int_V d^3x \bar{j}^0$$

$$\frac{d\bar{Q}}{dt} = \int_V d^3x \partial_t \bar{j}^0 =$$

$$= \int d^3x (\partial_t j^0 + \partial_t \partial_\sigma f^{\sigma 0}) =$$

$$= \frac{dQ}{dt} + \int d^3x (\partial_t \partial_t f^{\sigma\sigma} + \partial_i \partial_t f^{i0}) =$$

$$= \frac{dQ}{dt} + \underbrace{\int_V \nabla \cdot \vec{F} d^3x}_0$$

$$\text{for } F^i \equiv \partial_t f^{i0}$$

The same works for the energy-momentum tensor

So we must satisfy two conditions:

$$\psi^{\sigma\mu\nu} + \psi^{\mu\sigma\nu} = -\theta^{\sigma\mu\nu} - \theta^{\mu\sigma\nu} \quad (*)$$

$$\psi^{\sigma\nu\mu} = \psi^{\sigma\mu\nu} \quad (**)$$

Let's guess ψ as $\psi^{\sigma\nu\mu} = -\theta^{\nu\sigma\mu} - \theta^{\mu\sigma\nu}$
and check $(*)$

$$\begin{aligned} \text{L.H.S.} &= \underbrace{-\theta^{\mu\sigma\nu} - \theta^{\nu\sigma\mu}}_{\psi^{\sigma\mu\nu}} - \underbrace{\theta^{\sigma\mu\nu} - \theta^{\mu\sigma\nu}}_{\psi^{\mu\sigma\nu}} = \\ &= -\theta^{\sigma\mu\nu} - \theta^{\mu\sigma\nu} - \underbrace{(\theta^{\nu\sigma\mu} + \theta^{\mu\sigma\nu})}_{\substack{0 \\ \text{since } \theta^{\sigma\nu\mu} = -\theta^{\sigma\mu\nu}}} = \text{R.H.S.} \end{aligned}$$

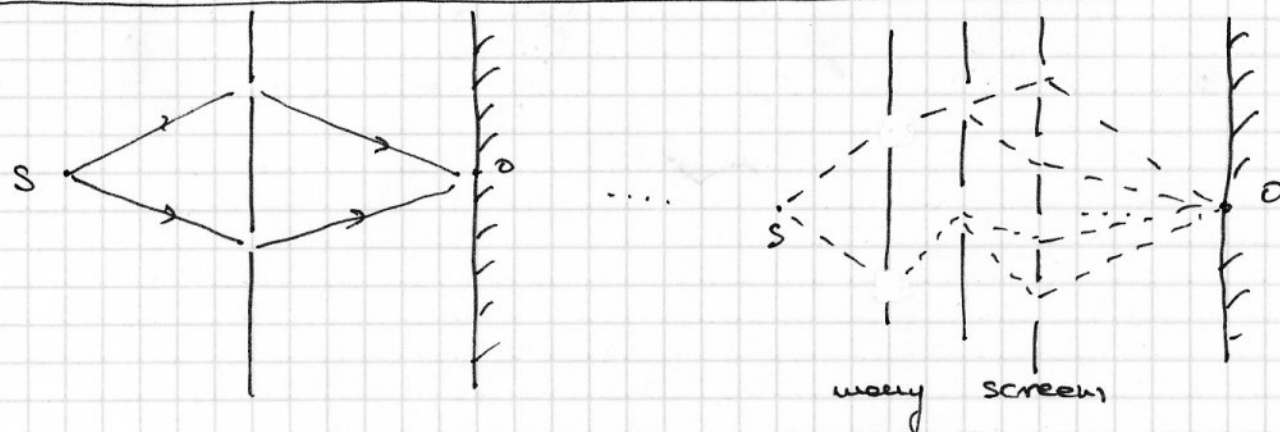
Obviously $(**)$ is satisfied. $\square$

$$\Omega^{\sigma\nu\mu} = \theta^{\sigma\nu\mu} - \theta^{\nu\sigma\mu} - \theta^{\mu\sigma\nu}, \quad \theta^{\sigma\nu\mu} := \frac{1}{2} (x^\mu T^{\sigma\nu} - x^\nu T^{\sigma\mu})$$

$$\bar{T}^{\mu\nu} = T^{\mu\nu} + \partial_\sigma \Omega^{\sigma\mu\nu} \quad \text{is symmetric}$$

$$\bar{T}^{\mu\nu} = \bar{T}^{\nu\mu}$$

Path Integral Formulation of Quantum Physics



$$A(\text{detection at } O) = \sum_{S \rightarrow O} A(S \rightarrow A_i \rightarrow O) \quad \text{the superposition principle}$$

- of the way and ^{all possible paths} we remove all the screens as they after drilling infinitely many holes are not really there

$$A(S \rightarrow O) = \sum_{\text{all paths}} A(S \xrightarrow{\text{path}} O)$$

Quantum mechanics

$$\rightarrow i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle_S = \hat{H}(\vec{p}, \vec{q}) |\psi(t)\rangle_S \Rightarrow |\psi(t)\rangle_S = e^{-\frac{i}{\hbar} \hat{H} t} |\psi(0)\rangle_S$$

Schrödinger picture (states evolve)

in the Heisenberg picture the operators are time dependent, so for the position operator

$$-i\hbar \dot{\hat{q}}_H = [\hat{H}, \hat{q}_H]$$

$$\text{and } \hat{q}_H(t) = e^{i\hat{H}t/\hbar} \hat{q}_S e^{-i\hat{H}t/\hbar}$$

The time-dependent operator $\hat{q}_H(t)$ has a complete set of eigenstates:

$$\hat{q}_H(t) |q, t\rangle = q, |q, t\rangle$$

$$|q, t\rangle_H = e^{i\hat{H}t/\hbar} |q\rangle; \hat{q}_s |q\rangle = q |q\rangle$$

- we will be interested in the following transition amplitude

$$\langle q_F, t_F | q_I, t_I \rangle_H = \langle q_F | e^{-i\hat{H}(t_F - t_I)/\hbar} | q_I \rangle =$$

"Feynman Kernel"

$$= \langle q_F | e^{-i\hat{H}T/\hbar} | q_I \rangle$$

- following Dirac we divide the time T into N segments $\delta t = \frac{T}{N}$, then (for $\hbar=1$)

$$\langle q_f | e^{-i\hat{H}T} | q_i \rangle = \langle q_f | e^{-i\hat{H}\delta t} \dots e^{-i\hat{H}\delta t} | q_i \rangle$$

$$\int dq |q\rangle \langle q| = 1$$

$$= \prod_{j=1}^{N-1} \int dq_j \langle q_f | e^{-i\hat{H}\delta t} | q_{N-1} \rangle \langle q_{N-1} | e^{-i\hat{H}\delta t} | q_{N-2} \rangle \dots \langle q_1 | e^{-i\hat{H}\delta t} | q_i \rangle$$

let us consider factor $\langle q_{j+1} | e^{-i\hat{H}\delta t} | q_j \rangle$
and assume free-particle Hamiltonian $H = \frac{\hat{p}^2}{2m}$

$$\hat{p} |p\rangle = p |p\rangle$$

- insert a complete set of states $1 = \int \frac{dp}{2\pi} |p\rangle \langle p|$

$$\langle q_{j+1} | e^{-i\hat{H}\delta t} | q_j \rangle = \langle q_{j+1} | e^{-i\frac{\hat{p}^2}{2m}\delta t} | q_j \rangle = \int \frac{dp}{2\pi} \langle q_{j+1} | e^{-i\frac{p^2}{2m}\delta t} | p \rangle \langle p | q_j \rangle$$

$$= \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}\delta t} \langle q_{j+1} | p \rangle \langle p | q_j \rangle = \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}\delta t} e^{ip(q_{j+1} - q_j)}$$

Gaussian integral

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2 + Jx} = \left(\frac{2\pi}{a}\right)^{1/2} e^{J^2/2a}$$

Then

$$\langle q_{j+1} | e^{-iH\delta t} | q_j \rangle = \frac{1}{2\pi} \left(\frac{2\pi m}{i\delta t} \right)^{1/2} e^{\frac{m}{2i\delta t} (q_{j+1} - q_j)^2} =$$

$$= \frac{i\delta t}{m} \quad \mathcal{J} = i(q_{j+1} - q_j)$$

$$= \left(\frac{-im}{2\pi\delta t} \right)^{1/2} e^{i\delta t \frac{m}{2} \left(\frac{q_{j+1} - q_j}{\delta t} \right)^2}$$

Putting all those factors together one gets

$$\langle q_f | e^{-iHT} | q_i \rangle = \left(\frac{-im}{2\pi\delta t} \right)^{N/2} \prod_{j=0}^{N-1} \int dq_j e^{i\delta t \frac{m}{2} \sum_{j=0}^{N-1} (q_{j+1} - q_j)^2 / \delta t^2}$$

with $q_0 \equiv q_i$, $q_N \equiv q_f$

Go to the limit $\delta t \rightarrow 0$:

$$\left(\frac{q_{j+1} - q_j}{\delta t} \right)^2 \rightarrow \dot{q}^2$$

$$\delta t \sum_{j=0}^{N-1} \rightarrow \int_0^T dt$$

Let's define the path integral

$$\int Dq(t) = \lim_{N \rightarrow \infty} \left(\frac{-im}{2\pi\delta t} \right)^{N/2} \prod_{j=0}^{N-1} \int dq_j$$

Then

$$\langle q_f | e^{-iHT} | q_i \rangle = \int Dq(t) e^{i \int_0^T dt \frac{1}{2} m \dot{q}^2}$$

The amplitude for each path recall the superposition principle

For $H = \frac{\hat{p}^2}{2m} + V(q)$

$$\langle q_f | e^{-iHT} | q_i \rangle = \int Dq(t) e^{i \int_0^T \underbrace{\left[\frac{1}{2} m \dot{q}^2 - V(q) \right]}_L dt}$$

$S[q(t)]$

For $Dq(t)$ to emerge it is crucial that $[p, q] \neq 0$.

In the field theory we will often be interested

in transition amplitudes between vacuum states

$$Z \equiv \langle 0 | e^{-iHT} | 0 \rangle = \int Dq(t) e^{i S[q(t)]}$$

vacuum $\equiv$ ground state $\equiv$ state of the lowest energy

$\langle 0 | \dots | 0 \rangle$ transitions are not the most interesting, we will consider next vacuum - vacuum transitions in the presence of some external disturbance (like an external force in mechanics: $J_e(t) q_e(t)$ extra in the Lagrangian):

$$Z = \int D\varphi(x) e^{i \int d^4x \left[\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - v(\varphi) + J(x) \varphi(x) \right]}$$

classical mechanics	field theory
$q_e(t)$	$\varphi(\vec{x}) = \varphi(x)$
e	$\vec{x}$
$\sum_e$	$\int d^D x$

let us consider the free scalar field theory

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$$

$$Z = \int D\varphi e^{i \int d^4x \left[-\frac{1}{2} \varphi (\partial^2 + m^2) \varphi + J \varphi \right]}$$

$\varphi \rightarrow 0$
 $x \rightarrow \infty$, therefore the boundary terms are neglected

let us imagine lattice (discretizing) space-time:
 — lattice spacing

$$\varphi(x) \rightarrow \varphi_i = \varphi(i\varepsilon)$$

$$\partial \varphi \rightarrow \frac{1}{\varepsilon} (\varphi_{i+\varepsilon} - \varphi_i) \equiv \sum_j \Pi_{ij} \varphi_j$$

$$\int d^4x J(x) \varphi(x) \rightarrow \varepsilon^4 \sum_j J_j \varphi_j$$

The integrals to perform:

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} dq_1 \dots dq_N e^{\frac{i}{2} q \cdot A \cdot q + i J q} = \left[\frac{(2\pi i)^N}{\det A} \right]^{\frac{1}{2}} e^{-\frac{i}{2} J \cdot A^{-1} \cdot J}$$

$$A A^{-1} = 11$$

$$A \leftrightarrow -(\partial^2 + m^2)$$

$$-(\partial^2 + m^2) D(x-y) = \delta^{(4)}(x-y)$$

$$-\frac{i}{2} \int d^4x d^4y J(x) D(x-y) J(y)$$

$$Z[J] = C e$$

$$e^{iW[J]} \equiv C e$$

$$iW[J]$$

$$Z[J] = Z[0] e$$

$$W[J] = -\frac{i}{2} \int d^4x d^4y J(x) D(x-y) J(y)$$

In order to have convergent integrals: $m^2 \rightarrow m^2 - i\varepsilon$
 then the factor $e^{-\frac{\varepsilon}{2} \int d^4x \varphi^2}$ ($\frac{\varepsilon}{2} \equiv \varepsilon$)

suppresses the integral if φ is large

$$\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)}$$

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} D(k) e^{ik(x-y)}$$

$$-(\partial^2 + m^2) D(x-y) = \delta^{(4)}(x-y)$$

$$-\int \frac{d^4k}{(2\pi)^4} D(k) (-k^2 + m^2 - i\varepsilon) e^{ik(x-y)} = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)}$$

$$D(k) = \frac{1}{k^2 - m^2 + i\varepsilon}$$

$$W[\mathcal{J}] = -\frac{1}{2} \int d^4x d^4y \mathcal{J}(x) D(x-y) \mathcal{J}(y)$$

$$\int d^4x e^{-ikx} \mathcal{J}(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \mathcal{J}(k)$$

$$\int d^4x e^{-ikx} \mathcal{J}(x) = \int \frac{d^4k}{(2\pi)^4} \int d^4x e^{i(k-k')x} \mathcal{J}(k') =$$

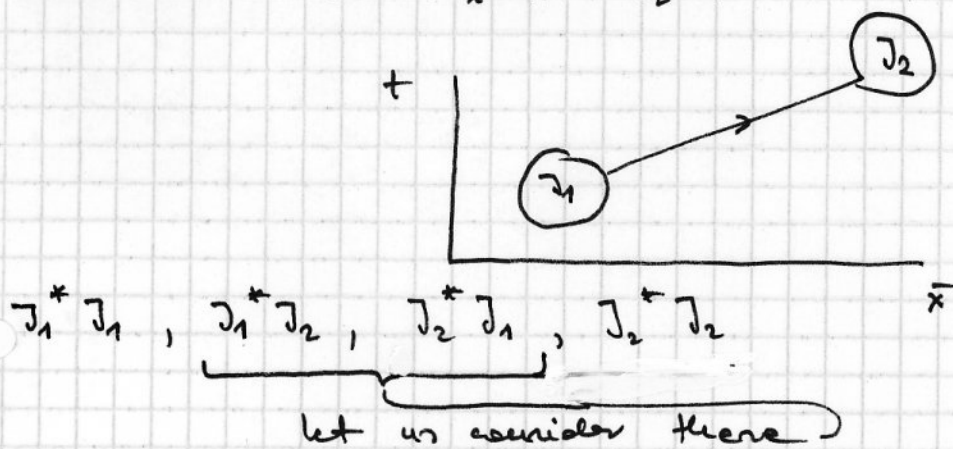
$$W[\mathcal{J}] = -\frac{1}{2} \int d^4x d^4y \int \frac{d^4k}{(2\pi)^4} e^{ikx} \mathcal{J}(k) \int \frac{d^4l}{(2\pi)^4} e^{il(x-y)} D(l) \int \frac{d^4p}{(2\pi)^4} e^{ip y} \mathcal{J}(p) =$$

$$= -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} d^4l d^4p \delta^{(4)}(k+l) \delta^{(4)}(p-l) \mathcal{J}(k) D(l) \mathcal{J}(p) =$$

$$= -\frac{1}{2} \int \frac{d^4l}{(2\pi)^4} \mathcal{J}(-l) D(l) \mathcal{J}(l) = -\frac{1}{2} \int \frac{d^4l}{(2\pi)^4} \mathcal{J}^*(l) D(l) \mathcal{J}(l)$$

since $\mathcal{J}(x)$ is real we have $\mathcal{J}(-l) = \mathcal{J}^*(l)$

Consider $\mathcal{J}(x) = \mathcal{J}_1(x) + \mathcal{J}_2(x)$



$$W[\mathcal{J}] = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \mathcal{J}_2^*(k) \frac{1}{k^2 - m^2 + i\epsilon} \mathcal{J}_1(k)$$

the main contribution

to the integral

comes from $k^2 \approx m^2$

so from an exchange of a real (on-shell) particle, which was

created at e.g. x_1 the region where $\mathcal{J}_1(x)$ is concentrated and annihilated at x_2 where $\mathcal{J}_2(x)$ is located

We choose $J_a(x) = \delta^{(3)}(\vec{x} - \vec{x}_a)$ $a = 1, 2$

since we are interested in interaction between the two sources we neglect $J_a^* J_a$ terms

$$W[J] = - \int d^4x^0 dy^0 \int \frac{d^3k}{(2\pi)^3} e^{ik^0(x^0 - y^0)} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}_1 - \vec{x}_2)}}{k^2 - m^2 + i\epsilon}$$

$J_2^* J_1 + J_1^* J_2$

$$J_2^*(x) J_1(x) = \int d^4x e^{ikx} \delta^{(3)}(\vec{x} - \vec{x}_2) \int d^4y e^{-iky} \delta^{(3)}(\vec{y} - \vec{x}_1) p(k) =$$

$$\int d^4x^0 dy^0 e^{ik^0(x^0 - y^0)} e^{-i\vec{k}(\vec{x}_2 - \vec{x}_1)}$$

$$\int dy^0 e^{-ik^0 y^0} = 2\pi \delta(k^0)$$

$$W[J] = + \left(\int d^4x^0 \right) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + m^2}$$

$i\epsilon$ not needed
since $\vec{k}^2 + m^2 \neq 0$

Remember that

$$Z[J] = c e^{iW[J]} = \langle 0 | e^{-iHT} | 0 \rangle =$$

see Weinberg
vol. II
sec. 10.3

$= e^{-iET}$ where E is the vacuum energy due to the presence of the two sources acting on each other

$$-iE \cancel{T} = iW[J] = i \left(\int d^4x^0 \right) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + m^2}$$

$$E = - \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + m^2}$$

E is the potential energy between two static sources

$$E = - \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{k^2 + m^2}$$

oscillations cut off the integral for large $|\vec{x}_1 - \vec{x}_2|$

since $\frac{1}{m}$ is the only scale in the problem we expect E to be strongly suppressed for $|\vec{x}_1 - \vec{x}_2| \gg \frac{1}{m}$

m determines the range of the attractive force

Direct calculation shows that

$$E = - \frac{1}{4\pi r} e^{-mr} \quad \text{for } r = |\vec{x}_1 - \vec{x}_2|$$

$$\frac{dE}{dr} > 0 \Rightarrow \text{attraction!}$$

Conclusion

An exchange of scalar quanta leads to an attractive force.

Electrodynamics : why like charges repel?

Between like objects Coulomb's force is repulsive.

Newtonian gravity : Newton's gravitational force is attractive

Can we understand?

that within a quantum

field theory

let us try.

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In order to simplify argument we will assume that photon has a tiny mass, then for QED

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + A_\mu J^\mu$$

in analogy with $\frac{1}{2} m^2 \phi^2$ in the scalar theory

external source: conserved current: $\partial_\mu J^\mu = 0$

$$Z \equiv e^{iW[J]} = \int \mathcal{D}A_\mu e^{iS[A_\mu]} \quad \text{for}$$

$$S[A_\mu] = \int d^4x \mathcal{L} = \int d^4x \left\{ \frac{1}{2} A_\mu [(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu] A_\nu + A_\mu J^\mu \right\}$$

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) =$$

$$= -\frac{1}{4} (2 \partial_\mu A_\nu \partial^\mu A^\nu - 2 \partial_\mu A_\nu \partial^\nu A^\mu) \rightarrow \text{by parts}$$

$$\rightarrow \frac{1}{2} \underbrace{A_\nu \square}_{\partial^2} A^\nu - \frac{1}{2} A_\nu \partial_\mu \partial^\mu A^\nu = \frac{1}{2} A_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu$$

Find the potential energy we need the photon propagator:

$$[(\partial^2 + m^2) g^{\mu\nu} - \partial^\mu \partial^\nu] D_{\nu\lambda}(x) = \delta^\mu_\lambda \delta^{(4)}(x)$$

in the momentum space we have

$$D_{\nu\lambda}(x) = \int \frac{d^4k}{(2\pi)^4} D_{\nu\lambda}(k) e^{ikx}$$

and therefore

$$[(-k^2 + m^2) g^{\mu\nu} + k^\mu k^\nu] D_{\nu\lambda}(k) = \delta^\mu_\lambda \quad \text{then } (*)$$

$$D_{\nu\lambda}(k) = \frac{-g_{\nu\lambda} + \frac{k_\nu k_\lambda}{m^2}}{k^2 - m^2}$$

inserting into we can see that this is indeed the photon propagator

So, for $W[J]$ we get

$$W[J] = -\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} J^\mu(k)^* \frac{-g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}}{k^2 - m^2 + i\epsilon} J^\nu(k)$$

The current J_μ is conserved, so $\partial^\mu J_\mu = 0$

$$\frac{k_\mu k_\nu}{m^2} \text{ could be dropped } \Leftarrow k^\mu J_\mu(k) = 0$$

$$W[J] = +\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} J^\mu(k)^* \frac{1}{k^2 - m^2 + i\epsilon} J_\mu(k)$$

the spin structure (Lorentz transformation properties) produce the extra

"- " sign!

$$J^\mu = (\partial^0, \vec{0})$$

$$E = \frac{\epsilon^2}{4\pi r} e^{-mr} \quad m \rightarrow 0$$

$$\frac{\epsilon^2}{4\pi r}$$

$$\frac{dE}{dr} < 0$$

The electromagnetic force between like charges is repulsive!

$$J^0(x) = \epsilon \delta^3(x - x_0)$$

↑
charge density

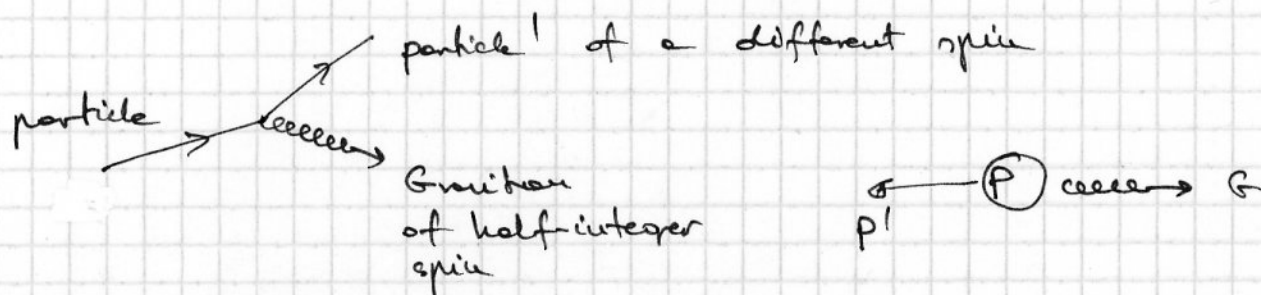
To accommodate positive and negative charges we shall

use $J^\mu = J_+^\mu - J_-^\mu$, then we will see that

for $J_+^0 J_-^0$ the energy is negative $E < 0$, so opposite charges attract each other

Let us try gravity now: the goal is to explain $F = - \frac{G m_1 m_2}{r}$

Feynman: "In order to produce a static force and not just scattering, the emission or absorption of a single graviton by either particle must leave both particles in the same internal state"



$\Downarrow$
gravity can't be mediated by exchange of half-integer graviton

$s=0 \Rightarrow$ attractive, so could ϕ mediate gravity?

assume that gravity couples to the energy-momentum tensor $T_{\mu\nu}$, then we would get

ϕT_{μ}^{μ} as the only Lorentz invariant possibility

however for QED we have

$$T_{\mu}^{\mu} = 0 \quad \Leftarrow \quad T_{\mu\nu} = - F_{\mu\lambda} F_{\nu}^{\lambda} + \frac{1}{4} \eta_{\mu\nu} F_{\lambda\delta} F^{\lambda\delta}$$

$$\sum^{\mu\nu} T_{\mu\nu} = - F_{\mu\lambda}^{\mu\lambda} + F_{\lambda\mu}^{\lambda\mu} = 0$$

$\Downarrow$

for ϕT_{μ}^{μ} there would be no gravity-light interactions, but bending of light is observed!
in vicinity of sun

Conclusion As electromagnetic interactions used $s=1$,
the next option for gravity is $s=2$

- to calculate the potential energy we need the graviton propagator!

- let us redetermine the propagator for a massive vector field

$$D_{\nu\lambda}(k) = \frac{-G_{\nu\lambda}}{k^2 - m^2} \quad G_{\nu\lambda} \equiv g_{\nu\lambda} - \frac{k_\nu k_\lambda}{m^2}$$

Let us understand the physics behind $G_{\nu\lambda}$.

A massive spin 1 particle has 3 polarization vectors:

$$k^\lambda = (m, 0, 0, 0)$$

(in the rest frame, the spin vector can point in three different directions)

$$k^\mu \epsilon_\mu^a = 0 \quad a = 1, 2, 3$$

$$\epsilon_\mu^1 = (0, 1, 0, 0)$$

$$\epsilon_\mu^2 = (0, 0, 1, 0)$$

$$\epsilon_\mu^3 = (0, 0, 0, 1)$$

possible choice

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{a=1}^3 \epsilon_\mu^a(k) \left(a(k) e^{-ikx} + a^\dagger(k) e^{ikx} \right)$$

$$\partial^\mu A_\mu = 0$$

consequence of the E-L equations or $m \neq 0$

- the amplitude for a

particle to be created at the source is proportional to ϵ_μ^a

- the same for the absorption $\propto \epsilon_\nu^a$

$$- \sum_a \epsilon_\nu^a \epsilon_\lambda^a \quad \leftarrow \text{sum (coherent) over } a$$

$\Downarrow$

$$\text{we expect that } \sum_a \epsilon_\nu^a \epsilon_\lambda^a \propto G_{\nu\lambda}$$

From the Lorentz covariance:

$$k^\nu \mid \sum_a \epsilon_\nu^a \epsilon_\lambda^a = A g_{\nu\lambda} + B k_\nu k_\lambda$$

$$\hookrightarrow \sum_a \epsilon_\nu^a \epsilon_\lambda^a = -B m^2 \left(g_{\nu\lambda} - \frac{k_\nu k_\lambda}{m^2} \right)$$

choose $\lambda = \nu = 1$

$$\sum_e \epsilon_1^e \epsilon_1^e = 1 = -B\omega^2 (-1 - 0) = B\omega^2$$

$$\sum_e \epsilon_\nu^e \epsilon_\lambda^e = -G_{\nu\lambda} = -\left(g_{\nu\lambda} - \frac{k_\nu k_\lambda}{\omega^2}\right)$$

Having the experience from the massive photon propagator let's consider the graviton propagator.

$$s=2 \Rightarrow 2s+1 = 5 \text{ degrees of freedom}$$

$\Downarrow$

5 polarization tensors $\epsilon_{\mu\nu}^a$ $a=1, \dots, 5$

- $\epsilon_{\mu\nu}^a = \epsilon_{\nu\mu}^a$ (symmetric tensor describes $s=2$)

- $k^\mu \epsilon_{\mu\nu}^a = 0$ (transverse)

- $g^{\mu\nu} \epsilon_{\mu\nu}^a = 0$ (traceless)

$\left. \begin{array}{l} \left[\begin{array}{l} k^\mu \epsilon_{\mu\nu}^a = 0 \\ g^{\mu\nu} \epsilon_{\mu\nu}^a = 0 \end{array} \right] \Leftarrow \text{equations of motion for spin 2, } h_{\mu\nu} \text{-graviton field with } m \neq 0$

degrees of freedom: $10 - 4 - 1 = 5$!

$\uparrow$ symmetric tensor $\left\{ \begin{array}{l} g^{\mu\nu} h_{\mu\nu} = 0 \\ g^{\mu\nu} h_{\mu\nu} = 0 \end{array} \right.$

The numerator of the propagator or (in analogy with $s=1$)

$$\sum_e \epsilon_{\mu\nu}^e(k) \epsilon_{\lambda\tau}^e(k) = G_{\mu\lambda} G_{\nu\tau} + G_{\mu\tau} G_{\nu\lambda} - \frac{2}{3} G_{\mu\nu} G_{\lambda\tau}$$

$\Downarrow$ $\Leftarrow$ homework

$$D_{\mu\nu, \lambda\tau}(k) = \frac{G_{\mu\lambda} G_{\nu\tau} + G_{\mu\tau} G_{\nu\lambda} - \frac{2}{3} G_{\mu\nu} G_{\lambda\tau}}{k^2 - \omega^2}$$

Assume that the gravity (gravitons) couples to $T^{\mu\nu}$

$$h^{\mu\nu} \underbrace{T_{\mu\nu}}_{\text{energy momentum tensor}}$$

Let us find the potential energy:

$$W[J] = -\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} T^{\mu\nu}(k) \star \frac{G_{\mu\nu} G_{\nu\sigma} + G_{\mu\sigma} G_{\nu\lambda} - \frac{2}{3} G_{\mu\nu} G_{\lambda\sigma}}{k^2 - m^2 + i\epsilon} T^{\lambda\sigma}(k)$$

$$\partial_\mu T^{\mu\nu} = 0 \Rightarrow G_{\mu\nu} \rightarrow g_{\mu\nu}$$

$$\hookrightarrow \int d^4 k T^{\mu\nu}(k) = 0$$

$$G_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}$$

For pure energy sources only $T^{00} \neq 0$ then

$$W[J] = -\frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} T^{00}(k) \star \frac{1 + 1 - \frac{2}{3}}{k^2 - m^2} T^{00}(k)$$

$\Downarrow$

since $1 + 1 - \frac{2}{3} > 0$ therefore two masses attract!

For a particle $T^{\alpha\beta} = \frac{p^\alpha p^\beta}{E} \delta^3(\vec{x} - \vec{x}_0)$
 at $\vec{x} = \vec{x}_0$

if the particle is at rest $T^{00} = p^0 \delta^3(\vec{x} - \vec{x}_0) =$
 $= m \delta^3(\vec{x} - \vec{x}_0)$

QED: $S_{\text{QED}} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right)$, spin 1 $\Rightarrow A_\mu$

- Gravity:
- spin 2 $\Rightarrow$ described by symmetric tensor $h_{\mu\nu}$
 - our guess (in analogy to $J^\mu A_\mu$) for the interaction term: $-\lambda h_{\mu\nu} T^{\mu\nu}$
 - kinetic term is however difficult to guess because of too many indices:

$$L = a \partial_\sigma h^{\mu\nu} \partial^\sigma h_{\mu\nu} + b \partial^\nu h^{\mu\sigma} \partial_\sigma h_{\mu\nu} + c \partial_\mu h^\mu{}_\nu \partial^\nu h^\sigma{}_\sigma + \dots$$

For QED we had

$$\delta S_{\text{QED}} = 0 \Rightarrow \partial^\nu F_{\nu\mu} = -J_\mu \quad \left\{ \begin{array}{l} \text{notation} \\ \partial^\nu \partial_\nu A_\mu - \partial^\nu \partial_\mu A_\nu = -J_\mu \end{array} \right. \quad A_{\mu,\nu}{}^{,\nu} - A_{\nu,\mu}{}^{,\nu} = -J_\mu$$

Note that $A_{\mu,\nu}{}^{,\nu} - A_{\nu,\mu}{}^{,\nu} = 0$ is an identity!



current conservation: $J_\mu{}^{,\mu} = 0$

For gravity a similar argument could be adopted to constrain the coefficients $a, b, c, \dots$ relative to each other:

- let's write gravitational e.o.m. as

from the kinetic term, contains $a, b, c, \dots$ $G_{\mu\nu} \propto T_{\mu\nu}$ from $\lambda h_{\mu\nu} T^{\mu\nu}$ in L_g

- since $T^{\mu\nu}{}_{,\nu} = 0$ (energy-momentum conservation)

therefore we expect that $G_{\mu\nu}{}^{,\nu}$ vanishes

$$e 2 h^{\alpha\beta,\sigma}{}_{\sigma\beta}{}^{,\sigma} + b (h^{\alpha\sigma,\beta}{}_{\sigma\beta}{}^{,\sigma} + h^{\beta\sigma,\alpha}{}_{\sigma\beta}{}^{,\sigma}) + c (h^{\sigma}{}_{\sigma,\alpha\beta} + \gamma_{\alpha\beta} h^{\mu\nu}{}_{,\nu\mu}) + \quad (*)$$

$$+ 2d \gamma_{\alpha\beta} h^{\sigma}{}_{\sigma,\mu}{}^{,\mu} = -\lambda T_{\alpha\beta} \quad \text{E-L equations for } h_{\mu\nu}$$

a-term: $\frac{\partial \mathcal{L}}{\partial h_{\mu\nu,\sigma}} \delta h_{\mu\nu,\sigma} = 2e h^{\mu\nu,\sigma} \delta h_{\mu\nu,\sigma}$

↓ by parts

b-term: $\frac{\partial}{\partial h^{\sigma}{}_{\mu,\sigma}} \left(b h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\mu,\sigma} \right) \cdot \delta h^{\sigma}{}_{\mu,\sigma} =$ by parts

$$= -2b h^{\mu\nu,\sigma}{}_{\sigma} \delta h_{\mu\nu} = -2e h^{\alpha\beta,\sigma}{}_{\sigma\beta}{}^{,\sigma} \delta h^{\alpha\beta}$$

$$= -2b h^{\mu\nu,\sigma}{}_{\sigma} \delta h_{\mu\nu} = -2b \frac{1}{2} \left(h^{\mu\nu,\sigma}{}_{\sigma} + h^{\nu\mu,\sigma}{}_{\sigma} \right) \delta h_{\mu\nu}$$

$(\mu,\nu) \rightarrow (\alpha,\beta)$

$\sigma \rightarrow \sigma$

$$= -b (h^{\alpha\sigma,\beta}{}_{\sigma\beta}{}^{,\sigma} + h^{\beta\sigma,\alpha}{}_{\sigma\beta}{}^{,\sigma}) \delta h^{\alpha\beta}$$

↑
only symmetric part contributes

etc.

Now let's calculate of (*) with respect to the index β and require that it vanishes identically:

$$2e h^{\alpha\beta,\sigma}{}_{\sigma\beta}{}^{,\sigma} + b h^{\alpha\sigma,\beta}{}_{\sigma\beta}{}^{,\sigma} + b h^{\beta\sigma,\alpha}{}_{\sigma\beta}{}^{,\sigma} + c h^{\sigma}{}_{\sigma,\alpha\beta} +$$

$$+ c h^{\mu\nu}{}_{,\nu\mu}{}^{\alpha\beta} + 2d h^{\sigma}{}_{\sigma,\mu}{}^{,\mu\alpha\beta} = 0 \quad \text{since } T^{\alpha\beta}{}_{\beta} = 0$$

$$(2e + b) h^{\alpha\beta,\sigma}{}_{\sigma\beta}{}^{,\sigma} + (c + 2d) h^{\sigma}{}_{\sigma,\alpha\beta} + (b + c) h^{\beta\sigma,\alpha}{}_{\sigma\beta}{}^{,\sigma} = 0$$

$$2e + b = 0$$

$$c + 2d = 0$$

independent tensors

$$d = -\frac{1}{2}c$$

$$\mathcal{L}_g = \frac{1}{2} h^{\mu\nu, \sigma} h_{\mu\nu, \sigma} - h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\mu, \sigma} + h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\sigma, \mu} - \frac{1}{2} h^{\nu}{}_{,\nu, \mu} h^{\sigma, \mu}{}_{\sigma} +$$

$$- \lambda T^{\mu\nu} h_{\mu\nu}$$

Let us define for arbitrary second rank tensor

$$\bar{X}_{\mu\nu} := \frac{1}{2} (X_{\mu\nu} + X_{\nu\mu}) - \frac{1}{2} \gamma_{\mu\nu} X^{\sigma}{}_{\sigma}$$

For a symmetric tensor such as $h_{\mu\nu}$ we have

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} h^{\sigma}{}_{\sigma} \quad \text{and} \quad \bar{\bar{h}}_{\mu\nu} = h_{\mu\nu}$$

$$\bar{\bar{h}}_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} \bar{h}^{\sigma}{}_{\sigma} = h_{\mu\nu}$$

Define $h = h^{\sigma}{}_{\sigma}$ (trace of $h^{\mu}{}_{\mu}$) $\left\{ \begin{array}{l} \text{"} \\ -h^{\sigma}{}_{\sigma} \end{array} \right.$

and note that $\bar{h}^{\sigma}{}_{\sigma} = -h$.

Let us rewrite e.o.m. in terms of the "bar" $T_{\alpha\beta}$

$$h_{\alpha\beta, \sigma}{}^{\sigma} - (h_{\alpha\sigma, \beta}{}^{\sigma} + h_{\beta\sigma, \alpha}{}^{\sigma}) + (h^{\sigma}{}_{\sigma, \alpha\beta} + \gamma_{\alpha\beta} h^{\mu\nu}{}_{,\nu, \mu}) +$$

$$- \gamma_{\alpha\beta} h^{\sigma}{}_{\sigma, \mu}{}^{\mu} = -\lambda T_{\alpha\beta} \quad (*)$$

$$\bar{h}_{\alpha\beta, \sigma}{}^{\sigma} - 2 h_{\alpha\sigma, \beta}{}^{\sigma} = -\lambda \bar{T}_{\alpha\beta} \quad \text{after "bar"}$$

Let us show that above is indeed correct, first subtract (*) with $\gamma^{\alpha\beta}$

$$h^{\alpha}{}_{\alpha, \sigma}{}^{\sigma} - 2 h_{\alpha\sigma}{}^{\sigma, \alpha} + h^{\sigma}{}_{\sigma, \alpha}{}^{\alpha} + 4 h^{\mu\nu}{}_{,\nu, \mu} - 4 h^{\sigma}{}_{\sigma, \mu}{}^{\mu} = -\lambda T_{\alpha}{}^{\alpha} \quad \left| \times (-\frac{1}{2} \gamma_{\alpha\beta}) \right.$$

$$h_{\alpha\beta, \sigma}{}^{\sigma} - h_{\alpha\sigma, \beta}{}^{\sigma} - h_{\beta\sigma, \alpha}{}^{\sigma} + h^{\sigma}{}_{\sigma, \alpha\beta} = -\lambda \bar{T}_{\alpha\beta}$$

and add to (*)

The graviton equation of motion

$$20 \quad h_{\alpha\beta,\sigma}{}^\sigma - (h_{\alpha\sigma,\beta}{}^\sigma + h_{\beta\sigma,\alpha}{}^\sigma) + h_{\sigma,\alpha\beta}^\sigma + \gamma_{\alpha\beta} h_{\mu\nu}{}^{\mu\nu} - \gamma_{\alpha\beta} h_{\sigma,\mu}{}^\mu = -\lambda T_{\alpha\beta} \quad | \gamma$$

$$h_{\alpha,\sigma}{}^\sigma - 2h_{\alpha\sigma}{}^{\sigma\sigma} + h_{\sigma,\alpha}{}^{\sigma\sigma} + 4h_{\mu\nu}{}^{\mu\nu} - 4h_{\sigma,\mu}{}^\mu = -\lambda T_{\alpha}{}^{\sigma} \quad | \times (-\frac{1}{2}\gamma_{\alpha\gamma})$$

$$-\frac{1}{2}\gamma_{\alpha\beta} h_{\mu,\sigma}{}^\sigma + \gamma_{\alpha\beta} h_{\mu\sigma}{}^{\mu\sigma} - \frac{1}{2}\gamma_{\alpha\beta} h_{\sigma,\mu}{}^\mu - 2\gamma_{\alpha\beta} h_{\mu\nu}{}^{\mu\nu} + 2\gamma_{\alpha\beta} h_{\sigma,\mu}{}^\mu = +\frac{1}{2}\lambda \gamma_{\alpha\beta} T_{\alpha}{}^{\sigma}$$

1 + 3 $\Downarrow$

$$h_{\alpha\beta,\sigma}{}^\sigma - \frac{1}{2}\gamma_{\alpha\beta} h_{\mu,\sigma}{}^\sigma - 2h_{\alpha\sigma,\beta}{}^\sigma + h_{\sigma,\alpha\beta}^\sigma - \frac{1}{2}\gamma_{\alpha\beta} h_{\sigma,\mu}{}^\mu - \gamma_{\alpha\beta} h_{\mu\nu}{}^{\mu\nu} + \gamma_{\alpha\beta} h_{\sigma,\mu}{}^\mu = -\lambda T_{\alpha\beta}$$

$$\begin{aligned} & -2h_{\alpha\sigma,\beta}{}^\sigma - \gamma_{\alpha\beta} h_{\mu\nu}{}^{\mu\nu} = -2 \left[\frac{1}{2} (h_{\alpha\sigma,\beta}{}^\sigma + h_{\beta\sigma,\alpha}{}^\sigma) - \frac{1}{2}\gamma_{\alpha\beta} h_{\alpha\sigma}{}^{\sigma\sigma} \right] - \gamma_{\alpha\beta} h_{\mu\nu}{}^{\mu\nu} \\ & = -h_{\alpha\sigma,\beta}{}^\sigma - h_{\beta\sigma,\alpha}{}^\sigma \end{aligned}$$

$$h_{\alpha\beta,\sigma}{}^\sigma - h_{\alpha\sigma,\beta}{}^\sigma - h_{\beta\sigma,\alpha}{}^\sigma + h_{\sigma,\alpha\beta}^\sigma = -\lambda T_{\alpha\beta}$$

gravity : $h_{\alpha\beta,\sigma}{}^{\sigma} - h_{\alpha\sigma,\beta}{}^{\sigma} - h_{\beta\sigma,\alpha}{}^{\sigma} + h^{\sigma}{}_{\sigma,\alpha\beta} = -\lambda \bar{T}_{\alpha\beta}$

QED : $A_{\alpha,\beta}{}^{\beta} - A^{\beta}{}_{,\alpha\beta} = -J_{\alpha}$ invariant under $A_{\alpha} \rightarrow A'_{\alpha} = A_{\alpha} + X_{,\alpha}$

For gravity we guess in analogy to

$$h_{\alpha\beta} \rightarrow h'_{\alpha\beta} = h_{\alpha\beta} + \underbrace{X_{\alpha,\beta} + X_{\beta,\alpha}}_{\text{symmetric}}$$

gauge transformation

Is the e.o.f. invariant under $h_{\alpha\beta} \rightarrow h'_{\alpha\beta}$?

$$h_{\alpha\beta,\sigma}{}^{\sigma} \rightarrow h_{\alpha\beta,\sigma}{}^{\sigma} + \underbrace{(X_{\alpha,\beta} + X_{\beta,\alpha})_{,\sigma}}_{= X_{\alpha,\beta\sigma} + X_{\beta,\alpha\sigma}}$$

$$-h_{\alpha\sigma,\beta}{}^{\sigma} \rightarrow -h_{\alpha\sigma,\beta}{}^{\sigma} - \underbrace{(X_{\alpha,\sigma} + X_{\sigma,\alpha})_{,\beta}}_{= -X_{\alpha,\sigma\beta} - X_{\sigma,\alpha\beta}}$$

$$-h_{\beta\sigma,\alpha}{}^{\sigma} \rightarrow -h_{\beta\sigma,\alpha}{}^{\sigma} - \underbrace{(X_{\beta,\sigma} + X_{\sigma,\beta})_{,\alpha}}_{= -X_{\beta,\sigma\alpha} - X_{\sigma,\beta\alpha}}$$

$$h^{\sigma}{}_{\sigma,\alpha\beta} \rightarrow h^{\sigma}{}_{\sigma,\alpha\beta} + \underbrace{(X^{\sigma}{}_{,\sigma} + X^{\sigma}{}_{,\sigma})_{,\alpha\beta}}_{= 2X^{\sigma}{}_{,\sigma\alpha\beta}}$$

$\Rightarrow$ e.o.m. is invariant!

QED : the Lorentz gauge : $A_{\alpha}{}^{,\alpha} = 0 \Rightarrow \square A_{\beta} = -J_{\beta}$

gravity : the Lorentz (harmonic) gauge : $\bar{h}_{\beta\alpha}{}^{,\alpha} = 0$
(de Donder) $\Downarrow h_{\beta\alpha}{}^{,\alpha} - \frac{1}{2} h^{\alpha}{}_{\alpha,\beta} = 0$

$$\square h_{\alpha\beta} = -\lambda \bar{T}_{\alpha\beta}$$

$(-h_{\alpha\sigma,\beta}{}^{\sigma} - h_{\beta\sigma,\alpha}{}^{\sigma} + h^{\sigma}{}_{\sigma,\alpha\beta} = 0)$

$$h_{\alpha\beta}^i = h_{\alpha\beta} + x_{\alpha 1\beta} + x_{\beta 1\alpha}$$

$$\bar{h}_{\beta\alpha} = h_{\beta\alpha} - \frac{1}{2} \eta_{\beta\alpha} h^\sigma{}_\sigma$$

$$\bar{h}_{\beta\alpha}^i = h_{\beta\alpha}^i - \frac{1}{2} \eta_{\beta\alpha} (h^\sigma{}_\sigma + 2 x^\sigma{}_{1\sigma})$$

$$\bar{h}_{\beta\alpha}^{i\alpha} = h_{\beta\alpha}^{i\alpha} - \frac{1}{2} (h^\sigma{}_{\sigma 1\beta} + 2 x^\sigma{}_{1\sigma\beta}) = 0$$

the gauge condition

$$\begin{aligned} x^\sigma{}_{1\sigma\beta} &= h_{\beta\alpha}^{i\alpha} - \frac{1}{2} h^\sigma{}_{\sigma 1\beta} = \\ &= h_{\beta\alpha}^{i\alpha} + x_{\alpha 1\beta} + x_{\beta 1\alpha} - \frac{1}{2} h^\sigma{}_{\sigma 1\beta} \end{aligned}$$

$$\overset{''}{x^\sigma{}_{1\sigma\beta}} \quad \overset{''}{\square} x_\beta$$

$$\square x_\beta = - \left(h_{\beta\alpha}^{i\alpha} - \frac{1}{2} h^\sigma{}_{\sigma 1\beta} \right) = - \bar{h}_{\beta\alpha}^{i\alpha}$$

$$\square x_\beta = - \bar{h}_{\beta\alpha}^{i\alpha}$$

$$h_{\alpha\beta,\sigma}{}^{\sigma} - (h_{\alpha\sigma,\beta}{}^{\sigma} + h_{\beta\sigma,\alpha}{}^{\sigma}) + h^{\sigma}{}_{\sigma,\alpha\beta} + \gamma_{\alpha\beta} h^{\mu\nu}{}_{,\mu\nu} - \gamma_{\alpha\beta} h^{\sigma}{}_{\sigma,\mu}{}^{\mu} = -\lambda T_{\alpha\beta}$$

Let's "bar" the above equation:

$$\bar{X}_{\mu\nu} := \frac{1}{2} (X_{\mu\nu} + X_{\nu\mu}) - \frac{1}{2} \gamma_{\mu\nu} X_{\alpha}{}^{\alpha}$$

$$\bar{\gamma}_{\alpha\beta} = \gamma_{\alpha\beta} - \frac{1}{2} \gamma_{\alpha\beta} \underbrace{\gamma^{\sigma}{}_{\sigma}}_4 = -\gamma_{\alpha\beta}$$

$$\overline{h_{\alpha\beta,\sigma}{}^{\sigma}} = h_{\alpha\beta,\sigma}{}^{\sigma} - \frac{1}{2} \gamma_{\alpha\beta} h^{\mu}{}_{\mu,\sigma}{}^{\sigma}$$

$$\overline{h_{\alpha\sigma,\beta}{}^{\sigma}} = \overline{h_{\beta\sigma,\alpha}{}^{\sigma}} = \frac{1}{2} (h_{\alpha\sigma,\beta}{}^{\sigma} + h_{\beta\sigma,\alpha}{}^{\sigma}) - \frac{1}{2} \gamma_{\alpha\beta} h_{\mu\sigma}{}^{\mu\sigma}$$

$$\overline{h^{\sigma}{}_{\sigma,\alpha\beta}} = h^{\sigma}{}_{\sigma,\alpha\beta} - \frac{1}{2} \gamma_{\alpha\beta} h^{\sigma}{}_{\sigma,\mu}{}^{\mu}$$

$\Downarrow$

$$h_{\alpha\beta,\sigma}{}^{\sigma} - \frac{1}{2} \gamma_{\alpha\beta} h^{\mu}{}_{\mu,\sigma}{}^{\sigma} - 2 \overline{h_{\alpha\sigma,\beta}{}^{\sigma}} + h^{\sigma}{}_{\sigma,\alpha\beta} - \frac{1}{2} \gamma_{\alpha\beta} h^{\sigma}{}_{\sigma,\mu}{}^{\mu} + \gamma_{\alpha\beta} h^{\mu\nu}{}_{,\mu\nu} + \gamma_{\alpha\beta} h^{\sigma}{}_{\sigma,\mu}{}^{\mu} = -\lambda \bar{T}_{\alpha\beta}$$

$$h_{\alpha\beta,\sigma}{}^{\sigma} - 2 \overline{h_{\alpha\sigma,\beta}{}^{\sigma}} + h^{\sigma}{}_{\sigma,\alpha\beta} - \gamma_{\alpha\beta} h^{\mu\nu}{}_{,\mu\nu} = -\lambda \bar{T}_{\alpha\beta}$$

Note that $(\bar{h}_{\alpha\sigma})_{,\beta}{}^{\sigma} = (h_{\alpha\sigma} - \frac{1}{2} \gamma_{\alpha\sigma} h^{\mu}{}_{\mu})_{,\beta}{}^{\sigma} = h_{\alpha\sigma,\beta}{}^{\sigma} - \frac{1}{2} \gamma_{\alpha\sigma} h^{\mu}{}_{\mu,\beta}{}^{\sigma} =$

$$= h_{\alpha\sigma,\beta}{}^{\sigma} - \frac{1}{2} h^{\mu}{}_{\mu,\beta\alpha}$$

$$(\bar{h}_{\beta\sigma})_{,\alpha}{}^{\sigma} = h_{\beta\sigma,\alpha}{}^{\sigma} - \frac{1}{2} h^{\mu}{}_{\mu,\alpha\beta}$$

Therefore

$$(\bar{h}_{\alpha\sigma})_{,\beta}{}^{\sigma} + (\bar{h}_{\beta\sigma})_{,\alpha}{}^{\sigma} = \underbrace{h_{\alpha\sigma,\beta}{}^{\sigma} + h_{\beta\sigma,\alpha}{}^{\sigma}}_{2 h_{\alpha\sigma,\beta}{}^{\sigma}} - h^{\mu}{}_{\mu,\alpha\beta}$$

$\Downarrow$

$$2 h_{\alpha\sigma,\beta}{}^{\sigma} + \gamma_{\alpha\beta} h_{\mu\sigma}{}^{\mu\sigma}$$

$$\boxed{h_{\alpha\beta,\sigma}{}^{\sigma} - \bar{h}_{\alpha\sigma,\beta}{}^{\sigma} - \bar{h}_{\beta\sigma,\alpha}{}^{\sigma} = -\lambda \bar{T}_{\alpha\beta}}$$

$$S_{\text{QED}} = \int d^4x \mathcal{L}_{\text{QED}} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \right] =$$

$$= \int d^4x \left[\frac{1}{2} A_\mu (\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu + A_\mu J^\mu \right]$$

$\mathcal{L}_{\text{QED}}$ is invariant under $A_\mu \rightarrow A_\mu - \partial_\mu \Lambda$

(new eigen-value)

$$Q^{\mu\nu} \partial_\nu \Lambda = 0$$

$\downarrow$

$\Downarrow$

$$I = \int DA_\mu e^{iS_{\text{QED}}}$$

sums over equivalent

there is no inverse

$$\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu = Q^{\mu\nu}$$

$\Leftarrow$

(connected by a gauge transformation)
field configurations

Feynman - Popov $\Rightarrow$

$$S_{\text{QED}} \rightarrow \int d^4x \mathcal{L}_{\text{QED}} - \frac{1}{2\xi} \int d^4x (\partial_\mu A^\mu)^2$$

the R_ξ gauge

$$Q_{\text{eff}}^{\mu\nu} = \partial^2 \eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \quad \text{has an inverse}$$

$$Q_{\text{eff}}^{\mu\nu} \left[-\frac{1}{k^2} + \left(1 - \frac{1}{\xi}\right) \frac{k_\mu k_\nu}{k^2} \right] \frac{1}{k^2} = \delta^\mu_\nu$$

$$-k^2 \eta^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu$$

The photon propagator :

$$= \frac{1}{k^2} \left[\eta_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \frac{k_\mu k_\nu}{k^2} \right]$$

$$\xi = 1 \quad \text{Feynman gauge}$$

$$S_g = \int d^4x \mathcal{L}_g = \int d^4x \left[\frac{1}{2} h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - h^{\mu\nu}_{;\mu} h^{\sigma}_{;\nu} + h^{\mu\nu}_{;\nu} h^{\sigma}_{;\mu} + \right. \\ \left. - \frac{1}{2} h^{\nu}_{;\nu\mu} h^{\sigma}_{;\sigma}{}^{\mu} - \lambda T^{\mu\nu} h_{\mu\nu} \right]$$

$\mathcal{L}_g$ invariant under $h_{\alpha\beta} \rightarrow h_{\alpha\beta} + X_{\alpha,\beta} + X_{\beta,\alpha}$

to fix the harmonic gauge: $\bar{h}_{\beta\alpha}{}^{,\alpha} = 0$
 $\hookrightarrow h_{\beta\alpha}{}^{,\alpha} - \frac{1}{2} h^{\alpha}_{;\alpha}{}_{,\beta} = 0$

we will add the gauge fixing term

$$\mathcal{L}_g \rightarrow \mathcal{L}_g + \underbrace{\left(h^{\mu\nu}{}^{,\mu} - \frac{1}{2} h^{\mu}{}_{;\mu}{}^{,\nu} \right) \left(h^{\nu\mu}{}^{,\nu} - \frac{1}{2} h^{\mu}{}_{;\mu}{}^{,\nu} \right)} = \mathcal{L}_{\text{eff}}$$

$$\left(h^{\mu\nu}{}^{,\mu} - \frac{1}{2} h^{\mu}{}_{;\mu}{}^{,\nu} \right)^2 = \left(\bar{h}^{\mu\nu}{}^{,\mu} \right)^2$$

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - h^{\mu\nu}_{;\mu} h^{\sigma}_{;\nu} + h^{\mu\nu}_{;\nu} h^{\sigma}_{;\mu} - \frac{1}{2} h^{\nu}_{;\nu\mu} h^{\sigma}_{;\sigma}{}^{\mu} + \\ + h^{\mu\nu}{}^{,\mu} h^{\nu\sigma}{}^{,\nu} + \frac{1}{4} h^{\mu}{}_{;\mu} h^{\sigma}{}_{;\sigma} - \frac{1}{2} h^{\mu}{}_{;\mu} h^{\nu\sigma}{}^{,\nu} - \frac{1}{2} h^{\mu}{}_{;\mu} h^{\sigma}{}_{;\sigma}{}^{\mu} + \dots \\ - h^{\mu\nu}_{;\mu} h^{\sigma}_{;\sigma}{}^{\mu}$$

$$= \frac{1}{2} \left[h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - \frac{1}{2} h^{\nu}_{;\nu\mu} h^{\sigma}_{;\sigma}{}^{\mu} \right] - \lambda h_{\mu\nu} T^{\mu\nu}$$

$$\underbrace{\quad}_{\text{III}} \\ h^{\mu\nu} K_{\mu\nu;\lambda\sigma} h^{\lambda\sigma}$$

let's determine the quadratic operator $K_{\mu\nu;\lambda\sigma}$,
 its inverse will give the graviton propagator

$$\int d^4x \left[h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - \frac{1}{2} h^{\nu}{}_{\nu,\mu} h^{\sigma}{}_{\sigma}{}^{,\mu} \right] = \text{by parts}$$

$$= \int d^4x \left[-h^{\mu\nu} \gamma_{\mu\lambda} \gamma_{\nu\sigma} \square h^{\lambda\sigma} + \frac{1}{2} \underbrace{h^{\nu}{}_{\nu} \square h^{\sigma}{}_{\sigma}}_{h^{\mu\nu} \gamma_{\mu\nu} \gamma_{\lambda\sigma} \square h^{\lambda\sigma}} \right] =$$

$$= \int d^4x \underbrace{h^{\mu\nu} \frac{1}{2} \left[\gamma_{\mu\lambda} \gamma_{\nu\sigma} + \gamma_{\mu\sigma} \gamma_{\nu\lambda} - \gamma_{\mu\nu} \gamma_{\lambda\sigma} \right]}_{\substack{\equiv \\ k_{\mu\nu;\lambda\sigma}}} (-\square) h^{\lambda\sigma}$$

Note that
(same work) $k_{\mu\nu;\lambda\sigma} k^{\lambda\sigma} = \frac{1}{2} \left(\gamma_{\mu\sigma} \gamma_{\nu\sigma} + \gamma_{\mu\sigma} \gamma_{\nu\sigma} \right) = \mathbb{I}_{\mu\nu;\sigma\sigma}$

identity acting in a
linear space spanned by
symmetric second rank tensors



the photon propagator in the harmonic gauge

$$D_{\mu\nu;\lambda\sigma}(k) = \frac{1}{2} \frac{\gamma_{\mu\lambda} \gamma_{\nu\sigma} + \gamma_{\mu\sigma} \gamma_{\nu\lambda} - \gamma_{\mu\nu} \gamma_{\lambda\sigma}}{k^2 + i\epsilon}$$

Massive gravitons

- Symmetric tensor $h_{\mu\nu}$ has 10 independent components
- Spin 2 field should have $2s+1 = 5$ components
- Vector and scalar components of $h_{\mu\nu}$ should be removed by the following Lorentz invariant conditions:

$$\partial^\nu h_{\mu\nu} = 0 \quad \text{and} \quad h_\nu{}^\nu = 0$$

- Lorentz invariant mass term:

$$+ \frac{1}{2} m^2 (e (h_\sigma{}^\sigma)^2 + f h_{\mu\nu} h^{\mu\nu})$$

QED with massive photons:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$$

$m \neq 0 \Rightarrow$ breaking of the gauge invariance

$$A_\mu \rightarrow A'_\mu = A_\mu + X_{,\mu}$$

$$A_\mu A^\mu \rightarrow A'_\mu A'^\mu = (A_\mu + X_{,\mu})(A^\mu + X^{,\mu}) = A_\mu A^\mu + A_\mu X^{,\mu} + X_{,\mu} A^\mu + X_{,\mu} X^{,\mu} \neq A_\mu A^\mu$$

Gravity with massive gravitons:

$$\mathcal{L} = \frac{1}{2} h^{\mu\nu,\sigma} h_{\mu\nu,\sigma} - h^{\mu\nu}{}_{;\sigma} h^\sigma{}_{\mu\nu} + h^{\mu\nu}{}_{;\sigma} h^\sigma{}_{\mu\nu} - \frac{1}{2} h^\nu{}_{\nu,\mu} h^\sigma{}_{\sigma}{}^{,\mu} + \frac{1}{2} m^2 (e h^2 + f h_{\mu\nu} h^{\mu\nu}) - \lambda T^{\mu\nu} h_{\mu\nu}$$

$h \equiv h_\sigma{}^\sigma$

gauge transformation

$$h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + X_{\mu\nu} + X_{\nu,\mu}$$

changes the

mass term $m^2 (e h^2 + f h_{\mu\nu} h^{\mu\nu})$

For instance

$$h^2 = (h_\sigma^\sigma)^2 \rightarrow (h_\sigma'^\sigma)^2 = (h_\sigma^\sigma + X_\sigma'^\sigma \cdot 2)^2 = \\ = h^2 + 4 h X_\sigma'^\sigma + 4 (X_\sigma'^\sigma)^2$$

We require that E-L equations derived from $\mathcal{L}$ (with $m^2 \neq 0$) will lead to the constraints that are necessary to remove vector and scalar components of symmetric $h_{\mu\nu}$, i.e.

$$\partial^\nu h_{\mu\nu} = 0 \quad \text{and} \quad h_\nu{}^\nu = 0$$

$$\delta S = 0$$

$\Downarrow$

$$h_{\mu\nu,\sigma}^\sigma - (h_{\mu\sigma,\nu}^\sigma + h_{\nu\sigma,\mu}^\sigma) + (h^\sigma_{\sigma,\mu\nu} + \gamma_{\mu\nu} h^{\sigma\lambda}_{,\sigma\lambda}) + \\ - \gamma_{\mu\nu} h^{\sigma\lambda}_{,\sigma\lambda} - m^2 (\gamma_{\mu\nu} e h + f h_{\mu\nu}) = -\lambda T_{\mu\nu}$$

Derivative (∂^ν) and contraction with $\gamma^{\mu\nu}$ should lead to $\partial^\nu h_{\mu\nu} = 0$ and $h_\nu{}^\nu = 0$.

Note that for $m^2 = 0$ the Lagrangian is identical as the one we discussed earlier where we demanded that $\partial^\nu (\text{left hand side of equation of motion})_{\mu\nu} = 0$ as a consequence of conservation of the energy momentum tensor i.e. $\partial^\nu T_{\mu\nu} = 0$.

$\Downarrow$ from ∂^ν

$$-m^2 (e h_{\sigma,\mu}^\sigma + f h_{\mu\nu}{}^{\nu}) = 0$$

Now let's take the trace

$$h^\mu_{\mu\sigma}{}^{1\sigma} - (h^\mu_{\mu\sigma}{}^{1\mu\sigma} + h^\mu_{\sigma,\mu}{}^{1\sigma}) + (h^\sigma_{\sigma,\mu}{}^{1\mu} + 4h^{\sigma\lambda}{}_{1\sigma\lambda}) +$$

$$- 4h^\sigma_{\sigma,\lambda}{}^{1\lambda} - m^2 (4eh^\sigma_{\sigma} + fh^\sigma_{\sigma}) = -\lambda T^\mu_\mu$$

Since we want to get $h_{\mu\nu}{}^{1\nu} = 0$ and $h_\sigma{}^\sigma = 0$
we have to assume $T^\mu_\mu = 0$ for consistency, then

$$- 2h^\mu_{\mu\sigma}{}^{1\sigma} + 2h^\mu_{\mu\sigma}{}^{1\mu\sigma} - m^2 (4e + f) h_\sigma{}^\sigma = 0$$

$$\left[\begin{aligned} & - 2(h^\sigma_{\sigma,\mu} - h_{\mu\nu}{}^{1\nu}){}^{1\mu} - m^2 (4e + f) h_\sigma{}^\sigma = 0 \\ & \underbrace{eh^\sigma_{\sigma,\mu} + fh_{\mu\nu}{}^{1\nu}} = 0 \end{aligned} \right] \quad (\text{from } \partial^\nu)$$

$$- 2(h^\sigma_{\sigma,\mu} + \frac{e}{f} h^\sigma_{\sigma,\mu}){}^{1\mu} - m^2 (4e + f) h_\sigma{}^\sigma = 0 \quad \Big| \quad \frac{1}{4e + f}$$

$$- \frac{1 + \frac{e}{f}}{4e + f} \Box h - m^2 h = 0 \quad \text{this equation leads to } h = 0$$

only if $1 + \frac{e}{f} = 0$

$\Downarrow$

$$e = -f = 1 \quad \leftarrow \text{choice of } m^2$$

$$\mathcal{L}_{\text{mass}} = + \frac{1}{2} m^2 (h^2 - h_{\mu\nu} h^{\mu\nu})$$

the Fierz - Pauli model

(see Nieuwenhuisen NPB 60 (1973) 478 for the
proof that $e = -f$ is the only choice which leads
to a theory free of tachyons)

Equation of motion for a massive graviton:

$$[(\gamma^{\mu\alpha} \gamma^{\nu\beta} - \gamma^{\mu\nu} \gamma^{\alpha\beta}) \Box + (\gamma^{\mu\nu} \partial^\alpha \partial^\beta + \gamma^{\alpha\beta} \partial^\mu \partial^\nu - \gamma^{\mu\alpha} \partial^\nu \partial^\beta - \gamma^{\nu\beta} \partial^\mu \partial^\alpha) +$$

$$+ m^2 (\gamma^{\mu\alpha} \gamma^{\nu\beta} - \gamma^{\mu\nu} \gamma^{\alpha\beta})] h_{\alpha\beta} = -\lambda T^{\mu\nu}$$