The graviton kinetic term : $\frac{1}{2}\, h_{\mu\nu}\, K^{\mu\nu;\alpha\beta}\, h_{\alpha\beta}$ for

$$K^{\mu\nu;\alpha\beta} = -\Big[\left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)\Box + \eta^{\mu\nu}\partial^\alpha\partial^\beta + \eta^{\alpha\beta}\partial^\mu\partial^\nu - \eta^{\mu\alpha}\partial^\nu\partial^\beta + - \eta^{\nu\beta}\partial^\mu\partial^\alpha\Big] - m^2\left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)$$

$$K^{\mu\nu;\alpha\beta}(x)\, D_{\alpha\beta}{}^{\rho\sigma}(x) = \underbrace{\frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)}_{= I^{\mu\nu;\rho\sigma}} \delta^{(4)}(x)$$

$$D^{\mu\nu;\alpha\beta}(x) = \int \frac{d^4k}{(2\pi)^4}\, D^{\mu\nu;\alpha\beta}(k)\, e^{ikx}$$

We are looking for the inverse of $K^{\mu\nu;\alpha\beta}$:

$$\boxed{K^{\mu\nu;\alpha\beta}\left(K^{-1}\right)_{\alpha\beta}{}^{\rho\sigma} = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right) = I^{\mu\nu;\rho\sigma}}$$

$$\Big[\left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)k^2 + \eta^{\mu\nu}k^\alpha k^\beta + \eta^{\alpha\beta}k^\mu k^\nu - \eta^{\mu\alpha}k^\nu k^\beta - \eta^{\nu\beta}k^\mu k^\alpha + - m^2\left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\nu}\eta^{\alpha\beta}\right)\Big]\, D_{\alpha\beta}{}^{\rho\sigma}(k) = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)$$

After some tedious algebra one finds

$$D^{\alpha\beta;\rho\sigma}(k) = \frac{1}{2}\, \frac{G^{\alpha\rho}G^{\beta\sigma} + G^{\alpha\sigma}G^{\beta\rho} - \frac{2}{3}\, G^{\alpha\beta}G^{\rho\sigma}}{k^2 - m^2 + i\varepsilon}$$ massive graviton

for $G^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}$

$$D^{\alpha\beta;\rho\sigma}(k) = \frac{1}{2}\, \frac{\eta^{\alpha\rho}\eta^{\beta\sigma} + \eta^{\alpha\sigma}\eta^{\beta\rho} - \eta^{\alpha\beta}\eta^{\rho\sigma}}{k^2 + i\varepsilon}$$ massless graviton

The deflection of light by massless and massive gravitons

$$\mathcal{L} = \frac{1}{2} h^{\mu\nu}{}_{,\sigma} h_{\mu\nu}{}^{,\sigma} - h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\mu,\sigma} + h^{\mu\nu}{}_{,\nu} h^{\sigma}{}_{\sigma,\mu} - \frac{1}{2} h^{\nu}{}_{\nu,\mu} h_{\sigma}{}^{\sigma,\mu} +$$

$$- \lambda T^{\mu\nu} h_{\mu\nu} + \begin{cases} \frac{1}{2} m^2 \left({h_\sigma{}^{\sigma}}^2 - h_{\mu\nu} h^{\mu\nu} \right) & \text{for massive graviton} \\ \left(h_{\mu\nu}{}^{,\mu} - \frac{1}{2} h^{\mu}{}_{\mu,\nu} \right)^2 & \text{for massless graviton (gauge fixing term)} \end{cases}$$

$$D_m^{\alpha\beta;\varsigma\sigma}(k) = \frac{1}{2} \frac{G^{\alpha\varsigma} G^{\beta\sigma} + G^{\alpha\sigma} G^{\beta\varsigma} - \frac{2}{3} G^{\alpha\beta} G^{\varsigma\sigma}}{k^2 - m^2 + i\varepsilon} \qquad m \neq 0 \qquad G^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}$$

$$D_0^{\alpha\beta;\varsigma\sigma}(k) = \frac{1}{2} \frac{\eta^{\alpha\varsigma} \eta^{\beta\sigma} + \eta^{\alpha\sigma} \eta^{\beta\varsigma} - \eta^{\alpha\beta} \eta^{\varsigma\sigma}}{k^2 + i\varepsilon} \qquad m = 0$$

$m = 0$:

$$W_0[T] = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \lambda T^{\mu\nu}(k)^* \, D_{0\,\mu\nu;\lambda\sigma}(k) \lambda T^{\lambda\sigma}(k)$$

$\Downarrow$ for $T^{\mu\nu}$ defined by stationary masses m_1, m_2

$$= -\frac{\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} T^{00\,*}(k) \frac{1}{2} \frac{1+1-1}{k^2 + i\varepsilon} T^{00}(k) = E_0 \int dx^0$$

$\Downarrow$ $\lambda \to \lambda_0$

$$E_0 = -\frac{\lambda_0^2}{4} \cdot 2 \; m_1 m_2 \frac{1}{4\pi r} = -G_N \frac{m_1 m_2}{r}$$

two mixed terms appear

$\Downarrow$

$$\lambda_0 = (8\pi G_N)^{1/2}$$

for $m = 0$

$m^2 \neq 0$

$$W_m[T] = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \lambda T^{\mu\nu *}(k) \underbrace{\frac{1}{2}\frac{G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda} - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}}{k^2 - m^2 + i\varepsilon}}_{D_{m\,\mu\nu;\lambda\sigma}(k)} \lambda T^{\lambda\sigma}(k) =$$

- again only $(0,0)$ components contribute
- note that $G_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}$ and the energy-momentum tensor is conserved : $k_\mu T^{\mu\nu} = 0$

$$\eta_{\mu\lambda} \leftarrow G_{\mu\lambda} \text{ effectively}$$

$$W_m[T] = -\frac{1}{2}\lambda^2 \int \frac{d^4k}{(2\pi)^4} T^{00*}(k) \frac{1}{2} \frac{1 + 1 - \frac{2}{3}}{k^2 - m^2 + i\varepsilon} T^{00}(k)$$

$$\Downarrow \quad \lambda \to \lambda_m$$

$$E_m = -\frac{\lambda_m^2}{4} \cdot 2 \, \frac{4}{3} \, m_1 m_2 \frac{1}{4\pi r} e^{-mr} = -G_N \frac{m_1 m_2}{r} \underbrace{e^{-mr}}_{\substack{\downarrow \\ 1 \\ \text{for } m \to 0}}$$

$$\lambda_m = (6\pi G_N)^{1/2}$$

<u>Conclusion</u>

In order to describe correctly the Newtonian gravity, not only $m \to 0$ in the massive case, but also the coupling constant λ must be different in massive and massless case.

let us now consider graviton – light interaction : $\lambda T^{\mu\nu} h_{\mu\nu}$

For QED we have

$$T^{\mu\nu}_{QED} = F^{\mu}{}_{\sigma} F^{\nu\sigma} - \frac{1}{4} \eta^{\mu\nu} F_{\sigma\lambda} F^{\sigma\lambda} \qquad \partial_\mu T^{\mu\nu} = 0$$

the amplitude for light scattering on a massive source (e.g. the Sun) will contain the following term

$$\mathcal{A} \sim D_{\alpha\beta;\mu\nu} \; T^{\mu\nu}_{QED} \cdot \lambda$$

Then for the massless case we have

$$\mathcal{A}_0 \sim \frac{1}{2} \frac{\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\alpha\beta}\eta_{\mu\nu}}{k^2 + i\varepsilon} T^{\mu\nu}_{QED} \cdot \lambda_0 \qquad m = 0$$

while for the massive case

$$\mathcal{A}_m \sim \frac{1}{2} \frac{\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \frac{2}{3}\eta_{\alpha\beta}\eta_{\mu\nu}}{k^2 - m^2 + i\varepsilon} T^{\mu\nu}_{QED} \cdot \lambda_m \qquad m \neq 0$$

Note that $\eta_{\mu\nu} T^{\mu\nu}_{QED} = 0$, so that the last term don't contribute , so therefore there is no way to compensate the difference in λ's !

$$\left(\frac{\mathcal{A}_0}{\mathcal{A}_m}\right)^2_{m \to 0} = \frac{\lambda_0^2}{\lambda_m^2} = \frac{8}{6} = \frac{4}{3}$$

The deflection of light by the Sun is measured (first 1919) with a precision ~ few percent and the measurement agrees with $\mathcal{A}_0$. So, the prediction for light bending in the massive theory is off by ~ 25% regardless how small is the graviton mass !

The Principle of Equivalence of Gravitation and Inertia

Consider a system of N particles moving with nonrelativistic velocities under the influence of forces $\bar{F}(\bar{x}_N - \bar{x}_M)$ and an external gravitational field $\bar{g}$ (static & homogeneous)

$$O: \qquad m_N \frac{d^2 \bar{x}_N}{dt^2} = m_N \bar{g} + \sum_{M \neq N} \bar{F}(\bar{x}_N - \bar{x}_M)$$

Suppose that we perform a non-Galilean space-time coordinate transformation

$$\bar{x}' = \bar{x} - \tfrac{1}{2} \bar{g} t^2 , \qquad t' = t$$

Then the "$\bar{g}$-part force" is cancelled by an inertial "force" and the e.o.m. become

$$O': \qquad m_N \frac{d^2 \bar{x}'_N}{dt'^2} = \sum_{M \neq N} \bar{F}(\bar{x}'_N - \bar{x}'_M)$$

Conclusions: 1°. Laws of mechanics are the same in O and O', <u>except that there is no gravitational force in O'</u>.

2° Gravitational forces are cancelled by inertial forces (and therefore equivalent) in any freely falling system, at least for static and homogenous gravitational field. If $\bar{g} = \bar{g}(\bar{x}, t)$ then e.g. inhomogenity of $\bar{g}$ is to be observed in freely falling systems. For instance observers in freely falling elevator would in principle be able to detect the earth's field, because objects falling freely would approach each other (because of radial direction of $\bar{g}$).

The equivalence principle:

At every space-time point in an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation.

- "the same form .." means laws of nature consistent with special relativity
- "a sufficiently small region" means in practice a region so small that the gravitational field could by considered as homogenous

Gravitational forces

Freely moving particle under the influence of purely gravitational forces:

$$\left.\begin{array}{l}\text{PoE} \\ \text{ur relativistic} \\ \text{.o.m}\end{array}\right\} \Rightarrow \quad \frac{d^2\xi^\alpha}{d\tau^2} = 0 \quad \text{in freely falling coordinate system } \xi^\alpha$$

where $d\tau^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$

Now suppose we want to use other coordinate system x^μ, e.g. Cartesian or curvilinear system at rest in the laboratory. Then

$$0 = \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \Bigg| \frac{\partial x^\lambda}{\partial \xi^\alpha}$$

we $\quad \frac{\partial \xi^\alpha}{\partial \xi^\mu}\frac{\partial x^\lambda}{\partial \xi^\alpha} = \delta_\mu^\lambda$

$$0 = \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

where $\Gamma^\lambda_{\mu\nu} \equiv \dfrac{\partial x^\lambda}{\partial \xi^\alpha} \dfrac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$ is the affine connection

the proper time τ can also be expressed in an arbitrary coordinate system:

$$d\tau^2 = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu \equiv g_{\mu\nu}\, dx^\mu dx^\nu$$

where

(*) $g_{\mu\nu} \equiv \eta_{\alpha\beta} \dfrac{\partial \xi^\alpha}{\partial x^\mu} \dfrac{\partial \xi^\beta}{\partial x^\nu}$ is the metric tensor

The values of the metric tensor $g_{\mu\nu}$ and the affine connection $\Gamma^\lambda_{\mu\nu}$ at a point X in an arbitrary coordinate system x^μ allow to determine the locally inertial coordinates $\xi^\alpha(x)$ in a neighborhood of X:

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \quad \Big| \quad \frac{\partial \xi^\beta}{\partial x^\lambda}$$

and use $\dfrac{\partial \xi^\beta}{\partial x^\lambda} \dfrac{\partial x^\lambda}{\partial \xi^\alpha} = \delta_\alpha{}^\beta$, then

$$\Gamma^\lambda_{\mu\nu} \frac{\partial \xi^\beta}{\partial x^\lambda} = \frac{\partial^2 \xi^\beta}{\partial x^\mu \partial x^\nu}$$

← differential equation for $\xi^\beta(x)$

the solution in the vicinity of X:

$$\xi^\alpha(x) = a^\alpha + b^\alpha{}_\mu (x^\mu - X^\mu) + \frac{1}{2} b^\alpha{}_\lambda \Gamma^\lambda_{\mu\nu} (x^\mu - X^\mu)(x^\nu - X^\nu) + \dots$$

$$a^\alpha = \xi^\alpha(X) \qquad b^\alpha{}_\lambda = \frac{\partial \xi^\alpha(x)}{\partial x^\lambda}\Big|_{x=X}$$

from (*) we have $g_{\mu\nu}(X) = \eta_{\alpha\beta}\, b^\alpha{}_\mu\, b^\beta{}_\nu$

So, if $\Gamma^\lambda_{\mu\nu}$ and $g_{\mu\nu}$ are given at X, then $\xi^\alpha(x)$ are determined to order $(x-X)^2$ except for the ambiguity in a^α and b^α_λ.

b^α_μ is determined by $g_{\mu\nu}(X) = \eta_{\alpha\beta}\, b^\alpha_\mu\, b^\beta_\nu$ up to a Lorentz transformation $b^\alpha_\mu \to \Lambda^\alpha{}_\beta\, b^\beta_\mu$:

$$\eta_{\alpha\beta}\, b^\alpha_\mu\, b^\beta_\nu \to \eta_{\alpha\beta}\, \Lambda^\alpha{}_\gamma\, b^\gamma_\mu\, \Lambda^\beta{}_\delta\, b^\delta_\nu =$$

$$= \underbrace{\eta_{\alpha\beta}\, \Lambda^\alpha{}_\gamma\, \Lambda^\beta{}_\delta}_{= \eta_{\gamma\delta}}\, b^\gamma_\mu\, b^\delta_\nu = \eta_{\gamma\delta}\, b^\gamma_\mu\, b^\delta_\nu$$

$$\Downarrow$$

- $\Gamma^\lambda_{\mu\nu}$ and $g_{\mu\nu}$ at X determines the locally inertial coordinate system ξ^α up to $\xi^\alpha \to \Lambda^\alpha{}_\beta\, \xi^\beta + c^\alpha$
- in other words, if ξ^α are locally inertial coordinates, then so are $\Lambda^\alpha{}_\beta\, \xi^\beta + c^\alpha$
- all effects of gravitation are comprised in $\Gamma^\lambda_{\mu\nu}$ and $g_{\mu\nu}$ (locally at X)

Relation between $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$

$\Gamma^\lambda_{\mu\nu} \longrightarrow$ the gravitational force

$g_{\mu\nu} \longrightarrow$ the proper time

$$\frac{d^2 x^\lambda}{d\tau^2} = -\,\Gamma^\lambda_{\mu\nu}\, \frac{dx^\mu}{d\tau}\, \frac{dx^\nu}{d\tau}$$

$$d\tau^2 = -\,g_{\mu\nu}\, dx^\mu dx^\nu$$

We will show now that $g_{\mu\nu}$ is also the gravitational potential ; i.e. its derivatives determine the affine connection $\Gamma^{\lambda}_{\mu\nu}$.

$$\frac{\partial}{\partial x^\lambda} \Big| \qquad g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} \qquad \text{def. of } g_{\mu\nu}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu} \eta_{\alpha\beta}$$

using the differential equation for ξ^α:

$$\Gamma^{\lambda}_{\mu\nu} \frac{\partial \xi^\beta}{\partial x^\lambda} = \frac{\partial^2 \xi^\beta}{\partial x^\mu \partial x^\nu}$$

we get

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^{\rho}_{\lambda\mu} \underbrace{\frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\nu} \eta_{\alpha\beta}}_{g_{\rho\nu}} + \Gamma^{\rho}_{\lambda\nu} \underbrace{\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\rho} \eta_{\alpha\beta}}_{g_{\mu\rho}}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} + \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu}$$

$\Downarrow$ add with $\mu \leftrightarrow \lambda$
subtract with $\nu \leftrightarrow \lambda$

$$\begin{aligned} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} &= g_{\rho\nu} \Gamma^{\rho}_{\lambda\mu} + g_{\rho\mu} \Gamma^{\rho}_{\lambda\nu} + \\ &+ g_{\rho\nu} \Gamma^{\rho}_{\mu\lambda} + g_{\rho\lambda} \Gamma^{\rho}_{\mu\nu} + \\ &- g_{\rho\lambda} \Gamma^{\rho}_{\nu\mu} - g_{\rho\mu} \Gamma^{\rho}_{\nu\lambda} = 2 g_{\rho\nu} \Gamma^{\rho}_{\lambda\mu} \end{aligned}$$

Define the inverse of $g_{\nu\sigma}$:

$$g^{\nu\sigma} g_{\rho\nu} = \delta^{\sigma}_{\rho} \qquad \text{then}$$

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \right]$$

this is called Christoffel symbol $\{ \sigma \}$

Let us consider a freely falling particle, the path is described by a parameter p. The proper time measured along the path reads

$$T_{BA} = \int_A^B \frac{d\tau}{dp}\, dp = \int_A^B \left(g_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp} \right)^{1/2} dp$$

$$d\tau = \left(g_{\mu\nu}\, dx^\mu\, dx^\nu \right)^{1/2}$$

$$\frac{d\tau}{dp} = \left(g_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp} \right)^{1/2}$$

Now vary the path from $x^\mu(p)$ to $x^\mu(p) + \delta x^\mu(p)$, keeping fixed the endpoints $(0 = \delta x^\mu(p_A) = \delta x^\mu(p_B))$:

$$\delta T_{BA} = \frac{1}{2} \int_A^B \left(g_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp} \right)^{-1/2} \left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda} \delta x^\lambda \frac{dx^\mu}{dp} \frac{dx^\nu}{dp} + 2 g_{\mu\nu} \frac{d\delta x^\mu}{dp} \frac{dx^\nu}{dp} \right] dp =$$

$$\equiv \int_A^B \left[\left(\; \right)^{1/2}\Big|_{x^\mu(p)+\delta x^\mu(p)} - \left(\; \right)^{1/2}\Big|_{x^\mu(p)} \right] dp$$

$$\int_A^B \frac{dp}{d\tau} \left[\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \delta x^\lambda \frac{dx^\mu}{dp} \frac{dx^\nu}{dp} + g_{\mu\nu} \frac{d\delta x^\mu}{dp} \frac{dx^\nu}{dp} \right] dp =$$

$$\llcorner = \frac{dp}{d\tau} d\tau$$

$$\int_A^B \left[\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \delta x^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{d\delta x^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] d\tau =$$

$\longleftarrow$ by parts with $\begin{cases} \delta x^\mu(p_B) = 0 \\ \delta x^\mu(p_A) = 0 \end{cases}$

$$\int_A^B \left[\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \underbrace{\frac{\partial g_{\lambda\nu}}{\partial x^\sigma} \frac{dx^\sigma}{d\tau}}_{(\sigma \to \mu)} \frac{dx^\nu}{d\tau} - g_{\lambda\nu} \frac{d^2 x^\nu}{d\tau^2} \right] \delta x^\lambda d\tau$$

use $\dfrac{\partial g_{\mu\nu}}{\partial x^\lambda} + \dfrac{\partial g_{\lambda\nu}}{\partial x^\mu} - \dfrac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2 g_{\sigma\nu} \Gamma^\sigma_{\lambda\mu}$

$(\sigma \leftrightarrow \nu)$ and symmetrie in $(\mu\nu)$

$$\int_A^B \left[- g_{\sigma\lambda} \Gamma^\sigma_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - g_{\lambda\nu} \frac{d^2 x^\nu}{d\tau^2} \right] \delta x^\lambda d\tau =$$

$$\delta T_{BA} = -\int_A^B \underbrace{\left(\Gamma^{\nu}_{\mu\sigma}\frac{dx^{\mu}}{d\tau}\frac{dx^{\sigma}}{d\tau} + \frac{d^2x^{\nu}}{d\tau^2}\right)}_{\substack{\shortparallel \\ 0}} g_{\lambda\nu}\,\delta x^{\lambda}\,d\tau$$

0 for a freely falling particle

$$\Downarrow$$

$$\delta T_{BA} = 0$$

Freely falling particle follows a path such that the proper time measured along the path is extremal ($\delta T_{BA} = 0$). Such paths are called geodesics.

The Newtonian limit

Let us consider a particle slowly moving in a weak stationary gravitational field.

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \underbrace{2\Gamma^{\mu}_{0i}\frac{dx^0}{d\tau}\frac{dx^i}{d\tau} + \Gamma^{\mu}_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}}_{} = 0$$

could be neglected for sufficiently slow particle $\left(\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau}\right)$

The gravitational field is stationary, so

$$\Gamma^{\mu}_{00} = \frac{1}{2} g^{\nu\mu}\Big[\underset{\substack{\shortparallel \\ 0}}{\frac{\partial g_{0\nu}}{\partial x^0}} + \underset{\substack{\shortparallel \\ 0}}{\frac{\partial g_{0\nu}}{\partial x^0}} - \frac{\partial g_{00}}{\partial x^{\nu}}\Big] = -\frac{1}{2}\frac{\partial g_{00}}{\partial x_{\mu}} g^{\nu\mu}$$

The field is weak:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad |h_{\alpha\beta}| \ll 1$$

to the first order

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\nu\mu}\frac{\partial h_{00}}{\partial x^{\nu}} +$$

Then $\displaystyle \frac{d^2 x^\mu}{d\tau^2} = \frac{1}{2}\,\eta^{\mu\nu}\,\frac{\partial h_{00}}{\partial x^\nu}\left(\frac{dt}{d\tau}\right)^2$, so

$$\frac{d^2 t}{d\tau^2} = 0 \qquad \& \qquad \frac{d^2 x^i}{d\tau^2} = -\frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \nabla_i h_{00} \quad \Big| \ \frac{1}{\left(\frac{dt}{d\tau}\right)^2}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\frac{dt}{d\tau} = \text{const} \qquad\qquad \frac{d^2 \bar{x}}{dt^2} = -\frac{1}{2}\,\bar{\nabla} h_{00}$$

The corresponding Newtonian result is

$$\frac{d^2 \bar{x}}{dt^2} = -\bar{\nabla}\phi$$

ϕ the gravitational potential

$\phi = -\dfrac{GM}{r}$ for spherical body of mass M

$$\phi = +\frac{1}{2} h_{00} + \text{const.}$$

$$h_{00} = +2\phi + \text{const.}$$

At large distances the coordinate system must become Minkowskian, so $h_{00} \underset{r\to\infty}{\to} 0$, therefore const. $= 0$

$$g_{00} = 1 + 2\phi$$

At the surface of a proton $\phi \sim 10^{-38}$

the earth 10^{-9}

the sun 10^{-6}

a white dwarf 10^{-4}

$\Downarrow$

gravitational distortion of $\eta_{\mu\nu}$ is very small

The Principle of General Covariance

as an alternative version of the Principle of Equivalence

To derive e.o.m. for a particle moving under the influence of gravity and electromagnetism one can write down the equation in a locally inertial coordinate system and then rewrite them in a general laboratory frame. That approach lead to tedious calculations. The Principle of General Covariance is provides an alternative but much more elegant and convenient way of the derivation.

A physical equation holds in a general gravitational field, if two conditions are met:

1. The equation holds in the absence of gravitation; that is, it agrees with the laws of special relativity when the metric tensor $g_{\alpha\beta}$ equals the Minkowski tensor $\eta_{\alpha\beta}$ and when the affine connection $\Gamma^{\alpha}_{\beta\gamma}$ vanishes.
2. The equation is generally covariant; that is, it preserves its form under a general coordinate transformation $x \to x'$.

The Principle of Equivalence

$\Downarrow$

The Principle of General Covariance

$$\underset{\substack{\Downarrow \\ \text{it is true} \\ \text{in locally} \\ \text{inertial frames}}}{1} \oplus \underbrace{2}_{\substack{\text{if it holds in} \\ \text{one frame it holds} \\ \text{in all}}} \Rightarrow \text{the equation is true in any frame in the presence of a general gravitational field}$$

Vectors and Tensors

Because of the Principle of General Covariance it is useful to know how various physical quantities transform under general coordinate transformations:

- sometimes (for those defined in terms of coordinate differentials) the transformation properties could be determined by a straightforward calculation
- sometimes, in part, the transformation properties are a matter of definition, e.g. electromagnetic field

Scalar: $\phi(x) \to \phi'(x') = \phi(x)$

for instance numbers: $\pi, \ldots$

$$d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

($g_{\mu\nu}$ is defined to transform in such a way that $d\tau^2$ is constant, see below)

Contravariant vectors:

$$V'^\mu(x') = V^\nu(x) \frac{\partial x'^\mu}{\partial x^\nu}$$

for instance, from the rules of partial differentiation we have

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \quad \Rightarrow \quad dx^\nu \text{ is a contravariant vector}$$

Covariant vectors:

$$U'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} U_\nu$$

for instance, if $\phi(x)$ is a scalar field then

$$\frac{\partial \phi}{\partial x'^\mu} = \frac{\partial \phi}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu}, \qquad \frac{\partial \phi}{\partial x^\mu} \to \left(\frac{\partial \phi}{\partial x^\mu}\right)' = \frac{\partial x^\nu}{\partial x'^\mu}\left(\frac{\partial \phi}{\partial x^\nu}\right)$$

Tensors:

e.g. $$T'^{\mu}{}_{\nu}{}^{\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\kappa}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} T^{\kappa}{}_{\rho}{}^{\sigma}$$

for instance the metric tensor is defined by

$$g_{\mu\nu} := \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}}$$

where ξ^{α} is a locally inertial coordinate system.
In a different system x'^{μ} the metric tensor is

$$g'_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x'^{\mu}} \frac{\partial \xi^{\beta}}{\partial x'^{\nu}} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} =$$

$$= g_{\rho\sigma} \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Rightarrow g_{\mu\nu} \text{ is a covariant tensor}$$

The inverse of the metric tensor $g^{\lambda\mu}$

$$g^{\lambda\mu} g_{\mu\nu} = \delta^{\lambda}_{\nu}$$

is contravariant:

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x'^{\mu}}{\partial x^{\sigma}} g^{\rho\sigma} g'_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x^{\kappa}}{\partial x'^{\mu}}}_{= \delta^{\kappa}_{\sigma}} g^{\rho\sigma} \frac{\partial x^{\eta}}{\partial x'^{\nu}} g_{\kappa\eta} = \delta^{\lambda}_{\nu}$$

$$g'_{\mu\nu} = g_{\kappa\eta} \frac{\partial x^{\kappa}}{\partial x'^{\mu}} \frac{\partial x^{\eta}}{\partial x'^{\nu}}$$

$$\delta^{\kappa}_{\sigma} \longrightarrow \delta^{\rho}_{\eta}$$

Therefore

$$g'^{\lambda\mu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x'^{\mu}}{\partial x^{\sigma}} g^{\rho\sigma} \qquad \square$$

For the Kronecker symbol we have

$$\delta^{\mu}_{\nu} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x'^{\sigma}} = \delta^{\rho}_{\sigma}$$

So δ^{μ}_{ν} is a mixed tensor whose components are the same in all coordinate system (aside from scalars and zero, it is the only tensor having that property)

Conclusion: Any equation will be invariant under general coordinate transformations if it states the equality of two tensors with the same upper and lower indices.

Linear combinations: $T^{\mu}{}_{\nu} \equiv a A^{\mu}{}_{\nu} + b B^{\mu}{}_{\nu}$, is a tensor if a, b are scalars and $A^{\mu}{}_{\nu}$, $B^{\mu}{}_{\nu}$ are tensors

- Direct products: $T^{\mu}{}_{\nu}{}^{\rho} = A^{\mu}{}_{\nu} B^{\rho}$ is a tensor if $A^{\mu}{}_{\nu}$ is a tensor and B^{ρ} is a vector (tensor)
- Contractions: $T^{\mu\rho} \equiv T^{\mu}{}_{\nu}{}^{\rho\nu}$ is a tensor if $T^{\mu}{}_{\nu}{}^{\rho\lambda}$ is a tensor

- Raising and lowering of indices:

tensors by the virtue of the above rules

$$S_{\nu}{}^{\sigma}{}_{\rho} \equiv g_{\mu\nu} T^{\mu\sigma}{}_{\rho} \qquad \text{lowering} \equiv T_{\nu}{}^{\sigma}{}_{\rho}$$

$$R^{\nu\rho}{}_{\sigma} = g^{\mu\nu} S_{\mu}{}^{\rho}{}_{\sigma} \qquad \text{raising} \equiv S^{\nu\rho}{}_{\sigma}$$

convention

Conclusion: The above rules of the tensor algebra are the same as those we have considered in special relativity (Lorentz transformation), exept of differentiation.

Tensor densities

An example of a nontensor

$$g \equiv - \mathrm{Det}\, g_{\mu\nu}$$

in a matrix form the transformation rules for $g_{\mu\nu}$

$$g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} g_{\rho\sigma} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Rightarrow g' = \left|\frac{\partial x}{\partial x'}\right|^2 g$$

$\left|\frac{\partial x}{\partial x'}\right|$ is the Jacobian of the transformation $x' \to x$

$= \mathrm{Det}\left(\frac{\partial x^\rho}{\partial x'^\mu}\right)$

A quantity that transforms as $g \to g' = \left|\frac{\partial x}{\partial x'}\right|^2 g$ is called a scalar density. In general for a tensor density $T^\mu{}_\nu{}^\lambda$:

$$T^\mu{}_\nu{}^\lambda \to \left(T^\mu{}_\nu{}^\lambda\right)' = \left|\frac{\partial x'}{\partial x}\right|^w \cdot \frac{\partial x'^\mu}{\partial x^\kappa}\frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x'^\lambda}{\partial x_\sigma} T^\kappa{}_\rho{}^\sigma$$

w – weight of the density

$\Downarrow$

since $\left|\frac{\partial x}{\partial x'}\right| = \left|\frac{\partial x'}{\partial x}\right|^{-1}$, so g is a density of $w = -2$

– Any tensor density of weight w can be expressed as an ordinary tensor times a factor $g^{-w/2}$.

For instance a tensor density $\mathcal{T}^\mu{}_\nu$ of weight w has the transformation rule

$$\mathcal{T}'^\mu{}_\nu = \left|\frac{\partial x'}{\partial x}\right|^w \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x^\kappa}{\partial x'^\nu}\mathcal{T}^\lambda{}_\kappa$$

$$g'^{w/2}\mathcal{T}'^\mu{}_\nu = \underbrace{g'^{w/2}\left|\frac{\partial x'}{\partial x}\right|^w}_{\left(g'\cdot\left|\frac{\partial x'}{\partial x}\right|^2\right)^{w/2} = g^{w/2}} \frac{\partial x'^\lambda}{\partial x^\lambda}\frac{\partial x^\kappa}{\partial x'^\nu}\mathcal{T}^\lambda{}_\kappa =$$

$$= \underbrace{\left(g^{w/2}\mathcal{T}^\lambda{}_\kappa\right)}_{\text{tensor}}\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x^\kappa}{\partial x'^\nu}$$

$$d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x$$

$\Downarrow$

d^4x . (tensor density of $w = -1$) is an ordinary tensor, e.g.

$\sqrt{g}\, d^4x$ is an invariant volume

– The Levi-Civita symbol $\varepsilon^{\mu\nu\lambda\kappa}$ is the only tensor density whose components are the same in all coordinate systems.

$$\varepsilon^{\mu\nu\lambda\kappa} = \begin{cases} +1 & \mu\nu\lambda\kappa \text{ even permutation of same reference sequence} \\ -1 & \mu\nu\lambda\kappa \text{ odd — " — " —} \\ 0 & \text{some indices are equal} \end{cases}$$

Homework is to show that

$$\frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial x'^{\eta}}{\partial x^{\lambda}} \frac{\partial x'^{\xi}}{\partial x^{\kappa}} \varepsilon^{\mu\nu\lambda\kappa} = \left| \frac{\partial x'}{\partial x} \right| \varepsilon^{\rho\sigma\eta\xi}$$

$$\Downarrow$$

$g^{-1/2} \varepsilon^{\mu\nu\lambda\kappa}$ is a tensor $\Leftarrow$ $\varepsilon^{\mu\nu\lambda\kappa}$ is a tensor density of weight $w=-1$, the corresponding tensor is the same in all coordinate systems:

$$\varepsilon^{\rho\sigma\eta\xi} = \left| \frac{\partial x'}{\partial x} \right|^{-1} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \cdots \varepsilon^{\mu\nu\lambda\kappa}$$

we say that the rhs $= \varepsilon'^{\rho\sigma\eta\xi}$

– covariant density:

$$\varepsilon_{\rho\sigma\eta\xi} \equiv g_{\rho\mu} g_{\sigma\nu} g_{\eta\lambda} g_{\xi\kappa} \varepsilon^{\mu\nu\lambda\kappa}$$

of weight $w=-1$

– $\varepsilon_{\rho\sigma\eta\xi}$ is totaly antisymmetric, so it must be proportional to $\varepsilon^{\rho\sigma\eta\xi}$, in fact one can show (homework) that

$$\varepsilon_{\rho\sigma\eta\xi} = +g\, \varepsilon^{\rho\sigma\eta\xi}$$

The rules of tensor algebra holds for densities as well.

Transformation of the Affine Connection

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$$

ξ ← locally inertial coordinate

In a different frame we obtain

$$\Gamma'^{\lambda}_{\mu\nu} \equiv \frac{\partial x'^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right) =$$

$$= \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial \xi^{\alpha}} \left[\frac{\partial x^{\sigma}}{\partial x'^{\nu}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau} \partial x^{\sigma}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu} \partial x'^{\nu}} \frac{\partial \xi^{\alpha}}{\partial x^{\sigma}} \right] =$$

$$= \underbrace{\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma}}_{\text{consistent with the tensor transformation rules}} + \underbrace{\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}}}_{\text{inhomogeneous term which makes the connection a nontensor}}$$

An alternative formula for the inhomogeneous term:

$$\frac{\partial}{\partial x'^{\mu}} \Bigg| \quad \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} = \delta^{\lambda}_{\nu}$$

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}} = - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\sigma} \partial x^{\rho}}$$

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} \frac{\partial^2 x'^{\lambda}}{\partial x^{\sigma} \partial x^{\rho}}$$

An alternative derivation of the equation of motion for a freely falling particle, as an illustration of the power of the Principle of General Covariance.

The equation to be derived:

$$(*) \qquad \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \qquad \text{for} \qquad d\tau^2 = g_{\mu\nu}\,dx^\mu dx^\nu$$

– in the absence of gravitation the above equations are indeed true:

$$\frac{d^2x^\mu}{d\tau^2} = 0 \qquad \text{and} \qquad d\tau^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu$$

as equation of motion for a free particle in special relativity

– let's check the covariance of (*) under a general coordinate transformation:

$$\frac{d^2x'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) = \frac{\partial x'^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau}$$

while from the transformation rule for the affine connection we get

$$\Gamma'^\mu_{\sigma\tau}\frac{dx'^\sigma}{d\tau}\frac{dx'^\tau}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\Gamma^\nu_{\lambda\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} - \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\lambda}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau}$$

$$\Downarrow$$

$$\frac{d^2x'^\mu}{d\tau^2} + \Gamma'^\mu_{\nu\lambda}\frac{dx'^\nu}{d\tau}\frac{dx'^\lambda}{d\tau} = \frac{\partial x'^\mu}{\partial x^\kappa}\left[\frac{d^2x^\kappa}{d\tau^2} + \Gamma^\kappa_{\sigma\rho}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau}\right]$$

Thus the equations (*) are generally covariant, then by the virtue of the Principle of General Covariance we can say that (*) holds in a general gravitational field.

Covariant derivative

Differentiation of a tensor does not generally yield another tensor, for instance consider a contravariant vector

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu} \qquad \Big| \ \frac{\partial}{\partial x'^{\lambda}}$$

$$\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} = \underbrace{\frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial V^{\nu}}{\partial x^{\rho}}}_{\text{a proper tensor transformation}} + \underbrace{\frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}}_{\text{this destroys the tensor behavior}}$$

Although $\frac{\partial V^{\mu}}{\partial x^{\lambda}}$ is not a tensor we will use it to construct a tensor:

$$\Gamma'^{\mu}_{\lambda\kappa} V'^{\kappa} = \left[\frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial x^{\sigma}}{\partial x'^{\kappa}} \Gamma^{\nu}_{\rho\sigma} - \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial x^{\sigma}}{\partial x'^{\kappa}} \right] \frac{\partial x'^{\kappa}}{\partial x^{\eta}} V^{\eta} =$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \Gamma^{\nu}_{\rho\sigma} V^{\sigma} - \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\sigma}$$

$$\Downarrow$$

$$\underbrace{\frac{\partial V'^{\mu}}{\partial x'^{\lambda}} + \Gamma'^{\mu}_{\lambda\kappa} V'^{\kappa}}_{\text{this is a tensor}} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \left(\frac{\partial V^{\nu}}{\partial x^{\rho}} + \Gamma^{\nu}_{\rho\sigma} V^{\sigma} \right)$$

$$V^{\mu}{}_{;\lambda} \equiv \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa} \qquad \text{covariant derivative}$$

$$V'^{\mu}{}_{;\lambda} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}{}_{;\rho} \qquad \leftarrow \text{this is a tensor}$$

A covariant derivative of a covariant vector:

$$V_{\mu;\nu} \equiv \frac{\partial V_{\mu}}{\partial x^{\nu}} - \Gamma^{\lambda}_{\mu\nu} V_{\lambda} \qquad \text{is a tensor} \quad \text{(homework)}$$

$$V'_{\mu;\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} V_{\rho;\sigma}$$

– Covariant derivative for a general tensor

$$T^{::}{}_{;\rho} = \frac{\partial}{\partial x^\rho} T^{::} + \dots + \Gamma^\mu_{\nu\rho}\, T^{:\overset{\nu}{\underset{\mu}{\uparrow}}:} + \dots$$

$$- \Gamma^\kappa_{\lambda\rho}\, T^{:}{}_{\underset{\kappa}{\overset{\lambda}{\downarrow}}} + \dots$$

For instance

$$(*) \qquad T^{\mu\sigma}{}_{\lambda;\rho} = \frac{\partial}{\partial x^\rho} T^{\mu\sigma}{}_\lambda + \Gamma^\mu_{\rho\nu} T^{\nu\sigma}{}_\lambda + \Gamma^\sigma_{\rho\nu} T^{\mu\nu}{}_\lambda + - \Gamma^\kappa_{\lambda\rho} T^{\mu\sigma}{}_\kappa$$

Homework: show that $T^{\mu\sigma}{}_{\lambda;\rho}$ is a tensor.

– Covariant derivative of a tensor density of weight W $\boxed{\mathfrak{F}}$

note that $g^{W/2}\,\mathfrak{F}$ is an ordinary tensor, so

$$\mathfrak{F}^{:::}{}_{;\rho} \equiv g^{-W/2}\left(g^{W/2}\,\mathfrak{F}^{:::}\right)_{;\rho}$$

– Properties of covariant differentiation:

a)
$$(\alpha A^\mu{}_\nu + \beta B^\mu{}_\nu)_{;\lambda} = \alpha A^\mu{}_{\nu;\lambda} + \beta B^\mu{}_{\nu;\lambda}$$

b) homework: prove the Leibniz rule, e.g.

$$(A^\mu{}_\nu B^\lambda)_{;\rho} = A^\mu{}_{\nu;\rho} B^\lambda + A^\mu{}_\nu B^\lambda{}_{;\rho}$$

c) The covariant derivative of a contracted tensor is the contraction of the covariant derivative, for instance let's contract (*) with δ^λ_σ

$$\delta^\lambda_\sigma T^{\mu\sigma}{}_{\lambda;\rho} = \frac{\partial}{\partial x^\rho} T^{\mu\lambda}{}_\lambda + \Gamma^\mu_{\rho\nu} T^{\nu\lambda}{}_\lambda + \cancel{\Gamma^\lambda_{\rho\nu} T^{\mu\nu}{}_\lambda} - \cancel{\Gamma^\kappa_{\lambda\rho} T^{\mu\lambda}{}_\kappa}$$

$$= \left(T^{\mu\lambda}{}_\lambda\right)_{;\rho}$$

– the covariant derivative of the metric tensor:

$$g_{\mu\nu;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \Gamma^{\varkappa}_{\mu\lambda} g_{\varkappa\nu} - \Gamma^{\varkappa}_{\nu\lambda} g_{\mu\varkappa}$$

We have shown that $\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} + \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu}$, so

$$\Downarrow$$

$$g_{\mu\nu;\lambda} = 0$$

$g_{\mu\nu;\lambda}$ should indeed vanish, since it is a covariant equation and there is a coordinate system where $\Gamma^{\mu}_{\nu\lambda} = 0$ and $\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = 0$ this is the locally inertial frame. Therefore, according to the Principle of General Covariance $g_{\mu\nu;\lambda} = 0$ in all system in the presence of general gravitational field.

Also we have: $g^{\mu\nu}{}_{;\lambda} = 0$ & $\delta^{\mu}{}_{\nu;\lambda} = 0$

$$g_{\mu\nu;\lambda} = 0 \quad \Downarrow$$

covariant differentiation commutes with raising and lowering indices e.g.

$$(g^{\mu\nu} V_\nu)_{;\lambda} = \underbrace{g^{\mu\nu}{}_{;\lambda}}_{=0} V_\nu + g^{\mu\nu} V_{\nu;\lambda}$$

Important properties of covariant differentiation:

The recipe for "gravitational interactions"

– $(\text{Tensor})_{;\lambda}$ is a tensor

– in the absence of gravity ($\Gamma^{\mu}_{\nu\lambda} = 0$) it reduces to ordinary differentiation

$\Rightarrow$ write the appropriate special-relativistic equations that hold in the absence of gravitation, replace η by g and ordinary derivatives with covariant derivatives

Gradient, Curl and Divergence

- for a scalar S, $\quad S_{;\mu} = \frac{\partial S}{\partial x^\mu}$ is a vector

- covariant curl ("rotation" or "circulation")

$$V_{\mu;\nu} \equiv \frac{\partial V_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} V_\lambda$$

$$V_{\mu;\nu} - V_{\nu;\mu} = \frac{\partial V_\mu}{\partial x^\nu} - \cancel{\Gamma^\lambda_{\mu\nu} V_\lambda} - \frac{\partial V_\nu}{\partial x^\mu} + \cancel{\Gamma^\lambda_{\nu\mu} V_\lambda} = \frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu}$$

covariant curl $\qquad$ as $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ $\qquad$ ordinary curl

- covariant divergence of a contravariant vector

$$V^\mu_{;\mu} = \frac{\partial V^\mu}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda} V^\lambda \quad \text{but} \quad \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{s\mu}\left\{ \frac{\partial g_{\lambda s}}{\partial x^\nu} + \frac{\partial g_{\nu s}}{\partial x^\lambda} - \frac{\partial g_{\lambda\nu}}{\partial x^s} \right\}$$

$$\Downarrow$$

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{s\mu}\left\{ \cancel{\frac{\partial g_{\lambda s}}{\partial x^\mu}} + \frac{\partial g_{\mu s}}{\partial x^\lambda} - \cancel{\frac{\partial g_{\lambda\mu}}{\partial x^s}} \right\} = \frac{1}{2} g^{s\mu} \frac{\partial g_{\mu s}}{\partial x^\lambda}$$

To calculate $\Gamma^\mu_{\mu\lambda}$ we use the following identity:

$$\mathrm{Tr}\left\{ H^{-1}(x) \frac{\partial}{\partial x^\lambda} H(x) \right\} = \frac{\partial}{\partial x^\lambda} \ln \mathrm{Det}\, H(x)$$

(homework is to prove it)

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} \frac{\partial}{\partial x^\lambda} \ln g = \qquad \left(\text{for } H(x) = i^{1/2} g_{s\mu} \right)$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\lambda} \sqrt{g}$$

$$\Downarrow$$

$$V^\mu_{;\mu} = \frac{\partial V^\mu}{\partial x^\mu} + \frac{1}{\sqrt{g}} \left(\frac{\partial}{\partial x^\lambda} \sqrt{g} \right) V^\lambda = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\lambda} \left(\sqrt{g}\, V^\lambda \right)$$

$\int d^4x \;\Big|\quad \sqrt{g}\, V^\mu{}_{;\mu} = \frac{\partial}{\partial x^\mu}\left(\sqrt{g}\, V^\mu\right)$

$$\int_M d^4x \sqrt{g}\, V^\mu{}_{;\mu} = \int_M d^4x\, \frac{\partial}{\partial x^\mu}\left(\sqrt{g} V^\mu\right) = \int_{\partial M} dS_\mu \sqrt{g}\, V^\mu$$

Gauss's theorem

if V^μ vanishes at infinity, then

$$\int d^4x \sqrt{g}\, V^\mu{}_{;\mu} = 0$$

— Let us consider $T^{\mu\nu}{}_{;\mu} = \frac{\partial T^{\mu\nu}}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda}$

but $\Gamma^\mu_{\mu\lambda} = \left(\frac{\partial}{\partial x^\lambda}\sqrt{g}\right)\frac{1}{\sqrt{g}}$, then

$$T^{\mu\nu}{}_{;\mu} = \frac{\partial T^{\mu\nu}}{\partial x^\mu} + \frac{1}{\sqrt{g}}\left(\frac{\partial}{\partial x^\lambda}\sqrt{g}\right) T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda} =$$

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{g}\, T^{\mu\nu}\right) + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda}$$

if $T^{\mu\lambda} = -T^{\lambda\mu}$ then $\quad T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{g}\, T^{\mu\nu}\right)$

— Let's now consider $A_{\mu\nu;\lambda} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} - \Gamma^\rho_{\mu\lambda} A_{\rho\nu} - \Gamma^\rho_{\nu\lambda} A_{\mu\rho}$

suppose $A_{\mu\nu} = -A_{\nu\mu}$, then by the virtue of the symmetry of $\Gamma^\rho_{\mu\nu}$ and the antisymmetry of $A_{\mu\nu}$ we find that

$$\underbrace{A_{\mu\nu;\lambda} + A_{\nu\lambda;\mu} + A_{\lambda\mu;\nu}}_{\text{cíclic permutation of } \mu\nu;\lambda} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A_{\nu\lambda}}{\partial x^\mu} + \frac{\partial A_{\lambda\mu}}{\partial x^\nu}$$

all Γ-terms cancel

The Electromagnetic Analogy

The general covariance is not an ordinary symmetry principle like the Lorentz invariance, but is rather a dynamical rule that tells us how to include effects of gravitational interaction. Therefore it is similar to local gauge invariance.

$$\Downarrow$$

e.o.m. should be invariant under

$$\phi(x) \to e^{ie\varphi(x)}\,\phi(x)$$

$$A_\alpha(x) \to A_\alpha(x) + \frac{\partial \varphi(x)}{\partial x^\alpha}$$

where $\varphi = \varphi(x)$ is an arbitrary function of x^α.

$$\frac{\partial \phi(x)}{\partial x^\alpha} \to \frac{\partial}{\partial x^\alpha}\left[e^{i\varphi(x)}\,\phi(x)\right] = e^{ie\varphi(x)}\left[\frac{\partial \phi}{\partial x^\alpha} + ie\,\phi(x)\frac{\partial \varphi}{\partial x^\alpha}\right]$$

Note that an ordinary derivative of a tensor doesn't behave like a tensor under general coordinate transformations.

$\to$ $(\Box + m^2)\,\phi(x) = 0 \qquad \Box \equiv \eta^{\alpha\beta}\dfrac{\partial}{\partial x^\alpha}\dfrac{\partial}{\partial x^\beta}$

is not invariant under a local gauge transformation, but symmetric under a global transformation

QED ——— Covariant derivative ——— GR

$$D_\alpha \phi(x) \equiv \frac{\partial \phi(x)}{\partial x^\alpha} - ie\,A_\alpha(x)\,\phi(x) \qquad\qquad V^\mu{}_{;\lambda} = \frac{\partial V^\mu(x)}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa}(x)\,V^\kappa(x)$$

$$\Downarrow$$

$$D_\alpha \phi(x) \to e^{ie\varphi(x)}\,D_\alpha \phi(x) \qquad\qquad V^\mu{}_{;\lambda} \longrightarrow \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\varsigma}{\partial x'^\lambda}\,V^\nu{}_{;\varsigma}$$

local gauge transformation — general coordinate transformation

$$A_\alpha(x) \longleftrightarrow \Gamma^\mu_{\lambda\kappa}(x)$$

global gauge transformation $\longleftrightarrow$ Lorentz transformation

QED — Covariant equations — GR

made of $\Phi(x)$ and $D_\alpha \Phi(x)$

made of tensors and their covariant derivatives

An equation invariant under [th]e global gauge transformation [w]ill be invariant under local transformations if

$$\frac{\partial \phi}{\partial x^\alpha} \to D_\alpha \phi$$

$$\Downarrow$$

$$\left(\eta^{\alpha\beta} D_\alpha D_\beta + m^2\right) \Phi(x) = 0$$

[loc]al gauge invariance

$$\Downarrow$$

$$J_\alpha(x) = -ie\left\{\Phi^{\dagger} D_\alpha \Phi - \Phi D_\alpha \Phi^{\dagger}\right\}$$

J_α is conserved

and gauge invariant (obvious)

An equation invariant under Lorentz transformations will be invariant under general transformations if ordinary derivatives are replaced by covariant derivatives

the gravitational analogy will be shown later.

Electrodynamics

Maxwell equations : $\qquad \dfrac{\partial}{\partial x^\alpha} F^{\alpha\beta} = - J^\beta$

(no gravitation !) $\qquad$ (*)

$$\frac{\partial}{\partial x_\alpha} F_{\beta\gamma} + \frac{\partial}{\partial x_\beta} F_{\gamma\alpha} + \frac{\partial}{\partial x_\gamma} F_{\alpha\beta} = 0$$

where

$$J^\alpha(x) = \int d\tau \sum_n e_n \, \delta^4(x - x_n(\tau)) \, \frac{d x_n^\alpha}{d\tau} \qquad \text{four-vector}$$

$$F^{12} = B_3 \; , \; F^{01} = E_1 \; , \; \ldots \qquad \text{field-strength tensor}$$

Let us define $F^{\mu\nu}$ and J^μ in general coordinates by the requirement that they reduce to $F^{\alpha\beta}$ and J^β in locally inertial coordinates and that they transform as tensors upon general coordinate transformations, i.e. if $\tilde{F}^{\alpha\beta}$ and $\tilde{J}^\alpha$ are the values measured in a locally inertial frame, then

$$F^{\mu\nu} \equiv \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta} \tilde{F}^{\alpha\beta} \; , \qquad J^\mu \equiv \frac{\partial x^\mu}{\partial \xi^\alpha} J^\alpha$$

definition

Then we make (*) generally covariant by replacing all derivatives by covariant derivatives :

$$(**) \qquad F^{\mu\nu}{}_{;\mu} = - J^\nu$$

$$F_{\mu\nu;\lambda} + F_{\nu\lambda;\mu} + F_{\lambda\mu;\nu} = 0 \qquad \text{and} \qquad F_{\lambda\kappa} \equiv g_{\lambda\mu} g_{\kappa\nu} F^{\mu\nu}$$

As $F_{\mu\nu} = - F_{\nu\mu}$ we can use the formula for a covariant divergence of a tensor :

$$A^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{g} \, A^{\mu\nu} \right) \qquad \text{for } A^{\mu\nu} = F^{\mu\nu}$$

then

$$\frac{\partial}{\partial x^\mu} \sqrt{g} F^{\mu\nu} = - \sqrt{g} \, J^\nu$$

We have also shown that for an antisymmetric tensor the following rule applies

$$A_{\mu\nu;\lambda} + A_{\nu\lambda;\mu} + A_{\lambda\mu;\nu} = \frac{\partial A_{\mu\nu}}{\partial x^\lambda} + \frac{\partial A_{\nu\lambda}}{\partial x^\mu} + \frac{\partial A_{\lambda\mu}}{\partial x^\nu}$$

Therefore we have in general relativity

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} = 0$$

Since the equations (**) are generally covariant and they hold in the absence of gravity, therefore according to the Principle of General Covariance, they are true for an arbitrary gravitational field.

- The electromagnetic force acting on a particle of charge e, in the absence of gravitation is given by

$$f^\alpha = e\, F^\alpha{}_\beta \frac{dx^\beta}{d\tau}$$

If $F^\alpha{}_\beta = g_{\beta\sigma} F^{\alpha\sigma}$, then f^α is generally covariant, since dx^β is a general vector and $d\tau = g_{\mu\nu}\, dx^\mu dx^\nu$ is a scalar. Since it holds in the absence of gravitation, the equation (again by the virtue of the PGC) is true for any gravitational fields.

- In special relativity the current vector J^ν is given by

$$J^\alpha = \sum_n e_n \int \delta^4(x - x_n)\, dx_n^\alpha$$ (along the trajectory of nth particle)

in a general coordinate system we define $\delta^4()$ as

$$\int d^4x\, \Phi(x)\, \delta^4(x-y) = \Phi(y)$$

since $\sqrt{g}\, d^4x$ is a general scalar therefore

$\frac{1}{\sqrt{g}} \delta^4(x-y)$ must be a scalar as well

The correct generalization of the current reads

$$J^\mu(x) = g^{-1/2}(x) \sum_n e_n \int d x_n^\mu \, \delta^4(x - x_n)$$

general vector

The special relativity conservation law

$$\frac{\partial J^\alpha}{\partial x^\alpha} = 0$$

is now replaced by $J_{\alpha;\alpha} = 0$ or

$$\frac{\partial}{\partial x^\alpha}\left(g^{1/2} J^\alpha\right) = 0$$

Energy-Momentum Tensor

In special relativity we had the following form of energy-momentum conservation:

$$\frac{\partial T^{\alpha\beta}}{\partial x^\alpha} = G^\beta$$

density of external force,

for isolated systems $G^\beta = 0$

As before we define $T^{\mu\nu}$ and G^ν as contravariant tensors that reduce to $T^{\alpha\beta}$ and G^β in the absence of gravitation, therefore for a general gravitation field we obtain

$$T^{\mu\nu}{}_{;\mu} = G^\nu$$

We have shown that for the covariant divergence of a tensor can be written as

$$T^{\mu\nu}{}_{;\mu} = \frac{\partial T^{\mu\nu}}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left(\sqrt{g}\, T^{\mu\nu}\right) + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda}$$

Therefore we obtain

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{g}\,T^{\mu\nu}\right) = G^\nu - \underbrace{\Gamma^\nu_{\mu\lambda}\,T^{\mu\lambda}}_{\text{gravitational force density}}$$

For a system of point particles we had in special relativity

$$T^{\alpha\beta} = \sum_n m_n \int \frac{dx_n^\alpha}{d\tau}\, dx_n^\beta\, \delta^4(x-x_n)$$ integrals along trajectories

For the same reasons as in the case of J^α we get in general relativity

$$T^{\mu\nu} = g^{-1/2}\sum_n m_n \int \frac{dx_n^\mu}{d\tau}\, dx_n^\nu\, \delta^4(x-x_n)$$

For an electromagnetic field we had

$$T^{\alpha\beta} = F^\alpha{}_\gamma F^{\beta\gamma} - \frac{1}{4}\eta^{\alpha\beta} F_{\gamma\delta}F^{\gamma\delta}$$

in GR

$$T^{\mu\nu} = F^\mu{}_\lambda F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F_{\gamma\delta}F^{\gamma\delta}$$

where $F^\mu{}_\lambda = g_{\lambda\rho}F^{\mu\rho}$ and $F_{\gamma\delta} = g_{\gamma\rho}\, g_{\delta\lambda}\, F^{\rho\lambda}$

Curvature

We are going to construct a tensor out of the metric tensor and its derivatives:

- no new tensor can be made using only first derivatives ($g_{\mu\nu;\lambda} = 0$)
- let's try second derivatives

Transformation rule for the affine connection

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x'^{\tau}} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \Gamma'^{\tau}_{\rho\sigma} + \frac{\partial x^{\lambda}}{\partial x'^{\tau}} \frac{\partial^2 x'^{\tau}}{\partial x^{\mu} \partial x^{\nu}}$$

Subtracting the above equation with ν and κ interchanged leads to

$$0 = \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \left(\frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu} \Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa} \Gamma^{\lambda}_{\nu\eta} \right) +$$

$$- \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial x'^{\eta}}{\partial x^{\kappa}} \left(\frac{\partial \Gamma'^{\tau}_{\rho\sigma}}{\partial x'^{\eta}} - \frac{\partial \Gamma'^{\tau}_{\rho\eta}}{\partial x'^{\sigma}} - \Gamma'^{\tau}_{\lambda\sigma} \Gamma'^{\lambda}_{\eta\rho} + \Gamma'^{\tau}_{\lambda\eta} \Gamma'^{\lambda}_{\sigma\rho} \right)$$

$$\Downarrow$$

$$R'^{\tau}{}_{\rho\sigma\eta} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} \frac{\partial x^{\kappa}}{\partial x'^{\eta}} R^{\lambda}{}_{\mu\nu\kappa}$$

for $R^{\lambda}{}_{\mu\nu\kappa} \equiv \dfrac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \dfrac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu} \Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa} \Gamma^{\lambda}_{\nu\eta}$

that is called the Riemann-Christoffel curvature tensor.

Homework: prove that $R^{\lambda}{}_{\mu\nu\kappa}$ is indeed a tensor.

The Ricci tensor: $R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa}$

The Ricci scalar (curvature scalar): $R \equiv g^{\mu\kappa} R_{\mu\kappa}$