$\boxed{\text{Free particle propagator}}$

$$\hat{H} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \qquad \dim = 1$$

$$\phi_p(x) = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \quad , \quad E_p = \frac{p^2}{2m}$$

$$G_T(x,y) = \sum_n \phi_n(x)\phi_n^*(y) e^{-iE_nT/\hbar} =$$

$$= \int \frac{dp}{2\pi\hbar} e^{\frac{i(x-y)p}{\hbar}} e^{-i\frac{p^2}{2m\hbar}T} \Rightarrow$$

$$\int dp\, e^{-ap^2+bp} = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$$

$$\boxed{G_T(x,y) = \left(\frac{m}{2\pi i\hbar T}\right)^{1/2} e^{i\frac{m(x-y)^2}{2\hbar T}}}$$

We compute the same using path integral method. We use discretized version and in the end take $N\to\infty$.

$$G_T(x,y) = \lim_{N\to\infty} B^{-N}\int dx_{N-1}\dots dx_1\, e^{-\frac{m}{2i\hbar\varepsilon}\sum_{n=1}^{N}(x_n - x_{n-1})^2}$$

$$B = \left(\frac{2\pi i\varepsilon\hbar}{m}\right)^{1/2}$$

Factors like $(x_n - x_{n-1})^2$ couple integrals together. To get rid of this we change variables

$$\underline{z_k = x_k - x_{k-1}} \qquad z_k \in \mathbb{R}$$

$$\sum_{k=1}^{N-1} z_k = \cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \dots\dots x_{N-1} - \cancel{x_{N-2}} =$$

$$= \boxed{x_{N-1} - y}$$

B.C.
$x = x_N$
$y = x_0$

let $a = \dfrac{m}{2i\varepsilon\hbar}$

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$$\boxed{G_T(x,y)} = \lim_{N\to\infty} B^{-N} \int dz_{N-1}\, dz_{N-2} \cdots dz_1\, e^{-a(x-x_{N-1})^2 - a\sum_{k=1}^{N-1} z_k^2} =$$

$$= \lim_{N\to\infty} B^{-N} \int dz_{N-1} \cdots dz_1\, e^{-a\sum_{k=1}^{N-1} z_k^2}\, e^{-a\left(x-y-\sum_{k=1}^{N-1} z_k\right)^2} =$$

$$= \lim_{N\to\infty} B^{-N} \int dz_{N-1} \cdots dz_1\, e^{-a\sum_{k=1}^{N-1} z_k^2} \int du\, e^{-a(x-y-u)^2}\, \delta\left(u - \sum_{k=1}^{N-1} z_k\right)^2 =$$

$$= \lim_{N\to\infty} B^{-N} \int dz_{N-1} \cdots dz_1\, e^{-a\sum_{k=1}^{N-1} z_k^2} \int du\, e^{-a(x-y-u)^2} \int \frac{dp}{2\pi}\, e^{ip\left(u - \sum_{k=1}^{N-1} z_k\right)} =$$

$$= \lim_{N\to\infty} B^{-N} \int \frac{dp}{2\pi} \int du\, e^{-a(x-u-y)^2}\, e^{ipu} \underbrace{\prod_{n=1}^{N-1} \int dz_k\, e^{-a z_k^2 - ip z_k}}_{N-1 \text{ Gaussian integrals}} =$$

$$= \lim_{N\to\infty} B^{-N} \int \frac{dp}{2\pi} \int du\, e^{-a(x-u-y)^2}\, e^{ipu} \left[\sqrt{\frac{\pi}{a}}\, e^{-p^2/4a}\right]^{N-1} =$$

$$= \lim_{N\to\infty} \left(\frac{\pi}{a}\right)^{\frac{N-1}{2}} B^{-N} \int \frac{dp}{2\pi} \int du\, e^{-a(x-u-y)^2}\, e^{ipu}\, e^{-(N-1)\frac{p^2}{4a}} =$$

$$= \lim_{N\to\infty} B^{-N} \left(\frac{\pi}{a}\right)^{\frac{N-1}{2}} \frac{1}{2\pi} \sqrt{\frac{\pi}{(N-1)/4a}} \int du\, e^{-a(x-u-y)^2}\, e^{-\frac{u^2}{4(N-1)/4a}} =$$

$$= \lim_{N\to\infty} B^{-N} \left(\frac{\pi}{a}\right)^{\frac{N-1}{2}} \frac{1}{2\pi} \sqrt{\frac{4a\pi}{N-1}} \sqrt{\frac{\pi}{a}}\, e^{-a(x-y)^2/(N-1)} =$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{N/2} \left(\frac{2\pi i\hbar\varepsilon}{m}\right)^{N/2} \sqrt{\frac{a}{\pi/(N-1)}}\, e^{-\frac{a(x-y)^2}{N-1}} =$$

$$= \lim_{N\to\infty} \sqrt{\frac{m}{2\pi i\hbar\varepsilon(N-1)}}\, e^{-\frac{m(x-y)^2}{2i\hbar\varepsilon(N-1)}} = \quad T = N\varepsilon$$

$$\boxed{= \frac{m}{2\pi i\hbar T}\, e^{i\frac{m(x-y)^2}{2\hbar T}}}$$