Grassmann – one-variable Gaussian integral

$$\int d\xi^* d\xi \, e^{-\xi^* a \xi} \underset{\xi^2 = \xi^{*2} = 0}{=} \int d\xi^* d\xi \, (1 - \xi^* a \xi) =$$

$$\int d\xi \, 1 = 0 \qquad \int d\xi^* 1 = 0$$

$$= -a \int d\xi^* d\xi \, \xi^* \xi = a \underbrace{\int d\xi^* \xi^*}_{1} \underbrace{\int d\xi \, \xi}_{1} = \boxed{a}$$

$$\xi \xi^* = -\xi^* \xi$$

•) compare with $z \in \mathbb{C}$

$$\int \frac{dz^* dz}{2\pi i} e^{-z^* a z} = \left\{ \begin{array}{l} z = x + iy \\ z^* = x - iy \\ z^* z = x^2 + y^2 \end{array} \right\} =$$

$$= \int \begin{vmatrix} \frac{\partial z^*}{\partial x} & \frac{\partial z^*}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{vmatrix} \frac{dx\,dy}{2\pi i} e^{-a(x^2+y^2)} =$$

$$= \int \underbrace{\left| \begin{vmatrix} 1 & -i \\ 1 & i \end{vmatrix} \right|}_{2i} \frac{dx\,dy}{2\pi i} e^{-a(x^2+y^2)} = \frac{1}{\pi} \underbrace{\left(\int dx \, e^{-ax^2} \right)}_{\sqrt{\pi/a}} \underbrace{\left(\int dx \, e^{-ay^2} \right)}_{\sqrt{\pi/a}} =$$

$$= \boxed{\frac{1}{a}}$$

•) change of variables

$$\eta = \alpha \xi$$

$$\int d\xi \, \xi = 1$$

$$1 = \int d\eta \, \eta = \int d(\alpha\xi) \, \alpha\xi = \alpha \int d(\alpha\xi) \, \xi \rightarrow \boxed{d(\alpha\xi) = \frac{1}{\alpha} d\xi}$$

$$\int d\xi^* d\xi \, e^{-\xi^* a \xi} = \left\{ \begin{array}{ll} \eta^* = \sqrt{a}\,\xi^*, & d\eta^* = \frac{1}{\sqrt{a}} d\xi^* \\ \eta = \sqrt{a}\,\xi, & d\eta = \frac{1}{\sqrt{a}} d\xi \end{array} \right\} =$$

$$= \int \sqrt{a}\, d\eta^* \sqrt{a}\, d\eta \, e^{-\eta^* \eta} = a \int d\eta^* d\eta \, (1 - \eta^* \eta) =$$

$$= -a \int d\eta^* d\eta \, \eta^* \eta = \boxed{a}$$

$$(-1)$$