Integration - change of variables

$$\int d\xi \, f(\xi)$$

change of variables $\xi = \xi' A + B$

•) preserving parity
$P(\xi) = -\xi$
$P(\xi') = -\xi' \Leftrightarrow \xi' \in \mathcal{A}^-$
$\Rightarrow A \in \mathcal{A}^+ \wedge B \in \mathcal{A}^-$

••) A is invertible $A A^{-1} = 1$

(Then ξ and ξ' are equivalent generators of $\mathcal{A}$)

$$\boxed{d\xi = d(\xi' A + B) = A^{-1} d\xi'}$$

check: $\int d\xi \, \xi = \int d(\xi' A + B)(\xi' A + B) = A^{-1} \int d\xi' (\xi' A + B) =$

$$= A^{-1} \underbrace{\int d\xi' \xi'}_{=1} A + A^{-1} \underbrace{\int d\xi'}_{=0} B = 1$$

$$\int d\xi \, f(\xi) = A^{-1} \int d\xi' f(\xi' A + B) = \int d\xi' \left(\frac{\partial \xi}{\partial \xi'} \right)^{-1} f(\xi(\xi'))$$

inverse of Jacobian!

<u>In general</u> $\xi_i = \xi_i(\xi')$, $\xi_i' \in \mathcal{A}^-$

where $\left(\frac{\partial \xi_i}{\partial \xi_j'} \right)$ invertible matrix.

$$d\xi_1 \cdots d\xi_n = d\xi_1' \cdots d\xi_n' \left| \frac{\partial \xi_i'}{\partial \xi_j} \right| =$$

$$= d\xi_1' \cdots d\xi_n' \left| \frac{\partial \xi_i}{\partial \xi_j'} \right|^{-1} =$$

$$= d\xi_1' \cdots d\xi_n' \, J(\xi')$$

inverse of Jacobian, determinant of Jacobi matrix.

<u>proof</u>: We consider a simple case

$$\xi_i = \sum_j M_{ij} \xi_j'$$

where
$$M_{ij} = \frac{\partial \xi_i}{\partial \xi_j'}$$

In the integral

$$\int d\xi_1 \dots d\xi_n \, f(\xi_1 \dots \xi_n)$$

the nonvanishing contributions arise from terms like
$$\int d\xi_1 \dots d\xi_n \; \xi_1 \dots \xi_n = \int d\xi_1 \dots d\xi_n \prod_{i=1}^{n} \xi_i$$

so we need to find J in

$$\underbrace{\int d\xi_1 \dots d\xi_n \prod_{i=1}^{n} \xi_i}_{(-1)^n} = J \underbrace{\int d\xi_1' \dots d\xi_n' \prod_{i=1}^{n} \Big(\sum_j M_{ij} \xi_j'\Big)}$$

non-vanishing contributions arise from $n!$ distinct permutations P of the $\{\xi_i'\}$ variables generated by the product.

$$(-1)^n = J \int d\xi_1' \dots d\xi_n' \prod_{i=1}^{n} \Big(\sum_j M_{ij} \xi_j'\Big) =$$

$$= J \int d\xi_1' \dots d\xi_n' \sum_P \prod_i M_{iP_i} \xi'_{P_i} =$$

$$= J \sum_P \prod_i M_{iP_i} (-1)^P \underbrace{\int d\xi_1' \dots d\xi_n' \; \xi_1' \dots \xi_n'}_{(-1)^n} =$$

$$= J (\det M)(-1)^n$$

$$\Rightarrow \boxed{J = (\det M)^{-1} = \left| \frac{\partial \xi_i'}{\partial \xi_j} \right|}$$

□

Proof (in general)

change variables one at a time and use

$$\int d\xi\, f(\xi) = \int d\xi' \left(\frac{\partial \xi'}{\partial \xi}\right) f(\xi(\xi'))$$

$$J = \left.\frac{\partial \xi_1'}{\partial \xi_1}\right|_{\xi_2 \cdots \xi_n} \cdot \left.\frac{\partial \xi_2'}{\partial \xi_2}\right|_{\xi_1' \xi_3 \cdots \xi_n} \cdots \left.\frac{\partial \xi_{n-1}'}{\partial \xi_{n-1}}\right|_{\xi_1' \cdots \xi_{n-2}' \xi_n} \left.\frac{\partial \xi_n'}{\partial \xi_n}\right|_{\xi_1' \cdots \xi_{n-1}'}$$

Introducing the matrices

$$M_{ij}^{(p)} = \frac{\partial \xi_i'}{\partial \xi_j} \qquad i,j \leq p \leq n$$

and using the chain rule

$$\frac{\partial}{\partial \theta} f(\sigma, \mu) = \frac{\partial \sigma}{\partial \theta}\frac{\partial f}{\partial \sigma} + \frac{\partial \mu}{\partial \theta}\frac{\partial f}{\partial \mu}$$

if $\sigma(\theta) \in A^-$, $\mu(\theta) \in A^+$

we get the recursion relation

$$\left.\frac{\partial \xi_n'}{\partial \xi_n}\right|_{\xi_1' \cdots \xi_{n-1}'} = \left.\frac{\partial \xi_n'}{\partial \xi_n}\right|_{\xi_1 \cdots \xi_{n-1}} - \sum_{i,j<n} \frac{\partial \xi_n'}{\partial \xi_i} \left[M^{(n-1)}\right]^{-1}_{ij} \frac{\partial \xi_j'}{\partial \xi_n}$$

$$= \det M^{(n)} \left[\det M^{(n-1)}\right]^{-1}$$

Hence,

$$d\xi_1 \cdots d\xi_n = d\xi_1' \cdots d\xi_n' \left(\det M^{(n)}\right)$$

$\square$

Example

$$1 = \int d\xi_1 \cdots d\xi_n\, \xi_n \cdots \xi_1$$

$$\xi_i = \sum_j a_{ij} \xi_j'$$

$$d\xi_1 \cdots d\xi_n = d\xi_1' \cdots d\xi_n' \left(\det a\right)$$