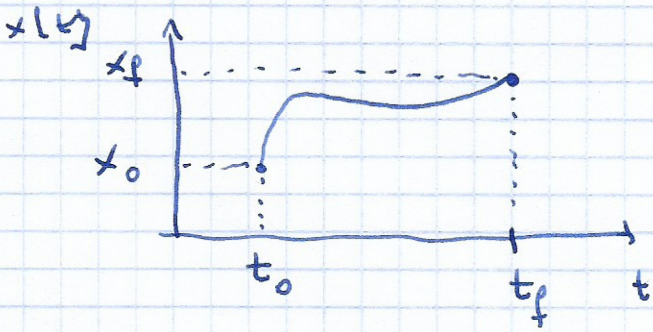
operators from Path integrals

For a given trajectory we can define a classical momentum $m\dot{x}(t)$. Let's define at t_f a quantum momentum operator as

$$\hat{p}\,\Psi(x_f,t_f) \equiv \int dy \int \mathcal{D}[x(t)]\; m\,\dot{x}(t_f)\; e^{\frac{i S[x]}{\hbar}}\; \Psi(y,t_0)$$



$t_0 < t_f$

Let $t = t_f$, $t_0 = t - \varepsilon$, and $\varepsilon \to 0$

$$\boxed{\hat{p}\,\Psi(x,t)} \equiv \lim_{\varepsilon\to 0} \int dy\; m\left(\frac{x-y}{\varepsilon}\right) G_\varepsilon(x,y)\,\Psi(y,t-\varepsilon) =$$

(problem 1 page 4 (*))

$$= \lim_{\varepsilon\to 0}\int dy\; m\left(\frac{x-y}{\varepsilon}\right)\left(\frac{m}{2\pi i\varepsilon\hbar}\right)^{1/2} e^{\,i\frac{m}{2\varepsilon\hbar}(x-y)^2 \underbrace{-\,i\frac{\varepsilon}{\hbar}V(x)}_{\downarrow\,\varepsilon\to 0}}\;\Psi(y,t-\varepsilon)$$

$$= \lim_{\varepsilon\to 0}\int dy\; m\left(\frac{x-y}{\varepsilon}\right)\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} e^{\,i\frac{m}{2\varepsilon\hbar}(x-y)^2}\;\Psi(y,t-\varepsilon) =$$

$\{z = y - x\}$

$$= \lim_{\varepsilon\to 0}\left(\frac{m}{2\pi i\varepsilon\hbar}\right)^{1/2}\frac{m}{\varepsilon}\int dz\,(-z)\,e^{\,i\frac{m}{2\varepsilon\hbar}z^2}\;\Psi(x+z,t-\varepsilon) \simeq$$

$$\simeq \lim_{\varepsilon\to 0}\left(\frac{m}{2\pi i\varepsilon\hbar}\right)^{1/2}\frac{m}{\varepsilon}\int dz\,(-z)\,e^{\,i\frac{m}{2\varepsilon\hbar}z^2}\left[\Psi(x,t-\varepsilon) + \Psi'(x,t-\varepsilon)\,z + \mathcal{O}(z^2)\ldots\right]$$

$\int dz\, e^{iaz^2} z = 0$

$$= -\lim_{\varepsilon\to 0}\left(\frac{m}{2\pi i\varepsilon\hbar}\right)^{1/2}\frac{m}{\varepsilon}\frac{d\Psi}{dx}\int dz\, z^2\, e^{\,i\frac{m}{2\varepsilon\hbar}z^2} =$$

$$= -\lim_{\varepsilon\to 0}\left(\frac{m}{2\pi i\varepsilon\hbar}\right)^{1/2}\frac{m}{\varepsilon}\frac{d\Psi}{dx}\left(-\frac{\pi}{i\,m/2\varepsilon\hbar}\right)^{1/2}\frac{i\varepsilon\hbar}{m} = \boxed{-i\hbar\,\frac{d\Psi(x,t)}{dt}}$$