$\boxed{\mathcal{A}\text{ - Grassmann algebra on } \mathbb{R} \text{ or } \mathbb{C}}$:

generated by a unit 1 and a set of generators $\{\xi_i\}$ satisfying

$$\boxed{\xi_i \xi_j + \xi_j \xi_i = 0 \qquad \forall i, j}$$

2^n - dimensional vector space if $i = 1, \dots, n$

$$\begin{aligned}
\{ & 1 \\
& \xi_1, \xi_2, \dots \xi_n \\
& \xi_1 \xi_2, \xi_1 \xi_3, \xi_1 \xi_4, \dots \xi_{n-1} \xi_n \\
& \xi_1 \xi_2 \xi_3, \xi_1 \xi_2 \xi_4, \dots, \\
& \vdots \\
& \xi_1 \xi_2 \xi_3 \dots \xi_n \}
\end{aligned}$$

$$\mathcal{A} \ni A = a_0 + a_1 \xi_1 + a_2 \xi_2 + \dots + a_{2^n} \xi_1 \xi_2 \dots \xi_n$$

A - is invertible if $a_0 \neq 0$.

e.g. $1 + \xi$, $1 - \xi$ $\qquad (1+\xi)(1-\xi) = 1 + \underbrace{\xi - \xi}_{0} - \underbrace{\xi^2}_{0} = 1$ $\square$

Reflection automorphism P

$$\begin{aligned}
P(A+B) &= P(A) + P(B) \qquad & A, B \in \mathcal{A} \\
P(AB) &= P(A) \cdot P(B) \\
P(\lambda A) &= \lambda P(A) & \lambda \in \mathbb{C}
\end{aligned}$$

and

$$P(\xi_i) = -\xi_i \quad \rightarrow \quad P^2 = 1$$

$$P(\xi_{i_1} \dots \xi_{i_p}) = (-1)^p \xi_{i_1} \dots \xi_{i_p}$$

$$\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^- \quad \text{with} \quad P(A^\pm) = \pm A^\pm$$

also

$$A \xi_i = \xi_i P(A)$$

$A_+ \in \mathcal{A}^+ \Rightarrow A_+ B = B A_+ \ \forall B$

A_- and $B_- \in \mathcal{A}^- \Rightarrow A_- B_- + B_- A_- = 0$

Bosonic conjugation

$$\xi_i \leftrightarrow \xi_i^*$$

$\lambda_i \in \mathbb{C}$

$A_1, A_2 \in \mathcal{A}$

$$(\lambda_1 A_1 + \lambda_2 A_2)^* = \lambda_1^* A_1^* + \lambda_2^* A_2^*$$

$$(A_1 A_2)^* = A_2^* A_1^*$$

if $A^* = A$ – bosonally real.

Differentiation – to the left

$$A = A_1 + A_2 \xi_i$$

Df. $\dfrac{\partial A}{\partial \xi_i} = A_2$ $\qquad \left(\dfrac{\partial}{\partial \xi_i}\right)^2 = 0$

$$\frac{\partial}{\partial \xi_i}(\lambda A + \mu B) = \lambda \frac{\partial A}{\partial \xi_i} + \mu \frac{\partial B}{\partial \xi_i}$$

$A, B \in \mathcal{A}$, $\mu, \lambda \in \mathbb{C}$

$$\frac{\partial}{\partial \xi_i}(AB) = \frac{\partial A}{\partial \xi_i} B + \gamma(A) \frac{\partial B}{\partial \xi_i}$$

not-Leibnitz rule

Integration – to the left

$$\int d\xi_i \, A \equiv \frac{\partial}{\partial \xi_i} A \qquad \forall A \in \mathcal{A}$$

- linear
- integral of total derivatives vanishes
- after integration over ξ_i result does not depend on ξ_i
- factorization

$$\int d\xi_i \; \xi_1 \cdots \xi_i \cdots \xi_j = \xi_1 \cdots \left(\int d\xi_i \, \xi_i\right) \cdots \xi_j$$

$\{1, \xi, \xi^*, \xi^*\xi\}$ - two generator algebra

$$\xi^2 = 0$$
$$\xi\xi^* = -\xi^*\xi$$

$$f(\xi) = f_0 + f_1\xi$$

$$A(\xi^*, \xi) = a_0 + a_1\xi + \bar{a}_1\xi^* + a_{12}\xi^*\xi$$

$$\frac{\partial}{\partial\xi}(\xi^*\xi) = -\frac{\partial}{\partial\xi}(\xi\xi^*) = -\xi^*$$

$$\boxed{\begin{gathered}\frac{\partial}{\partial\xi}\xi = 1 \\ \frac{\partial}{\partial\xi}1 = 0\end{gathered}}$$

derivation

$$\frac{\partial}{\partial\xi}A(\xi^*,\xi) = a_1 - a_{12}\xi^*$$

$$\frac{\partial}{\partial\xi^*}A(\xi^*,\xi) = \bar{a}_1 + a_{12}\xi$$

$$\frac{\partial}{\partial\xi^*}\frac{\partial}{\partial\xi}A(\xi^*,\xi) = -a_{12} = -\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi^*}A(\xi^*,\xi)$$

integration

$$\boxed{\int d\xi\, 1 = 0 \qquad \int d\xi\, \xi = 1}$$

$$\int d\xi^*\, 1 = 0 \qquad \int d\xi^*\xi^* = 1$$

$$\int d\xi \frac{\partial}{\partial\xi}\xi = \xi\Big| \overset{df.}{=} 0$$

$$\int d\xi\, f(\xi) = f_1$$

$$\int d\xi\, A(\xi^*,\xi) = \int d\xi\,(a_0 + a_1\xi + \bar{a}_1\xi^* + a_{12}\xi^*\xi) = a_1 - a_{12}\xi^*$$

$$\int d\xi^*\, A(\xi^*,\xi) = \bar{a}_1 + a_{12}\xi$$

$$\int d\xi^* d\xi\, A(\xi^*,\xi) = -a_{12} = -\int d\xi\, d\xi^*\, A(\xi^*,\xi)$$

δ - Dirac function

$$\delta(\xi,\xi') = \int d\eta\, e^{-\eta(\xi-\xi')} = \int d\eta\,(1-\eta(\xi-\xi')) = -(\xi-\xi')$$

$$\int d\xi'\,\delta(\xi,\xi')f(\xi') = -\int d\xi'\,(\xi-\xi')f(\xi') = -\int d\xi'\,(\xi-\xi')(f_0 + f_1\xi') =$$
$$= f_0 + f_1\xi = f(\xi)$$

scalar product $\qquad g(\xi) = g_0 + g_1\xi$

$$\langle f|g\rangle = \int d\xi^* d\xi\, e^{-\xi^*\xi} f^*(\xi)\, g(\xi^*) =$$
$$= \int d\xi^* d\xi\,(1-\xi^*\xi)(f_0^* + f_1^*\xi)(g_0 + g_1\xi^*) = -\int d\xi^* d\xi\, f_0^* g_0\, \xi^*\xi +$$
$$+ \int d\xi^* d\xi\, \xi\xi^* f_1^* g_1 = f_0^* g_0 + f_1^* g_1$$