AGQM problems

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A number of problems, prepared by dr A. Kamada you can download from:

`https://www.dropbox.com/s/fu1z307frptpge5/homeworks.pdf?dl=0`

1 Problems

1. Consider a particle moving in a one-dimension in a periodic potential
$$V(x) = V_0 \cos(\frac{2\pi x}{a}).$$
Is the momentum conserved? Why? Recollect the Bloch theorem form earlier classes. Which is the quantity, with the same unit as the momentum, is conserved in this potential and in which sense?

2. Let at $t = 0$ the system is in a even (odd) wave function state. Show that this is conserved in later (earlier) times t if the Hamiltonian is parity symmetric.

3. Space is invariant under the (scale) transformation $\vec{r} \to \vec{r}' = e^c \vec{r}$, where c is a parameter. The corresponding Wigner operator is $\hat{U}_c = e^{-ic\hat{D}}$, where $\hat{D}$ is dilation generator. Find $\hat{D}$ in position representation and determine the commutator $[\hat{D}, \hat{\vec{P}}]$, where $\hat{\vec{P}}$ is a momentum operator.

4. The unitary operator $\hat{U}(\vec{V}) = e^{i\vec{V}\cdot\hat{\vec{G}}}$ describes the instantaneous ($t = 0$) effect of a transformation to a frame of reference moving at the velocity $\vec{V}$ with respect to the laboratory frame, i.e.
$$\hat{U}\vec{v}\hat{U}^{-1} = \vec{v} - \vec{V}\hat{I},$$
$$\hat{U}\vec{r}\hat{U}^{-1} = \vec{r}.$$
Find an operator $\hat{\vec{G}}_t$ such that the unitary operator $\hat{U}(\vec{V}, t) = e^{i\vec{V}\cdot\,\hat{\vec{G}}_t}$ will yield the full Galilei transformation
$$\hat{U}\vec{v}\hat{U}^{-1} = \vec{v} - \vec{V}\hat{I},$$
$$\hat{U}\vec{r}\hat{U}^{-1} = \vec{r} - \vec{V}t\hat{I}.$$

5. Gauge transformation is given by
$$\Psi'(\mathbf{r}, t) = U(\chi)\Psi(\mathbf{r}, t) = e^{\frac{iq}{\hbar}\chi(\mathbf{r},t)}\Psi(\mathbf{r}, t),$$
$$\mathbf{A}'(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \nabla\chi(\mathbf{r}, t),$$
$$\Phi'(\mathbf{r}, t) = \Phi(\mathbf{r}, t) - \frac{\partial\chi(\mathbf{r}, t)}{\partial t}.$$
a) Prove explicitly that the Schrodinger equation is gauge invariant.
b) Physical observables are Hermitian operators $\hat{O}$ and gauge covariant, i.e. under a gauge transformation they obey
$$\hat{O}' = U\hat{O}U^\dagger.$$
Explain why why gauge covariant observable have gauge invariant expectation values
$$\langle\Psi'|\hat{O}'|\Psi'\rangle = \langle\Psi|\hat{O}|\Psi\rangle.$$
Answer with brief explanations:
- Is the sum of gauge covariant operators a gauge covariant operator?
- Is the product of gauge covariant operators a gauge covariant operator?
- Is $\hat{x}_i$ (position operator) gauge covariant?
- Is $\hat{p}_i$ (momentum operator) gauge covariant?
- Is $\hat{v}_i = \frac{1}{m}(\hat{p}_i - qA_i)$ gauge covariant?

6. The probability current, in the presence of an electromagnetic field, is given by
$$\mathbf{J}(\mathbf{r}, t) = \frac{1}{m}\Re\left[\Psi^*\left(-i\hbar\nabla - q\mathbf{A}\right)\Psi\right] \equiv \Re(\Psi^*\hat{\mathbf{v}}\Psi).$$
a) Derive the continuity equation
$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0,$$
where $\rho = |\Psi|^2$.
b) Show that $\mathbf{j} = q\mathbf{J}$ has units of the electric current density.
c) Show that $\mathbf{J}(\mathbf{r}, t)$ is gauge invariant.

7. Imagine a particle constrained to move on a circle of radius b. Along the axis of the circle there runs a solenoid of radius $a < b$, carrying a magnetic induction $\mathbf{B} = (0, 0, B)$. The field inside the solenoid is uniform and zero outside it.
a) In cylindrical coordinates the vector potential has form $A_r = A_z = 0$ and $A_\phi = \alpha(r)$. Find the function $\alpha(r)$.
b) Find the appropriate boundary conditions on the wave functions and the corresponding Hilbert space. Solve the Schrodinger equation (determining eigenvalues and eigenfunctions) in this potential for a particle constrained to move on a circle of radius b.
c) Plot and discuss the energy eigenvalues as a function of a magnetic flux Φ. Finds parallels with the Aharonov-Bohm effect.
d) Find the probability current density for a given

eigenfunction and discuss the result.
e) Suppose that on the circle of radius b there exists a trapping potential such that the wave function $\psi(\phi)$ has a finite support for $\phi \in (-\phi_0, \phi_0)$, with $\phi_0 < \pi$. Show that the binding energy does not depend on the existence of a solenoid.

8. Consider a single level system with the Green's function
$$G_0(z) = \frac{1}{z - \epsilon}.$$
a) Find the density of states and the occupation of the state with a given chemical potential μ in case of the Fermi-Dirac statistics (hint: $n = \int \rho(\omega) f_{FD}(\omega) d\omega$). Discuss $T = 0$ case separately.
The system is perturbed by the external potential V shifting the energy level and giving the Green's function
$$G(z) = \frac{1}{z - \epsilon - V}.$$
b) Find the change in the density of states $\Delta\rho(\omega)$ and in the occupation.
c) Express the change $\Delta\rho(\omega)$ in terms of the T-matrix, which you need to find.
d) Express the change $\Delta\rho(\omega)$ in terms of the scattering phase shift, which you need to find.
e) Repeat c) and d) for the change in the occupation.
Hint: $x \pm i0^+ = |x| e^{\pm i\pi\Theta(-x)}$.

9. A two-site problem is described by the Hamiltonian
$$\hat{H} = t(|r\rangle\langle l| + h.c.).$$
The system is perturbed by the potential
$$\hat{V} = -v|l\rangle\langle l|.$$
a) Find the Green's functions for the unperturbed and perturbed problems.
b) Find the T-matrix and the scattering phase shift matrix for the perturbed problem.
c) Express the change in the density of states and the level occupations (in case of fermions with the chemical potential μ and the temperature T and discuss $T = 0$ case explicitely) in terms of the T-matrix and in terms of the phase shift matrix elements.

10. A spin 1/2 particle in one-dimension is described by the following Pauli Hamiltonian
$$\hat{H} = -\mathbf{I}\frac{\hbar^2}{2m}\frac{d^2}{dz^2} + \begin{pmatrix} V_1(z) & 0 \\ 0 & V_2(z) \end{pmatrix} - \mu B \sigma_x,$$
where μ is the magnetic dipole moment, B is the magnetic field oriented into the x-axis, σ_x is the x-Pauli matrix and $\mathbf{I}$ is the 2×2 identity matrix. Assuming $V_1(z) = -g_1\delta(z)$ and $V_2(z) = -g_2\delta(z)$, with positive and different g_1 and g_2 solve the problem for bound states, continuum states and for resonant states. Consider three energy sectors:
i) $E < 0$ both channels are closed;
ii) $0 < E < 2\mu B$ one channel is closed the other is open;
iii) $E > 2\mu B$ both channels are open.
Discuss results and limits.
Hint: See Am J. Phys. **81**, 603 (2013) for solutions.

11. A particle is moving in one-dimension in the presence of two-Dirac-delta potentials, i.e.
$$\hat{V} = v\delta(x - a) + v\delta(x + a),$$
with $v > 0$. Solve the Schrodinger equation of this problem for normal boundary conditions, i.e. initially the particle is approaching from the left side, and for outgoing boundary conditions. Discuss existence and origin of the resonance states.
Hint: Use Mathematica and see for solutions arXiv:0705.1388.