Feynman Path integral in QM

position representation

$$\hat{x}|x\rangle = x|x\rangle$$
$$\int dx\, |x\rangle\langle x| = \mathbb{1}$$

concept of propagator

$$|\psi(t')\rangle = \hat{U}(t', t)\, |\psi(t)\rangle \qquad \Big/ \langle x|$$

unitary evolution operator

$$\hat{U}(t', t) = e^{-i\frac{\hat{H}(t'-t)}{\hbar}}$$

$$\underbrace{\langle x|\psi(t')\rangle}_{\psi(x,t')} = \int dy\, \underbrace{\langle x|\hat{U}(t',t)|y\rangle}_{G(xt',\, yt)} \underbrace{\langle y|\psi(t)\rangle}_{\psi(y,t)}$$

propagator

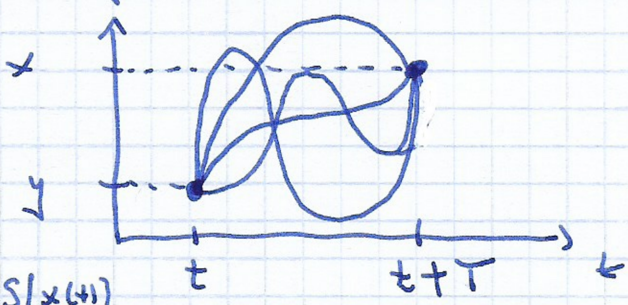
$$\boxed{\psi(x,t') = \int dy\ G(xt', yt)\, \psi(y,t)}$$

consider $t' = t + T$ and $G_T(x,y) \equiv G(x, t+T, y, t)$

since $\hat{H}\, \phi_n(x) = E_n\, \phi_n(x)$ we have

$$G_T(x,y) = \sum_n \phi_n(x)\, \phi_n^*(y)\, e^{-i\frac{E_n T}{\hbar}}$$

In 1948 Feynman discovered that this propagator can be expressed as a sum over all paths



$$G_T(x,y) = \int \mathcal{D}[x(t)]\, e^{i\frac{S[x(t)]}{\hbar}}$$

Derivation via Feynmann

take $\underline{T = \varepsilon}$ - small

$$\Psi(x, t+\varepsilon) = \int dy\, G_\varepsilon(x,y)\, \Psi(y,t)$$

$\Rightarrow$

$$\Psi(x, t+2\varepsilon) = \int dy \int dx_1\, G_\varepsilon(x,x_1)\, G_\varepsilon(x_1,y)\, \Psi(y,t)$$

$$\Psi(x, t+3\varepsilon) = \int dy \int dx_2\, dx_1\, G_\varepsilon(x,x_2)\, G_\varepsilon(x_2,x_1)\, G_\varepsilon(x_1,y)\, \Psi(y,t)$$

$$\vdots$$

by induction

$$\Psi(x, t+N\varepsilon) = \int dy \int \left(\prod_{i=1}^{N-1} dx_i\right) G_\varepsilon(x, x_{N-1})\, G_\varepsilon(x_{N-1}, x_{N-2}) \cdots$$

$$\cdots G_\varepsilon(x_1, y)\, \Psi(y,t)$$

Hence, the propagator at $\underline{T = N\varepsilon}$ is

$$G_T(x,y) = \int \left(\prod_{i=1}^{N-1} dx_i\right) G_\varepsilon(x, x_{N-1})\, G_\varepsilon(x_{N-1}, x_{N-2}) \cdots G_\varepsilon(x_1, y)$$

y at time t

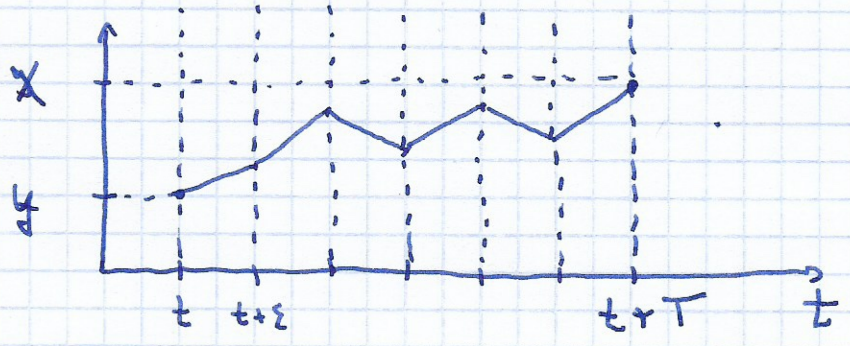
x_1 at time $t+\varepsilon$

x_2 at time $t+2\varepsilon$

$\vdots$

x_{N-1} at time $t+(N-1)\varepsilon$

x at time $t+T$



when $N \to \infty$ we get a smooth path $x(t)$

How to find $G_\varepsilon(x,y)$ without solving $\hat{H}\phi_n = E_n\phi_n$?

Schrödinger equation at small ε

$$i\hbar \frac{\Psi(x,t+\varepsilon) - \Psi(x,t)}{\varepsilon} \simeq \hat{H}\,\Psi(x,t)$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

Then

$$\Psi(x,t+\varepsilon) = \left(1 + \frac{\varepsilon}{i\hbar}\hat{H}\right)\Psi(x,t) + \mathcal{O}(\varepsilon^2) =$$

$$= \left(1 - i\frac{\varepsilon}{\hbar}V(x) + \frac{i\varepsilon\hbar}{2m}\frac{d^2}{dx^2} + \mathcal{O}(\varepsilon^2)\right)\Psi(x,t) =$$

$$= \underline{e^{-i\frac{\varepsilon V(x)}{\hbar}}\left(\Psi(x,t) + \frac{i\varepsilon\hbar}{2m}\Psi''(x,t)\right)} + \underline{\underline{\mathcal{O}(\varepsilon^2)}}$$

Let $y = x + \eta$ and expand $\Psi(y,t)$

$$\Psi(x,t+\varepsilon) = \int dy\, G_\varepsilon(x,y)\,\Psi(y,t) =$$

$$= \underline{\int d\eta\, G_\varepsilon(x,x+\eta)\left[\Psi(x,t) + \Psi'(x,t)\eta + \frac{1}{2}\Psi''(x,t)\eta^2 + \dots\right]}$$

Comparing these two results

$$\int d\eta\, G_\varepsilon(x,x+\eta) = e^{-i\frac{\varepsilon V(x)}{\hbar}} \qquad (1)$$

$$\int d\eta\, G_\varepsilon(x,x+\eta)\,\eta = 0 \qquad (2)$$

$$\int d\eta\, G_\varepsilon(x,x+\eta)\,\eta^2 = \frac{i\varepsilon\hbar}{2m} \qquad (3)$$

and the boundary condition

$$\lim_{\varepsilon\to 0} G_\varepsilon(x,x+\eta) = \delta(\eta) \quad \text{Dirac delta}$$

We take a Gaussian delta-sequence

$$G_\varepsilon(x, x+\eta) = e^{-i\frac{\varepsilon V(x)}{\hbar}} \sqrt{\frac{A}{\pi}}\, e^{-A\eta^2}$$

Dirac delta in the limit $A \to \infty$, $\varepsilon \to 0$

- Show that it satisfies (1) and (2).

From condition (3) we find A (in $\varepsilon \to 0$ limit) (*)

$$\frac{i\varepsilon\hbar}{m} = \int d\eta \sqrt{\frac{A}{\pi}}\, e^{-A\eta^2}\, \eta^2 = \frac{1}{2A} \to \boxed{A = \frac{m}{2i\varepsilon\hbar}}$$

$$G_\varepsilon(x, x+\eta) = e^{-i\frac{\varepsilon V(x)}{\hbar}} \sqrt{\frac{m}{2\pi i\varepsilon\hbar}}\, e^{-\frac{m}{2i\varepsilon\hbar}\eta^2}$$

recalling $\eta = x - y$ we get up to the first order in ε

$$\boxed{G_\varepsilon(x,y) = \left(\frac{m}{2i\pi\varepsilon\hbar}\right)^{1/2} e^{\,i\frac{\varepsilon}{\hbar}\left[\frac{1}{2}m\left(\frac{x-y}{\varepsilon}\right)^2 - V(x)\right]}}$$

(*) $$\frac{i\varepsilon\hbar}{m} = \frac{1}{2A}\, e^{-i\frac{\varepsilon V}{\hbar}} \to A = \frac{m}{2i\varepsilon\hbar}\, e^{-i\frac{\varepsilon V}{\hbar}} =$$

$$= \frac{m}{2i\varepsilon\hbar}\left(1 - i\frac{\varepsilon V}{\hbar} - \frac{\varepsilon^2 V^2}{2\hbar^2} + \dots\right)$$

most obviously $\xrightarrow[\varepsilon\to 0]{} \dfrac{m}{2i\varepsilon\hbar}$ □

To get the exact result we take $N \to \infty$ limit with $\varepsilon = \frac{T}{N} \to 0$

$$G_T(x,y) = \lim_{N\to\infty} \int \left(\prod_{i=1}^{N-1} dx_i\right) G_\varepsilon(x, x_{N-1})\, G_\varepsilon(x_{N-1}, x_{N-2}) \cdots G_\varepsilon(x_1, y)$$

$$= \lim_{N\to\infty} \int \left(\prod_{i=1}^{N-1} dx_i\right) \left(\frac{m}{2i\pi\varepsilon\hbar}\right)^{N/2} e^{\frac{i}{\hbar}\sum_{n=1}^{N} \varepsilon\left[\frac{1}{2}m\left(\frac{x_n - x_{n-1}}{\varepsilon}\right)^2 - V(x_n)\right]}$$

where $x_0 = y$ and $x_N = x$.

Notation

$$\int \mathcal{D}[x(t)] = \lim_{N\to\infty} \int \left(\prod_{i=1}^{N-1} dx_i\right) \left(\frac{m}{2i\pi\varepsilon\hbar}\right)^{N/2}$$

and in the $\varepsilon \to 0$ limit

$$\lim_{N\to\infty} \sum_{n=1}^{N} \varepsilon\left[\frac{1}{2}m\left(\frac{x_n - x_{n-1}}{\varepsilon}\right)^2 - V(x_n)\right] = \int_t^{t+T} dt \underbrace{\left(\frac{1}{2}m\dot{x}^2 - V(x)\right)}_{L\text{ - Lagrangian}} =$$

$$= S[x(t)] \quad \text{- classical action}$$

$$\boxed{\begin{array}{l}\text{Feynmann path integral} \\ G_T(x,y) = \int \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar} S[x(t)]}\end{array}}$$