Grassmann Gaussian integral

$$\boxed{\int \prod_{i=1}^{n} d\eta_i^* d\eta_i \, e^{-\sum_{ij} \eta_i^* H_{ij} \eta_j + \sum_i \xi_i^* \eta_i + \xi_i \eta_i^*} = [\det H] \, e^{\sum_{ij} \xi_i^* (H^{-1})_{ij} \xi_j}}$$

<u>Proof</u>: we define a diagonalizing transformation

$$\theta_i = \eta_i - \sum_j H_{ij}^{-1} \xi_j$$

$$\theta_i^* = \eta_i^* - \sum_j H_{ij}^{-1} \xi_j^* \qquad \text{shift}$$

and

$$\sigma_i = \sum_j U_{ij}^{-1} \theta_j \qquad \text{unitary}$$

$$\sigma_i^* = \sum_j U_{ij}^{-1*} \theta_j^*$$

Einstein convention

$$\int \prod_{i=1}^{n} d\eta_i^* d\eta_i \, e^{-\eta_i^* H_{ij} \eta_j + \xi_i^* \eta_i + \xi_i \eta_i^*} =$$

$$= \int \prod_{i=1}^{n} d\theta_i^* d\theta_i \, e^{-\theta_i^* H_{ij} \theta_j} \; e^{\sum_{ij} \xi_i^* (H_{ij}^{-1}) \xi_j}$$

$$= \int \prod_{i=1}^{n} d\sigma_i^* d\sigma_i \, e^{-\sum_i h_i \sigma_i^* \sigma_i} \overbrace{= \prod_{i=1}^{n} h_i}^{\cdot e^{\sum_{ij} \xi_i^* H_{ij}^{-1} \xi_j}} \overbrace{= [\det H]}^{e^{\sum_{ij} \xi_i^* H_{ij}^{-1} \xi_j}} e^{\sum_{ij} \xi_i H_{ij}^{-1} \xi_j}$$

*) all Jacobians are unity. $\square$

Show that

- $$\int \prod_{i=1}^{n} d\eta_i^* d\eta_i \; \eta_k \eta_l^* \, e^{-\sum_{ij} \eta_i^* H_{ij} \eta_j} = (\det H)(H^{-1})_{kl}$$